Weak Collocation Regression For Parameter Identification Chaotic Dynamics Via Invariant Mesure

BY ZHIJUN ZENG

Date:03/22

1 Dynamic System Setting

Noisy observations are denoted as

$$\mathbf{X}^* = (\mathbf{x}^*(t_0) + \eta_0, \mathbf{x}^*(t_1) + \eta_1, \cdots, \mathbf{x}^*(t_n) + \eta_n) \tag{1}$$

where $x^*$is the solution of the autonomous dynamical system of $\dot{x} = v(x, \theta^*)$, and $\{\eta_0, \eta_1, \ldots, \eta_n\}$are the measurement errors or uncertainties.

We suppress the time variable and consider the state-space distribution of data

$$\rho^* = \frac{\sum_{i=0}^{n} \delta_{x^*(t_i)}}{n+1} \tag{2}$$

The dynamic system admits a physical measure $\rho(\theta)$ if for a Lebesgue positive set of initial condition $x(0) = x,$ one has that

$$\rho(\theta) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \delta_{x(t)} \mathrm{dt} \tag{3}$$

Mathematically, statistical properties of can be characterized by the occupation measure $\rho_{x,T}$ defined as

$$\rho_{x,T}(B) = \frac{1}{T} \int_0^T \mathbb{1}_B(\mathbf{x}(s))\, d\,s = \frac{\int_0^T \mathbb{1}_B(\mathbf{x}(s))\, d\,s}{\int_0^T \mathbb{1}_{\mathbb{R}^d}(\mathbf{x}(s))\, d\,s} \tag{4}$$

where B is any Borel Measurable set.If there exist an invariant measure ρ such that aull $\rho_{x,T}$ weakly converges to ρ for all initial condition, then ρ is an physical measure .

By definition of physical measures μ^*, we have

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f(x(t))\, d\,t = \int_{\mathbb{R}^d} f(x)\, d\,\mu^*(x), \quad f \in C_c^\infty(\mathbb{R}^d) \tag{5}$$

By taking $f(x) = \nabla \phi(x) \cdot v(x)$ for some $\phi \in C_c^\infty(\mathbb{R}^d)$

$$\begin{aligned}\int_{\mathbb{R}^d} \nabla\, \phi(x) \cdot v(x)\, d\,\mu^*(x) &= \lim_{T \to \infty} \frac{1}{T} \int_0^T \nabla\, \phi(x(t)) \cdot v(x(t))\, d\,t = \lim_{T \to \infty} \frac{1}{T} \int_0^T \nabla\, \phi(x(t)) \cdot \dot{x}(t)\, d\,t \\ &= \lim_{T \to \infty} \frac{1}{T} (\phi(x(T)) - \phi(x(0))) = 0\end{aligned} \tag{6}$$

This shows that μ^* is the stationary distributional solution

$$\nabla \cdot (v(x, \theta) \rho(x, \theta)) = 0. \tag{7}$$

2 Weak Collocation Regression Method

Using Galerkin method, one can derivate the weak form of (7)

$$\int_\Omega v(x,\theta)\cdot\nabla\phi(x)\,\rho(x,\theta)\mathrm{dx}=\mathbb{E}_{\rho(x,\theta)}(v(x,\theta)\cdot\nabla\phi(x))=0,\forall\phi\in C^\infty(\Omega)$$

Taking $\phi(x)=\{\rho_{\mu,\Sigma}(x)$: the density function of $\{N(\mu_i,\Sigma_i)\}_{i=1}^N\}$, we can construct N nonlinear(linear in Linear ODE systems) equations

$$F_j(\theta)=0, j=1,2,\ldots N$$

The equations are solved by Newton method or Gradient descent method. In the second case, the loss function is chosen as $\|\boldsymbol{F}(\theta)\|_2^2$.

Taking Lorentz equation as an example

$$\left\{\begin{array}{ccc}\dot{x} & = & \sigma(y-x)\\ \dot{y} & = & x(\rho-z)-y\\ \dot{z} & = & \mathrm{xy}-\beta z\end{array}\right., v\,(x,y,z)=\left[\begin{array}{c}\sigma(y-x)\\ x(\rho-z)-y\\ \mathrm{xy}-\beta z\end{array}\right]$$

the equation $\mathbb{E}_{\rho(x,\theta)}(v(x,\theta)\cdot\nabla\phi(x))=0$ result in

$$\sigma\left(\int_\Omega y\phi_{x_1}-x\phi_{x_1}\right)+\rho\int_\Omega x\phi_{x_2}-\beta\int_\Omega z\phi_{x_3}=\int_\Omega(\mathrm{xz}+y)\phi_{x_2}-\int_\Omega \mathrm{xy}\phi_{x_3}$$

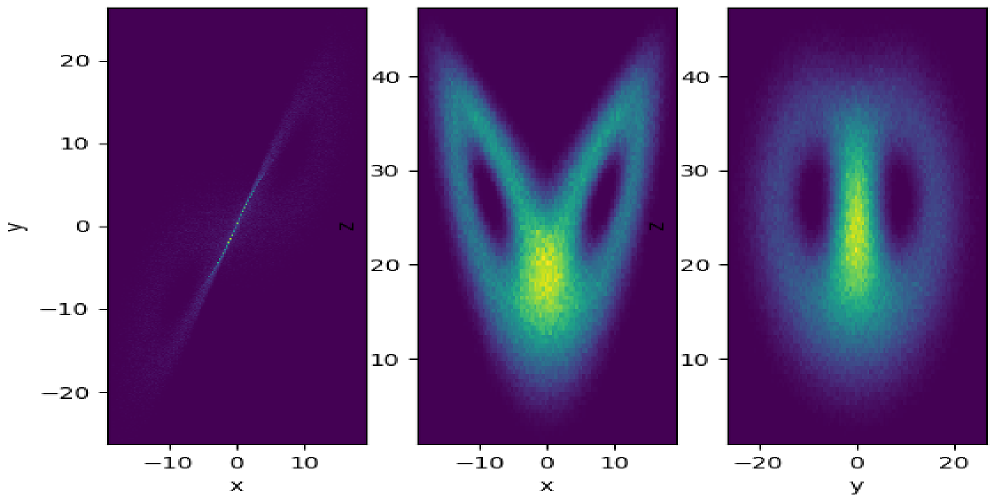
which is indeed a linear equation for each test function ϕ.

3 Examples

3.1 Lorentz System

$$\left\{\begin{array}{ccc}\dot{x} & = & \sigma(y-x)\\ \dot{y} & = & x(\rho-z)-y\\ \dot{z} & = & \mathrm{xy}-\beta z\end{array}\right.,$$

Here $\sigma=10,\rho=28,\sigma=8/3$, initial condition is [1,1,5],$T=20000$, $\mathrm{dt}=0.01$.the projection of density map wrt xy,xz,yz direction is below.



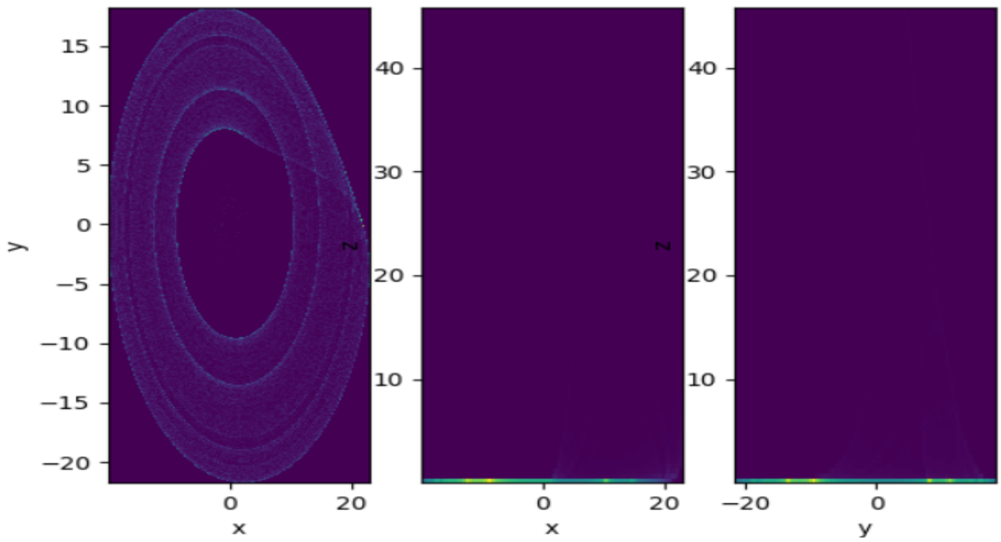
To reconstruct the parameter, we choose 1000 gaussion whose mean is generated by LHS sampling in the domain of data.Using Gradient method with lr = 0.01 and 30000 epoch and Least square method , the reconstruction result is

	σ	ρ	β
GD	10.0567	28.0346	2.71
Error	0.0567	0.0346	0.0453
LS	10.0573	28.0351	2.7115
Error	0.0573	0.0351	0.0448

3.2 Rossler System

$$\begin{cases} \dot{x} &= & -y-z \\ \dot{y} &= & x+\mathrm{ay} \\ \dot{z} &= & b+z(x-c) \end{cases},$$

Here $a=0.1, b=0.1, c=14$. initial condition is [2,2,5],$T=20000$, dt $=0.01$..the projection of density map wrt xy,xz,yz direction is below.



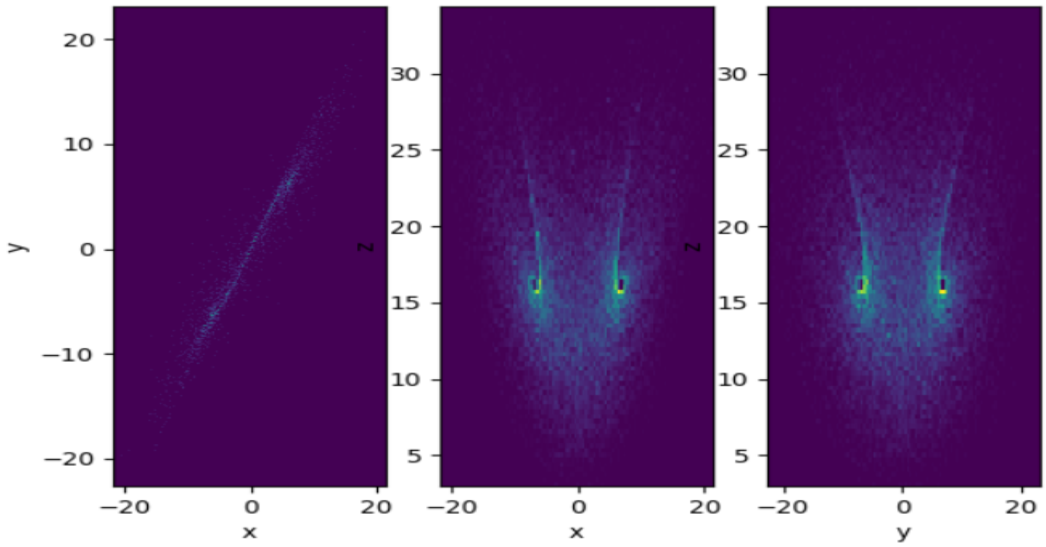
To reconstruct the parameter, we choose 1000 gaussion whose mean is generated by LHS sampling in the domain of data.Using Gradient method with lr = 0.01 and 30000 epoch and Least square method , the reconstruction result is

	a	b	c
GD	0.1051	0.080	13.9053
Error	0.0051	-0.0199	-0.0947
LS	0.1051	0.0801	13.9053
Error	0.0051	-0.0199	-0.0947

3.3 Chen System

$$\begin{cases} \dot{x} &= & a(y-x) \\ \dot{y} &= & (c-a)x-\mathrm{xz}+\mathrm{cy} \\ \dot{z} &= & \mathrm{xy}-\mathrm{bz} \end{cases},$$

Here $a=40, b=3, c=28$. initial condition is [2,2,5],$T=20000$, dt $=0.01$.the projection of density map wrt xy,xz,yz direction is below.

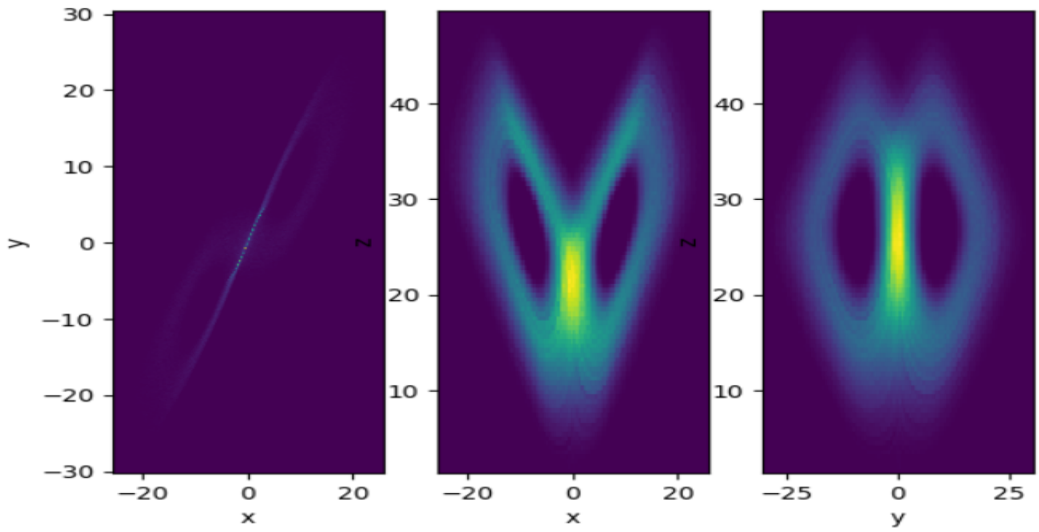


	a	b	c
GD	40.2535	3.0110	28.1462
Error	0.2535	0.0110	0.1462
LS	40.2535	3.0110	28.1462
Error	0.2535	0.0110	0.1462

3.4 Arctan Lorentz System

$$\begin{cases} \dot{x} &= 50\text{arctan}\left(\dfrac{\sigma(y-x)}{50}\right) \\ \dot{y} &= 50\text{arctan}\left(\dfrac{x(\rho-z)-y}{50}\right) \\ \dot{z} &= 50\text{arctan}\left(\dfrac{(\text{xy}-\beta z)}{50}\right) \end{cases},$$

Here $\sigma=10, \rho=28, \sigma=8/3$, initial condition is [1,1,5],$T=20000$, dt$=0.01$.the projection of density map wrt xy,xz,yz direction is below.



Here we use stochastic gradient descent, for each batch we sample 50 gaussian from the 1000 LHS gaussian list.The learning rate is 0.1 and a total 3000 epoch is used in training.

	a	b	c
GD	10.0177	28.0473	2.7052
Error	0.0177	0.0473	0.0386

Also we can see that the system converge in around 700 epoch.

```
step = 0, loss = 13606, W = [[0.2606559097766876], [-0.18592116236686707], [-0.29381808638572693]]
step = 50, loss = 7330.8, W = [[0.23338229954242706], [4.244386672973633], [-1.6133419275283813]]
step = 100, loss = 3267.05, W = [[-0.7432518601417542], [8.373147010803223], [-1.1452692747116089]]
step = 150, loss = 1874.38, W = [[-0.776563286781311], [13.598703384399414], [-1.652927279472351]]
step = 200, loss = 9870.32, W = [[0.17139899730682373], [16.284543991088867], [-0.4884900450706482]]
step = 250, loss = 1133.35, W = [[1.718989610671997], [18.993850708007812], [-0.689951479434967]]
step = 300, loss = 3519.36, W = [[2.7393362522125244], [21.802255630493164], [0.09377888590097427]]
step = 350, loss = 4682.72, W = [[4.563082218170166], [24.928869247436523], [1.011198878288269]]
step = 400, loss = 1050.5, W = [[7.475602149963379], [27.00672149658203], [2.163640260696411]]
step = 450, loss = 180.075, W = [[10.267330169677734], [28.045682907104492], [2.724902629852295]]
step = 500, loss = 229.029, W = [[10.095870018005371], [27.876361846923828], [2.7024264335632324]]
step = 550, loss = 198.879, W = [[9.993453025817871], [28.066232681274414], [2.586441993713379]]
step = 600, loss = 215.098, W = [[9.981377601623535], [28.20126724243164], [2.7300398349761963]]
step = 650, loss = 83.5705, W = [[9.908446311950684], [27.98430824279785], [2.6761767864227295]]
step = 700, loss = 127.472, W = [[10.000760078430176], [28.072101593017578], [2.6652183532714844]]
step = 750, loss = 12.7585, W = [[10.049041748046875], [27.909833908081055], [2.651001214981079]]
step = 800, loss = 34.0581, W = [[10.044719696044922], [28.067684173583984], [2.6968798637390137]]
step = 850, loss = 96.3634, W = [[10.046515464782715], [27.925172805786133], [2.69292950630188]]
step = 900, loss = 24.1463, W = [[9.920768737792969], [28.056562423706055], [2.671236276626587]]
step = 950, loss = 20.2743, W = [[9.971670150756836], [27.984514236450195], [2.6705563068389893]]
step = 1000, loss = 20.6455, W = [[9.991427421569824], [28.039653778076172], [2.6724250316619873]]
step = 1050, loss = 137.069, W = [[10.036661148071289], [28.008699417114258], [2.706580877304077]]
step = 1100, loss = 32.3733, W = [[10.027220726013184], [27.92790985107422], [2.6619715690612793]]
step = 1150, loss = 70.1727, W = [[10.046793937683105], [27.960403442382812], [2.617043972015381]]
step = 1200, loss = 30.4325, W = [[10.12477970123291], [27.953689575195312], [2.6405186653137207]]
step = 1250, loss = 64.829, W = [[9.998767852783203], [27.955413818359375], [2.72802686912842]]
step = 1300, loss = 11.7296, W = [[9.963363647460938], [27.978424072265625], [2.66508412361145]]
step = 1350, loss = 29.1377, W = [[9.940669059753418], [27.8826904296875], [2.657597064971924]]
step = 1400, loss = 50.655, W = [[10.066084861755371], [27.976806640625], [2.71030330657959]]
step = 1450, loss = 32.8353, W = [[9.989806175231934], [28.007421493530273], [2.665088415145874]]
step = 1500, loss = 8.36017, W = [[10.032815933227539], [27.968191146850586], [2.657179594039917]]
step = 1550, loss = 13.306, W = [[10.052788734436035], [28.055137634277344], [2.643028736114502]]
step = 1600, loss = 64.1716, W = [[10.022038459777832], [28.030010223388672], [2.657855987548828]]
step = 1650, loss = 13.7256, W = [[9.959837913513184], [27.931169509887695], [2.730159044265747]]
```

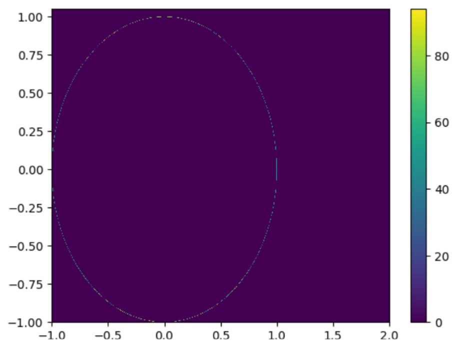
3.5 2D Periodic Attractor

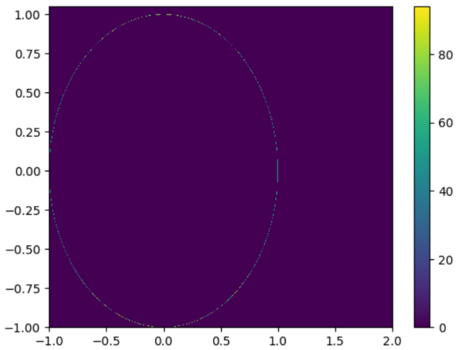
$$\begin{cases} \dot{x} &= x - y - (x^2 + y^2)x \\ \dot{y} &= x + y - (x^2 + y^2)y \end{cases},$$

This is a planar system which has a unique closed orbit γ and is a periodic attractor.

Note that the phase chart of such equation is almost the same as

$$\begin{cases} \dot{x} &= x - 0.3\,y - (x^2 + y^2)x \\ \dot{y} &= 0.3x + y - (x^2 + y^2)y \end{cases},$$





we parametrize the equation as follow

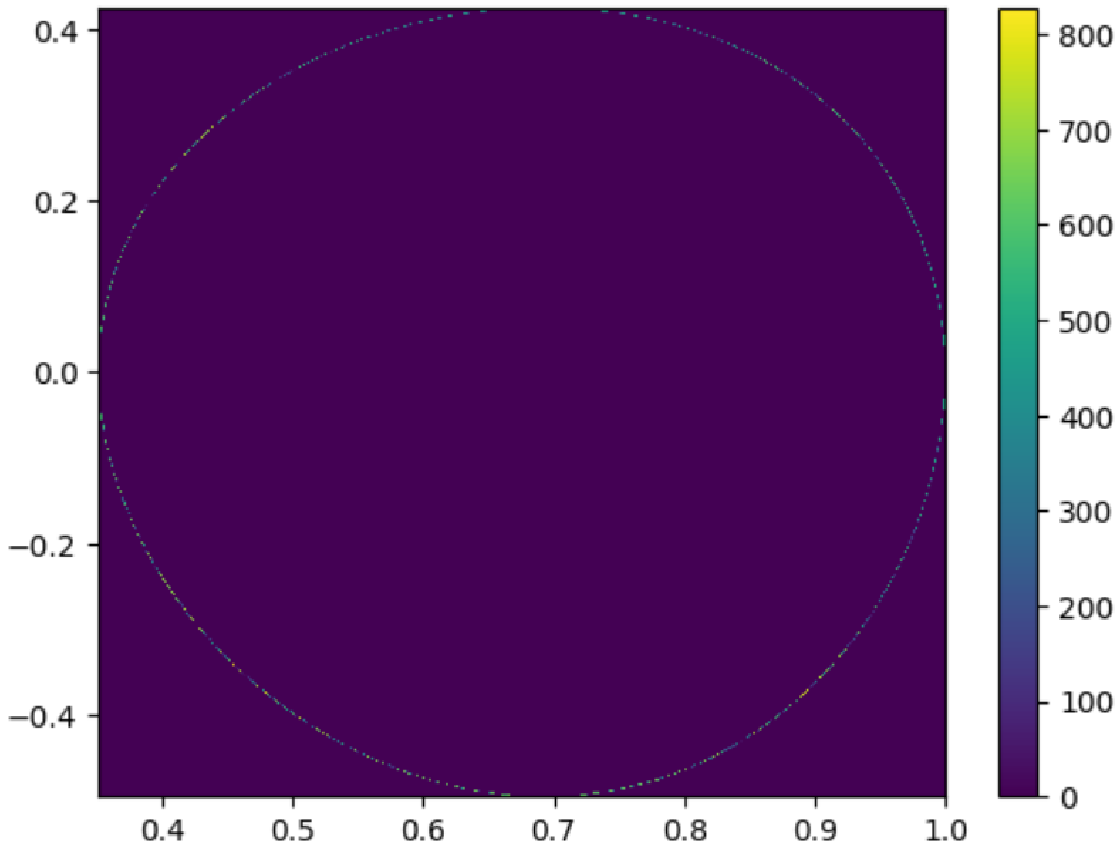
$$\begin{cases} \dot{x} &= x - y - \mathrm{a}x^3 - \mathrm{bxy}^2 \\ \dot{y} &= x + y - \mathrm{c}x^2 y - y^3 \end{cases},$$

The reference result is [1,1,1,1]

	a	b	c	d
LS	0.9982	1.0031	0.9954	1.0021
Error	0.0018	0.0031	0.0046	0.0021

3.6 Lotka-Volterra System

$$\begin{cases} \dot{x} &= 1 - \exp y \\ \dot{y} &= \exp x - 2 \end{cases},$$



we parametrize the equation as follow

$$\begin{cases} \dot{x} &= A - \mathrm{B}\exp y \\ \dot{y} &= \mathrm{C}\exp x - 2 \end{cases},$$

The reference result is [1,1,1]

	A	B	C
LS	1.0001	1.0001	1.000
Error	0.0001	0.0001	0.000