Computer Science

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Contents

Chapter 912

Power series and generating functions

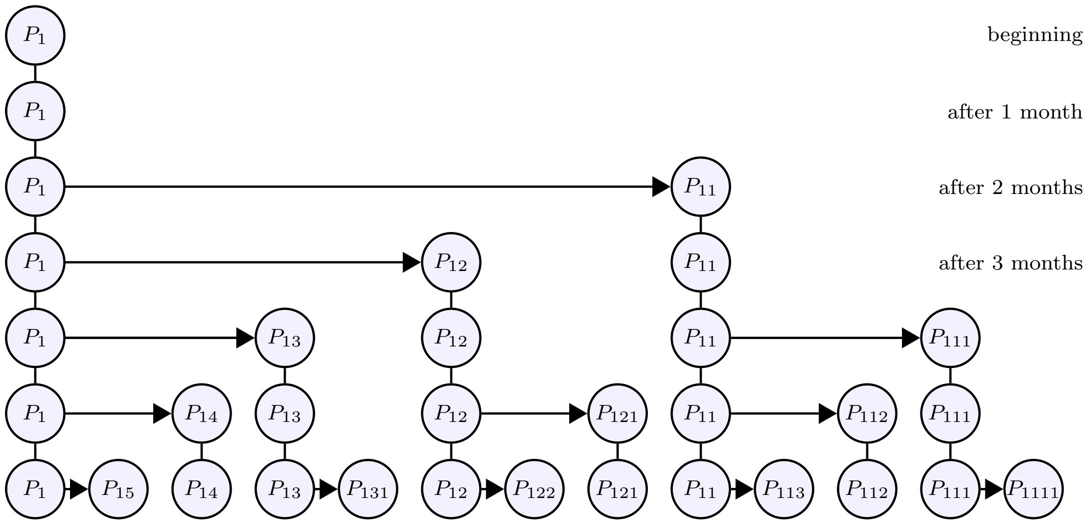
912.1 Recurrence relations recurrence-relations.tex

912.1.1 Fibonacci sequence

You have just received a birthday present. It's a pair of rabbits from the Neverdie Island. Let's call this pair of rabbits P_1. At the end of the first month, nothing happens – you still have P_1. Then suddenly at the end of the second month, the pair P_1 produces a pair of male and female baby rabbits P_{11}.

You see, every pair of male and female rabbits from Neverdie Island will start producing a pair of male and female rabbits after 2 months and continue to do so every month.

At the beginning there's 1 pair. After one month, there's still 1 pair (the same P_1). After two months, there are 2 pairs – there's P_1 and the offspring pair P_{11} of P_1. After 3 months, there are 3 pairs – there's P_1 and first offspring pair P_{11} and the new pair P_{12}. Note that P_{11} don't have offsprings yet. The following diagram might be more systematic:



The number of pairs at the beginning and at the end of each month give rise to the following sequence

$$1, 1, 2, 3, 5, 8, 13, ...$$

This is called the **Fibonacci sequence**. Let F_n be the number of pairs after n month. Then $F_0 = 1, F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, F_5 = 8, ...$

The question is this: How many pairs of rabbits do you have at the end of 12 months? Continuing the above "accounting" diagram is painful. So ... show that

$$F_n = F_{n-1} + F_{n-2}$$

if $n \geq 2$. And then compute F_{12}. (This problem appeared in the book Liber Abaci written by Fibonacci in 1202.)

(Hint: It's basic accounting. You want to keep track of which pair can reproduce. So I suggest you keep track of the ages of the pairs. Spoiler (hint) on next page ...)

WARNING ... INCOMING SPOILER ...

Hint.

You might want to keep track on the number of pairs of rabbits for three age groups: just born, one year old, at least two years old.

Spoiler (solution) on next page ...

WARNING ... INCOMING SPOILER ...

Solution.

Our goal is to prove $F_n = F_{n-1} + F_{n-2}$.

The proof is derived by basically keeping track of numbers (i.e. accounting). At time n, I have F_n pairs. I want to count the number of pairs F_{n+1} (number of pairs at time $n+1$) and also at time $n+2$. The number of pairs at time $n+1$ (and also $n+2$) depends on how many pairs can reproduce (right?). So I break up my pairs at time n by age:

```
Fn   = A (age 0) + B (age 1) + C (age >= 2)
```

where there are A pairs of age 0, B pairs of age 1, C pairs at age 2 and older. What happens at time $n+1$? I have this:

```
Fn   = A (age 0) + B (age 1)             + C (age >= 2)
       |           |                       |
       |           +-----------+           +-------------+
       |           |           |           |             |
Fn+1 = A (age 1) + B (age 2) + B (age 0) + C (age >=2) + C (age=0)
```

Correct? I'll let you do some basic accounting check on the above. Reorganizing by age groups, I get this:

```
Fn   = A (age 0) + B (age 1) + C (age >= 2)
Fn+1 = A (age 1) + B (age 2) + B (age 0) + C (age >=2) + C (age=0)
     = (B + C) (age 0) + A (age 1) + (B + C) (age >= 2)
```

OK. Now what happens at time $n+2$? I get this:

```
Fn   = A (age 0) + B (age 1) + C (age >= 2)
Fn+1 = A (age 1) + B (age 2) + B (age 0) + C (age >=2) + C (age=0)
     = (B + C) (age 0) + A (age 1)             + (B + C) (age >= 2)
       |                 |                       |
       |                 +-----------+           +-------------------+
       |                 |           |           |                   |
Fn+2 = (B + C) (age 1)   A (age 2) + A (age 0)   (B + C) (age >= 2)  (B + C) (age = 0)
```

Yes? Reorganizing, I get this

```
Fn   = A (age 0) + B (age 1) + C (age >= 2)
Fn+1 = A (age 1) + B (age 2) + B (age 0) + C (age >=2) + C (age=0)
     = (B + C) (age 0) + A (age 1) + (B + C) (age >= 2)
Fn+2 = (B + C) (age 1) + A (age 2) + A (age 0) + (B + C) (age >= 2)  (B + C) (age = 0)
     = (A + B + C) (age 0) + (B + C) (age 1) + (A + B + C) (age >= 2)
```

Ignoring the age data, I have this:

$$\begin{aligned} F_n &= A + B + C \\ F_{n+1} &= B + C + A + B + C = A + 2(B + C) \\ F_{n+2} &= A + B + C + B + C + A + B + C = 2A + 3(B + C) \end{aligned}$$

Therefore

$$F_n + F_{n+1} = (A + B + C) + (A + 2(B + C)) = 2A + 3(B + C) = F_{n+2}$$

Voilà! QED. $\square$

The above notation

```
Fn   = A (age 0) + B (age 1) + C (age >= 2)
Fn+1 = A (age 1) + B (age 2) + B (age 0) + C (age >=2) + C (age=0)
     = (B + C) (age 0) + A (age 1) + (B + C) (age >= 2)
Fn+2 = (B + C) (age 1) + A (age 2) + A (age 0) + (B + C) (age >= 2)  (B + C) (age = 0)
     = (A + B + C) (age 0) + (B + C) (age 1) + (A + B + C) (age >= 2)
```

is not very mathematical, hard to read, and messy.

To use proper mathematical notation, I will rewrite the above using a 3-tuple (A, B, C) to describe the number of pairs of age 0, age 1, and age ≥ 2. Let G_n be the 3-tuple of pairs at time n. For instance $G_0 = (1, 0, 0)$, $G_1 = (0, 1, 0)$, and $G_2 = (1, 0, 1)$. You then get this:

$$\text{if } G_n = (A, B, C), \text{ then } G_{n+1} = (B + C, A, B + C)$$

And now applying this fact to G_{n+1}, you get

$$\begin{aligned} G_{n+1} = (B + C, A, B + C) \implies G_{n+2} &= (A + (B + C), B + C, A + (B + C)) \\ &= (A + B + C, B + C, A + B + C) \end{aligned}$$

Clearly the relation between F_n and G_n is

$$\text{if } G_n = (A, B, C), \text{ then } F_n = A + B + C$$

and you'll arrive at pretty much the same proof before but using a cleaner and a more mathematical presentation. Now write up a proper proof using LaTeX.

Cleanness and clarity is important for both math and writing code.

Exercise 912.1.1. What if you received a pair of mice and they reproduce 2 pairs after 2 months? What is the recurrence relation? How many pairs do you have after 12 months? (Go to solution, page **??**) □

Exercise 912.1.2. What if you received a pair of dogs and they reproduce 4 pairs after 3 years and every pair they produce also reproduces 4 pairs after 3 years? What is the recurrence relation? How many pairs do you have after 15 years? (Go to solution, page **??**) □

Exercise 912.1.3. * What if you receive a pair of guinea pigs and they reproduce a pairs after b years and every pair they produce also reproduce a pairs after b years? What is the recurrence relation? This one is not that easy: And what if a pair only lives up to exactly c years? (Go to solution, page **??**) □

CS, math, physics, engineering is heavily involved in the analysis of functions. (Right?) For instance I can ask "if an object mass m and starts moving on a straight line at velocity v, where is it at time t?" Here I am asking for the position function $p(t)$, where $t \in \mathbb{R}$. In the case of the Fibonacci sequence, the function is $F(n)$ (or F_n) where $n \in \mathbb{N}$. A function with a domain of $\{0, 1, 2, ...\}$ is very common in CS, engineering, and math. So a sequence of numbers is really just a function with domain $\{0, 1, 2, ...\}$.

Frequently, the sequence of numbers we want to study actually obey a recurrence relation. What do I mean? Well take for instance the famous Fibonacci sequence

$$1, 1, 2, 3, 5, \ldots$$

Now by definition if we denote the sequence by F_n, we see that

$$F_n = F_{n-1} + F_{n-2}$$

for $n \geq 2$. Once two base conditions are given, the F_n's are well-defined.

In general, given a sequence $a_0, a_1, a_2, ...$ (or equivalently a function with domain $\{0, 1, 2, ...\}$), a **recurrence relation** is an algebraic expression of a_n in terms of $n, a_0, ..., a_{n-1}$ whenever $n \geq d$ for some $d \geq 0$. The values for $a_0, ..., a_{d-1}$ are called **base cases/conditions**.

For our F_n's, I'm going to change the first two numbers from $1, 1$ to $0, 1$:

$$F_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F_{n-1} + F_{n-2} & \text{if } n \geq 2 \end{cases}$$

Instead of the sequence

$$1, 1, 2, 3, 5, 8, 13, ...$$

The sequence is now

$$0, 1, 1, 2, 3, 5, 8, ...$$

Here are a few more recurrence relations:

$$\begin{aligned}
a_n &= 2a_{n-3} \\
b_n &= 3b_{n-3} + 4b_{n-7} \\
c_n &= c_{n-3} + 7c_{n-7} + n^2 \\
d_n &= d_{n-3}d_{n-4} \\
e_n &= 42e_{n-3} + e_{n-4}^2 \\
f_n &= f_{n-1}\sqrt{f_{n-2} + f_{n-3}} \\
g_n &= g_{\lfloor n/2 \rfloor} + n \\
h_n &= \frac{1}{n}\sum_{k=0}^{n-1} h_k \\
j_n &= j_{n-1}j_0 + j_{n-2}j_1 + j_{n-3}j_2 + \cdots + j_1j_{n-2} + j_0j_{n-1}
\end{aligned}$$

Of course you can also write the numbers in a sequence using function notation. So instead of writing F_n, sometimes I can also write $F(n)$. This is especially the case when the subscripts/indexes are complex such as

$$g_n = g_{\lfloor n/2 \rfloor} + n$$

which will sometimes be written

$$g(n) = g(\lfloor n/2 \rfloor) + n$$

or say something more complex like

$$g(n) = g(n - \lfloor n/2 \rfloor) + n$$

Otherwise you'll have to squint your eyes at the subscripts:

$$g_n = g_{n-\lfloor n/2 \rfloor} + n$$

Suppose a problem gives rise to a sequence a_n (or function with domain of $\{0, 1, 2, ...\}$). Frequently you want a way to be able to compute a_n when given a value for n. For instance in the Fibonacci problem, you want F_{12}.

A recurrence relation on a_n, if found, will let you compute a_n for any n using the recurrence relation, but usually this is very slow (see CISS350). In the case of computing F_n (the n–th Fibonacci number) using the Fibonacci recurrence relation, the time taken (the runtime) is exponential! For instance it would be impossible for you to compute F_{60} by hand or by using your laptop. (Do ahead and try it.) So F_{12} is very tedious (but the recurrence relation is still better than drawing pictures to do rabbit accounting).

To speed up the computation, one way would be to find a closed form for a_n, i.e., a formula for a_n in terms of n. Just like

$$s_n = 1 + 2 + \ldots + n$$

is quickly evaluated using the closed-form

$$s_n = \frac{n(n+1)}{2}$$

A **closed form** for a sequence $a_0, a_1, a_2, ...$ is a formula for a_n in terms of n (and not a_i's). Sometimes the formula might not apply to the first few terms of the sequence (the base cases).

Not only that: a closed-form is also easier to use for approximation, especially in the case of asymptotic approximations. For instance

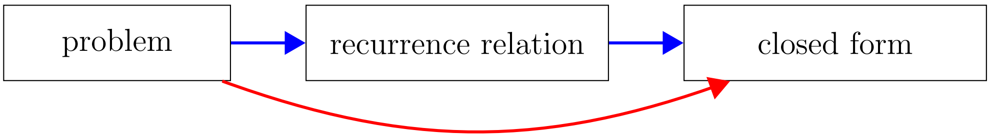
$$s_n = \frac{n(n+1)}{2}$$

is asymptotically bounded above by n^2, i.e.

$$s_n = \frac{n(n+1)}{2} = O(n^2)$$

This is the situation where you want to compare the growth of two functions for large n and you are not interested in specific values such as s_{1000}.

If you have a recurrence for a sequence, there are techniques to compute a closed form, or if that fails, one can try to compute approximations, or if that also fails, then one can try to compute asymptotic approximations:



Although the ultimate goal is the closed form, frequently the intermediate step of finding recurrence relation cannot be avoided. In fact it's a crucial step.

We say that the Fibonacci sequence F_n satisfies a **degree** 2 recurrence relation. The "2" is due to the fact that in the recurrence relation

$$F_n = F_{n-1} + F_{n-2}$$

if you look at the subscripts/indices:

$$F_{\boxed{n}} = F_{\boxed{n-1}} + F_{\boxed{n-2}}$$

i.e.,

$$n, \;\; n-1, \;\; n-2$$

the maximum absolute difference is 2. In general $a_0, a_1, \ldots$ is a **degree** k relation if there is a formula $\mathcal{F}(x_0, \ldots, x_k)$ such that

$$\mathcal{F}(a_{n-k}, \ldots, a_n) = 0$$

In the case of the Fibonacci recurrence relation, the formula is

$$\mathcal{F}(x_0, x_1, x_2) = x_2 - x_1 - x_0$$

since if you substitute x_2 by F_n, x_1 by F_{n-1}, x_0 by F_{n-2}, you get

$$\mathcal{F}(F_{n-2}, F_{n-1}, F_n) = F_n - F_{n-1} - F_{n-2}$$

and

$$\mathcal{F}(F_{n-2}, F_{n-1}, F_n) = 0$$

is exactly our Fibonacci recurrence relation $F_n = F_{n-1} + F_{n-2}$. It should be obvious that the higher the degree, the more complex is the recurrence relation and the longer it will take to compute a specific term of the sequence.

(Aside: Of course if $F_n = F_{n-1} + F_{n-2}$, then it's also true that $F_n = F_{n-1} + F_{n-2} + 0 \cdot F_{n-3}$, but you don't consider terms like $0 \cdot F_{n-3}$ and say the Fibonacci sequence has degree 3! Please don't do that!)

Note that the worst case is where the computation of a_n depends on *all* the previous a_i:

$$a_n = \frac{1}{n} \sum_{i=0}^{n-1} a_i$$

and the degree is $n+1$, i.e., the degree is not constant. Here's an example where b_n depends on half of the previous b_i's: or at least the degree is not constant and grows with n:

$$b_n = \sum_{i=0}^{\lfloor n/2 \rfloor} b_i$$

The degree is again not a constant.

In general a constant degree recurrence is easier to handle. The cases of non-constant degree like the above are harder to handle. They are important because they occur in "average" computations (such as average runtime analysis).

Exercise 912.1.4. What is the degree of each of the following recurrence relations:

(a) $a_n = 12a_{n-1} + 42a_{n-7}$
(b) $b_n = 12n^3 b_{n-7}$
(c) $c_{n+3} = -2c_{n-2}/c_{n-3}$
(d) $d_{n+3}d_{n-3} = \sqrt{d_n}$

(Go to solution, page **??**) □

Furthermore we say that the Fibonacci recurrence relation is linear. It is a **linear recurrence relation** because F_n, F_{n-1} and F_{n-2} occurs as a sum of constant multiple (i.e. linear combination): $F_n = F_{n-1} + F_{n-2}$ is the same as

$$1 \cdot F_n - 1 \cdot F_{n-1} - 1 \cdot F_{n-2} = 0$$

The following is also a linear recurrence (of degree 5):

$$5 \cdot G_n = 7 \cdot G_{n-2} - 13 \cdot G_{n-5}$$

However the following is nonlinear:

$$a_n = na_{n-1} - 3a_{n-2}$$

because the coefficient in front of a_{n-1} is n which is not constant. This is also not linear:

$$a_n = a_{n-1}^2 - a_{n-2}$$

because in front of a_{n-1} is a_{n-1} which is not a constant. This is also not linear:

$$a_n = \sqrt{a_{n-1}} - a_{n-2}$$

because of $\sqrt{a_{n-1}}$.

Exercise 912.1.5.

Let $s_n = 0 + 1 + 2 + \cdots + n$. Is there a recurrence relation for $s_0, s_1, s_2, ...$? If you can find one, is that linear? homogeneous? What is the degree? State the base case. (Go to solution, page ??) □

Exercise 912.1.6. Let $a_n = n^2$. Is there a recurrence relation for $a_0, a_1, a_2, ...$? If you can find one, is that linear? homogeneous? What is the degree? (Go to solution, page ??) □

Exercise 912.1.7. * Let ϕ be the Euler ϕ–function. This defines a sequence $\phi(1)$, $\phi(2)$, $\phi(3)$, ... Does this sequence satisfy a recurrence relation? If you can find one, is that linear? homogeneous? What is the degree? (Go to solution, page ??) □

One of the most important type of nonlinear recurrences is of the form

$$a(n) = a(\lfloor n/2 \rfloor) + 3n + 4$$

or

$$a(n) = 2a(\lfloor n/2 \rfloor) + 3n \lg n + 4$$

or

$$a(n) = a(\lfloor n/2 \rfloor) + a(n - \lfloor n/2 \rfloor) + n$$

More generally such recurrences can be

$$a(n) = \alpha a(\lfloor n/\beta \rfloor) + f(n)$$

or

$$a(n) = \alpha_1 a(\lfloor n/\beta \rfloor) + \alpha_2 a(n - \lfloor n/\beta \rfloor) + f(n)$$

where α, α_i, β are constants and $f(n)$ is a formula in n. These are called **divide-and-conquer recurrences**. In CS and engineering, divide-and-conquer recurrences appear in the runtime analysis of recursive algorithms that involves the divide-and-conquer algorithm design technique. These includes binary search, mergesort, quicksort, etc.

A recurrence relation in $a_0, a_1, a_2, ...$ is **homogeneous** if a_n is expressed as a sum of expressions that involve some a_i. For instance

$$F_n = F_{n-1} + F_{n-2}$$

and

$$G_n = G_{n-1} + G_{n-2}G_{n-3}$$

are homogeneous. However

$$H_n = H_{n-1} + H_{n-2} + 42$$

is not homogeneous because the term 42 in the sum on the right does not involve any H_i's. The following is also not homogeneous:

$$a_n = 2a_{n-1} - 3a_{n-2}a_{n-3} + 42n^3 + \sqrt{n+1}$$

because of the term $42n^3$ and also of the term $\sqrt{n+1}$.

There are many techniques for studying recurrence relations. Here are some

1. Algebraic manipulations that involves various "substitutions"
2. Generating functions.
3. Master theorem.

Earlier I said that the computation of F_n is extremely slow when you use the recurrence relation $F_n = F_{n-1} + F_{n-2}$. Here's a program (in Python) that prints F_n for $n = 0, 1, \ldots 100$:

```
def f(n):
    if n == 0: return 0
    elif n == 1: return 1
    else: return f(n-1) + f(n-2)

for n in range(100):
    print(f(n))
```

You'll see that it grinds to a halt as the computational cost of computation explodes exponentially. You can rewrite it in your favorite programming lang and you'll see that it'll still grind to a halt when n is "sufficiently large". For my computer it takes about 6 seconds to compute `f(33)`. By the time I reach `f(55)` it pretty much grinds to a halt.

The point is that to compute `f(20)`, you need to compute `f(19)` and `f(18)`; the computation of `f(19)` requires the computation of `f(18)` and `f(17)` and the computation of `f(18)` requires the computation of `f(17)` and `f(16)`; etc. You notice that the computation of `f(18)` is carried out twice; other `f(n)` might be computed even more times.

One way to prevent this is to store up your computations. This program:

```
table = {}
table[0] = 0
table[1] = 1
def f(n):
    if not table.has_key(n):
        table[n] = f(n-1) + f(n-2)
    return table[n]

for n in range(100): print(f(n))
```

stores computations in a table and print all F_n for $n = 0, \ldots, 99$ in a split second:

```
0
1
1
2
3
5
8
13
21
34
55
89
144
233
377
610
987
1597
2584
4181
6765
10946
17711
28657
46368
75025
121393
196418
317811
514229
```

```
832040
1346269
2178309
3524578
5702887
9227465
14930352
24157817
39088169
63245986
102334155
165580141
267914296
433494437
701408733
1134903170
1836311903
2971215073
4807526976
7778742049
12586269025
20365011074
32951280099
53316291173
86267571272
139583862445
225851433717
365435296162
591286729879
956722026041
1548008755920
2504730781961
4052739537881
6557470319842
10610209857723
17167680177565
27777890035288
44945570212853
72723460248141
117669030460994
190392490709135
308061521170129
498454011879264
806515533049393
1304969544928657
2111485077978050
3416454622906707
5527939700884757
8944394323791464
14472334024676221
23416728348467685
```

```
37889062373143906
61305790721611591
99194853094755497
160500643816367088
259695496911122585
420196140727489673
679891637638612258
1100087778366101931
1779979416004714189
2880067194370816120
4660046610375530309
7540113804746346429
12200160415121876738
19740274219868223167
31940434634990099905
51680708854858323072
83621143489848422977
135301852344706746049
218922995834555169026
```

Note that the improvement in speed comes with a cost: memory. There are several other techniques of computation that does not require that much memory.

Besides the above, which requires you to build a table, one can try to find a closed form for F_n. You'll see later that from the recurrence, we can derive this closed form for F_n:

$$F_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right)$$

YIKES! Given that F_n is a whole number, the above expression should surprise you because it contains $\sqrt{5}$!!! Go ahead and substitute n with say 5 and use a calculator to compute F_5. You do get a whole number right? (Well, with floating point numbers, you will get some small error. So you'll be very very very close to an integer.)

Now how do you compute this monster closed form?

912.1.2 Closed form computation

I'm going to take a digression to show you how to compute the closed form using the very powerful technique of generating functions. Later, I'll go over this technique all over again in detail.

Consider the follwing recurrence relation:

$$F_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F_{n-1} + F_{n-2} & \text{if } n > 1 \end{cases}$$

This is the same as the original Fibonacci except I have $F_0 = 0$ instead of 1. Let's consider the **generating function** of F_n ($n = 0, 1, 2, \ldots$):

$$F(x) = \sum_{n=0}^{\infty} F_n x^n$$

In other words, you create a power series using the sequence $F_0, F_1, ...$ We have

$$\begin{aligned} F(x) &= \sum_{n=0}^{\infty} F_n x^n \\ &= F_0 + F_1 x + \sum_{n=2}^{\infty} F_n x^n \\ &= x + \sum_{n=2}^{\infty} F_n x^n \end{aligned}$$

We've just used the two base conditions. Now we use the recursive condition:

$$\begin{aligned} F(x) &= x + \sum_{n=2}^{\infty} F_n x^n \\ &= x + \sum_{n=2}^{\infty} (F_{n-1} + F_{n-2}) x^n \end{aligned}$$

We've used all the properties of F_n ($n = 0, 1, 2, \ldots$). So here comes the magic of generating functions. Watch this carefully. Continuing the above computation, the point is that the recurrence relation will actually produce new $F(x)$'s on the right-hand side:

$$\begin{aligned}
F(x) &= x + \sum_{n=2}^{\infty} (F_{n-1} + F_{n-2}) x^n \\
&= x + \sum_{n=2}^{\infty} F_{n-1} x^n + \sum_{n=2}^{\infty} F_{n-2} x^n \\
&= x + x \sum_{n=2}^{\infty} F_{n-1} x^{n-1} + x^2 \sum_{n=2}^{\infty} F_{n-2} x^{n-2} \\
&= x + x \sum_{n=1}^{\infty} F_n x^n + x^2 \sum_{n=0}^{\infty} F_n x^n
\end{aligned}$$

The first series on the right $\sum_{n=1}^{\infty} F_n x^n$ is almost $F(x) = \sum_{n=0}^{\infty} F_n x^n$:

$$\begin{aligned}
F(x) &= x + x \left(\sum_{n=0}^{\infty} F_n x^n - F_0 x^0 \right) + x^2 \sum_{n=0}^{\infty} F_n x^n \\
&= x + x \left(\sum_{n=0}^{\infty} F_n x^n - 0 \right) + x^2 \sum_{n=0}^{\infty} F_n x^n \\
&= x + x \sum_{n=0}^{\infty} F_n x^n + x^2 \sum_{n=0}^{\infty} F_n x^n \\
&= x + xF(x) + x^2 F(x)
\end{aligned}$$

The crucial thing to remember is the recurrence relation gives us an algebraic relationship on $F(x)$. Now we use this algebraic relationship in a crucial way:

$$\begin{aligned}
F(x) &= x + xF(x) + x^2 F(x) \\
\therefore \quad F(x) - xF(x) - x^2 F(x) &= x \\
\therefore \quad (1 - x - x^2) F(x) &= x \\
\therefore \quad F(x) &= \frac{x}{1 - x - x^2} \\
&= -\frac{x}{x^2 + x - 1}
\end{aligned}$$

We have now rewritten $F(x) = \sum_{n=0}^{\infty} F_n x^n$ (a power series) as a **rational expression**, i.e., a fraction of polynomials. Let me factorize the denominator of the rational expression of $F(x)$. The roots of $x^2 + x - 1$ are

$$\frac{-1 \pm \sqrt{5}}{2}$$

Therefore

$$x^2 + x - 1 = \left(x - \frac{-1-\sqrt{5}}{2}\right)\left(x - \frac{-1+\sqrt{5}}{2}\right)$$

and hence

$$\begin{aligned} F(x) &= -\frac{x}{x^2+x-1} \\ &= -\frac{x}{\left(x - \frac{-1-\sqrt{5}}{2}\right)\left(x - \frac{-1+\sqrt{5}}{2}\right)} \\ &= -\frac{x}{\left(\frac{-1-\sqrt{5}}{2} - x\right)\left(\frac{-1+\sqrt{5}}{2} - x\right)} \end{aligned}$$

The theory of partial fractions tells us that there are constants A and B such that

$$\frac{x}{\left(\frac{-1-\sqrt{5}}{2} - x\right)\left(\frac{-1+\sqrt{5}}{2} - x\right)} = \frac{A}{\left(\frac{-1-\sqrt{5}}{2} - x\right)} + \frac{B}{\left(\frac{-1+\sqrt{5}}{2} - x\right)}$$

i.e., our rational expression can be rewritten as a linear sum of simpler rational expressions. These simpler rational expressions are called **partial fractions**. The point of doing this is that the partial fractions can then be converted to power series. For now, let me compute A and B.

Multiplying

$$\frac{x}{\left(\frac{-1-\sqrt{5}}{2} - x\right)\left(\frac{-1+\sqrt{5}}{2} - x\right)} = \frac{A}{\left(\frac{-1-\sqrt{5}}{2} - x\right)} + \frac{B}{\left(\frac{-1+\sqrt{5}}{2} - x\right)}$$

with $\left(\frac{-1-\sqrt{5}}{2} - x\right)\left(\frac{-1+\sqrt{5}}{2} - x\right)$, I get

$$x = A\left(\frac{-1+\sqrt{5}}{2} - x\right) + B\left(\frac{-1-\sqrt{5}}{2} - x\right)$$

There are two unknowns A and B. I will substitute two values for x into this equation and find my A and B. If I let $x = \frac{-1-\sqrt{5}}{2}$, I will get

$$\begin{aligned} \frac{-1-\sqrt{5}}{2} &= A\left(\frac{-1+\sqrt{5}}{2} - \frac{-1-\sqrt{5}}{2}\right) + 0 \\ &= \sqrt{5}A \end{aligned}$$

and therefore

$$A = \frac{-1-\sqrt{5}}{2\sqrt{5}}$$

And if I set $x = \frac{-1+\sqrt{5}}{2}$ in the above equation I will get

$$\begin{aligned}
\frac{-1+\sqrt{5}}{2} &= 0 + B\left(\frac{-1-\sqrt{5}}{2} - \frac{-1+\sqrt{5}}{2}\right) \\
&= -\sqrt{5}B \\
\therefore \quad B &= \frac{1-\sqrt{5}}{2\sqrt{5}}
\end{aligned}$$

Notice that in the above I picked values for x to get the simplest possible equations in A and B. Specifically, I use the two roots from earlier. Therefore I now know that

$$F(x) = -\left(\frac{\frac{-1-\sqrt{5}}{2\sqrt{5}}}{\left(\frac{-1-\sqrt{5}}{2} - x\right)} + \frac{\frac{1-\sqrt{5}}{2\sqrt{5}}}{\left(\frac{-1+\sqrt{5}}{2} - x\right)}\right)$$

I now rewrite this as a power series:

$$\begin{aligned}
F(x) &= -\left(\frac{\frac{-1-\sqrt{5}}{2\sqrt{5}}}{\frac{-1-\sqrt{5}}{2}} \cdot \frac{1}{\left(1 - x/\frac{-1-\sqrt{5}}{2}\right)} + \frac{\frac{1-\sqrt{5}}{2\sqrt{5}}}{\frac{-1+\sqrt{5}}{2}} \cdot \frac{1}{\left(1 - x/\frac{-1+\sqrt{5}}{2}\right)}\right) \\
&= -\left(\frac{1}{\sqrt{5}}\frac{1}{\left(1 - x/\frac{-1-\sqrt{5}}{2}\right)} - \frac{1}{\sqrt{5}}\frac{1}{\left(1 - x/\frac{-1+\sqrt{5}}{2}\right)}\right) \\
&= -\frac{1}{\sqrt{5}}\left(\sum_{n=0}^{\infty}\left(x\middle/\frac{-1-\sqrt{5}}{2}\right)^n - \sum_{n=0}^{\infty}\left(x\middle/\frac{-1+\sqrt{5}}{2}\right)^n\right) \\
&= -\frac{1}{\sqrt{5}}\sum_{n=0}^{\infty}\left(\left(\frac{2}{-1-\sqrt{5}}\right)^n - \left(\frac{2}{-1+\sqrt{5}}\right)^n\right)x^n
\end{aligned}$$

Since $F(x) = \sum_{n=0}^{\infty} F_n x^n$, I now have

$$F(x) = \sum_{n=0}^{\infty} F_n x^n = \frac{1}{\sqrt{5}}\sum_{n=0}^{\infty}\left(\left(\frac{2}{-1-\sqrt{5}}\right)^n - \left(\frac{2}{-1+\sqrt{5}}\right)^n\right)x^n$$

which means that F_n is this hideous looking thing:

$$F_n = -\frac{1}{\sqrt{5}}\left(\left(\frac{2}{-1-\sqrt{5}}\right)^n - \left(\frac{2}{-1+\sqrt{5}}\right)^n\right)$$

I tidy it up a bit to get

$$
\begin{aligned}
& -\frac{1}{\sqrt{5}}\left(\left(\frac{2}{-1-\sqrt{5}}\right)^n - \left(\frac{2}{-1+\sqrt{5}}\right)^n\right) \\
&= -\frac{1}{\sqrt{5}}\left(\left(\frac{2(-1+\sqrt{5})}{(-1-\sqrt{5})(-1+\sqrt{5})}\right)^n - \left(\frac{2(-1-\sqrt{5})}{(-1+\sqrt{5})(-1-\sqrt{5})}\right)^n\right) \\
&= -\frac{1}{\sqrt{5}}\left(\left(\frac{2(-1+\sqrt{5})}{-4}\right)^n - \left(\frac{2(-1-\sqrt{5})}{-4}\right)^n\right) \\
&= -\frac{1}{\sqrt{5}}\left(\left(\frac{1-\sqrt{5}}{2}\right)^n - \left(\frac{1+\sqrt{5}}{2}\right)^n\right) \\
&= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)
\end{aligned}
$$

Hence the n–th Fibonacci number is

$$
F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)
$$

TADA! ... well this is kind of a surprise that the number is so complicated. Not only that: the formula involves square roots. But if you apply the binomial theorem, you will see easily that the square roots actually cancels. You can of course try out the formula with say $n = 0, 1, 2, 3, 4$. You'll see that it does work.

Of course with this you can now write a third program to compute F_n:

```
from math import sqrt

r = sqrt(5)
c = 1 / r
a = (1 + r) / 2
b = (1 - r) / 2

def f2(n):
    return round(c * (a**n - b**n))
```

The number

$$
\phi = \frac{1+\sqrt{5}}{2}
$$

that appears in the closed form of F_n is a very important constant in math, CS, physics, etc. It's called the golden ratio. Note that

$$\begin{aligned}
-\phi^{-1} &= -\frac{2}{1+\sqrt{5}} \cdot \frac{1-\sqrt{5}}{1-\sqrt{5}} \\
&= -\frac{2(1-\sqrt{5})}{1-5} \\
&= \frac{1-\sqrt{5}}{2}
\end{aligned}$$

Note also that

$$\begin{aligned}
1-\phi &= 1-\frac{1+\sqrt{5}}{2} \\
&= \frac{2-1-\sqrt{5}}{2} \\
&= \frac{1-\sqrt{5}}{2}
\end{aligned}$$

Therefore you can also express F_n as

$$F_n = \frac{\phi^n - (-1/\phi)^n}{\sqrt{5}} = \frac{\phi^n - (1-\phi)^n}{\sqrt{5}}$$

Here's a summary of the steps in computing a closed form for our F_n:

1. Define $F(x) = \sum_{n=0}^{\infty} F_n x^n$.
2. Rewrite $F(x)$ as a rational expression $F(x) = -\frac{x}{x^2+x-1}$.
3. Rewrite $F(x)$ as a sum of partial fractions $F(x) = \frac{A}{r_1-x} + \frac{B}{r_1-x}$ where r_1, r_2 are roots of x^2+x-1.
4. Rewrite $F(x)$ as a power series. F_n is the coefficient of x^n of this final power series.

Exercise 912.1.8 (REMOVE?)**.** You may skip this exercise. The above expression for F_n is

$$F_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right)$$

and it involves $\sqrt{5}$. Of course we know that F_n is an integer. So the $\sqrt{5}$ must some how disappear. Let's expand the terms of the n–th powers using binomial theorem and verify that *all* the $\sqrt{5}$ disappears. (There's also the mysterious division by a very high power of 2.)

$$\begin{aligned}
F_n &= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right) \\
&= \frac{1}{2^n\sqrt{5}}\left((1+\sqrt{5})^n - (1-\sqrt{5})^n\right) \\
&= \frac{1}{2^n\sqrt{5}}\left(\sum_{i=0}^{n}\binom{n}{i}\sqrt{5}^i - \sum_{i=0}^{n}(-1)^i\binom{n}{i}\sqrt{5}^i\right) \\
&= \frac{1}{2^n\sqrt{5}}\sum_{i=0}^{n}\left(\binom{n}{i} - (-1)^i\binom{n}{i}\right)\sqrt{5}^i \\
&= \frac{1}{2^n\sqrt{5}}\sum_{i=0}^{n}(1-(-1)^i)\binom{n}{i}\sqrt{5}^i
\end{aligned}$$

Of course $1-(-1)^i$ is 0 when i is even. Therefore

$$\begin{aligned}
F_n &= \frac{1}{2^n\sqrt{5}}\sum_{\substack{i=0 \\ i \text{ odd}}}^{n} 2\binom{n}{i}\sqrt{5}^i \\
&= \frac{2}{2^n}\sum_{\substack{i=0 \\ i \text{ odd}}}^{n}\binom{n}{i}\sqrt{5}^{i-1} \\
&= \frac{1}{2^{n-1}}\sum_{\substack{i=0 \\ i \text{ odd}}}^{n}\binom{n}{i}5^{\frac{i-1}{2}}
\end{aligned}$$

Note that since i is odd, $i-1$ is even and hence $(i-1)/2$ is an integer. Therefore we know now that the expression of F_n in fact does not involve the mysterious $\sqrt{5}$.

In particular for the case of $n = 10$ we have

$$F_{10} = \frac{1}{2^9}\left(\binom{10}{1}5^0 + \binom{10}{3}5^1 + \binom{10}{5}5^2 + \binom{10}{7}5^3 + \binom{10}{9}5^4\right)$$

which can be "folded":

$$F_{10} = \frac{1}{2^9}\left(2\binom{10}{1}5^0 + 2\binom{10}{3}5^1 + \binom{10}{5}5^2\right)$$

Likewise for odd n, say $n = 9$,

$$F_9 = \frac{1}{2^8}\left(2\binom{9}{1}5^0 + 2\binom{9}{3}5^1\right)$$

Note however that the closed form will evaluate faster than the form that uses a summation. Furthermore it's also easier to see the big-O of F_n using the closed form. □

Exercise 912.1.9. Let a_n be defined by the following recurrence relation:

$$a_n = \begin{cases} 3 & \text{if } n = 0 \\ 5 & \text{if } n = 1 \\ a_{n-1} + a_{n-2} & \text{if } n > 1 \end{cases}$$

Compute a_2, a_3, a_4 by hand from the recurrence relation. Derive a closed form for a_n and verify $a_0, a_1, \ldots, a_4$ fit your closed form. (Go to solution, page **??**) □

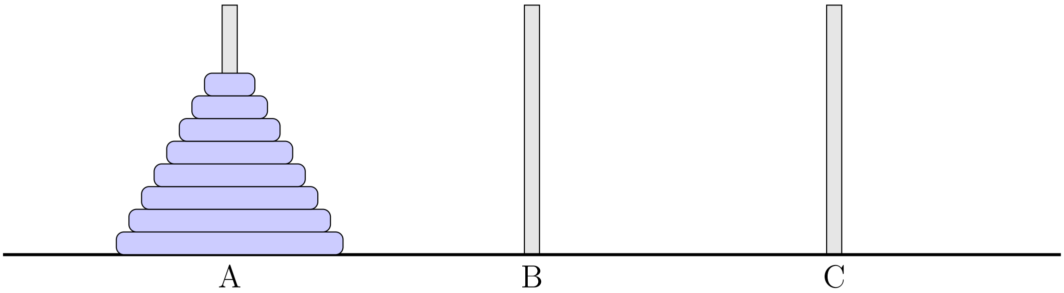
Exercise 912.1.10. Let b_n be defined by the following recurrence relation:

$$b_n = \begin{cases} 3 & \text{if } n = 0 \\ 5 & \text{if } n = 1 \\ 7b_{n-1} + 11b_{n-2} & \text{if } n > 1 \end{cases}$$

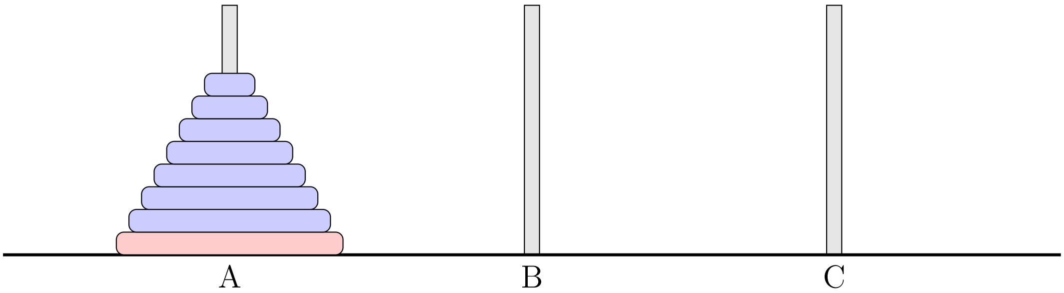
Derive a closed form for b_n. Again make sure that you check your closed form is correct for several cases of b_n computed by hand. (Go to solution, page **??**) □

912.1.3 Tower of Hanoi

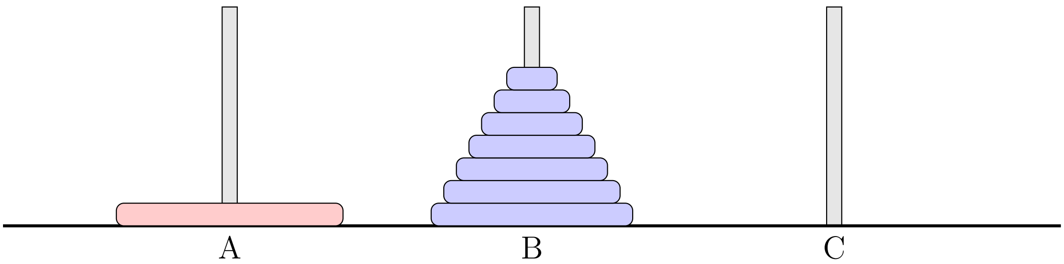
Can the above method be used for a degree one recurrence relation? Remember the Tower of Hanoi problem? Let t_n be the number of steps to solve the problem. Recall that we solve the problem by providing a recursive procedure. Here's the problem again. You have n disks that you want to move from A to C.



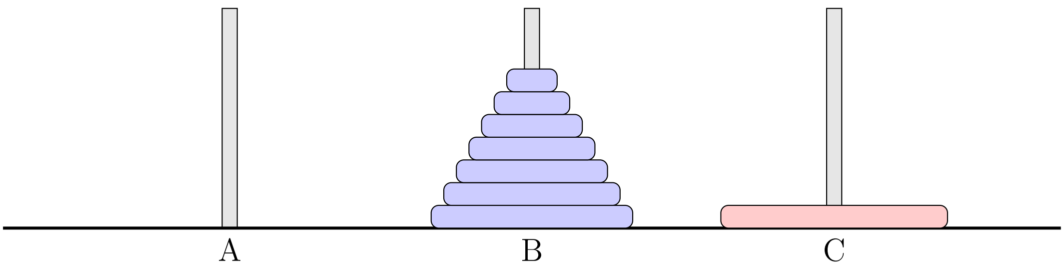
We think of the n-disk problem in terms of the $n-1$ disk problem:



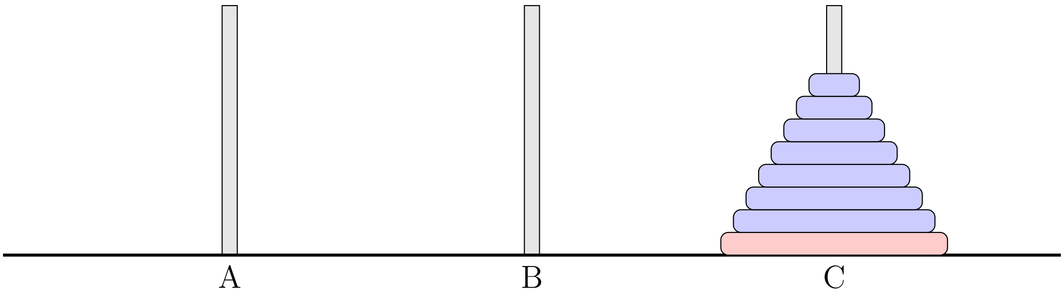
We ignore disk n for the time being and apply our procedure to move the top $n-1$ disks from A to B:



This should take t_{n-1} steps. Now we move disk n from A to C:



That takes 1 step. And finally we apply our procedure to move the $n-1$ disks from B to C:



That takes t_{n-1} steps. Altogther we took $t_{n-1} + 1 + t_{n-1}$ steps. Hence

$$\begin{aligned} t_n &= t_{n-1} + 1 + t_{n-1} \\ &= 2t_{n-1} + 1 \end{aligned}$$

We need a base condition. So what's t_0? That's the problem with 0 disks. It should probably be 0 step: $t_0 = 0$. But vacuous problems are sometimes dangeourous. So let's consider t_1. Clearly $t_1 = 1$. Now since we want $t_1 = 2t_0 + 1$, we have

$$1 = 2t_0 + 1$$

and, yes, we do get $t_0 = 0$. Altogether we have

$$t_n = \begin{cases} 0 & \text{if } n = 0 \\ 2t_{n-1} + 1 & \text{if } n > 0 \end{cases}$$

Furthermore note that the recurrence relation is not just defined in terms of a linear combination of t_n's for small n: There's a "+ 1" in the recurrence relation:

$$t_n = 2t_{n-1} \underline{+1}$$

This is a degree 1 nonhomogeneous recurrence relation.

For this recurrence relation, it's so simple that you can actually find a closed form quickly, using "substitutions". Here's how you would do it.

$$\begin{aligned} t_n &= 2t_{n-1} + 1 \\ &= 2(2t_{n-2} + 1) + 1 = 4t_{n-2} + 2 + 1 \\ &= 4(2t_{n-3} + 1) + 2 + 1 = 8t_{n-3} + 4 + 2 + 1 \end{aligned}$$

All the above assume that $n \geq 3$. At this point you see a pattern:

$$t_n = 2^3 t_{n-3} + 2^2 + 2^1 + 2^0$$

To check on the pattern, you do one more step (assuming $n \geq 4$):

$$\begin{aligned} t_n &= 2^3 t_{n-3} + 2^2 + 2^1 + 2^0 \\ &= 2^3(2t_{n-4} + 1) + 2^2 + 2^1 + 2^0 = 2^4 t_{n-4} + 2^3 + 2^2 + 2^1 + 2^0 \end{aligned}$$

i.e.,

$$\begin{aligned} t_n &= 2^3 t_{n-3} + 2^2 + 2^1 + 2^0 \\ &= 2^4 t_{n-4} + 2^3 + 2^2 + 2^1 + 2^0 \\ &= ... \\ &= 2^k t_{n-k} + 2^{k-1} + 2^{k-2} + \cdots + 2^3 + 2^2 + 2^1 + 2^0 \end{aligned}$$

At some point you'd reach the base case, i.e., when $n - k = 1$,

$$\begin{aligned} t_n &= 2^{n-1} t_1 + 2^{n-2} + 2^{n-3} + \cdots + 2^3 + 2^2 + 2^1 + 2^0 \\ &= 2^{n-1} + 2^{n-2} + 2^{n-3} + \cdots + 2^3 + 2^2 + 2^1 + 2^0 \\ &= 2^n - 1 \end{aligned}$$

by the geometric sum formula. TADA!

So immediately, you know that to solve the tower of hanoi problem you need to make $2^{32} - 1 = 4294967295$ move.

Now to be absolutely mathematically correct, the following from the above:

$$\begin{aligned} t_n &= 2^3 t_{n-3} + 2^2 + 2^1 + 2^0 \\ &= 2^4 t_{n-4} + 2^3 + 2^2 + 2^1 + 2^0 \\ &= ... \\ &= 2^k t_{n-k} + 2^{k-1} + 2^{k-2} + \cdots + 2^3 + 2^2 + 2^1 + 2^0 \end{aligned}$$

is not absolutely rigorous. Why? Because it this:

$$\begin{aligned} t_n &= ...(\leftarrow \text{look at the missing steps described by "..."}) \\ &= 2^k t_{n-k} + 2^{k-1} + 2^{k-2} + \cdots + 2^3 + 2^2 + 2^1 + 2^0 \end{aligned}$$

The above leads to

$$t_n = 2^n - 1$$

Generally, this is what will happen in the mathematical derivation of $t_n =$

$2^n - 1$. The above is OK, as long as it is to derive a plausible closed form for t_n. After that you prove $t_n = 2^n - 1$ is indeed true by induction, using the recurrence relation.

Exercise 912.1.11. Prove $t_n = 2^n - 1$ by induction.

Another thing to note is this very importact fact:

> a recursive procedure gives rise to a recurrence relation on the runtime of the procedure

Exercise 912.1.12. The above proves that the recursive procedure I used will take up $2^n - 1$ moves. But how do you know that the procedure is the *best* strategy? Is there another strategy that moves with fewer steps? Our strategy is in fact the best as in it uses the least number of moves, so prove that our strategy is optimal.

INCOMING SPOILER ALERT ... SOLUTION ON NEXT PAGE

SOLUTION.

Let our original procedure be $P(n, A, B, C)$. I have already proved that the number of moves made by $P(n, A, B, C)$ is $T(n) = 2^n - 1$. Now suppose there's another procedure $P'(n, A, B, C)$ that uses $T'(n)$ moves. I will prove by induction that $T'(n) = T(n)$. (Yes, it's that "induction" thing again.)

First of all if you have one disk (i.e., $n = 1$), then of course $P'(1, A, B, C)$, being optimal, will execute $A \to C$. That means $T'(1) = 1 = T(1)$.

Now suppose $T'(k) = T(k)$ for $k = 1, 2, 3, ..., n-1$ and we consider the moves made by $P'(n, A, B, C)$. Disk n (the largest) has to move from A to either B or C. Note that this is the first move made by disk n. (Of course in the end it will land in C, but I'm not even assuming that yet. I'm just saying this disk has to move. If this disk does not move, there's no way it's going to land in C!) So P' at this point will either execute $A \to B$ or $A \to C$. Remember that P' is optimal.

CASE: THE MOVE IS $A \to C$. This costs 1 step. After this, I use the optimal strategy to move the first $n-1$ disks from B to C and I'm done. But by induction, the optimal strategy takes $T'(n-1) = 2^{n-1} + 1$ moves. Therefore altogether the number of numbers is $T'(n-1) + 1 + T'(n-1) = 2^n - 1 = T(n)$.

CASE: THE MOVE IS $A \to B$. At this point, $T'(n-1) + 1$ moves has been made. Consider what will happen next. At some point (in the future), after $\alpha \geq 0$ moves, disk n has to land in C – that cost at least one step. This move for disk n is either $A \to C$ or $B \to C$. If it's $A \to C$, then the first $n-1$ disks must be at B. If it's $B \to C$, then the first $n-1$ disks must be at A. This means that the first $n-1$ disks has to be moved (in the future) to B or A. But the number of moves has to be $\geq T'(n-1)$. So for this case, the total number of moves is $\geq T'(n-1) + 1 + \alpha + T'(n-1) + 1 = 2T'(n-1) + 2 + \alpha$ where $\alpha \geq 0$. By inductive hypothesis $T'(n-1) = 2^{n-1} - 1$ which means that the number of moves is at least

$$2T'(n-1) + 2 + \alpha = 2(2^{n-1} - 1) + 2 + \alpha = 2^n + \alpha$$

But this is greater than the first case. And since we are using the optimal strategy P', only the first case occurs – the second case does not happen.

We conclude that $T'(n) = 2^n - 1 = T(n)$.

By inductive hypothesis, we have shown that any optimal strategy will make $2^n - 1$ moves. In particular, our earlier strategy is the optimal strategy. $\square$

tower-of-hanoi-variations.tex

Exercise 912.1.13. Here's a variation of the original Tower of Hanoi problem. The new rule is this: No disk can be moved from A to C.

(a) To familiarize yourself with this problem, compute a_1, a_2, a_3.
(b) Design a recursive procedure and compute the recurrence relation on the number of moves in this case.
(c) If the recurrence relation on the runtime is easy (fingers crossed), attempt to find a closed form.

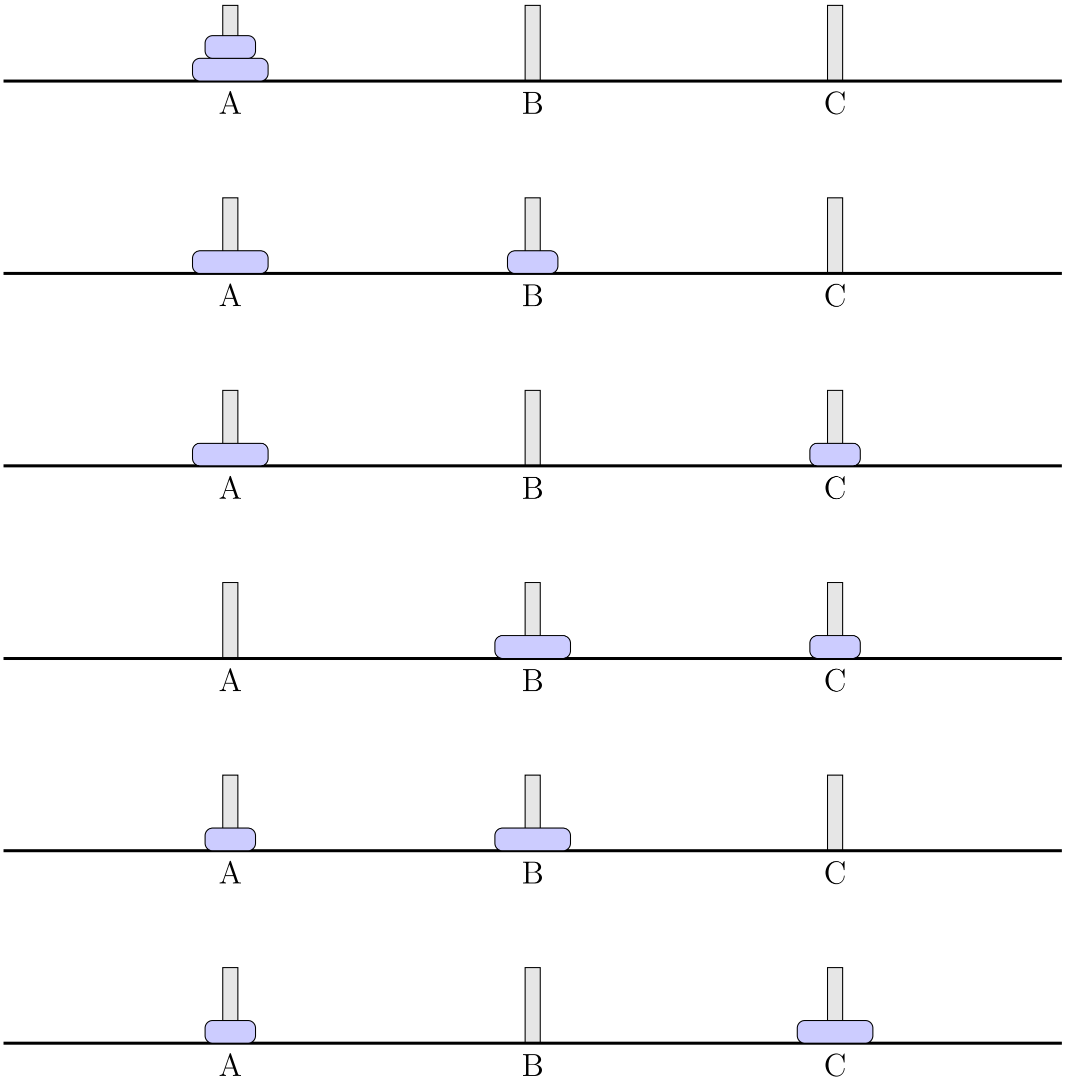
INCOMING SPOILER ALERT ... PARTIAL SOLUTION ON NEXT PAGE

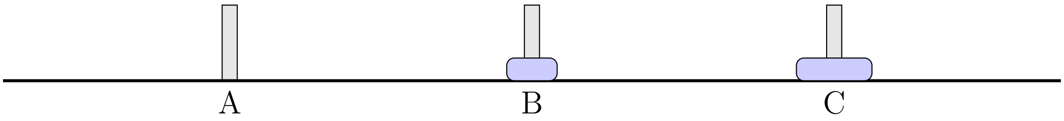
SOLUTION.

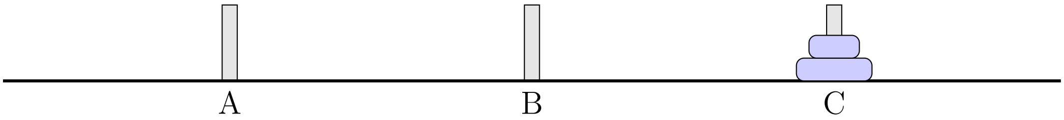
For the case of $n = 1$, we have to move disk 1 from A to B to C. Therefore if a_n is the number of moves, then

$$a_1 = 2$$

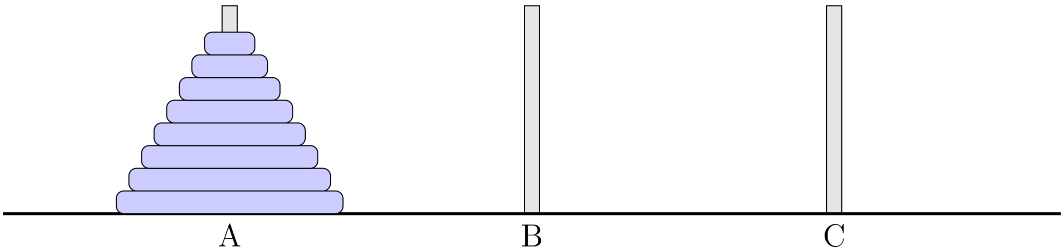
Note that in the original problem the number of steps for $n = 1$ is 1. It's easy (although tedious) to check that $a_2 = 8$ and $a_3 = 26$.



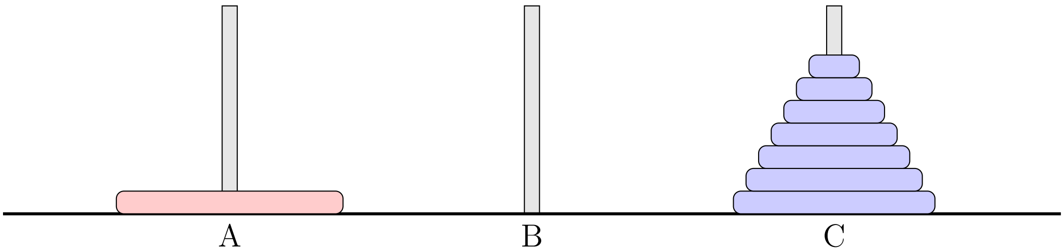




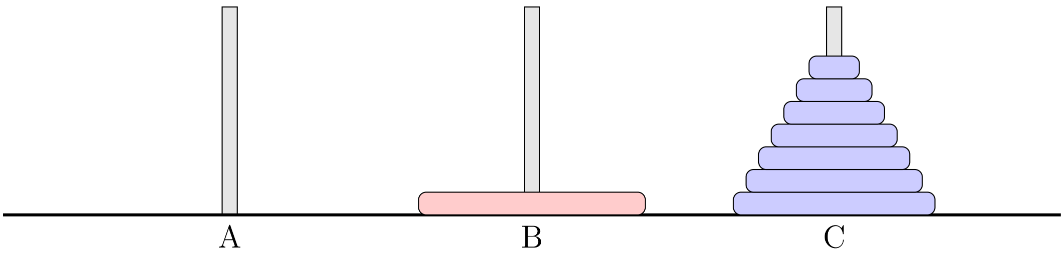
Here's the recursive procedure. Again we solve the problem with n disks by assuming that we can solve the problem for $n-1$ disk. We are given this:



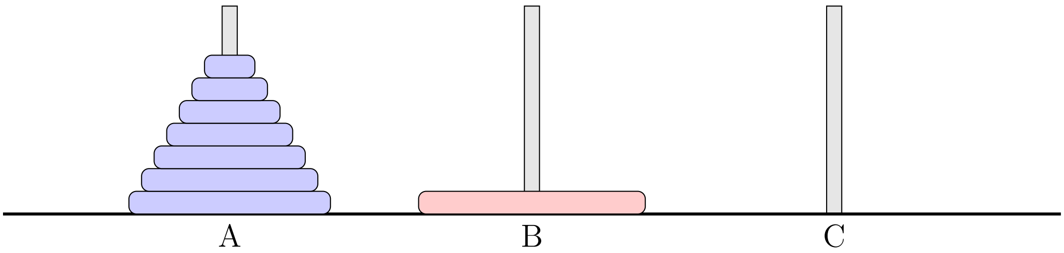
We cannot move the first $n-1$ disks to B as before. Why? Because the last disk, disk n, must move from A to B and then to C. Therefore we move the first $n-1$ disks to C:



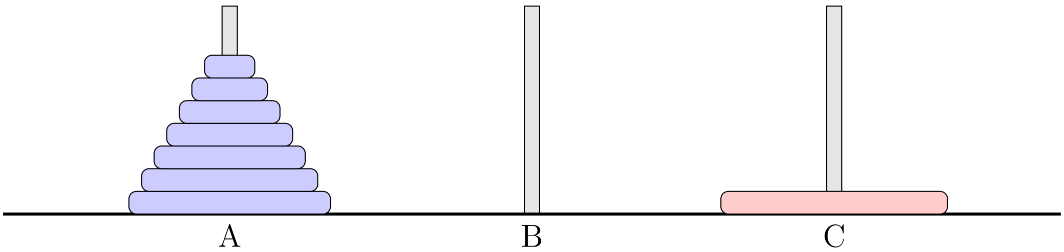
and then move disk n from A to B:



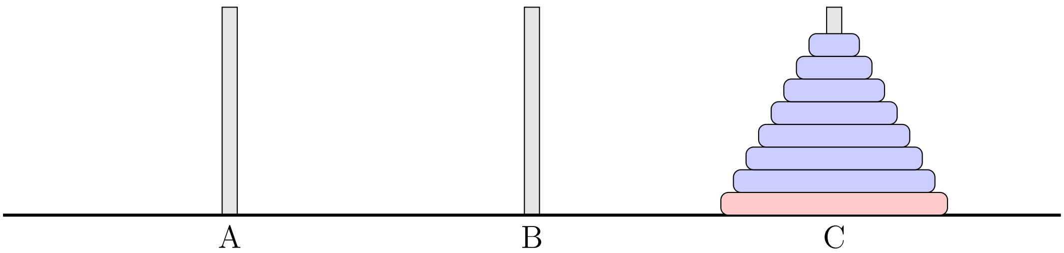
then move the first $n-1$ disks to A:



and move disk n to C:



and then move the first $n-1$ disks to C:



Altogether the number of steps, i.e. a_n is

$$a_n = 3a_{n-1} + 2$$

I'll leave you to compute a_0 and give a complete description for a_n. $\square$

Exercise 912.1.14. Here are two variations of the tower of Hanoi. Suppose that instead of n disks, you have $2n$ disks of n different sizes, i.e. there are exactly two disks with the same size.

1. Assume two disks of the same sizes are indistinguishable.
2. * Assume two disks of the same sizes are distinguishable and their relative position is retained when all disks are moved to the target peg.

Exercise 912.1.15. * What if the original tower of Hanoi problem is now changed so that you have 4 pegs? This is called Reve's puzzle. There's a Frank-Stewart algorithm since 1941, but it is not known if this algorithm is optimal. This is a good research project. You should of course find an algorithm that runs faster that $2^n - 1$. What about 5 pegs? Etc.

Exercise 912.1.16. What if the original tower of Hanoi problem is now changed so that you have 4 pegs A,B,C,D. and if the disks are numbered 1, 2, 3, ..., n where n is the largest disk, and only even numbered disk can be placed at B and only odd numbered disk can be placed at C.

Exercise 912.1.17. Let a_n $(n = 0, 1, 2, \ldots)$ satisfy

$$a_n = \begin{cases} 1 & \text{if } n = 0 \\ 2 & \text{if } n = 1 \\ a_{n-1} + a_{n-2} + 3 & \text{if } n > 1 \end{cases}$$

Let $a(x) = \sum_{n=0}^{\infty} a_n x^n$. Find a rational function for $a(x)$. Find a closed form for a_n. □

Exercise 912.1.18. Let a_n $(n = 0, 1, 2, \ldots)$ satisfy

$$a_n = \begin{cases} 1 & \text{if } n = 0 \\ 2 & \text{if } n = 1 \\ a_{n-1} + a_{n-2} + n & \text{if } n > 1 \end{cases}$$

Let $a(x) = \sum_{n=0}^{\infty} a_n x^n$. Find a rational function for $a(x)$. Find a closed form for a_n. $\square$

Exercise 912.1.19. Do the same for:

$$a_n = \begin{cases} 1 & \text{if } n = 0 \\ 2 & \text{if } n = 1 \\ 3a_{n-2} + n^2 + 6n + 1 & \text{if } n > 1 \end{cases}$$

$\square$

Exercise 912.1.20. It's time to prove your own theorem: Can you find a rational expression for the generating function of a_n where

$$a_n = \begin{cases} a & \text{if } n = 0 \\ b & \text{if } n = 1 \\ ca_{n-1} + da_{n-2} & \text{if } n > 1 \end{cases}$$

□

Exercise 912.1.21. What is a_n where

$$a_n = \begin{cases} 1 & \text{if } n = 0 \\ na_{n-1} & \text{if } n > 1 \end{cases}$$

□

Exercise 912.1.22. What can you tell me about a_n $(n = 0, 1, 2, \ldots)$ where

$$a_n = \begin{cases} 1 & \text{if } n = 0 \\ 2 & \text{if } n = 1 \\ na_{n-1} + a_{n-2} & \text{if } n > 1 \end{cases}$$

□

tower-of-hanoi-generating-function.tex

Can generating functions be used to derive a closed form for t_n? Let's go ahead and try it. Again we define the generating function for t_n:

$$t(x) = \sum_{n=0}^{\infty} t_n x^n$$

Hence

$$\begin{aligned} t(x) &= t_0 + \sum_{n=1}^{\infty} t_n x^n \\ &= 0 + \sum_{n=1}^{\infty} (2t_{n-1} + 1)x^n \\ &= 2\sum_{n=1}^{\infty} t_{n-1} x^n + \sum_{n=1}^{\infty} x^n \\ &= 2x\sum_{n=1}^{\infty} t_{n-1} x^{n-1} + \sum_{n=1}^{\infty} x^n \\ &= 2x\sum_{n=0}^{\infty} t_n x^n + \sum_{n=1}^{\infty} x^n \end{aligned}$$

Now note that we see that $t(x)$ pops up and ... a geoemtric series appears as well (with the x^0 missing)! But that's no big deal. Remember that ultimately we want to write $t(x)$ as a rational function. Geometric series are rational functions too. Having one around doesn't make the solution harder. So we just plow ahead.

$$
\begin{aligned}
t(x) &= 2x \sum_{n=0}^{\infty} t_n x^n + \sum_{n=0}^{\infty} x^n - 1 \\
&= 2xt(x) + \frac{1}{1-x} - 1 \\
\therefore t(x) - 2xt(x) &= \frac{1}{1-x} - 1 = \frac{1-1+x}{1-x} \\
\therefore (1-2x)t(x) &= \frac{x}{1-x} \\
\therefore t(x) &= \frac{x}{(1-x)(1-2x)}
\end{aligned}
$$

Exercise 912.1.23. Check that

$$\frac{1}{(1-x)(1-2x)} = \frac{-1}{1-x} + \frac{2}{1-2x}$$

□

Hence

$$\begin{aligned}
t(x) &= x\frac{1}{(1-x)(1-2x)} \\
&= x\left(\frac{-1}{1-x} + \frac{2}{1-2x}\right) \\
&= x\left(-\sum_{n=0}^{\infty} x^n + 2\sum_{n=0}^{\infty} 2^n x^n\right) \\
&= \sum_{n=0}^{\infty} \left(-1 + 2^{n+1}\right) x^{n+1} \\
&= \sum_{n=1}^{\infty} \left(-1 + 2^{n}\right) x^{n}
\end{aligned}$$

Therefore we have

$$t_n = \begin{cases} 0 & \text{if } n = 0 \\ 2^n - 1 & \text{if } n > 0 \end{cases}$$

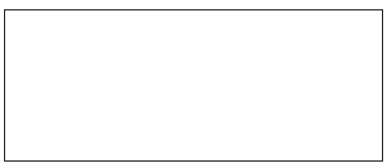
Note that $2^0 - 1 = 0$. Therefore

$$t_n = 2^n - 1$$

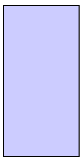
for $n \geq 0$. So it's clear that the method of generating function can solve recurrences of degree one as well. Note that the "+ 1" in the recurrence relation, the nonhomogeneous term, is not a problem either.

912.1.4 Tiling Problems

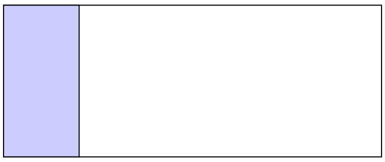
Example 912.1.1. Suppose you are given a 2-by-n area



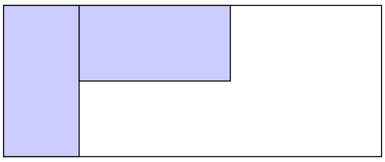
and you need to tile it with 2-by-1 tiles that look like this:



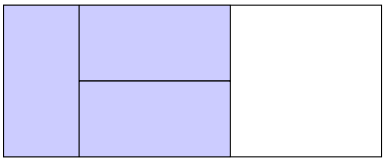
For instance here's a tiling of a 2-by-5 area: I put a vertical tile:



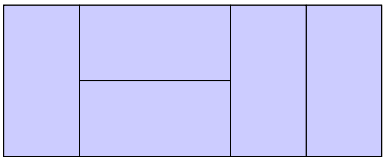
And a horizontal tile:



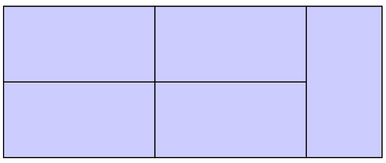
And another horizontal tile:



And two vertical tiles:



There. That's one tiling. Here's another:



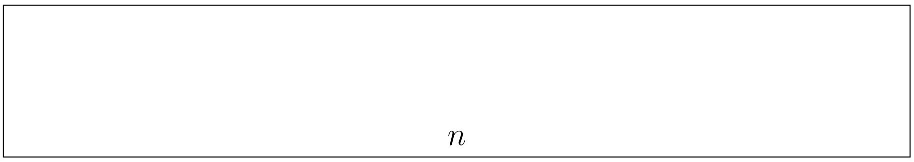
Now let a_n be the number of tilings for a 2-by-n area. Compute a_{20} Of course instead of trying to draw all the diagram up to $n = 20$, you probably want to find a recurrence relation and then use the recurrence relation to compute a_{20}. It's even better if you can find a closed form for a_n – but we'll do that later. Before first you should do some experiments.

(a) Draw all the tilings for $n = 1, 2, 3, 4, 5$.
(b) Find a recurrence relation for a_n.

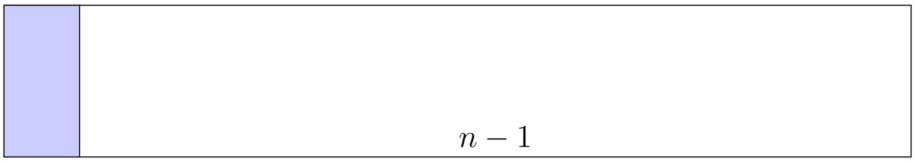
Solution.

(a) is simple. I'll leave that to you.

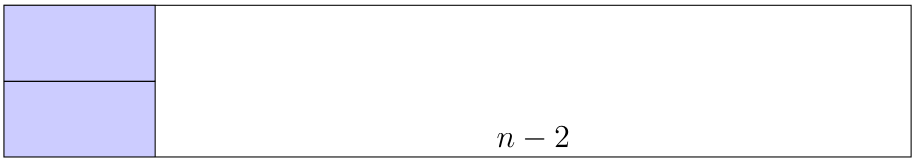
(b) This problem is recursive, i.e., the problem of tiling the 2-by-n area contains the same problem(s), but with a smaller area. Here's a 2-by-n area:



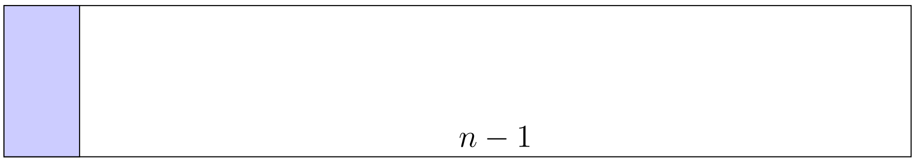
The left side of this area is either tiled by a vertical tile:



or by two horizontal tiles:

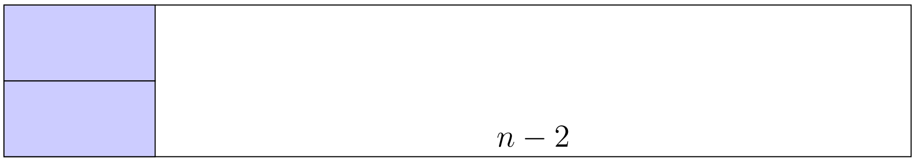


In the first case, the resulting area to to be tiled is a 2-by-$(n - 1)$ area.



and clearly the number of ways to tile the remaining untiled area is the number of ways to tile a 2-by-$(n-1)$ area. Hence there are a_{n-1} to tile the 2-by-n area assuming that we start with a vertical tile.

Likewise, in the second case, the resulting area to be tiled is a 2-by-$(n-2)$ area.



and therefore the number of ways to the 2-by-n area assuming that we start with two horizontal tiles is a_{n-2}.

Altogether, since a tiling of the 2-by-n area must be one of the two cases above and since the above two cases of tilings do not have any tiling in common, by the addition principle, the total number of ways to tile the 2-by-n area is

$$a_n = a_{n-1} + a_{n-2}$$

for $n \geq 2$. Note that

$$a_1 = 1$$

Also, it's easy to see that

$$a_2 = 2$$

Hence from $a_2 = a_1 + a_0$, we get

$$2 = 1 + a_0$$

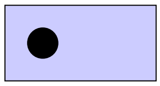
i.e. $a_0 = 1$. Therefore we have

$$a_n = \begin{cases} 1 & \text{if } n = 0, 1 \\ a_{n-1} + a_{n-2} & \text{if } n \geq 2 \end{cases}$$

□

Does it surprise you that the above is the Fibonacci recurrence relation?

Exercise 912.1.24. Here's a variation of the above problem: Suppose the 2-by-1 tiles look like this:

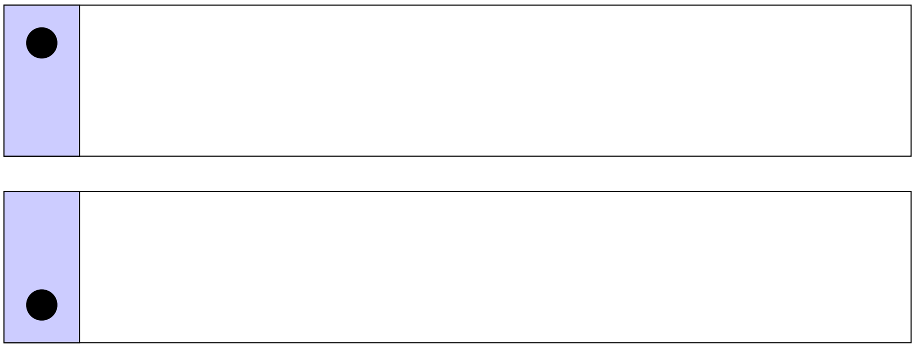


(i.e. there's a black dot on one side.) Let a_n be the number of tilings for this problem. Find a recurrence relation for a_n and include some base cases.

INCOMING SPOILER ALERT ... SOLUTION ON NEXT PAGE

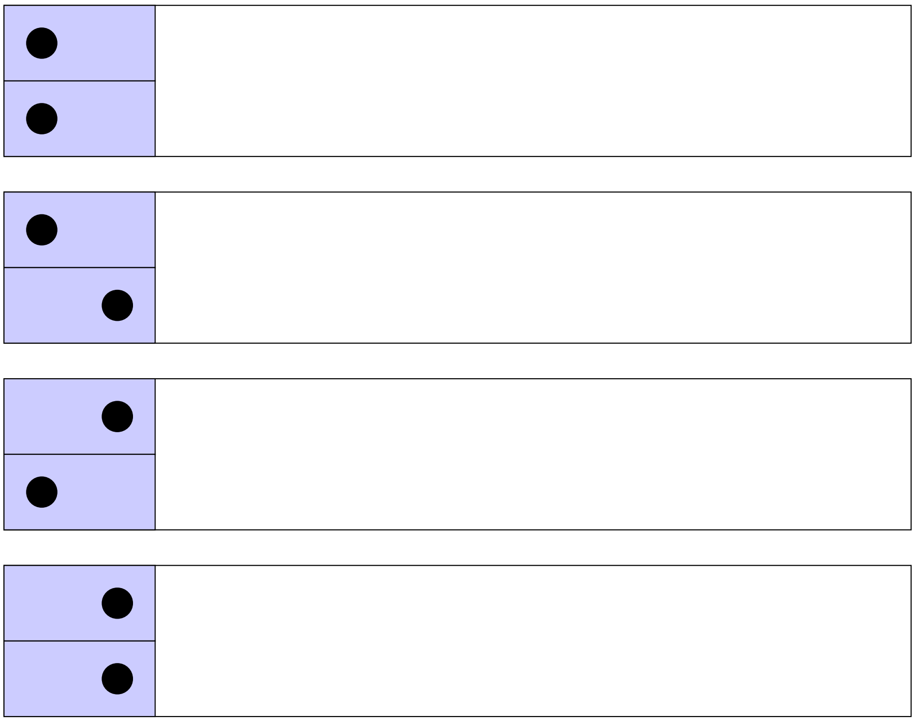
Solution.

The following



can be completed with tilings of 2-by-$(n-1)$ tilings. All these complete tilings are distinct. The number of such tilings is $2a_{n-1}$.

The following



can be completed with tilings of 2-by-$(n-2)$ tilings. All these complete tilings are distinct. The number of such tilings is $4a_{n-2}$.

Hence

$$a_n = 2a_{n-1} + 4a_{n-2} \quad \text{if } n \geq 2$$

Furthermore, from the above, we see that $a_1 = 2$ and $a_2 = 4$. We can compute a_0 from the recurrence relation:

$$a_2 = 2a_1 + 4a_0$$

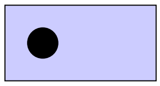
Therefore $a_0 = 0$. Hence we have

$$a_n = \begin{cases} 0 & \text{if } n = 0 \\ 2 & \text{if } n = 1 \\ 2a_{n-1} + 4a_{n-2} & \text{if } n \geq 2 \end{cases}$$

□

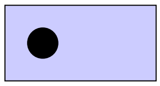
Exercise 912.1.25. Here's another variation: Suppose your 2-by-1 tiles include the unmarked ones and

and ones with a dot at one side

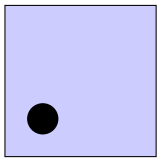


(Go to solution, page ??) □

Exercise 912.1.26. Here's another variation: Suppose you have 2-by-1 and 2-by-2 tiles of the form

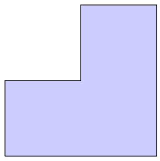


and



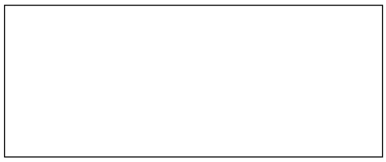
Find a closed form for the number of tilings. (Go to solution, page **??**) □

Exercise 912.1.27. What if the tiles look like this:

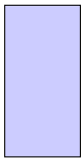


(This tile has as area of 3.) (Go to solution, page ??) □

Exercise 912.1.28. Let's go back to the original tiling problem: you are given a 2-by-n area

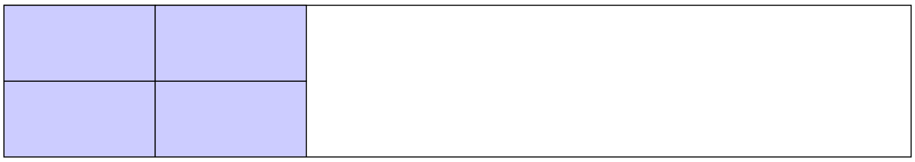


and you need to tile it with 2-by-1 tiles that look like this:



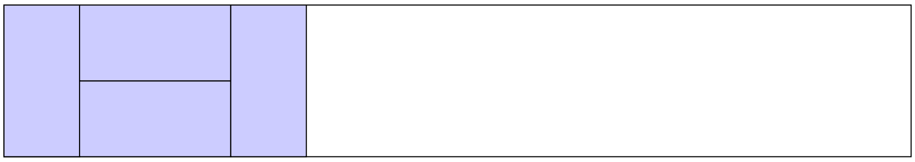
What if I say to you that I only want to consider the case where the tiling does not have two consecutive verticals in the tiling? What is the recurrence relation? (Go to solution, page ??) □

Exercise 912.1.29. What about the case where you cannot have 4 horizontal tiles side by side, i.e. the following is not allowed:



whether it is at the beginning, at the end, or somewhere in the middle.

Exercise 912.1.30. What about the case where the following is not allowed:



whether it is at the beginning, at the end, or somewhere in the middle.

912.1.5 String problems

Exercise 912.1.31. Let a_n be the number of bit patterns of length n where two consecutive 0s are not allowed. As an example, for the case of $n = 4$, we have

0000 not allowed
0001 not allowed
0010 not allowed
0011 not allowed
0100 not allowed
0101
0110
0111
1000 not allowed
1001 not allowed
1010
1011
1100 not allowed
1101
1110
1111

i.e., $a_4 = 8$.

Solutions

Solution to Exercise **??**.

Let $G_n = (A, B, C)$ where n is the number of months (since the beginning), A is the number of pairs with age 0, B is the number of pairs with age 1, C is the number of pairs with age ≥ 2. Then

$$G_{n+1} = (2(B + C), A, B + C)$$

and

$$G_{n+2} = (2(A + B + C), 2(B + C), A + B + C)$$

Altogether we have

$$\begin{aligned} G_n &= (A, B, C) \\ G_{n+1} &= (2(B + C), A, B + C) \\ G_{n+2} &= (2(A + B + C), 2(B + C), A + B + C) \end{aligned}$$

For $G_n = (A, B, C)$, let $F_n = A + B + C$. Suppose $F_{n+2} = aF_{n+1} + bF_n$. Then

$$(2(A+B+C)+2(B+C)+(A+B+C) = a((2(B+C)+A+B+C))+b(A+B+C)$$

i.e.,

$$3A + 5B + 5C = (a + b)A + (3a + b)B + (3a + b)C$$

Hence $a = 1, b = 2$. Hence

$$F_{n+2} = F_{n+1} + 2F_n$$

We also have $F_0 = 1 = F_1$. Altogether we have

$$F_n = \begin{cases} 1 & \text{if } n = 0, 1 \\ F_{n-1} + 2F_{n-2} & \text{if } n \geq 3 \end{cases}$$

Hence

$$\begin{aligned}
F_0 &= 1 \\
F_1 &= 1 \\
F_2 &= 3 \\
F_3 &= 5 \\
F_4 &= 11 \\
F_5 &= 21 \\
F_6 &= 43 \\
F_7 &= 85 \\
F_8 &= 171 \\
F_9 &= 341 \\
F_{10} &= 683 \\
F_{11} &= 1365 \\
F_{12} &= 2731
\end{aligned}$$

(One can draw a diagram similar to the original Fibonacci problem and verify the first few numbers are indeed correct.) □

Solution to Exercise **??**.

exercises/dog/ans

Solution is not provided.

Solution to Exercise **??**.

exercises/guinea-pigs/answer.t

Solution is not provided.

Solution to Exercise ??.

exercises/ans

(a) 7

(b) 7

(c) 6

(d) 6 □

Solution to Exercise **??**.

exercises/sum
n/answer.tex

Note that

$$\begin{aligned} s_n &= 0 + 1 + 2 + \cdots + n \\ &= s_{n-1} + n \end{aligned}$$

for $n \geq 1$. Clearly $s_0 = 0$. This is a nonhomogeneous linear recurrence relation of degree 1. Here's a complete description of s_n:

$$s_n = \begin{cases} 0 & \text{if } n = 0 \\ s_{n-1} + n & \text{f } n \geq 1 \end{cases}$$

□

Solution to Exercise ??.

exer-
cises/n2/ques

We have

$$(n+1)^2 = n^2 + 2n + 1$$

Hence

$$a_{n+1} = a_n + 2n + 1$$

or

$$a_n = a_{n-1} + 2(n-1) + 1 = a_{n-1} + 2n - 1$$

for $n \geq 1$. Hence

$$a_n = \begin{cases} 0 & \text{if } n = 0 \\ a_{n-1} + 2n - 1 & \text{if } n > 0 \end{cases}$$

This is a nonhomogeneous linear recurrence of degree 1. □

Solution to Exercise ??.

exercises/euler-totient-recurrence/an

Solution not provided.

Solution to Exercise ??.

exercises/recurrence-to-closed-form-1/answer.tex

Let

$$a(x) = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$\begin{aligned}
a(x) &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_0 + a_1 x + \sum_{n=2}^{\infty} a_n x^n \\
&= 3 + 5x + \sum_{n=2}^{\infty} (a_{n-1} + a_{n-2}) x^n \\
&= 3 + 5x + \sum_{n=2}^{\infty} a_{n-1} x^n + \sum_{n=0}^{\infty} a_{n-2} x^n \\
&= 3 + 5x + x \sum_{n=2}^{\infty} a_{n-1} x^{n-1} + x^2 \sum_{n=2}^{\infty} a_{n-2} x^{n-2} \\
&= 3 + 5x + x \sum_{p=1}^{\infty} a_p x^p + x^2 \sum_{q=0}^{\infty} a_q x^q \quad \text{(let } p = n-1 \text{ and } q = n-2\text{)} \\
&= 3 + 5x + x \left(\sum_{p=0}^{\infty} a_p x^p - a_0 \right) + x^2 a(x) \\
&= 3 + 5x + x\left(a(x) - 3\right) + x^2 a(x) \\
&= 3 + 5x + x a(x) - 3x + x^2 a(x) \\
&= 3 + 2x + x a(x) + x^2 a(x) \\
\therefore \quad (1 - x - x^2) a(x) &= 3 + 2x \\
\therefore \quad a(x) &= \frac{3 + 2x}{1 - x - x^2} \\
&= \frac{-3 - 2x}{x^2 + x - 1}
\end{aligned}$$

The roots of $x^2 + x - 1$ are

$$\frac{1 \pm \sqrt{5}}{2}$$

Let

$$r_1 = \frac{-1 + \sqrt{5}}{2}, \quad r_2 = \frac{-1 - \sqrt{5}}{2}$$

By the theory of partial fractions, let

$$\frac{-3-2x}{x^2+x-1} = \frac{-3-2x}{(x-r_1)(x-r_2)} = \frac{A}{x-r_1} + \frac{B}{x-r_2}$$

where A, B are constants. Then

$$-3-2x = A(x-r_2) + B(x-r_1)$$

If $x = r_1$, we get

$$\begin{aligned}
-3-2r_1 &= A(r_1-r_2) \\
\therefore \quad A &= \frac{-3-2r_1}{r_1-r_2} \\
&= \frac{-3-2r_1}{\sqrt{5}} \\
&= \frac{(-3-2r_1)\sqrt{5}}{5}
\end{aligned}$$

And if $x = r_2$, we get

$$\begin{aligned}
-3-2r_2 &= B(r_2-r_1) \\
\therefore \quad B &= \frac{-3-2r_2}{r_2-r_1} \\
&= \frac{3+2r_2}{\sqrt{5}} \\
&= \frac{(3+2r_2)\sqrt{5}}{5}
\end{aligned}$$

Hence

$$a(x) = \frac{A}{x-r_1} + \frac{B}{x-r_2}$$

where

$$\begin{aligned}
A &= \frac{(-3-2r_1)\sqrt{5}}{5} \\
B &= \frac{(3+2r_2)\sqrt{5}}{5} \\
r_1 &= \frac{-1+\sqrt{5}}{2} \\
r_2 &= \frac{-1-\sqrt{5}}{2}
\end{aligned}$$

Let's check with a program:

```
from math import sqrt

def an(n):
    if n == 0: return 3
    elif n == 1: return 5
    else: return an(n - 1) + an(n - 2)

def a(x, N=20):
    s = 0
    for n in range(N + 1):
        s += an(n) * x**n
    return s

def a1(x): # rational express
    return (-3 - 2 * x) / (x**2 + x - 1)

def a2(x): # partial fractions
    r = sqrt(5)
    r1 =  (-1 + r) / 2
    r2 =  (-1 - r) / 2
    A = (-3 - 2 * r1) * sqrt(5) / 5
    B = (3 + 2 * r2) * sqrt(5) / 5
    return A / (x - r1) + B / (x - r2)

for x in range(0, 10):
    x = x / 100.0
    print(a(x), a1(x), a2(x))
```

The output is

```
3.0 3.0 3.0
3.0508132134559043 3.0508132134559047 3.0508132134559047
3.10330747243773 3.1033074724377294 3.10330747243773
3.1575688783407285 3.1575688783407285 3.1575688783407285
3.2136894824707842 3.2136894824707847 3.2136894824707847
3.2717678100263856 3.271767810026385 3.2717678100263856
3.3319094404100813 3.3319094404100813 3.3319094404100813
3.3942276510647504 3.3942276510647496 3.39422765106475
3.458844133099826 3.458844133099825 3.4588441330998245
3.525889788224859 3.5258897882248585 3.525889788224858
```

Now we compute the coefficient of x^n from the partial fractions:

$$\begin{aligned}
a(x) &= \frac{A}{x - r_1} + \frac{B}{x - r_2} \\
&= -\frac{A}{r_1 - x} - \frac{B}{r_2 - x} \\
&= -\frac{A}{r_1}\frac{1}{1 - x/r_1} - \frac{B}{r_2}\frac{1}{1 - x/r_2} \\
&= -\frac{A}{r_1}\sum_{n=0}^{\infty}\left(\frac{x}{r_1}\right)^n - \frac{B}{r_2}\sum_{n=0}^{\infty}\left(\frac{x}{r_2}\right)^n \\
&= -\frac{A}{r_1}\sum_{n=0}^{\infty}\frac{1}{r_1^n}x^n - \frac{B}{r_2}\sum_{n=0}^{\infty}\frac{1}{r_2^n}x^n \\
&= \sum_{n=0}^{\infty}\left(-\frac{A}{r_1^{n+1}} - \frac{B}{r_2^{n+1}}\right)x^n
\end{aligned}$$

Hence

$$a_n = -\frac{A}{r_1^{n+1}} - \frac{B}{r_2^{n+1}}$$

Let's check with a program:

```
from math import sqrt

def an(n):
    if n == 0: return 3
    elif n == 1: return 5
    else: return an(n - 1) + an(n - 2)

def bn(n):
    r = sqrt(5)
    r1 =  (-1 + r) / 2
    r2 =  (-1 - r) / 2
    A = (-3 - 2 * r1) * sqrt(5) / 5
    B = (3 + 2 * r2) * sqrt(5)/ 5
    return -A/r1 ** (n + 1) - B/r2 ** (n + 1)

for n in range(10):
    print(an(n), bn(n))
```

The output is

```
3 3.0
5 4.999999999999999
8 7.999999999999998
13 12.999999999999996
21 20.999999999999993
34 33.999999999999986
55 54.99999999999997
89 88.99999999999994
144 143.9999999999999
233 232.99999999999983
```

Simplifying a_n, we get

$$\begin{aligned}
a_n &= -\frac{A}{r_1^{n+1}} - \frac{B}{r_2^{n+1}} \\
&= -A\left(\frac{r_2}{r_1 r_2}\right)^{n+1} - B\left(\frac{r_1}{r_1 r_2}\right)^{n+1} \\
&= \frac{1}{(r_1 r_2)^{n+1}}\left(-A r_2^{n+1} - B r_1^{n+1}\right) \\
&= \frac{1}{(-1)^{n+1}}\left(-\frac{(-3-2r_1)\sqrt{5}}{5} r_2^{n+1} - \frac{(3+2r_2)\sqrt{5}}{5} r_1^{n+1}\right) \\
&= \frac{(-1)^{n+1}\sqrt{5}}{5}\left((3+2r_1)r_2^{n+1} - (3+2r_2)r_1^{n+1}\right) \\
&= \frac{(-1)^{n+1}\sqrt{5}}{5}\left(\left(3+2\frac{-1+\sqrt{5}}{2}\right) r_2^{n+1} - \left(3+2\frac{-1-\sqrt{5}}{2}\right) r_1^{n+1}\right) \\
&= \frac{(-1)^{n+1}\sqrt{5}}{5}\left(\left(2+\sqrt{5}\right) r_2^{n+1} - \left(2-\sqrt{5}\right) r_1^{n+1}\right)
\end{aligned}$$

Let's check with a program:

```
from math import sqrt

def an(n):
    if n == 0: return 3
    elif n == 1: return 5
    else: return an(n - 1) + an(n - 2)

def bn(n):
    r = sqrt(5)
    r1 =  (-1 + r) / 2
```

```
    r2 =  (-1 - r) / 2
    A = (-3 - 2 * r1) * sqrt(5) / 5
    B = (3 + 2 * r2) * sqrt(5)/ 5
    return -A/r1 ** (n + 1) - B/r2 ** (n + 1)

def cn(n):
    r = sqrt(5)
    r1 =  (-1 + r) / 2
    r2 =  (-1 - r) / 2
    return (((-1)**(n+1) * sqrt(5))/5) * \
        ((2 + r) * r2**(n+1) - \
         (2 - r) * r1**(n+1))

for n in range(10):
    print(an(n), bn(n), cn(n))
```

The output is

```
3 3.0 3.0000000000000004
5 4.999999999999999 5.000000000000001
8 7.999999999999998 8.000000000000002
13 12.999999999999996 13.000000000000004
21 20.999999999999993 21.000000000000004
34 33.999999999999986 34.00000000000001
55 54.9999999999997 55.000000000000014
89 88.99999999999994 89.00000000000004
144 143.9999999999999 144.00000000000006
233 232.99999999999983 233.00000000000009
```

Hence

$$\begin{aligned} a_n &= \frac{(-1)^{n+1}\sqrt{5}}{5}\left(\left(2+\sqrt{5}\right)r_2^{n+1} - \left(2-\sqrt{5}\right)r_1^{n+1}\right) \\ &= \frac{(-1)^{n+1}\sqrt{5}}{5}\left(\left(2+\sqrt{5}\right)\left(\frac{-1-\sqrt{5}}{2}\right)^{n+1} - \left(2-\sqrt{5}\right)\left(\frac{-1+\sqrt{5}}{2}\right)^{n+1}\right) \end{aligned}$$

for $n \geq 0$. $\square$

Solution to Exercise ??.

exercises/recurrence-to-closed-form-0/answer.tex

Let

$$b(x) = \sum_{n=0}^{\infty} b_n x^n$$

Then

$$\begin{aligned}
b(x) &= \sum_{n=0}^{\infty} b_n x^n \\
&= b_0 + b_1 x + \sum_{n=2}^{\infty} b_n x^n \\
&= 3 + 5x + \sum_{n=2}^{\infty} (7b_{n-1} + 11b_{n-2}) x^n \\
&= 3 + 5x + \sum_{n=2}^{\infty} 7b_{n-1} x^n + \sum_{n=0}^{\infty} 11b_{n-2} x^n \\
&= 3 + 5x + 7x \sum_{n=2}^{\infty} b_{n-1} x^{n-1} + 11x^2 \sum_{n=2}^{\infty} b_{n-2} x^{n-2} \\
&= 3 + 5x + 7x \sum_{p=1}^{\infty} b_p x^p + 11x^2 \sum_{q=0}^{\infty} b_q x^q \ \text{ (let } p = n-1 \text{ and } q = n-2) \\
&= 3 + 5x + 7x \left(\sum_{p=0}^{\infty} b_p x^p - b_0 \right) + 11x^2 b(x) \\
&= 3 + 5x + 7x\,(b(x) - 3) + 11x^2 b(x) \\
&= 3 + 5x + 7xb(x) - 21x + 11x^2 b(x) \\
&= 3 - 16x + 7xb(x) + 11x^2 b(x) \\
\therefore \ \ (1 - 7x - 11x^2) b(x) &= 3 - 16x \\
\therefore \ \ b(x) &= \frac{3 - 16x}{1 - 7x - 11x^2} \\
&= \frac{-3 + 16x}{11x^2 + 7x - 1}
\end{aligned}$$

The roots of $11x^2 + 7x - 1$ are

$$\frac{-7 \pm \sqrt{49 + 44}}{22} = \frac{-7 \pm \sqrt{93}}{22}$$

Let

$$r_1 = \frac{-7 + \sqrt{93}}{22}, \ \ r_2 = \frac{-7 - \sqrt{93}}{22}$$

By the theory of partial fractions, let

$$\frac{-3+16x}{11x^2+7x-1} = \frac{-3+16x}{11(x-r_1)(x-r_2)} = \frac{A}{x-r_1} + \frac{B}{x-r_2}$$

where A, B are constants. Then

$$-3+16x = 11A(x-r_2) + 11B(x-r_1)$$

If $x = r_1$, we get

$$\begin{aligned} -3+16r_1 &= 11A(r_1-r_2) \\ \therefore \quad A &= \frac{-3+16r_1}{11(r_1-r_2)} \\ &= \frac{-3+16r_1}{11(2\sqrt{93}/22)} \\ &= \frac{-3+16r_1}{\sqrt{93}} \\ &= \frac{(-3+16r_1)\sqrt{93}}{93} \end{aligned}$$

And if $x = r_2$, we get

$$\begin{aligned} -3+16r_2 &= 11B(r_2-r_1) \\ \therefore \quad B &= \frac{-3+16r_2}{11(r_2-r_1)} \\ &= \frac{-3+16r_2}{11(-2\sqrt{93}/22)} \\ &= \frac{3-16r_2}{\sqrt{93}} \\ &= \frac{(3-16r_2)\sqrt{93}}{93} \end{aligned}$$

Hence

$$b(x) = \frac{A}{x-r_1} + \frac{B}{x-r_2}$$

where

$$A = \frac{(-3 + 16r_1)\sqrt{93}}{93}$$
$$B = \frac{(3 - 16r_2)\sqrt{93}}{93}$$
$$r_1 = \frac{-7 + \sqrt{93}}{22}$$
$$r_2 = \frac{-7 - \sqrt{93}}{22}$$

Let's check with a program:

```
from math import sqrt

def bn(n):
    if n == 0: return 3
    elif n == 1: return 5
    else: return 7 * bn(n - 1) + 11 * bn(n - 2)

def b(x, N=20):
    s = 0
    for n in range(N + 1):
        s += bn(n) * x**n
    return s

def b1(x): # rational express
    return (-3 + 16 * x) / (11 * x**2 + 7 * x - 1)

def b2(x): # partial fractions
    r = sqrt(93)
    r1 =  (-7 + r) / 22
    r2 =  (-7 - r) / 22
    A = (-3 + 16 * r1) * sqrt(93) / 93
    B = (3 - 16 * r2) * sqrt(93) / 93
    return A / (x - r1) + B / (x - r2)

for x in range(0, 10):
    x = x / 100.0
    print(b(x), b1(x), b2(x))
```

The output is

```
3.0 3.0 3.0
3.0573796964151145 3.0573796964151145 3.057379696415114
3.1323048153342685 3.1323048153342685 3.132304815334268
3.2303550826814327 3.2303550826817076 3.230355082681707
3.3599088836974675 3.3599088838268796 3.359908883826879
3.53413653015422 3.5341365461847394 3.5341365461847385
3.774980635127497 3.774981495188749 3.774981495188749
4.12187682786023 4.121903091427319 4.121903091427319
4.653137982649629 4.653679653679655 4.653679653679654
5.545021611066334 5.553577785688857 5.553577785688857
```

Now we compute the coefficient of x^n from the partial fractions:

$$\begin{aligned}
b(x) &= \frac{A}{x - r_1} + \frac{B}{x - r_2} \\
&= -\frac{A}{r_1 - x} - \frac{B}{r_2 - x} \\
&= -\frac{A}{r_1}\frac{1}{1 - x/r_1} - \frac{B}{r_2}\frac{1}{1 - x/r_2} \\
&= -\frac{A}{r_1}\sum_{n=0}^{\infty}\left(\frac{x}{r_1}\right)^n - \frac{B}{r_2}\sum_{n=0}^{\infty}\left(\frac{x}{r_2}\right)^n \\
&= -\frac{A}{r_1}\sum_{n=0}^{\infty}\frac{1}{r_1^n}x^n - \frac{B}{r_2}\sum_{n=0}^{\infty}\frac{1}{r_2^n}x^n \\
&= \sum_{n=0}^{\infty}\left(-\frac{A}{r_1^{n+1}} - \frac{B}{r_2^{n+1}}\right)x^n
\end{aligned}$$

Hence

$$b_n = -\frac{A}{r_1^{n+1}} - \frac{B}{r_2^{n+1}}$$

Let's check with a program:

```
from math import sqrt

def bn(n):
    if n == 0: return 3
    elif n == 1: return 5
    else: return 7 * bn(n - 1) + 11 * bn(n - 2)

def cn(n):
```

```
    r = sqrt(93)
    r1 =  (-7 + r) / 22
    r2 =  (-7 - r) / 22
    A = (-3 + 16*r1) * sqrt(93) / 93
    B = (3 - 16 * r2) * sqrt(93)/ 93
    return -A/r1 ** (n + 1) - B/r2 ** (n + 1)

for n in range(10):
    print(bn(n), cn(n))
```

The output is

```
3 3.0
5 5.0
68 68.0
531 531.0000000000001
4465 4465.0
37096 37096.00000000001
308787 308787.00000000006
2569565 2569565.0000000005
21383612 21383612.000000004
177950499 177950499.00000003
```

Simplifying b_n, we get

$$\begin{aligned}
b_n &= -\frac{A}{r_1^{n+1}} - \frac{B}{r_2^{n+1}} \\
&= -A\left(\frac{r_2}{r_1 r_2}\right)^{n+1} - B\left(\frac{r_1}{r_1 r_2}\right)^{n+1} \\
&= \frac{1}{(r_1 r_2)^{n+1}}\left(-A r_2^{n+1} - B r_1^{n+1}\right) \\
&= \frac{1}{(-44/22^2)^{n+1}}\left(-\frac{(-3+16r_1)\sqrt{93}}{93} r_2^{n+1} - \frac{(3-16r_2)\sqrt{93}}{93} r_1^{n+1}\right) \\
&= (-1)^{n+1} 11^{n+1}\left(-\frac{(-3+16r_1)\sqrt{93}}{93} r_2^{n+1} - \frac{(3-16r_2)\sqrt{93}}{93} r_1^{n+1}\right) \\
&= \frac{(-1)^{n+1} 11^{n+1}\sqrt{93}}{93}\left((3-16r_1)\, r_2^{n+1} + (-3+16r_2)\, r_1^{n+1}\right) \\
&= \frac{(-1)^{n+1} 11^{n+1}\sqrt{93}}{93}\left(\left(3 - 16\cdot\frac{-7+\sqrt{93}}{22}\right)\left(\frac{-7-\sqrt{93}}{22}\right)^{n+1}\right. \\
&\qquad\qquad \left. + \left(-3 + 16\cdot\frac{-7-\sqrt{93}}{22}\right)\left(\frac{-7+\sqrt{93}}{22}\right)^{n+1}\right) \\
&= \frac{(-1)^{n+1} 11^{n+1}\sqrt{93}}{93}\left(\left(3 - 8\cdot\frac{-7+\sqrt{93}}{11}\right)\left(\frac{-7-\sqrt{93}}{22}\right)^{n+1}\right. \\
&\qquad\qquad \left. + \left(-3 + 8\cdot\frac{-7-\sqrt{93}}{11}\right)\left(\frac{-7+\sqrt{93}}{22}\right)^{n+1}\right) \\
&= \frac{(-1)^{n+1} 11^{n+1}\sqrt{93}}{93}\left(\left(\frac{89-8\sqrt{93}}{11}\right)\left(\frac{-7-\sqrt{93}}{22}\right)^{n+1}\right. \\
&\qquad\qquad \left. + \left(\frac{-89-8\sqrt{93}}{11}\right)\left(\frac{-7+\sqrt{93}}{22}\right)^{n+1}\right) \\
&= \frac{(-1)^{n+1} 11^{n}\sqrt{93}}{93}\left(\left(89-8\sqrt{93}\right)\left(\frac{-7-\sqrt{93}}{22}\right)^{n+1}\right. \\
&\qquad\qquad \left. + \left(-89-8\sqrt{93}\right)\left(\frac{-7+\sqrt{93}}{22}\right)^{n+1}\right)
\end{aligned}$$

for $n \geq 0$.

Let's check with a program:

```
from math import sqrt

def bn(n):
    if n == 0: return 3
    elif n == 1: return 5
    else: return 7 * bn(n - 1) + 11 * bn(n - 2)

def cn(n):
    r = sqrt(93)
    r1 =  (-7 + r) / 22
    r2 =  (-7 - r) / 22
    A = (-3 + 16*r1) * sqrt(93) / 93
    B = (3 - 16 * r2) * sqrt(93)/ 93
    return -A/r1 ** (n + 1) - B/r2 ** (n + 1)

def dn(n):
    r = sqrt(93)
    r1 =  (-7 + r) / 22
    r2 =  (-7 - r) / 22
    return ((-1)**(n + 1) * 11**(n) * r/93) * \
        ((89 - 8*r) * r2**(n + 1) + (-89 - 8*r) * r1**(n + 1))

for n in range(10):
    print(bn(n), cn(n), dn(n))
```

The output is

```
3 3.0 3.0
5 5.0 4.999999999999998
68 68.0 67.99999999999997
531 531.0000000000001 530.9999999999999
4465 4465.0 4464.999999999998
37096 37096.00000000001 37095.999999999985
308787 308787.00000000006 308786.9999999998
2569565 2569565.0000000005 2569564.999999998
21383612 21383612.000000004 21383611.999999985
177950499 177950499.00000003 177950498.99999985
```

Solution to Exercise ??.

tiling-0/answ

Solution not provided.

Solution to Exercise **??**.

tiling-1/answ

Solution not provided.

Solution to Exercise **??**.

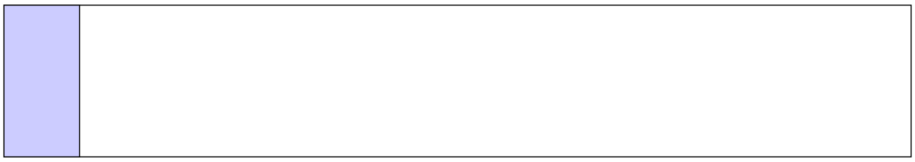
exercises/tilin
2/answer.tex

Solution not provided.

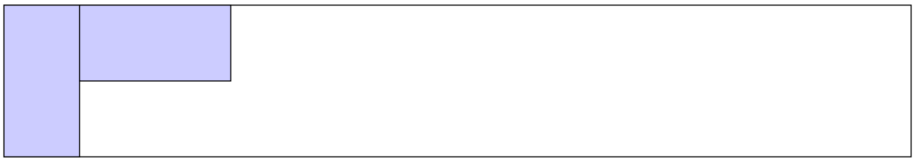
Solution to Exercise ??.

exercises/tilin
3/answer.tex

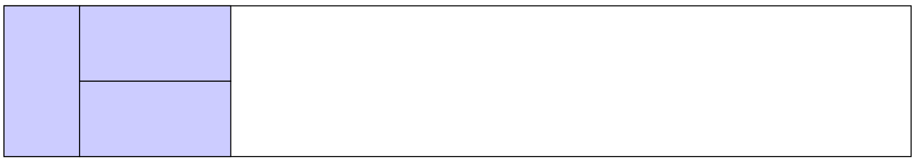
If we start our tiling with



we cannot complete this with any tiling from 2-by-$(n-1)$ since a 2-by-$(n-1)$ might begin with a vertical tailing. So we continue our tiling with

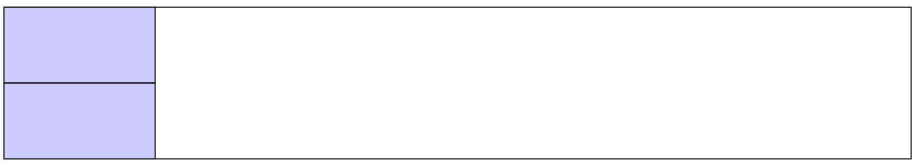


(since a vertical tiling is not allow), and this must be completed as follows:



At this point we can use any 2-by-$(n-3)$ tiling. There are a_{n-3} such tilings.

If we start our 2-by-n tiling like this:



we can complete it with any 2-by-$(n-2)$ tiling. There are a_{n-2} such tilings.

Altogether, we have

$$a_n = a_{n-2} + a_{n-3}$$

for $n \geq 3$. Clearly $a_1 = 1$, $a_2 = 1$ and $a_3 = 2$. Substituting these into the recurrence $a_3 = a_1 + a_0$ we get $2 = 1 + a_0$, i.e., $a_0 = 1$. Hence

$$a_n = \begin{cases} 1 & \text{if } n = 0, 1, 2 \\ a_{n-2} + a_{n-3} & \text{if } n \geq 3 \end{cases}$$

□

912.2 Power series power-series.tex

A **power series** is an expression of the form

$$\sum_{n=0}^{\infty} a_n x^n$$

where a_n is an expression in n. $\sum_{n=0}^{\infty} a_n x^n$ is just a short-hand notation:

$$\sum_{n=0}^{\infty} a_n x^n = a_0 x^0 + a_1 x^1 + a_2 x^2 + \cdots$$

Here's an example:

$$\sum_{n=0}^{\infty} \frac{1}{2^n} x^n$$

In this example,

$$a_n = \frac{1}{2^n}$$

If you like, you can think of a power series as a polynomial that "goes on forever", i.e. a polynomial with infinite degree.

A series need not start with $n = 0$. For instance this is a perfectly good series:

$$\sum_{n=5}^{\infty} \frac{1}{2^n} x^n$$

And it doesn't have to go on forever. This is OK too:

$$\sum_{n=5}^{1000000} \frac{1}{2^n} x^n$$

Of course this means

$$\sum_{n=5}^{1000000} \frac{1}{2^n} x^n = \frac{1}{2^5} x^5 + \frac{1}{2^6} x^6 + \frac{1}{2^7} x^7 + \cdots + \frac{1}{2^{999999}} x^{999999} + \frac{1}{2^{1000000}} x^{1000000}$$

Of course this *is* a polynomial (of degree 1000000).

More generally, power series are examples of series of functions:

$$\sum_{n=0}^{\infty} A_n(x)$$

where each $A_n(x)$ is a function of x. One can also talk about $\sum_{n=0}^{\infty} c_n$ where each c_n are constants and of course when you substitute a value for x in

$$\sum_{n=0}^{\infty} A_n(x)$$

say $x = 3.1415$, you *do* get a series of constants:

$$\sum_{n=0}^{\infty} A_n(3.1415)$$

At this point, of course you have to think about the serious consequence of adding infinitely many numbers. For instance when you substitute $x = 1$ into

$$\sum_{n=0}^{\infty} x^n$$

you get

$$\sum_{n=0}^{\infty} 1^n = \sum_{n=0}^{\infty} 1 = 1 + 1 + 1 + \cdots$$

which is infinite! There are however cases where the sum is not infinite: When $x = 1/2$,

$$\sum_{n=0}^{\infty} (1/2)^n$$

is finite. If you run this program

```
s = 0.0
term = 1.0
while 1:
    s = s + term
    print(s)
    term = term / 2.0
```

(rewritten in your favorite programming lang) you will see that `s` seems to stop growing after some time. In fact from your Pre-calc you already know that

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

This is the famous **geometric series**. When you substitute $x = 1/2$ into the above, you get

$$\sum_{n=0}^{\infty} (1/2)^n = \frac{1}{1-1/2} = 2$$

I already mentioned that if you substitute $x = 1$, the above series explodes in your face. In fact for the geometric series identity to be correct, you can only substitute x when $|x| < 1$. Only then will the series *converge*. For us however, we frequently do not substitute values for x in a power series. You can think of the powers of x as place holders for values (i.e. coefficients).

Here are some basic facts about series. The following facts are easy since they mirror the same facts for polynomials:

Proposition 912.2.1. *Let $\sum_{n\in I} A_n$, $\sum_{n\in I} B_n$ be two series; here $\sum_{n\in I}$ means "sum over all i in index set I" and the A_i and B_i can be functions of x. c can be a constant, a polynomial, or a power series.*

(a) $c\sum_{n\in I} A_n = \sum_{n\in I} cA_n$
(b) $\sum_{n\in I} A_n + \sum_{n\in I} B_n = \sum_{n\in I}(A_n + B_n)$

Note that the above applies only when the index sets are all the same. In terms of power series we have the following:

Proposition 912.2.2. *Let $\sum_{n\in I} a_n x^n$, $\sum_{n\in I} b_n x^n$ be two series.*

(a) $c\sum_{n\in I} a_n x^n = \sum_{n\in I} ca_n x^n$
(b) $\sum_{n\in I} a_n x^n + \sum_{n\in I} b_n x^n = \sum_{n\in I}(a_n + b_n)x^n$

For instance

$$3\sum_{n=0}^{\infty} \frac{1}{2^n} x^n = \sum_{n=0}^{\infty} 3 \cdot \frac{1}{2^n} x^n = \sum_{n=0}^{\infty} \frac{3}{2^n} x^n$$

and

$$(3 + 2x + x^2)\sum_{n=0}^{\infty} \frac{1}{2^n} x^n = \sum_{n=0}^{\infty} (3 + 2x + x^2)\frac{1}{2^n} x^n$$

Note that when working with power series, we are usually interested in extracting the coefficient of x^n from a power series. In the case of

$$3\sum_{n=0}^{\infty} \frac{1}{2^n} x^n = \sum_{n=0}^{\infty} \frac{3}{2^n} x^n$$

we see that the coefficient of x^n is

$$\frac{3}{2^n}$$

However for the case of

$$(3+2x+x^2)\sum_{n=0}^{\infty}\frac{1}{2^n}x^n = \sum_{n=0}^{\infty}(3+2x+x^2)\frac{1}{2^n}x^n$$

the coefficient of x^n is not obvious. In fact we have to do this:

$$\begin{aligned}
(3+2x+x^2)\sum_{n=0}^{\infty}\frac{1}{2^n}x^n &= 3\sum_{n=0}^{\infty}\frac{1}{2^n}x^n + 2x\sum_{n=0}^{\infty}\frac{1}{2^n}x^n + x^2\sum_{n=0}^{\infty}\frac{1}{2^n}x^n \\
&= \sum_{n=0}^{\infty}3\frac{1}{2^n}x^n + \sum_{n=0}^{\infty}2x\frac{1}{2^n}x^n + \sum_{n=0}^{\infty}x^2\frac{1}{2^n}x^n \\
&= \sum_{n=0}^{\infty}\frac{3}{2^n}x^n + \sum_{n=0}^{\infty}\frac{2}{2^n}x^{n+1} + \sum_{n=0}^{\infty}\frac{1}{2^n}x^{n+2}
\end{aligned}$$

If you stare at the above carefully you should be able to see the coefficient of x^n. If you don't see it you can write down a few terms:

$$\begin{aligned}
(3+2x+x^2)\sum_{n=0}^{\infty}\frac{1}{2^n}x^n &= \sum_{n=0}^{\infty}\frac{3}{2^n}x^n + \sum_{n=0}^{\infty}\frac{2}{2^n}x^{n+1} + \sum_{n=0}^{\infty}\frac{1}{2^n}x^{n+2} \\
&= \left(\frac{3}{2^0}x^0 + \frac{3}{2^1}x^1 + \frac{3}{2^2}x^2 + \cdots\right) \\
&\qquad + \left(\frac{2}{2^0}x^1 + \frac{2}{2^1}x^2 + \frac{2}{2^2}x^3 + \cdots\right) \\
&\qquad\qquad + \left(\frac{1}{2^0}x^2 + \frac{1}{2^1}x^3 + \frac{1}{2^2}x^4 + \cdots\right)
\end{aligned}$$

Do you see that when $n \geq 2$, the coefficient is

$$\frac{3}{2^n} + \frac{2}{2^{n-1}} + \frac{1}{2^{n-2}}$$

For the remaining coefficients (i.e. for x^0 and x^1) we have to handle them separately. They are respectively

$$\frac{3}{2^0}, \quad \frac{3}{2^1} + \frac{2}{2^0}$$

In order not to rely on all the above term-by-term expansion, we can formalize the computations as follows.

Let's go back to our series:

$$(3 + 2x + x^2) \sum_{n=0}^{\infty} \frac{1}{2^n} x^n = \sum_{n=0}^{\infty} \frac{3}{2^n} x^n + \sum_{n=0}^{\infty} \frac{2}{2^n} x^{n+1} + \sum_{n=0}^{\infty} \frac{1}{2^n} x^{n+2}$$

For the second term of the sum of three series:

$$\sum_{n=0}^{\infty} \frac{2}{2^n} x^{n+1}$$

we really want x^n and not x^{n+1}. Again by expanding out a few terms you can see what it should be. Another way to do this is to do a substitution to replace the index variable n. Let

$$p = n + 1$$

Note that the sum is from $n = 0$ to ∞. In terms of p, this means the sum is from $p = 0 + 1$ to ∞. Therefore we have

$$\sum_{n=0}^{\infty} \frac{2}{2^n} x^{n+1} = \sum_{p=1}^{\infty} \frac{2}{2^{p-1}} x^p$$

Of course p is just a "dummy" variable; you can replace it with n:

$$\sum_{n=0}^{\infty} \frac{2}{2^n} x^{n+1} = \sum_{p=1}^{\infty} \frac{2}{2^{p-1}} x^p = \sum_{n=1}^{\infty} \frac{2}{2^{n-1}} x^n$$

Now for the third term:

$$\sum_{n=0}^{\infty} \frac{1}{2^n} x^{n+2}$$

Let

$$q = n + 2$$

Then

$$\sum_{n=0}^{\infty} \frac{1}{2^n} x^{n+2} = \sum_{q=2}^{\infty} \frac{1}{2^{q-2}} x^q$$

Therefore

$$\begin{aligned}
(3+2x+x^2)\sum_{n=0}^{\infty} \frac{1}{2^n} x^n &= \sum_{n=0}^{\infty} \frac{3}{2^n} x^n + \sum_{n=0}^{\infty} \frac{2}{2^n} x^{n+1} + \sum_{n=0}^{\infty} \frac{1}{2^n} x^{n+2} \\
&= \sum_{n=0}^{\infty} \frac{3}{2^n} x^n + \sum_{n=1}^{\infty} \frac{2}{2^{n-1}} x^n + \sum_{n=2}^{\infty} \frac{1}{2^{n-2}} x^n
\end{aligned}$$

You see that all the power series have the same power of x for each of their terms. We are now ready to combine the three power series into one. But note that their index values run over different values. Note that if I and J are two disjoint index sets,

$$\sum_{i \in I \cup J} A_i = \sum_{i \in I} A_i + \sum_{i \in J} A_i$$

This implies for instance that

$$\sum_{n=0}^{\infty} A_i = A_0 + \sum_{n=1}^{\infty} A_i$$

and

$$\sum_{n=0}^{\infty} A_i = A_0 + A_1 + \sum_{n=2}^{\infty} A_i$$

In other words you can split up your series into parts. This is useful for instance when you have the sum of two series and their indices run over different values. You simply split out the values not common to both. For instance

$$\begin{aligned}
\sum_{n=1}^{\infty} A_n + \sum_{n=3}^{\infty} B_n &= A_1 + A_2 + \sum_{n=3}^{\infty} A_n + \sum_{n=3}^{\infty} B_n \\
&= A_1 + A_2 + \sum_{n=3}^{\infty} (A_n + B_n)
\end{aligned}$$

Now let's go back to our series:

$$\begin{aligned}(3+2x+x^2)\sum_{n=0}^{\infty}\frac{1}{2^n}x^n &= \sum_{n=0}^{\infty}\frac{3}{2^n}x^n + \sum_{n=0}^{\infty}\frac{2}{2^n}x^{n+1} + \sum_{n=0}^{\infty}\frac{1}{2^n}x^{n+2} \\ &= \sum_{n=0}^{\infty}\frac{3}{2^n}x^n + \sum_{n=1}^{\infty}\frac{2}{2^{n-1}}x^n + \sum_{n=2}^{\infty}\frac{1}{2^{n-2}}x^n\end{aligned}$$

The common index values for all three series are $n \geq 2$. Therefore

$$\begin{aligned}(3+2x+x^2)\sum_{n=0}^{\infty}\frac{1}{2^n}x^n &= \sum_{n=0}^{\infty}\frac{3}{2^n}x^n + \sum_{n=0}^{\infty}\frac{2}{2^n}x^{n+1} + \sum_{n=0}^{\infty}\frac{1}{2^n}x^{n+2} \\ &= \sum_{n=0}^{\infty}\frac{3}{2^n}x^n + \sum_{n=1}^{\infty}\frac{2}{2^{n-1}}x^n + \sum_{n=2}^{\infty}\frac{1}{2^{n-2}}x^n \\ &= \frac{3}{2^0}x^0 + \frac{3}{2^1}x^1 + \sum_{n=2}^{\infty}\frac{3}{2^n}x^n + \frac{2}{2^{1-1}}x^1 + \sum_{n=2}^{\infty}\frac{2}{2^{n-1}}x^n + \sum_{n=2}^{\infty}\frac{1}{2^{n-2}}x^n\end{aligned}$$

Note that from the first series, I split off two terms ($n = 0, 1$) and from the second I split off one term ($n = 1$). Continuing the computation we have

$$\begin{aligned}(3+2x+x^2)\sum_{n=0}^{\infty}\frac{1}{2^n}x^n &= \sum_{n=0}^{\infty}\frac{3}{2^n}x^n + \sum_{n=0}^{\infty}\frac{2}{2^n}x^{n+1} + \sum_{n=0}^{\infty}\frac{1}{2^n}x^{n+2} \\ &= \frac{3}{2^0}x^0 + \frac{3}{2^1}x^1 + \sum_{n=2}^{\infty}\frac{3}{2^n}x^n \\ &\quad + \frac{2}{2^{1-1}}x^1 + \sum_{n=2}^{\infty}\frac{2}{2^{n-1}}x^n \\ &\quad + \sum_{n=2}^{\infty}\frac{1}{2^{n-2}}x^n \\ &= \frac{3}{2^0} + \left(\frac{3}{2^1} + \frac{2}{2^0}\right)x + \sum_{n=2}^{\infty}\left(\frac{3}{2^n} + \frac{2}{2^{n-1}} + \frac{1}{2^{n-2}}\right)x^n\end{aligned}$$

And we're done! You can read off the coefficients very quickly from the above: the coefficient of x^n is

$$\begin{cases} 3 & \text{if } n = 0 \\ \frac{7}{2} & \text{if } n = 1 \\ \dfrac{3}{2^n} + \dfrac{2}{2^{n-1}} + \dfrac{1}{2^{n-2}} & \text{if } n \geq 2 \end{cases}$$

We see that for a constant c (and k any positive integer):

$$c \sum_{n=k}^{\infty} a_n x^n = \sum_{n=k}^{\infty} c a_n x^n$$

which does not change the power of the x in the general term of the power series. However

$$x \sum_{n=k}^{\infty} a_n x^n = \sum_{n=k}^{\infty} a_n x^{n+1}$$

In general

$$x^\ell \sum_{n=k}^{\infty} a_n x^n = \sum_{n=k}^{\infty} a_n x^{n+\ell}$$

This results in a change of the power of x in the general term of the power series. A change of index variable

$$p = n + \ell$$

will allow you to change the power of x so that the power is the same as the index variable of the summation.

If you list the the coefficient of a power series $\sum_{n=0}^{\infty} a_n x^n$ as an infinite-dimensional vector:

$$(a_0, a_1, a_2, a_3, \ldots)$$

for every x^n and you do likewise for the coefficients for $x^2 \sum_{n=0}^{\infty} a_n x^n$:

$$(0, 0, a_0, a_1, a_2, a_3, \ldots)$$

you see that "multiplication-by-x^2" acts like a "shift-by-2" operator. In general "multiplication-by-x^ℓ" acts like a "shift-by-ℓ" operator. The multiplication-by-c is like a scalar multiplication: the coefficients of $c \sum_{n=0}^{\infty} a_n x^n$ is

$$(ca_0, ca_1, ca_2, ca_3, \ldots)$$

Exercise 912.2.1. Rewrite

$$(1 + 3x^2 + 5x^4) \sum_{n=1}^{\infty} \frac{2^n}{3^n} x^n$$

as a power series. What is the coefficient of x^n? (Go to solution, page **??**)
□

Exercise 912.2.2. Rewrite

$$(1 - 3x^2 + 5x^7) \sum_{n=1}^{\infty} \frac{2^n}{3^n} x^n + \sum_{n=100}^{\infty} \frac{1}{5^n} x^n$$

as a power series. What is the coefficient of x^n?

Exercise 912.2.3. Rewrite

$$(1 - 3x^{110} + 5x^{200}) \sum_{n=1}^{\infty} \frac{2^n}{3^n} x^n + \sum_{n=100}^{\infty} \frac{1}{11^n - 1} x^n$$

as a power series. What is the coefficient of x^n?

Exercise 912.2.4. Rewrite

$$(1 - 3x^{110} + 5x^{200}) \sum_{n=1}^{\infty} \frac{2^n}{3^n} x^n + \sum_{n=100}^{\infty} \frac{1}{5^n} x^n + \sum_{n=5}^{90} \frac{1}{5^n} x^n$$

as a power series. What is the coefficient of x^n?

Now that you have a couple of warmups you can prove a couple of little theorems of your own.

Exercise 912.2.5. Rewrite

$$x^\ell \sum_{n=0}^{\infty} a_n x^n$$

as a power series. What is the coefficient of x^n?

Exercise 912.2.6. Rewrite

$$cx^\ell \sum_{n=0}^{\infty} a_n x^n$$

as a power series. What is the coefficient of x^n?

Exercise 912.2.7. Rewrite

$$(c_0 + c_1 x + c_2 x^2) \cdot \sum_{n=0}^{\infty} a_n x^n$$

as a power series. What is the coefficient of x^n?

Exercise 912.2.8. Rewrite

$$(c_0 + c_1 x + c_2 x^2 + \cdots + c_d x^d) \cdot \sum_{n=0}^{\infty} a_n x^n$$

as a power series. What is the coefficient of x^n?

One last fact about power series:

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n \text{ iff } a_n = b_n \text{ for } n \geq$$

Solutions

Solution to Exercise **??**.

power-series-0/answer.tex

$$
\begin{aligned}
&(1 + 3x^2 + 5x^4) \sum_{n=1}^{\infty} \frac{2^n}{3^n} x^n \\
&= \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n x^n + 3x^2 \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n x^n + 5x^4 \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n x^n \\
&= \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n x^n + \sum_{n=1}^{\infty} 3 \left(\frac{2}{3}\right)^n x^{n+2} + \sum_{n=1}^{\infty} 5 \left(\frac{2}{3}\right)^n x^{n+4} \\
&= \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n x^n + \sum_{p=3}^{\infty} 3 \left(\frac{2}{3}\right)^{p-2} x^p + \sum_{q=5}^{\infty} 5 \left(\frac{2}{3}\right)^{q-4} x^q \ \ (\text{let } p = n+2,\ q = n+4) \\
&= \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n x^n + \sum_{n=3}^{\infty} 3 \left(\frac{2}{3}\right)^{n-2} x^n + \sum_{n=5}^{\infty} 5 \left(\frac{2}{3}\right)^{n-4} x^n \ \ (\text{replace } p \text{ by } n,\ q \text{ by } n) \\
&= \left(\frac{2}{3}\right)^1 x^1 + \left(\frac{2}{3}\right)^2 x^2 + \left(\frac{2}{3}\right)^3 x^3 + \left(\frac{2}{3}\right)^4 x^4 + \sum_{n=5}^{\infty} \left(\frac{2}{3}\right)^n x^n \\
&\qquad + 3\left(\frac{2}{3}\right)^1 x^3 + 3\left(\frac{2}{3}\right)^2 x^4 + \sum_{n=5}^{\infty} 3 \left(\frac{2}{3}\right)^{n-2} x^n \\
&\qquad + \sum_{n=5}^{\infty} 5 \left(\frac{2}{3}\right)^{n-4} x^n \\
&= \frac{2}{3}x + \frac{4}{9}x^2 + \frac{62}{27}x^3 + \frac{124}{81}x^4 + \sum_{n=5}^{\infty} \left(\left(\frac{2}{3}\right)^n + 3\left(\frac{2}{3}\right)^{n-2} + 5\left(\frac{2}{3}\right)^{n-4} \right) x^n \\
&= \frac{2}{3}x + \frac{4}{9}x^2 + \frac{62}{27}x^3 + \frac{124}{81}x^4 + \sum_{n=5}^{\infty} \left(\left(\frac{2}{3}\right)^4 + 3\left(\frac{2}{3}\right)^2 + 5 \right) \left(\frac{2}{3}\right)^{n-4} x^n \\
&= \frac{2}{3}x + \frac{4}{9}x^2 + \frac{62}{27}x^3 + \frac{124}{81}x^4 + \sum_{n=5}^{\infty} \frac{529}{81} \left(\frac{2}{3}\right)^{n-4} x^n
\end{aligned}
$$

Therefore the coefficient of x^n is

$$\begin{cases} 0 & \text{if } n = 0 \\ 2/3 & \text{if } n = 1 \\ 4/9 & \text{if } n = 2 \\ 62/27 & \text{if } n = 3 \\ 124/81 & \text{if } n = 4 \\ \dfrac{529}{81}\left(\dfrac{2}{3}\right)^{n-4} & \text{if } n \geq 5 \end{cases}$$

$\square$

912.3 Rational expressions and rational functions

rational-functions.tex

A **rational expression** is a fraction of polynomials:

$$\frac{P(x)}{Q(x)}$$

where $P(x)$ and $Q(x)$ are polynomials. For instance

$$\frac{x^{42}+1}{5x^3+2x^2+67x}$$

is a rational expression. A **rational function** is when you view the rational expression as a function. This is similar to polynomials. You can write down a polynomial such as

$$p(x) = x^2 + x + 1$$

without ever evaluating it at $x = 2.5$ or any other $x \in \mathbb{R}$. In that case, you are viewing $x^2 + x + 1$ as a polynomial expression. And if you compute $p(x)$ when $x \in \mathbb{R}$, then you are viewing $p(x)$ as a polynomial function. The distinction is not that important for us right now. So I will view polynomials as expressions or functions interchangeably. Likewise I will use rational expressions and rational functions interchangeably. Note that a rational function is not defined at zeroes of its denominator. For instance $\frac{1}{(x-1)(x-2)}$ as a rational function is not defined at $x = 1, 2$.

Here are some formulas that translate between certain power series and rational functions. For $x \in \mathbb{R}$ with $|x| < 1$, we have the following:

Geometric series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

Power of geometric series

$$\left(\frac{1}{1-x}\right)^k = \sum_{n=0}^{\infty} \binom{k+n-1}{n} x^n$$

Note the requirement that $|x| < 1$. This number "1" is called the **radius of convergence** of $1/(1-x)$. For instance for $x = 1/2$, since $|x| = |1/2| < 1$, i.e., I would say that $x = 1/2$ is "in the radius of convergence",

$$\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} (1/2)^n = 1 + 1/2 + 1/4 + 1/8 + \cdots$$

and if you have time and compute 1000 terms, $1+1/2+1/4+1/8+\cdots+1/2^{999}$ will be very close to $1/(1-(1/2)) = 2$. And if you compute $1 + 1/2 + 1/4 + 1/8 + \cdots + 1/2^{999999}$, you'll see that it will be even closer to 2. We say that $\sum_{n=0}^{\infty} (1/2)^n$ is **convergent** and it **converges** to 2. For instance the geometric series formula does not even make sense when $x = 1$ since

$$\frac{1}{1-x} = \frac{1}{1-1} = \frac{1}{0}$$

is not defined and

$$\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} 1^n = 1 + 1 + 1 + 1 + \cdots$$

blows up to infinity! And when x is beyond the radius of convergence, say $x = 10$, then

$$\frac{1}{1-x} = \frac{1}{1-10} = -\frac{1}{9}$$

is defined while

$$\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} 10^n = 1 + 10 + 100 + 1000 + \cdots$$

blows up to infinity.

(ASIDE. In general, for a power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$, there is the concept of the radius of convergence, R, of $f(x)$. Assuming R exists and is finite, if $|x| < R$, $f(x)$ converges to some real number and if $|x| > R$, $f(x)$ diverges. More details in Calculus.)

Note that (b) includes (a) (when $k = 1$). Note also that the x appears in the above as a variable. In particular on replacing x in (a) with $4x$ you can

$$\frac{1}{1-4x} = \sum_{n=0}^{\infty} (4x)^n = \sum_{n=0}^{\infty} 4^n x^n$$

Note that in this case the equality holds for $x \in \mathbb{R}$ such that

$$|4x| < 1$$

i.e., when

$$|x| < 1/4$$

So I would say that the radius of convergence of $\sum_{n=0}^{\infty} 4^n x^n$ is $1/4$.

You can also handle this:

$$\frac{1}{3-x}$$

All you need to do is to force the 3 to become 1 like this:

$$\frac{1}{3-x} = \frac{1}{3} \cdot \frac{1}{1-x/3}$$

In this case the radius of convergence is $1/3$, i.e., you should only substitute values of $x \in \mathbb{R}$ into the power series if $|x/3| < 1$, i.e., $|x| < 3$. And for $|x| < 3$, you get

$$\begin{aligned}
\frac{1}{3-x} &= \frac{1}{3} \cdot \frac{1}{1-x/3} \\
&= \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n \\
&= \sum_{n=0}^{\infty} \frac{1}{3} \left(\frac{1}{3}\right)^n x^n \\
&= \sum_{n=0}^{\infty} \frac{1}{3^{n+1}} x^n
\end{aligned}$$

And if you have a $+$ instead of a $-$:

$$\frac{1}{3+x}$$

you just force a $-$ to appear:

$$\frac{1}{3+x} = \frac{1}{3-(-x)}$$

and then proceed as above:

$$\begin{aligned}
\frac{1}{3+x} &= \frac{1}{3-(-x)} \\
&= \frac{1}{3} \cdot \frac{1}{1-(-x/3)}
\end{aligned}$$

At this point, to compute the radius of convergence, use $|-x/3| < 1$ to get $|x| < 3$. Continuing the above, if $|x| < 3$,

$$\begin{aligned}
\frac{1}{3+x} &= \frac{1}{3} \cdot \frac{1}{1-(-x/3)} \\
&= \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{-x}{3} \right)^n \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{1}{3^{n+1}} x^n
\end{aligned}$$

Whenever you're doing computations, it's always a good idea to check your work. In the case of series, it is easy to write simple programs to verify your computations or by doing some computations by hand. For instance the above computations claim that

$$\frac{1}{3+x} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{3^{n+1}} x^n \tag{$*$}$$

So let's choose a value for x and see if the right side of the identity equals the left. For this case, the radius of convergence is $|x| < 3$. Of course the easiest number to use is $x = 0$. When $x = 0$,

$$\begin{aligned}
\frac{1}{3+x} &= \frac{1}{3} \\
\sum_{n=0}^{\infty} (-1)^n \frac{1}{3^{n+1}} x^n &= (-1)^0 \frac{1}{3} = \frac{1}{3}
\end{aligned}$$

Therefore $(*)$ holds for $x = 0$. For other values you just need to approximate your series with sufficiently many terms. For instance this program computes the series up to $n = 1000$:

Student ques[t…]
you using 0^0[…]
that undefine[…]
Instructor: N[…]
You're not re[…]
this carefully[…]
this: $(\sum_{n=0}^{0}$ […]
$(x^0)(0) = (1)$[…]

```
def f(x):
    s = 0
    for n in range(1001):
        s += (-1)**n * 1.0/3.0**(n+1) * x**n
    return s
```

(Use your fav programming lang.) The value of `f(1.0)` is

```
0.75000000000000033
```

We then quickly check that the value of $1/(3+1.0) = 1/4 = 0.75$. Or you can check 10 cases of x for both expressions like this:

```
def f(x):
    s = 0
    for n in range(1001):
        s += (-1)**n * 1.0/3.0**(n+1) * x**n
        return s

def g(x):
    return 1.0 / (3 + x)

for x in range(0, 20, 2):
    x = x / 10.0
    print(f(x), g(x), abs(f(x) - g(x)))
```

Exercise 912.3.1. Rewrite

$$f(x) = \frac{1}{3 - 4x}$$

as a power series. What is the coefficient of x^n in the power series of $f(x)$?
(Go to solution, page **??**) $\square$

Exercise 912.3.2. Rewrite

$$f(x) = \frac{2}{3 + 4x}$$

as a power series. What is the coefficient of x^n in the power series of $f(x)$?
(Go to solution, page **??**) $\square$

Exercise 912.3.3. Rewrite

$$f(x) = 1 + 2x + \frac{2}{3 + 4x} + \frac{5}{7x - 1}$$

as a power series. What is the coefficient of x^n in the power series of $f(x)$?
(Go to solution, page **??**) $\square$

Exercise 912.3.4. Now that you've confidently worked through the above you can prove your own little theorems: Rewrite

$$f(x) = \frac{a}{b - cx}$$

as a power series where a, b, c are constants. What is the coefficient of x^n in the power series of $f(x)$? (Go to solution, page ??) $\square$

Going the other way around of course it's possible to rewrite certain power series as rational functions. For instance consider the function

$$f(x) = \sum_{n=0}^{\infty} \frac{2}{3^n} x^n$$

as a rational function. At this point we only know two formulas for translating power series into rational functions (see above). This doesn't look like any of the power series in the above formulas. However if we massage the power series a little we see that:

$$f(x) = 2 \sum_{n=0}^{\infty} \left(\frac{x}{3} \right)^n$$

we see that the power series is just the geometric series with x replaced by $x/3$. Therefore

$$\begin{aligned}
f(x) &= 2 \sum_{n=0}^{\infty} \left(\frac{x}{3} \right)^n \\
&= 2 \cdot \frac{1}{1 - x/3} \\
&= \frac{6}{3 - x}
\end{aligned}$$

Exercise 912.3.5. Rewrite

$$\sum_{n=0}^{\infty} \frac{1}{2^n} x^n$$

as a rational function. (Go to solution, page **??**) □

Exercise 912.3.6. Rewrite

$$f(x) = \sum_{n=0}^{\infty} \left(1 + \frac{2^n}{3^n}\right) x^n$$

as a rational function. (Go to solution, page ??) □

Exercise 912.3.7. Rewrite

$$\sum_{n=0}^{\infty} \left(\frac{2^n + (-1)^n}{3^n} \right) x^n$$

as a rational function. (Go to solution, page ??) $\square$

Exercise 912.3.8. Rewrite

$$\sum_{n=0}^{\infty}\left(\frac{2^n + (-1)^n \cdot 5}{7^{n+1}}\right)x^n$$

as a rational function. (Go to solution, page ??) $\square$

Exercise 912.3.9. Rewrite

$$\sum_{n=0}^{\infty} \left(\frac{2^{n-1}x + (-1)^n \cdot 5}{7^{n+1}} \right) x^n$$

as a rational function. (Go to solution, page ??) $\square$

Exercise 912.3.10. Rewrite

$$(1+5x^2)\sum_{n=2}^{\infty}\left(\frac{2^{n-1}x+(-1)^n\cdot 5}{7^{n+1}}\right)x^{n+1}$$

as a rational function. (Go to solution, page **??**) □

So far we've been replacing the x in the geometric series with a linear multiple of x. For instance on replacing x with $(-4/5) \cdot x$ we get

$$\sum_{n=0}^{\infty} \left(\frac{-4x}{5} \right)^n = \frac{1}{1 - 4x/5}$$

In fact there's nothing to prevent us from substituting x^2 for x into the geometric series to obtain

$$\sum_{n=0}^{\infty} (x^2)^n = \frac{1}{1 - x^2}$$

i.e.

$$\sum_{n=0}^{\infty} x^{2n} = \frac{1}{1 - x^2}$$

Note that in this case the coefficient of x^n is

$$\begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

Why? The power series

$$\sum_{n=0}^{\infty} x^{2n}$$

only has x^m where m is even, and when m is even, say $m = 2n$, the coefficient of x^m is 1. When m is odd, x^m does not appear in the power series, which is the same as saying that in this case x^m appears in the power series as $0x^m$.

And of course we can be really adventurous and do this:

$$\sum_{n=0}^{\infty} (5x^3)^n = \frac{1}{1 - 5x^3}$$

i.e.

$$\sum_{n=0}^{\infty} 5^n x^{3n} = \frac{1}{1 - 5x^3}$$

In this case the coefficient of x^n in the power series of $\dfrac{1}{1 - 5x^3}$ is

$$\begin{cases} 0 & \text{if } 3 \nmid n \\ 5^{n/3} & \text{if } 3 \mid n \end{cases}$$

The reasoning is the same as earlier. If you look for x^m in the power series, x^m appears only when $m = 3n$ and in this case the coefficient of x^m or x^{3n} is 5^n or $5^{m/3}$, i.e., if m is divisible by 3, then the coefficient for x^m is $5^{m/3}$. Otherwise, if m is not divisible by 3, coefficient of x^m is 0. You can verify this by writing down some terms of the power series:

$$\begin{aligned}
\frac{1}{1-5x^3} &= 1 + (5x^3) + (5x^3)^2 + (5x^3)^3 + (5x^3)^4 + (5x^3)^5 + \cdots \\
&= 1 + 5x^3 + 5^2x^6 + 5^3x^9 + 5^4x^{12} + 5^5x^{15} + \cdots \\
&= 1 + 5^{3/3}x^3 + 5^{6/3}x^6 + 5^{9/3}x^9 + 5^{12/3}x^{12} + 5^{15/3}x^{15} + \cdots
\end{aligned}$$

This is a *verification up to* x^{15} whereas the argument above is *for all* x^m.

By the way instead of

$$\sum_{n=0}^{\infty} 5^n x^{3n}$$

it's also OK to write

$$\sum_{\substack{n=0 \\ 3|n}}^{\infty} 5^{n/3} x^n$$

This summation means "run n through $0, 1, 2, ...$ and only include term $5^{n/3}x^n$ if $3 \mid n$". Sometimes it's also common to simply write

$$\sum_{3|n} 5^{n/3} x^n$$

to mean the same thing.

Exercise 912.3.11. Rewrite

$$\frac{1}{3+5x^4}$$

as a power series. What is the coefficient of x^n? (Go to solution, page ??)

$\square$

Exercise 912.3.12. Rewrite

$$\frac{2}{5x^4 - 7}$$

as a power series. What is the coefficient of x^n? (Go to solution, page **??**)

$\square$

Exercise 912.3.13. Prove your own theorem: Rewrite

$$\frac{a}{b - cx^d}$$

as a power series where a, b, c, d are integers with $b \neq 0$ and $d > 0$. What is the coefficient of x^n? What about

$$\frac{a}{b + cx^d}$$

Exercise 912.3.14. Rewrite

$$\sum_{n=0}^{\infty} x^n + 2 \cdot \sum_{n=0}^{\infty} x^{2n}$$

as a rational function.

Exercise 912.3.15. Rewrite

$$\sum_{n=0}^{\infty}\left(\frac{1+2^n x^n}{5^n}\right)x^n$$

as a rational function.

Exercise 912.3.16. What is the coefficient of x^n of the power series of the function

$$\frac{2x}{7 - 3x^5}$$

Exercise 912.3.17. What is the coefficient of x^n of the power series of the function

$$\frac{2x}{7x^3 - 11}$$

Exercise 912.3.18. What is the coefficient of x^n of the power series of the function

$$\frac{2x}{7x^3 + 11x}$$

All the above are variations on the geometric series:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

There is also a "finite" version of the geometric series and we might as well include the famous binomial theorem:

GEOMETRIC SUM

$$\sum_{n=0}^{N} x^n = \frac{1-x^{N+1}}{1-x}$$

BINOMIAL THEOREM

$$\sum_{n=0}^{N} \binom{N}{n} x^n = (1+x)^N$$

Proof of geometric sum formula. This can be proven using induction (probably in Math225). There are other proofs.

Proof of geometric series formula. For $|x| < 1$, on taking the limit as $N \to \infty$, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} x^n &= \lim_{N\to\infty} \sum_{n=0}^{N} x^n \\
&= \lim_{N\to\infty} \frac{1-x^N}{1-x} \\
&= \frac{1}{1-x} \lim_{N\to\infty} \left(1-x^N\right) \\
&= \frac{1}{1-x} \left(1 - \lim_{N\to\infty} x^N\right) \\
&= \frac{1}{1-x}
\end{aligned}$$

(This is probably proven in calculus.)

Proof of binomial theorem. This can be proven by mathematical induction and by using Pascal's triangle (probably in Math225).

Exercise 912.3.19.

What is the coefficient of x^n in the power series of

$$\frac{1 + x + x^2 + \cdots + x^{100}}{(1-x)^{200}}$$

(Go to solution, page **??**) □

Exercise 912.3.20. What is the coefficient of x^n in the power series of

$$\frac{x^2 + 2x^3 + \cdots + 2^{98}x^{100}}{(2+3x)^{200}}$$

Exercise 912.3.21. What is the coefficient of x^n in the power series of

$$\frac{\sum_{n=0}^{\infty} x^n}{\sum_{n=0}^{N} x^n}$$

Exercise 912.3.22. What is the coefficient of x^n in the power series of

$$\frac{\sum_{n=0}^{N} x^n}{\sum_{n=0}^{\infty} x^n}$$

Exercise 912.3.23. What is the coefficient of x^n in the power series of

$$\frac{1 - x^{100}}{1 - x}$$

Exercise 912.3.24. What is the coefficient of x^n in the power series of

$$\frac{1 - x^{100}}{1 - x^2}$$

Exercise 912.3.25. What is the coefficient of x^n in the power series of

$$\frac{1+x^{100}}{1-x^2}$$

Exercise 912.3.26. Prove the geometric sum formula by induction.

Exercise 912.3.27.

(a) Prove Pascal's triangle algebraically:

$$\binom{n}{r} = \binom{n-1}{n-r} + \binom{n-1}{r}$$

(b) Prove Pascal's triangle using a combinatorial argument.
(c) Prove the binomial theorem by induction.
(d) Prove that if $x, y \in \mathbb{R}$, then

$$(x+y)^N = \sum_{n=0}^{N} \binom{N}{n} x^n y^{N-n}$$

This is also called the **binomial theorem**. (To be proper, historically, this version is called the binomial theorem, but both versions of binomial are equivalent to each other. The reason this is called the "binomial" theorem is because this allows you to expand powers of $x+y$ and $x-y$, "polynomials in two variables".)
(e) Prove that

$$\sum_{n=0}^{N} \binom{N}{n} = 2^N$$

using the binomial theorem and then by using a combinatorial argument.
(f) Prove that

$$\sum_{n=0}^{N} \binom{N}{n} (-1)^n = 0$$

Solutions

Solutions

Solution to Exercise **??**.

power-series-1/answer.tex

As a power series,

$$\begin{aligned}
\frac{1}{3-4x} &= \frac{1}{3} \cdot \frac{1}{1-4x/3} \\
&= \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{4}{3}x\right)^n \\
&= \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{4}{3}\right)^n x^n \\
&= \sum_{n=0}^{\infty} \frac{1}{3} \left(\frac{4}{3}\right)^n x^n
\end{aligned}$$

The coefficient of x^n is $\frac{1}{3}\left(\frac{4}{3}\right)^n$. □

Solution to Exercise **??**.

power-series-2/answer.tex

As a power series,

$$\begin{aligned}
\frac{2}{3+4x} &= \frac{2}{3} \cdot \frac{1}{1-(-4x/3)} \\
&= \frac{2}{3} \sum_{n=0}^{\infty} \left(-\frac{4}{3} x \right)^n \\
&= \frac{2}{3} \sum_{n=0}^{\infty} (-1)^n \left(\frac{4}{3} \right)^n x^n \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{2}{3} \left(\frac{4}{3} \right)^n x^n
\end{aligned}$$

The coefficient of x^n is $(-1)^n \dfrac{2}{3} \left(\dfrac{4}{3} \right)^n$. $\square$

Solution to Exercise ??.

power-series-3/answer.tex

As a power series,

$$\begin{aligned}
&1 + 2x + \frac{2}{3+4x} + \frac{5}{7x-1} \\
&= 1 + 2x + \frac{2}{3} \cdot \frac{1}{1-(-4x/3)} - 5 \cdot \frac{1}{1-7x} \\
&= 1 + 2x + \frac{2}{3} \sum_{n=0}^{\infty} \left(\frac{-4x}{3} \right)^n - 5 \cdot \sum_{n=0}^{\infty} (7x)^n \\
&= 1 + 2x + \frac{2}{3} \sum_{n=0}^{\infty} (-1)^n \left(\frac{4}{3} \right)^n x^n - 5 \cdot \sum_{n=0}^{\infty} 7^n x^n \\
&= 1 + 2x + \sum_{n=0}^{\infty} (-1)^n \frac{2}{3} \left(\frac{4}{3} \right)^n x^n - \sum_{n=0}^{\infty} 5 \cdot 7^n x^n \\
&= 1 + 2x + \sum_{n=0}^{\infty} \left((-1)^n \frac{2}{3} \left(\frac{4}{3} \right)^n - 5 \cdot 7^n \right) x^n \\
&= 1 + 2x + \left((-1)^0 \frac{2}{3} \cdot \left(\frac{4}{3} \right)^0 - 5 \cdot 7^0 \right) + \left((-1)^1 \frac{2}{3} \cdot \left(\frac{4}{3} \right)^1 - 5 \cdot 7^1 \right) x \\
&\qquad + \sum_{n=2}^{\infty} \left((-1)^n \frac{2}{3} \left(\frac{4}{3} \right)^n - 5 \cdot 7^n \right) x^n \\
&= 1 + 2x - \frac{13}{3} - \frac{323}{9} x + \sum_{n=2}^{\infty} \left((-1)^n \frac{2}{3} \left(\frac{4}{3} \right)^n - 5 \cdot 7^n \right) x^n \\
&= -\frac{10}{3} - \frac{305}{9} x + \sum_{n=2}^{\infty} \left((-1)^n \frac{2}{3} \left(\frac{4}{3} \right)^n - 5 \cdot 7^n \right) x^n
\end{aligned}$$

The coefficient of x^n is

$$\begin{cases} -10/3 & \text{if } n = 0 \\ -305/9 & \text{if } n = 1 \\ (-1)^n \frac{2}{3} \left(\frac{4}{3} \right)^n - 5 \cdot 7^n & \text{if } n \geq 2 \end{cases}$$

Check: The following program

```
def f(x):
    return 1 + 2 * x + 2 / (3 + 4 * x) + 5 / (7 * x - 1)
```

```
def g(x):
    s = (-10/3) - (305/9) * x
    for n in range(2, 100):
        s += ((-1)**n * (2/3) * (4/3)**n - 5 * 7**n) * x**n
    return s

for x in range(10):
    x = x / 100
    print(f(x), g(x))
```

gives the following output:

```
-3.3333333333333335 -3.3333333333333335
-3.6984493491794006 -3.6984493491794006
-4.124602839021444 -4.124602839021443
-4.628088283024992 -4.628088283024993
-5.2315330520393815 -5.231533052039383
-5.967307692307694 -5.967307692307695
-6.883405704555129 -6.883405704555128
-8.054165471066476 -8.054165471066476
-9.601226725082148 -9.60122672508214
-11.73827541827542 -11.738275418275414
```

□

Solution to Exercise ??.

power-series-4/answer.tex

As a power series,

$$\begin{aligned} f(x) &= \frac{a}{b-cx} \\ &= \frac{a}{b} \cdot \frac{1}{1-cx/b} \\ &= \frac{a}{b} \sum_{n=0}^{\infty} \left(\frac{cx}{b}\right)^n \\ &= \frac{a}{b} \sum_{n=0}^{\infty} \left(\frac{c}{b}\right)^n x^n \\ &= \sum_{n=0}^{\infty} \frac{a}{b} \left(\frac{c}{b}\right)^n x^n \end{aligned}$$

The coefficient of x^n is

$$\frac{a}{b}\left(\frac{c}{b}\right)^n$$

for $n \geq 0$.

(Check: To test the above, you give a, b, c some values and then write a program that test the rational function against the power series for some values of x. For instance if you substitute $a = b = c = 1$, the above gives you

$$\frac{1}{1-1x} = \sum_{n=0}^{\infty} \frac{1}{1}\left(\frac{1}{1}\right)^n x^n = \sum_{n=0}^{\infty} x^n$$

which you already know is true. You can then try using $a = 2, b = 3, c = 4$ and test if

$$\frac{2}{3-4x} = \sum_{n=0}^{\infty} \frac{2}{3}\left(\frac{4}{3}\right)^n x^n$$

is true.) $\square$

Solution to Exercise ??.

power-series-5/answer.tex

As a rational function

$$\sum_{n=0}^{\infty} \frac{1}{2^n} x^n = \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$$
$$= \frac{1}{1 - x/2}$$
$$= \frac{2}{2 - x}$$

Solution to Exercise ??.

power-series-6/answer.tex

As a rational function

$$\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} \left(1 + \frac{2^n}{3^n}\right) x^n \\
&= \sum_{n=0}^{\infty} \left(x^n + \frac{2^n}{3^n} x^n\right) \\
&= \sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} \frac{2^n}{3^n} x^n \\
&= \frac{1}{1-x} + \sum_{n=0}^{\infty} \left(\frac{2}{3}x\right)^n \\
&= \frac{1}{1-x} + \frac{1}{1-2x/3} \\
&= \frac{1}{1-x} + \frac{3}{3-2x} \\
&= \frac{3-2x+3(1-x)}{(1-x)(3-2x)} \\
&= \frac{6-5x}{3-5x+2x^2}
\end{aligned}$$

Solution to Exercise ??.

power-series-7/answer.tex

As a rational function

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left(\frac{2^n+(-1)^n}{3^n}\right)x^n \\
&= \sum_{n=0}^{\infty}\left(\frac{2^n}{3^n}+\frac{(-1)^n}{3^n}\right)x^n \\
&= \sum_{n=0}^{\infty}\left(\frac{2^n}{3^n}x^n+\frac{(-1)^n}{3^n}x^n\right) \\
&= \sum_{n=0}^{\infty}\frac{2^n}{3^n}x^n+\sum_{n=0}^{\infty}\frac{(-1)^n}{3^n}x^n \\
&= \sum_{n=0}^{\infty}\left(\frac{2}{3}x\right)^n+\sum_{n=0}^{\infty}\left(\frac{-1}{3}x\right)^n \\
&= \frac{1}{1-2x/3}+\frac{1}{1-(-x/3)} \\
&= \frac{3}{3-2x}+\frac{3}{3+x} \\
&= \frac{3(3+x)+3(3-2x)}{(3-2x)(3+x)} \\
&= \frac{18-3x}{9-3x-2x^2}
\end{aligned}$$

Check: The following program

```
def f(x):
    s = 0
    for n in range(100):
        s += (2**n + (-1)**n) / 3**n * x**n
    return s

def g(x):
    return (18 - 3 * x) / (9 - 3 * x - 2 * x * x)


for x in range(10):
    x = x / 100
    print(f(x), g(x))
```

```
2.0 2.0
2.003389150259761 2.00338915025976
2.0068909969572224 2.0068909969572224
2.010507173166296 2.010507173166296
2.0142393655371302 2.0142393655371302
2.0180893159977398 2.0180893159977393
2.0220588235294117 2.0220588235294117
2.0261497460194526 2.026149746019453
2.0303640021949874 2.0303640021949882
2.0347035736418095 2.0347035736418095
```

Solution to Exercise **??**.

exercises/pow
series-12/ans

Solution not provided.

Solution to Exercise **??**.

exercises/pow
series-13/ans

Solution not provided.

Solution to Exercise **??**.

exercises/pow
series-14/ans

Solution not provided.

Solution to Exercise ??.

power-series-8/answer.tex

As a power series

$$\begin{aligned}
\frac{1}{3+5x^4} &= \frac{1}{3} \cdot \frac{1}{1+5x^4/3} \\
&= \frac{1}{3} \cdot \frac{1}{1-(-5x^4/3)} \\
&= \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{-5x^4}{3} \right)^n \\
&= \frac{1}{3} \sum_{n=0}^{\infty} (-1)^n \left(\frac{5}{3} \right)^n x^{4n} \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{1}{3} \left(\frac{5}{3} \right)^n x^{4n}
\end{aligned}$$

Now we consider the coefficient of x^m where $m \geq 0$ is an integer. The only x^m that appears in the power series is when $m = 4n$ for some integer n, i.e., when m is divisible by 4. If m is divisible by 4, say $m = 4n$, then coefficient of x^m is $(-1)^n \frac{1}{3} \left(\frac{5}{3} \right)^n$, i.e., since $n = m/4$, this coefficient is $(-1)^{m/4} \frac{1}{3} \left(\frac{5}{3} \right)^{m/4}$. If m is not divisible by 4, the coefficient of x^m is 0.

Altogether the coefficient of x^n is

$$\begin{cases} (-1)^{n/4} \frac{1}{3} \left(\frac{5}{3} \right)^{n/4} & \text{if } 4 \mid n \\ 0 & \text{otherwise} \end{cases}$$

Check: The radius of convergence is $|5/3^{1/4} x| < 1$, i.e., $|x| < (3/5)^{0.25} \approx 0.8801...$. I'll test it with $x = 0.0, 0.1, ..., 0.8$. The output of this program

```
def f(x):
    return 1.0 / (3 + 5 * x**4)

def g(x):
    s = 0
    for n in range(100):
        s += (-1)**n * (1.0/3) * (5.0/3)**n * x**(4*n)
    return s
```

```
for x in range(9):
    x = x / 10
    print(f(x), g(x))
```

is

```
0.3333333333333333 0.3333333333333333
0.33327778703549404 0.3332777870354941
0.3324468085106383 0.33244680851063824
0.328893274132544 0.328893274132544
0.319693094629156 0.319693094629156
0.3018867924528302 0.3018867924528301
0.2741228070175439 0.2741228070175438
0.23806689679800025 0.23806689679800025
0.19809825673534073 0.19809825673534082
```

□

Solution to Exercise ??.

power-series-9/answer.tex

As a power series

$$\begin{aligned}\frac{2}{5x^4-7} &= -\frac{2}{7-5x^4} \\ &= -\frac{2}{7}\cdot\frac{1}{1-5x^4/7} \\ &= -\frac{2}{7}\sum_{n=0}^{\infty}\left(\frac{5x^4}{7}\right)^n \\ &= -\frac{2}{7}\sum_{n=0}^{\infty}\left(\frac{5}{7}\right)^n x^{4n} \\ &= \sum_{n=0}^{\infty}\left(-\frac{2}{7}\right)\left(\frac{5}{7}\right)^n x^{4n}\end{aligned}$$

Now we consider the coefficient of x^m where $m \geq 0$ is an integer. The only x^m that appears in the power series is when $m = 4n$ for some integer n, i.e., when m is divisible by 4. If m is divisible by 4, say $m = 4n$, then coefficient of x^m is $\left(-\frac{2}{7}\right)\left(\frac{5}{7}\right)^n$, i.e., since $n = m/4$, this coefficient is $\left(-\frac{2}{7}\right)\left(\frac{5}{7}\right)^{m/4}$. If m is not divisible by 4, the coefficient of x^m is 0.

Altogether the coefficient of x^n is

$$\begin{cases}\left(-\frac{2}{7}\right)\left(\frac{5}{7}\right)^{n/4} & \text{if } 4 \mid n \\ 0 & \text{otherwise}\end{cases}$$

Check: The radius of convergence is $|(5x/7)^4| < 1$, i.e., $|x| < (7/5)^{1.4} \approx 1.0877....$ I'll test with $x = 0.0, ..., 1.0$. The output of this program

```
def f(x):
    return 2.0 / (5 * x**4 - 7)

def g(x):
    s = 0
    for n in range(100):
        s += (-2.0/7)  * (5.0/7)**n * x**(4*n)
    return s

for x in range(11):
    x = x / 10
```

```
    print(f(x), g(x))
```

is

```
-0.2857142857142857 -0.2857142857142857
-0.2857346953353811 -0.2857346953353811
-0.28604118993135014 -0.28604118993135014
-0.28737696673611607 -0.287376966736116
-0.2910360884749709 -0.29103608847497087
-0.29906542056074764 -0.2990654205607476
-0.3148614609571788 -0.3148614609571788
-0.3448573152858005 -0.3448573152858003
-0.4038772213247173 -0.4038772213247173
-0.5377066810055114 -0.5377066810055112
-1.0 -0.9999999999999974
```

□

Solution to Exercise ??.

power-series-10/answer.tex

You can do this:

$$
\begin{aligned}
&\frac{1+x+x^2+\cdots+x^{100}}{(1-x)^{200}} \\
&= \frac{1}{(1-x)^{200}} + x\frac{1}{(1-x)^{200}} + x^2\frac{1}{(1-x)^{200}} + \cdots + x^{100}\frac{1}{(1-x)^{200}} \\
&= \sum_{n=0}^{\infty} \binom{200+n-1}{n} x^n + x\sum_{n=0}^{\infty} \binom{200+n-1}{n} x^n + x^2\sum_{n=0}^{\infty} \binom{200+n-1}{n} x^n + \cdots \\
&\qquad + x^{100}\sum_{n=0}^{\infty} \binom{200+n-1}{n} x^n
\end{aligned}
$$

Stop this madness! There are 101 terms! This is a lot of work! The following is better ...

$$
\begin{aligned}
\frac{1+x+x^2+\cdots+x^{100}}{(1-x)^{200}} &= (1+x+x^2+\cdots+x^{100})\frac{1}{(1-x)^{200}} \\
&= \frac{1-x^{101}}{1-x}\frac{1}{(1-x)^{200}} \\
&= \frac{1-x^{101}}{(1-x)^{201}}
\end{aligned}
$$

Checkmate! There are now fewer terms. In fact the numerator has only two terms instead of 101!!! Continuing,

$$
\begin{aligned}
\frac{1+x+x^2+\cdots+x^{100}}{(1-x)^{200}} &= \frac{1-x^{101}}{(1-x)^{201}} \\
&= (1-x^{101})\frac{1}{(1-x)^{201}} \\
&= (1-x^{101})\sum_{n=0}^{\infty}\binom{201+n-1}{n}x^n \\
&= (1-x^{101})\sum_{n=0}^{\infty}\binom{n+200}{n}x^n \\
&= (1-x^{101})\sum_{n=0}^{\infty}\binom{n+200}{200}x^n \\
&= \sum_{n=0}^{\infty}\binom{n+200}{n}x^n - x^{101}\sum_{n=0}^{\infty}\binom{n+200}{200}x^n \\
&= \sum_{n=0}^{\infty}\binom{n+200}{200}x^n - \sum_{n=0}^{\infty}\binom{n+200}{200}x^{n+101} \\
&= \sum_{n=0}^{\infty}\binom{n+200}{200}x^n - \sum_{p=101}^{\infty}\binom{p+99}{p-101}x^p \quad \text{(let } p = n+101\text{)} \\
&= \sum_{n=0}^{\infty}\binom{n+200}{200}x^n - \sum_{n=101}^{\infty}\binom{n+99}{n-101}x^n \\
&= \sum_{n=0}^{100}\binom{n+200}{200}x^n + \sum_{n=101}^{\infty}\binom{n+200}{200}x^n - \sum_{n=101}^{\infty}\binom{n+99}{n-101}x^n \\
&= \sum_{n=0}^{100}\binom{n+200}{200}x^n + \sum_{n=101}^{\infty}\left(\binom{n+200}{200} - \binom{n+99}{n-101}\right)x^n
\end{aligned}
$$

Hence the coefficient of x^n is

$$
\begin{cases}
\binom{n+200}{200} & \text{if } 0 \le n \le 100 \\
\binom{n+200}{200} - \binom{n+99}{n-101} & \text{if } n \ge 101
\end{cases}
$$

$\square$

Solutions

Solution to Exercise **??**.

power-series-1/answer.tex

As a power series,

$$\begin{aligned}
\frac{1}{3-4x} &= \frac{1}{3} \cdot \frac{1}{1-4x/3} \\
&= \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{4}{3} x \right)^n \\
&= \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{4}{3} \right)^n x^n \\
&= \sum_{n=0}^{\infty} \frac{1}{3} \left(\frac{4}{3} \right)^n x^n
\end{aligned}$$

The coefficient of x^n is $\dfrac{1}{3} \left(\dfrac{4}{3} \right)^n$. □

Solution to Exercise **??**.

power-series-2/answer.tex

As a power series,

$$\begin{aligned}\frac{2}{3+4x} &= \frac{2}{3}\cdot\frac{1}{1-(-4x/3)}\\ &= \frac{2}{3}\sum_{n=0}^{\infty}\left(-\frac{4}{3}x\right)^n\\ &= \frac{2}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{4}{3}\right)^n x^n\\ &= \sum_{n=0}^{\infty}(-1)^n\frac{2}{3}\left(\frac{4}{3}\right)^n x^n\end{aligned}$$

The coefficient of x^n is $(-1)^n\frac{2}{3}\left(\frac{4}{3}\right)^n$. $\square$

Solution to Exercise ??.

power-series-3/answer.tex

As a power series,

$$\begin{aligned}
&1 + 2x + \frac{2}{3+4x} + \frac{5}{7x-1} \\
&= 1 + 2x + \frac{2}{3} \cdot \frac{1}{1-(-4x/3)} - 5 \cdot \frac{1}{1-7x} \\
&= 1 + 2x + \frac{2}{3} \sum_{n=0}^{\infty} \left(\frac{-4x}{3} \right)^n - 5 \cdot \sum_{n=0}^{\infty} (7x)^n \\
&= 1 + 2x + \frac{2}{3} \sum_{n=0}^{\infty} (-1)^n \left(\frac{4}{3} \right)^n x^n - 5 \cdot \sum_{n=0}^{\infty} 7^n x^n \\
&= 1 + 2x + \sum_{n=0}^{\infty} (-1)^n \frac{2}{3} \left(\frac{4}{3} \right)^n x^n - \sum_{n=0}^{\infty} 5 \cdot 7^n x^n \\
&= 1 + 2x + \sum_{n=0}^{\infty} \left((-1)^n \frac{2}{3} \left(\frac{4}{3} \right)^n - 5 \cdot 7^n \right) x^n \\
&= 1 + 2x + \left((-1)^0 \frac{2}{3} \cdot \left(\frac{4}{3} \right)^0 - 5 \cdot 7^0 \right) + \left((-1)^1 \frac{2}{3} \cdot \left(\frac{4}{3} \right)^1 - 5 \cdot 7^1 \right) x \\
&\qquad + \sum_{n=2}^{\infty} \left((-1)^n \frac{2}{3} \left(\frac{4}{3} \right)^n - 5 \cdot 7^n \right) x^n \\
&= 1 + 2x - \frac{13}{3} - \frac{323}{9} x + \sum_{n=2}^{\infty} \left((-1)^n \frac{2}{3} \left(\frac{4}{3} \right)^n - 5 \cdot 7^n \right) x^n \\
&= -\frac{10}{3} - \frac{305}{9} x + \sum_{n=2}^{\infty} \left((-1)^n \frac{2}{3} \left(\frac{4}{3} \right)^n - 5 \cdot 7^n \right) x^n
\end{aligned}$$

The coefficient of x^n is

$$\begin{cases} -10/3 & \text{if } n = 0 \\ -305/9 & \text{if } n = 1 \\ (-1)^n \frac{2}{3} \left(\frac{4}{3} \right)^n - 5 \cdot 7^n & \text{if } n \geq 2 \end{cases}$$

Check: The following program

```
def f(x):
    return 1 + 2 * x + 2 / (3 + 4 * x) + 5 / (7 * x - 1)
```

```
def g(x):
    s = (-10/3) - (305/9) * x
    for n in range(2, 100):
        s += ((-1)**n * (2/3) * (4/3)**n - 5 * 7**n) * x**n
    return s

for x in range(10):
    x = x / 100
    print(f(x), g(x))
```

gives the following output:

```
-3.3333333333333335 -3.3333333333333335
-3.6984493491794006 -3.6984493491794006
-4.124602839021444 -4.124602839021443
-4.628088283024992 -4.628088283024993
-5.2315330520393815 -5.231533052039383
-5.967307692307694 -5.967307692307695
-6.883405704555129 -6.883405704555128
-8.054165471066476 -8.054165471066476
-9.601226725082148 -9.60122672508214
-11.73827541827542 -11.738275418275414
```

□

Solution to Exercise ??.

power-series-4/answer.tex

As a power series,

$$\begin{aligned}
f(x) &= \frac{a}{b - cx} \\
&= \frac{a}{b} \cdot \frac{1}{1 - cx/b} \\
&= \frac{a}{b} \sum_{n=0}^{\infty} \left(\frac{cx}{b}\right)^n \\
&= \frac{a}{b} \sum_{n=0}^{\infty} \left(\frac{c}{b}\right)^n x^n \\
&= \sum_{n=0}^{\infty} \frac{a}{b} \left(\frac{c}{b}\right)^n x^n
\end{aligned}$$

The coefficient of x^n is

$$\frac{a}{b} \left(\frac{c}{b}\right)^n$$

for $n \geq 0$.

(Check: To test the above, you give a, b, c some values and then write a program that test the rational function against the power series for some values of x. For instance if you substitute $a = b = c = 1$, the above gives you

$$\frac{1}{1 - 1x} = \sum_{n=0}^{\infty} \frac{1}{1} \left(\frac{1}{1}\right)^n x^n = \sum_{n=0}^{\infty} x^n$$

which you already know is true. You can then try using $a = 2, b = 3, c = 4$ and test if

$$\frac{2}{3 - 4x} = \sum_{n=0}^{\infty} \frac{2}{3} \left(\frac{4}{3}\right)^n x^n$$

is true.) □

Solution to Exercise **??**.

power-series-5/answer.tex

As a rational function

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{1}{2^n} x^n &= \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n \\ &= \frac{1}{1 - x/2} \\ &= \frac{2}{2 - x}\end{aligned}$$

Solution to Exercise ??.

power-series-6/answer.tex

As a rational function

$$
\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} \left(1 + \frac{2^n}{3^n}\right) x^n \\
&= \sum_{n=0}^{\infty} \left(x^n + \frac{2^n}{3^n} x^n\right) \\
&= \sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} \frac{2^n}{3^n} x^n \\
&= \frac{1}{1-x} + \sum_{n=0}^{\infty} \left(\frac{2}{3} x\right)^n \\
&= \frac{1}{1-x} + \frac{1}{1-2x/3} \\
&= \frac{1}{1-x} + \frac{3}{3-2x} \\
&= \frac{3-2x+3(1-x)}{(1-x)(3-2x)} \\
&= \frac{6-5x}{3-5x+2x^2}
\end{aligned}
$$

Solution to Exercise ??.

power-series-7/answer.tex

As a rational function

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\left(\frac{2^n+(-1)^n}{3^n}\right)x^n \\
&= \sum_{n=0}^{\infty}\left(\frac{2^n}{3^n}+\frac{(-1)^n}{3^n}\right)x^n \\
&= \sum_{n=0}^{\infty}\left(\frac{2^n}{3^n}x^n+\frac{(-1)^n}{3^n}x^n\right) \\
&= \sum_{n=0}^{\infty}\frac{2^n}{3^n}x^n+\sum_{n=0}^{\infty}\frac{(-1)^n}{3^n}x^n \\
&= \sum_{n=0}^{\infty}\left(\frac{2}{3}x\right)^n+\sum_{n=0}^{\infty}\left(\frac{-1}{3}x\right)^n \\
&= \frac{1}{1-2x/3}+\frac{1}{1-(-x/3)} \\
&= \frac{3}{3-2x}+\frac{3}{3+x} \\
&= \frac{3(3+x)+3(3-2x)}{(3-2x)(3+x)} \\
&= \frac{18-3x}{9-3x-2x^2}
\end{aligned}
$$

Check: The following program

```
def f(x):
    s = 0
    for n in range(100):
        s += (2**n + (-1)**n) / 3**n * x**n
    return s

def g(x):
    return (18 - 3 * x) / (9 - 3 * x - 2 * x * x)


for x in range(10):
    x = x / 100
    print(f(x), g(x))
```

```
2.0 2.0
2.003389150259761 2.00338915025976
2.0068909969572224 2.0068909969572224
2.010507173166296 2.010507173166296
2.0142393655371302 2.0142393655371302
2.0180893159977398 2.0180893159977393
2.0220588235294117 2.0220588235294117
2.0261497460194526 2.026149746019453
2.0303640021949874 2.0303640021949882
2.0347035736418095 2.0347035736418095
```

Solution to Exercise **??**.

exercises/pow
series-12/ansv

Solution not provided.

Solution to Exercise ??.

exercises/power-series-13/ansv

Solution not provided.

Solution to Exercise ??.

exercises/pow
series-14/ans

Solution not provided.

Solution to Exercise ??.

power-series-8/answer.tex

As a power series

$$\begin{aligned}
\frac{1}{3+5x^4} &= \frac{1}{3} \cdot \frac{1}{1+5x^4/3} \\
&= \frac{1}{3} \cdot \frac{1}{1-(-5x^4/3)} \\
&= \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{-5x^4}{3} \right)^n \\
&= \frac{1}{3} \sum_{n=0}^{\infty} (-1)^n \left(\frac{5}{3} \right)^n x^{4n} \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{1}{3} \left(\frac{5}{3} \right)^n x^{4n}
\end{aligned}$$

Now we consider the coefficient of x^m where $m \geq 0$ is an integer. The only x^m that appears in the power series is when $m = 4n$ for some integer n, i.e., when m is divisible by 4. If m is divisible by 4, say $m = 4n$, then coefficient of x^m is $(-1)^n \frac{1}{3} \left(\frac{5}{3} \right)^n$, i.e., since $n = m/4$, this coefficient is $(-1)^{m/4} \frac{1}{3} \left(\frac{5}{3} \right)^{m/4}$. If m is not divisible by 4, the coefficient of x^m is 0.

Altogether the coefficient of x^n is

$$\begin{cases} (-1)^{n/4} \frac{1}{3} \left(\frac{5}{3} \right)^{n/4} & \text{if } 4 \mid n \\ 0 & \text{otherwise} \end{cases}$$

Check: The radius of convergence is $|5/3^{1/4} x| < 1$, i.e., $|x| < (3/5)^{0.25} \approx 0.8801...$. I'll test it with $x = 0.0, 0.1, ..., 0.8$. The output of this program

```
def f(x):
    return 1.0 / (3 + 5 * x**4)

def g(x):
    s = 0
    for n in range(100):
        s += (-1)**n * (1.0/3) * (5.0/3)**n * x**(4*n)
    return s
```

```
for x in range(9):
    x = x / 10
    print(f(x), g(x))
```

is

```
0.3333333333333333 0.3333333333333333
0.33327778703549404 0.3332777870354941
0.3324468085106383 0.33244680851063824
0.328893274132544 0.328893274132544
0.319693094629156 0.319693094629156
0.3018867924528302 0.3018867924528301
0.2741228070175439 0.2741228070175438
0.23806689679800025 0.23806689679800025
0.19809825673534073 0.19809825673534082
```

□

Solution to Exercise ??.

power-series-9/answer.tex

As a power series

$$\begin{aligned}\frac{2}{5x^4-7} &= -\frac{2}{7-5x^4}\\ &= -\frac{2}{7}\cdot\frac{1}{1-5x^4/7}\\ &= -\frac{2}{7}\sum_{n=0}^{\infty}\left(\frac{5x^4}{7}\right)^n\\ &= -\frac{2}{7}\sum_{n=0}^{\infty}\left(\frac{5}{7}\right)^n x^{4n}\\ &= \sum_{n=0}^{\infty}\left(-\frac{2}{7}\right)\left(\frac{5}{7}\right)^n x^{4n}\end{aligned}$$

Now we consider the coefficient of x^m where $m \geq 0$ is an integer. The only x^m that appears in the power series is when $m = 4n$ for some integer n, i.e., when m is divisible by 4. If m is divisible by 4, say $m = 4n$, then coefficient of x^m is $\left(-\frac{2}{7}\right)\left(\frac{5}{7}\right)^n$, i.e., since $n = m/4$, this coefficient is $\left(-\frac{2}{7}\right)\left(\frac{5}{7}\right)^{m/4}$. If m is not divisible by 4, the coefficient of x^m is 0.

Altogether the coefficient of x^n is

$$\begin{cases}\left(-\frac{2}{7}\right)\left(\frac{5}{7}\right)^{n/4} & \text{if } 4 \mid n\\ 0 & \text{otherwise}\end{cases}$$

Check: The radius of convergence is $|(5x/7)^4| < 1$, i.e., $|x| < (7/5)^{1.4} \approx 1.0877....$ I'll test with $x = 0.0, ..., 1.0$. The output of this program

```
def f(x):
    return 2.0 / (5 * x**4 - 7)

def g(x):
    s = 0
    for n in range(100):
        s += (-2.0/7)  * (5.0/7)**n * x**(4*n)
    return s

for x in range(11):
    x = x / 10
```

```
    print(f(x), g(x))
```

is

```
-0.2857142857142857 -0.2857142857142857
-0.2857346953353811 -0.2857346953353811
-0.28604118993135014 -0.28604118993135014
-0.28737696673611607 -0.287376966736116
-0.2910360884749709 -0.29103608847497087
-0.29906542056074764 -0.2990654205607476
-0.3148614609571788 -0.3148614609571788
-0.3448573152858005 -0.3448573152858003
-0.4038772213247173 -0.4038772213247173
-0.5377066810055114 -0.5377066810055112
-1.0 -0.9999999999999974
```

□

Solution to Exercise ??.

power-series-10/answer.tex

You can do this:

$$\begin{aligned}
&\frac{1 + x + x^2 + \cdots + x^{100}}{(1-x)^{200}} \\
&= \frac{1}{(1-x)^{200}} + x\frac{1}{(1-x)^{200}} + x^2\frac{1}{(1-x)^{200}} + \cdots + x^{100}\frac{1}{(1-x)^{200}} \\
&= \sum_{n=0}^{\infty} \binom{200+n-1}{n} x^n + x\sum_{n=0}^{\infty} \binom{200+n-1}{n} x^n + x^2\sum_{n=0}^{\infty} \binom{200+n-1}{n} x^n + \cdots \\
&\qquad + x^{100}\sum_{n=0}^{\infty} \binom{200+n-1}{n} x^n
\end{aligned}$$

Stop this madness! There are 101 terms! This is a lot of work! The following is better ...

$$\begin{aligned}
\frac{1 + x + x^2 + \cdots + x^{100}}{(1-x)^{200}} &= (1 + x + x^2 + \cdots + x^{100})\frac{1}{(1-x)^{200}} \\
&= \frac{1 - x^{101}}{1-x}\frac{1}{(1-x)^{200}} \\
&= \frac{1 - x^{101}}{(1-x)^{201}}
\end{aligned}$$

Checkmate! There are now fewer terms. In fact the numerator has only two terms instead of 101!!! Continuing,

$$
\begin{aligned}
\frac{1+x+x^2+\cdots+x^{100}}{(1-x)^{200}} &= \frac{1-x^{101}}{(1-x)^{201}} \\
&= (1-x^{101})\frac{1}{(1-x)^{201}} \\
&= (1-x^{101})\sum_{n=0}^{\infty}\binom{201+n-1}{n}x^n \\
&= (1-x^{101})\sum_{n=0}^{\infty}\binom{n+200}{n}x^n \\
&= (1-x^{101})\sum_{n=0}^{\infty}\binom{n+200}{200}x^n \\
&= \sum_{n=0}^{\infty}\binom{n+200}{n}x^n - x^{101}\sum_{n=0}^{\infty}\binom{n+200}{200}x^n \\
&= \sum_{n=0}^{\infty}\binom{n+200}{200}x^n - \sum_{n=0}^{\infty}\binom{n+200}{200}x^{n+101} \\
&= \sum_{n=0}^{\infty}\binom{n+200}{200}x^n - \sum_{p=101}^{\infty}\binom{p+99}{p-101}x^p \quad \text{(let } p = n+101\text{)} \\
&= \sum_{n=0}^{\infty}\binom{n+200}{200}x^n - \sum_{n=101}^{\infty}\binom{n+99}{n-101}x^n \\
&= \sum_{n=0}^{100}\binom{n+200}{200}x^n + \sum_{n=101}^{\infty}\binom{n+200}{200}x^n - \sum_{n=101}^{\infty}\binom{n+99}{n-101}x^n \\
&= \sum_{n=0}^{100}\binom{n+200}{200}x^n + \sum_{n=101}^{\infty}\left(\binom{n+200}{200} - \binom{n+99}{n-101}\right)x^n
\end{aligned}
$$

Hence the coefficient of x^n is

$$
\begin{cases}
\binom{n+200}{200} & \text{if } 0 \leq n \leq 100 \\
\binom{n+200}{200} - \binom{n+99}{n-101} & \text{if } n \geq 101
\end{cases}
$$

□

912.4 Homogeneous linear recurrence relations

linear-recurrence-relations-homogeneous-case.tex

The examples in the previous section seems to indicate that much of the computation doesn't really depend on the specific recurrence relation. So with some experience with recurrence relations, we now move on to generalizing some of the results.

Note that I'm just showing you that many of the results in the previous section can be generalized. I'm not implying that you should memorize the results. In other words the point is to show you that the method of generating functions is powerful and can be applied to many cases. When solving recurrences, I expect *you* to *derive* a rational function for the generating function and then rewrite it as a power series and collect the general coefficient.

OK. Now let's take a look at the general linear homogeneous recurrence relations on sequences. Suppose a_n is a linear homogeneous recurrence relation of degree d:

$$a_n = c_1 a_{n-1} + \cdots + c_d a_{n-d}$$

for $n \geq d$. Of course we proceed as before: Let

$$a(x) = \sum_{n=0}^{\infty} a_n x^n$$

Hence

$$
\begin{aligned}
a(x) &= a_0 + a_1 x + \cdots + a_{d-1}x^{d-1} + \sum_{n=d}^{\infty} a_n x^n \\
&= a_0 + a_1 x + \cdots + a_{d-1}x^{d-1} + \sum_{n=d}^{\infty} \big(c_1 a_{n-1} + \cdots + c_d a_{n-d}\big) x^n \\
&= a_0 + a_1 x + \cdots + a_{d-1}x^{d-1} + \sum_{n=d}^{\infty} c_1 a_{n-1} x^n + \cdots + \sum_{n=d}^{\infty} c_d a_{n-d} x^n \\
&= a_0 + a_1 x + \cdots + a_{d-1}x^{d-1} + c_1 x \sum_{n=d}^{\infty} a_{n-1} x^{n-1} + \cdots + c_d x^d \sum_{n=d}^{\infty} a_{n-d} x^{n-d} \\
&= a_0 + a_1 x + \cdots + a_{d-1}x^{d-1} + c_1 x \sum_{n=d-1}^{\infty} a_n x^n + \cdots + c_d x^d \sum_{n=0}^{\infty} a_n x^n \\
&= a_0 + a_1 x + \cdots + a_{d-1}x^{d-1} \\
&\quad + c_1 x \left(-a_0 - a_1 x - \cdots - a_{d-2}x^{d-2} + a_0 + a_1 x + \cdots a_{d-2}x^{d-2} + \sum_{n=d-1}^{\infty} a_n x^n \right) \\
&\quad + c_2 x^2 \left(-a_0 - a_1 x - \cdots - a_{d-3}x^{d-3} + a_0 + a_1 x + \cdots a_{d-2}x^{d-3} + \sum_{n=d-2}^{\infty} a_n x^n \right) \\
&\quad + \cdots \\
&\quad + c_d x^d a(x) \\
&= a_0 + a_1 x + \cdots + a_{d-1}x^{d-1} \\
&\quad + c_1 x \big(-a_0 - a_1 x - \cdots - a_{d-2}x^{d-2} \big) + c_1 x a(x) \\
&\quad + c_2 x^2 \big(-a_0 - a_1 x - \cdots - a_{d-3}x^{d-3} \big) + c_2 x^2 a(x) \\
&\quad + \cdots \\
&\quad + c_d x^d a(x) \\
&= a_0 + (a_1 - c_1 a_0)x + (a_2 - c_1 a_1 - c_2 a_0)x^2 + (a_3 - c_1 a_2 - c_2 a_1 - c_3 a_0)x^3 \\
&\quad + \cdots + (a_{d-1} - c_1 a_{d-2} - c_2 a_{d-3} - \cdots - c_d a_0)x^{d-1} \\
&\quad + (c_1 x + c_2 x^2 + \cdots + c_d x^d) a(x)
\end{aligned}
$$

Hence

$$
a(x) = \frac{a_0 + (a_1 - c_1 a_0)x + \cdots + (a_{d-1} - c_1 a_{d-2} - c_2 a_{d-3} - \cdots - c_d a_0)x^{d-1}}{1 - (c_1 x + c_2 x^2 + \cdots + c_d x^d)}
$$

Not too bad. Tedious ... but not theoretically deep. You just have to be careful.

The case of $d = 1$ (degree 1 recurrence relation) is

$$a(x) = \frac{a_0}{1 - c_1 x}$$

Of course I can now immediate solve the general case for $d = 1$:

$$\begin{aligned} a(x) &= \frac{a_0}{1 - c_1 x} \\ &= \sum_{n=0}^{\infty} a_0 c_1^n x^n \end{aligned}$$

i.e. ...

Theorem 912.4.1. *If*

$$a_n = c_1 a_{n-1} \textit{ for } n \geq 1$$

then

$$a_n = a_0 c_1^n \textit{ for } n \geq 0$$

Exercise 912.4.1. Let

$$a_n = \begin{cases} 42 & \text{if } n = 0 \\ 3a_{n-1} & \text{if } n > 0 \end{cases}$$

Find a closed form for a_n using the above theorem. Next, derive the closed form without using the theorem.

That was easy ... and we've solve every single linear recurrence relation of degree 1! Note further that solving the general homogeneous case of degree 1 is not really any more difficult than solving a specific homogeneous case of degree 1 case.

Onward! ... on to $d = 2$. If

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}$$

for $n \geq 2$, then

$$a(x) = \frac{a_0 + (a_1 - c_1 a_0)x}{1 - (c_1 x + c_2 x^2)}$$

Of course in order to move forward, we have to factorize

$$1 - (c_1 x + c_2 x^2)$$

in order to apply the theory of partial fractions. (What else can you do anyway?)

$$1 - (c_1 x + c_2 x^2) = 1 - c_1 x - c_2 x^2 = -c_2 \left(x^2 + \frac{c_1}{c_2} x - \frac{1}{c_2} \right)$$

The roots of

$$x^2 + \frac{c_1}{c_2} x - \frac{1}{c_2}$$

are

$$x = \frac{-\frac{c_1}{c_2} \pm \sqrt{\left(\frac{c_1}{c_2}\right)^2 + 4\frac{1}{c_2}}}{2} = \frac{-c_1 \pm \sqrt{c_1^2 + 4c_2}}{2c_2}$$

Note that It's possible to have complex roots (i.e. $c_1^2 + 4c_2 < 0$) and it's possible to have repeated roots (i.e. $c_1^2 + 4c_2 = 0$).

Let's consider the case $c_1^2 + 4c_2 = 0$ first. Of course in this case the roots are both

$$-\frac{c_1}{2c_2}$$

$$\begin{aligned} 1 - (c_1 x + c_2 x^2) &= -c_2 \left(x^2 + \frac{c_1}{c_2} x - \frac{1}{c_2} \right) \\ &= -c_2 \left(x - \frac{c_1}{2c_2} \right)^2 \end{aligned}$$

Therefore our generating function becomes

$$\begin{aligned}
a(x) &= \frac{a_0 + (a_1 - c_1 a_0)x}{1 - (c_1 x + c_2 x^2)} \\
&= \frac{a_0 + (a_1 - c_1 a_0)x}{-c_2\left(x - \frac{c_1}{2c_2}\right)^2} \\
&= \frac{a_0 + (a_1 - c_1 a_0)x}{-c_2} \cdot \frac{1}{\left(\frac{c_1}{2c_2} - x\right)^2}
\end{aligned}$$

At this point it's already clear that we can rewrite the rational function very easily into a power series. (The computation is tedious but not deep.) We just basically need to rewrite a rational function of the form

$$(A + Bx) \cdot \frac{1}{(C - x)^2}$$

as a power series and then substitute the following:

$$A = -\frac{a_0}{c_2}, B = -\frac{a_1 - c_1 a_0}{c_2}, C = \frac{c_1}{2c_2}$$

We have

$$\begin{aligned}
(A + Bx) \cdot \frac{1}{(C - x)^2} &= \frac{A + Bx}{C^2} \sum_{n=0}^{\infty} (n+1) \frac{1}{C^n} x^n \\
&= \frac{A}{C^2} \sum_{n=0}^{\infty} (n+1) \frac{1}{C^n} x^n + \frac{B}{C^2} \sum_{n=0}^{\infty} (n+1) \frac{1}{C^n} x^{n+1} \\
&= \frac{A}{C^2} \sum_{n=0}^{\infty} (n+1) \frac{1}{C^n} x^n + \frac{B}{C^2} \sum_{n=1}^{\infty} n \frac{1}{C^{n-1}} x^n
\end{aligned}$$

The coefficient of x^0 is

$$\frac{A}{C^2}$$

and the coefficient of x^n for $n \geq 1$ is

$$\frac{A}{C^2}(n+1)\frac{1}{C^n} + \frac{B}{C^2} n \frac{1}{C^{n-1}} = \frac{1}{C^{n+1}} \left(\frac{(n+1)A}{C} + nB \right)$$

Now we replace A, B, C with the constants from our original problem:

$$\frac{A}{C^2} = \frac{-\frac{a_0}{c_2}}{(\frac{c_1}{2c_2})^2} = -\frac{a_0}{c_2} \cdot \frac{4c_2^2}{c_1^2}$$

But wait a minute: isn't a_0 the coefficient of x^0 for $a(x) = \sum_{n=0} a_n x^n$?!? Not to worry: Remember that we're in the case of repeated roots and the condition guaranteeing that is $c_1^2 + 4c_2 = 0$. This means that $c_1^2 = -4c_2$. Therefore

$$\frac{A}{C^2} = -\frac{a_0}{c_2} \cdot \frac{4c_2^2}{c_1^2} = -\frac{a_0}{c_2} \cdot \frac{4c_2^2}{-4c_2} = a_0$$

Phew! Check that the coefficient for x^1 does simplify to a_1.

Finally compute and simplify the closed form for a_n for $n \geq 1$ and complete the statement of the homogeneous degree-2 recurrence relation when the roots of the denominator of the rational function for $a(x)$ has repeated roots:

Theorem 912.4.2. *Let a_n $(n = 0, 1, 2, \ldots)$ be a sequence satisyfing the following degree 2 recurrence relation:*

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}, \quad n \geq 2$$

Furthermore assume that

$$c_1^2 + 4c_2 = 0$$

Then a_n $(n = 0, 1, 2, \ldots)$ has the following closed form: XXX

Now let's tackle the case where $c_1^2 + 4c_2 \neq 0$, i.e. the denominator of the rational function for $a(x)$ has two distinct roots. Recall that following facts: Our recurrence relation is this:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}, \quad (n \geq 2)$$

and the rational function for $a(x) = \sum_{n=0}^{\infty} a_n x^n$ is

$$\frac{a_0 + (a_1 - c_1 a_0)x}{1 - (c_1 x + c_2 x^2)}$$

The denominator of the rational function is

$$1 - (c_1 x + c_2 x^2) = -c_2 \left(x^2 + \frac{c_1}{c_2} x - \frac{1}{c_2} \right)$$

The roots of

$$x^2 + \frac{c_1}{c_2}x - \frac{1}{c_2}$$

are

$$x = \frac{-c_1 \pm \sqrt{c_1^2 + 4c_2}}{2c_2}$$

Of course putting everything together we have this:

$$\begin{aligned} a(x) &= \frac{a_0 + (a_1 - c_1 a_0)x}{-c_2} \cdot \frac{1}{x^2 + \frac{c_1}{c_2}x - \frac{1}{c_2}} \\ &= (A + Bx)\frac{1}{(x - C)(x - D)} \end{aligned}$$

where A and B are as before for the case of repeated roots and for C and D are distinct roots of the denominator of the rational function of $a(x)$:

$$C = \frac{-c_1 - \sqrt{c_1^2 + 4c_2}}{2c_2}, D = \frac{-c_1 - \sqrt{c_1^2 + 4c_2}}{2c_2}$$

Note that it's possible to have complex roots. However you don't have to worry about it since the algebra will still work, i.e. just ignore the fact that C and D can be complex and simply perform the computation. However when C and D are complex there are certain steps that you can take to speed up the computations of powers. I'll handle the degree 2 complex roots case later.

Now go ahead and rewrite

$$(A + Bx)\frac{1}{(x - C)(x - D)}$$

as a power series and finish this case of your theorem:

Theorem 912.4.3. *Let a_n $(n = 0, 1, 2, \ldots)$ be a sequence satisyfing the following degree 2 recurrence relation:*

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}, \quad n \geq 2$$

Furthermore assume that

$$c_1^2 + 4c_2 \neq 0$$

Then a_n $(n = 0, 1, 2, \ldots)$ has the following closed form:

Solutions

912.5 Linear homogeneous recurrence relations of degree 2: complex case

power-series-recurrence-relation-homogeneous-degree-2-complex.tex

Recall that following facts derived for the homogeneous recurrence relation of degree 2 : Our recurrence relation is this:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}, \quad (n \geq 2)$$

and the rational function for $a(x) = \sum_{n=0}^{\infty} a_n x^n$ is

$$\frac{a_0 + (a_1 - c_1 a_0)x}{1 - (c_1 x + c_2 x^2)}$$

The denominator of the rational function is

$$1 - (c_1 x + c_2 x^2) = -c_2 \left(x^2 + \frac{c_1}{c_2} x - \frac{1}{c_2} \right)$$

The roots of

$$x^2 + \frac{c_1}{c_2} x - \frac{1}{c_2}$$

are

$$x = \frac{-c_1 \pm \sqrt{c_1^2 + 4c_2}}{2c_2}$$

Therefore $a(x)$ looks like this:

$$\begin{aligned} a(x) &= \frac{a_0 + (a_1 - c_1 a_0)x}{-c_2} \cdot \frac{1}{x^2 + \frac{c_1}{c_2} x - \frac{1}{c_2}} \\ &= (A + Bx) \frac{1}{(x - C)(x - D)} \end{aligned}$$

where

$$A = -\frac{a_0}{c_2}, \quad B = -\frac{a_1 - c_1 a_0}{c_2}$$

and for C and D are distinct roots of the denominator of the rational function of $a(x)$:

$$C = \frac{-c_1 - \sqrt{c_1^2 + 4c_2}}{2c_2}, D = \frac{-c_1 - \sqrt{c_1^2 + 4c_2}}{2c_2}$$

Note that it's possible to have complex roots.

Let's pause here to try an example where the roots are complex. Consider the case where

$$a_n = \begin{cases} 1 & \text{if } n = 0, 1 \\ -a_{n-2} & \text{if } n \geq 2 \end{cases}$$

In this case

$$a(x) = \frac{1+x}{1+x^2} = \frac{1+x}{(x-i)(x+i)}$$

Now

$$\frac{1}{(x-i)(x+i)} = \frac{1}{2i}\left(\frac{1}{x-i} - \frac{1}{x+i}\right)$$

(make sure you derive this) and hence

$$\begin{aligned}
a(x) &= \frac{1+x}{1+x^2} = \frac{1+x}{(x-i)(x+i)} = (1+x)\cdot\frac{1}{2i}\left(\frac{1}{x-i} - \frac{1}{x+i}\right) \\
&= (1+x)\cdot\frac{1}{2i}\left(-\frac{1}{i-x} - \frac{1}{i-(-x)}\right) \\
&= (1+x)\cdot\frac{1}{2i}\left(-\frac{1}{i}\frac{1}{1-x/i} - \frac{1}{i}\frac{1}{1-(-x)/i}\right) \\
&= (1+x)\cdot\frac{1}{2i}\cdot\frac{-1}{i}\left(\sum_{n=0}^{\infty}\left(\frac{x}{i}\right)^n + \sum_{n=0}^{\infty}\left(\frac{-x}{i}\right)^n\right) \\
&= (1+x)\cdot\frac{1}{2}\sum_{n=0}^{\infty}\left(\left(\frac{1}{i}\right)^n + \left(\frac{-1}{i}\right)^n\right)x^n \\
&= (1+x)\cdot\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1+(-1)^n}{i^n}\right)x^n \\
&= \frac{1}{2}\left(\sum_{n=0}^{\infty}\frac{1+(-1)^n}{i^n}x^n + \sum_{n=0}^{\infty}\frac{1+(-1)^n}{i^n}x^{n+1}\right) \\
&= \frac{1}{2}\left(\sum_{n=0}^{\infty}\frac{1+(-1)^n}{i^n}x^n + \sum_{n=1}^{\infty}\frac{1+(-1)^{n-1}}{i^{n-1}}x^n\right) \\
&= \frac{1}{2}\left(\frac{1+(-1)^0}{i^0} + \sum_{n=1}^{\infty}\frac{1+(-1)^n}{i^n}x^n + \sum_{n=1}^{\infty}\frac{1+(-1)^{n-1}}{i^{n-1}}x^n\right) \\
&= 1 + \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1+(-1)^n}{i^n} + \frac{1+(-1)^{n-1}}{i^{n-1}}\right)x^n
\end{aligned}$$

Hence

$$a_n = \begin{cases} 1 & \text{if } n = 0 \\ \frac{1}{2}\left(\frac{1+(-1)^n}{i^n} + \frac{1+(-1)^{n-1}}{i^{n-1}}\right) & \text{if } n > 0 \end{cases}$$

Note that $i^0 = 1, i^1 = i, i^2 = -1, i^3 = -i$. In general

$$i^{4m} = 1, \quad i^{4m+1} = i, \quad i^{4m+2} = -1, \quad i^{4m+3} = -i$$

Therefore for a_n with $n \neq 0$,

$$\begin{aligned} a_{4m} &= \frac{1}{2}\left(\frac{1+1}{1} + \frac{1+(-1)}{-i}\right) = \frac{1}{2}(2+0) = 1 \\ a_{4m+1} &= \frac{1}{2}\left(\frac{1+(-1)}{i} + \frac{1+1}{1}\right) = \frac{1}{2}(0+2) = 1 \\ a_{4m+2} &= \frac{1}{2}\left(\frac{1+1}{-1} + \frac{1+(-1)}{i}\right) = \frac{1}{2}(0-2) = -1 \\ a_{4m+3} &= \frac{1}{2}\left(\frac{1+(-1)}{-i} + \frac{1+1}{-1}\right) = \frac{1}{2}(0-2) = -1 \end{aligned}$$

In summary we have

$$a_n = \begin{cases} 1 & \text{if } n = 4m \\ 1 & \text{if } n = 4m+1 \\ -1 & \text{if } n = 4m+2 \\ -1 & \text{if } n = 4m+3 \end{cases}$$

But ... wait a minute ...

In fact we already know how to compute the coefficients of x^n in the power series of $a(x)$ without going with the use of complex numbers!!! We don't even need to factorize $x^2 + 1$ since

$$\begin{aligned} a(x) = \frac{1+x}{1+x^2} &= (1+x)\frac{1}{1-(-x^2)} \\ &= (1+x)\sum_{n=0}^{\infty}(-x^2)^n \\ &= (1+x)\sum_{n=0}^{\infty}(-1)^n x^{2n} \\ &= \sum_{n=0}^{\infty}(-1)^n x^{2n} + \sum_{n=0}^{\infty}(-1)^n x^{2n+1} \end{aligned}$$

Now we go through several cases to compute the coefficient of x^k. (It's possible but messier to combine the two power algebraically.) For the case when $k = 4m$, the first power series, with

$$2n = 4m$$

implies

$$n = 2m$$

hence the coefficient is

$$(-1)^{2m} = 1$$

The second power series only have off x-powers and does not have an x^{4m} term. Altogether the coefficient of x^{4m} of $a(x)$ is 1.

Now let's consider the case $k = 4m + 1$. In this case the power is odd. The first power series only has even x–powers. Therefore the first power series does not contribute a coefficient for x^k. For the second power series, with

$$4m + 1 = 2n + 1$$

we have

$$n = 2m$$

Hence the coefficient is

$$(-1)^n = (-1)^{2m} = 1$$

Altogether the coefficient of x^{4m+1} in $a(x)$ is 1.

Now for x^k with $k = 4m + 2$. For the first power series

$$2n = 4m + 2$$

implies that

$$n = 2m + 1$$

Hence the coefficient of x^k from the first power series is

$$(-1)^{2m+1} = -1$$

The second power series only contains odd x–powers therefore does not contribute to the coefficient of x^k. Hence the coefficient of x^{4m+2} in $a(x)$ is -1.

Lastly (phew!) we compute the coefficient of x^k for $k = 4m + 3$. The first power series does not have odd x–powers therefore cannot contribute toward x^{4m+3}. As for the second power series,

$$2n + 1 = 4m + 3$$

implies

$$n = 2m + 1$$

Therefore the coefficient in this case is

$$(-1)^n = (-1)^{3m+1} = -1$$

Altogether we have shown that the coefficient of x^n is $a(x)$ is

$$a_n = \begin{cases} 1 & \text{if } n = 4m \\ 1 & \text{if } n = 4m + 1 \\ -1 & \text{if } n = 4m + 2 \\ -1 & \text{if } n = 4m + 3 \end{cases}$$

Note that in both cases we get the same result. The first method factorize $1/(x^2 + 1)$ into linear factors containing complex numbers

$$\frac{1}{x^2 + 1} = \frac{1}{(x - i)(x + i)}$$

and breaks it up using theory of partial fractions. The second method uses

$$\frac{1}{1 + x^2} = \frac{1}{1 - (-x^2)}$$

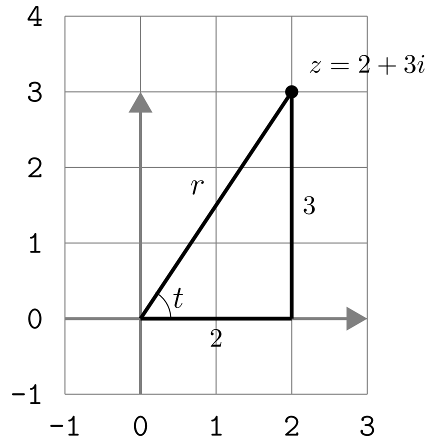
and uses the geometric series.

The above example looks more generally involves a quadratic that looks like this
$$\frac{1}{ax^2 + b}$$
You can always use
$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$
to rewrite $1/(ax^2 + b)$ as a power series. In the case where we need to work with a rational function that involves
$$\frac{1}{ax^2 + bx + c}$$
where b is not zero and the roots are not real.

Note that in the case of using complex numbers we have to compute powers of a complex number. In the above example we're spared lots of pain because powers of i is easy. The general case can be pretty bad. For instance try z^2, z^3, z^4 where
$$z = \frac{1}{4} + \frac{3}{4}i$$
(Yes I mean it ... do the powers.)

Well there's another way to compute powers for complex numbers. That's de Moivres theorem. Recall that you can rewrite any complex number in polar coordinates:
$$z = r(\cos t + i \sin t)$$
Now de Moivre's theorem tells us that
$$z^n = \left(r(\cos t + i \sin t) \right)^n = r^n \big(\cos(nt) + i \sin(nt)\big)$$

How do you convert a complex number into polar coordinates form? Let's consider an example. Say $z = 2 + 3i$. We need to find r and t such that
$$2 + 3i = r(\cos t + i \sin t)$$
The r and t can be described pictorially as follows. You draw z in the complex plane:



I've added the distance of z from 0, i.e. r and the angle the line from 0 to z makes with the real axis, i.e. t. Again, r is the distance from z to 0 and t is the angle made by the line $\overline{0z}$ with the positive real axis. The distance of z from 0 is written

$$|z|$$

OK, that great. But we want to avoid drawing pictures because we might not be absolutely accurate in measuring r and t. By Pythagorus theorem

$$r = \sqrt{2^2 + 3^2} = \sqrt{13}$$

The angle t is just

$$\tan t = \frac{3}{2}$$

i.e.

$$t = \tan^{-1} \frac{3}{2}$$

In this case t is approximately 0.9828 in radians (or 56.31 degrees). Hence

$$z = \sqrt{13}(\cos 0.9828 + i \sin 0.9828)$$

Hence

$$z^n = \sqrt{13}^n (\cos 0.9828n + i \sin 0.9828n)$$

Note that taking powers is now taking powers of a real number (i.e. the $\sqrt{13}$) and computing nt. Of course you still need to compute sine and cosine of the angle nt.

Of course the problem is that we need to approximate angles and sines and cosines for the above example. There are however standard angles that you should be aware of. This includes $t = 0, \pi/6, \pi/4, \pi/3, \pi/2$ (corresponding to

$0, 30, 45, 60, 90$ degrees). In these cases we can be exact.

Here's an example where you can compute the angle exactly. Let $z = 1 + \sqrt{3}i$. In this case

$$r = \sqrt{1+3} = 2$$

and

$$\tan t = \frac{\sqrt{3}}{1} = \sqrt{3}$$

i.e.

$$t = \frac{\pi}{3}$$

(i.e. 60 degrees). Hence

$$z^n = 2^n \left(\cos \frac{n\pi}{3} + i \sin \frac{n\pi}{3} \right)$$

For instance

$$z^{1000} = 2^{1000} \left(\cos \frac{1000\pi}{3} + i \sin \frac{1000\pi}{3} \right)$$

Now note that

$$\frac{1000}{3}\pi = 333\pi + \frac{1}{3}\pi$$

Of course you know that the sine and cosine functions have period of 2π:

$$\frac{1000}{3}\pi = 166(2\pi) + \frac{4}{3}\pi$$

Therefore

$$\begin{aligned} \cos \frac{1000}{3}\pi &= \cos(4\pi/3) = -1/2 \\ \sin \frac{1000}{3}\pi &= \sin(4\pi/3) = -\sqrt{3}/2 \end{aligned}$$

Therefore

$$z^{1000} = 2^{1000} \left(\frac{-1}{2} + i \frac{-\sqrt{3}}{2} \right)$$

Let's try an example. Derive a closed form for a_n where

$$a_n = \begin{cases} 1 & \text{if } n = 0 \\ 2 & \text{if } n = 1 \\ -a_{n-1} - a_{n-2} & \text{if } n > 1 \end{cases}$$

I'll let you derive the rational function for $a(x) = \sum_{n=0}^{\infty} a_n x^n$. It's

$$a(x) = \frac{1+3x}{1+x+x^2}$$

The roots of $1 + x + x^2$ are (using the quadratic equation formula for roots)

$$\frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$$

Let

$$z_1 = \frac{-1+i\sqrt{3}}{2}, \quad z_2 = \frac{-1-i\sqrt{3}}{2}$$

Going back to our rational function for $a(x)$ we obtain

$$\begin{aligned} a(x) &= \frac{1+3x}{1+x+x^2} \\ &= \frac{1+3x}{(x-z_1)(x-z_2)} \end{aligned}$$

Of course we now need to break up the rational function into two pieces using partial fractions. Let

$$\frac{1}{(x-z_1)(x-z_2)} = \frac{A}{x-z_1} + \frac{B}{x-z_2}$$

therefore

$$1 = A(x-z_2) + B(x-z_1)$$

Let $x = z_1$ and we get

$$\begin{aligned} 1 &= A(z_1 - z_2) \\ \therefore A &= \frac{1}{z_1 - z_2} \end{aligned}$$

Note that

$$z_1 - z_2 = \frac{-1+i\sqrt{3}}{2} - \frac{-1-i\sqrt{3}}{2} = \sqrt{3}i$$

Therefore

$$A = \frac{1}{\sqrt{3}i} \cdot \frac{-\sqrt{3}i}{-\sqrt{3}i} = \frac{-\sqrt{3}}{3}i$$

Now for B ... if we let $x = z_2$ in

$$1 = A(x - z_2) + B(x - z_1)$$

we get

$$1 = B(z_2 - z_1)$$

and therefore

$$B = \frac{1}{z_2 - z_1}$$

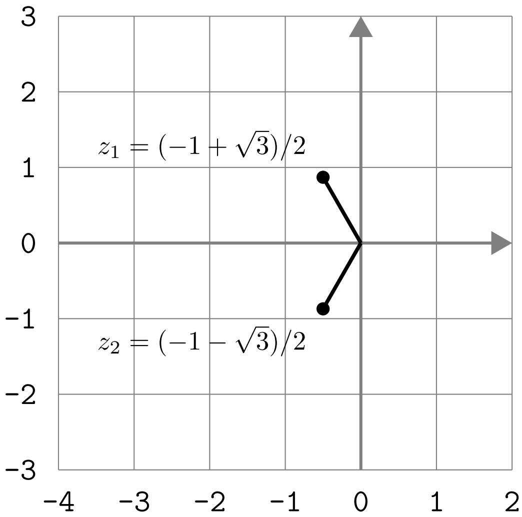
If you're sharp, you'd see that $B = -A$ and therefore save you time in compute B from scratch. Therefore

$$B = \frac{\sqrt{3}}{3}i$$

Hence

$$\begin{aligned}
\frac{1}{(x-z_1)(x-z_2)} &= \frac{A}{x-z_1} + \frac{B}{x-z_2} \\
&= \frac{-i\sqrt{3}}{3}\frac{1}{x-z_1} + \frac{i\sqrt{3}}{3}\frac{1}{x-z_2} \\
&= \frac{i\sqrt{3}}{3}\frac{1}{z_1-x} - \frac{i\sqrt{3}}{3}\frac{1}{z_2-x} \\
&= \frac{i\sqrt{3}}{3}\frac{1}{z_1}\frac{1}{1-x/z_1} - \frac{i\sqrt{3}}{3}\frac{1}{z_2}\frac{1}{1-x/z_2} \\
&= \frac{i\sqrt{3}}{3}\frac{1}{z_1}\sum_{n=0}^{\infty}\left(\frac{x}{z_1}\right)^n - \frac{i\sqrt{3}}{3}\frac{1}{z_2}\sum_{n=0}^{\infty}\left(\frac{x}{z_2}\right)^2 \\
&= \frac{i\sqrt{3}}{3}\sum_{n=0}^{\infty}\left(\frac{1}{z_1}\left(\frac{1}{z_1}\right)^n - \frac{1}{z_2}\left(\frac{1}{z_2}\right)^n\right)x^n \\
&= \frac{i\sqrt{3}}{3}\sum_{n=0}^{\infty}\left((z_1^{-1})^{n+1} - (z_2^{-1})^{n+1}\right)x^n
\end{aligned}$$

Uh-oh ... now we need to compute powers of z_1 and z_2. We can use binomial theorem ... but we're going to use de Moivre's theorem instead. If I draw z_1 and z_2 on the complex plane I get

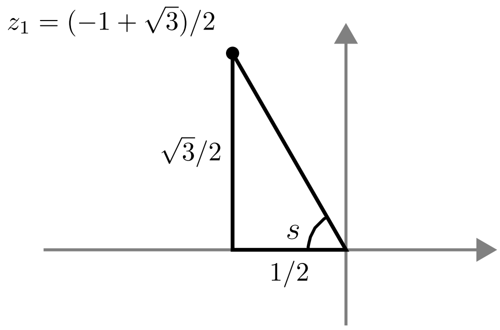


Let's focus on z_1. The distance of z_1 from 0, let's call it r_1 is

$$|z_1| = r_1 = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{-1}{2}\right)^2} = 1$$

Exercise 912.5.1. Show that $|z_2| = 1$. □

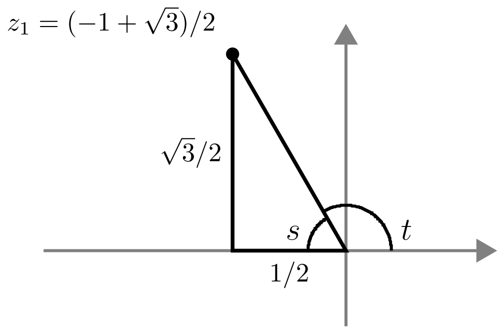
Now for the angle.



Angle s satisfies

$$\tan s = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$$

This is one of the standard angles: $s = \pi/3$ (i.e. 60 degrees). However remember that the angle we want is the angle that z_1 makes with the positive real axis:



We have

$$t = \pi - s = \frac{2\pi}{3}$$

Therefore z_1 in polar coordinates is $1(\cos(2\pi/3) + i\sin(2\pi/3))$, i.e.

$$z_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)$$

I'll leave it to you to show

$$z_2 = \cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)$$

Let's collect all the facts together just to know where we are. First we have the rational function of our generating function:

$$a(x) = \frac{1+3x}{1+x+x^2}$$

The denominator is rewritten as

$$\frac{1}{1+x+x^2} = \frac{i\sqrt{3}}{3}\sum_{n=0}^{\infty}\left((z_1^{-1})^{n+1} - (z_2^{-1})^{n+1}\right)x^n$$

where

$$z_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)$$

and

$$z_2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i = \cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)$$

Now

$$\begin{aligned} z_1^{-1} &= \frac{1}{\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)} \cdot \frac{\cos\left(\frac{2\pi}{3}\right) - i\sin\left(\frac{2\pi}{3}\right)}{\cos\left(\frac{2\pi}{3}\right) - i\sin\left(\frac{2\pi}{3}\right)} \\ &= \frac{\cos\left(\frac{2\pi}{3}\right) - i\sin\left(\frac{2\pi}{3}\right)}{1} \\ &= z_2 \end{aligned}$$

and therefore $z_2^{-1} = z_1$. Hence

$$\begin{aligned} \frac{1}{1+x+x^2} &= \frac{i\sqrt{3}}{3} \sum_{n=0}^{\infty} \left((z_1^{-1})^{n+1} - (z_2^{-1})^{n+1} \right) x^n \\ &= \frac{i\sqrt{3}}{3} \sum_{n=0}^{\infty} \left(z_2^{n+1} - z_1^{n+1} \right) x^n \end{aligned}$$

By de Moivre's theorem,

$$z_1^{n+1} = \cos\left(\frac{2(n+1)\pi}{3}\right) + i\sin\left(\frac{2(n+1)\pi}{3}\right)$$

and

$$\begin{aligned} z_2^{n+1} &= \cos\left(-\frac{2(n+1)\pi}{3}\right) + i\sin\left(-\frac{2(n+1)\pi}{3}\right) \\ &= \cos\left(\frac{2(n+1)\pi}{3}\right) - i\sin\left(\frac{2(n+1)\pi}{3}\right) \end{aligned}$$

Together we have

$$\begin{aligned} z_2^{n+1} - z_1^{n+1} &= \cos\left(\frac{2(n+1)\pi}{3}\right) - i\sin\left(\frac{2(n+1)\pi}{3}\right) \\ &\quad - \cos\left(\frac{2(n+1)\pi}{3}\right) - i\sin\left(\frac{2(n+1)\pi}{3}\right) \\ &= -2i\sin\frac{2(n+1)\pi}{3} \end{aligned}$$

Hence our rational $\dfrac{1}{1+x+x^2}$ becomes

$$\begin{aligned}
\frac{1}{1+x+x^2} &= \frac{i\sqrt{3}}{3}\sum_{n=0}^{\infty}\left(-2i\sin\frac{2(n+1)\pi}{3}\right)x^n \\
&= \frac{2\sqrt{3}}{3}\sum_{n=0}^{\infty}\left(\sin\frac{2(n+1)\pi}{3}\right)x^n
\end{aligned}$$

I want to consider cases for $n = 3m, 3m+1, 3m+2$ (... why? to kill the denominator 3 of course!) Let $n = 3m+k$. Then

$$\begin{aligned}
\sin\frac{2(3m+k+1)\pi}{3} &= \sin\left(2m\pi + \frac{2(k+1)}{3}\pi\right) \\
&= \sin\left(\frac{2(k+1)}{3}\pi\right) \\
&= \begin{cases} \sin\left(\frac{2}{3}\pi\right) & \text{if } k=0 \\ \sin\left(\frac{4}{3}\pi\right) & \text{if } k=1 \\ \sin\left(\frac{6}{3}\pi\right) & \text{if } k=2 \end{cases} \\
&= \begin{cases} \sin\left(\frac{1}{3}\pi\right) & \text{if } k=0 \\ -\sin\left(\frac{1}{3}\pi\right) & \text{if } k=1 \\ 0 & \text{if } k=2 \end{cases} \\
&= \begin{cases} \frac{\sqrt{3}}{2} & \text{if } k=0 \\ -\frac{\sqrt{3}}{2} & \text{if } k=1 \\ 0 & \text{if } k=2 \end{cases}
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{2\sqrt{3}}{3}\sin\frac{2(n+1)\pi}{3} &= \begin{cases} \frac{2\sqrt{3}}{3}\cdot\frac{\sqrt{3}}{2} & \text{if } n=3m \\ -\frac{2\sqrt{3}}{3}\cdot\frac{\sqrt{3}}{2} & \text{if } n=3m+1 \\ 0 & \text{if } n=3m+2 \end{cases} \\
&= \begin{cases} 1 & \text{if } n=3m \\ -1 & \text{if } n=3m+1 \\ 0 & \text{if } n=3m+2 \end{cases}
\end{aligned}$$

Recall that

$$\frac{1}{1+x+x^2} = \sum_{n=0}^{\infty} \frac{2\sqrt{3}}{3} \left(\sin \frac{2(n+1)\pi}{3} \right) x^n$$

Instead of trying to force the series to contain all three cases, I'll write it as 3 separate series:

$$\begin{aligned}
\frac{1}{1+x+x^2} &= \sum_{m=0}^{\infty} x^{3m} + \sum_{m=0}^{\infty} (-1) x^{3m+1} \sum_{m=0}^{\infty} 0 x^{3m+2} \\
&= \sum_{m=0}^{\infty} x^{3m} + \sum_{m=0}^{\infty} (-1) x^{3m+1}
\end{aligned}$$

At this point it's a good idea to check the computations numerically with a program:

```
>>> def f(x): return 1.0/(1+x+x*x)
...
>>> def g(x):
...     s = 0.0
...     for n in range(1000000):
...         import math
...         if n % 3 == 0: s += x**n
...         elif n % 3 == 1: s += -x**n
...     return s
...
>>> f(0.5)
0.5714285714285714
>>> g(0.5)
0.5714285714285714
>>> f(0.25)
0.76190476190476186
>>> g(0.25)
0.76190476190476186
>>>
```

Good! Onward!

And now (finally),

$$
\begin{aligned}
a(x) &= (1+3x) \cdot \frac{1}{1+x+x^2} \\
&= (1+3x)\left(\sum_{m=0}^{\infty} x^{3m} + \sum_{m=0}^{\infty} (-1)x^{3m+1}\right) \\
&= \sum_{m=0}^{\infty} x^{3m} + \sum_{m=0}^{\infty} (-1)x^{3m+1} \\
&\qquad + 3x\left(\sum_{m=0}^{\infty} x^{3m} + \sum_{m=0}^{\infty} (-1)x^{3m+1}\right) \\
&= \sum_{m=0}^{\infty} x^{3m} + \sum_{m=0}^{\infty} (-1)x^{3m+1} \\
&\qquad + \sum_{m=0}^{\infty} 3x^{3m+1} + \sum_{m=0}^{\infty} (-3)x^{3m+2} \\
&= \sum_{m=0}^{\infty} x^{3m} + \sum_{m=0}^{\infty} 2x^{3m+1} \sum_{m=0}^{\infty} (-3)x^{3m+2}
\end{aligned}
$$

i.e.

$$
a_n = \begin{cases} 1 & \text{if } n = 3m \\ 2 & \text{if } n = 3m+1 \\ -3 & \text{if } n = 3m+2 \end{cases}
$$

Whoa!!! Can it be that simple?!? Let's check it with the definition of our a_n via recurrences

$$
a_n = \begin{cases} 1 & \text{if } n = 0 \\ 2 & \text{if } n = 1 \\ -a_{n-1} - a_{n-2} & \text{if } n > 1 \end{cases}
$$

Well ... *is* true for a_0, a_1. What about the rest?

$$
\begin{aligned}
a_2 &= -a_1 - a_0 = -2 - 1 = -3 \\
a_3 &= -a_2 - a_1 = -(-3) - 1 = 2 \\
a_4 &= -a_3 - a_2 = -2 + 3 = 1 \\
a_5 &= -a_4 - a_3 = -1 - 2 = -3 \\
a_6 &= -a_5 - a_4 = -(-3) - 1 = 2 \\
a_7 &= -a_6 - a_5 = -2 - (-3) = 1 \\
a_8 &= -a_7 - a_6 = -1 - 2 = -3 \\
a_9 &= -a_8 - a_7 = -(-3) - 1 = -2
\end{aligned}
$$

and now note that the last two lines of checks begin to repeat earlier ones.

Neat right?

Exercise 912.5.2.

Solutions

912.6 Linear nonhomogeneous recurrence relations recurrence-relations-linear-non-homogeneous-case.tex

The general linear non-homogeneous case looks like this:

$$a_n = c_1 a_{n-1} + \cdots c_d a_{n-d} + f(n)$$

where $f(n)$ is a nonzero function in n. Here's one example:

$$a_n = 2a_{n-1} + n$$

and here's another:

$$a_n = a_{n-1} + 3a_{n-2} + 2n^3 + 1$$

and yet another:

$$a_n = a_{n-1} + 3a_{n-2} + \lfloor n \log n \rfloor$$

Let's consider the a degree 2 nonhomogeneous recurrence relation:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + f(n), \quad n \geq 2$$

Of course using our method of generating functions, in order to compute a closed form for a_n, we do this: We let $a(x) = \sum_{n=0}^{\infty} a_n x^n$. From

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + f(n), \quad n \geq 2$$

we get

$$\begin{aligned}
a(x) &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_0 + a_1 x + \sum_{n=0}^{\infty} a_n x^n \\
&= a_0 + a_1 x + \sum_{n=2}^{\infty} (c_1 a_{n-1} + c_2 a_{n-2} + f(n)) x^n \\
&= a_0 + a_1 x + \sum_{n=2}^{\infty} (c_1 a_{n-1} + c_2 a_{n-2}) x^n + \sum_{n=2}^{\infty} f(n) x^n
\end{aligned}$$

Without going into details, we know that had the recurrence been this:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}, \quad n \geq 2$$

we would have a rational function:

$$a(x) = \frac{P(x)}{Q(x)}$$

With the nonhomogeneous factor $f(n)$ in the recurrence relation we get this:

$$a(x) = \frac{P(x) + \sum_{n=2}^{\infty} f(n)x^n}{Q(x)}$$

Therefore as long as we can express

$$\sum_{n=2}^{\infty} f(n)x^n$$

as a rational function then there is hope of getting the coefficient of x^n of the power series of $a(x)$.

Let's try an example:

$$a_n = 2a_{n-1} + n$$

for $n \geq 1$. Of course we let $a(x) = \sum_{n=0}^{\infty} a_n x^n$. Then

$$\begin{aligned}
a(x) &= \sum_{n=0} a_n x^n \\
&= a_0 + \sum_{n=1}^{\infty} a_n x^n \\
&= a_0 + \sum_{n=1}^{\infty} (2a_{n-1} + n)x^n \\
&= a_0 + 2\sum_{n=1}^{\infty} a_{n-1}x^n + \sum_{n=1}^{\infty} nx^n \\
&= a_0 + 2x\sum_{n=1}^{\infty} a_{n-1}x^{n-1} + \sum_{n=0}^{\infty} nx^n
\end{aligned}$$

Notice that I've secretly actually added $0x^0$. Continuing the computation we have

$$\begin{aligned} a(x) &= a_0 + 2x \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} n x^n \\ &= a_0 + 2x a(x) + \sum_{n=0}^{\infty} n x^n \\ \therefore \quad (1-2x)a(x) &= a_0 + \sum_{n=0}^{\infty} n x^n \\ \therefore \quad a(x) &= \frac{a_0 + \sum_{n=0}^{\infty} n x^n}{1-2x} \end{aligned}$$

And by the way ... we can rewrite $\sum_{n=0}^{\infty} n x^n$ as a rational function! It's just

$$x \frac{d}{dx} \sum_{n=0}^{\infty} x^n$$

remember? Since

$$\begin{aligned} x \frac{d}{dx} \sum_{n=0}^{\infty} x^n &= x \frac{d}{dx} \frac{1}{1-x} \\ x \frac{d}{dx} \sum_{n=0}^{\infty} x^n &= x \frac{d}{dx} \frac{1}{1-x} \\ &= x \frac{1}{(1-x)^2} \end{aligned}$$

Hence

$$\begin{aligned} a(x) &= \frac{a_0 + \sum_{n=0}^{\infty} n x^n}{1-2x} \\ &= \frac{a_0 + \frac{x}{(1-x)^2}}{1-2x} \\ &= \frac{a_0(1-x)^2 + x}{(1-2x)(1-x)^2} \end{aligned}$$

Now we use the theory of partial fractions: There are constants A, B, C such that

$$\frac{1}{(1-2x)(1-x)^2} = \frac{A}{1-2x} + \frac{B}{1-x} + \frac{C}{(1-x)^2}$$

We have

$$1 = A(1-x)^2 + B(1-x)(1-2x) + C(1-2x)$$

When $x = 1$, we get $C = -1$. When $x = 1/2$, we obtain $A = 4$. Finally when $x = 0$, $1 = A + B + C$, and hence $B = 1 - 4 + 1 = -2$. Hence

$$\frac{1}{(1-2x)(1-x)^2} = \frac{4}{1-2x} + \frac{-2}{1-x} + \frac{-1}{(1-x)^2}$$

Hence

$$\begin{aligned}
a(x) &= (a_0(1-x)^2 + x)\left(\frac{4}{1-2x} + \frac{-2}{1-x} + \frac{-1}{(1-x)^2}\right) \\
&= (a_0 + (1-2a_0)x + a_0x^2)\left(4\sum_{n=0}^{\infty} 2^n x^n - 2\sum_{n=0}^{\infty} x^n - \sum_{n=0}^{\infty}(n+1)x^n\right) \\
&= (a_0 + (1-2a_0)x + a_0x^2)\sum_{n=0}^{\infty}(4 \cdot 2^n - 2 - (n+1))x^n
\end{aligned}$$

Hence

$$a_n = \begin{cases} a_0 & \text{if } n = 0 \\ a_0(4 \cdot 2 - 2 - (1+1)) + (1-2a_0)(4 - 2 - (0+1)) & \text{if } n = 1 \\ a_0(4 \cdot 2^n - 2 - (n+1)) + (1-2a_0)(4 \cdot 2^{n-1} - 2 - n) + a_0(4 \cdot 2^{n-2} - 2 - (n-1)) & \text{if } n \geq 2 \end{cases}$$

Simplifying the above horrendous closed forms we get:

$$a_n = \begin{cases} a_0 & \text{if } n = 0 \\ 1 + 2a_0 & \text{if } n = 1 \\ (2 + a_0)2^n - n - 2 & \text{if } n \geq 2 \end{cases}$$

Let's check that the first few terms are correct. First of all, $a_n = a_0$ for $n = 0$. That's good. For $n = 1$, from the closed form we have $a_1 = 1 + 2a_0$. Now the recurrence we started with is

$$a_n = 2a_{n-1} + n$$

Therefore from the recurrence relation, $a_1 = 2a_0 + 1$. That's also good. One last check: Let's look at $n = 2$. From the recurrence relation we have

$$\begin{aligned} a_2 &= 2a_1 + 2 \\ a_1 &= 2a_0 + 1 \end{aligned}$$

Hence

$$a_2 = 2(2a_0 + 1) + 2 = 4a_0 + 4$$

From the derived closed form we have

$$(2 + a_0)2^2 - 2 - 2 = 8 + 4a_0 - 4 = 4a_0 + 4$$

Exercise 912.6.1. Find a closed form for a_n where

$$a_n = 2a_{n-1} + a_{n-2} + n^2 + 1$$

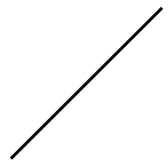
Exercise 912.6.2. Solve completely the degree 2 linear nonhomogenous case where the nonhomogenous factor is a polynomial of degree 1: Find a closed form for a_n where

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + d_1 x + d_2 x^2$$

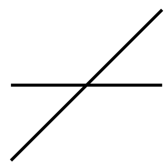
Solutions

912.7 Space subdivision problems space-subdivision-problems.tex

What is the maximum number of regions do you get when you cut up a 2-dimensional space by n lines. With one line:



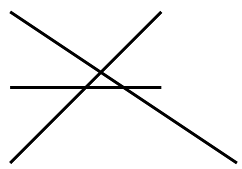
the 2-D space is cut up into two pieces. Let a_n be the number of regions cut out by n lines. With two lines we get 4.



So far we have

$$a_0 = 1, \quad a_1 = 2, \quad a_2 = 4$$

To simplify the problem, a line cannot go through a point of intersection. With three:



you get 7. Note that the last line above, L_3, cuts through 3 regions, and therefore adds 3 more regions to the regions already obtained with the first two lines. The next line will cut through 4 regions. Come to think of it, L_1 (the first line) cuts through 1 region – the uncut region, L_2 (the second line) cuts through 2 regions. In general line L_n cuts through n regions. Now before L_n, therefore a_{n-1} regions. With L_n, the number of regions is a_{n-1} and n regions are cut up by L_n. Therefore

$$a_n = a_{n-1} + n$$

for $n \geq 1$.

Exercise 912.7.1. What is a_0? □

Let $a(x) = \sum_{n=0}^{\infty} a_n x^n$.

$$
\begin{aligned}
a(x) &= a_0 + \sum_{n=1}^{\infty} (a_{n-1} + n) x^n \\
&= a_0 + \sum_{n=1}^{\infty} a_{n-1} x^n + \sum_{n=0}^{\infty} n x^n \\
&= a_0 + x a(x) + \sum_{n=0}^{\infty} n x^n \\
\therefore \quad a(x) &= \frac{1 + x \frac{d}{dx} \sum_{n=0}^{\infty} x^n}{1 - x} \\
&= \frac{1 + x \frac{d}{dx} \frac{1}{1-x}}{1 - x} \\
&= \frac{1 + x \frac{1}{(1-x)^2}}{1 - x} \\
&= \frac{(1-x)^2 + x}{(1-x)^2} \\
&= \frac{1 - x + x^2}{(1-x)^3} \\
&= (1 - x + x^2) \sum_{n=0}^{\infty} \binom{3 + n - 1}{n} x^n \\
&= (1 - x + x^2) \sum_{n=0}^{\infty} \binom{n+2}{2} x^n \\
&= \sum_{n=0}^{\infty} \binom{n+2}{2} x^n - \sum_{n=0}^{\infty} \binom{n+2}{2} x^{n+1} + \sum_{n=0}^{\infty} \binom{n+2}{2} x^{n+2} \\
&= \sum_{n=0}^{\infty} \binom{n+2}{2} x^n - \sum_{n=1}^{\infty} \binom{n+1}{2} x^n + \sum_{n=2}^{\infty} \binom{n}{2} x^n
\end{aligned}
$$

Hence

$$
a_0 = 1, \quad a_1 = \binom{0+2}{2} + \binom{1+1}{2} = 2
$$

and for $n \geq 2$

$$\begin{aligned}
a_n &= \binom{n+2}{2} + -\binom{n+1}{2} + \binom{n}{2} \\
&= \frac{1}{2}((n+2)(n+1) - (n+1)n + n(n-1)) \\
&= \frac{1}{2}(n^2 + n + 2)
\end{aligned}$$

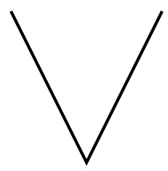
and note that this closed form can also be used for a_0, a_1. Hence for $n \geq 0$

$$a_n = \frac{1}{2}(n^2 + n + 2)$$

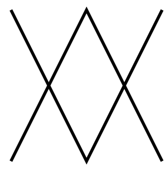
or if you like

$$a_n = \frac{n(n+1)}{2} + 1$$

Exercise 912.7.2. Instead of straight lines, what if we use lines bent at exactly one point:



For instance with two such lines



we can have 5 regions. However this is not the maximum possible. Show that it should be 7. (HINT: See it? No? Try to rotate one of the bent lines.) Find a closed form for this problem. □

Exercise 912.7.3. What if instead of lines or lines with one bend we have lines with 2 bends, i.e. something that looks like z and its reflection? What about lines with k bends where k is fixed? □

Exercise 912.7.4. What if for the original problem we only count finite regions? (i.e. we only count regions which are bounded on all sides by lines.)
□

Exercise 912.7.5. What if instead of lines (or bent lines) we use triangles?

□

Exercise 912.7.6. What if instead of lines we use circles? □

Solutions

912.7.1 Euler-Catalan recurrence

Here are two very famous problems: the triangulation problem and the parenthesizing problem. The two problems are related.

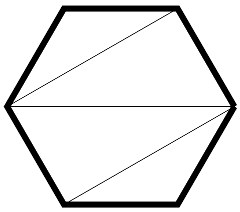
Historically, the triangulation problem appears first, i.e., the counting of the number of ways t_n to triangulate a convex n–sided polygon, and was first proposed by Euler in letter dated 1751 to Goldbach. (Goldbach was Euler's mentor.) The whole subject surrounding counting triangulations has a long and interesting history.

Euler found a closed form for t_n back in 1751. This was purely from computing t_n up to $n = 25$. However he couldn't prove its correctness. Later, in 1758, Segner discovered a recurrence relation for t_n but he couldn't find a connection between his recurrence relation to Euler's closed form. In 1838, Lamé finally proved Euler's closed was correct. So altogether from Euler's discovery of a possible closed form for triangulation counting to the discovery of proof of correctness took more than 80 years!

Back in 1758, after Segner found his recurrence for t_n, Segner computed t_n up to $n = 20$. Euler checked Segner's calculation and found that Segner made an error in his computation at $n = 15$. (Which is why I already said that computation with recurrences is slower and and hence very error prone – one would prefer to use a closed form!) But since Segner's calculation of agreeing with Euler's computation of t_n up to $n = 14$, it's very likely that Euler realized the value of Segner's recurrence relation. Using Segner's recurrence relation, it's very likely that in 1758 Euler was able to prove that his closed form is correct by using the method of generating functions (which requires Segner's recurrence). If he did, then he probably never found time to publish his proof.

The numbers C_n in the second problem are called **Catalan numbers** and are closely related to the triangulation numbers. Catalan's contribution to these two problems is actually minimal when compared to Euler, Segner, and Lamé. Research triangulations and the Catalan numbers is still very active to this day.

Exercise 912.7.7. Let $n \geq 3$ and let t_n be the number of ways to triangulate a convex polygon with n vertices. A convex polygon is a polygon where if you draw a line between any two vertices of the polygon, this line is in the polygon (and not outside). Here's a triangulation for the case of $n = 6$:



(Triangulation is a very important step in geometric computations and appears in areas such as computer graphics, computer vision, geographic information systems, etc.)

Exercise 912.7.8. let C_n be the number of ways to fully parenthesize a product of $n+1$ symbols. For instance when $n = 4$, here's one way to parenthesize $x_0x_1x_2x_3x_4$:

$$x_0((x_1(x_2x_3))x_4)$$

The fully parenthesized expression will then allow you to compute the product. (C_n will then tell you how many possible ways to compute the product, which will tell you how many cases you would need to analyze before actually caring out the multiplications. This is important for instance in matrix chain multiplications where different parenthesizing gives rise to very different runtimes. See CISS358.)

Solutions

912.8 Characteristic equation: homogeneous case

char-eq-homogeneous.tex

Now that you're an expert on generating functions and recurrences, let's step back and look at our computations again. For instance you notice that in many of the computations, it seems like there's a pattern.

Suppose we go back to the original homogeneous degree 2 case:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}$$

Then if we let $a(x) = \sum_{n=0}^{\infty} a_n x^n$, then

$$\begin{aligned} a(x) &= a_0 + a_1 x + \sum_{n=2}^{\infty} a_n x^n \\ &= a_0 + a_1 x + \sum_{n=2}^{\infty} (c_1 a_{n-1} + c_2 a_{n-2}) x^n \\ &= a_0 + a_1 x + c_1 x(-a_0 + a(x)) + c_2 x^2 a(x) \\ \therefore \quad a(x) &= \frac{a_0 + (a_1 - c_1 a_0)x}{1 - c_1 x - c_2 x^2} \end{aligned}$$

Now if you step back and look at the above, you see that the next step is determined by the factorization of $1 - c_1 x - c_2 x^2$ and furthermore that if the roots of $c_2 x^2 + c_1 x - 1$ factorization is say

$$(x - r_1)(x - r_2)$$

with $r_1 \neq r_2$, then the rational function for $a(x)$ has the form

$$a(x) = (A + Bx)\left(\frac{C}{x - r_1} + \frac{D}{x - r_2}\right)$$

which would lead to

$$a(x) = (A + Bx)\left(\frac{-C}{r_1} \cdot \sum_{n=0}^{\infty} \frac{1}{r_1^n} x^n + \frac{-D}{r_2} \cdot \sum_{n=0}^{\infty} \frac{1}{r_2^n} x^n\right)$$

which implies that $a(x)$ has the following form:

$$a(x) = (A + Bx)\left(E \cdot \sum_{n=0}^{\infty} \frac{1}{r_1^n} x^n + F \cdot \sum_{n=0}^{\infty} \frac{1}{r_2^n} x^n\right)$$

and if we stare harder:

$$\begin{aligned}
a(x) &= AE \cdot \sum_{n=0}^{\infty} \frac{1}{r_1^n} x^n + BF \cdot \sum_{n=0}^{\infty} \frac{1}{r_1^n} x^{n+1} \\
&\quad + AE \cdot \sum_{n=0}^{\infty} \frac{1}{r_2^n} x^n + BE \cdot \sum_{n=0}^{\infty} \frac{1}{r_2^n} x^{n+1} \\
&= AE + \sum_{n=1}^{\infty} \frac{AE + BEr_1}{r_1^n} x^n + \\
&\quad + AF + \sum_{n=1}^{\infty} \frac{AF + BFr_2}{r_2^n} x^n \\
&= -BEr_1 + \frac{AE + BEr_1}{r_1^0} + \sum_{n=1}^{\infty} \frac{AE + BEr_1}{r_1^n} x^n + \\
&\quad + -BFr_2 + \frac{AF + BFr_2}{r_2^0} + \sum_{n=1}^{\infty} \frac{AF + BFr_2}{r_2^n} x^n \\
&= -BEr_1 + \sum_{n=0}^{\infty} \frac{AE + BEr_1}{r_1^n} x^n \\
&\quad - BFr_2 + \sum_{n=0}^{\infty} \frac{AF + BFr_2}{r_2^n} x^n
\end{aligned}$$

Now let's look at

$$-BEr_1 - BFr_2$$

If you go back a couple of steps you will see that

$$E = \frac{-C}{r_1}, \quad F = \frac{-D}{r_2}$$

Therefore

$$Er_1 + Fr_2 = -C - D$$

and that C and D are constants such that

$$\frac{1}{(x - r_1)(x - r_2)} = \frac{C}{x - r_1} + \frac{D}{x - r_2}$$

From the partial fraction decomposition we get

$$1 = C(x - r_2) + D(x - r_1) = (C + D)x + (-Cr_2 - Dr_1)$$

By the comparing the coefficients on both sides of this identity we see that

$$C + D = 0$$

Hence

$$Er_1 + Fr_2 = -C - D = 0$$

Amazing! This means that

$$a(x) = \sum_{n=0}^{\infty} \frac{AE + BEr_1}{r_1^n} x^n + \sum_{n=0}^{\infty} \frac{AF + BFr_2}{r_2^n} x^n$$

Now you might say: so what? The coefficients for x^n still have to be determined ... and the above doesn't simplify the process since we still have to compute the constants A, B, E, F, etc ... blah, blah, blah.

But wait! Hang on there!

The above shows that

$$a(x) = \sum_{n=0}^{\infty} \frac{C_1}{r_1^n} x^n + \sum_{n=0}^{\infty} \frac{C_2}{r_2^n} x^n$$

where C_1, C_2 are constants and r_1, r_2 are roots of $1 - c_1 x - c_2 x^2$. The computation of r_1 and r_2 is unavoidable. In any case that's just an application of the quadratic equation formula. How hard is that?

What about C_1 and C_2? Does the above form of $a(x)$ simplify the computation of C_1 and C_2? Well from

$$a(x) = \sum_{n=0}^{\infty} \frac{C_1}{r_1^n} x^n + \sum_{n=0}^{\infty} \frac{C_2}{r_2^n} x^n$$

we know that

$$a_n = C_1 \frac{1}{r_1^n} + C_2 \frac{1}{r_1^n}$$

Well we can use the values of a_0 and a_1 (or any two values from the sequence of a_n that we know) and get two linear equations:

$$\begin{aligned} a_0 &= C_1 + C_2 \\ a_1 &= \frac{1}{r_1} \cdot C_1 + \frac{1}{r_2} \cdot C_2 \end{aligned}$$

and then solve for C_1 and C_2. This is just solving a system of two linear equations. No big deal.

So let's summarize what we know: Suppose we have the following homogeneous degree 2 recurrence relation:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}$$

then

- First we find the roots of $1 - c_1 x - c_2 x^2$. Say the roots r_1, r_2 are distinct.
- Then we know that there are constants such that

$$a_n = C_1 \frac{1}{r_1^n} + C_2 \frac{1}{r_2^n}$$

- We substitute two values of n into the above closed form for a_n to get two linear equations for C_1 and C_2 and solve for these constants.

Note that for $r = r_1$ or r_2:

$$1 - c_1 r - c_2 r^2 = 0$$

And if we multiply this equation by $1/r^2$ we get

$$\left(\frac{1}{r}\right)^2 - c_1\left(\frac{1}{r}\right) - c_2 = 0$$

Therefore if we look at this equation

$$x^2 - c_1 x - c_2 = 0$$

the roots we get are reciprocals of our original r_1 and r_2, i.e. if we let s_1, s_2 be roots

$$x^2 - c_1 x - c_2 = 0$$

then $s_1 = 1/r_1, s_2 = 1/r_2$. And with these roots instead of writing

$$a_n = C_1 \frac{1}{r_1^n} + C_2 \frac{1}{r_2^n}$$

we write

$$a_n = C_1 s_1^n + C_2 s_2^n$$

In summary: If we're given this recurrence relation:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}$$

then we do the following:

Assume $a_n = s^n$ is a solution where $s \in \mathbb{R}$, i.e.,

$$s^n = c_1 s^{n-1} + c_2 s^{n-2}$$

Crossing out s^{n-2}, the above equation becomes the following quadratic equation in s:

$$s^2 = c_1 s + c_2$$

i.e.,

$$s^2 - c_1 s - c_2 = 0$$

Find roots s_1, s_2 of the quadratic equation

$$x^2 - c_1 x - c_2 = 0$$

- Suppose $s_1 \neq s_2$. Then the general closed form of a_n must be

  $$a_n = C_1 s_1^n + C_2 s_2^n$$

  for some constants C_1 and C_2.
- Suppose $s_1 \neq s_2$. Then the general closed form of a_n must be

  $$a_n = C_1 s_1^n + C_2 n s_2^n$$

  for some constants C_1 and C_2.

In both cases, to compute C_1 and C_2, substitute two values for n in the above closed form and solve for C_1 and C_2.

The quadratic equation

$$x^2 - c_1 x - c_2 = 0$$

is called the **characteristic equation** of the recurrence relation.

Let's use the above method to find the closed form for the Fibonacci sequence

$$F_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F_{n-1} + F_{n-2} & \text{if } n \geq 2 \end{cases}$$

The recurrence relation is

$$F_n = F_{n-1} + F_{n-2}$$

i.e.

$$F_n - F_{n-1} - F_{n-2} = 0$$

The quadratic equation to solve is

$$x^2 - x - 1 = 0$$

The roots are

$$\frac{1+\sqrt{5}}{2}, \quad \frac{1-\sqrt{5}}{2}$$

Hence the closed form must be

$$F_n = C_1\left(\frac{1+\sqrt{5}}{2}\right)^n + C_2\left(\frac{1-\sqrt{5}}{2}\right)^n$$

To solve for C_1 and C_2, let $n = 0, 1$ and we get

$$\begin{aligned} 0 &= F_0 = C_1 + C_2 \\ 1 &= F_1 = C_1\frac{1+\sqrt{5}}{2} + C_2\frac{1-\sqrt{5}}{2} \end{aligned}$$

The first equation implies that $C_2 = -C_1$ and hence we get

$$\begin{aligned} 1 &= C_1\frac{1+\sqrt{5}}{2} - C_1\frac{1-\sqrt{5}}{2} \\ \therefore \quad 1 &= C_1\sqrt{5} \\ \therefore \quad C_1 &= \frac{1}{\sqrt{5}} \end{aligned}$$

Therefore $C_2 = -\frac{1}{\sqrt{5}}$. Hence

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

which we already know from a previous section.

Note that the whole point of this computational technique is to *delay* the computation of certain constants until after we have a closed form for our sequence. The closed form has unknown constants (made up of constants which were left unresolved). Finally the constants in the closed form are resolved by writing down linear equations for the constants using known values from the sequence.

Example 912.8.1. Consider the homogeneous degree 2 linear recurrence

$$a_n = 5a_{n-1} - 6a_{n-2}$$

(a) What is the characteristic equation?
(b) What are the roots of the characteristic equation?
(c) What is the general closed form for a_n?
(d) If $a_0 = 1, a_1 = 3$, what is the closed form for a_n?

SOLUTION. (a) Let $a_n = r^n$ be a solution of the recurrence relation. Then

$$r^n = 5r^{n-1} - 6r^{n-2}$$

Hence

$$r^2 - 5r + 6 = 0$$

Therefore the characteristic equation of the recurrence relation is

$$x^2 - 5x + 6 = 0$$

(b) Using the qudratic equation formula, the roots of the characteristic equation from (a) are

$$r = \frac{5 \pm \sqrt{5^2 - 4 \cdot 1 \cdot 6}}{2} = 2, 3$$

(c) The general closed form for a_n is

$$a_n = C_1 2^n + C_2 3^n$$

(d) From $a_0 = 1, a_1 = 3$, we have

$$\begin{aligned} 1 &= a_0 = C_1 + C_2 \\ 3 &= a_1 = 2C_1 + 3C_2 \end{aligned}$$

Hence $C_1 = 0, C_2 = 1$. Therefore

$$a_n = 3^n$$

for $n \geq 0$.

(Check: $3^0 = 1 = a_0$, $3^1 = 3 = a_1$. So 3^n matches the two base cases. For $n = 2$, $3^2 = 9$ and $a_2 = 5a_1 - 6a_0 = 5 \cdot 3 - 6 \cdot 1 = 9$.) □

Exercise 912.8.1. Consider the homogeneous degree 2 linear recurrence

$$b_n = 4b_{n-1} - 4b_{n-2}$$

(a) What is the characteristic equation?
(b) What are the roots of the characteristic equation?
(c) What is the general closed form for a_n?
(d) If $b_0 = 1, b_1 = 1$, what is the closed form for b_n?

SOLUTION. (a) Let $b_n = r^n$ be a solution of the recurrence relation. Then

$$r^n = 4r^{n-1} - 4r^{n-2}$$

Hence

$$r^2 - 4r + 4 = 0$$

Therefore the characteristic equation of the recurrence relation is

$$x^2 - 4x + 4 = 0$$

(b) Using the quadratic equation formula, the roots of the characteristic equation from (a) are

$$r = \frac{4 \pm \sqrt{4^2 - 4 \cdot 1 \cdot 4}}{2} = 2, 2$$

(c) The general closed form for b_n is

$$b_n = C_1 2^n + C_2 n 2^n$$

(d) From $b_0 = 1, b_1 = 1$, we have

$$\begin{aligned} 1 &= b_0 = C_1 \\ 1 &= b_1 = 2C_1 + 2C_2 \end{aligned}$$

Hence $C_1 = 1, C_2 = -1/2$. Therefore

$$b_n = 2^n - \frac{1}{2} n 2^n$$

for $n \geq 0$.

(Check: $2^0 - \frac{1}{2} 0 \cdot 2^0 = 1 = b_0$, $2^1 - \frac{1}{2} 1 \cdot 2^1 = 1 = b_1$. So $2^n - \frac{1}{2} n 2^n$ matches the two base cases. For $n = 2$, $2^2 - \frac{1}{2} 2 \cdot 2^2 = 0$ and $b_2 = 4b_1 - 4b_0 = 4 \cdot 1 - 4 \cdot 1 = 0$.) $\square$

In general if you are given a homogeneous linear recurrence relation of degree d:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_d a_{n-d}$$

it can be shown that the following works: Find all the roots of

$$x^d - c_1 x^{d-1} - c_2 x^{d-2} - \cdots - c_d = 0$$

- Suppose the roots $r_1, \ldots, r_d$ are distinct. Then there are constants $C_1, \ldots, C_d$ such that

$$a_n = C_1 r_1^n + C_2 r_2^n + \cdots + C_d r_d^n$$

- Now suppose some roots repeat. Suppose r_1 appear exactly 5 times in the factorization of the characteristic equation, then the part of the closed form of a_n that involves r_1 will contain a linear combination of the following 5 terms: r_1^n, nr_1^n, $n^2 r_1^n, n^3 r_1^n, n^4 r_1^n$. In other words a_n will look like this:

$$a_n = \left(C_1 r_1^n + C_2 n r_1^n + C_3 n^2 r_1^n + C_4 n^3 r_1^n + C_5 n^4 r_1^n \right) + \cdots$$

  where $C_1, ..., C_5$ are constants. Now suppose root r_2, which is different from r_1, appears 3 times. Then

$$\begin{aligned} a_n = &\left(C_1 r_1^n + C_2 n r_1^n + C_3 n^2 r_1^n + C_4 n^3 r_1^n + C_5 n^4 r_1^n \right) \\ &+ \left(C_6 r_2^n + C_7 n r_2^n + C_8 n^2 r_2^n \right) + \cdots \end{aligned}$$

  where $C_1, ..., C_8$ are constants. Now suppose root r_3, which is different from r_1 and r_2, appears 2 times. Then

$$\begin{aligned} a_n = &\left(C_1 r_1^n + C_2 n r_1^n + C_3 n^2 r_1^n + C_4 n^3 r_1^n + C_5 n^4 r_1^n \right) \\ &+ \left(C_6 r_2^n + C_7 n r_2^n + C_8 n^2 r_2^n \right) \\ &+ \left(C_9 r_3^n + C_{10} n r_3^n \right) + \cdots \end{aligned}$$

  where $C_1, ..., C_{10}$ are constants. And you keep going until all roots are accounted for.

[Aside: Those of you who took Diff Eq, why does this feel like de jevu? Check your notes on homogeneous differential equations.]

For instance suppose that the characteristic equation looks like this:

$$\left(x - \frac{1}{2}\right)\left(x - \frac{1}{3}\right)\left(x - \frac{5}{7}\right)^4\left(x - \frac{2}{13}\right)^2 = 0$$

Then we must have

$$\begin{aligned} a_n = {} & A\left(\frac{1}{2}\right)^n \\ & + B\left(\frac{1}{3}\right)^n \\ & + C\left(\frac{5}{7}\right)^n + Dn\left(\frac{5}{7}\right)^n + En^2\left(\frac{5}{7}\right)^n + Fn^3\left(\frac{5}{7}\right)^n \\ & + G\left(\frac{2}{13}\right)^n + Hn\left(\frac{2}{13}\right)^n \end{aligned}$$

[EXERCISES]

Aside.

Why is the quadratic equation of a degree 2 recurrence relation called the characteristic equation? And if you've taken linear algebra you should be scratching your head and asking yourself why does this sounds so familiar.

I've already mentioned that you can view $\sum_{n=0}^{\infty} a_n x^n$ as an infinite dimensional vector:

$$(a_0, a_1, a_2, \ldots)$$

Recall that multiplication by x^ℓ acts as a shift-by-ℓ operator on $\sum_{n=0}^{\infty} a_n x^n$. Let me write T for *left* shift operator.

$$T((a_0, a_1, a_2, \ldots)) = (a_1, a_2, a_3, \ldots)$$

The corresponding action on a power series is

$$\sum_{n=0}^{\infty} a_n x^n \to \frac{\sum_{n=0}^{\infty} a_n x^n - a_0}{x}$$

Let T^n be applying T n times, i.e., the left-shift-by-n operator. Suppose we look at the case of Fibonacci sequence:

$$F_n = F_{n-1} + F_{n-2}$$

Then I have the following:

$$\begin{aligned} T^2((F_0, F_1, F_2, \ldots)) &= (F_2, F_3, F_4, \ldots) \\ T^1((F_0, F_1, F_2, \ldots)) &= (F_1, F_2, F_3, \ldots) \\ T^0((F_0, F_1, F_2, \ldots)) &= (F_0, F_1, F_2, \ldots) \end{aligned}$$

and therefore

$$(T^2 - T^1 - T^0)((F_0, F_1, F_2, \ldots)) = (F_2 - F_1 - F_0, F_3 - F_2 - F_1, F_4 - F_3 - F_2, \ldots)$$

Since $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$, the above is in fact

$$(T^2 - T^1 - T^0)((F_0, F_1, F_2, \ldots)) = (0, 0, 0, \ldots)$$

In other words the $T^2 - T^1 - T^0$ acts like the multiplication by 0 operator on $(F_0, F_1, F_2, \ldots)$:

$$T^2 - T^1 - T^0 = 0$$

The corresponding polynomial

$$x^2 - x - 1$$

is called the characteristic polynomial of T.

Solutions

912.9 Characteristic Equation: Non-homogeneous Case – ??

912.10 Complex numbers complex.tex

When factoring your characteristic equation, you might get complex roots. Here's a quick intro or recap of complex numbers.

A complex number is a number of form

$$z = a + bi$$

where a, b are real numbers. The i is a shorthand for $\sqrt{-1}$. We say that the real part of z is a and the complex part of z is b. The set of complex numbers include the set of real numbers since if a is a real number then

$$a = a + 0i$$

You can add complex numbers $z = a + bi$ and $z' = a' + b'i$ like this:

$$z + z' = (a + a') + (b + b')i$$

For instance $(1 + 3i) + (0.2 + 1.7i) = 1.2 + 4.7i$. You can subtract likewise

$$z - z' = (a - a') + (b - b')i$$

You can also multiply complex numbers:

$$zz' = (a + bi)(a' + b'i) = aa' + ab'i + a'bi + bb'i^2$$

Note that since $i = \sqrt{-1}$, we have $i^2 = -1$,

$$zz' = aa' + ab'i + a'bi + bb'(-1) = (aa' - bb') + (ab' + a'b)i$$

The multiplication of complex numbers seems to be the most complicated. But the thing to remember is this: You should not memorize the multiplication formula. You only need to remember $i^2 = -1$. In fact a good way to remember everything above is to think of the i as a variable X and you are sort of like working with polynomials. For instance the above then becomes clear:

$$(a + bX) + (a' + b'X) = (a + a') + (b + b')X$$

and

$$(a + bX)(a' + b'X) = (a'b') + (ab' + a'b)X + (bb')X^2$$

The only thing different between the variable X and i is that $i^2 = -1$. Therefore in the above you remember to replace X by i and X^2 by -1. So remember this: think of i as a variable X except that this variable has the rule

$X^2 = -1$. Not only that since $X^2 = -1$, you have $X^3 = X^2 \cdot X = -X$, and $X^4 = X^3 \cdot X = (-X)X = -X^2 = -(-1) = 1$. In terms of i, you have the following integer powers of i:

$$i, \quad i^2 = -1, \quad i^3 = -i, \quad i^4 = 1,$$

Of course after that it repeats:

$$i^5 = i^4 \cdot i, 1 \cdot i = i, \quad i^6 = i^4 \cdot i^2, 1 \cdot i^2 = -1, \quad \text{etc.}$$

As you multiply more and more complex numbers you might accumulate higher and higher powers of i. But remember that you never need more than i^1 because all higher powers of i can be rewritten in terms of $\pm i$ or ± 1. In general

$$\begin{aligned} i^{4n} &= i \\ i^{4n+1} &= i \\ i^{4n+2} &= -1 \\ i^{4n+3} &= -i \end{aligned}$$

where n is an integer. (There's no reason to memorize this. Just go through the earlier computation of power of i and you'll see it.) In hifaluting jargon, the integral powers of i has a period of 4, just like the sine function has a period of 2π since $\sin(x + 2\pi) = \sin(x)$.

Now I need to tell you how to divide complex numbers. Note that if z and z' are complex numbers, we obviously want $z/z' = z \cdot (1/z')$. Therefore it's enough for me to tell you how to compute $1/z'$ given z'. If $z' = a' + b'i$ then

$$\begin{aligned} \frac{1}{z'} &= \frac{1}{a' + b'i} \cdot \frac{a' - b'i}{a' - b'i} \\ &= \frac{a' - b'i}{a'^2 + b'^2} \end{aligned}$$

The key thing to remember about the above is that you're basically trying to *force the denominator to be a real number*. And to do that you need to remember that for any complex number $a' - b'i$,

$$(a' - b'i) \cdot (a' + b'i)$$

will always be a real numbers. The two complex numbers $a' - b'i$ and $a' + b'i$ are said to be *complex conjugates*. This is not as foreign as you think. If you're experiencing a de jevu, then that's because of examples like this ("rationalization") in high school:

$$\frac{1}{2+3\sqrt{7}} \cdot \frac{2-3\sqrt{7}}{2-3\sqrt{7}} = \frac{2-3\sqrt{7}}{2^2-9^2\cdot 7}$$

So just like multiplying $a'-b'\sqrt{7}$ with $a'+b'\sqrt{7}$ makes the $\sqrt{7}$ go away, multiplying $a'-b'i$ with $a'+b'i$ makes the $i=\sqrt{-1}$ go away. That's all there is to it. Note that the method of rationalizing it not specific to $\sqrt{7}$ or $\sqrt{-1}$. So for an expression of the form $a+b\sqrt{c}$ where a,b,c comes from a set of numbers where $\sqrt{c}$ is not in this set of numbers, multiplying this number with $a-b\sqrt{c}$ will remove $\sqrt{c}$. In the case of $a+b\sqrt{7}$, the set is usually the rationals which is why you want to get rid of $\sqrt{7}$ by rationalization since $\sqrt{7}$ is not rational (right?) For the case of $1+2i$, the set is real numbers. We say that $a+b\sqrt{c}$ and $a-b\sqrt{c}$ are conjugates.

There are two main reasons why we rationalize such expressions. First it makes computation easier. Without calculator, an expression like $1/(2+3\sqrt{7})$ would require us to check tables for $\sqrt{7}$, multiply it with 3 (easy), add 2 to it (also easy) and then computing the reciprocal, which is tedious because you're doing division with a number with lots of decimal places (depending on how accurate you want your number to be). If you're doing an engineering project and the required accuracy is 10 decimal places, then your division would be really terrible (without a calculator that it.) So instead of that we rational it to get $\frac{2-3\sqrt{7}}{-563}$ Note that this expression is a lot easier to compute without calculators. You again check the tables for $\sqrt{7}$, multiply it with -3 (easy), add 2 to it (easy), and divide by -563. Compare the above two different ways of getting an approximation of $1/(2+3\sqrt{7})$.

Another reason why we rationalize is that by doing so, it changes an expression to a standard form that allows us to add, subtract, and multiply easily. (So again this has to do with easy computation.) For instance if we want to add $(2+3\sqrt{7})+\frac{1}{2-3\sqrt{7}}$, you would be forced to approximate the second number if you do not know how to rationalize it. However with rationalization, you're basically adding

$$\begin{aligned}
(2+3\sqrt{7})+\left(\frac{1}{2-3\sqrt{7}}\right) &= (2+3\sqrt{7})+\left(\frac{2-3\sqrt{7}}{-563}\right) \\
&= (2+3\sqrt{7})+\left(\frac{2}{-563}-\frac{3}{-563}\sqrt{7}\right) \\
&= \left(2+\frac{2}{-563}\right)+\left(3-\frac{3}{-563}\right)\sqrt{7}
\end{aligned}$$

and the above computation is exact. (Of course if you're only interesting in an approximation and you have a calculator, rationalizing is redundant.)

Exercise 912.10.1. How do you rationalize $\frac{1}{1+2^{1/3}}$? (See the *cube* root? It's not a square root.)

Let's try an example.

$$\begin{aligned}\frac{1+2i}{3+4i} &= \frac{(1+2i)(3-4i)}{(3+4i)(3-4i)} \\ &= \frac{3-4i+6i-8i^2}{3^2+4^2} \\ &= \frac{3+2i+8}{9+16} \\ &= \frac{11+2i}{25} \\ &= \frac{11}{25}+\frac{2}{25}i\end{aligned}$$

Let's check that

$$1+2i = \left(\frac{11}{25}+\frac{2}{25}i\right)(3+4i)$$

The right hand side is

$$\begin{aligned}\left(\frac{11}{25}+\frac{2}{25}i\right)(3+4i) &= \frac{11}{25}\cdot 3+\frac{11}{25}\cdot 4i+\frac{2}{25}i\cdot 3+\frac{2}{25}i\cdot 4i \\ &= \left(\frac{11}{25}\cdot 3-\frac{2}{25}\cdot 4\right)+\left(\frac{11}{25}\cdot 4+\frac{2}{25}\cdot 3\right)i \\ &= \left(\frac{33-8}{25}\right)+\left(\frac{44+6}{25}\right)i \\ &= \left(\frac{25}{25}\right)+\left(\frac{50}{25}\right)i\end{aligned}$$

which of course does give me $1+2i$.

The important thing about the complex numbers is not that it's just a bunch of abstract symbols that we write down. The important thing is that the complex numbers include the real numbers not just as a set – the operation (addition, subtraction, multiplication, division) on the complex numbers extend the operation on real numbers. This means for instance that when you look at the above addition of complex numbers you see that when the complex numbers you're adding are actually real numbers (i.e. their complex parts are actually zero) then the addition is in fact the addition from real numbers. For instance the addition formula for complex numbers gives

$$(3 + 0i) + (2 + 0i) = (3 + 2) + (0 + 0)i = 5 + 0i$$

which in this case is the same as the addition of *real* numbers

$$3 + 2 = 5$$

This is the same for multiplication:

$$(3 + 0i)(2 + 0i) = (3 \cdot 2 - 0 \cdot 0) + (3 \cdot 0 + 0 \cdot 2)i = (6 - 0) + (0 + 0)i = 6 + 0i$$

which is just the following multiplication of *real* numbers:

$$3 \cdot 2 = 6$$

Now why do we need to extend the real numbers to get the complex numbers? The reason is that when we have a question regarding real numbers, the question can be viewed as a question regarding complex numbers. How so? Well take for instance look at the equation

$$x^2 + 1 = 0$$

viewed as an equation in real numbers, i.e., we want to find real numbers x satisfying

$$x^2 + 1 = 0$$

Of course there are no such real numbers! (One property of real numbers is that $x^2 \geq 0$ for any real number x. Therefore $x^2 + 1 \geq 0 + 1 = 1$.) However when you view this as an equation in the world of complex numbers, if you like you can write

$$(1 + 0i)x^2 + (1 + 0i) = (0 + 0i)$$

but we usually don't include the complex part of a complex number is it's zero; likewise we leave out the real part of a complex number when it's 0. So I'll just write

$$x^2 + 1 = 0$$

but remember that all numbers are complex. Now in this case (i.e., viewing the equation in the complex world), we see that there are now solutions!! In fact is $x = i$ (i.e. $\sqrt{-1}$), we get

$$i^2 + 1 = (-1) + 1 = 0$$

i.e., in the complex world this equation $x^2 + 1 = 0$ now has solutions!!! How many solutions are there? Just one? No, there are two. The other one is $x = -i$:

$$(-i)^2 + 1 = (-i)(-i) + 1 = -1 + 1 = 0$$

(Use the multiplication rule for complex number to compute $(-i)(-i)$.) Therefore in the real world

$$x^2 + 1 = 0$$

has no solutions while in the *complex* world

$$x^2 + 1 = 0$$

has two solutions. In fact there are exactly two solutions.

With complex numbers, any equation involving a quadratic equation with real coefficients, say

$$x^2 + 1 = 0$$

will have two roots, i.e. $x = \pm i = \pm\sqrt{-1}$. But what about

$$x^2 + 3 = 0$$

Since we have

$$x^2 = -3$$

and hence

$$x = \pm\sqrt{-3}$$

do we now need to also add numbers like $\sqrt{-3}$ to our already expanded real numbers, i.e., complex numbers? NO! The reason is that

$$\sqrt{-3} = \sqrt{(-1)(3)} = \sqrt{-1} \cdot \sqrt{3} = 3i$$

So we do not need to introduce more exotic complex numbers

$$\sqrt{-2}, \sqrt{-3}, \sqrt{-1.46}, \ldots$$

because already in our complex numbers we can already handle these numbers, which are

$$i\sqrt{2}, i\sqrt{3}, i\sqrt{1.46}, \ldots$$

With i we can now say that the solutions to

$$ax^2 + bx + c = 0$$

are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

without lawsuits as long as we allow include complex numbers. (See the square root? The $b^2 - 4ac$ can be negative – and we are not worried with that because we have complex numbers.) Therefore with complex numbers and multiplicity we can say that every quadratic polynomial has exactly two roots. We don't have special cases: real distinct roots, one real root, no real solutions.

Everything is going real well with quadratic polynomials ... but what about other polynomials? Do we need to add some kind of cube root of our complex numbers to handle cubic (i.e. degree 3) polynomials? Take a look at this equation:

$$x^3 + 1 = 0$$

i.e.

$$x^3 = -1$$

We know that -11 is a solution. Can we add something (some funky super-duper complex numbers) to our complex numbers to increase the number of solutions? Surprisingly there are two solutions not among the real numbers, but which are in the complex numbers. You don't need extra super-duper complex numbers. Look at this complex number $z = \frac{1}{2} + \frac{\sqrt{3}}{2}i$.

$$z^3 = \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^3$$

Now we use binomial theorem to get

$$z^3 = \frac{1}{2}^3 + 3\left(\frac{1}{2}\right)^2 \frac{\sqrt{3}}{2}i + 3\frac{1}{2}\left(\frac{\sqrt{3}}{2}i\right)^2 + \left(\frac{\sqrt{3}}{2}i\right)^3$$

Looks bad ... but it's just a tedious computation. Let me do it slowly.

$$\begin{aligned}
z^3 &= \frac{1}{8}\left(1 + 3\sqrt{3}i + 3(\sqrt{3}i)^2 + (\sqrt{3}i)^3\right) \\
&= \frac{1}{8}\left(1 + 3\sqrt{3}i + 3 \cdot \sqrt{3}^2 i^2 + \sqrt{3}^3 i^3\right) \\
&= \frac{1}{8}\left(1 + 3\sqrt{3}i + 9i^2 + 3\sqrt{3}i^3\right)
\end{aligned}$$

Now we lower the powers of i in the expression:

$$z^3 = \frac{1}{8}\left(1 + 3\sqrt{3}i + 9(-1) + 3\sqrt{3}(-i)\right)$$

Now you see that some terms combine:

$$\begin{aligned}
z^3 &= \frac{1}{8}\left((1 + 9(-1)) + (3\sqrt{3}i + 3\sqrt{3}(-i))\right) \\
&= \frac{1}{8}(-8 + 0) \\
&= -1
\end{aligned}$$

Vóila!

Exercise 912.10.2. The above is an exponentiation computation on a complex number. The above computation uses binomial theorem. Later you'll learn the de Moivre's theorem, a very powerful tool for exponentiation. After you learn de Moivre's theorem, come back to this problem and compute $\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^3$ using de Moivre's theorem. □

It's curious that a cubic polynomial can have extra roots just like the quadratic polynomial simply because we've "repaired" the real numbers just by handling only the quadratic case.

But are there any other solutions?

Exercise 912.10.3. Your turn ... show that

$$z' = \frac{1}{2} - \frac{\sqrt{3}}{2}i$$

is another solution to $x^3 = -1$.

It turns out that the i, which is inherently a creature of the *square root*, is actually enough to solution *every* polynomial, cubic, quartic (degree 4), quintic (degree 5), sextic (degree 6), whatever-tic.

In fact in the complex world any polynomial of degree d has *exactly* d roots. You have to be careful here with how you count roots. For instance

$$(x-1)^2 = 0$$

has two roots (or we say that root 1 occurs with multiplicity of 2.) In order words you can't just find some number a and say that a is a root of polynomial $p(x)$ simply because $p(a) = 0$, and happy go on the the next number. Once you've found a root say a you need to factor out as many $x - a$'s from your polynomial as you can to get $p(x) = (x-a)^n q(x)$ such that $q(a) \neq 0$ then conclude that a occurs n times (or we say it has multiplicity n, then find other numbers for satisfying $q(x) = 0$, etc., until you get something like

$$p(x) = c(x-a)^n (x-b)^m ...$$

then you say that you've found all the roots, i.e. a occurs n times, b occurs m times, etc.

The big theorem regarding complex numbers is that you can *always* find a complex root for your polynomial $p(x)$ as long as the degree is ≥ 1 and with that root, say a, you can factorize $p(x)$ as $(x-a)^n q(x)$. You then repeat the process with $q(x)$ to get $q(x) = (x-b)^m r(x)$, and hence $p(x) = (x-a)^n (x-b)^m r(x)$, etc., until you've found all the roots. The other part of this big theorem is that not can you *always* find a root, the number of roots (again counting repeats) will *always* be the degree of the polynomial.

This is called the fundamental theorem of algebra. Once again this says that any polynomial of degree d viewed as a polynomial in complex numbers will always have exactly d roots.

Another important thing about the factorization of your polynomial $p(x)$ into something like $c(x-a)^n(x-b)^m \cdots$ is that the factorization is unique. OK what do I mean by that? Well suppose instead factorizing it "root by root, what if you manage to factor a degree 100 polynomial into two with degree 40 and 60? say $p(x) = u(x)v(x)$. If you carry out the factorization on $u(x)$ and $v(x)$, you will still get them into the form $c_1(x-a_1)^{n_1} \cdots$ and $c_2(x-a_2)^{n_2} \cdots$. Putting these back into $p(x) = c_1(x-a_1)^{n-1} \cdots c_2(x-a_2)^{n_2} \cdots$, you'll find that the linear terms, i.e., the $x - \alpha$ terms that appear will always be the same, it doesn't matter how you carry out the factorization. Again, once you finish factorization into linear terms, you get the same thing.

This is similar to the integers. For instance look at the number 100. If you factorize it by factoring out the smallest primes, you can the following:

$$100 = 2 \cdot 50 = 2 \cdot 2 \cdot 25 = 2 \cdot 2 \cdot 5 \cdot 5$$

If however you factorize it by break off a 10 first:

$$100 = 10 \cdot 10$$

and then factorize the 10's separately to get $10 = 2 \cdot 5$ and then put everything together to get:

$$100 = 10 \cdot 10 = 2 \cdot 5 \cdot 2 \cdot 5$$

you see the the "smallest" nontrivial factors (i.e. primes), i.e. 2, 2, 5, 5, are the same in both cases. If you prefer a picture here's one:

```
   100               100
  /   \             /   \
 2     50          10    10
      /  \        /  \  /  \
     2    25     2    5 2   5
         /  \
        5    5
```

The analogy is more than a mere curiosity. There is a deep connection between the math of integers and the math of polynomials.

By the way the corresponding theorem for integer is called the fundamental theorem of arithmetic. So you have now seen:

- The fundamental theorem of arithmetic
- The fundamental theorem of algebra

Let's go back to computations. I've given you the solutions to $x^3 + 1 = 0$. But ... how did I come up with the solutions

$$z = \frac{1}{2} + \frac{\sqrt{3}}{2} i$$

and

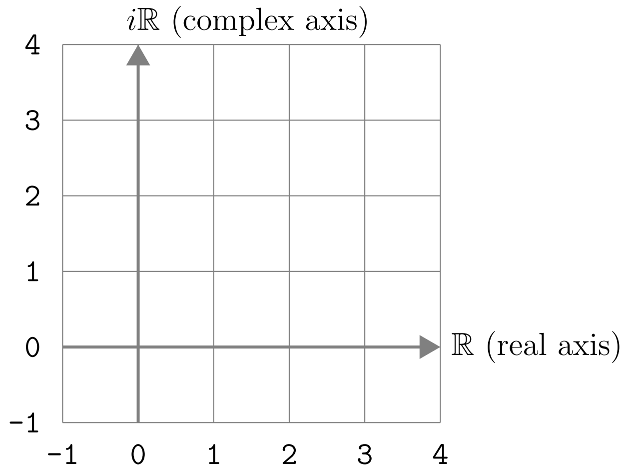
$$z' = \frac{1}{2} - \frac{\sqrt{3}}{2} i$$

for $x^3 = -1$?!? Furthermore ... something's fishy here ... z, z' are conjugates.

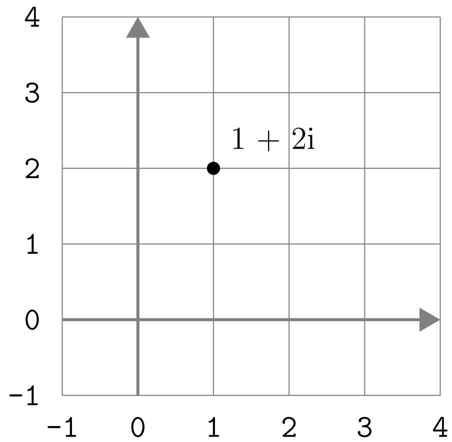
First of all, note that the fundamental theory of algebra is useful because it tells us that $x^3 + 1$ has 3 roots. So at least we know that together with -1 there are no other roots: the fundamental theorem of algebra tells us when we can stop the search for roots. But it doesn't tell us how to find the roots!

Here's where trigonometry and geometry come to the rescue!

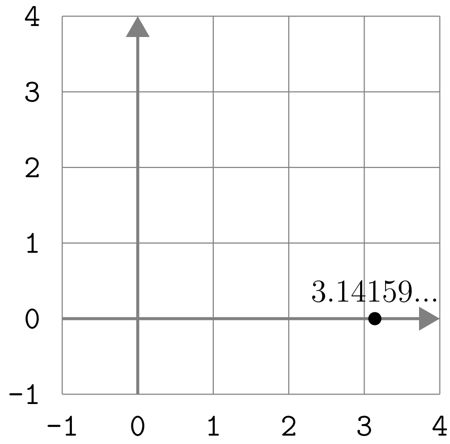
First you can draw complex numbers on a 2-D plane called the complex plane:



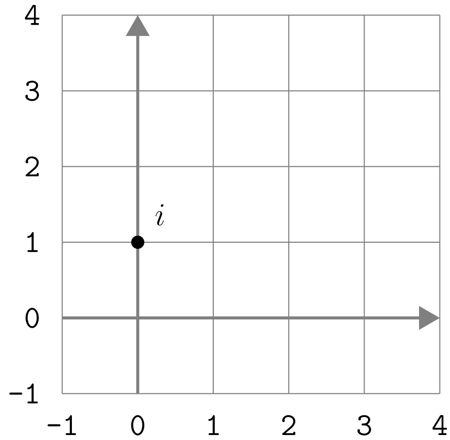
For instance here's $1 + 2i$:



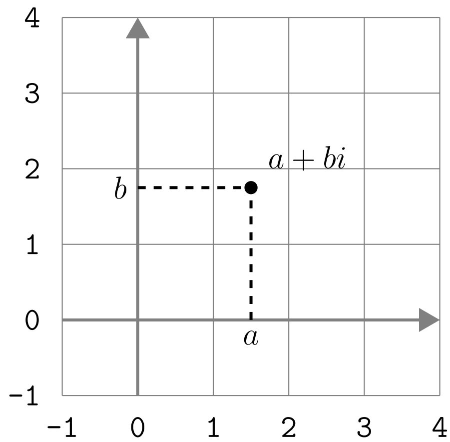
Note in particular that real numbers are entirely on the real axis. For instance here's $\pi = 3.14159...$ approximately:



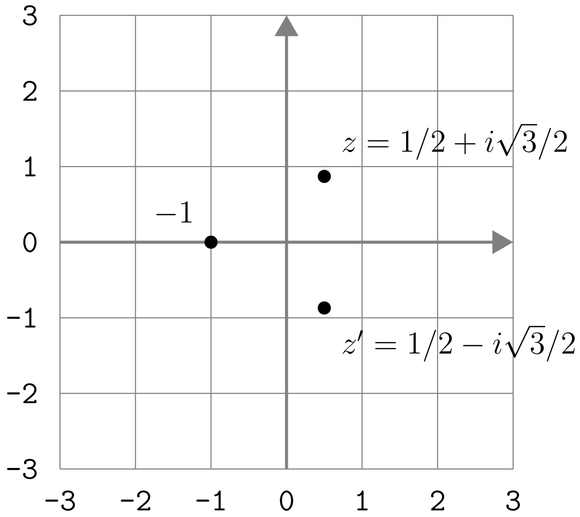
Here's i:



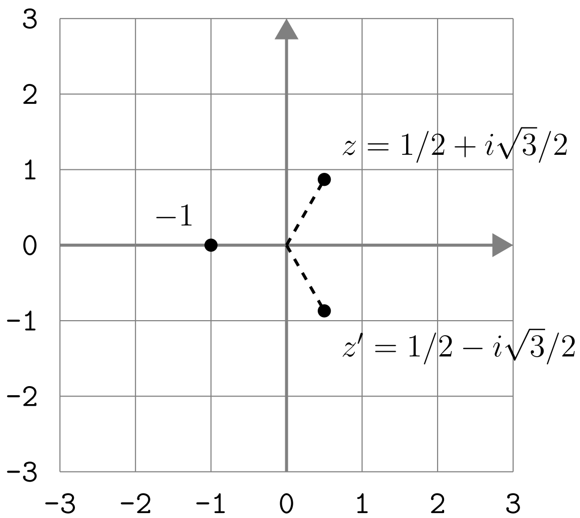
In general $a + bi$ is at the point (a, b) of the 2-D plane.



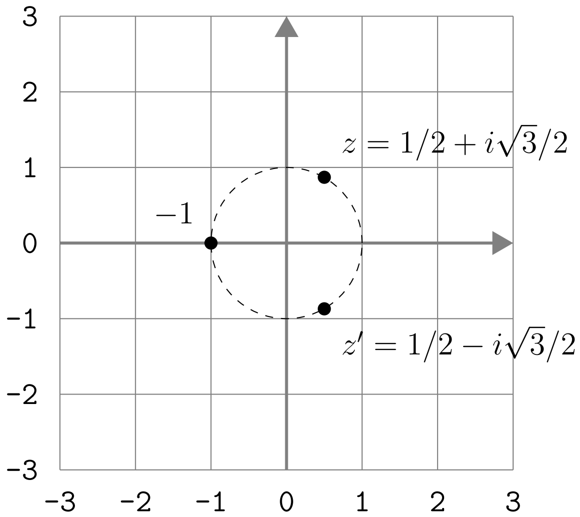
Now I'm going to put our solutions of $x^3 + 1 = 0$ into the complex plane:



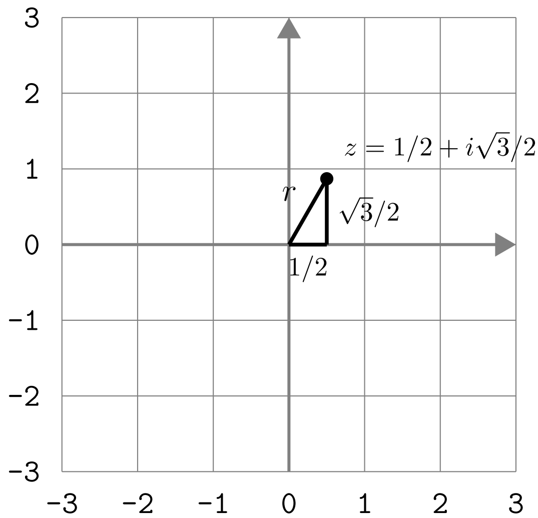
Now I'm going to draw lines from 0 to the roots:



There's a very high level of symmetric here!!! The real axis is like a mirror, reflecting z to get z' and vice versa. It's not just something curious about z and z'. In fact there's something about *all* three solutions: they are all on a circle of radius 1 about center 0!



For instance to show that $z = 1/2 + \sqrt{3}/2i$ is on a circle of radius 1, you only need to show that the distance from 0 to z in the above picture is 1. For that we can use Pythagorus theorem on this triangle to compute the distance from 0 to z which I'm marking as r:

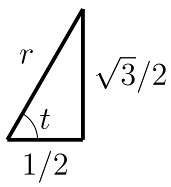


With the help of Mr Pythagorus we have:

$$r = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$$

Exercise 912.10.4. Show that z' is also on the circle of radius 1 and center 0.

Furthermore, the three lines from 0 to the roots divide up the circle into three equal parts. In other words the angles between the three lines are exactly $360/3 = 120$ degrees. Why? Let's look at the above picture again. The angle z makes with the real axis is marked t:



From trigo you know that

$$\tan t = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$$

and since this angle is acute (between 0 and 90 degrees) we see that

$$t = 60^\circ$$

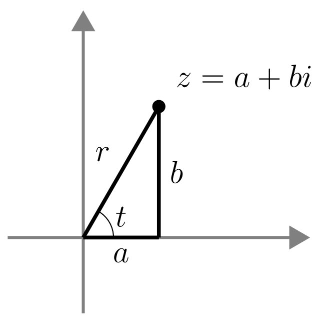
which in radians is

$$t = 60 \cdot \frac{\pi}{180} = \frac{\pi}{3}$$

I'll let you check that all the three angles are indeed the same (in fact they are 120 degrees).

I won't talk about solving the general polynomials for roots. (In fact it's a difficult subject.) But there's something that's very useful about describing complex numbers in terms of it's distance from 0 and the angle it makes with the positive real axis.

Suppose you have a complex number z and you draw it on the complex plane and you've already computed it's distance from 0, which I will write as r, and the angle it makes with the positive real axis, which I will call t:



then from trigo you know that, if $z = a + bi$,

$$a = r\cos t$$
$$b = r\sin t$$

In other words

$$z = (r\cos t) + i(\sin t)$$
$$= r(\cos t + i\sin t)$$

The complex number

$$\cos t + i\sin t$$

has length 1 from 0. When written this way

$$z = (r\cos t) + i(\sin t)$$

we say that z is written in **polar coordinates**. When we write z as

$$z = a + bi$$

we say that z is written in **rectangle coordinates**.

Proposition 912.10.1. *If $z = a + bi$, then*

$$z = r(\cos t + i\sin t)$$

where

$$r = \sqrt{x^2 + y^2}$$
$$\tan t = \frac{b}{a}$$

□

So what?

There's something special about taking powers of a complex number written in polar coordinates. First of all, focusing only on the

$$\cos t + i\sin t$$

de Moivre's theorem says that

Theorem 912.10.1. (**de Moivre's theorem**)

$$(\cos t + i \sin t)^n = \cos(nt) + i \sin(nt)$$

When applied to z it means that we have a very fast method for computing powers:

$$\begin{aligned} z^n &= (r(\cos t + i \sin t))^n \\ &= r^n(\cos(nt) + i \sin(nt)) \end{aligned}$$

Exercise 912.10.5. Compute $(1 + \sqrt{3}i)^{100}$ in two different ways:

(a) Use binomial theorem and then simplify.
(b) Find r and t such that $z = r(\cos t + i \sin t)$. Compute z^{100} using de Moivre's theorem.

□

Here's another thing you can do with de Moivre's theorem. Remember those pesky trigo identities involving multiple angles? For instance here are two double angle formulas:

$$\begin{aligned} \cos 2x &= \cos^2 x - \sin^2 x \\ \sin(2x) &= 2 \sin x \cos x \end{aligned}$$

Here's one way of remember it. By de Moivre's theorem

$$(\cos x + i \sin x)^2 = \cos(2x) + i \sin(2x)$$

But by the binomial theorem

$$\begin{aligned} (\cos x + i \sin x)^2 &= (\cos x)^2 + 2(\cos x)(i \sin x) + (i \sin x)^2 \\ &= \cos^2 x + 2(\cos x)(i \sin x) + - \sin^2 x \\ &= \cos^2 x + 2(\cos x)(i \sin x) - \sin^2 x \\ &= (\cos^2 x - \sin^2 x) + 2i \cos x \sin x \end{aligned}$$

This means that

$$\cos(2x) + i \sin(2x) = (\cos^2 x - \sin^2 x) + 2i \cos x \sin x$$

Now on comparing the real and complex part of the equation you get

$$\begin{aligned}\cos 2x &= \cos^2 x - \sin^2 x\\ \sin 2x &= 2\sin x \cos x\end{aligned}$$

Tada! All you need it binomial theorem and de Moivre's theorem to remember the double angle formula for sine and cosine.

Of course if you just want the formula for $\sin 2x$ you can just ignore the real part of the equation. You can even do this quickly in your head:

$$\begin{aligned}(...) + i \sin 2x &= (\cos x + i \sin x)^2\\ &= (...) + 2\cos x i \sin x\end{aligned}$$

i.e.

$$\sin 2x = 2\sin x \cos x$$

Now let me derive the formulas for $\cos 3x$ again using binomial theorem and de Moivre's theorem:

$$\begin{aligned}\cos 3x + i \sin 3x &= (\cos x + i \sin x)^3\\ &= (\cos x + i \sin x)^3\\ &= \cos^3 x + (...) + 3\cos^1 x (i \sin x)^2 + (...)\end{aligned}$$

(ignore the complex part – do you see why the second and fourth terms are complex?) i.e.

$$\cos 3x = \cos^3 x - 3\cos x \sin^2 x$$

now use Pythagorus' theorem to convert $\sin^2 x$ to $\cos x$

$$\sin^2 x + \cos^2 x = 1$$

and you'll get

$$\begin{aligned}\cos 3x &= \cos^3 x - 3\cos x(1 - \cos^2 x)\\ &= 4\cos^3 x - 3\cos x\end{aligned}$$

Exercise 912.10.6. Derive $\sin 3x$ in terms of $\sin x$. □

I can even do $\cos 4x$ without too much pain:

$$\begin{aligned}
\cos 4x &= (\cos x)^4 + 6(\cos x)^2(i\sin x)^2 + (i\sin x)^4 \\
&= \cos^4 x - 6\cos^2 x\sin^2 x + \sin^4 x \\
&= \cos^4 x - 6\cos^2 x(1-\cos^2 x) + (1-\cos^2 x)^2 \\
&= 7\cos^4 x - 6\cos^2 x + (1 - 2\cos^2 x + \cos^4 x) \\
&= 8\cos^4 x - 8\cos^2 x + 1
\end{aligned}$$

Exercise 912.10.7. Your turn ... derive $\sin 4x$ in terms of $\sin x$.

Exercise 912.10.8. Factorize $x^2 = 1$ using de Moivre's theorem. (Of course you know that the roots are $1, -1$. Just pretend you don't. Let $z = r(\cos t + i\sin t)$ be a complex number satisfying $x^2 = 1$. Use de Moivre's theorem and solve for r and t.) □

Exercise 912.10.9. Factorize $x^2 = -1$ using de Moivre's theorem. □

Exercise 912.10.10. Factorize $x^3 = 1$ using de Moivre's theorem, i.e., find all the roots of $x^3 = 1$. □

Exercise 912.10.11. Factorize $x^3 = 4$. (Use previous exercise.) □

Exercise 912.10.12. Factorize $x^4 = 1$. □

Exercise 912.10.13. Factorize $x^n = 1$ where n is a positive integer. □

Solutions

912.11 Exponential Generating Functions egf.tex

If you look at all the above examples involving generating functions, you notice that they all involve selecting objects. What if you're interested in selecting *and* permuting objects – i.e. order is important.

Now if a_n is the number of ways to select n objects from a pool under some constraint, why do we use $\sum_{n=0}^{\infty} a_n x^n$ as the generating function? Why not $\sum_{n=0}^{\infty} a_n \sin(nx)$?

Because if b_n is the number of ways to select from another set then

$$\sum_{n=0}^{\infty} a_n x^n \cdot \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} (a_0 b_n + \cdot + a_n b_0) x^n$$

gives the generating function for selecting n objects from both sets.

Now let's go back to permutations.

Suppose a_n is the number of ways to select n objects (under some constraint). Then the number of ways to select n objects and to permute the objects is

$$n! a_n$$

(right?) The function $\sum_{n=0}^{\infty} a_n x^n$ can be written as

$$\sum_{n=0}^{\infty} (n! a_n) \frac{x^n}{n!}$$

In general, if $b_0, b_1, b_2, \cdots$ is a sequence of numbers, then the **exponential generating function** of $b_0, b_1, b_2, \ldots$ is

$$\sum_{n=0}^{\infty} b_n \frac{x^n}{n!}$$

(More generally, if you have a bunch of functions written $K_n(x)$, you can form generating functions $\sum_{n=0}^{\infty} a_n K_n(x)$. Up to this point we've considered $K_n(x) = x^n$ for ordinary generating functions and $K_n(x) = x^n/n!$ for exponential generating functions.)

Does multiplication work for exponential generating functions? Let $a = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$ and $b = \sum_{n=0}^{\infty} b_n \frac{x^n}{n!}$ be exponential generating functions where a_n is the number of ways of selecting n objects from the a first set and permutating them and b_n is the number of ways of selecting and permutating n objects from a second pool. Consider the product of the i–th term from a and j–th term from b, i.e. the product

$$a_i \frac{x^i}{i!} \cdot b_j \frac{x^j}{j!}$$

We have

$$a_i \frac{x^i}{i!} \cdot b_j \frac{x^j}{j!} = \frac{a_i}{i!} \cdot \frac{b_j}{j!} \cdot x^{i+j}$$

Not very helpful! But if I rewrite the above like this:

$$a_i \frac{x^i}{i!} \cdot b_j \frac{x^j}{j!} = a_i \cdot b_j \cdot \frac{(i+j)!}{i!j!} \cdot \frac{x^{i+j}}{(i+j)!}$$

you see that the coefficient in front of $x^{i+j}/(i+j)!$ is

$$\frac{a_i}{i!} \cdot \frac{b_j}{j!} \cdot (i+j)!$$

which is the number of ways of selecting i objects from the first pool, followed by the number of ways of selecting j objects from the second pool, followed by permutating $i+j$ objects. (if a_i is the number of ways to select i objects and permutating them, then $a_i/i!$ is the number of ways to select i objects – see earlier remarks.)

Therefore, on multiplying the two series, we have

$$\begin{aligned}
a \cdot b &= \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} \cdot \sum_{n=0}^{\infty} b_n \frac{x^n}{n!} \\
&= \sum_{n=0}^{\infty} \Bigg(\frac{a_0 b_n}{0!n!} (0+n)! \frac{x^{0+n}}{(0+n)!} \\
&\qquad + \frac{a_1 b_{n-1}}{1!(n-1)!} (1+(n-1))! \frac{x^{1+(n-1)}}{(1+(n-1))!} \\
&\qquad + \cdots \\
&\qquad + \frac{a_n b_0}{n!0!} (n+0)! \frac{x^{n+0}}{(n+0)!} \Bigg) \\
&= \sum_{n=0}^{\infty} \left(\frac{a_0 b_n}{0!n!} + \frac{a_1 b_{n-1}}{1!(n-1)!} + \cdots + \frac{a_n b_0}{n!0!} \right) n! \cdot \frac{x^n}{n!}
\end{aligned}$$

The coefficient of $x^n/n!$ is the number of ways of 0 object from the first pool and n objects from the second pool and permuting them, or 1 object from the first pool and $n-1$ objects from the second pool and permuting them, or ... which is the same as the number of ways of selecting n objects from the first and second pool and permuting them.

Note that while the "basic" (ordinary) generating function for $1, 1, 1, \ldots$ is

$$\sum_{n=0} 1 \cdot x^n = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

the exponential generating function for $1, 1, 1, \ldots$ is

$$\sum_{n=0} 1 \cdot \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

Therefore, whereas mathematical tools/formulas for rational functions are crucial in order to work effectively with ordinary generating functions, for the case of exponential functions, we must know how to handle exponential functions well.

[facts on and related to e^x, etc.]

Example 912.11.1. How many ways are there to arrange 4 letters in ENGINE?

Now the word “arrange” means that order does mattter. So don’t use ordinary generating functions!

First let’s think about the exponential generating function for selecting and Es.

$$\begin{aligned}
\text{number of ways to select 0 E's and permute them} &= 1 \\
\text{number of ways to select 1 E's and permute them} &= 1 \\
\text{number of ways to select 2 E's and permute them} &= 1 \\
\text{number of ways to select 3 E's and permute them} &= 0 \\
\text{number of ways to select 4 E's and permute them} &= 0 \\
&\dots
\end{aligned}$$

Therefore the exponential generating function, say E, is

$$E = 1\frac{x^0}{0!} + 1\frac{x^1}{1!} + 1\frac{x^2}{2!} + 0\frac{x^3}{3!} + 0\frac{x^3}{4!} + \cdots = 1 + x + \frac{x^2}{2!}$$

Write N, G, and I for the exponential generating function for selecting and permuting N’s, G’s, and I’s respectively. Then

$$\begin{aligned}
N &= 1 + x + \frac{x^2}{2!} \\
G &= 1 + x \\
I &= 1 + x
\end{aligned}$$

Altogether the required number must be coefficient of $x^4/4!$ of $ENGI$. Let’s compute that:

$$\begin{aligned}
ENGI &= \left(1 + x + \frac{x^2}{2!}\right)^2 (1 + x) \\
&= 1 + x + 4x + 7x^2 + 7x^3 + \frac{17}{4}x^4 + \frac{3}{2}x^5 + \frac{1}{4}x^6
\end{aligned}$$

Now watchout! The coefficient of x^4 is $17/4$. The coefficient we want is the coefficient of $x^4/4!$. The term containing x^4 is

$$\frac{17}{4}x^4 = \frac{17}{4}4! \cdot \frac{x^4}{4!}$$

Hence the required number is

$$\frac{17}{4}4!$$

or $17 \cdot 3!$.

Example 912.11.2. How many arrangements of 12 symbols are there out of 12 R's, 12 W's, 12 B's, 12 C's. if

(a) We have an even number of B's, odd number of C's.

(b) We have at least 3 W's or no W's

(a) The exponential generating function is

$$f(x) = \left(1 + x + \frac{x^2}{2!} + \cdots\right)^2 \cdot \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} \cdots\right) \cdot \left(x + \frac{x^3}{3!} + \frac{x^5}{5!} \cdots\right)^2$$

Recall that

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Now watch the following tricks. First of all from the above power series for e^x, on replacing x with $-x$ we get

$$e^{-x} == 1 - x + \frac{x^2}{2!} - \cdots = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}$$

On adding the power series of e^x and e^{-x} we get

$$\begin{aligned} e^x + e^{-x} &= 2 + 2\frac{x^2}{2!} + 2\frac{x^4}{4!} + 2\frac{x^6}{6!} + \cdots \\ &= 2\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots\right) \\ &= 2\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \end{aligned}$$

and therefore

$$\frac{e^x + e^{-x}}{2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

If we subtract

$$
\begin{aligned}
e^x - e^{-x} &= 2\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots\right) \\
&= 2\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}
\end{aligned}
$$

Going back to our problem the exponential generating function becomes

$$
f(x) = (e^x)^2 \cdot \frac{e^x + e^{-x}}{2} \cdot \frac{e^x - e^{-x}}{2}
$$

The point of changing the form of $f(x)$ is that we can now work with a finite number of terms in the products. We now simplify our $f(x)$:

$$
\begin{aligned}
f(x) &= (e^x)^2 \cdot \frac{e^x + e^{-x}}{2} \cdot \frac{e^x - e^{-x}}{2} \\
&= \frac{1}{4} e^{2x}(e^x + e^{-x})(e^x - e^{-x}) \\
&= \frac{1}{4} e^{2x}(e^x e^x - e^x e^{-x} + e^{-x} e^x - e^{-x} e^{-x}) \\
&= \frac{1}{4} e^{2x}(e^{2x} - e^{-2x}) \\
&= \frac{1}{4}(e^{4x} - 1)
\end{aligned}
$$

And now we rewrite this as a power series:

$$
\begin{aligned}
f(x) &= \frac{1}{4}(e^{4x} - 1) \\
&= \frac{1}{4}\left(\sum_{n=0}^{\infty} \frac{(4x)^n}{n!} - 1\right) \\
&= \frac{1}{4}\left(\sum_{n=0}^{\infty} 4^n \frac{x^n}{n!} - 1\right) \\
&= \frac{1}{4}\sum_{n=1}^{\infty} 4^n \frac{x^n}{n!} \\
&= \sum_{n=1}^{\infty} 4^{n-1} \frac{x^n}{n!}
\end{aligned}
$$

The required number is the coefficient of $fracx^{12}12!$ which is

$$4^{12-1} = 4^{11}$$

(b) The exponential generating function

$$g(x) = e^x\left(1 + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)e^x e^x \qquad = e^{3x}\left(1 + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)$$

The troublesome factor is this:

$$1 + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

Well ... it's not as bad as it seems. This is almost like e^x, right? There is a finite number of terms missing. So ... we just repair it!

$$\begin{aligned}1 + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots &= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) - x - \frac{x^2}{2!} \\ &= e^x - x - \frac{x^2}{2}\end{aligned}$$

AHA! A finite of terms! Going back to our $g(x)$:

$$\begin{aligned}g(x) &= e^{3x}\left(e^x - x - \frac{x^2}{2}\right) \\ &= e^{4x} - xe^{3x} - \frac{x^2}{2}e^{3x}\end{aligned}$$

Now we expand this into a power series:

$$\begin{aligned}g(x) &= \sum_{n=0}^{\infty} \frac{(4x)^n}{n!} - x\sum_{n=0}^{\infty} \frac{(3x)^n}{n!} - \frac{x^2}{2}\sum_{n=0}^{\infty} \frac{(3x)^n}{n!} \\ &= \sum_{n=0}^{\infty} 4^n \frac{x^n}{n!} - x\sum_{n=0}^{\infty} 3^n \frac{x^n}{n!} - \frac{x^2}{2}\sum_{n=0}^{\infty} 3^n \frac{x^n}{n!} \\ &= \sum_{n=0}^{\infty} 4^n \frac{x^n}{n!} - \sum_{n=0}^{\infty} 3^n \frac{x^{n+1}}{n!} - \sum_{n=0}^{\infty} \frac{3^n}{2} \frac{x^{n+2}}{n!}\end{aligned}$$

Remember that we want a power series in $x^n/n!$ and not x^n!!!

$$\begin{aligned}
g(x) &= \sum_{n=0}^{\infty} 4^n \frac{x^n}{n!} - \sum_{n=0}^{\infty} 3^n (n+1) \frac{x^{n+1}}{(n+1)!} - \sum_{n=0}^{\infty} \frac{3^n}{2} (n+1)(n+2) \frac{x^{n+2}}{(n+2)!} \\
&= \sum_{n=0}^{\infty} 4^n \frac{x^n}{n!} - \sum_{n=1}^{\infty} 3^{n-1} n \frac{x^n}{n!} - \sum_{n=2}^{\infty} \frac{3^{n-2}}{2} (n-1) n \frac{x^n}{n!} \\
&= 4^0 + 4^1 x + \sum_{n=2}^{\infty} 4^n \frac{x^n}{n!} \\
&\qquad - 3^0 x - \sum_{n=2}^{\infty} 3^{n-1} n \frac{x^n}{n!} \\
&\qquad\qquad - \sum_{n=2}^{\infty} \frac{3^{n-2}}{2} (n-1) n \frac{x^n}{n!} \\
&= 1 + 3x + \sum_{n=2}^{\infty} \left(4^n - 3^{n-1} n - \frac{3^{n-2}(n-1)n}{2} \right) \frac{x^n}{n!}
\end{aligned}$$

The required number is the coefficient of $x^{12}/12!$ which is

$$4^{12} - 3^{11} 12 - \frac{3^{10}(11)12}{2}$$

The process of computing the relevnat coefficient is similar to the technique used in the "usual" generating function:

- First write down the power series
- Simplify possibly using e^x and its variants
- Get the relevant coefficient of $x^n/n!$ either directly from the power series of maybe by doing some manipulation.

The basic "trick" (or formulas):

- The most important power series for exponential generating function is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

  From that you immediately get this

$$e^{ax} = \sum_{n=0}^{\infty} \frac{(ax)^n}{n!} = \sum_{n=0} a^n \frac{x^n}{n!}$$

In particular you get

$$e^{2x} = \sum_{n=0}^{\infty} 2^n \frac{x^n}{n!}$$

and the following that looks like e^x except for the alternating signs:

$$e^{-x} = \sum_{n=0} (-1)^n \frac{x^n}{n!} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots$$

- You frequently need this basic formula:

$$a^{x+y} = a^x a^y$$

For instance

$$e^{x+x} = e^x e^x$$

- When a power series is like e^x except for a finite number of missing terms, you repair. For instance look at this:

$$f(x) = 2 + 3x^3 + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots$$

which is of course the same as this (using the summation notation):

$$f(x) = 2 + 3x^3 + \sum_{n=4}^{\infty} \frac{x^n}{n!}$$

It's almost e^x. So we repair it like this:

$$\begin{aligned}
f(x) &= 2 + 3x^3 + \sum_{n=4}^{\infty} \frac{x^n}{n!} \\
&= 2 + 3x^3 + \sum_{n=0}^{\infty} \frac{x^n}{n!} - \sum_{n=0}^{3} \frac{x^n}{n!} \\
&= 2 + 3x^3 + \sum_{n=0}^{\infty} \frac{x^n}{n!} - \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}\right) \\
&= 1 - x - \frac{x^2}{2} + \frac{17}{6}x^3 + e^x
\end{aligned}$$

- When the power series looks like e^x except that it includes only odd powers of x or it is includes only even powers of x, then you should look at both e^x and x^{-x}.

Example 912.11.3 (Grimaldi p439 Example 9.29)**.** 11 new employees are to be assigned to 4 subdivisions. Each subdivision gets at least one new employee. How many possible assignments are there?

This is the same as the number of arrangements of 11 symbols taken from A's, B's, C's, D's where each arrangement has at least one A, one B, one C, and one D.

The exponential generating function is

$$\begin{aligned}
f(x) &= \left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)^4 \\
&= \sum_{n=1}^{\infty} \frac{x^n}{n!} \\
&= \left(\sum_{n=0}^{\infty} \frac{x^n}{n!} - 1\right)^4 \\
&= (e^x - 1)^4 \\
&= (e^x)^4 - 4(e^x)^3 + 6(e^x)^2 - 4(e^x) + 1 \\
&= e^{4x} - 4e^{3x} + 6e^{2x} - 4e^x + 1 \\
&= \sum_{n=0}^{\infty} \frac{(4x)^n}{n!} - 4\sum_{n=0}^{\infty} \frac{(3x)^n}{n!} + 6\sum_{n=0}^{\infty} \frac{(2x)^n}{n!} - 4\sum_{n=0}^{\infty} \frac{x^n}{n!} + 1 \\
&= \sum_{n=0}^{\infty} 4^n \frac{x^n}{n!} - 4\sum_{n=0}^{\infty} 3^n \frac{x^n}{n!} + 6\sum_{n=0}^{\infty} 2^n \frac{x^n}{n!} - 4\sum_{n=0}^{\infty} \frac{x^n}{n!} + 1 \\
&= \sum_{n=0}^{\infty} \left(4^n - 4 \cdot 3^n + 6 \cdot 2^n - 4\right) \frac{x^n}{n!} + 1
\end{aligned}$$

The required number is the coefficient of $x^{11}/11!$ which is

$$4^{11} - 4 \cdot 3^{11} + 6 \cdot 2^{11} - 4$$

Note that the problem is the same as counting the number of functions f

```
employee1                A
employee2        f
employee3      ----->    B
.
```

```
.                          C
.
employee11                 D
```

such that f is onto (... "at least one A, one B, one C, one D").

Of course you already know that such problems can be solved using the inclusion-exclusion principle.

Exercise 912.11.1. Solve the above problem where "at least one" is replaced by "at least two". [Try the same with inclusion-exclusion principle.]

Exercise 912.11.2. Solve the above problem where "at least one for each subdivision" is replaced by "at least one for the first and second subdivision and at least two for the third and fourth subdivision". [Try the same with inclusion-exclusion principle.]

Solutions

912.12 More non-linear recurrences non-linear.tex

The next recurrence is different from ones you have seen earlier ... do you see the *main* difference?

Example 912.12.1. Consider this:

$$a_n = \frac{1}{n} a_{n-1} + n$$

for $n \geq 1$ and $a_0 = 10$. Find a closed form for a_n.

Quick pause ... previous recurrences look like, for instance, the following:

$$a_n = 3a_{n-1} + 4a_{n-2}$$

which is linear and homogeneous, or this

$$a_n = -2a_{n-1} + 10a_{n-2} + n^2 + 1$$

which is linear and nonhomogeneous, or

$$a_n = na_{n-1} + 4a_{n-2} + n^2 + 1$$

which is nonlinear and nonhomogeneous. In the above example, the recurrence is nonlinear and nonhomogenous, but the multiplier in front of a_{n-1} is $1/n$. In previous nonlinear cases, the multiplier in front of a_{n-1} looks like n or $(n+1)$ or $(n^2 + 3n + 5)$, etc. – they are polynomials in n.

Before find the closed form, of course you must do some experiments to get some data for later testing. Here are some test data that I'll use later:

$$\begin{aligned}
a_0 &= 10 \\
a_1 &= 11 \\
a_2 &= \frac{1}{2} \cdot 11 + 2 = \frac{15}{2} \\
a_3 &= \frac{1}{3} \cdot \frac{15}{2} + 3 = \frac{33}{6} \\
a_4 &= \frac{1}{4} \cdot \frac{33}{6} + 4 = \frac{129}{24}
\end{aligned}$$

Let

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$\begin{aligned}
f(x) &= a_0 + \sum_{n=1}^{\infty} a_n x^n \\
&= a_0 + \sum_{n=1}^{\infty} \left(\frac{1}{n} a_{n-1} + n \right) x^n \\
&= a_0 + \sum_{n=1}^{\infty} \frac{1}{n} a_{n-1} x^n + \sum_{n=1}^{\infty} n x^n \\
&= a_0 + \sum_{n=1}^{\infty} a_{n-1} \cdot \frac{1}{n} x^n + x \sum_{n=1}^{\infty} n x^{n-1} \\
&= a_0 + \sum_{n=1}^{\infty} a_{n-1} \cdot \int_0^x t^{n-1} \, dt + x \sum_{n=1}^{\infty} \frac{d}{dx} x^n \\
&= a_0 + \int_0^x \left(\sum_{n=1}^{\infty} a_{n-1} t^{n-1} \right) \, dt + x \frac{d}{dx} \sum_{n=1}^{\infty} x^n \\
&= a_0 + \int_0^x f(t) \, dt + x \frac{d}{dx} \left(\sum_{n=0}^{\infty} x^n - 1 \right) \\
&= a_0 + \int_0^x f(t) \, dt + x \frac{d}{dx} \left(\frac{1}{1-x} - 1 \right) \\
&= a_0 + \int_0^x f(t) \, dt + \frac{x}{(1-x)^2}
\end{aligned}$$

Study the above very carefully. In particular, why did I introduce the integral? Where does the *idea* come from? If you really understood the stuff I talked about when we find closed forms earlier, you would really know why I made the above attack on the problem. If you still don't see the intuition/idea: pause and compute the closed form for this:

$$b_n = n b_{n-1} + n$$

for $n \geq 1$ and $b_0 = 10$ using ideas from previous sections. OK ... move on ...

At this point we have

$$f(x) - \int_0^x f(t)\ dt = a_0 + \frac{x}{(1-x)^2}$$

This is an integral equation – it contains the unknown $f(x)$ and an integral of $f(x)$. I want to convert this to a differential equation. Taking the derivative and replacing $f(x)$ by y (to make it more recognizable):

$$y' - y = \frac{d}{dx}\left(a_0 + \frac{x}{(1-x)^2}\right) = \frac{(1-x)^2 - x \cdot 2(1-x)(-1)}{(1-x)^4} = \frac{1+x}{(1-x)^3}$$

I get this horrific looking thingy:

$$y' - y = \frac{1+x}{(1-x)^3}$$

Well ... don't shout and scream ... it's actually not that bad because this is a Bernoulli differential equation which is documented in all standard textbooks on differential equations and you can also find it on the web. The solution to the above differential equation is of the form

$$y = e^x \left(\int e^{-x} \frac{1+x}{(1-x)^3}\ dx + C\right)$$

where C is a constant. OK ... so we now need to compute

$$\int e^{-x} \frac{1+x}{(1-x)^3}\ dx$$

The next thing to do is of course to check a table of integrals. Unfortunately you (probably) won't find the above in standard integral tables.

Wait ... in fact note that the original differential equation looks like this:

$$y' - y = g'(x)$$

where $g(x) = a_0 + x/(1-x)^2$; i.e., the right-hand-side of the differential equation is not just any function, it's a *derivative*. Why is that important? Because the differential equation looks like this

$$y = e^x \left(\int e^{-x} g'(x)\ dx + C\right)$$

and, *using integration by parts*,

$$\begin{aligned}\int e^{-x} g'(x)\ dx &= e^{-x} g(x) - \int g(x)(-e^{-x})\ dx \\ &= e^{-x} g(x) + \int e^{-x} g(x)\ dx\end{aligned}$$

which in our case is

$$\begin{aligned}\int e^{-x} g'(x)\ dx &= e^{-x}\left(a_0 + \frac{x}{(1-x)^2}\right) + \int e^{-x}\left(a_0 + \frac{x}{(1-x)^2}\right)\ dx \\ &= e^{-x}\left(a_0 + \frac{x}{(1-x)^2}\right) + a_0(-e^{-x}) + \int e^{-x}\frac{x}{(1-x)^2}\ dx \\ &= e^{-x}\frac{x}{(1-x)^2} + \int e^{-x}\frac{x}{(1-x)^2}\ dx\end{aligned}$$

Now why is this a good thing? Because we've moved from computing this:

$$\int e^{-x}\frac{1+x}{(1-x)^3}\ dx$$

to computing this:

$$\int e^{-x}\frac{x}{(1-x)^2}\ dx$$

I don't know about you, but the second integral definitely looks easier to me.

Now, again I don't see

$$\int e^{-x}\frac{x}{(1-x)^2}\ dx$$

in standard integration tables. But I notice that the derivative of $\frac{1}{1-x}$ is $\frac{1}{(1-x)^2}$. (Do you know why I do not try to think of "the derivative of $-e^{-x}$ is e^{-x}"?)

So I'm going to try integration by parts again:

$$\begin{aligned}
\int e^{-x} \frac{x}{(1-x)^2} \, dx &= \int xe^{-x} \cdot \frac{1}{(1-x)^2} \, dx \\
&= \int xe^{-x} \cdot d\left(\frac{1}{1-x}\right) \\
&= xe^{-x} \cdot \frac{1}{1-x} - \int \frac{1}{1-x} \cdot d\left(xe^{-x}\right) \\
&= xe^{-x} \cdot \frac{1}{1-x} - \int \frac{1}{1-x} \cdot \left(e^{-x} - xe^{-x}\right) dx \\
&= xe^{-x} \cdot \frac{1}{1-x} - \int \frac{1}{1-x} \cdot e^{-x} \left(1-x\right) dx \\
&= xe^{-x} \cdot \frac{1}{1-x} - \int \frac{1}{1-x} \cdot e^{-x} \left(1-x\right) dx \\
&= xe^{-x} \cdot \frac{1}{1-x} - \int e^{-x} dx && \text{well ... what do you know ...} \\
&= xe^{-x} \cdot \frac{1}{1-x} + e^{-x} \\
&= e^{-x} \cdot \frac{x + (1-x)}{1-x} \\
&= e^{-x} \frac{1}{1-x}
\end{aligned}$$

Putting the above together I get this:

$$\begin{aligned}
\int e^{-x} g'(x) \, dx &= e^{-x} \frac{x}{(1-x)^2} + e^{-x} \frac{1}{1-x} \\
&= e^{-x} \frac{x + (1-x)}{(1-x)^2} \\
&= e^{-x} \frac{1}{(1-x)^2}
\end{aligned}$$

Let's check:

$$\begin{aligned}
\frac{d}{dx}\left(\frac{e^{-x}}{(x-1)^2}\right) &= \frac{(-e^{-x}(1-x)^2 - e^{-x}\cdot 2(1-x)(-1)}{(1-x)^4} \\
&= e^{-x}\frac{-(1-x)+2}{(1-x)^4} \\
&= e^{-x}\frac{1+x}{(1-x)^4} \\
&= e^{-x}g'(x)
\end{aligned}$$

Looks like everything is good. Therefore

$$\begin{aligned}
f(x) = y &= e^x\left(\int e^{-x}g'(x)\ dx + C\right) \\
&= e^x\left(e^{-x}\frac{1}{(1-x)^2} + C\right) \\
&= \frac{1}{(1-x)^2} + Ce^x
\end{aligned}$$

Phew! Now ... where are we?!?

Recall that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and the goal is to find a closed form for a_n. First we compute C.

$$10 = a_0 = f(0) = \frac{1}{(1-0)^2} + Ce^0 = 1 + C$$

i.e., $C = 9$. Therefore

$$f(x) = \frac{1}{(1-x)^2} + 9e^x$$

Since we have well–known power series for $1/(1-x)^2$ and e^x, we're now in the final lap! Onward!!!

$$\begin{aligned} f(x) &= \frac{1}{(1-x)^2} + 9e^x \\ &= \sum_{n=0}^{\infty} \binom{2+n-1}{n} x^n + 9 \sum_{n=0}^{\infty} \frac{1}{n!} x^n \\ &= \sum_{n=0}^{\infty} \binom{n+1}{n} x^n + \sum_{n=0}^{\infty} 9 \frac{1}{n!} x^n \\ &= \sum_{n=0}^{\infty} \binom{n+1}{1} x^n + \sum_{n=0}^{\infty} \frac{9}{n!} x^n \\ &= \sum_{n=0}^{\infty} \left((n+1) + \frac{9}{n!} \right) x^n \end{aligned}$$

Hence

$$a_n = n + 1 + \frac{9}{n!}$$

for $n \geq 0$. Of course we have to check this closed form:

$$\begin{aligned} a_0 &= 0 + 1 + \frac{9}{1} = 10 \\ a_1 &= 1 + 1 + \frac{9}{1} = 11 \\ a_2 &= 2 + 1 + \frac{9}{2} = \frac{15}{2} \\ a_3 &= 3 + 1 + \frac{9}{6} = \frac{33}{6} \\ a_4 &= 4 + 1 + \frac{9}{24} = \frac{129}{24} \end{aligned}$$

and they all agree with the values computed using the recursive form of a_n. TADA!

I'm going to check the closed form for $n = 0, ..., 17$:

```
def f(n):
    if n == 0: return 1
    else: return n * f(n - 1)

def b(n):
    return n + 1 + 9.0 / f(n)

def a(n):
    if n == 0: return 10
```

```
    else: return a(n - 1) / float(n) + n

for n in range(18):
    print('%.30f' % a(n), '%.30f' % b(n), a(n) == b(n), a(n) - b(n))
```

Here's the output:

```
10.000000000000000000000000000000 10.000000000000000000000000000000 True 0.0
11.000000000000000000000000000000 11.000000000000000000000000000000 True 0.0
7.500000000000000000000000000000 7.500000000000000000000000000000 True 0.0
5.500000000000000000000000000000 5.500000000000000000000000000000 True 0.0
5.375000000000000000000000000000 5.375000000000000000000000000000 True 0.0
6.075000000000000177635683940025 6.075000000000000177635683940025 True 0.0
7.012500000000000177635683940025 7.012500000000000177635683940025 True 0.0
8.001785714285714945503968920093 8.001785714285714945503968920093 True 0.0
9.000223214285714590232601040043 9.000223214285714590232601040043 True 0.0
10.000024801587301226390991359949 10.000024801587301226390991359949 True 0.0
11.000002480158730833181834896095 11.000002480158730833181834896095 True 0.0
12.000000225468975045828301517759 12.000000225468975045828301517759 True 0.0
13.000000018789080513670342043042 13.000000018789080513670342043042 True 0.0
14.000000001445313202452780387830 14.000000001445313202452780387830 True 0.0
15.000000000103236530435424356256 15.000000000103236530435424356256 True 0.0
16.000000000006881606395836570300 16.000000000006881606395836570300 True 0.0
17.000000000000429878355134860612 17.000000000000429878355134860612 True 0.0
18.000000000000024868995751603507 18.000000000000024868995751603507 True 0.0
```

Yes!

Now I'm ready to present the solution ...

SOLUTION.

Let
$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$
We then have
$$\begin{aligned}
f(x) &= a_0 + \sum_{n=1}^{\infty} a_n x^n \\
&= a_0 + \sum_{n=1}^{\infty} \left(\frac{1}{n} a_{n-1} + n \right) x^n \\
&= a_0 + \sum_{n=1}^{\infty} \frac{1}{n} a_{n-1} x^n + \sum_{n=1}^{\infty} n x^n \\
&= a_0 + \sum_{n=1}^{\infty} a_{n-1} \cdot \frac{1}{n} x^n + x \sum_{n=1}^{\infty} n x^{n-1} \\
&= a_0 + \sum_{n=1}^{\infty} a_{n-1} \cdot \int_0^x t^{n-1} \, dt + x \sum_{n=1}^{\infty} \frac{d}{dx} x^n \\
&= a_0 + \int_0^x \left(\sum_{n=1}^{\infty} a_{n-1} t^{n-1} \right) \, dt + x \frac{d}{dx} \sum_{n=1}^{\infty} x^n \\
&= a_0 + \int_0^x f(t) \, dt + x \frac{d}{dx} \left(\sum_{n=0}^{\infty} x^n - 1 \right) \\
&= a_0 + \int_0^x f(t) \, dt + x \frac{d}{dx} \left(\frac{1}{1-x} - 1 \right) \\
&= a_0 + \int_0^x f(t) \, dt + \frac{x}{(1-x)^2}
\end{aligned}$$
Therefore
$$f(x) - \int_0^x f(t) \, dt = a_0 + \frac{x}{(1-x)^2}$$
Taking the derivative and replacing $f(x)$ by y, we obtain the following differential equation:
$$y' - y = g'(x)$$
where $g(x) = a_0 + x/(1-x)^2$. Viewing the above as a Bernoulli differential

equation, the solution to the above differential equation is of the form

$$y = e^{x}\left(\int e^{-x} g'(x)\ dx + C\right) \tag{1}$$

where C is a constant. Using integration by parts,

$$\begin{aligned}
\int e^{-x} g'(x)\ dx &= e^{-x} g(x) - \int g(x)(-e^{-x})\ dx \\
&= e^{-x} g(x) + \int e^{-x} g(x)\ dx \\
&= e^{-x}\left(a_0 + \frac{x}{(1-x)^2}\right) + \int e^{-x}\left(a_0 + \frac{x}{(1-x)^2}\right)\ dx \\
&= e^{-x}\left(a_0 + \frac{x}{(1-x)^2}\right) + a_0(-e^{-x}) + \int e^{-x} \frac{x}{(1-x)^2}\ dx \\
&= e^{-x} \frac{x}{(1-x)^2} + \int e^{-x} \frac{x}{(1-x)^2}\ dx
\end{aligned} \tag{2}$$

We now compute $\int e^{-x} \frac{x}{(1-x)^2}\ dx$ using integration by parts:

$$\begin{aligned}
\int e^{-x} \frac{x}{(1-x)^2}\ dx &= \int x e^{-x} \cdot \frac{1}{(1-x)^2}\ dx \\
&= \int x e^{-x} \cdot d\left(\frac{1}{1-x}\right) \\
&= x e^{-x} \cdot \frac{1}{1-x} - \int \frac{1}{1-x} \cdot d\left(x e^{-x}\right) \\
&= x e^{-x} \cdot \frac{1}{1-x} - \int \frac{1}{1-x} \cdot \left(e^{-x} - x e^{-x}\right) dx \\
&= x e^{-x} \cdot \frac{1}{1-x} - \int \frac{1}{1-x} \cdot e^{-x} \left(1-x\right) dx \\
&= x e^{-x} \cdot \frac{1}{1-x} - \int e^{-x} dx \\
&= x e^{-x} \cdot \frac{1}{1-x} + e^{-x} \\
&= e^{-x} \cdot \frac{x + (1-x)}{1-x} \\
&= e^{-x} \frac{1}{1-x}
\end{aligned} \tag{3}$$

Substituting (3) into (2), we obtain:

$$\begin{aligned}\int e^{-x}g'(x)\ dx &= e^{-x}\frac{x}{(1-x)^2} + e^{-x}\frac{1}{1-x} \\ &= e^{-x}\frac{x+(1-x)}{(1-x)^2} \\ &= e^{-x}\frac{1}{(1-x)^2} \qquad (4)\end{aligned}$$

Substituting (4) into (2), we have

$$\begin{aligned}f(x) = y &= e^x\left(\int e^{-x}g'(x)\ dx + C\right) \\ &= e^x\left(e^{-x}\frac{1}{(1-x)^2} + C\right) \\ &= \frac{1}{(1-x)^2} + Ce^x\end{aligned}$$

Using $a_0 = 10$, we compute C:

$$10 = a_0 = f(0) = \frac{1}{(1-0)^2} + Ce^0 = 1 + C$$

i.e., $C = 9$. Therefore

$$\begin{aligned}f(x) &= \frac{1}{(1-x)^2} + 9e^x \\ &= \sum_{n=0}^{\infty}\binom{2+n-1}{n}x^n + 9\sum_{n=0}^{\infty}\frac{1}{n!}x^n \\ &= \sum_{n=0}^{\infty}\binom{n+1}{n}x^n + \sum_{n=0}^{\infty}9\frac{1}{n!}x^n \\ &= \sum_{n=0}^{\infty}\binom{n+1}{1}x^n + \sum_{n=0}^{\infty}\frac{9}{n!}x^n \\ &= \sum_{n=0}^{\infty}\left((n+1) + \frac{9}{n!}\right)x^n\end{aligned}$$

Hence the following is a closed form for a_n:

$$a_n = n + 1 + \frac{9}{n!}$$

for $n \geq 0$. □

Exercise 912.12.1. Note that the above recurrence looks like this:

$$a_n = \frac{1}{n} a_{n-1} + n$$

In particular, the multiple in front of a_{n-1} is $\frac{1}{n}$ which is not constant (and therefore the recurrence is not linear) and more importantly it's not a polynomial ... that's the whole point of this section. The appearance of $\frac{1}{n}$ can appear in cases where there's an "average" computation. This happens, for instance, when a_n is the average runtime of quicksort. What do I mean by "average" here? Let's look at quicksort (see section on quicksort algorithms).

Suppose the array to be sorted is `x[0..(n-1)]`. The pivot can be at `x[0]` or it can be at `x[1]` or ... In any case the array is re–organized using the pivot. Specifically, the array is partitioned into two parts using the pivot so that the array is re-organized into "left-hand-side array, pivot, right-hand-side array". There are several cases. The left-hand-side array of values $\leq$ the pivot can have 0 elements. This is followed by the pivot. The right-hand-side array of values $>$ the pivot can have $n-1$ values. That's one case. For this case, the time taken (recursively) is

$$An + B + T(0) + T(n-1)$$

where $T(n)$ is the time taken to quicksort an array of size n. The $An + B$ is time to scan the array and place the values of the array into the structure of "left-hand side array, pivot, right-hand side array". Another case is when the left-hand-side array has 1 value. This is followed by the pivot and then followed by the right-hand-side array which has $n-2$ values. In this case, the time taken (recursively) is

$$An + B + T(1) + T(n-2)$$

The time taken when the left-hand-side array has 2 values is

$$An + B + T(2) + T(n-3)$$

Etc. When we take the average over all n cases, we get

$$\begin{aligned} T(n) &= An + B \\ &\quad + \frac{1}{n}\Big((T(0) + T(n-1)) + (T(1) + T(n-2)) + \cdots + (T(n-1) + T(0)) \Big) \\ &= An + B + \frac{1}{n} \sum_{i=0}^{n-1} (T(i) + T(n-1-i)) \end{aligned}$$

Of course $T(0) = T(1) = C$ where C is a constant. Note that, if you're sharp, you realized that I'm assuming that the values in the array `x` are unique. Furthermore, I'm assuming all the above cases are equally likely.

In the above, you can see that each $T(i)$ appears twice. So I can simplify it a little like this:

$$T(n) = An + B + \frac{2}{n} \sum_{i=0}^{n-1} T(i)$$

As you can see this recursion involves $\frac{1}{n}$ as coefficient In fact the above is more complicated than the first example of this section. Why? Because

$$T(n) = An + B + \frac{2}{n} \sum_{i=0}^{n-1} T(i)$$

does not have a (fixed) degree. $\square$

Exercise 912.12.2. In the above, I computed

$$\int e^{-x} g'(x)\ dx$$

using (basically) two integration by parts. Is it possible to simplify the above by doing one integration by parts? □

Exercise 912.12.3. One of the above computation involves an integral of the form

$$\int e^{-x} \frac{?}{(1-x)^2}\, dx$$

How far can you go in generalizing the earlier computation? Find all possible ? so that the above integral can be computed. What about

$$\int e^{-x} \frac{?}{(?-x)^2}\, dx$$

Or this

$$\int e^{?x} \frac{?}{(?-x)^2}\, dx$$

Or this

$$\int e^{?x} \frac{?}{(?-?x)^2}\, dx$$

Or how about this:

$$\int e^{?x} \frac{?}{(?-?x)^?}\, dx$$

State the most general version of your theorem ... and prove it. □

Exercise 912.12.4. The differential equation

$$y' - y = g'(x) = \frac{1+x}{(1-x)^3}$$

is a degree one linear nonhomogeneous differential equation. Can you compute the solution viewing the differential equation this way? (In other words don't view the equation as a Bernoulli differential equation.) $\square$

Exercise 912.12.5. Study this:

$$a_n = \frac{c}{n} a_{n-1} + dn$$

where $a_0 = e$. Here c, d, e are constants. □

Exercise 912.12.6. What about this:

$$a_n = \frac{1}{n+1} a_{n-1} + n^2 + 1$$

where $a_0 = 1$? $\square$

Exercise 912.12.7. Or this:

$$a_n = \frac{1}{n-1} a_{n-1} + n^2 + n$$

where $a_0 = 1$? □

Exercise 912.12.8. Or this:

$$a_n = \frac{1}{n^2} a_{n-1} + 3$$

where $a_0 = 2$? □

Exercise 912.12.9. Now try this:

$$a_n = \frac{1}{n^2 - 1} a_{n-1} + 5n$$

where $a_0 = 3$? □

Exercise 912.12.10. And this:

$$a_n = \frac{1}{n^2+1} a_{n-1} + 5n^2$$

where $a_0 = 7$? □

Exercise 912.12.11. Analyze this problem: Find a closed form for a_n where

$$a_n = \frac{p(n)}{q(n)} a_{n-1} + r(n)$$

where $p(n), q(n), r(n)$ are polynomials (in n). The method used in the first problem of this section relies heavily on your ability (luck?) of integrating complex looking functions. The smart (and deep) question is to ask what functions can be integrated? What functions cannot be integrated? In fact what do you mean by "can be integrated"? Do some research on this and write a paper. □

Exercise 912.12.12. What about something like this:

$$a_n = \frac{2}{n} \sum_{i=0}^{n-1} a_i$$

□

Exercise 912.12.13. Can you handle this:

$$a_n = An + B + \frac{2}{n} \sum_{i=0}^{n-1} a_i$$

where a_0 is given? □

Let

$$a_n = An + B + \frac{2}{n}\sum_{i=0}^{n-1} a_i$$

for $n \geq 1$ and $a_0 = C$.

Numerically, for testing purposes let $A = B = C = 1$. Then

$$\begin{aligned}
a_0 &= 1 \\
a_1 &= 1 + 1 + \frac{2}{1}(1) = 4 \\
a_2 &= 2 + 1 + \frac{2}{2}(1 + 4) = 8
\end{aligned}$$

Algebraically, we have the following cases:

$$\begin{aligned}
a_1 &= A + B + \frac{2}{1}(a_0) \\
&= A + B + 2a_0 \\
a_2 &= 2A + B + \frac{2}{2}(a_0 + a_1) \\
&= 2A + B + (a_0 + A + B + 2a_0) \\
&= 3A + 2B + 3a_0
\end{aligned}$$

Recall that

$$\frac{1}{1-x}\sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n} c_i\right) x^n$$

Note that for $n > 0$

$$a_n = An + B + \frac{2}{n}\sum_{i=0}^{n-1} a_i = \sum_{i=0}^{n-1}\left(A + \frac{B}{n} + \frac{2a_i}{n}\right)$$

Let

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$\begin{aligned}
f(x) = a_0 + \sum_{n=1}^{\infty} a_n x^n &= a_0 + \sum_{n=1}^{\infty} \left(\sum_{i=0}^{n-1} \left(A + \frac{B}{n} + \frac{2a_i}{n} \right) \right) x^n \\
&= a_0 + x \sum_{n=1}^{\infty} \left(\sum_{i=0}^{n-1} \left(A + \frac{B}{n} + \frac{2a_i}{n} \right) \right) x^{n-1} \\
&= a_0 + x \sum_{m=0}^{\infty} \left(\sum_{i=0}^{m} \left(A + \frac{B}{m+1} + \frac{2a_i}{m+1} \right) \right) x^m \\
&= a_0 + x \frac{1}{1-x} \sum_{m=0}^{\infty} \left(A + \frac{B}{m+1} + \frac{2a_m}{m+1} \right) x^m \\
&= a_0 + \frac{x}{1-x} \left(\sum_{m=0}^{\infty} A x^m + \sum_{m=0}^{\infty} \frac{B}{m+1} x^m + \sum_{m=0}^{\infty} \frac{2a_m}{m+1} x^m \right) \\
&= a_0 + \frac{x}{1-x} \left(A \sum_{n=0}^{\infty} x^n + \frac{1}{x} \sum_{m=0}^{\infty} \frac{B}{m+1} x^{m+1} + \frac{1}{x} \sum_{m=0}^{\infty} \frac{2a_m}{m+1} x^{m+1} \right) \\
&= a_0 + \frac{x}{1-x} \left(A \frac{1}{1-x} + \frac{1}{x} \sum_{m=0}^{\infty} B \int_{t=0}^{x} t^m \; dt + \frac{1}{x} \sum_{m=0}^{\infty} 2a_m \int_{t=0}^{x} t^m \; dt \right) \\
&= a_0 + \frac{x}{1-x} \left(\frac{A}{1-x} + \frac{B}{x} \int_{t=0}^{x} \sum_{m=0}^{\infty} t^m \; dt + \frac{2}{x} \int_{t=0}^{x} \sum_{m=0}^{\infty} a_m t^m \; dt \right) \\
&= a_0 + \frac{x}{1-x} \left(\frac{A}{1-x} + \frac{B}{x} \int_{t=0}^{x} \frac{1}{1-t} \; dt + \frac{2}{x} \int_{t=0}^{x} f(t) \; dt \right) \\
&= a_0 + A \frac{x}{(1-x)^2} + \frac{B}{1-x} \int_{t=0}^{x} \frac{1}{1-t} \; dt + \frac{2}{1-x} \int_{t=0}^{x} f(t) \; dt
\end{aligned}$$

Therefore

$$(1-x) f(x) - 2 \int_{t=0}^{x} f(t) \; dt = a_0(1-x) + A \frac{x}{1-x} + B \int_{t=0}^{x} \frac{1}{1-t} \; dt$$

Taking derivative, we get

$$\begin{aligned}
-f(x) + (1-x)f'(x) - 2f(x) &= -a_0 + A\frac{1(1-x) - x(-1)}{(1-x)^2} + B\frac{1}{1-x} \\
\therefore\ (1-x)f'(x) - 3f(x) &= -a_0 + A\frac{1}{(1-x)^2} + B\frac{1}{1-x} \\
&= \frac{-a_0(1-x)^2 + A + B(1-x)}{(1-x)^2} \\
&= \frac{-a_0x^2 + (2a_0 - B)x + (-a_0 + A + B)}{(1-x)^2} \\
f'(x) - \frac{3}{1-x}f(x) &= \frac{-a_0x^2 + (2a_0 - B)x + (-a_0 + A + B)}{(1-x)^3}
\end{aligned}$$

Hence $f(x)$ is the solution to the following Bernoulli differential equation:

$$y' - \frac{3}{1-x}y = \frac{-a_0x^2 + (2a_0 - B)x + (-a_0 + A + B)}{(1-x)^3}$$

Therefore

$$f(x) = \frac{\displaystyle\int m(x)\left(\frac{-a_0x^2 + (2a_0 - B)x + (-a_0 + A + B)}{(1-x)^3}\right)\,dx + C}{m(x)}$$

where C is a constant. and

$$m(x) = e^{\int\left(-\frac{3}{1-x}\right)\,dx} = e^{3\ln(1-x)} = (1-x)^3$$

So

$$\begin{aligned}
f(x) &= \frac{\displaystyle\int (1-x)^3\left(\frac{-a_0x^2 + (2a_0 - B)x + (-a_0 + A + B)}{(1-x)^3}\right)\,dx + C}{(1-x)^3} \\
&= \frac{\displaystyle\int \left(-a_0x^2 + (2a_0 - B)x + (-a_0 + A + B)\right)\,dx + C}{(1-x)^3} \\
&= \frac{1}{(1-x)^3}\left(-\frac{a_0}{3}x^3 + \frac{2a_0 - B}{2}x^2 + (-a_0 + A + B)x + C\right)
\end{aligned}$$

To solve for C, we use $f(0) = a_0$:

$$a_0 = f(0) = \frac{1}{(1-0)^3}\left(0 + 0 + 0 + C\right)$$

i.e., $C = a_0$.

Now we expand the right-hand-side into power series:

$$\begin{aligned}
f(x) &= \left(-\frac{a_0}{3}x^3 + \frac{2a_0 - B}{2}x^2 + (-a_0 + A + B)x + a_0\right)\sum_{n=0}^{\infty}\binom{3+n-1}{n}x^n \\
&= \left(-\frac{a_0}{3}x^3 + \frac{2a_0 - B}{2}x^2 + (-a_0 + A + B)x + a_0\right)\sum_{n=0}^{\infty}\binom{n+2}{n}x^n \\
&= \left(-\frac{a_0}{3}x^3 + \frac{2a_0 - B}{2}x^2 + (-a_0 + A + B)x + a_0\right)\sum_{n=0}^{\infty}\binom{n+2}{2}x^n \\
&= -\frac{a_0}{3}\sum_{n=0}^{\infty}\binom{n+2}{2}x^{n+3} + \frac{2a_0 - B}{2}\sum_{n=0}^{\infty}\binom{n+2}{2}x^{n+2} \\
&\qquad + (-a_0 + A + B)\sum_{n=0}^{\infty}\binom{n+2}{2}x^{n+1} + a_0\sum_{n=0}^{\infty}\binom{n+2}{2}x^n \\
&= -\frac{a_0}{3}\sum_{n=3}^{\infty}\binom{n-1}{2}x^n + \frac{2a_0 - B}{2}\sum_{n=2}^{\infty}\binom{n}{2}x^n \\
&\qquad + (-a_0 + A + B)\sum_{n=1}^{\infty}\binom{n+1}{2}x^n + a_0\sum_{n=0}^{\infty}\binom{n+2}{2}x^n \\
&= a_0\binom{2}{2} + \left(a_0\binom{3}{2} + (-a_0 + A + B)\binom{2}{2}\right)x \\
&\qquad + \left(a_0\binom{4}{2} + (-a_0 + A + B)\binom{3}{2} + \frac{2a_0 - B}{2}\binom{2}{2}\right)x^2 \\
&\qquad + \sum_{n=3}^{\infty}\left(-\frac{a_0}{3}\binom{n-1}{2} + \frac{2a_0 - B}{2}\binom{n}{2} + (-a_0 + A + B)\binom{n+1}{2} + a_0\binom{n+2}{2}\right)x^n \\
&= a_0 + (3a_0 + (-a_0 + A + B))\,x \\
&\qquad + \left(6a_0 + 3(-a_0 + A + B) + \frac{2a_0 - B}{2}\right)x^2 \\
&\qquad + \sum_{n=3}^{\infty}\left(-\frac{a_0}{3}\binom{n-1}{2} + \frac{2a_0 - B}{2}\binom{n}{2} + (-a_0 + A + B)\binom{n+1}{2} + a_0\binom{n+2}{2}\right)x^n \\
&= a_0 + (A + B + 2a_0)\,x \\
&\qquad + \left(3A + \frac{5}{2}B + 4a_0\right)x^2 \\
&\qquad + \sum_{n=3}^{\infty}\left(-\frac{a_0}{3}\binom{n-1}{2} + \frac{2a_0 - B}{2}\binom{n}{2} + (-a_0 + A + B)\binom{n+1}{2} + a_0\binom{n+2}{2}\right)x^n
\end{aligned}$$

At this point, notice that the closed form derived is incorrect. Your job is to find the error.

SOLUTION. .

The above is incorrect because of the following step:

$$\begin{aligned}
f(x) &= ... \\
&= a_0 + x \sum_{m=0}^{\infty} \left(\sum_{i=0}^{m} \left(A + \frac{B}{m+1} + \frac{2a_i}{m+1} \right) \right) x^m \\
&= a_0 + x \frac{1}{1-x} \sum_{m=0}^{\infty} \left(A + \frac{B}{m+1} + \frac{2a_m}{m+1} \right) x^m \\
&= ...
\end{aligned}$$

Basically

$$c_{i,m} = A + \frac{B}{m+1} + \frac{2a_i}{m+1}$$

is a term of two variables. The following is correct

$$\frac{1}{1-x} \sum_{n=0}^{\infty} c_{n,n} x^n = \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} c_{i,i} \right) x^n$$

But the following is wrong

$$\frac{1}{1-x} \sum_{n=0}^{\infty} c_{n,n} x^n = \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} c_{i,n} \right) x^n$$

This incorrect version is used in the computations above.

Let's analyze the following recurrence carefully:

$$a_n = An + B + \frac{2}{n}\sum_{i=0}^{n-1} a_i$$

I will assume that there's only one base condition: a_0 is given. For instance a_1 is computed using the above recurrence like this:

$$a_1 = A + B + \frac{2}{1}a_0 = A + B + 2a_0$$

(For the case of quicksort, there are usually *two* base conditions, i.e., a_0 and a_1 are given – this will be an exercise later.) Multiplying the above recurrence by n, I get

$$na_n = An^2 + Bn + 2\sum_{i=0}^{n-1} a_i \tag{1}$$

Here's the crucial step: We also have

$$(n-1)a_{n-1} = A(n-1)^2 + B(n-1) + 2\sum_{i=0}^{n-2} a_i \tag{2}$$

Using that, (1) − (2) gives us

$$\begin{aligned} na_n - (n-1)a_{n-1} &= A(n^2 - (n-1)^2) + B(n - (n-1)) + 2a_{n-1} \\ &= (2n-1)A + B + 2a_{n-1} \end{aligned}$$

Why is this step a reasonable strategy? Because the sum $\sum_{i=0}^{n-1} a_i$ has too many a_i's – I want to get rid of them as fast as possible. And you also now understand why I had to multiple the first recurrence by n. (What will happen if I don't?)

(ASIDE: In case you're thinking I'm just pulling algebraic tricks like that out of thin air, then think again: it's not entirely new. The above trick is analogous to the method used in one of the standard derivations of the geometric sum formula. Here's the derivation. Let $S = 1 + r + r^2 + \cdots + r^n$. Notice that there's a summation. Then $rS = r + r^2 + r^3 + \cdots r^{n+1}$. This is also a summation. But notice that many of the terms in the two summations are the same. On subtraction, we get

$$rS - S = r^{n+1} - 1$$

Notice that the summation disappears - get it now? After that it's easy: we get

$$(r-1)S = r^{n+1} - 1$$

and therefore

$$S = \frac{r^{n+1} - 1}{r - 1}$$

i.e.,

$$1 + r + r^2 + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1}$$

See the magic now? This is a very important trick in removing summations ... and it's not new.)

And now finally I get this:

$$\begin{aligned} na_n &= (2n-1)A + B + 2a_{n-1} + (n-1)a_{n-1} \\ &= (2n-1)A + B + (n+1)a_{n-1} \\ \therefore \quad a_n &= \left(2 - \frac{1}{n}\right) A + \frac{1}{n} B + \left(1 + \frac{1}{n}\right) a_{n-1} \\ &= 2A + \frac{1}{n}(B - A) + \left(1 + \frac{1}{n}\right) a_{n-1} \end{aligned}$$

To make life easier, I'm going to let $C = B - A$ and $D = 2A$. Then I get

$$a_n = C\frac{1}{n} + D + \left(1 + \frac{1}{n}\right) a_{n-1}$$

Note that this is a recurrence of degree 1. The original recurrence

$$a_n = An + B + \frac{2}{n} \sum_{i=0}^{n-1} a_i$$

has degree n.

Let

$$f(x) = \sum_{n=0}^{\infty} a_n$$

Then

$$
\begin{aligned}
f(x) &= a_0 + \sum_{n=1}^{\infty} a_n x^n \\
&= a_0 + \sum_{n=1}^{\infty} \left(C\frac{1}{n} + D + \left(1 + \frac{1}{n}\right) a_{n-1} \right) x^n \\
&= a_0 + C\sum_{n=1}^{\infty} \frac{1}{n} x^n + D\sum_{n=1}^{\infty} x^n + \sum_{n=1}^{\infty} a_{n-1} x^n + \sum_{n=1}^{\infty} a_{n-1} \frac{1}{n} x^n \\
&= a_0 + C\sum_{n=1}^{\infty} \int_{t=0}^{x} t^{n-1}\ dt + D\left(\frac{1}{1-x} - 1\right) + x\sum_{n=1}^{\infty} a_{n-1} x^{n-1} + \sum_{n=1}^{\infty} a_{n-1} \int_{t=0}^{x} t^{n-1}\ dt \\
&= a_0 + C\int_{t=0}^{x} \left(\sum_{n=1}^{\infty} t^{n-1}\right)\ dt + D\frac{x}{1-x} + x\sum_{n=0}^{\infty} a_n x^n + \int_{t=0}^{x} \left(\sum_{n=1}^{\infty} a_{n-1} t^{n-1}\right)\ dt \\
&= a_0 + C\int_{t=0}^{x} \left(\sum_{n=0}^{\infty} t^{n}\right)\ dt + \frac{Dx}{1-x} + xf(x) + \int_{t=0}^{x} \left(\sum_{n=0}^{\infty} a_n t^{n}\right)\ dt \\
&= a_0 + C\int_{t=0}^{x} \frac{1}{1-t}\ dt + \frac{Dx}{1-x} + xf(x) + \int_{t=0}^{x} f(t)\ dt
\end{aligned}
$$

Therefore we get this

$$
(1-x)f(x) - \int_{t=0}^{x} f(t)\ dt = a_0 + C\int_{t=0}^{x} \frac{1}{1-t}\ dt + \frac{Dx}{1-x}
$$

Taking derivatives on both sides:

$$
\begin{aligned}
-f(x) + (1-x)f'(x) - f(x) &= \frac{C}{1-x} + D\frac{(1-x) - x(-1)}{(1-x)^2} \\
\therefore\ \ (1-x)f'(x) - 2f(x) &= \frac{C}{1-x} + D\frac{1}{(1-x)^2} \\
\therefore\ \ f'(x) - \frac{2}{1-x}f(x) &= \frac{C}{(1-x)^2} + \frac{D}{(1-x)^3}
\end{aligned}
$$

Therefore viewing $f(x)$ as a solution to the following Bernoulli differential equation,

$$
y' - \frac{2}{1-x}y = \frac{C}{(1-x)^2} + \frac{D}{(1-x)^3}
$$

we get

$$
f(x) = \frac{\int m(x)q(x)\ dx + E}{m(x)}
$$

where E is a constant,

$$m(x) = e^{\int \frac{-2}{1-x} \, dx}$$

and $q(x) = \frac{C}{(1-x)^2} + D\frac{1}{(1-x)^3}$. We have

$$m(x) = e^{\int \frac{-2}{1-x} \, dx} = e^{2\int \frac{1}{1-x} d(1-x)} = e^{2\ln(1-x)} = (1-x)^2$$

Therefore

$$\begin{aligned}
f(x) &= \frac{1}{(1-x)^2}\left(\int (1-x)^2 \left(\frac{C}{(1-x)^2} + \frac{D}{(1-x)^3}\right) \, dx + E\right) \\
&= \frac{1}{(1-x)^2}\left(\int \left(C + \frac{D}{1-x}\right) \, dx + E\right) \\
&= \frac{1}{(1-x)^2}\left(Cx - D\ln(1-x) + E\right) \\
&= \frac{Cx}{(1-x)^2} - D\frac{\ln(1-x)}{(1-x)^2} + \frac{E}{(1-x)^2} \\
&= \frac{Cx+E}{(1-x)^2} - D\frac{\ln(1-x)}{(1-x)^2}
\end{aligned}$$

The constant E can be computed easily: Substituting $x = 0$, we get

$$a_0 = f(0) = 0 + E - 0$$

i.e.,

$$E = a_0$$

The following is a well–known power series for $\ln(1-x)$:

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{1}{n} x^n$$

Therefore

$$\begin{aligned}
-\frac{\ln(1-x)}{(1-x)^2} &= (-\ln(1-x)) \cdot \frac{1}{(1-x)^2} \\
&= -\left(-\sum_{n=1}^{\infty} \frac{1}{n} x^n\right) \cdot \sum_{n=0}^{\infty} (n+1) x^n \\
&= \sum_{n=1}^{\infty} \frac{1}{n} x^n \cdot \sum_{n=0}^{\infty} (n+1) x^n
\end{aligned}$$

I'll show you that the big-Θ of the coefficients of the power series of this function is $n \ln n$; all the other functions have power series with "smaller" coefficients. In fact I'll do better than that: I'll derive an exact closed form for coefficients of the power series for $-\ln(1-x)/(1-x)^2$ and then for $f(x)$.

The n–th coefficient of the above power series

$$\sum_{n=1}^{\infty} \frac{1}{n} x^n \cdot \sum_{n=0}^{\infty} (n+1) x^n$$

is

$$b_n = 0 \cdot (n+1) + \frac{1}{1} \cdot n + \frac{1}{2} \cdot (n-1) + \cdots + \frac{1}{n} \cdot (1) = \frac{n}{1} + \frac{n-1}{2} + \cdots + \frac{1}{n}$$

Finding an exact closed form for this is difficult ... to say the least. Also, note that in the section on products of power series, as much as possible we in fact do *not* want to compute the above expression doing term-by-term multiplication. For instance if you have the following product of power series:

$$\sum_{n=1}^{\infty} x^n \sum_{n=0}^{\infty} (n+1) x^n$$

you should not multiply out term-by-term. You should do this instead:

$$\begin{aligned}
\sum_{n=1}^{\infty} x^n \sum_{n=0}^{\infty} (n+1) x^n &= x \sum_{n=0}^{\infty} x^n \sum_{n=0}^{\infty} (n+1) x^n \\
&= x \frac{1}{1-x} \frac{1}{(1-x)^2} \\
&= x \frac{1}{(1-x)^3} \\
&= x \sum_{n=0}^{\infty} \binom{n+2}{2} x^n \\
&= ...
\end{aligned}$$

Right?

However in the expression above

$$-\frac{\ln(1-x)}{(1-x)^2}$$

there's no easy way to write $\ln(1-x)$ into something that will multiply easily with $\frac{1}{(1-x)^2}$ or vice versa. So we are forced to compute the coefficient of x^n "by hand", term-by-term. It's not really as hopeless as you think. Watch this:

$$\begin{aligned}
b_n &= \frac{n}{1} + \frac{n-1}{2} + \cdots + \frac{1}{n} \\
&= \frac{n-0}{1} + \frac{n-1}{2} + \frac{n-2}{3} + \cdots + \frac{n-(n-1)}{n} \\
&= \left(\frac{n}{1} + \frac{n}{2} + \frac{n}{3} + \ldots + \frac{n}{n}\right) - \left(\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{n-1}{n}\right) \\
&= \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right) n - \left(\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{n-1}{n}\right)
\end{aligned}$$

The sequence

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$$

is well–known – this sum is called the harmonic sum. In particular

$$H_n - \ln n \to \gamma$$

as $n \to \infty$ where γ is a constant of value 0.577...; γ is called the Euler's constant for harmonic sums. Even more precisely, it's known that

$$H_n = \ln n + \gamma + \epsilon_n$$

where ϵ_n is approximately $1/(2n) - 1/(12n^2)$. (See below.) As for

$$\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{n-1}{n}$$

again, the harmonic sum is actually lurking around:

$$\begin{aligned}
\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{n-1}{n} &= \frac{2-1}{2} + \frac{3-1}{3} + \frac{4-1}{4} + \cdots + \frac{n-1)}{n} \\
&= \left(1 - \frac{1}{2}\right) + \left(1 - \frac{1}{3}\right) + \left(1 - \frac{1}{4}\right) + \cdots + \left(1 - \frac{1}{n}\right) \\
&= (n-1) - (H_n - 1) \\
&= n - H_n
\end{aligned}$$

Therefore altogether here's what I can tell you about b_n:

$$\begin{aligned}
b_n = &= nH_n - (n - H_n) \\
&= (n+1)H_n - n \\
&= (n+1)(\ln n + \gamma + \epsilon_n) - n \\
&= n \ln n + (\gamma - 1)n + \ln n + n\epsilon_n + \gamma + \epsilon_n
\end{aligned}$$

which has a big-Θ of $n \ln n$.

Let's get on with the power series of $f(x)$.

$$\begin{aligned}
f(x) &= \frac{Cx+E}{(1-x)^2} + D\sum_{n=1}^{\infty} b_n x^n \\
&= (Cx+E)\sum_{n=0}^{\infty}(n+1)x^n + D\sum_{n=1}^{\infty} b_n x^n \\
&= \sum_{n=0}^{\infty} C(n+1)x^{n+1} + \sum_{n=0}^{\infty} E(n+1)x^n + \sum_{n=1}^{\infty} Db_n x^n \\
&= \sum_{n=1}^{\infty} Cnx^n + E + \sum_{n=1}^{\infty} E(n+1)x^n + \sum_{n=1}^{\infty} Db_n x^n \\
&= E + \sum_{n=1}^{\infty} \left(Cn + E(n+1) + Db_n\right) x^n
\end{aligned}$$

The n–th coefficient for $n > 0$ is

$$\begin{aligned}
&Cn + E(n+1) + Db_n \\
&= (C+E)n + E + D\left(n\ln n + (\gamma-1)n + \ln n + n\epsilon_n + \gamma + \epsilon_n\right) \\
&= Dn\ln n + (C+E+D(\gamma-1))n + D\ln n + (E + D(n\epsilon_n + \gamma)) + D\epsilon_n
\end{aligned}$$

Let's not forget that $C = B - A$, $D = 2A$, and $E = a_0$:

$$\begin{aligned}
An+B + \frac{2}{n}\sum_{i=0}^{n-1} a_i \\
= 2An\ln n \\
&+ (B - A + a_0 + 2A(\gamma-1))n \\
&+ 2A\ln n \\
&+ (a_0 + 2A(n\epsilon_n + \gamma)) \\
&+ 2A\epsilon_n
\end{aligned}$$

Note that this is *not* an approximation or some big-O/big-Θ thingy – it's exact. Even if we repllace ϵ_n with $1/(2n)$, the right-hand side is still a very good approximation to the right. In particular, note that

$$An + B + \frac{2}{n}\sum_{i=0}^{n-1} a_i = \Theta(n\ln n)$$

In fact we even know the constant of the big-Θ. Therefore I can be even more precise:

$$An + B + \frac{2}{n}\sum_{i=0}^{n-1} a_i = 2An \ln n + \Theta(n)$$

Here's a test:

```
import math

e = math.e
def ln(x):
    x = float(x)
    return math.log(x, e)

a0 = 1.0
A =  1.0
B = -1.0
gamma = 0.5772156649

table_a = {0:a0}
def a(n):
    if not table_a.has_key(n):
        s = 0.0
        for i in range(n):
            s += a(i)
        table_a[n] = A * n + B + (2.0 / n) * s
    return table_a[n]

def ep(n):
    return 1.0/(2.0*n) - 1.0/(12.0*n*n) + 1.0/(120.0*n**4)

def b(n):
    if n == 0: return a0
    t0 = 2 * A * n * ln(n)
    t1 = (B - A + a0 + 2 * A * (gamma - 1)) * n
    t2 = 2 * A * ln(n)
    t3 = (a0 + 2 * A * (n * ep(n) + gamma))
    t4 = 2 * A * ep(n)
    return t0 + t1 + t2 + t3 + t4

for n in range(1, 100):
    print('%5s' % n, '%10.5f' % a(n), '%10.5f' % b(n), '%15.12f' % (a(n)-b(n)))
```

Here's the output

```
    1    2.00000    2.00886 -0.008862659600
    2    4.00000    4.00030 -0.000302072760
    3    6.66667    6.66671 -0.000039266405
```

```
   4    9.83333    9.83334 -0.000009114366
   5   13.40000   13.40000 -0.000002928009
   6   17.30000   17.30000 -0.000001157628
   7   21.48571   21.48571 -0.000000528491
   8   25.92143   25.92143 -0.000000268102
   9   30.57937   30.57937 -0.000000147415
  10   35.43730   35.43730 -0.000000086367
snipped
  30  158.68920  158.68920 -0.000000000242
  31  165.74369  165.74369 -0.000000000188
  32  172.86068  172.86068 -0.000000000143
  33  180.03828  180.03828 -0.000000000105
  34  187.27470  187.27470 -0.000000000072
  35  194.56826  194.56826 -0.000000000045
  36  201.91738  201.91738 -0.000000000021
  37  209.32055  209.32055 -0.000000000001
  38  216.77636  216.77636  0.000000000017
  39  224.28344  224.28344  0.000000000032
  40  231.84053  231.84053  0.000000000046
  41  239.44640  239.44640  0.000000000059
  42  247.09988  247.09988  0.000000000070
snipped
  86  619.50162  619.50162  0.000000000265
  87  628.59934  628.59934  0.000000000268
  88  637.71979  637.71979  0.000000000271
  89  646.86271  646.86271  0.000000000275
  90  656.02785  656.02785  0.000000000278
  91  665.21497  665.21497  0.000000000281
  92  674.42383  674.42383  0.000000000284
  93  683.65419  683.65419  0.000000000287
  94  692.90583  692.90583  0.000000000290
  95  702.17852  702.17852  0.000000000293
  96  711.47205  711.47205  0.000000000296
  97  720.78620  720.78620  0.000000000299
  98  730.12075  730.12075  0.000000000302
  99  739.47550  739.47550  0.000000000306
```

OK ... let's tidy up what we have ...

First I'm going to assume the following without proof:

Theorem 912.12.1. *Let H_n be the n–th harmonic sum, i.e.,*

$$H_n = 1 + \frac{1}{2} + \cdots \frac{1}{n}$$

γ be Euler's constant for harmonic series, i.e.,

$$\gamma = \lim_{n \to \infty} \left(\sum_{i=1}^{n} \frac{1}{k} - \ln n \right) = 0.577...$$

and ϵ_n be the n–th error term of the harmonic series according to the following:

$$H_n - \ln n = \gamma + \epsilon_n$$

Then, asymptotically,

$$\epsilon_n \sim \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} - \frac{1}{252n^6} - \cdots$$

□

(The infinite sum on the right can actually be specified explicitly, but I'm not going to. Don't get me wrong: it's an interesting series. However I won't be using it in this set of notes. To find out more, refer to my notes on analytic number theory.)

When I write

$$f(n) \sim g(n)$$

and say that $f(n)$ and $g(n)$ are **asymptotically equivalent**, I mean "$f(n)$ and $g(n)$ are more or less the same when $n \to \infty$". The intuition is that when n gets larger and larger, $f(n)$ and $g(n)$ get closer and closer to each other. To be precise (mathematically speaking), $f(n) \sim g(n)$ means

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 1$$

Like other asymptotics (big-O, big-Θ, etc.) asymptotic equivalence was originally used by people studying number theory, especially analytic number theory. Concretely speaking, here's how to think about asymptotic equivalence. You know that

$$n^2 = O(n^3)$$

In fact

$$5n^2 = O(1.5n^3)$$

The tightest power you can get for the upper bound is 2:

$$5n^2 = O(1.5n^2)$$

As for big–Θ we have

$$5n^2 = \Theta(10.5n^1)$$

The tightest you can get for the lower bound is 2:

$$5n^2 = \Theta(10.5n^2)$$

Combining the two,

$$5n^2 = \Theta(n^2)$$

However Θ ignores constants so that the following are true:

$$5n^2 + 10n + 1 = \Theta(2.3n^2 + 0.1n + 3.5)$$

and

$$2.3n^2 + 10n + 1 = \Theta(2.3n^2 + 0.1n + 3.5)$$

Asymptotic equivalence is stronger in the sense that

$$2.3n^2 + 10n + 1 \sim 2.3n^2 + 0.1n + 3.5$$

So asymptotic equivalence focuses on the "most significant term" of the sum *and* its coefficient as well. The above is an over–simplification but will be helpful in seeing the big picture.

OK .... let's get back to the main theorem:

Theorem 912.12.2. *Let*

$$a_n = An + B + \frac{2}{n} \sum_{i=0}^{n-1} a_i$$

for $n > 0$. *Then*

$$\begin{aligned} a_n = 2An \ln n &+ (B - A + a_0 + 2A(\gamma - 1))n \\ &+ 2A \ln n + (a_0 + 2A(n\epsilon_n + \gamma)) + 2A\epsilon_n \end{aligned}$$

In particular

$$a_n = 2An \ln n + \Theta(n)$$

Proof.

Exercise 912.12.14. What happens if there are *two* base considiions, i.e., a_0 and a_1 are given and

$$a_n = An + B + \frac{2}{n} \sum_{i=0}^{n-1} a_i$$

for $n \geq 2$? □

Exercise 912.12.15. Let's see if you can find the closed form or big-Θ or big-O of this:

$$a_n = An + B + n \sum_{i=0}^{n-1} a_i$$

for $n \geq 1$ where a_0 is given. □

Exercise 912.12.16. Can you handle this:

$$a_n = An^2 + Bn + C + \frac{2}{n}\sum_{i=0}^{n-1} a_i, \quad n > 0$$

where a_0 is given. □

Exercise 912.12.17. Can you handle this:

$$a_n = An + B + \frac{2}{n} \sum_{i=0}^{n-1} (i+1) a_i$$

where $n > 0$. Assume that a_0 is given. □

Exercise 912.12.18. Suppose you have a recurrence of the form

$$a_n = f(n) + g(n) \sum_{i=0}^{n-1} h(i)a(i)$$

for $n > n_0$ where $f(n), g(n), h(n)$ are functions with $g(n) \neq 0$ for all $n > n_0$. Convert the recurrence to another of degree 1. □

SOLUTION.

Let $n > n_0$. Dividing the given recurrence by $g(n)$, we get

$$\frac{1}{g(n)} a_n = \frac{f(n)}{g(n)} + \sum_{i=0}^{n-1} h(i)a(i) \tag{1}$$

We also have

$$\frac{1}{g(n-1)} a_{n-1} = \frac{f(n-1)}{g(n-1)} + \sum_{i=0}^{n-2} h(i)a(i) \tag{2}$$

(1) − (2) give us

$$\begin{aligned}
\frac{1}{g(n)} a_n - \frac{1}{g(n-1)} a_{n-1} &= \frac{f(n)}{g(n)} - \frac{f(n-1)}{g(n-1)} + h(n-1)a_{n-1} \\
\therefore \quad \frac{1}{g(n)} a_n &= \frac{f(n)}{g(n)} - \frac{f(n-1)}{g(n-1)} + h(n-1)a_{n-1} + \frac{1}{g(n-1)} a_{n-1} \\
&= \frac{f(n)}{g(n)} - \frac{f(n-1)}{g(n-1)} + \left(h(n-1) + \frac{1}{g(n-1)} \right) a_{n-1} \\
\therefore \quad a_n &= f(n) - \frac{f(n-1)g(n)}{g(n-1)} + g(n) \left(h(n-1) + \frac{1}{g(n-1)} \right) a_{n-1}
\end{aligned}$$

which is clearly a degree 1 recurrence. □

Exercise 912.12.19. Design a quicksort that partitions not into twwo parts but into three. Program it (and test it of course). Compute the average runtime. □

Solutions

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