STAT 5010 Tutorial 1*

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Definitions

1. For X, $\{X_n\}$ random variables, F, $\{F_n\}$ the corresponding distribution functions,
   - $X_n \xrightarrow{a.s.} X$: $\mathbf{P}(\lim_{n\to\infty} X_n = X) = 1$
   - $X_n \xrightarrow{L^r} X$: $\lim_{n\to\infty} \mathbf{E}|X_n - X|^r = 0$
   - $X_n \xrightarrow{p} X$: $\forall\, \delta > 0, \quad \lim_{n\to\infty} \mathbf{P}(|X_n - X| > \delta) = 0$
   - $F_n \xrightarrow{w} F$: $\forall\, x$ continuity point of F, $\lim_{n\to\infty} F_n(x) = F(x)$
   - $X_n \xrightarrow{d} X$: $F_n \xrightarrow{w} F$
2. Let $X_1, X_2, \ldots$ be random vectors and $Y_1, Y_2, \ldots$ be random variables defined on a common probability space.
   - $X_n = O_p(Y_n)$ if and only if, $\forall\, \epsilon > 0$, $\exists\, C_\epsilon > 0$ such that $\sup_n \mathbf{P}(||X_n|| \geq C_\epsilon |Y_n|) < \epsilon$.
   - $X_n = o_p(Y_n)$ if and only if $X_n/Y_n \xrightarrow{p} 0$

Propositions and Theorems

1. Chebyshev's Inequality: $\forall\, t > 0$, $\mathbf{P}(|X - EX| \geq t) \leq \frac{Var(X)}{t^2}$

   Proof. $t^2\mathbf{P}(|X - EX| \geq t) = t^2\mathbf{P}(|X - EX|^2 \geq t^2) = t^2\mathbf{E}[I_{|X-EX|^2 \geq t^2}] \leq E[|X - EX|^2] = Var(X)$ ∎

2. $X_n \xrightarrow{a.s.} X \Leftrightarrow \forall\, \epsilon > 0, \quad \lim\limits_{k\to\infty} \mathbf{P}(\bigcup\limits_{n=k}^{\infty} \{||X_n - X|| > \epsilon\}) = 0$

   Proof. $\mathbf{P}(\lim_{n\to\infty} X_n = X) = \mathbf{P}(\bigcap\limits_{k=1}^{\infty} \bigcup\limits_{N=1}^{\infty} \bigcap\limits_{n=N}^{\infty} \{|X_n - X| < \frac{1}{k}\}) = \lim\limits_{k\to\infty} \mathbf{P}(\bigcup\limits_{N=1}^{\infty} \bigcap\limits_{n=N}^{\infty} \{|X_n - X| < \frac{1}{k}\}) = 1$.
   Note that $A_k \triangleq \bigcup\limits_{N=1}^{\infty} \bigcap\limits_{n=N}^{\infty} \{|X_n - X| < \frac{1}{k}\}$ are nonincreasing in k. The equations above hold if and only if $\mathbf{P}(A_k) = 1$ for all k, which is equivalent to the statement in the right hand side. ∎

3. $X_n \xrightarrow{L^r} X \Rightarrow X_n \xrightarrow{p} X$

   Proof. Similar to the proof of Chebyshev's Inequality. ∎

4. $X_n \xrightarrow{a.s.} X \Rightarrow X_n \xrightarrow{p} X$

   Proof. The result follows directly from the proposition 2 above. ∎

5. $X_n \xrightarrow{p} X \Rightarrow X_n \xrightarrow{d} X$

   Proof. Let x be a continuity point of F, $\epsilon > 0$ given,

   $$\begin{aligned} F(x - \epsilon) &= \mathbf{P}(X \in (-\infty, x - \epsilon]) \\ &\leq \mathbf{P}(X \in (-\infty, x - \epsilon],\ X_n \notin (-\infty, x]) + \mathbf{P}(X_n \in (-\infty, x]) \\ &\leq F_n(x) + \mathbf{P}(|X_n - X| > \epsilon) \end{aligned}$$

   Letting $n \to \infty$, we obtain that $F(x - \epsilon) \leq \liminf_n F_n(x)$.
   Similarly, we have $F(x + \epsilon) \geq \limsup_n F_n(x)$. Since ϵ is arbitrary and F is continuous at x,

   $$F(x) = \lim_{n\to\infty} F_n(x)$$

   ∎

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6. The corresponding converses of propositions 3-5 are all incorrect.

   Proof. Construction of counter examples. ∎

7. $X_n \xrightarrow{p} c \Leftrightarrow X_n \xrightarrow{d} c$, here c is a constant.

8. Slutsky's Theorem: For random variables X_n, Y_n on a probability space, $X_n \xrightarrow{d} c$, $\quad Y_n \xrightarrow{d} Y$, where c is a fixed real number. Then $X_n + Y_n \xrightarrow{d} c + Y$.

Question 1

1. If random variables $X_n = O_p(1)$, $Y_n = o_p(1)$, prove that $X_nY_n = o_p(1)$.

Solution:

Proof. $\forall\ \epsilon,\ \delta > 0$, by definition, one can find M s.t. $\sup_n \mathbf{P}(|X_n| > M) < \frac{\epsilon}{2}$ and $N_0 \in \mathbf{N}^+$ s.t. $\mathbf{P}(|Y_n| > \frac{\delta}{M}) < \frac{\epsilon}{2}$ holds for all $n > N_0$. Then, for all $n > N_0$,

$$\begin{aligned}\mathbf{P}(|X_nY_n| > \delta) &= \mathbf{P}(|X_n| > M,\ |X_nY_n| > \delta) + \mathbf{P}(|X_n| \le M,\ |X_nY_n| > \delta)\\ &\le \mathbf{P}(|X_n| > M) + \mathbf{P}(|Y_n| > \frac{\delta}{M})\\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\end{aligned}$$

Therefore, $X_nY_n = o_p(1)$. ∎

Question 2

1. $X_1, X_2, \ldots, X_n, \ldots$ are a series of i.i.d. random variables following $Uniform[0,1]$ and defined on the same probability space. Give the asymptotic distribution of $n(1 - X_{(n)})$.

Solution: For $\forall\, t > 0$ fixed, $n > t$ eventually,

$$\begin{aligned}\mathbf{P}(n(1 - X_{(n)}) \le t) &= \mathbf{P}(X_{(n)} \ge 1 - \frac{t}{n}) = 1 - \mathbf{P}(X_{(n)} \le 1 - \frac{t}{n})\\ &= 1 - (1 - \frac{t}{n})^n \xrightarrow{n\to\infty} 1 - e^{-t}.\end{aligned}$$

Therefore, $n(1 - X_{(n)})$ is asymptotically distributed as $exp(1)$.

2. Is it possible to have a random variable Y s.t. $n(1 - X_{(n)}) \xrightarrow{a.s.} Y$?

Solution: NO! For simplicity, use notation: $P_n \triangleq \max_{1\le i\le n} X_i$, $Q_n \triangleq \max_{1\le i\le 2n} X_i$. Suppose there exists such Y, we naturally have $n(1 - X_{(n)}) \xrightarrow{p} Y$ and $Y \sim exp(1)$. Hence,

$$2n(1 - P_n) \xrightarrow{p} 2Y, \qquad 2n(1 - Q_n) \xrightarrow{p} Y.$$

Thus, take difference and we get

$$2n(P_n - Q_n) \xrightarrow{p} Y.$$

Note that $\mathbf{P}(P_n = Q_n) \ge \frac{1}{2}$, we have $\mathbf{P}(Y = 0) \ge \frac{1}{2}$ and we have contradiction here given that $Y \sim exp(1)$.