Identification of Time label in dynamical system

1 Results

We demonstrate the methods on a variaty of dynamic systems, ranging from simple linear and nonlinear oscillators, to noisy measurements of the fully chaotic Lorenz system.

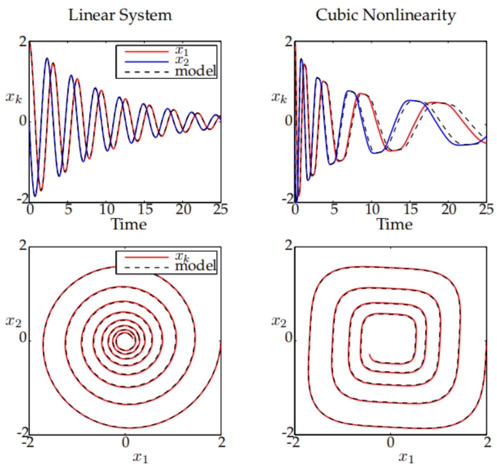
1.1 Example 1:Simple illustrative system

1.1.1 Example 1a: Two-dimensional damped oscillator (linear vs. nonlinear)

In this example , we consider the two-dimensional damped hamonic oscillator with either linear or cubic dynamic, as in

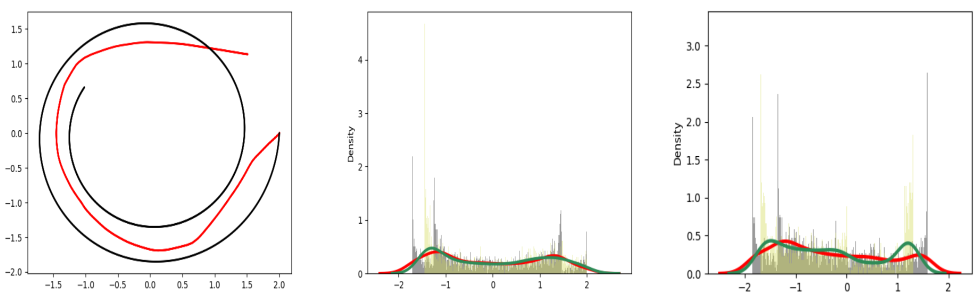
$$\frac{d}{\mathrm{dt}}\begin{bmatrix} q \\ p \end{bmatrix} = \begin{bmatrix} -0.1 & 2 \\ -2 & -0.1 \end{bmatrix}\begin{bmatrix} q \\ p \end{bmatrix} \tag{1}$$

$$\frac{d}{\mathrm{dt}}\begin{bmatrix} q \\ p \end{bmatrix} = \begin{bmatrix} -0.1 & 2 \\ -2 & -0.1 \end{bmatrix}\begin{bmatrix} q^3 \\ p^3 \end{bmatrix} \tag{2}$$

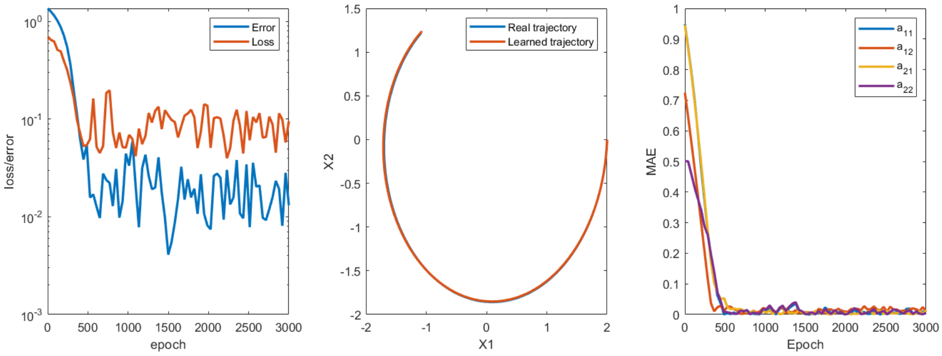


For Linear 2D system and Cubic 2D system, the initial condition is taken as $(p_0, q_0) = [2, 0]$.

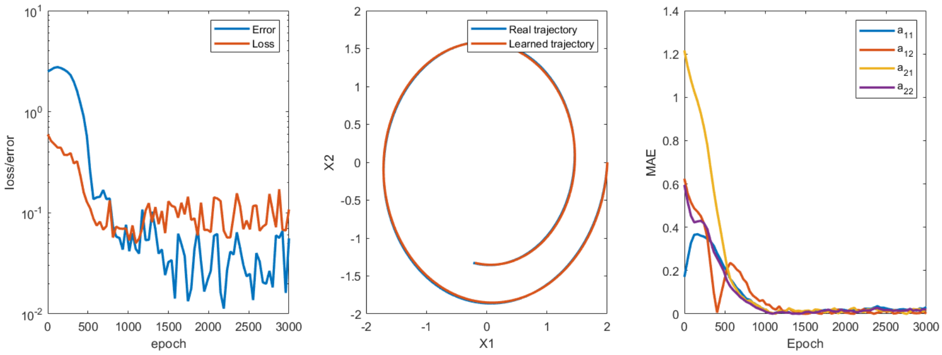
For linear2D system , a large terminal time $t = 2$ may cause non uniqueness of solution:



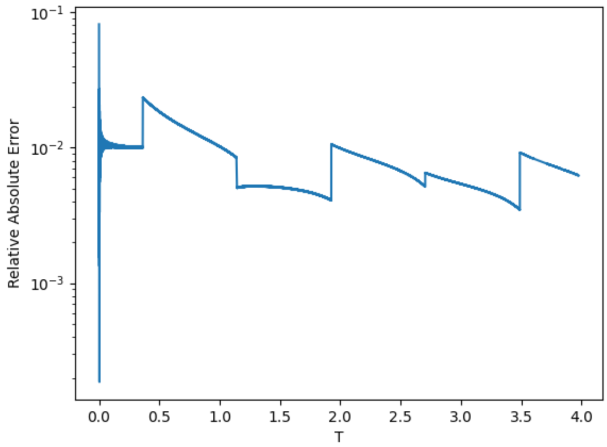
Here the first picture is the phase diagram of linear 2D system(Black is the real system,red is the learned system), while the second and the third picture is the density function of learned system and real system in dimension 1 and 2.



Above is T=2 Case

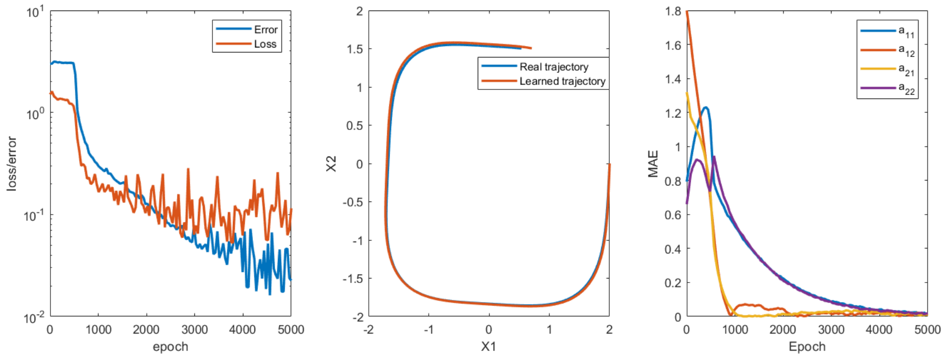


Above is T=4 Case, the time reconstruction result is



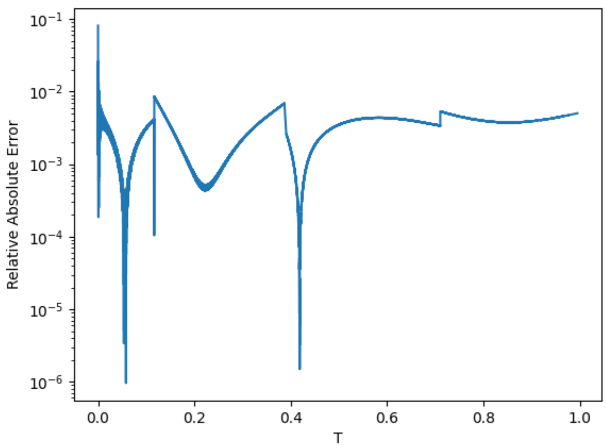
The sindy reconstruction result is [-0.119,2.009;-2.019,-0.084] for order 2 dictionary using STLSQ.

Cubic 2D:



Theta = [-0.119,-2.0221;1.9986,-0.0800];

The time reconstruction result is



The sindy reconstruction result is

$$d_t x_0 = 0.041 x_1 - 0.099 x_0^3 + 1.995 x_1^3$$

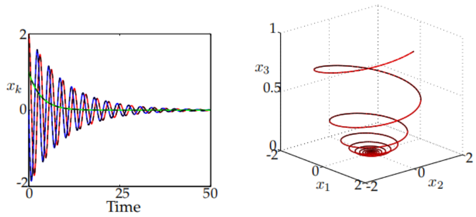
$$d_t x_1 = -0.052 x_0 - 1.992 x_0^3 - 0.102 x_1^3$$

for order 3 dictionary using STLSQ.

1.1.2 Example 1b: Three-dimensional linear system

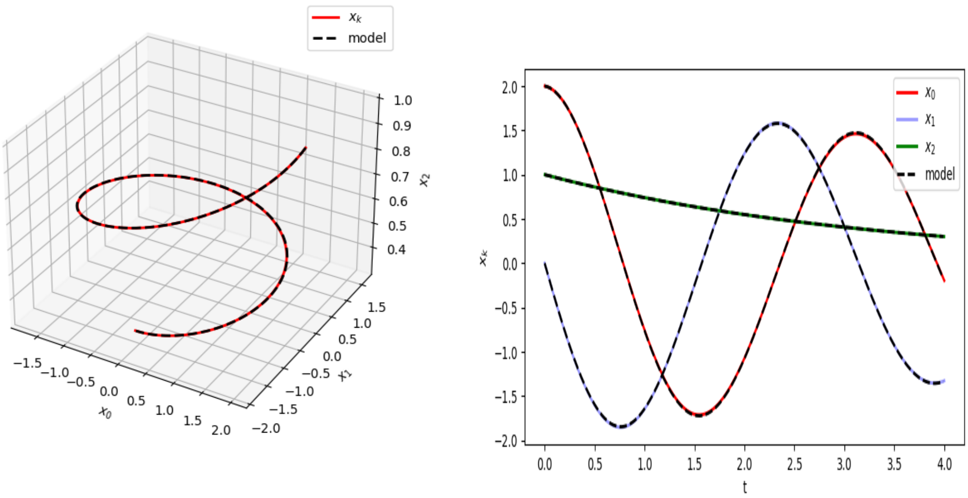
A linear system with three variables and the sparse approximation are shown below. In this case, the dynamics are given by

$$\frac{d}{\mathrm{dt}}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -0.1 & -2 & 0 \\ 2 & -0.1 & 0 \\ 0 & 0 & -0.3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{3}$$



The original terminal time is $T = 50$, initial condition is $x_1, x_2, x_3 = [2{,}0{,}1]$.

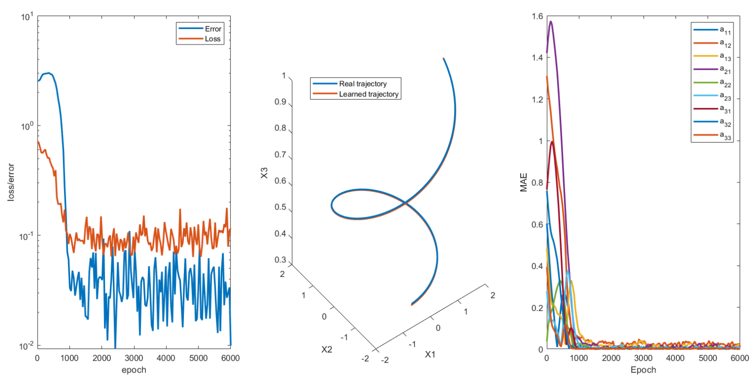
Here we take terminal time to be $T = 4$, the phase and time-value diagram is



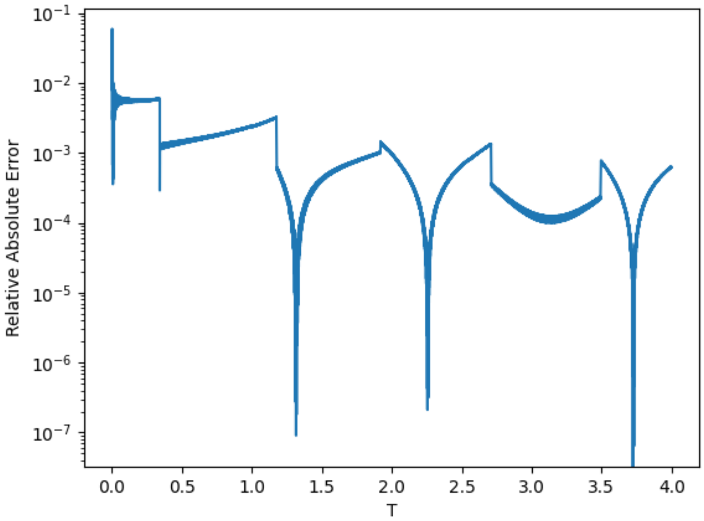
The result is [[-1.0330e-01, -1.9849e+00, -9.5114e-04],

[2.0151e+00, -9.2906e-02, 1.0624e-03],

[6.0379e-03, -5.6568e-03, -2.9889e-01]]



The time reconstruction result is



The sindy reconstruction result is

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(x0)' = -0.101 x0 + 2.009 x1
(x1)' = -1.999 x0 + -0.098 x1
(x2)' = -0.300 x2
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for order 1 dictionary using STLSQ.

2 Direct Reconstruct

2.1 Problem Setting

Given an nonlinear ODE system

$$\begin{aligned} \frac{\mathrm{dy}}{\mathrm{dt}} &= f(y,t) \\ y(0) &= y_0 \end{aligned} \tag{4}$$

The solution of this system in interval $t \in [0,T]$ is given by $y(t)$. Suppose one sample observation in a uniform distributional manner, that is $X \sim y(\mathcal{U}[0,T])$, the total observation then become the emperical distribution $\{x_i\}_{i=1}^N \sim X$. So the question is can we recover $y(t)$ or $f(y,t)$ from such observation.

2.2 Theoretical Analysis

定理 1. Any non-descreasing left-continuous normal Function on $\mathbb{R}$ is a distribution function of a random variable.

证明. Let probability space be $(\Omega, \mathcal{F}, P)$, $Y \sim \mathcal{U}[0,1]$, then consider a random variable $X = F^{-1}(Y)$, we then prove that the distribution function of X is $F(x)$.

First,

$$F(x) > u \Leftrightarrow x > F^{-1}(u) \tag{5}$$

and for any $x \in \mathbb{R}$, $F(x) \in \mathbb{R}$, thus

$$(X < x) = (F^{-1}(Y) < x) = (Y < F(x)) \in \mathcal{F} \tag{6}$$

thus X is a random variable , notice that Y is a uniform distribution,

$$P(X < x) = P(F^{-1}(Y) < x) = P(Y < F(x)) = F(x) \tag{7}$$

□

This theorem shows that in 1-dimensional observation ,if the distributional function is observed as $F(x)$ then the transformation will be $y(x) = F^{-1}(x)$

More generally, suppose we have a random variable X and a transformation function $F(\cdot)$, set $Y = F(X)$. suppose F is 1-1 map, moreover which is strictly increasing or descreasing which implies that

$$P(Y \leqslant F(x)) = P(X \leqslant x) \tag{8}$$

therefore for $y = F(x)$

$$P_Y(y) = P_Y(F(x)) = P_X(x) \tag{9}$$

This relationship between CDFs leads directly to the relationship between their PDFs. If we assume this derivative is greater than 0, differentiating gives

$$p_Y(y)\frac{\mathrm{dF}}{\mathrm{dx}}(x) = p_X(x) \tag{10}$$

Then

$$p_Y(y) = \left|\frac{\mathrm{dF}}{\mathrm{dx}}\right|^{-1} p_x(x) = \left|\frac{\mathrm{dF}}{\mathrm{dx}}\right|^{-1} (F^{-1}(y)) p_x(F^{-1}(y)) \tag{11}$$

Here for X is uniformly distributed over $[0,1]$ then $p_x = 1$, for previous theorem the $Y = F^{-1}(X)$ follows $P_Y(y) = F(y)$

$$1 = p_X(x) = \frac{\mathrm{dF}}{\mathrm{dx}}\frac{\mathrm{dF}^{-1}}{\mathrm{dx}} \tag{12}$$

定理 2. Let X be random variable with distributional function $F_X(x)$ and probability density function $p_X(x)$, the transformation $u = g(t)$ is a strictly increasing or decreasing , whose inverse function is given by $h(u) = g^{-1}(u)$ is a smooth function (at least first order differentiable) in $[\alpha, \beta]$, then $Y = g(X)$ has density function

$$p_Y(x) = p_X(h(x))|h'(x)| = p_X(g^{-1}(x))|g^{-1\prime}(x)|, x \in [\alpha, \beta] \tag{13}$$

证明. Assuming that g is strictly increasing, then

$$F_Y(x) = P(Y < x) = P(g(X) < x) = P(X < h(x)) = F_X(h(x)) \tag{14}$$

then

$$p_Y(x) = \frac{d}{\mathrm{dx}} F_X(h(x)) = p_X(h(x)) h'(x) \tag{15}$$

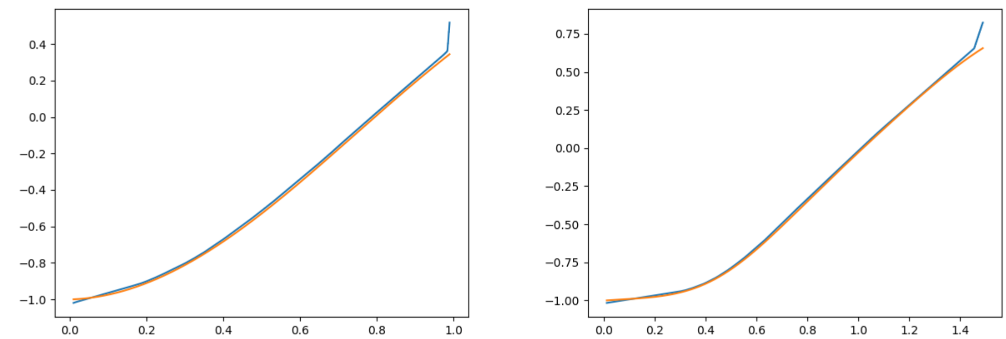
□

So our problem is we have $p_X(x)$ and $p_X(h(x))h'(x)$ we want to recover $g(x)$ or $h(x) = g^{-1}(x)$.

2.3 Numerical Example

Take Cubic2D for example ,

1. First estimate F(x) using emperical method.
2. Learning $F^{-1}(x)$ using Neural network.
3. Evaluating t at F^{-1}

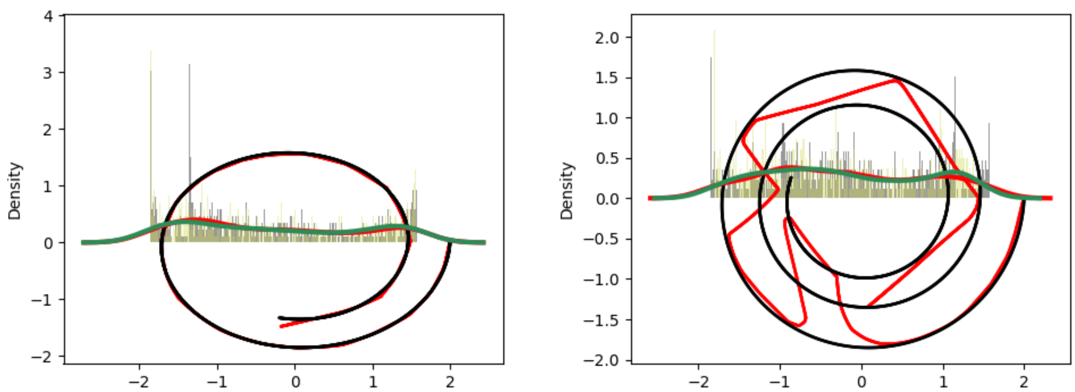


The left result is the reconstruction of y(t) for dimension 0 for $t \in [0, 1]$, while the right are $t \in [0, 1.5]$.

3 Direct Learning $y(t)$ map using Neural network

Using Deep Neural network as an approximator to learn the map $y : t \to y(t)$, the loss function is wasserstein distance and all other setting remains the same.

To enforce the uniqueness of solution , we add boundary loss to make sure the initial condition are obeyed.



The left are linear2D example for $t \in [0,4]$, after 30000 epoch training the Neural network successfully converge to the right distribution. The right are $t \in [0,8]$, the Neural Network failed here.

4 To be finished

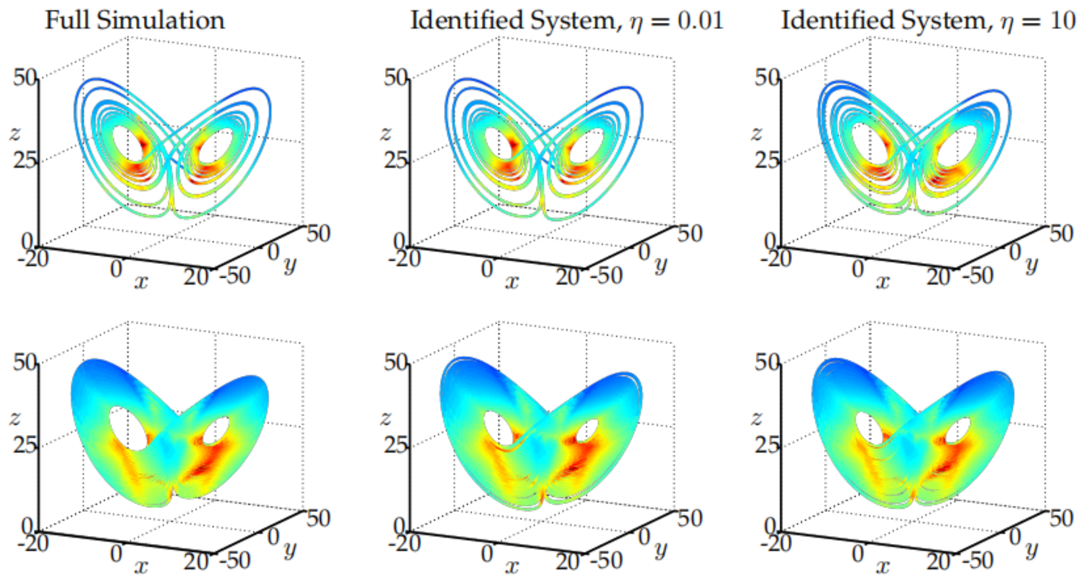
4.1 Example 2: Lorenz system(Nonlinear ODE)

Here we consider the nonlinear Lorenz system to explore the identification of chaotic dynamics evolving on an attractor, shown in

$$\begin{aligned} \dot{x} &= \sigma(y-x) \\ \dot{y} &= x(\rho - z) - y \\ \dot{z} &= \text{xy} - \beta z \end{aligned} \tag{16}$$

Although these equations give rise to rich and chaotic dynamics that evolve on an attractor, there are only a few terms in the right-hand side of the equations.Here the right-hand side dynamics are identified in the space of polynomials $\Theta(\boldsymbol{X})$ in (x, y, z) up to fifth order.

Zero-mean Gaussian measurement noise with variance η is added to the derivative calculation to investigate the effect of noisy derivatives. The short-time $t=0$ to $t=20$ and long-time $t=0$ to $t=250$ system reconstruction is shown for two different noise values, $\eta = 0.01$ and $\eta = 10$.



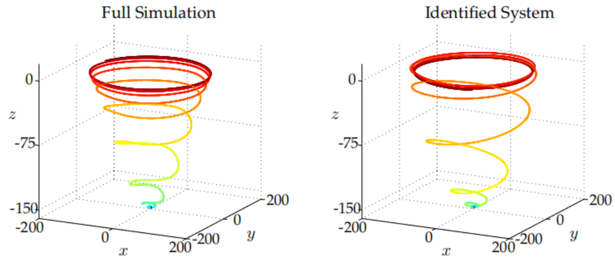
For this example, we use the standard parameters $\sigma = 10$, $\beta = \frac{8}{3}$, $\rho = 28$, with an initial condition $[x,y,z]^T = [-8,7,27]^T$, original data sample size 100000.

4.2 Example 3: Mean field model of Fluid wake behind a cylinder

In the three-dimensional coordinate system described above, the mean-fifield model for the cylinder dynamics are given by:

$$\begin{aligned} \dot{x} &= \mu x - \omega y + \text{Axz} \\ \dot{y} &= \omega x + \mu y + \text{Ayz} \\ \dot{z} &= -\lambda(z - x^2 - y^2) \end{aligned} \tag{17}$$

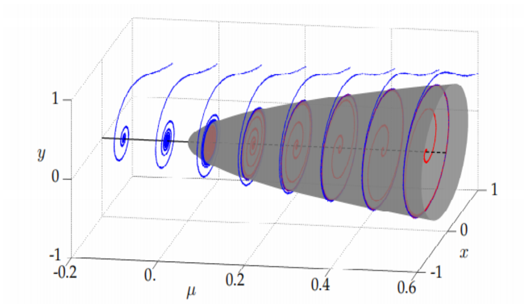
If λ is large, so that the z-dynamics are fast, then the mean flow rapidly corrects to be on the (slow)manifold $z = x^2 + y^2$ given by the amplitude of vortex shedding. When substituting this algebraic relationship , we recover the Hopf normal form on the slow manifold.



4.3 Example: Hopf normal form

The fifinal example illustrating the ability of the sparse dynamics method to identify parameterized normal forms is the Hopf normal form. Noisy data is collected from the Hopf system

$$\begin{aligned} \dot{x} &= \mu x + \omega y - \mathrm{Ax}(x^2 + y^2) \\ \dot{y} &= -\omega x + \mu y - \mathrm{Ay}(x^2 + y^2) \end{aligned} \quad (18)$$



The initial condition is $x = 0, y = 0$.