# TP3 CompStat - LACOMBE Yoach

## Exercise 1

### Question 1

To recall the notations, as fixed parameters, we have $\sigma_{t_0}, \sigma_{v_0}$ (the fixed parameters of the latent variables) AND $\bar{\bar{t}}_0, \bar{\bar{v}}_0, s_{t_0}, s_{v_0}, m_\xi, v_\xi, m_\tau, v_\tau, v, m$ (the fixed parameters of the a priori).

The parameters θ are $(\bar{t}_0, \bar{v}_0, \sigma_\xi, \sigma_\tau, \sigma)$, following the model:

$$\begin{cases} \bar{t}_0 \sim \mathcal{N}(\bar{\bar{t}}_0, s_{t_0}^2) \\ \bar{v}_0 \sim \mathcal{N}(\bar{\bar{v}}_0, s_{v_0}^2) \\ \sigma^2 \sim \mathcal{W}^{-1}(v, m) \quad \text{and same for } \sigma_\xi, \sigma_\tau \end{cases}$$

The latent variables are $\xi_i, \tau_i, t_0, v_0$ which respects:

$$\begin{cases} \alpha_i = e^{\xi_i}, \qquad\qquad \text{where } \xi_i \overset{i.i.d}{\sim} \mathcal{N}(0, \sigma_\xi^2) \\ \tau_i \overset{i.i.d}{\sim} \mathcal{N}(0, \sigma_\tau^2) \\ t_0 \sim \mathcal{N}(\bar{t}_0, \sigma_{t_0}^2) \\ v_0 \sim \mathcal{N}(\bar{v}_0, \sigma_{v_0}^2) \end{cases}$$

Complete log-likelihood:

$$log[q(y, z, \theta)] = log[q(y|z, \theta)] + log[q(z|\theta)] + log[q(\theta)]$$

where [**note**: I don't write additive constants in the following results]:

---

$$\begin{aligned} log[q(y|z, \theta)] &= -\sum_{i,j}[||y_{ij} - d_i(t_{ij})||^2/(2\sigma^2) + log\sigma^2/2] \\ &= S_1\phi_1 + \psi_1 \end{aligned}$$

---

$$\begin{aligned} log[q(z|\theta)] = &-\sum_i[||\tau_i||^2/(2\sigma_\tau^2) + log\sigma_\tau^2/2] \\ &-\sum_i[||\epsilon_i||^2/(2\sigma_\epsilon^2) + log\sigma_\epsilon^2/2] \\ &-||t_0 - \bar{t}_0||^2/(2\sigma_{t_0}^2) - ||v_0 - \bar{v}_0||^2/(2\sigma_{v_0}^2) \\ = &\ S_2\phi_2 + \psi_2 \\ &+S_3\phi_3 + \psi_3 \\ &+S_4\phi_4 + \psi_5 + S_5\phi_5 + \psi_5 + R(y, z) \end{aligned}$$

---

$$\begin{aligned} log[q(\theta)] = &-||\bar{t}_0 - \bar{\bar{t}}_0||^2/(2s_{t_0}^2) - ||\bar{v}_0 - \bar{\bar{v}}_0||^2/(2s_{v_0}^2) \\ &-(1 + m_\xi/2)log\sigma_\xi^2 - v_\xi^2/2\sigma_\xi^2 \\ &-(1 + m_\tau/2)log\sigma_\tau^2 - v_\tau^2/2\sigma_\tau^2 \end{aligned}$$

$$-(1+m/2)log\sigma^2 - v^2/2\sigma^2$$
$$= \phi_6$$

---

## Curved exponential family

**Note:** Let N be the number of individuals and $N_{tot} = \sum_{i=1}^{N} k_i$

We identify with:

$$\begin{cases} S_1 = \sum_{i,j}[||y_{ij} - d_i(t_{ij})||^2/(N_{tot}) \\ \phi_1 = -N_{tot}/(2\sigma^2) \end{cases}$$
$$\begin{cases} S_2 = \sum_i[||\tau_i||^2/N \\ \phi_2 = -N/(2\sigma_\tau^2) \end{cases}$$
$$\begin{cases} S_3 = \sum_i[||\xi_i||^2/N \\ \phi_3 = -N/(2\sigma_\xi^2) \end{cases}$$
$$\begin{cases} S_4 = t_0 \\ \phi_4 = \bar{t}_0/(\sigma_{t_0}^2) \end{cases}$$
$$\begin{cases} S_5 = v_0 \\ \phi_5 = \bar{v}_0/(\sigma_{v_0}^2) \end{cases}$$

And finally:

$$\psi = -N_{tot}log\sigma^2/2$$
$$-Nlog\sigma_\tau^2/2$$
$$-Nlog\sigma_\xi^2/2$$
$$-\bar{v}_0^2/(2\sigma_{v_0}^2)$$
$$-\bar{t}_0^2/(2\sigma_{t_0}^2)$$
$$-||\bar{t}_0 - \bar{\bar{t}}_0||^2/(2s_{t_0}^2) - ||\bar{v}_0 - \bar{\bar{v}}_0||^2/(2s_{v_0}^2)$$
$$-(1+m_\xi/2)log\sigma_\xi^2 - v_\xi^2/2\sigma_\xi^2$$
$$-(1+m_\tau/2)log\sigma_\tau^2 - v_\tau^2/2\sigma_\tau^2$$
$$-(1+m/2)log\sigma^2 - v^2/2\sigma^2$$

# Question 2

Following the probability model, I will sample first from θ, then from z|θ, then from y|z,θ.

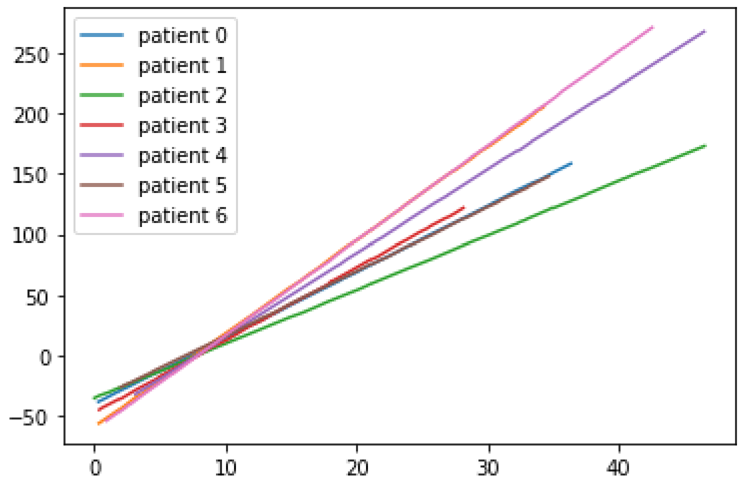
```
from scipy.stats import invwishart, norm, gamma
import scipy
import numpy as np

def draw_from_invwishart(shape, v, m):
  # v = scale > 0, m = df (degree of freedom, int > 0 )
  rv = scipy.stats.invwishart(df=m, scale=v)
  return rv.rvs(shape)

def draw_from_normal_distr(shape, mu, sigma):
```

```
  rv = scipy.stats.norm(loc = mu, scale = sigma)
  return rv.rvs(shape)

def synthetic_data_generator(N, k, sigma_t0, sigma_v0, t0_bar_bar, v0_bar_bar,s_t0, s_v0,
  '''
  N: nb of patients. here only works with integer >0.
  k: nb of measurements per patients. (Should be a different integer per patients but I to
  '''
  #draw theta
  t0_bar = draw_from_normal_distr(1, t0_bar_bar, s_t0)
  v0_bar = draw_from_normal_distr(1, v0_bar_bar, s_v0)

  sigma_2 = draw_from_invwishart(1, v, m)
  sigma_xi2 = draw_from_invwishart(1, v_xi, m_xi)
  sigma_tau2 = draw_from_invwishart(1, v_tau, m_tau)


  epsilon = draw_from_normal_distr((N,k), 0, np.sqrt(sigma_2))

  #draw latent variables
  xi = draw_from_normal_distr((N,1), 0, np.sqrt(sigma_xi2))
  alpha = np.exp(xi)

  tau = draw_from_normal_distr((N,1), 0, np.sqrt(sigma_tau2))

  t0 = draw_from_normal_distr(1, t0_bar, np.sqrt(sigma_t0))
  v0 = draw_from_normal_distr(1, v0_bar, np.sqrt(sigma_v0))

  if verbose:
    print("sigma, sigma_xi, sigma_tau", sigma_2, sigma_xi2, sigma_tau2)
    print("t0_bar, t0", t0_bar, t0)
    print("vo_bar, v0", v0_bar, v0)

  T = np.zeros((N,1))
  rv = scipy.stats.gamma(1)
  T = rv.rvs((N,k))
  T = np.cumsum(T, axis = 1)




  y = v0*(alpha*(T - t0 - tau)) + epsilon

  theta = [t0_bar, v0_bar, (sigma_xi2), (sigma_tau2), (sigma_2)]
  latent = [xi, tau, v0, t0]
  return T, y, theta, latent


#parameters·setting

N·=·7
k·=·40

sigma_t0·=·1
```

```
sigma_v0 = 2 

t0_bar_bar = 2

v0_bar_bar = 3

s_t0 = 5
s_v0 = 5

m_xi = 3
v_xi = .1
m_tau = 3
v_tau = .2
m = 3 
v = .1


samples_time, samples, theta_true, latent_true = synthetic_data_generator(N, k, sigma_t0, 
```

```
sigma, sigma_xi, sigma_tau 0.08613635820211252 0.025457933192624425 0.12408839147619
t0_bar, t0 [6.63474285] [7.61670864]
vo_bar, v0 [6.09501178] [6.23906862]
```

```
import matplotlib.pyplot as plt

for i in range(N):
  plt.plot(samples_time[i,:], samples[i,:], label = "patient " + str(i))

plt.legend()
plt.show()
```




## Question 3

As indicated in the statement, and after simplification,

$$\alpha_k = min[1, q(z^*, y, \theta)/q(z^{(k)}, y, \theta)]$$

Moreover, we'll take $z^* \sim \mathcal{N}(z^{(k)}, \sigma_{prop})$.

```
def log_likelihood(T, t0, v0, xi, tau, observations, theta, compute_log_theta = True, thet
  '''
  T : times t_ij
  theta = [to_bar, v0_bar, sigma_xi, sigma_tau, sigma]
  theta_parameters = t0_bar_bar, v0_bar_bar,s_t0, s_v0, m_xi, v_xi, m_tau,v_tau, m, v
  #Beware, sigma is already squared (sigma = sigma^2 instead of sigma)
  '''
  lg = 0

  t0_bar, v0_bar, sigma_xi, sigma_tau, sigma = theta
  if compute_log_theta:
    t0_bar_bar, v0_bar_bar,s_t0, s_v0, m_xi, v_xi, m_tau,v_tau, m, v = theta_parameters
    lg += - (t0_bar_bar -t0_bar)**2/(2*s_t0) - (v0_bar_bar -v0_bar)**2/(2*s_v0)
    lg += -(1 + m/2)*np.log(sigma) - v**2/(2*sigma)
    lg += -(1 + m_xi/2)*np.log(sigma_xi) - v_xi**2/(2*sigma_xi)
    lg += -(1 + m_tau/2)*np.log(sigma_tau) - v_tau**2/(2*sigma_tau)


  #log q (z|theta)
  lg += - (t0_bar -t0)**2/(2*sigma_t0) - (v0_bar -v0)**2/(2*sigma_v0)
  lg += -len(tau)*(np.log(sigma_tau) + np.log(sigma_xi))/2
  lg += -np.sum(tau*tau)/ (2*sigma_tau) - np.sum(xi*xi)/ (2*sigma_xi)

  #log q(y|z,theta)
  predictions = v0*(np.exp(xi)*(T - t0 - tau))
  lg += -np.sum( np.square(observations-predictions)/(2*sigma) + np.log(sigma)/2)

  return lg




class MH_sampler():
  def __init__(self, sigma_prop, N, observations, theta, T, theta_parameters, sigma_t0 = 2
    '''
    sigma_prop: trade-off between exploration and exploitation
    theta: as in the enonce
    theta_parameters: the hyperparameters defining the law of theta
    '''
    self.sigma_prop = sigma_prop
    self.log_likelihoods = []
    self.alphas = []
    self.theta_parameters = theta_parameters

    self.N = N
    #2N + 2 = size of latent variable

    self.observations = observations
    self.theta = theta
    self.sigma_t0 = sigma_t0
    self.sigma_v0 = sigma_v0

    self.compteur_acceptance = 0
```

```
    self.T = T

  def log_likelihood(self, t0, v0, xi, tau):

    return log_likelihood(self.T, t0, v0, xi, tau, self.observations, self.theta, compute_

  def initialize_algo(self, t0 = None, v0 = None, xi = None, tau = None):
    #latent variables = t0, v0, xi, tau

    if t0 is not None:
      self.t0 = t0
      self.v0 = v0
      self.xi = xi
      self.alpha = np.exp(self.xi)
      self.tau = tau
    else:
      self.t0 = 0
      self.v0 = 1
      self.xi = np.zeros((self.N,1))
      self.alpha = np.exp(self.xi)
      self.tau = np.random.normal(size = (self.N,1))


  def sample(self):
    t0 = draw_from_normal_distr(1, self.t0, self.sigma_prop)
    v0 = draw_from_normal_distr(1, self.v0, self.sigma_prop)
    xi = [draw_from_normal_distr(1, x, self.sigma_prop) for x in self.xi]
    xi = np.array(xi)

    tau = [draw_from_normal_distr(1, x, self.sigma_prop) for x in self.tau]
    tau = np.array(tau)

    return t0, v0, xi, tau

  def step(self):
    #proposal
    t0, v0, xi, tau = self.sample()

    #acceptance-rejection
    lg_k = self.log_likelihood(self.t0, self.v0, self.xi, self.tau)
    self.log_likelihoods.append(lg_k)
    alpha = min(0, (self.log_likelihood(t0, v0, xi, tau) - lg_k)[0])

    self.alphas.append(alpha)

    u = np.log(np.random.uniform())
    #acceptance
    if u<=alpha:
      self.compteur_acceptance += 1
      #print("acceptance", alpha, u, self.log_likelihood(t0, v0, xi, tau), lg_k)
      self.t0 = t0
      self.v0 = v0
      self.xi = xi
      self.tau = tau
```

```
    #else:
    #  print("reject", alpha, u, self.log_likelihood(t0, v0, xi, tau), lg_k)


    #rejectance - do nothing
```

## Question 4

Thanks to question 1, we have $\Phi$ and $\Psi$.

Coordinate-wise, the function we want to maximize goes to $-\infty$ at its extremities and has one critical point, so it has one maximum coordinate-wise, so one maximum everywhere. We can compute the gradient coordinate-wise and set it equals to 0 to find the optimal parameters.

Let's denote $Q(\theta) = \Phi(\theta) + S_k^T \Psi(\theta)$.

$\nabla_{\bar{t}_0} Q(\theta) = S_4/(\sigma_{t_0}^2) - \bar{t}_0/(\sigma_{t_0}^2) - (\bar{t}_0 - \bar{\bar{t}}_0)/(s_{t_0}^2) = 0$

So $\bar{t}_0 = \frac{S_4/\sigma_{t_0}^2 + \bar{\bar{t}}_0/s_{t_0}^2}{1/\sigma_{t_0}^2 + 1/s_{t_0}^2}$.

In the same manner, $\bar{v}_0 = \frac{S_5/\sigma_{v_0}^2 + \bar{\bar{v}}_0/s_{v_0}^2}{1/\sigma_{v_0}^2 + 1/s_{v_0}^2}$.

For $\sigma, \sigma_\tau, \sigma_\xi$, we can compute in the same manner:
$\nabla_{\sigma^2} Q(\theta) = +S_1 N_{tot}/(2\sigma^4) - N_{tot}/(2\sigma^2) - (1 + m/2)/\sigma^2 + v^2/2\sigma^4 = 0$

That is $\sigma^2 = \frac{N_{tot} S_1 + v^2}{N_{tot} + 2 + m}$.

To resume,

$$\begin{cases} \bar{t}_0 = \frac{S_4/\sigma_{t_0}^2 + \bar{\bar{t}}_0/s_{t_0}^2}{1/\sigma_{t_0}^2 + 1/s_{t_0}^2} \\ \bar{v}_0 = \frac{S_5/\sigma_{v_0}^2 + \bar{\bar{v}}_0/s_{v_0}^2}{1/\sigma_{v_0}^2 + 1/s_{v_0}^2} \\ \sigma^2 = \frac{N_{tot} S_1 + v^2}{N_{tot} + 2 + m} \\ \sigma_\tau^2 = \frac{N S_2 + v_\tau^2}{N + 2 + m_\tau} \\ \sigma_\xi^2 = \frac{N S_3 + v_\xi^2}{N + 2 + m_\xi} \end{cases}$$

Recall that at step k:

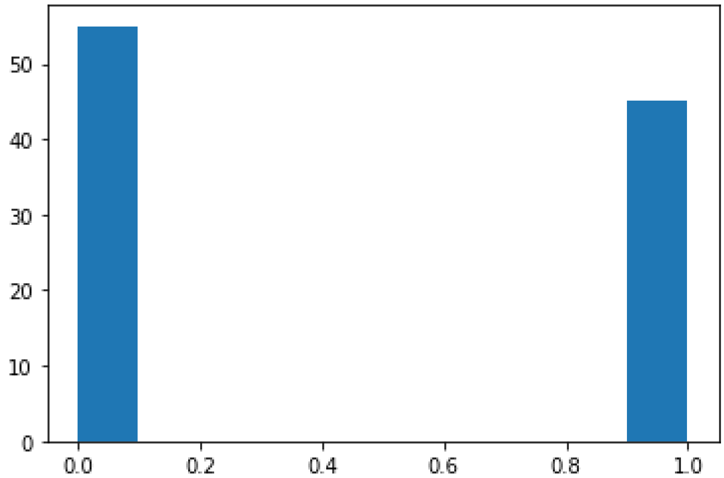
$$\begin{cases} S_1 = \sum_{i,j} [||y_{ij} - d_i(t_{ij})||^2/(N_{tot}) \\ S_2 = \sum_i [||\tau_i||^2/N \\ S_3 = \sum_i [||\xi_i||^2/N \\ S_4 = t_0 \\ S_5 = v_0 \end{cases}$$

```
class MCMC_SAEM():
  def __init__(self, sigma_prop, N, observations, T, theta_parameters, sigma_t0 = 2, sigma
    self.sigma_prop = sigma_prop
    self.log_likelihoods = []
    self.alphas = []
    self.theta_parameters = theta_parameters

    self.N = N
    #2N + 2 = size of latent variable

    self.observations = observations
    self.sigma_t0 = sigma_t0
    self.sigma_v0 = sigma_v0
    self.compteur = 0

    self.T = T

  def initialize_algo(self, initial_guess_theta, maxIter = 5000, alpha_burn_in = 0.8):
    self.theta = initial_guess_theta
    self.t0 = 0
    self.v0 = 0
    self.xi = np.zeros((self.N,1))
    self.alpha = np.exp(self.xi)
    self.tau = np.zeros((self.N,1))

    self.maxIter = maxIter
    self.burn_in = maxIter//2
    self.alpha_burn_in = alpha_burn_in

    self.S1 = 0
    self.S2 = 0
    self.S3 = 0
    self.S4 = 0
    self.S5 = 0

  def sample(self, maxIterSampling):
    #Simulation
    sampler = MH_sampler(self.sigma_prop, self.N, self.observations, self.theta, self.T,th

    sampler.initialize_algo(t0 = self.t0, v0 = self.v0, xi = self.xi, tau = self.tau)
    for i in range(maxIterSampling):
      sampler.step()

    return sampler


  def step(self, maxIterSampling = 10):
    #Simulation
    sampler = self.sample(maxIterSampling)

    self.t0 = sampler.t0
    self.v0 = sampler.v0
    self.xi = sampler.xi
    self.tau = sampler.tau
```

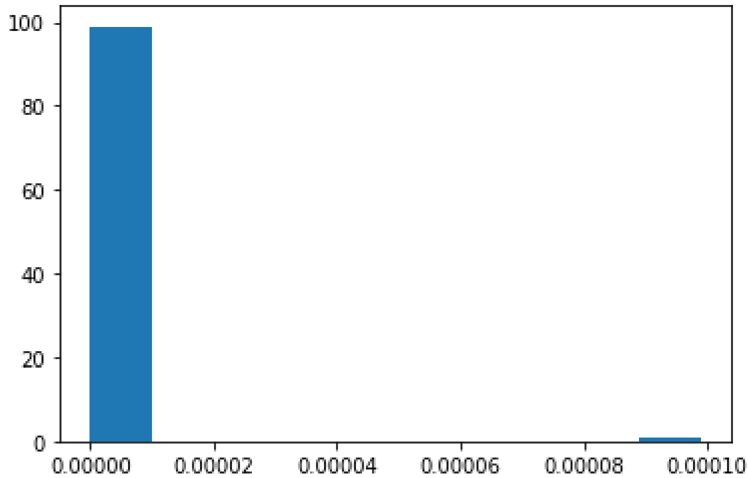
```
    #plot and verbose
    if self.compteur == self.burn_in:
      print("------Start Learning (eps<1) ---")

    if (self.compteur%50 == 0 and self.compteur>=self.burn_in) or self.compteur == 0:
      print('---iteration number', self.compteur, '-----')
      plt.hist(np.exp(sampler.alphas))
      print("acceptance counter: ",sampler.compteur_acceptance)
      plt.show()


    #Stochastic approximation
    if self.compteur <= self.burn_in:
      eps = 1
    else:
      eps = (self.compteur-self.burn_in)**(-self.alpha_burn_in)
    self.compteur+=1

    predictions = self.v0*(np.exp(self.xi)*(self.T - self.t0 - self.tau))
    self.S1 = self.S1*(1-eps) + eps*np.mean(np.square(self.observations - predictions))
    self.S2 = self.S2*(1-eps) + eps*np.mean(self.tau*self.tau)
    self.S3 = self.S3*(1-eps) + eps*np.mean(self.xi*self.xi)
    self.S4 = self.S4*(1-eps) + eps*self.t0
    self.S5 = self.S5*(1-eps) + eps*self.v0


    #Maximization

    t0_bar_bar, v0_bar_bar,s_t0, s_v0, m_xi, v_xi, m_tau,v_tau, m, v = self.theta_paramete

    #theta = [to_bar, v0_bar, sigma_xi, sigma_tau, sigma]
    #theta_parameters = t0_bar_bar, v0_bar_bar,s_t0, s_v0, m_xi, v_xi, m_tau,v_tau, m, v

    t0_bar = (self.S4/(self.sigma_t0**2) + t0_bar_bar/(s_t0**2))/ (1/(self.sigma_t0**2) +
    v0_bar = (self.S5/(self.sigma_v0**2) + v0_bar_bar/(s_v0**2))/ (1/(self.sigma_v0**2) +

    sigma = (self.T.size*self.S1 + v**2)/(self.T.size + 2 + m)
    sigma_tau = (self.tau.size*self.S2 + v_tau**2)/(self.tau.size + 2 + m_tau)
    sigma_xi = (self.xi.size*self.S3 + v_xi**2)/(self.xi.size + 2 + m_xi)

    self.theta = [t0_bar, v0_bar, sigma_xi, sigma_tau, sigma]


sigma_prop = 0.01
observations = samples
T = samples_time
theta_parameters = t0_bar_bar, v0_bar_bar,s_t0, s_v0, m_xi, v_xi, m_tau,v_tau, m, v

maxIter = 102
maxIterSampling = 100



initial_guess_theta = [-5, 2, .5, .6, 1]
```

```
solver = MCMC_SAEM(sigma_prop, N, observations, T, theta_parameters, sigma_t0 = sigma_t0,
solver.initialize_algo(initial_guess_theta, maxIter = maxIter, alpha_burn_in = 0.6)


for j in range(maxIter):
  solver.step(maxIterSampling)
```

```
---iteration number 0 -----
acceptance counter:  45
```

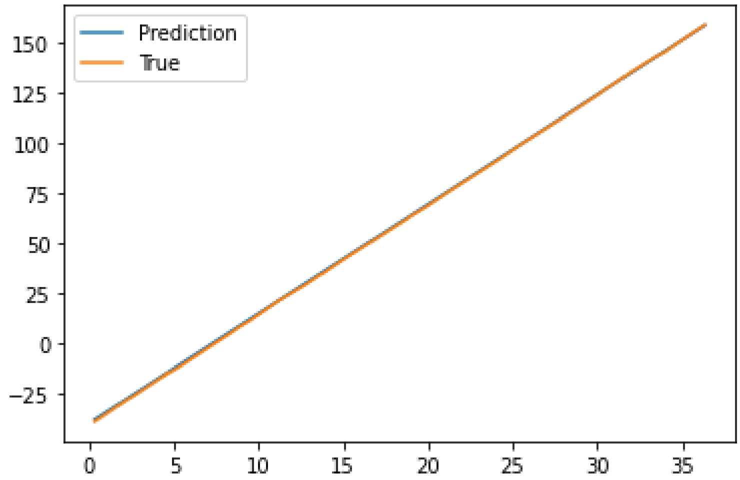


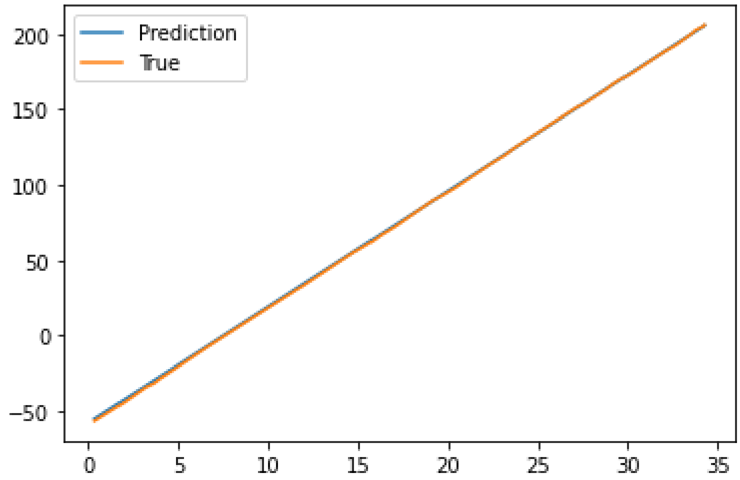

```
------Start Learning (eps<1) ---
---iteration number 100 -----
acceptance counter:  0
```

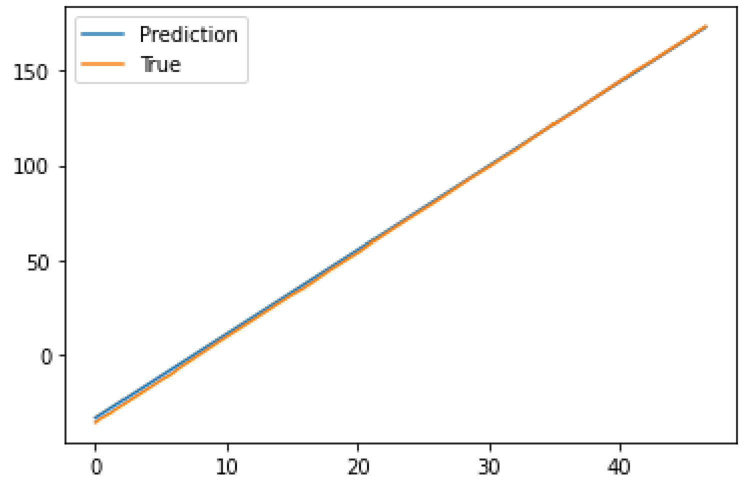


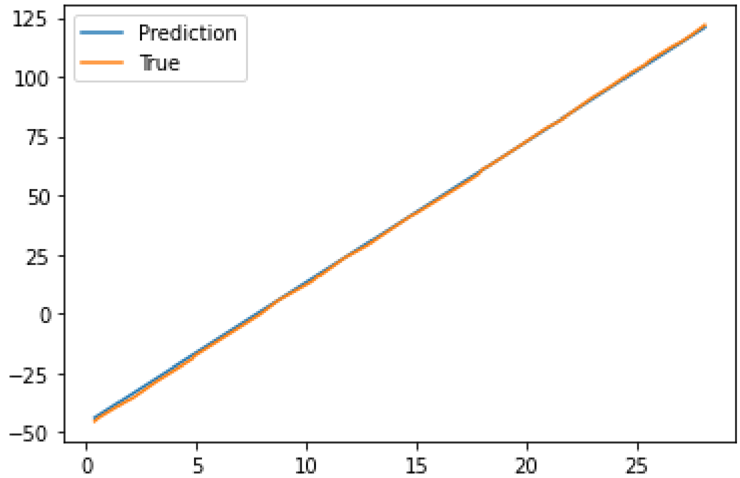

```
print(solver.theta)
print(theta_true)

predictions =solver.v0*(np.exp(solver.xi)*(solver.T - solver.t0 - solver.tau))
for i in range(N):
  plt.plot(T[i,:], predictions[i,:], label = "Prediction")
  plt.plot(T[i,:], samples[i,:], label = "True")
  plt.legend()
  plt.show()
```

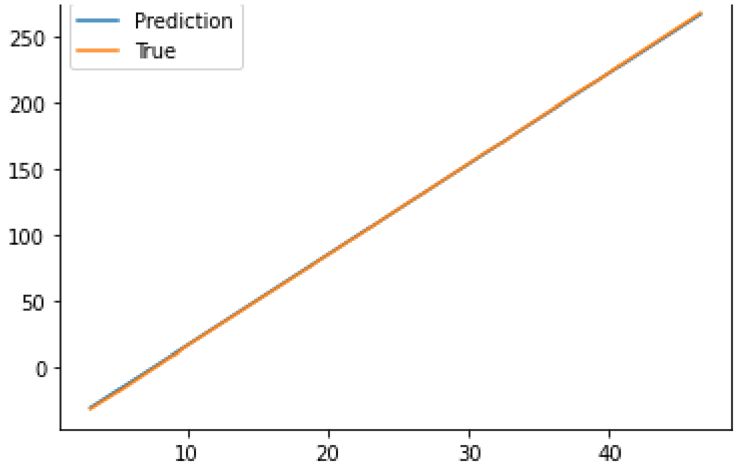
```
[array([6.81665826]), array([3.37105436]), 0.2173695740726256, 0.1902948202173876, 1
[array([6.63474285]), array([6.09501178]), 0.025457933192624425, 0.1240883914761929,
```

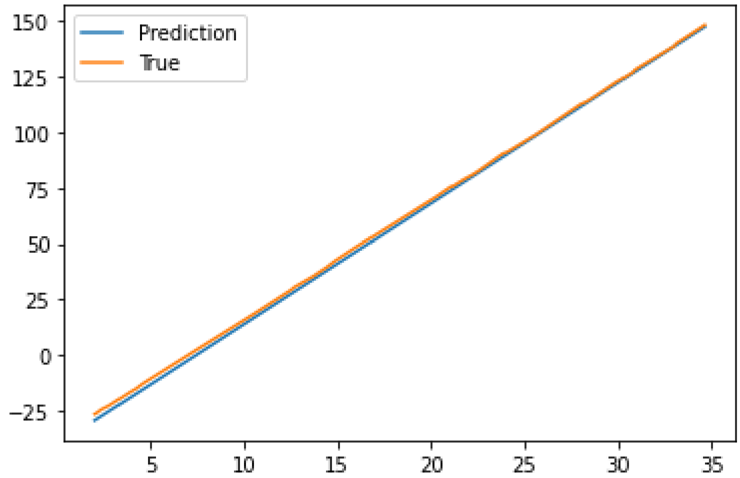


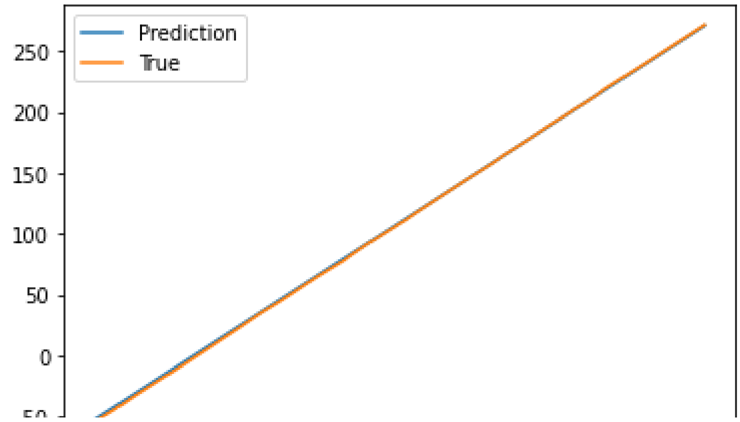






## Question 5 and 6

We use the same algorithm than in Algorithm 2. The only difference rests in the **#simulation** step. Indeed instead of sampling directly $z^{(k+1)}$ at once, we sample each composant $z_i^{(k+1)}$ following the Gibbs sampler algorithm.

To draw $z_i^{(k+1)}$, you simply use the second algorithm, except that $z_{-l}^{(k+1)}$ is fixed everywhere.

So to resume, for each i, we draw $z_i^*$ from $\mathcal{N}(z_i^{(k)}, \sigma_{prop})$ and accept the $z^*$ defined as $(z_1^{(k+1)}, \dots z_{i-1}^{(k+1)}, z_i^*, z_{i+1}^{(k)}, \dots, z_{2N+2}^{(k)})$ with probability $\alpha$ with:

$$\alpha = min(1, q(z^*, y, \theta)/q((z_1^{(k+1)}, \dots z_{i-1}^{(k+1)}, z_i^{(k)}, z_{i+1}^{(k)}, \dots, z_{2N+2}^{(k)}), y, \theta))$$

.

## Question 7

```
class MHwG_sampler(MH_sampler):
  def __init__(self, sigma_prop, N, observations, theta, T, theta_parameters, sigma_t0 = 2
    super(MHwG_sampler, self).__init__(sigma_prop, N, observations, theta, T, theta_parame

  def sample_xi(self,index):
    return draw_from_normal_distr(1, self.xi[index], self.sigma_prop)

  def sample_tau(self,index):
    return draw_from_normal_distr(1, self.tau[index], self.sigma_prop)

  def sample_t0(self):
    return draw_from_normal_distr(1, self.t0, self.sigma_prop)

  def sample_v0(self):
    return draw_from_normal_distr(1, self.v0, self.sigma_prop)

  def acceptance_rejection(self, t0, v0, xi, tau):
    #acceptance-rejection
    lg_k = self.log_likelihood(self.t0, self.v0, self.xi, self.tau)
    self.log_likelihoods.append(lg_k)
    alpha = min(0, (self.log_likelihood(t0, v0, xi, tau) - lg_k)[0])

    self.alphas.append(alpha)

    u = np.log(np.random.uniform())
    #acceptance
    if u<=alpha:
      self.compteur_acceptance += 1
      self.t0 = t0
      self.v0 = v0
      self.xi = xi
      self.tau = tau
    #rejectance - do nothing

  def step_v0(self):
    #proposal
    v0 = self.sample_v0()
    self.acceptance_rejection(self.t0, v0, self.xi, self.tau)

  def step_t0(self):
    #proposal
    t0 = self.sample_t0()
    self.acceptance_rejection(t0, self.v0, self.xi, self.tau)

  def step_xi(self, index):
    #proposal
    xi_i = self.sample_xi(index)
    proposition_xi = self.xi.copy()
    proposition_xi[index] = xi_i
    self.acceptance_rejection(self.t0, self.v0, proposition_xi, self.tau)
```

```
  def step_tau(self, index):
    #proposal
    tau_i = self.sample_tau(index)
    proposition_tau = self.tau.copy()
    proposition_tau[index] = tau_i
    self.acceptance_rejection(self.t0, self.v0, self.xi, proposition_tau)
```

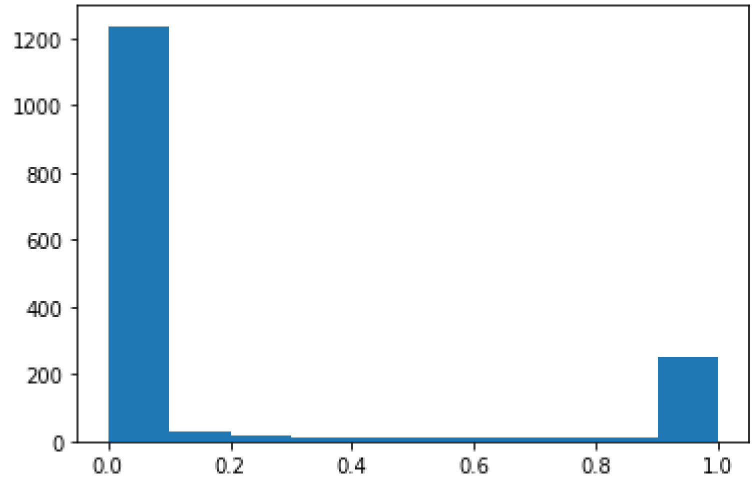
```
class MCMC_SAEM_with_MHwG_sampler(MCMC_SAEM):
  def __init__(self, sigma_prop, N, observations, T, theta_parameters, sigma_t0 = 2, sigma
    super(MCMC_SAEM_with_MHwG_sampler, self).__init__(sigma_prop, N, observations, T, thet

  def sample(self, maxIterSampling):
    #Simulation
    sampler = MHwG_sampler(self.sigma_prop, self.N, self.observations, self.theta, self.T,
    sampler.initialize_algo(t0 = self.t0, v0 = self.v0, xi = self.xi, tau = self.tau)

    #t0
    for i in range(maxIterSampling):
      sampler.step_t0()

    #v0
    for i in range(maxIterSampling):
      sampler.step_v0()

    #xi
    for j in range(len(self.xi)):
      for i in range(maxIterSampling):
        sampler.step_xi(j)

    #tau
    for j in range(len(self.tau)):
      for i in range(maxIterSampling):
        sampler.step_tau(j)

    return sampler
```

```
initial_guess_theta = [-5, 2, .5, .6, 1]
sigma_prop = 0.4
```

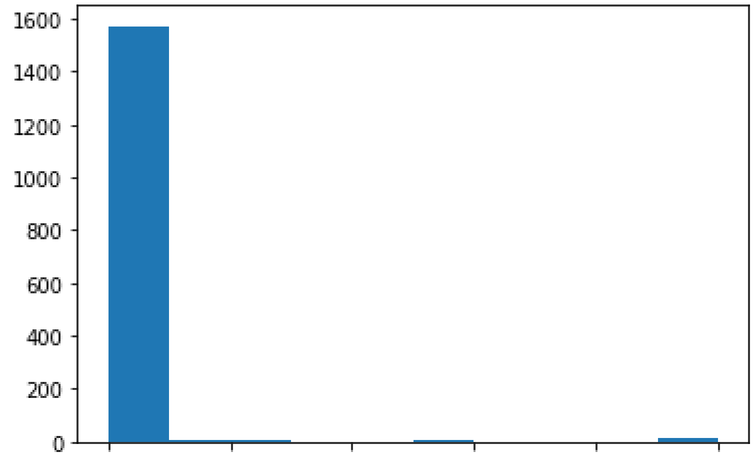
```
solver = MCMC_SAEM_with_MHwG_sampler(sigma_prop, N, observations, T, theta_parameters, sig
solver.initialize_algo(initial_guess_theta, maxIter = maxIter, alpha_burn_in = 0.6)
```

```
for j in range(maxIter):
  solver.step(maxIterSampling = maxIterSampling)
```

```
---iteration number 0 -----
acceptance counter:  299
```

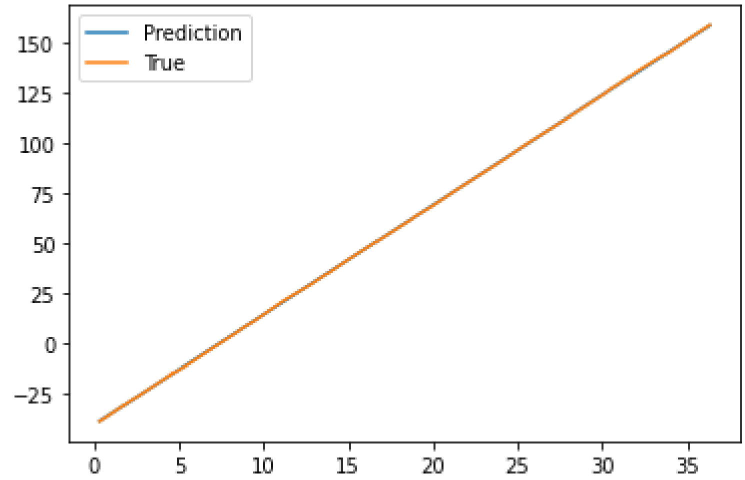


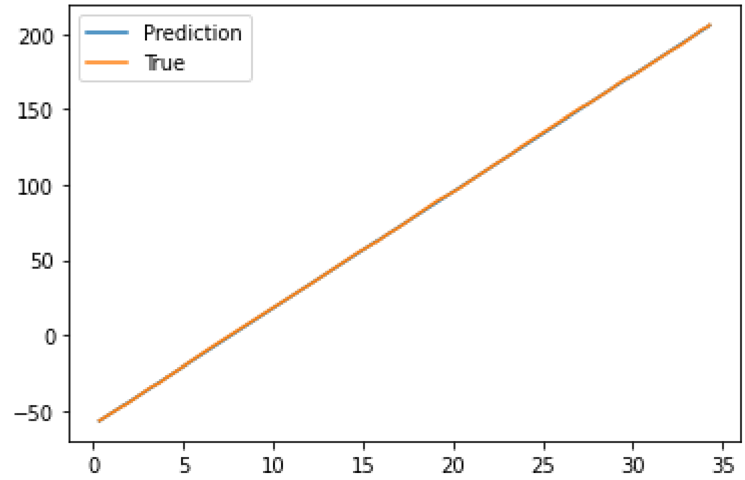

```
------Start Learning (eps<1) ---
---iteration number 100 -----
acceptance counter:  20
```

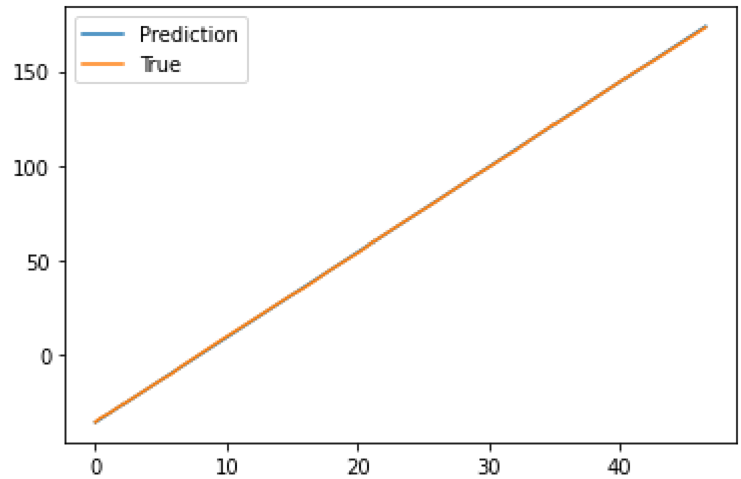


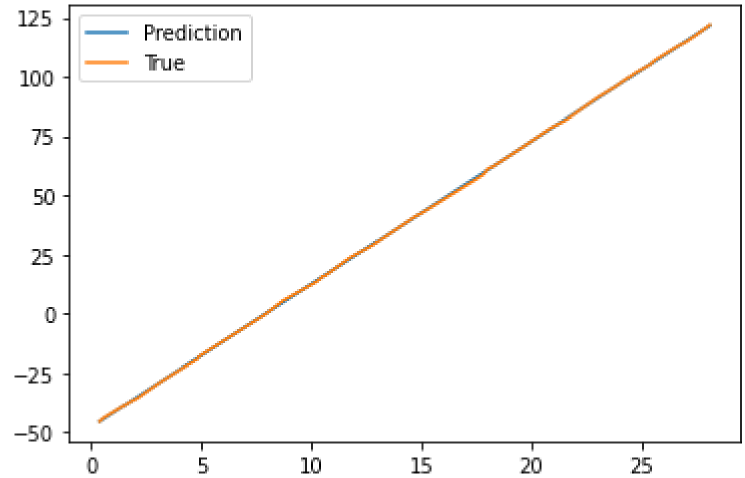

```
print(solver.theta)
print(theta_true)

predictions =solver.v0*(np.exp(solver.xi)*(solver.T - solver.t0 - solver.tau))
for i in range(N):
  plt.plot(T[i,:], predictions[i,:], label = "Prediction")
  plt.plot(T[i,:], samples[i,:], label = "True")
  plt.legend()
  plt.show()
```

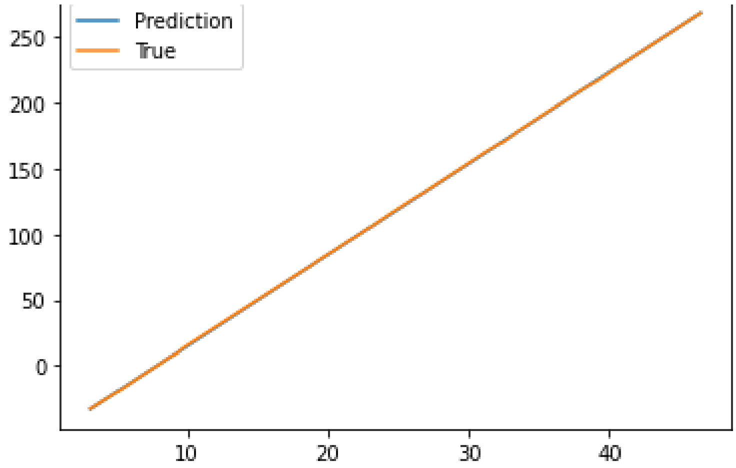
```
[array([-4.81335093]), array([5.71581212]), 0.022361392579823405, 94.98118308963917,
[array([6.63474285]), array([6.09501178]), 0.025457933192624425, 0.1240883914761929,
```

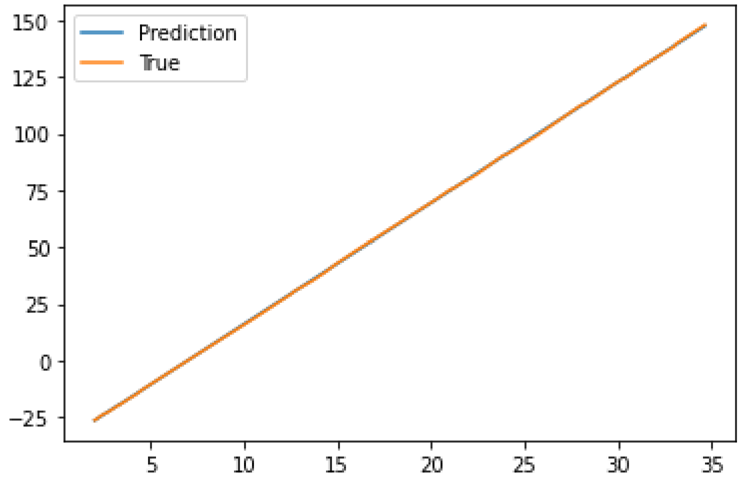





## Question 8

Advantages:

- Sampling and updating variable by variable is time consuming. Grouping by block makes it faster.
- It might be more efficient to update small block per small block, ex: in 2d, using 1d Gibbs sampling is not time reversing, whereas using MH on the two dimensions is (you can move diagonaly and revert).

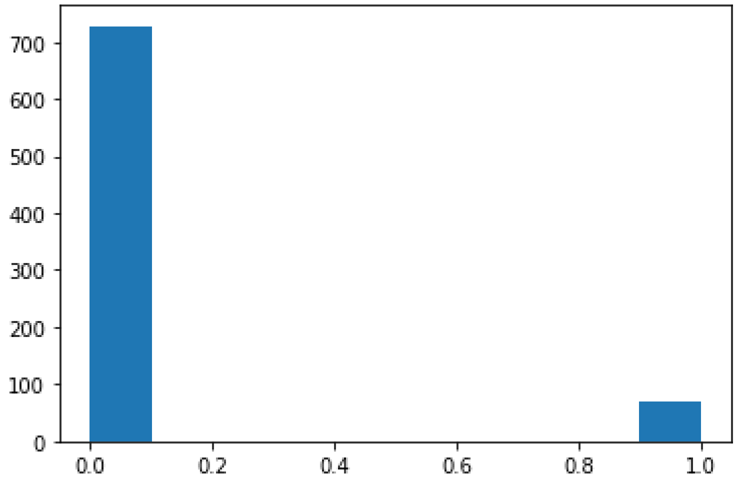
## Question 9

```
class MHwG_sampler_per_block(MH_sampler):
  def __init__(self, sigma_prop, N, observations, theta, T, theta_parameters, sigma_t0 = 2
    super(MHwG_sampler_per_block, self).__init__(sigma_prop, N, observations, theta, T, th

  def sample_xi(self,index):
    return draw_from_normal_distr(1, self.xi[index], self.sigma_prop)

  def sample_tau(self,index):
    return draw_from_normal_distr(1, self.tau[index], self.sigma_prop)

  def sample_t0(self):
```

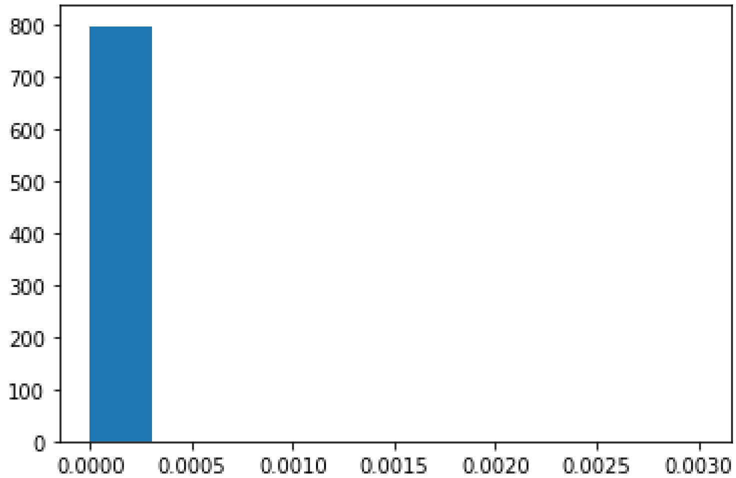
```
      return draw_from_normal_distr(1, self.t0, self.sigma_prop)

    def sample_v0(self):
      return draw_from_normal_distr(1, self.v0, self.sigma_prop)

    def acceptance_rejection(self, t0, v0, xi, tau):
      #acceptance-rejection
      lg_k = self.log_likelihood(self.t0, self.v0, self.xi, self.tau)
      self.log_likelihoods.append(lg_k)
      alpha = min(0, (self.log_likelihood(t0, v0, xi, tau) - lg_k)[0])

      self.alphas.append(alpha)

      u = np.log(np.random.uniform())
      #acceptance
      if u<=alpha:
        self.compteur_acceptance += 1
        self.t0 = t0
        self.v0 = v0
        self.xi = xi
        self.tau = tau
      #rejectance - do nothing

    def step_fixed_effects(self):
      #proposal
      v0 = self.sample_v0()
      t0 = self.sample_t0()
      self.acceptance_rejection(t0, v0, self.xi, self.tau)

    def step_per_individuals(self, index):
      #proposal
      xi_i = self.sample_xi(index)
      proposition_xi = self.xi.copy()
      proposition_xi[index] = xi_i

      tau_i = self.sample_tau(index)
      proposition_tau = self.tau.copy()
      proposition_tau[index] = tau_i

      self.acceptance_rejection(self.t0, self.v0, proposition_xi, proposition_tau)




  class MCMC_SAEM_with_MHwG_sampler_per_block(MCMC_SAEM):
    def __init__(self, sigma_prop, N, observations, T, theta_parameters, sigma_t0 = 2, sigma
      super(MCMC_SAEM_with_MHwG_sampler_per_block, self).__init__(sigma_prop, N, observation

    def sample(self, maxIterSampling):
      #Simulation
      sampler = MHwG_sampler_per_block(self.sigma_prop, self.N, self.observations, self.thet
      sampler.initialize_algo(t0 = self.t0, v0 = self.v0, xi = self.xi, tau = self.tau)

      #fixed effects
      for i in range(maxIterSampling):
```

```
        sampler.step_fixed_effects()

      #individuals
      for j in range(len(self.xi)):
        for i in range(maxIterSampling):
          sampler.step_per_individuals(j)


      return sampler





initial_guess_theta = [-5, 2, .5, .6, 1]


solver = MCMC_SAEM_with_MHwG_sampler_per_block(sigma_prop, N, observations, T, theta_param
solver.initialize_algo(initial_guess_theta, maxIter = maxIter, alpha_burn_in = 0.6)


for j in range(maxIter):
  solver.step(maxIterSampling = maxIterSampling)
```

```
---iteration number 0 -----
acceptance counter:  71
```




```
------Start Learning (eps<1) ---
---iteration number 100 -----
acceptance counter:  0
```



```
print(solver.theta)
print(theta_true)

predictions =solver.v0*(np.exp(solver.xi)*(solver.T - solver.t0 - solver.tau))
for i in range(N):
  plt.plot(T[i,:], predictions[i,:], label = "Prediction")
  plt.plot(T[i,:], samples[i,:], label = "True")
  plt.legend()
  plt.show()
```

Exercice 3 :

(1) Puisque $X_{n+1} \sim f_{X|Y}(\cdot, Y_n)$,

$X_{n+1}$ est $\sigma(Y_n)$ - mesurable.

De même, $Y_{n+1}$ est $\sigma(X_{n+1})$-mesurable donc $\sigma(Y_n)$- mesurable.

Ainsi $(X_{n+1}, Y_{n+1})$ est $\sigma(Y_n)$-mesurable donc $\sigma((X_n, Y_n))$ - mesurable

Donc puisque $\sigma(X_n, Y_n) \subset \mathcal{F}_n$,

$$\mathbb{P}((X_{n+1}, Y_{n+1}) \in A \mid \mathcal{F}_n) = \mathbb{P}((X_{n+1}, Y_{n+1}) \in A \mid X_n, Y_n)$$

Donc $(X_n, Y_n)_{n \in \mathbb{N}}$ est une chaîne de Markov.

Soit $A \in \mathcal{B}(\mathbb{R}^{p+q})$, on note $A = A^p \times A^q$.

$$\mathbb{P}((X_{n+1}, Y_{n+1}) \in A \mid X_n, Y_n) = \int_{A^p} f_{X|Y}(x, Y_n)\, dx \times \int_{A^q} f_{Y|X}(x, y)\, dy$$

$$= \int_{x \in A^p, y \in A^q} f_{X|Y}(x, Y_n)\, f_{Y|X}(x, y)\, dy\, dx$$

(2) J'ai montré plus haut que $Y_{n+1}$ est $\sigma(Y_n)$ -mesurable donc $(Y_n)_n$ est une chaîne de Markov.

On trouve son transition kernel grâce à la question précédente.

$$\mathbb{P}(Y_{n+1} \in A^q \mid Y_n) = \mathbb{P}(Y_{n+1} \in A^q \cap X_{n+1} \in \mathbb{R}^p \mid Y_n)$$

$$= \int_{y \in A^q} \int_{x \in \mathbb{R}^p} f_{X|Y}(x, Y_n) f_{Y|X}(x, y)\, dx\, dy$$

Let's note $P_Y$ this kernel ( $P_Y(y, A) = \mathbb{P}(Y_{n+1} \in A \mid Y_n = y)$ )

(2) Let's note $\mu$ such that $\forall A \in \mathcal{B}(\mathbb{R}^1), \mu(A) = \int_A f_Y(y)\,dy$.

Avec $A \in \mathcal{B}(\mathbb{R}^1)$

$$\mu P_Y(A) = \int_{\mathbb{R}^p} P_Y(y, A) f_Y(y)\,dy$$

$$= \int_{y \in \mathbb{R}^p} \int_{y' \in A} \int_{x \in \mathbb{R}^q} \underbrace{f_Y(y) f_{X|Y}(x,y)}_{= f(x,y)} \underbrace{f_{Y|X}(x,y')}_{= \frac{f(x,y')}{f_X(x)}} \,dx\,dy\,dy'$$

en intégrant par rapport à $y$

$$= \int_{y' \in A} \int_{x \in \mathbb{R}^q} \frac{f_X(x)}{f_X(x)} f(x,y')\,dx\,dy'$$

$$= \int_{y' \in A} f_Y(y')\,dy = \mu(A)$$

Notons que je peux intégrer comme je veux grâce à Fubini parce que toutes les quantités sont positives

Notons aussi que j'ai contourné le problème de définition de $f_{X|Y}$ et $f_{X|Y}$ en considérant que les quantités sont nulles en dehors de $Y$ et $X$.

(3) Avec la loi $\Gamma(\alpha, \beta)$ de pdf $: \Gamma_{\alpha,\beta}(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$,

on reconnaît que $f(x,y) \propto \Gamma_{\alpha,\beta}(y)$ avec $\begin{cases} \alpha = \frac{5}{2} \\ \beta = \frac{x^2}{2} + 2 \end{cases}$,

Donc $y|x \sim \Gamma\left(\frac{5}{2}, \frac{x^2}{2} + 2\right)$.

De même, $x|y \sim \mathcal{N}\left(0, \frac{1}{\sqrt{y}}\right)$.

Donc pour l'algo de Gibbs :

- Initialisation : $y^{(0)} > 0$, $x^{(0)} \in \mathbb{R}$
- Pour $k$ de $1$ à $n$ :

sample $x^{(k)} \sim \mathcal{N}\left(0, \frac{1}{\sqrt{y^{(k-1)}}}\right)$

sample $y^{(k)} \sim \Gamma\left(\frac{5}{2}, \frac{x^{(k-1)2}}{2} + 2\right)$