MATH 2010?

$\vec{x} = (x_1, x_2, \cdots, x_n)$

$f(\vec{x}) = \sum_{i=1}^{n} x_i$.

$f(\vec{x}) = x_1 + x_2 + \cdots + x_n = f\left(\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}\right)$.

$\boxed{\text{matrix calculus}}$. (wikipedia).

n- layout notation. d- layout notation

$$\frac{d}{d\vec{x}} f(\vec{x}) = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

$f(\vec{x}) = x^T A x$. A is fixed, known.

$g(x) = ax^2 \Rightarrow g'(x) = \frac{d}{dx}(ax \cdot x) = a \cdot x + ax \cdot 1 = 2ax$.

$$\frac{d}{d\vec{x}} f(\vec{x}) = \frac{d}{d\vec{x}} (\overset{\textcircled{2}}{x^T \underline{Ax}}) = x^T A^T + x^T A = 2x^T A.$$

①

A is symmetric. $x^T A x = (x^T A x)^T = x^T A^T x \in \mathbb{R}$. $a \in \mathbb{R}$. $a^T = a$.

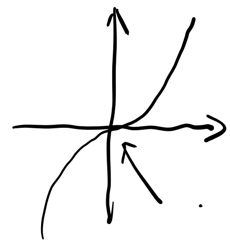
$A^* = \frac{1}{2}(A + A^T)$.

$\rightarrow x^T A x = x^T A^T x = x^T A^* x$. $\leftarrow$ A is symmetric.

(new A) $= \frac{1}{2}(A + A^T)$

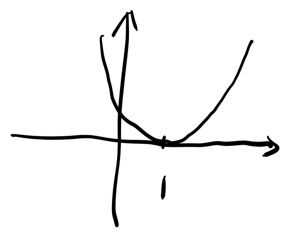
Second order condition. $f(\vec{x})$.

$\min_x f(x)$. $f'(x) = 0$. $f(x) = x^3$.



$f'(x) = 3x^2 \quad f'(0) = 0$.

$f''(x) > 0$. $f(x) = (x-1)^2$.



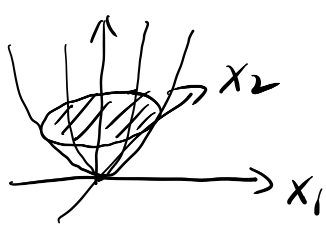
$f'(x) = 2(x-1)$. $f'(x) = 0 \Rightarrow x = 1$.

$f''(x) = 2 > 0$.

Hessian matrix.

$f''(\vec{x}) > 0$.

$$f(\vec{x}) = f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = x_1^2 + x_2^2.$$



$$f'(\vec{x}) = \begin{pmatrix} \boxed{2x_1} \\ 2x_2 \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{pmatrix}$$

$$f''(\vec{x}) = \begin{pmatrix} \frac{\partial}{\partial x_1}\left(\frac{\partial f}{\partial x_1}\right) & \frac{\partial}{\partial x_2}\left(\frac{\partial f}{\partial x_1}\right) \\ \frac{\partial}{\partial x_1}\left(\frac{\partial f}{\partial x_2}\right) & \frac{\partial}{\partial x_2}\left(\frac{\partial f}{\partial x_2}\right) \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} > 0. \qquad \text{positive definite.}$$

$v \in \mathbb{R}^2$. $\quad v^T f''(\vec{x}) v > 0$. for $\forall$ $v \neq 0$.

$\uparrow \quad (=0)$.

$$(v_1 \ v_2) \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = (2v_1, 2v_2) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 2(v_1^2 + v_2^2) > 0.$$

$v = (v_1, v_2) \neq \vec{0}$.

satisfying 2-order condition. $\leftarrow$ f. differentiable.

$$\boxed{\frac{d}{d\vec{x}} f} = \vec{0}. \qquad \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

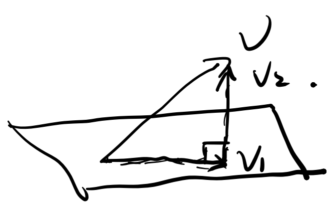
$H = X(X^T X)^{-1} X^T$.

v. $\quad f(v) = Hv$. $\quad$ $f(v)$ projects v onto $C(X)$.

$Y = X\beta + \varepsilon$. $\quad \hat{\beta} = (X^T X)^{-1} X^T Y$. $\quad \hat{y} = X\hat{\beta} = X(X^T X)^{-1} X^T Y = HY$.

① $v \perp C(X)$. $\quad Hv = 0$. ✓

② $v \in C(X)$. $\quad Hv = v$. ✓



$v = v_1 + v_2$.

$\quad \in C(X) \quad \perp C(X)$.

$H(v) = v_1 + 0 = v_1$.

① $v \perp C(X)$ $\quad v^T X = 0. \Rightarrow X^T v = 0.$

$$Hv = X(X^TX)^{-1}\underbrace{X^T v}_{=0} = 0.$$

② $v \in C(X)$ $\quad v = X\vec{\lambda}. \quad X_{n\times p}. \quad \vec{\lambda}_{p\times 1}.$

$$\Rightarrow Hv = X(X^TX)^{-1}X^T v = X\underbrace{\boxed{(X^TX)^{-1}X^TX}}_{I}\lambda.$$

$$= X\lambda$$

$$= v.$$