Computer Science

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Contents

Chapter 912

Power series and generating functions

912.1 Products of power series products-of-power-series.tex

Note that from the previous section, sums of two power series can be easily calculated to form a single power series. We also talked about multiplying a *polynomial* with a power series. What about the product of a *power series* with another power series? We have this:

$$\begin{aligned}\sum_{n=0}^{\infty} a_n x^n \cdot \sum_{n=0}^{\infty} b_n x^n &= \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots\right) \cdot \left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots\right) \\ &= a_0 b_0 \\ &\quad + (a_0 b_1 + a_1 b_0) x \\ &\quad + (a_0 b_2 + a_1 b_1 + a_2 b_0) x^2 \\ &\quad + (a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0) x^3 \\ &\quad + (a_0 b_4 + a_1 b_3 + a_2 b_2 + a_3 b_1 + a_4 b_0) x^4 \\ &\quad + \cdots\end{aligned}$$

Note that the x^n–term looks like this:

$$\left(a_0 b_n + a_1 b_{n-1} + \cdots a_n b_0\right) x^n = \left(\sum_{k=0} a_k b_{n-k}\right) x^n$$

Some people also write the summation on the right this way:

$$\sum_{k=0}^{n} a_k b_{n-k} = \sum_{k+\ell=n} a_k b_\ell$$

where the summation on the right "$\sum_{k+\ell=n}(...)$" means "sum over all terms (...) where integers $k \geq 0$ and $\ell \geq 0$ such that $k + \ell = n$". Let me state the above as a small proposition:

Proposition 912.1.1.

$$\sum_{n=0}^{\infty} a_n x^n \cdot \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k}\right) x^n$$

By the way, I hope it's clear that the above also applies to the product of two polynomials. Given two polynomials to multiply, you can extend them to power series by including terms of the form $0x^n$ to each of them:

$$(1+2x+5x^2)(3+x-7x^2) = (1+2x+5x^2+0x^3+0x^4+\cdots)(3+x-7x^2+0x^3+0x^4+\cdots)$$

As an aside, if you are given two finite sequences of the same length, $(a_0, ..., a_n)$ and $(b_0, ..., b_n)$, there are two "products" for these two sequences that you will see frequently in CS, math, engineering, physics, etc. The **dot product** of $(a_0, ..., a_n)$ and $(b_0, ..., b_n)$ is defined to be

$$(a_0, ..., a_n) \cdot (b_0, ..., b_n) = a_0 b_0 + \cdots a_n b_n = \sum_{k=0}^{n} a_k b_k$$

The **convolution** of $(a_0, ..., a_n)$ and $(b_0, ..., b_n)$ is defined to be

$$(a_0, ..., a_n) * (b_0, ..., b_n) = a_0 b_n + a_1 b_{n-1} + \cdots + a_{n-1} b_1 + a_n b_0 = \sum_{k+\ell=n} a_k b_\ell$$

For dot products, the sequences come from vectors and can be used to detect if two vectors are parallel and can be used to define the angle between the two vectors. Convolutions come from many areas, not just in the coefficients of products of power series. If you want to analyze some image data and $(b_0, ..., b_n)$ is a collection of adjacent pixels, then for a specially chosen $(a_0, ..., a_n)$, the convolution will give you some useful about $(b_0, ..., b_n)$. These special $(a_0, ..., a_n)$ are called filters and appear in image processing and computer vision such as cleaning up an image, blurring an image, sharpening objects in the image, edge detection of objects in the image, facial detection, etc. Convolution is also used in analyzing audio streams and video streams.

Exercise 912.1.1.

Some simple drills for you. Answer the following using Proposition **??** (on p **??**).

(a) What is the coefficient of x^3 of

$$\sum_{n=0}^{\infty} \frac{1}{2^n} x^n \cdot \sum_{n=0}^{\infty} \frac{1}{2^n} x^n$$

What about the coefficient x^n? Simplify it (write it in closed form) and write the above product as a power series.

(b) Do the same for

$$\sum_{n=0}^{\infty} \frac{1}{2^n} x^n \cdot \sum_{n=0}^{\infty} \frac{1}{3^n} x^n$$

(c) Is it possible to compute the coefficient of x^3 of

$$(1 + 5x - 3x^2 + 2x^3 + 3x^4 + \cdots) \cdot (1 - 2x + 3x^2 + 4x^3 - 3x^4 + \cdots)$$

if I only give you the first 5 terms of each power series? (The rest were lost in transmission.) Which coefficients can you determine correctly?

(Go to solution, page **??**) □

Let's look at one of the simplest power series:

$$\sum_{n=0}^{\infty} x^n$$

What if you need to multiply two of them?

$$\sum_{n=0}^{\infty} x^n \cdot \sum_{n=0}^{\infty} x^n$$

Of course you can expand the series out and multiply term-by-term:

$$\begin{aligned}
&\sum_{n=0}^{\infty} x^n \cdot \sum_{n=0}^{\infty} x^n \\
&= (1 + x + x^2 + x^3 + \cdots)(1 + x + x^2 + x^3 + \cdots) \\
&= 1 \cdot 1 + (1 \cdot x + x \cdot 1) + (1 \cdot x^2 + x \cdot x + x^2 \cdot 1) + (1 \cdot x^3 + x \cdot x^2 + x^2 \cdot x + x^3 \cdot 1) + \cdots \\
&= 1 + 2x + 3x^2 + 4x^3 + \cdots
\end{aligned}$$

Once you've done enough terms (say up to x^5), you suspect the product is

$$\sum_{n=0}^{\infty} (n+1)x^n$$

But a verification up $n = 5$ does not mean a conjecture is true for all $n \geq 0$. Furthermore, what if the product is more complex:

$$\sum_{n=0}^{\infty} \frac{1}{4^n} x^n \cdot \sum_{n=0}^{\infty} \frac{2^n}{3^n} x^n$$

so that making a guess is not that easy, plus the verification is usually too time consuming and error prone. So what we should we do? Is there a faster and easier way to compute the product?

If we go back to the first example:

$$\sum_{n=0}^{\infty} x^n \cdot \sum_{n=0}^{\infty} x^n$$

The trick is to note that

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

and hence

$$\sum_{n=0}^{\infty} x^n \cdot \sum_{n=0}^{\infty} x^n = \left(\frac{1}{1-x}\right) \cdot \left(\frac{1}{1-x}\right) = \left(\frac{1}{1-x}\right)^2$$

And using the formula for powers of the geometric series (case of $k = 2$):

$$\left(\frac{1}{1-x}\right)^k = \sum_{n=0}^{\infty} \binom{k+n-1}{n} x^n$$

we have

$$\begin{aligned}
\sum_{n=0}^{\infty} x^n \cdot \sum_{n=0}^{\infty} x^n &= \left(\frac{1}{1-x}\right)^2 \\
&= \sum_{n=0}^{\infty} \binom{2+n-1}{n} x^n \\
&= \sum_{n=0}^{\infty} \binom{n+1}{n} x^n \\
&= \sum_{n=0}^{\infty} \binom{n+1}{1} x^n \\
&= \sum_{n=0}^{\infty} (n+1) x^n
\end{aligned}$$

The above does not involve verification and making a guess on coefficients up to $n = 5$. The computation is simply true *for all* $n \geq 0$.

I hope you realize the reason the second computation is simpler than the first attempt is because this computation

$$\begin{aligned}
&\sum_{n=0}^{\infty} x^n \cdot \sum_{n=0}^{\infty} x^n \\
&= (1 + x + x^2 + x^3 + \cdots)(1 + x + x^2 + x^3 + \cdots) \\
&= 1 \cdot 1 + (1 \cdot x + x \cdot 1) + (1 \cdot x^2 + x \cdot x + x^2 \cdot) + (1 \cdot x^3 + x \cdot x^2 + x^2 \cdot x + x^3 \cdot 1) + \cdots
\end{aligned}$$

involves pairing *infinitely* many x^n from the left power series with *infinitely* many x^n from the right power series. But the second computation

$$\sum_{n=0}^{\infty} x^n \cdot \sum_{n=0}^{\infty} x^n = \left(\frac{1}{1-x}\right) \cdot \left(\frac{1}{1-x}\right)$$

converts the power series to rational expressions $\dfrac{1}{1-x}$ which are finitary objects. See it?

Exercise 912.1.2. What is the coefficient of x^n in the power series of

$$\left(\sum_{n=0}^{\infty} x^n \right)^{100}$$

(Go to solution, page **??**) □

Exercise 912.1.3. What is the coefficient of x^n in the power series of

$$\left(2 + 5x + \frac{7}{1-x}\right)\left(\sum_{n=0}^{\infty} x^n\right)^{100}$$

(Go to solution, page **??**) □

Of course this helps only when the series we're multiplying together are all

$$\sum_{n=0}^{\infty} x^n$$

What if they are different? What about

$$\sum_{n=0}^{\infty} \frac{1}{4^n} x^n \cdot \sum_{n=0}^{\infty} \frac{2^n}{3^n} x^n$$

Then using the geometric series, with x replaced by $x/4$,

$$\sum_{n=0}^{\infty} \left(\frac{x}{4}\right)^n = \frac{1}{1 - x/4} = \frac{4}{4 - x}$$

Likewise

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{2^n}{3^n} x^n &= \sum_{n=0}^{\infty} \left(\frac{2x}{3}\right)^n \\ &= \frac{1}{1 - 2x/3} \\ &= \frac{3}{3 - 2x} \end{aligned}$$

So we now know that

$$\sum_{n=0}^{\infty} \frac{1}{4^n} x^n \cdot \sum_{n=0}^{\infty} \frac{2^n}{3^n} x^n = \frac{4}{4 - x} \cdot \frac{3}{3 - 2x} = \frac{12}{(4 - x)(3 - 2x)}$$

Of course we can try to apply the Maclaurin/Taylor series to develop a power series for the function

$$\frac{12}{(4 - x)(3 - 2x)}$$

But let's not forget that coefficients provided by the Maclaurin/Taylor series is

$$\frac{f^{(n)}(0)}{n!}$$

which requires you to compute the n–th derivative until you see a pattern. That's time-consuming and also error-prone.

We'll solve this problem via a different route: We'll use the theory of partial fractions. According to the theory of partial fractions, there are constants A

and B such that

$$\frac{1}{(4-x)(3-2x)} = \frac{A}{4-x} + \frac{B}{3-2x}$$

More generally, if $ax+b$ and $cx+d$ are degree 1 polynomials with distinct roots (i.e., $-b/a \neq -d/c$), then

$$\frac{1}{(ax+b)(cx+d)} = \frac{A}{ax+b} + \frac{B}{cx+d}$$

where A, B are constants.

Assuming we can find the constants A and B, and here's the kicker, notice that ... **the theory of partial fractions has changed the product problem into an addition problem!** Get it? Look at this:

$$\frac{12}{(4-x)(3-2x)} = 12\left(\frac{A}{4-x} + \frac{B}{3-2x}\right)$$

If we have A, we can definitely write

$$\frac{A}{4-x}$$

as a power series. Likewise for $\dfrac{B}{3-2x}$. We then add up the series to get a new series. We'll get back to the general theory of partial fractions later. Right now you only need to know that this process of breaking up product of functions into sums of functions does not work for all functions. For instance it is *not* true that you can find constants A and B such that

$$\frac{1}{\sin x \cdot \cos x} = \frac{A}{\sin x} + \frac{B}{\cos x}$$

The theory of partial fractions applies only to rational functions, i.e. functions which are fractions of polynomials. Examples are

$$\frac{x}{(2x+1)(3x+5)}, \quad \frac{2x+1}{(2x+1)^2(3x^2+5)}, \quad \text{etc.}$$

OK, let's get back to finding A and B. Here's what the theory of partial fractions says:

$$\frac{1}{(4-x)(3-2x)} = \frac{A}{4-x} + \frac{B}{3-2x}$$

There are two standard ways to find A and B. I'll show you one of them.

You first clear denominators by multiplying both sides of the equation with $(4-x)(3-2x)$ to get

$$1 = A(3-2x) + B(4-x)$$

This is an identity involving two unknowns. All you need to do is to substitute two values of x into the identity to get two equations. You can then solve for A and B quickly. I'll pick the easiest values for x, i.e., x values that will force some of the terms to be zero. First I'll choose $x = 4$. This gives me

$$1 = A(3-8) + 0$$

which gives me

$$A = -\frac{1}{5}$$

Next I'll choose $x = 3/2$. This gives us

$$1 = 0 + B(4 - 3/2)$$

i.e.

$$B = \frac{2}{5}$$

Substituting the values of A and B we get

$$\sum_{n=0}^{\infty} \frac{1}{4^n} x^n \cdot \sum_{n=0}^{\infty} \frac{2^n}{3^n} x^n = \frac{12}{(4-x)(3-2x)} = 12\left(-\frac{1}{5} \cdot \frac{1}{4-x} + \frac{2}{5} \cdot \frac{1}{3-2x}\right)$$

Now we simply note that

$$\begin{aligned}
\frac{1}{4-x} &= \frac{1}{4} \cdot \frac{1}{1-x/4} \\
&= \frac{1}{4} \sum_{n=0}^{\infty} \left(\frac{x}{4}\right)^n \\
&= \sum_{n=0}^{\infty} \frac{1}{4} \left(\frac{1}{4}\right)^n x^n
\end{aligned}$$

and

$$\begin{aligned}\frac{1}{3-2x} &= \frac{1}{3}\cdot\frac{1}{1-2x/3}\\ &= \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{2x}{3}\right)^n\\ &= \sum_{n=0}^{\infty}\frac{1}{3}\left(\frac{2}{3}\right)^n x^n\end{aligned}$$

Putting these series into the above equation we get

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{1}{4^n}x^n\cdot\sum_{n=0}^{\infty}\frac{2^n}{3^n}x^n &= 12\left(-\frac{1}{5}\cdot\frac{1}{4-x}+\frac{2}{5}\cdot\frac{1}{3-2x}\right)\\ &= 12\left(-\frac{1}{5}\sum_{n=0}^{\infty}\frac{1}{4}\left(\frac{1}{4}\right)^n x^n+\frac{2}{5}\sum_{n=0}^{\infty}\frac{1}{3}\left(\frac{2}{3}\right)^n x^n\right)\\ &= \sum_{n=0}^{\infty}12\left(-\frac{1}{5}\cdot\frac{1}{4}\left(\frac{1}{4}\right)^n+\frac{2}{5}\cdot\frac{1}{3}\left(\frac{2}{3}\right)^n\right)x^n\\ &= \sum_{n=0}^{\infty}\left(-\frac{3}{5}\left(\frac{1}{4}\right)^n+\frac{8}{5}\left(\frac{2}{3}\right)^n\right)x^n\end{aligned}$$

I'll check my work with this program

```
def f(x):
    s = 0.0
    for n in range(100):
        s += (1.0/4**n) * x**n
    return s

def g(x):
    s = 0.0
    for n in range(100):
        s += (2.0/3)**n * x**n
    return s

def h(x):
    s = 0.0
    for n in range(100):
        s += ((-3.0/5) * (1.0/4)**n + (8.0/5) * (2.0/3)**n) * x**n
    return s

for x in range(10):
    x = x/ 100
    print(f(x) * g(x), h(x))
```

The output is

```
1.0 1.0
1.009234495635061 1.009234495635061
1.0186065462447367 1.0186065462447371
1.0281190561867066 1.0281190561867066
1.0377750103777503 1.0377750103777499
1.047577477084243 1.0475774770842428
1.0575296108291032 1.057529610829103
1.067634655420915 1.0676346554209148
1.0778959471112388 1.0778959471112393
1.0883169178864889 1.088316917886488
```

Voilà! As you can see I've bypassed the method of multiplying lots of terms and then try to guess the formula for the coefficients (more or less the brute-force method). Note that we use the theory of partial fractions to break up a complex rational expression into into a linear sum of simpler rational expressions (called partial fractions).

Make sure you get the big picture: You want to multiply two power series to get a power series. Instead of multiplying the two power series, you take a detour, and rewrite the two power series as rational expressions and multiply the two rational expressions because multiplying two rational expressions is easier. Here's a summary of the above computations:

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \frac{1}{4^n} x^n \cdot \sum_{n=0}^{\infty} \frac{2^n}{3^n} x^n \\
&\overset{(a)}{=\!=} \frac{4}{4-x} \cdot \frac{3}{3-2x} && \text{convert power series to rational expression} \\
&\overset{(b)}{=\!=} \frac{12}{(4-x)(3-2x)} && \text{multiply rational expressions (instead of power series)} \\
&\overset{(c)}{=\!=} 12\left(-\frac{1}{5} \cdot \frac{1}{4-x} + \frac{2}{5} \cdot \frac{1}{3-2x}\right) && \text{write rational expression as linear sum of rational expressions} \\
&\overset{(d)}{=\!=} 12\left(-\frac{1}{5} \sum_{n=0}^{\infty} \frac{1}{4}\left(\frac{1}{4}\right)^n x^n + \frac{2}{5} \sum_{n=0}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^n x^n\right) && \text{convert rational expressions to power series} \\
&\overset{(d)}{=\!=} \sum_{n=0}^{\infty} \left(-\frac{3}{5}\left(\frac{1}{4}\right)^n + \frac{8}{5}\left(\frac{2}{3}\right)^n\right) x^n && \text{tidy up to one power series}
\end{aligned}
$$

This section is a quick introduction to (c), one special case of writing a complex rational expression to a linear sum of rational functions: if $ax + b$ and $cx + d$ are linear polynomials with distinct roots, then there are constants A, B such that

$$\frac{1}{(ax+b)(cx+d)} = \frac{A}{ax+b} + \frac{B}{cx+d}$$

In the next section, I'll go into details about partial fractions decomposition to handle more cases. And of course if the product is a product of the form

$$\frac{1}{1-x} \cdot \frac{1}{1-x}$$

then you do not need partial fractions, since you can use the formula for powers of geometric series.

Exercise 912.1.4. In a previous exercise you computed the power series of the following power series products:

(a)
$$\sum_{n=0}^{\infty} \frac{1}{2^n} x^n \cdot \sum_{n=0}^{\infty} \frac{1}{2^n} x^n$$

(b)
$$\sum_{n=0}^{\infty} \frac{1}{2^n} x^n \cdot \sum_{n=0}^{\infty} \frac{1}{3^n} x^n$$

using Proposition **??** (on page **??**). Now do it again by first converting the power series involved into rational expressions and then by using the theory of partial fractions. (Go to solution, page **??**) □

Solutions

Solution to Exercise **??**.

exercises/pow
series-11/ansv

(a) From

$$\sum_{n=0}^{\infty} \frac{1}{2^n} x^n \cdot \sum_{n=0}^{\infty} \frac{1}{2^n} x^n = \left(1 + \frac{1}{2}x + \frac{1}{4}x^2 + \frac{1}{8}x^3 + \cdots\right)\left(1 + \frac{1}{2}x + \frac{1}{4}x^2 + \frac{1}{8}x^3 + \cdots\right)$$

the coefficient of x^3 is

$$1 \cdot \frac{1}{8} + \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{2} + \frac{1}{8} \cdot 1 = 4 \cdot \frac{1}{8} = \frac{1}{2}$$

The coefficient of x^n is

$$\sum_{k=0}^{n} \frac{1}{2^k} \cdot \frac{1}{2^{n-k}} = \sum_{k=0}^{n} \frac{1}{2^k \cdot 2^{n-k}} = \sum_{k=0}^{n} \frac{1}{2^n} = \frac{n+1}{2^n}$$

(b) First let's derive the coefficient of x^n in general. The coefficient of x^n is

$$\begin{aligned}
\sum_{k=0}^{n} \frac{1}{2^k} \cdot \frac{1}{3^{n-k}} = \sum_{k=0}^{n} \frac{1}{2^k} \cdot \frac{3^k}{3^n} &= \frac{1}{3^n} \sum_{k=0}^{n} \left(\frac{3}{2}\right)^k \\
&= \frac{1}{3^n} \cdot \frac{1 - (3/2)^{n+1}}{1 - 3/2} \\
&= \frac{1}{3^n} \cdot \frac{1 - (3/2)^{n+1}}{-1/2} \\
&= \frac{1}{3^n} \cdot \frac{(3/2)^{n+1} - 1}{1/2} \\
&= \frac{2}{3^n} \cdot \left(\frac{3^{n+1}}{2^{n+1}} - 1\right) \\
&= 2 \cdot \left(\frac{3^{n+1} - 2^{n+1}}{2^{n+1} 3^n}\right) \\
&= \frac{3^{n+1} - 2^{n+1}}{6^n}
\end{aligned}$$

The coefficient of x^3 is

$$\frac{3^4 - 2^4}{6^4} = \frac{65}{216}$$

Solution to Exercise **??**.

exercises/power-series-15/answer

We have

$$
\begin{aligned}
\left(\sum_{n=0}^{\infty} x^n\right)^{100} &= \left(\frac{1}{1-x}\right)^{100} \\
&= \sum_{n=0}^{\infty} \binom{100+n-1}{n} x^n \\
&= \sum_{n=0}^{\infty} \binom{n+99}{n} x^n \\
&= \sum_{n=0}^{\infty} \binom{n+99}{99} x^n
\end{aligned}
$$

Hence the coefficient of x^n is $\binom{n+99}{99} x^n$ for $n \geq 0$.

Solution to Exercise ??.

exercises/power-series-16/answer

We have

$$\left(2+5x+\frac{7}{1-x}\right)\left(\sum_{n=0}^{\infty} x^n\right)^{100}$$
$$= \left(2+5x+\frac{7}{1-x}\right)\left(\frac{1}{1-x}\right)^{100}$$
$$= 2\left(\frac{1}{1-x}\right)^{100} + 5x\left(\frac{1}{1-x}\right)^{100} + \frac{7}{1-x}\left(\frac{1}{1-x}\right)^{100}$$
$$= 2\sum_{n=0}^{\infty}\binom{100+n-1}{n}x^n + 5x\sum_{n=0}^{\infty}\binom{100+n-1}{n}x^n + 7\left(\frac{1}{1-x}\right)^{101}$$
$$= 2\sum_{n=0}^{\infty}\binom{n+99}{n}x^n + 5x\sum_{n=0}^{\infty}\binom{n+99}{n}x^n + 7\sum_{n=0}^{\infty}\binom{101+n-1}{n}$$
$$= \sum_{n=0}^{\infty}2\binom{n+99}{99}x^n + \sum_{n=0}^{\infty}5\binom{n+99}{99}x^{n+1} + \sum_{n=0}^{\infty}7\binom{n+100}{n}x^n$$
$$= \sum_{n=0}^{\infty}2\binom{n+99}{99}x^n + \sum_{p=1}^{\infty}5\binom{p+98}{99}x^p + \sum_{n=0}^{\infty}7\binom{n+100}{n}x^n \quad (\text{let } p=n+1)$$
$$= \sum_{n=0}^{\infty}2\binom{n+99}{99}x^n + \sum_{n=1}^{\infty}5\binom{n+98}{99}x^n + \sum_{n=0}^{\infty}7\binom{n+100}{100}x^n \quad (\text{replace } p \text{ by } n)$$
$$= 2\binom{99}{99} + \sum_{n=1}^{\infty}2\binom{n+99}{99}x^n + \sum_{n=1}^{\infty}5\binom{n+98}{99}x^n + 7\binom{100}{100} + \sum_{n=1}^{\infty}7\binom{n+100}{100}$$
$$= 9 + \sum_{n=1}^{\infty}\left(2\binom{n+99}{99} + 5\binom{n+98}{99} + 7\binom{n+100}{100}\right)x^n$$

Hence the coefficient of x^n is

$$\begin{cases} 9 & \text{if } n=0 \\ 2\binom{n+99}{99} + 5\binom{n+98}{99} + 7\binom{n+100}{100} & \text{if } n>0 \end{cases}$$

Solution to Exercise **??**.

exercises/pow series-17/ansv

(a)

$$
\begin{aligned}
\sum_{n=0}^{\infty} \frac{1}{2^n} x^n \cdot \sum_{n=0}^{\infty} \frac{1}{2^n} x^n &= \left(\sum_{n=0}^{\infty} \frac{1}{2^n} x^n \right)^2 \\
&= \left(\sum_{n=0}^{\infty} \left(\frac{x}{2} \right)^n \right)^2 \\
&= \left(\sum_{n=0}^{\infty} \left(\frac{x}{2} \right)^n \right)^2 \\
&= \left(\frac{1}{1-(x/2)} \right)^2 \\
&= \sum_{n=0}^{\infty} \binom{2+n-1}{n} \left(\frac{x}{2} \right)^n \\
&= \sum_{n=0}^{\infty} \binom{n+1}{n} \left(\frac{1}{2} \right)^n x^n
\end{aligned}
$$

$$
\sum_{n=0}^{\infty} \left(\frac{n+1}{2^n} \right) x^n
$$

(b)

$$
\sum_{n=0}^{\infty} \frac{1}{2^n} x^n \cdot \sum_{n=0}^{\infty} \frac{1}{3^n} x^n
$$

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