Asymptotic Variability Analysis for Tandem Queue

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Abstract

We study a multi-stage tandem queue and develop the law of the iterated logarithm (LIL) for the performance measures including the queue length, workload, busy time, idle time and departure processes. All the LILs obtained are presented through the cases with only one critically loaded stage or no such stages in the system. For the case without critically loaded stages, all stages are assumed to be in the underloaded regime or overloaded regime or both. For the case with only one critically loaded stage, except it all other stages are assumed to be in the underloaded regime or overloaded regime. A strong approximation method, established on the fluid and strong approximations and Brownian motions, is adopted to find all the LILs, which are expressed as some simple functions and compact sets of continuous functions based on the primitive data: means and variances of the interarrival and service times, respectively. Some interesting and counter-intuitive insights are given through the limits obtained, such as, the asymptotic variability associated with a service station only depends on the exogenous arrival (service time of the stage in its front) and its service time when all stages are in the underloaded (overloaded) regime; a stage in the critically loaded or overloaded regimes affects its next adjacent stage (sequent stages) in overloaded (underloaded) regime only through the variability of its service time, and has no influence on the second overloaded stage behind it and the following ones; the Little's law in the view of LIL holds in the underloaded and critically loaded regimes and fails in the overloaded regime.

Keywords: tandem queue; the law of the iterated logarithm (LIL); the strong approximation method; Brownian motion

1 Introduction

We develop a *law of the iterated logarithm* (LIL) to characterize the asymptotic variability for a K-stage tandem queue denoted by $GI/GI^K/1/\text{FCFS}$, in which all customers come from outside by a renewal process (the first GI) to stage-1, and leave the system after successive service completions at K stages from 1 to K, service times at K stages are

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mutually independent and are assumed to be K sequences of *independent and identically distributed* (iid) random variables (the second GI), service discipline in force for all stages is assumed to be first-come first-served (FCFS). Suppose that there is only one single-server station providing service at every stage. The asymptotic variability is represented through the following five performance measures: queue length, workload, busy time, idle time and departure processes.

Tandem queue is a special generalized Jackson network ([22, 23, 24]), in which customers are served along a pre-designed straight forward route and leave the system without feedback after the last service completion. In literature, tandem queue has been widely focused by researchers for many years because its simple topology makes the corresponding results be intuitive and enlightening, and it has enormous applications in manufacture system and wireless communication networks too, see [9, 20, 25]. Many results are obtained for tandem queue, see [32] for transient behaviour analysis, [2, 4] for joint queue length distribution, [3] for the sojourn time distribution, [12] for the fluid approximation, [35] for moments of performance measures, [17, 18, 33] for diffusion approximation, [26, 36] for applications in inventory system, [1] for analysis on throughput maximization, [15] for asymptotic analysis for departure process, [38] for large deviations of queue length and departure processes.

The LIL is classical problem in probability theory and stochastic process. The type of the LIL in consideration originates from the definition on the BM in [10, 11]. Let $W(t)$ is a one-dimensional standard Brownian motion (BM), with probability one (w.p.1), the LIL

$$\limsup_{t\to\infty} \frac{\sup_{0\le s\le t} |W(s))|}{\sqrt{2t\log\log t}} = 1, \tag{1.1}$$

where the explicit number "1" tells us the fluctuation of BM around its mean value 0. For queueing network, the LIL in (1.1) is usually used to characterize the asymptotic variability of performance measures around their fluid approximations. Che and Yao stated in chapter 7 of their monograph [8] that it is interesting to find the LIL in the type (1.1) for queue length, workload, busy time, idle time and departure processes. We try to partially answer this question through a multi-stage tandem queue. For the type of the LIL in (1.1), Guo and Liu [16] developed the LILs for the performance measures for a multi-class queueing system under preemptive static priority service discipline.

There are another types of the LIL in the literature. For the BM $W(t)$ defined above, they are, w.p.1,

$$\limsup_{t\to\infty} \frac{W(t)}{\sqrt{2t\log\log t}} = -\liminf_{t\to\infty} \frac{W(t)}{\sqrt{2t\log\log t}} = 1, \tag{1.2}$$

$$\sup_{0\le s\le t} |W(s))| = O(\sqrt{2t\log\log t}), \tag{1.3}$$

where the function O means that $f(t) = O(g(t))$ if $\limsup_{t\to\infty} |f(t)/g(t)| \le M$ for some $M > 0$. In general, (1.2) is thought of as from [27, 28], and it characterizes the fluctuation of BM through constant "1" similarly with (1.1). However, (1.3) is a relatively weak type of the LIL because it uses a trend of same order $O(\sqrt{2t\log\log t})$ to show us the fluctuation. For queues, Iglehart [21] got the LILs in type (1.2) for the performance measures through a functional LIL for a multi-channel queue, in which new stochastic processes based on renewal process are constructed. Minkevičius etc. [29, 30, 31] adopted a method of probability definition of the LIL (1.2) to obtained the LIL for the tandem queue, single-server and multi-server station generalized Jackson networks under strict heavy traffic condition, respectively. For the type (1.3) of the LIL, Chen and Yao [8] summarized all the LILs for GI/G/1 queue, generalized Jackson network and multiclass feedforward networks.

We use the strong approximation method to find all the LILs in type (1.1) for the tandem queue in consideration. Next, we simply show the strong approximation method with the aid of a renewal process $N(t)$ whose interarrival times are iid sequence of nonnegative random variables with mean $1/\alpha > 0$ and variance $\sigma^2 > 0$. Let $\sigma > 0$ be the standard deviation. So doing, the corresponding fluid approximation is $\bar{N}(t) = \alpha t$ and the strong approximation is $\widetilde{N}(t) = \bar{N}(t) + \alpha^{3/2}\sigma W(t)$ satisfying

$$\sup_{0\le s\le t}\left|\widetilde{N}(s) - N(t)\right| = o(t^{1/r}), \quad \text{w.p.1 for some} \quad r > 2,$$

where W is a one-dimensional standard Brownian motion. Notice that, for the LIL: for $t > \mathrm{e}$ (Euler constant),

$$\frac{\sup\limits_{0\le s\le t}\left|N(t) - \bar{N}(s)\right|}{\sqrt{2t\log\log t}} \quad \text{and} \quad \frac{\sup\limits_{0\le s\le t}\left|\widetilde{N}(s) - \bar{N}(t)\right|}{\sqrt{2t\log\log t}} = \alpha^{3/2}\sigma\cdot\frac{\sup\limits_{0\le s\le t}|W(t)|}{\sqrt{2t\log\log t}} \tag{1.4}$$

have the same limit w.p.1, where the front one is the original problem and the latter one is its corresponding transformed problem. So, it suffices for us to consider the continuous Brownian problem on the right of the later one in (1.4). In doing so, we summarize the strong approximation method in the following three steps. First, we formulate the variability of performance measures into the LIL in the type (1.1), which is a discrete and stochastic problem as (1.4), and is generally difficult to find. Second, we transform the original LIL into continuous problem of BM. In this step, the corresponding fluid and strong approximations are needed to constructed in advance, which are the basis of transformation. Third, we aim to deal with the continuous BM problems with the help of properties of BM. Here, we emphasize that the most related to the strong approximation method is the theory of strong approximation for queueing networks, see [5, 19, 39] for generalized Jackson network, [7] for multiclass feedforward queueing networks, [14] for tandem-queue networks. In addition, readers can refer to [10, 11] for fundamental results of strong approximation for renewal process, which can be see as the basis for strong approximation of queueing networks.

We summarize our contributions as follows.

1. First, we study the asymptotic variability for a stage-K tandem queue and present it through five performance measures including the queue length, workload, busy time, idle time and departure processes.

2. Second, we formulate the variability into the LIL in type (1.2), and express all the LILs as analytical functions, which are constructed on data of the expectations and variances of interarrival times and service times.

3. Three, some interesting or counterintuitive insights are obtained, such as, the asymptotic variability associated with a service station only depends on the exogenous arrival (service times of the up-stream stage) and its service time when all stages are in the underloaded (overloaded) regime; a critically loaded or overloaded stage affects its first down-stream stage (all down-stages) in overloaded (underloaded) regime only through the variability of its service time, and has no influence on all down-stream overloaded stages from the second one; the Little's law in the view of LIL and FLIL holds in the underloaded and critically loaded regimes and fails in the overloaded regime; both the LILs and FLILs for the busy times and idle times are identical, which is consistent with our intuition because they increase and decrease like a see-saw.

We next give some notations appeared in the rest of the paper. Suppose that all the random variables and stochastic processes are defined on the probability space $(\Omega, \mathcal{F}, \mathsf{P})$. E is the expectation operator. Let $\mathbb{R}(\mathbb{R}_+)$ be the space of (non-negative) real numbers. Let $\mathbb{D}^k(H)$ be the space of right-continuous function with left limit defined on $H \subset \mathbb{R}$, and $\mathbb{D}_0^k(H) \equiv \{x \in \mathbb{D}^k(H) : x(0) \geq 0\}$ and $\mathbb{C}^k(H) \equiv \{x \in \mathbb{D}^k(H) : x(t) \text{ is continuous}\} \subset \mathbb{D}^k(H)$. When $k = 1$, let $\mathbb{D} \equiv \mathbb{D}^1$ and $\mathbb{C} \equiv \mathbb{C}^1$. For $f \in \mathbb{C}_0([0,\infty))$ and $f_n \in \mathbb{D}_0([0,\infty))$, $n = 1, 2, \ldots,$ if $\sup_{0\leq t\leq\infty} |f_n(t) - f(t)| \to 0$ as $n \to \infty$, we say $f_n \to f$ uniformly on compact set (u.o.c.). Let $\mathcal{K}_f \subset \mathbb{C}^k$, we say $f_n \rightrightarrows \mathcal{K}_f$ if $\{f_n\}$ is a functional sequence of relatively compact (i.e., every subsequence has a convergent subsubsequence) and the set of all limits of convergent subsubsequences is the compact set $\mathcal{K}_f$. Suppose that $\mathbf{1}$ is an indicator function satisfying $\mathbf{1}_C(t) = 1$ if $t \in C$ and 0 otherwise. Let η is a zero function defined on $\mathbb{R}$ satisfying $\eta(t) = 0$ for all $t \in \mathbb{R}$. For real numbers x and y in $\mathbb{R}$, denote $x \vee y = \max\{x, y\}$, $x \wedge y = \min\{x, y\}$ and $[x]^+ = \max\{x, 0\}$. For 1-dimensional real function $f(t)$, define norm $||f||_L = \sup_{0\leq t\leq L} |f(t)|$ for all $L \geq 0$. Let $\varphi(t) = \sqrt{2t \log\log t}$ for $t >$ e Euler constant. Let $\stackrel{\mathrm{d}}{=}$ denote equality in distribution.

2 The $GI/GI^K/1$/FCFS Queueing Model

The $GI/GI^K/1$/FCFS queueing system is composed of K single-server stations, $K \geq 2$. All customers are assumed to arrive at the first station for their first service through a unique exogenous arrival process, and go to the $(k+1)^{\text{st}}$ station after service completion at the k^{th} station, and leave the system after completion of service at the K^{th} station, $k = 1, 2, \ldots, K-1$. We call a customer in stage-k if she is waiting for or in service at station k, which is supposed to have an infinite buffer for customers waiting for service. At every stage, a first-come first-service (FCFS) service discipline is enforced, that is, all customers at every stage are served in the order of arrival. The FCFS service discipline is work-conserving, i.e., every station can not stay idle if there are customers waiting for service.

Primitive assumption. Let $v_0(n)$ be the interarrival time between the n^{th} and $(n+1)^{\text{st}}$ customers arriving from outside, and $v_k(n)$ be the n^{th} service time at stage-k, $k =$

$1,2,\ldots,K$. Let $v_k = \{v_k(n), n = 1,2,\ldots\}$ for $k = 0,1,\ldots,K$. Suppose that v_k are $K+1$ mutually independent sequences of non-negative and iid random variables, having means $\mathsf{E}[v_k(1)] \equiv 1/\mu_k$, variances $Var[v_k(1)]$ and *squared coefficients of variation* (SCVs) $c_k^2 \equiv Var[v_k(1)]/(\mathsf{E}[v_k(1)])^2$, respectively. Let

$$\mathcal{D} \equiv \left(\mu_0, \mu_k, c_0^2, c_k^2, 1 \le k \le K\right) \tag{2.1}$$

be the primitive data. Define partial sums, for $k = 0,1,\ldots,K$,

$$V_k(n) \equiv \sum_{i=1}^{n} v_k(i), \quad n = 1,2,\ldots, \tag{2.2}$$

and their corresponding renewal processes

$$S_k(t) \equiv \max\{n \ge 0 : V_k(n) \le t\}, \quad t \ge 0, \tag{2.3}$$

where $S_0(t)$ counts the total number of arrivals from outside in $[0,t]$ and $S_k(t)$ counts the number of service completions that the station at stage-k potentially serves in $[0,t]$, $k = 1,2,\ldots,K$. Define the traffic intensity at stage-k:

$$\rho_k = \frac{\min\{\mu_0, \mu_1, \ldots, \mu_{k-1}\}}{\mu_k}, \quad k = 1,2,\ldots,K. \tag{2.4}$$

We call stage-k in the underloaded regime if $\rho_k < 1$, the critically loaded regime if $\rho_k = 1$, and the overloaded regime if $\rho_k > 1$.

Performance measures for stage-k. At first, we let $k = 1,2,\ldots,K$ and $t \ge 0$. let $Q_k(t)$ be the queue length of stage-k and denote the total number of customers at stage-k at time t. Suppose $Q_k(0) = 0$ for all k and it means that the system is empty initially. Let $Z_k(t)$ be the workload of stage-k and denote the workload at stage-k at time t, and in words it is the total amount of work (measured in time) required to serve the current customers (including waiting for and in service) at stage-k assuming no future arrivals after time t. Let $T_k(t)$ be the busy time of the station at stage-k and denote the total amount of time the server is busy by time t at stage-k and is defined as:

$$T_k(t) = \int_0^t \mathbf{1}_{\{Q_k(s)>0\}} \mathsf{d}s, \quad k = 1,2,\ldots,K. \tag{2.5}$$

Let $I_k(t)$ be the idle time of station at stage-k in $[0,t]$, and satisfies

$$I_k(t) = t - T_k(t) \quad \text{for all} \quad t \ge 0. \tag{2.6}$$

Let $D_k(t) \equiv S_k(T_k(t))$ be the departure process of stage-k and count the total number of service completions of stage-k in $[0,t]$. Let $T_0(t) \equiv t$ for all $t \ge 0$. We have the following dynamic performance equations: for $k = 1,2,\ldots,K$,

$$Q_k(t) = S_{k-1}(T_{k-1}(t)) - S_k(T_k(t)) \ge 0, \tag{2.7}$$

$$Z_k(t) = V_k(S_{k-1}(T_{k-1}(t))) - T_k(t), \tag{2.8}$$

$$0 = \int_0^t Q_k(t) \mathsf{d}I_k(t), \tag{2.9}$$

where (2.7) holds by flow conservation, (2.8) holds because $V_k(S_{k-1}(T_{k-1}(t)))$ represents the total amount of work of all arrivals at stage-k in $[0,t]$, and (2.9) tells us that the idle time $I_k(t)$ increases only when $Q_k(t)=0$ under the FCFS service discipline.

Fluid approximation. For the performance measures above, the fluid approximation is used to characterize their mean values and then is involved in defining the LILs and the functional LILs. So, we present the fluid approximation of the $GI/GI^K/1$/FCFS queueing model as applications below. Define the following scaling process:

$$\bar{Q}_k^{(n)}(t) = \frac{1}{n} Q_k(nt), \quad k = 1, 2, \ldots, K.$$

Similarly, with the same token as $\bar{Q}_k^{(n)}(t)$ above we define scaling processes $\bar{Z}_k^{(n)}(t)$, $\bar{T}_k^{(n)}(t)$, $\bar{I}_k^{(n)}(t)$, $\bar{D}_k^{(n)}(t)$, respectively. Next, we show the results of the fluid approximation as follows and omit its proof. Readers can refer to [6, 12] for details if needed.

Lemma 2.1 (Fluid approximation)**.** *Suppose that the system is initially empty, and* $\mathsf{E}[v_k(1)] < \infty$*,* $k = 0, 1, \ldots, K$*, we have, for* $k = 1, 2, \ldots, K$*,*

$$\left(\bar{Q}_k^{(n)}, \bar{Z}_k^{(n)}, \bar{T}_k^{(n)}, \bar{I}_k^{(n)}, \bar{D}_k^{(n)}\right) \to \left(\bar{Q}_k, \bar{Z}_k, \bar{T}_k, \bar{I}_k, \bar{D}_k\right) \equiv \bar{\mathbb{X}}_k, \quad \textit{u.o.c., w.p.1,} \quad \textit{as } n \to \infty,$$

where

$$\begin{aligned}
&\bar{Q}_k(t) = \mu_k Z_k(t) = \bar{X}_k(t) + \bar{Y}_k(t) \geq 0, \\
&\bar{X}_k(t) = \mu_{k-1}\bar{T}_{k-1}(t) - \mu_k t, \quad \bar{Y}_k(t) = \Psi(\bar{X}_k)(t), \quad \bar{D}_k(t) = \mu_k \bar{T}_k(t), \\
&\bar{T}_0(t) = t, \quad \bar{T}_k(t) = t - \bar{I}_k(t), \quad \bar{I}_k(t) = \frac{1}{\mu_k}\Psi(\bar{X}_k)(t),
\end{aligned} \tag{2.10}$$

and functions Φ *and* Ψ *are defined on the functional space* $\mathbb{D}_0([0,\infty))$*, for all* $t \geq 0$*,*

$$\Psi(x)(t) \equiv \sup_{0\leq s\leq t}\{-x(s)\}^+ \quad \textit{and} \quad \Phi(x)(t) \equiv x(t) + \sup_{0\leq s\leq t}\{-x(s)\}^+. \tag{2.11}$$

Remark 2.1. (*Oblique reflection mapping*) *The mapping* (Ψ, Φ) *is known as the one dimensional oblique reflection mapping, and is Lipschitz continuous under the uniform topology, see Chapter 6 in [8]. In Lemma 2.1, the assumption that the system is initially empty implies that* $\bar{X}_k(0) = 0$*, then* (2.11) *is equivalent to, for* $\bar{X}_k$*,*

$$\Psi(\bar{X}_k)(t) \equiv \sup_{0\leq s\leq t}\{-\bar{X}_k(s)\} \quad \textit{and} \quad \Phi(\bar{X}_k)(t) \equiv \bar{X}_k(t) + \sup_{0\leq s\leq t}\{-\bar{X}_k(s)\}. \tag{2.12}$$

In addition, we note that the Lipschitz constant for Φ *is 2: for any given* $\bar{X}$ *and* $\bar{X}'$*,*

$$\begin{aligned}
\left|\left|\Phi(\bar{X}) - \Phi(\bar{X}')\right|\right|_L &\leq \sup_{0\leq t\leq L}\left|\bar{X}(t) - \bar{X}'(t)\right| + \left|\sup_{0\leq s\leq t}\{-\bar{X}(s)\}^+ - \sup_{0\leq s\leq t}\{-\bar{X}'(s)\}^+\right| \\
&\leq \left|\left|\bar{X} - \bar{X}'\right|\right|_L + \left|\sup_{0\leq s\leq t}\{\bar{X}(s) - \bar{X}'(s)\}\right| \\
&= 2\left|\left|\bar{X} - \bar{X}'\right|\right|_L.
\end{aligned} \tag{2.13}$$

From now, our objective is to find three types of LILs and one functional LIL for performance measures $(Q_k, Z_k, B_k, I_k, D_k)$ and identify them as simple analytic functions and compact sets of continuous functions, respectively, which are all composed of the primitive data $\mathcal{D}$ in (2.1).

3 Main results

We develop the following LILs for The $GI/GI^K/1/\text{FCFS}$ queueing system:

$$\begin{aligned} Q_k^* &= \limsup_{L\to\infty} \frac{\left|\left|Q_k - \bar{Q}_k\right|\right|_L}{\varphi(L)}, \quad Z_k^* = \limsup_{L\to\infty} \frac{\left|\left|Z_k - \bar{Z}_k\right|\right|_L}{\varphi(L)}, \\ T_k^* &= \limsup_{L\to\infty} \frac{\left|\left|T_k - \bar{T}_k\right|\right|_L}{\varphi(L)}, \quad I_k^* = \limsup_{L\to\infty} \frac{\left|\left|I_k - \bar{I}_k\right|\right|_L}{\varphi(L)}, \\ D_k^* &= \limsup_{L\to\infty} \frac{\left|\left|D_k - \bar{D}_k\right|\right|_L}{\varphi(L)}, \quad k = 1, 2, \ldots, K. \end{aligned} \tag{3.1}$$

For convenience, we let $\mathcal{X}_k^* \equiv (Q_k^*, Z_k^*, T_k^*, I_k^*, D_k^*)$, and formulate them by the model data in (2.1).

At first, we give the following hypothesis throughout the rest of the paper. For some $r > 2$,

$$\mathsf{E}\left[v_k(1)^r\right] < \infty \quad \text{for all} \quad k = 0, 1, \ldots, K. \tag{3.2}$$

Next, we give our main results. Theorem 3.1 shows us the LILs for the case without the critically loaded stages, in words, all the stages are in the underloaded regime or overloaded regime or both; Theorem 3.2 corresponds to the case with only one critically loaded stage, more specifically, the other stages are in the underloaded regime or overloaded regime.

Theorem 3.1 (Without critically loaded stages)**.** *Suppose that* (3.2) *holds. Given data* $\mathcal{D}$ *in* (2.1)*, we have the following LILs in three cases.*
Case 1. *All stages are in the underloaded regime, that is,* $\rho_k < 1$ *for all* $k = 1, 2, \ldots, K$*, we have*

$$\mathcal{X}_k^* = \left(0, 0, \frac{\sqrt{\mu_0(c_0^2 + c_k^2)}}{\mu_k}, \frac{\sqrt{\mu_0(c_0^2 + c_k^2)}}{\mu_k}, \mu_0^{1/2} c_0\right). \tag{3.3}$$

Case 2. *All stages are in the overloaded regime, that is,* $\rho_k > 1$ *for all* $k = 1, 2, \ldots, K$*, we have*

$$\mathcal{X}_k^* = \left(\sqrt{\mu_{k-1} c_{k-1}^2 + \mu_k c_k^2}, \frac{\sqrt{\mu_{k-1}(c_{k-1}^2 + c_k^2)}}{\mu_k}, 0, 0, \mu_k^{1/2} c_k\right). \tag{3.4}$$

Case 3. *Some stages are in the underloaded regime and the others are in the overloaded regime. In words, there exist* l_0 *integers* $k'_1, k'_2, \ldots, k'_{l_0} (l_0 < K) : 0 < k'_1 \le k'_2 \le \cdots \le k'_{l_0} \le K$ *such that* $\rho_k > 1$ *for all* $k = k'_1, k'_2, \ldots, k'_{l_0}$ *and* $\rho_k < 1$ *for all* $k \neq k'_1, k'_2, \ldots, k'_{l_0}$*. We have the following LILs for stages with* $l = 1, 2, \ldots, l_0$*:*

$$\mathcal{X}_{k'_l}^* = \left(\sqrt{\mu_{k'_{l-1}} c_{k'_{l-1}}^2 + \mu_{k'_l} c_{k'_l}^2}, \frac{\sqrt{\mu_{k'_{l-1}}(c_{k'_{l-1}}^2 + c_{k'_l}^2)}}{\mu_{k'_l}}, 0, 0, \mu_{k'_l}^{1/2} c_{k'_l}\right). \tag{3.5}$$

The LIL for $k = 1, 2, \ldots, k'_1 - 1$ satisfy (3.3). For $k = k'_l + 1, k'_l + 2, \ldots, k'_{l+1} - 1$ and $l = 1, 2, \ldots, l_0$,

$$\mathcal{X}_k^* = \left(0, 0, \frac{\sqrt{\mu_{k'_l}(c_{k'_l}^2 + c_k^2)}}{\mu_k}, \frac{\sqrt{\mu_{k'_l}(c_{k'_l}^2 + c_k^2)}}{\mu_k}, \mu_{k'_l}^{1/2} c_{k'_l}\right). \tag{3.6}$$

In order to understand Theorem 3.1 intuitively, we give the following Remarks.

Remark 3.1 (Understanding the underloaded case)**.** *When all stages are in the underloaded regime, that is, $\rho_k < 1$ for all $k = 1, 2, \ldots, K$, the system is in light traffic. It follows from Dai [12] that the Markov process describing the dynamic performance has a stability distribution. This implies that the LIL for the queue length is zero. As a result, the LIL for the workload is zero by the Little's law. An interesting phenomenon appears in the asymptotic variability of the busy time, idle time and departure processes. In light traffic, the departure process of a given stage inherits the variability from its arrival process, which is the variance parameter $\mu_0^{1/2} c_0$ in (3.3). So, every stage seems like an $GI/G/1$ queue with arrival process $S_0(t)$. The replacement of $S_0(t)$ as arrival process to every stage is well reflected in the LILs of busy and idle times for stage-k, which are composed of μ_0 and μ_k, and have nothing to do with μ_{k-1} for $k = 2, 3, \ldots, K$.*

Remark 3.2 (Understanding the overloaded case)**.** *When every stage is in the overloaded regime, that is, $\rho_k > 1$ for all $k = 1, 2, \ldots, K$, every stage is very heavy. Intuitively, each stage is almost busy all the time, the variability of the departure almost depends on its own service times. This explains the LIL of the k-th departure is $\mu_k^{1/2} c_k$ very well. Since all stages almost keep busy, the busy time $T_k(t) \approx t$ for large t and then has a little variability. This follow a LIL zero for the busy time. So, it does for the idle time. A counter-intuitive phenomenon appears in the LILs for the queue length and workload processes. Different from the fluid and diffusion approximations for tandem queue, their relationship reveals a failed Little's law in the version of LIL, that is, the LIL $Q_k^* \neq \mu_k Z_k^*$. This inequality holds because Z_k^* captures the variability of the workload of customers waiting for or in service at a stage, and Q_k^* describes the variability of number of these customers. These customers will "never" be served in the overloaded regime, and have a huge impact on the workload if their service times are highly variable, however, they do not affect the queue length.*

Remark 3.3 (Understanding the hybrid case with the underload and overloaded)**.** *As described in* **Case 3** *in Theorem 3.1, stages $k'_1, k'_2, \ldots, k'_{l_0}$ are very heavy, the others are in light traffic. The tandem queue can be broken into l_0+1 sub-tandem queues: the subsystem-l consisting of stages $k'_{l-1} + 1, k'_{l-1} + 2, \ldots, k'_l$ for $l = 1, 2, \ldots, l_0$, and the subsystem-$(l_0 + 1)$ consisting of stages $k'_{l_0} + 1, k'_{l_0} + 2, \ldots, K$. The first l_0 subsystems operate similarly because, for every subsystem, the last stage is congested and the others are in light traffic. Since the stage-k'_{l-1} is very heavy, its contribution to the variability of subsystem-l is the variance parameter $c_{k'_{l-1}}$, appeared in $\sqrt{\mu_{k'_{l-1}}(c_{k'_{l-1}}^2 + c_k^2)}$ (3.6), $k = k'_{l-1}+1, k'_{l-1}+2, \ldots, k'_l - 1$. For stage-$k'_l$ in the subsystem-l, the Little's law fails because of the same reason as in Remark*

3.2. Notice that stages $k'_{l-1}+1, k'_{l-1}+2, \ldots, k'_l-1$ are in light traffic, their variabilities of service times have no influence on the variability of stage-k'_l. This interesting result is embodied in LILs in (3.5), which have the variance parameters $c_{k'_{l-1}}$ and $c_{k'_l}$ only. For subsystem-(l_0+1), it is almost the same as an underloaded system except the exogenous arrival process $S_{k'_{l_0}}(T_{k'_{l_0}}(t))$, operating similarly with renewal process $S_{k'_{l_0}}(t)$ given that stage-k'_{l_0} is in heavy traffic. So, the corresponding LILs satisfy (3.3) with $\mu_{k'_{l_0}}$ and $c_{k'_{l_0}}$ replacing μ_0 and c_0, respectively.

Next, we give the LILs with one critically loaded stage.

Theorem 3.2 (With one critically loaded stage)**.** *Suppose that (3.2) holds and $\rho_{k_0}=1$ for some k_0. Given the model data (2.1), we have the following LILs.*
Case 1. *If $\rho_k<1$ for all $k\neq k_0$, then the LILs $\mathcal{X}^*_k$ for stages $1,2,\ldots,k_0-1$ are given in (3.3). For stage k_0,*

$$\mathcal{X}^*_{k_0}=\left(\sqrt{\mu_0}C_{0,k_0},\frac{C_{0,k_0}}{\sqrt{\mu_0}},\frac{C_{0,k_0}}{\sqrt{\mu_0}},\frac{C_{0,k_0}}{\sqrt{\mu_0}},\sqrt{\mu_0}C_{0,k_0}\right) \tag{3.7}$$

where $C_{0,k_0}=\sqrt{c_0^2+c_{k_0}^2}$. For stage k_0+l with $l=1,2,\ldots,K-k_0$,

$$\mathcal{X}^*_{k_0+l} \;=\; \left(0,0,\frac{\sqrt{\mu_0(c_0^2\vee c_{k_0}^2+c_{k_0+l}^2)}}{\mu_{k_0+l}},\frac{\sqrt{\mu_0(c_0^2\vee c_{k_0}^2+c_{k_0+l}^2)}}{\mu_{k_0+l}},\mu_0^{1/2}(c_0\vee c_{k_0})\right). \tag{3.8}$$

Case 2. *If $\rho_k>1$ for all $k\neq k_0$, then the LILs $\mathcal{X}^*_k$ for stages 1 to k_0-1 are given by (3.4). For stage k_0,*

$$\mathcal{X}^*_{k_0}=\left(\sqrt{\mu_{k_0}}C_{k_0-1,k_0},\frac{C_{k_0-1,k_0}}{\sqrt{\mu_{k_0}}},\frac{C_{k_0-1,k_0}}{\sqrt{\mu_{k_0}}},\frac{C_{k_0-1,k_0}}{\sqrt{\mu_{k_0}}},\mu_{k_0}^{1/2}(c_{k_0-1}\vee c_{k_0})\right). \tag{3.9}$$

where $C_{k_0-1,k_0}=\sqrt{c_{k_0-1}^2+c_{k_0}^2}$. For stage k_0+1, $\mathcal{X}^*_{k_0+1}=$

$$\left(\sqrt{\mu_{k_0}(c_{k_0-1}^2\vee c_{k_0}^2)+\mu_{k_0+1}c_{k_0+1}^2},\frac{\sqrt{\mu_{k_0}(c_{k_0-1}^2\vee c_{k_0}^2+c_{k_0+1}^2)}}{\mu_{k_0+1}},0,0,\mu_{k_0+1}^{1/2}c_{k_0+1}\right). \tag{3.10}$$

For stages k_0+l, $l=2,3,\ldots,K-k_0$, $\mathcal{X}^*_{k_0+l}=$

$$\left(\sqrt{\mu_{k_0+l-1}c_{k_0+l-1}^2+\mu_{k_0+l}c_{k_0+l}^2},\frac{\sqrt{\mu_{k_0+l-1}(c_{k_0+l-1}^2+c_{k_0+l}^2)}}{\mu_{k_0+l}},0,0,\mu_{k_0+l}^{1/2}c_{k_0+l}\right). \tag{3.11}$$

We give the following two Remarks to help readers intuitively understand Theorem 3.2.

Remark 3.4 (Understanding the hybrid case with the critically loaded and underload)**.** *In **Case 1**, all stages except k_0 are in the underloaded regime, thus the first k_0-1 stages can be thought of as an independent (k_0-1)-stage tandem queue in light traffic, and then* *have limits in* (3.3)*. However, stages $k_0+1, k_0+2, \ldots, K$ are all in the influence of stage-k_0, and operate very differently. Because the first $k_0 - 1$ stages are in light traffic, they intuitively affect stage k_0 through the exogenous arrival process as if there is only stage-*1 *before stage-k_0. In words, it seems that stage k_0 is a $GI/G/1$ queue with the arrival process $S_0(t)$, which is embodied by the variance parameter c_0 of term $C_{0,k_0} = \sqrt{c_0^2 + c_{k_0}^2}$ in* (3.7)*, $k = 2, 3, \ldots, k_0-1$. Another interesting phenomenon is the variability associated with stages $k_0+1, k_0+2, \ldots, K$. That stage-k_0 operates like an independent $GI/G/1$ with arrival $S_0(t)$ is strongly perceived by stages after k_0. In words, stages $k_0+1, k_0+2, \ldots, K$ do not realize the existence of stages* 1 *to $k_0 - 1$. For stages after k_0, the LILs for the queue length and workload is negligible because all stages are essentially in light traffic. For busy or idle times of stage-(k_0+l), the variability effect from stage-k_0 is embodied by the sum of two terms* (a) *the variability $c_{k_0+l}^2$ of service times of stage-(k_0+l) and* (b) *the maximum value $c_0^2 \vee c_{k_0}^2$ of the variance parameters of "its arrival" and service time: only one, whichever is bigger, plays a role. For the LIL of departure, since the service time has no influence on them when the system is in underloaded regime, all the LILs depend on the arrival variability of stage-k_0, i.e., $c_0^2 \vee c_{k_0}^2$.*

Remark 3.5 (Understanding the hybrid case with the critically loaded and overload)**.** *In* **Case 2***, all stages except k_0 are in the overloaded regime, the first $k_0 - 1$ stages consist of an independent tandem queue. As a result, the corresponding LILs are given in* (3.4) *with K replaced by $k_0 - 1$. Since the upstream stage-$(k_0 - 1)$ is in heavy traffic, its departure process $S_{k_0-1}(T_{k_0-1}(t))$, which is exactly the arrival process to stage-k_0, almost depends on its own service times. So, station-k_0 is like a $GI/G/1$ in critically loaded regime, and it follows the LILs in* (3.9)*, which distinguishes from* (3.7) *through SCV $c_{k_0-1}^2$ in $C_{k_0-1,k_0} = \sqrt{c_{k_0-1}^2 + c_{k_0}^2}$. Different from* **Case 1***, the influence of the overloaded stage-k_0 on stage-(k_0+1) and the later stages is different. For stage-(k_0+1), the influence of stage-k_0 on the queue length and workload is the sum of two terms (a) the variability of service time at stage-(k_0+1), and (b) the maximum value $c_{k_0-1}^2 \vee c_{k_0}^2$ of the variance parameters of "its arrival" and service time: only one, whichever is bigger, plays a role. We also note the reason that the Little's law fails for stage-(k_0+1) is due to the overloaded regime. It is also intuitive that the LILs of the departure process for stage-(k_0+1) only depend on the service times. For stages after k_0+1, the variability influence from stage-k_0 disappears due to the overloaded regime. This explains why the LILs in* (3.4) *and* (3.11) *are similar.*

4 Proofs of Main Results

We prove Theorems 3.1 and 3.2 in this section. Before the proofs, we present some Lemmas needed, and put their proofs except short ones at the back of proofs of two Theorems.

In Subsection 4.1, we give some results on strong approximation; In Subsection **??**, we give a result on the underloaded regime and another result on the continuous mapping of relatively compact; We prove Theorem 3.1 in Subsection 4.2, Theorem 3.2 in Subsection 4.3, and Lemma 4.3 in Subsection 4.4.

4.1 Strong Approximations

Strong approximation approximates discrete stochastic process by a function of Brownian motion, and shows us an asymptotic error which is a basis supporting the proof. Next, we give the strong approximations for the tandem queue, and omit its proof because the tandem queue is a special case of generalized Jackson network and feedforward queueing network. Readers can refer to [7, 8] for more general proofs.

Lemma 4.1 (**Strong approximations for $GI/GI^K/1$ queue).** *If* (3.2) *holds, then, for some $r > 2$,*

$$\left\|Q_k - \widetilde{Q}_k\right\|_L = o(T^{1/r}), \quad \left\|Z_k - \widetilde{Z}_k\right\|_L = o(T^{1/r}), \quad \left\|T_k - \widetilde{T}_k\right\|_L = o(T^{1/r}),$$
$$\left\|I_k - \widetilde{I}_k\right\|_L = o(T^{1/r}), \quad \left\|D_k - \widetilde{D}_k\right\|_L = o(T^{1/r}), \quad \textit{w.p.1}, \quad k = 1, 2, \ldots, K, \tag{4.1}$$

where $\widetilde{I}_0(t) \equiv 0$ and for all $k = 1, 2, \ldots, K$,

$$\begin{aligned}
&(\widetilde{Q}_k, \widetilde{Y}_k) = (\Phi, \Psi)(\widetilde{X}_k), \quad \widetilde{I}_k(t) = \frac{1}{\mu}\widetilde{Y}_k(t), \quad \widetilde{T}_k(t) = t - \widetilde{I}_k(t),\\
&\widetilde{X}_k(t) = (\mu_{k-1} - \mu_k)t - \mu_{k-1}\widetilde{I}_{k-1}(t) + \mu_{k-1}^{1/2}c_{k-1}W_{k-1}(\bar{T}_{k-1}(t)) - \mu_k^{1/2}c_kW_k(\bar{T}_k(t)),\\
&\widetilde{Z}_k(t) = \frac{1}{\mu_k}\left[\widetilde{Q}_k(t) + \mu_k^{1/2}c_k(W_k(\bar{T}_k(t)) - W_k(\rho_k t))\right],\\
&\widetilde{D}_k(t) = \mu_k\widetilde{T}_k(t) + \mu_k^{1/2}c_kW_k(\bar{T}_k(t)),
\end{aligned} \tag{4.2}$$

and W_0 and W_k are mutually independent Brownian motions associated with the exogenous arrival process and service process at stage-k, respectively, Φ and Ψ are defined in (2.11).

We develop $\mathcal{X}_k^*$ through the LILs associated with the BM system approximated by the strong approximation. So, we defined the LILs for the performance measures in (4.2):

$$\begin{aligned}
&\widetilde{Q}_k^* = \limsup_{L\to\infty} \frac{\left\|\widetilde{Q}_k - \bar{Q}_k\right\|_L}{\varphi(L)}, \quad \widetilde{Z}_k^* = \limsup_{L\to\infty} \frac{\left\|\widetilde{Z}_k - \bar{Z}_k\right\|_L}{\varphi(L)},\\
&\widetilde{T}_k^* = \limsup_{L\to\infty} \frac{\left\|\widetilde{T}_k - \bar{T}_k\right\|_L}{\varphi(L)}, \quad \widetilde{I}_k^* = \limsup_{L\to\infty} \frac{\left\|\widetilde{I}_k - \bar{I}_k\right\|_L}{\varphi(L)},\\
&\widetilde{D}_k^* = \limsup_{L\to\infty} \frac{\left\|\widetilde{D}_k - \bar{D}_k\right\|_L}{\varphi(L)}, \quad k = 1, 2, \ldots, K.
\end{aligned} \tag{4.3}$$

Next, we give an elementary Lemma, which transfers the original limits into a continuous problem associated with the fluid and strong approximations.

Lemma 4.2. *Suppose that* (3.2) *holds, we have $\mathcal{X}_k^* = \left(\widetilde{Q}_k^*, \widetilde{Z}_k^*, \widetilde{T}_k^*, \widetilde{I}_k^*, \widetilde{D}_k^*\right)$ w.p.1.*

Proof. All proofs are similar, we only give a proof for queue length. Firstly, we note that $T^{1/r} = o(\varphi(T))$ for $r > 2$, and

$$\left|\frac{\left\|\widetilde{Q}_k - \bar{Q}_k\right\|_L}{\varphi(L)} - \frac{\left\|Q_k - \widetilde{Q}_k\right\|_L}{\varphi(L)}\right| \leq \frac{\left\|Q_k - \bar{Q}_k\right\|_L}{\varphi(L)} \leq \frac{\left\|\widetilde{Q}_k - \bar{Q}_k\right\|_L}{\varphi(L)} + \frac{\left\|Q_k - \widetilde{Q}_k\right\|_L}{\varphi(L)}.$$

Lemma 4.1 implies that $\lim_{L\to\infty} \left|\left|Q_k - \widetilde{Q}_k\right|\right|_L / \varphi(L) = 0$ w.p.1. As a result, $\left|\left|Q_k - \bar{Q}_k\right|\right|_L / \varphi(L)$ and $\left|\left|Q_k - \bar{Q}_k\right|\right|_L / \varphi(L)$ have the same limits if they exist as $L \to \infty$. In other words, $Q_k^* = \widetilde{Q}_k^*$ for $k = 1, 2, \ldots, K$. Hence, all results are proved. $\square$

By Lemma (4.2), it suffices to compute the LILs in (4.3) if we plan to find (3.1). Finally, we give the following classical result on the LIL for the BM. For a one-dimensional standard Brownian motion $\widehat{W}$, we have

$$\limsup_{L\to\infty} \frac{\left|\left|\widehat{W}\right|\right|_L}{\varphi(L)} = 1, \quad \text{w.p.1.} \tag{4.4}$$

Next, we give the following Lemmas 4.3 and 4.4 and postpone their proofs to Subsections 4.4 and 4.5, respectively.

Lemma 4.3. *If $\rho_k < 1$ for some stage-k, then, $Q_k^* = Z_k^* = 0$ w.p.1.*

Lemma 4.4. *If $\rho_k > 1$ for some stage-k which is in cases without and with one critically loaded stage, corresponding to Theorems 3.1 and 3.2 respectively, then, $I_k^* = T_k^* = 0$ w.p.1.*

4.2 Proof of Theorem 3.1

We prove all cases one by one.

Case 1. Since $\rho_k < 1$ for all $k = 1, 2, \ldots, K$, then $\mu_0 < \mu_1 < \cdots < \mu_K$ and $\rho_k = \mu_0/\mu_k$, the corresponding fluid solution:

$$\bar{\mathbb{X}}_k(t) \;=\; (\eta, \eta, \rho_k t, (1-\rho_k)t, \mu_0 t), \quad k = 1, 2, \ldots, K. \tag{4.5}$$

By Lemma 4.3, we note that $Q_k^* = Z_k^* = 0$ for all $k = 1, 2, \ldots, K$, w.p.1. For idle time I_k, by (4.2) and (4.5),

$$\begin{aligned}
\widetilde{I}_k(t) - \bar{I}_k(t) &= \frac{1}{\mu_k}\left[\widetilde{Y}_k(t) - \bar{Y}_k(t)\right] \\
&= \frac{1}{\mu_k}\widetilde{Q}_k(t) - \frac{1}{\mu_k}\left[\widetilde{X}_k(t) - \bar{X}_k(t)\right],
\end{aligned}$$

where, because $\bar{X}_k(t) = (\mu_0 - \mu_k)t$ under condition $\rho_k < 1$ for all $k = 1, 2, \ldots, K$,

$$\begin{aligned}
& \widetilde{X}_k(t) - \bar{X}_k(t) \\
= \;& \mu_{k-1}\widetilde{T}_{k-1}(t) - \mu_0 t + \mu_{k-1}^{1/2} c_{k-1} W_{k-1}(\bar{T}_{k-1}(t)) - \mu_k^{1/2} c_k W_k(\bar{T}_k(t)) \\
= \;& -\widetilde{Y}_{k-1}(t) - \bar{X}_{k-1}(t) + \mu_{k-1}^{1/2} c_{k-1} W_{k-1}(\bar{T}_{k-1}(t)) - \mu_k^{1/2} c_k W_k(\bar{T}_k(t)) \\
= \;& \left[\widetilde{X}_{k-1}(t) - \bar{X}_{k-1}(t)\right] - \widetilde{Q}_{k-1}(t) + \mu_{k-1}^{1/2} c_{k-1} W_{k-1}(\bar{T}_{k-1}(t)) - \mu_k^{1/2} c_k W_k(\bar{T}_k(t)) \\
= \;& \left[\widetilde{X}_1(t) - \bar{X}_1(t)\right] - \sum_{l=1}^{k-1} \widetilde{Q}_l(t) + \mu_1^{1/2} c_1 W_1(\bar{T}_1(t)) - \mu_k^{1/2} c_k W_k(\bar{T}_k(t)) \\
= \;& -\sum_{l=1}^{k-1} \widetilde{Q}_l(t) + \mu_0^{1/2} c_0 W_0(t) - \mu_k^{1/2} c_k W_k(\bar{T}_k(t)),
\end{aligned} \tag{4.6}$$

so,

$$\begin{aligned}\widetilde{I}_k(t)-\bar{I}_k(t) &= \frac{1}{\mu_k}\left[\widetilde{Y}_k(t)-\bar{Y}_k(t)\right] \\ &= \frac{1}{\mu_k}\sum_{l=1}^{k}\widetilde{Q}_l(t)-\frac{1}{\mu_k}\left[\mu_0^{1/2}c_0W_0(t)-\mu_k^{1/2}c_kW_k(\bar{T}_k(t))\right].\end{aligned}$$

Notice that $Q_k^*=0$ w.p.1 on $[0,1]$, and $\bar{T}_k(t)=(\mu_0/\mu_k)t$ for all $t\geq 0$, it says that $\mu_0^{1/2}c_0W_0(t)-\mu_k^{1/2}c_kW_k(\bar{T}_k(t))$ is a driftless Brownian motion with variance parameter $\mu_0(c_0^2+c_k^2)$. As a result, the LIL $I_k^*=\sqrt{\mu_0(c_0^2+c_k^2)}/\mu_k$. For busy time T_k, by (2.10) and (4.2), $\widetilde{T}_k(t)-\bar{T}_k(t)=-\left[\widetilde{I}_k(t)-\bar{I}_k(t)\right]$. This follows that the LIL $T_k^*=\sqrt{\mu_0(c_0^2+c_k^2)}/\mu_k$. For departure process D_k, by (2.10), (4.2) and (4.6),

$$\begin{aligned}\widetilde{D}_k(t)-\bar{D}_k(t) &= \mu_k\left[\widetilde{T}_k(t)-\bar{T}_k(t)\right]+\mu_k^{1/2}c_kW_k(\bar{T}_k(t)) \\ &= -\mu_k\left[\widetilde{I}_k(t)-\bar{I}_k(t)\right]+\mu_k^{1/2}c_kW_k(\bar{T}_k(t)) \\ &= -\widetilde{Q}_k(t)+\left[\widetilde{X}_k(t)-\bar{X}_k(t)\right]+\mu_k^{1/2}c_kW_k(\bar{T}_k(t)) \\ &= -\sum_{l=1}^{k}\widetilde{Q}_l(t)+\mu_0^{1/2}c_0W_0(t).\end{aligned}$$

This, together with $Q_k^*=0$, implies that the norm-LIL $D_k^*=\mu_0^{1/2}c_0$ w.p.1.

Hence, we get all the LILs in (3.3).

Case 2. Suppose that $\rho_k>1$ for all $k=1,2,\ldots,K$. We have $\mu_0>\mu_1>\cdots>\mu_K$ and $\rho_k=\mu_{k-1}/\mu_k$, the corresponding fluid solution:

$$\bar{\mathbb{X}}_k(t) = ((\mu_{k-1}-\mu_k)t,(\rho_k-1)t,t,\eta,\mu_kt), \quad k=1,2,\ldots,K. \tag{4.7}$$

By (2.10) and (4.2), $\bar{X}_k(t)=(\mu_{k-1}-\mu_k)t$ for all $k=1,2,\ldots,K$ and $t\geq 0$.

At first, By Lemma 4.4, we note that $I_k^*=T_k^*=0$ for all $k=1,2,\ldots,K$, w.p.1.

For queue length Q_k, $k=1,2,\ldots,K$, by (2.10) and (4.2) we have $\bar{Q}_k(t)=\bar{X}_k(t)=(\mu_{k-1}-\mu_k)t$ for all $t\geq 0$, and

$$\begin{aligned}\widetilde{Q}_k(t)-\bar{Q}_k(t) &= \widetilde{X}_k(t)-\bar{X}_k(t)+\widetilde{Y}_k(t) \\ &= \mu_k\widetilde{I}_k(t)-\mu_{k-1}\widetilde{I}_{k-1}(t)+\mu_{k-1}^{1/2}c_{k-1}W_{k-1}(t)-\mu_k^{1/2}c_kW_k(t).\end{aligned} \tag{4.8}$$

Since $I_k^*=0$ w.p.1 for all $k=1,2,\ldots,K$, we have $Q_k^*=\sqrt{\mu_{k-1}c_{k-1}^2+\mu_kc_k^2}$ w.p.1 for all $k=1,2,\ldots,K$.

For workload Z_k, by (2.10) and (4.2), $\bar{Z}_k(t)=\bar{Q}_k(t)/\mu_k=(\rho_k-1)t$ for all $t\geq 0$, this, together with (4.8), implies that

$$\begin{aligned}\widetilde{Z}_k(t)-\bar{Z}_k(t) &= \frac{1}{\mu_k}\left\{\left[\widetilde{Q}_k(t)-\bar{Q}_k(t)\right]+\mu_k^{1/2}c_k(W_k(t)-W_k(\rho_kt))\right\} \\ &= \frac{1}{\mu_k}\left[\widetilde{X}_k(t)-\bar{X}_k(t)+\widetilde{Y}_k(t)+\mu_k^{1/2}c_k(W_k(t)-W_k(\rho_kt))\right] \\ &= \widetilde{I}_k(t)-\frac{\mu_{k-1}}{\mu_k}\widetilde{I}_{k-1}(t)+\frac{1}{\mu_k}\left[\mu_{k-1}^{1/2}c_{k-1}W_{k-1}(t)-\mu_k^{1/2}c_kW_k(\rho_kt)\right].\end{aligned}$$

Notice that $I_k^*=0$ w.p.1 by Lemma 4.4, we have $Z_k^*=\sqrt{\mu_{k-1}(c_{k-1}^2+c_k^2)/\mu_k}$ w.p.1 for all $k=1,2,\ldots,K$.

For departure D_k, by (2.10) and (4.2),

$$\widetilde{D}_k(t)-\bar{D}_k(t)=\mu_k\left[\widetilde{T}_k(t)-\bar{T}_k(t)\right]+\mu_k^{1/2}c_kW_k(t)=-\mu_k\widetilde{I}_k(t)+\mu_k^{1/2}c_kW_k(t).$$

By Lemma 4.4, we have $D_k^*=\mu_k^{1/2}c_k$ for all $k=1,2.\ldots,K$, w.p.1.

Hence, we get all the LILs in (3.4).

Case 3. Since it is assumed that $k_1',k_2',\ldots,k_{l_0}' : 0<k_1'\le k_2'\le\cdots\le k_{l_0}'\le K$ satisfy that $\rho_k>1$ for all $k=k_1',k_2',\ldots,k_{l_0}'$ and $\rho_k<1$ for all $k\ne k_1',k_2',\ldots,k_{l_0}'$, then by the definition of traffic intensity (2.4),

$$\begin{aligned}&\mu_{k_i'}<\mu_{k_{i-1}'}<\mu_k, \quad k=k_{i-1}'+1,k_{i-1}'+2,\ldots,k_i'-1, \quad i=1,2,\ldots,l_0,\\ &\mu_{k_{l_0}'}<\mu_k, \quad k=k_{l_0'}+1,k_{l_0'}+2,\ldots,K.\end{aligned}\tag{4.9}$$

We firstly proceed to find the LIL and the functional LIL for all $k=k_1',k_2',\ldots,k_{l_0}'$ and then limits for $k\ne k_1',k_2',\ldots,k_{l_0}'$.

By Lemma 4.4, we note that $I_k^*=T_k^*=0$ for all $k:\rho_k>1$, w.p.1.

For Q_k, $k=k_1',k_2',\ldots,k_{l_0}'$, $\widetilde{Q}_{k_l'}(t)=\widetilde{X}_{k_l'}(t)+\widetilde{Y}_{k_l'}(t)$ by (4.2). Similarly with (4.46) and (4.47), we have, for all $l=1,2,\ldots,$

$$\begin{aligned}\widetilde{Q}_{k_l'}(t)-\bar{Q}_{k_l'}(t) &= -\mu_{k_{l-1}'}\widetilde{I}_{k_{l-1}'}(t)+\mu_{k_l'}\widetilde{I}_{k_l'}(t)-\sum_{i=k_{l-1}'+1}^{k_l'-1}\widetilde{Q}_i(t)\\ &\quad+\mu_{k_{l-1}'}^{1/2}c_{k_{l-1}'}W_{k_{l-1}'}(\bar{T}_{k_{l-1}'}(t))-\mu_{k_l'}^{1/2}c_{k_l'}W_{k_l'}(\bar{T}_{k_l'}(t)),\end{aligned}\tag{4.10}$$

where $k_0'\equiv 0$. Notice that $I_k^*=0$ for all $k=k_1',k_2',\ldots,k_{l_0}'$ by Lemma 4.4, and $Q_k^*=0$ for all $k\ne k_1',k_2',\ldots,k_{l_0}'$ by Lemma 4.3, w.p.1. Then, for all $l=1,2,\ldots,l_0$, $Q_{k_l'}^*=\sqrt{\mu_{k_{l-1}'}c_{k_{l-1}'}^2+\mu_{k_l'}c_{k_l'}^2}$, w.p.1, because $\mu_{k_{l-1}'}^{1/2}c_{k_{l-1}'}W_{k_{l-1}'}(\bar{T}_{k_{l-1}'}(t))-\mu_{k_l'}^{1/2}c_{k_l'}W_{k_l'}(\bar{T}_{k_l'}(t))$ has variance parameter $\mu_{k_{l-1}'}c_{k_{l-1}'}^2+\mu_{k_l'}c_{k_l'}^2$.

For Z_k, $k=k_1',k_2',\ldots,k_{l_0}'$, by (4.2) and (4.10),

$$\begin{aligned}\widetilde{Z}_{k_l'}(t)-\bar{Z}_{k_l'}(t) &= \frac{1}{\mu_{k_l'}}\left[\widetilde{Q}_{k_l'}(t)-\bar{Q}_{k_l'}(t)+\mu_{k_l'}^{1/2}c_{k_l'}(W_{k_l'}(\bar{T}_{k_l'}(t))-W_{k_l'}(\rho_{k_l'}t))\right]\\ &= \frac{1}{\mu_{k_l'}}\Bigg[-\mu_{k_{l-1}'}\widetilde{I}_{k_{l-1}'}(t)+\mu_{k_l'}\widetilde{I}_{k_l'}(t)-\sum_{i=k_{l-1}'+1}^{k_l'-1}\widetilde{Q}_i(t)\\ &\quad+\mu_{k_{l-1}'}^{1/2}c_{k_{l-1}'}W_{k_{l-1}'}(\bar{T}_{k_{l-1}'}(t))-\mu_{k_l'}^{1/2}c_{k_l'}W_{k_l'}(\rho_{k_l'}t)\Bigg]\end{aligned}$$

for $l=1,2,\ldots,l_0$. Notice that $\rho_{k_l'}=\mu_{k_{l-1}'}/\mu_{k_l'}$ by (2.4), and $\bar{T}_{k_l'}(t)=t$ for all $t\ge 0$ and $l=1,2,\ldots,l_0$, then $\mu_{k_{l-1}'}^{1/2}c_{k_{l-1}'}W_{k_{l-1}'}(\bar{T}_{k_{l-1}'}(t))-\mu_{k_l'}^{1/2}c_{k_l'}W_{k_l'}(\rho_{k_l'}t)$ in the above equation is a driftless Brownian motion with variance parameter $\mu_{k'_{l-1}}(c^2_{k'_{l-1}} + c^2_{k'_l})$. So, as the LIL for $Q_{k'_l}$, we have

$$Z^*_{k'_l} = \frac{\sqrt{\mu_{k'_{l-1}}(c^2_{k'_{l-1}} + c^2_{k'_l})}}{\mu_{k'_l}}, \quad \text{w.p.1}$$

for all $l = 1, 2, \ldots, l_0$.

For D_k, $k = k'_1, k'_2, \ldots, k'_{l_0}$, by (2.10) and (4.2),

$$\widetilde{D}_{k'_l}(t) - \bar{D}_{k'_l}(t) = \mu_{k'_l}\left[\widetilde{T}_{k'_l}(t) - \bar{T}_{k'_l}(t)\right] + \mu^{1/2}_{k'_l} c_{k'_l} W_{k'_l}(\bar{T}_{k'_l}(t)).$$

Since $T^*_{k'_l} = 0$ for all $l = 1, 2, \ldots, l_0$, w.p.1, we have $D^*_{k'_l} = \mu^{1/2}_{k'_l} c_{k'_l}$ for all $l = 1, 2, \ldots, l_0$, w.p.1.

So, we obtain all the LILs in (3.5) for stages $k'_1, k'_2, \ldots, k'_{l_0}$, and now proceed to find the LILs for stages $k \neq k'_1, k'_2, \ldots, k'_{l_0}$.

At first, similarly with **Case 1**, we note that all the LILs satisfy (3.3) for $k = 1, 2, \ldots, k'_1 - 1$. Now, we consider the LILs for $k = k'_l + 1, k'_l + 2, \ldots, k'_{l+1} - 1$ with $l = 1, 2, \ldots, l_0$, where $k'_{l_0+1} \equiv K$. For all $k = k'_l + 1, k'_l + 2, \ldots, k'_{l+1} - 1$, we note by Lemma 2.1 and (2.4) that $\rho_k = \mu_{k'_l}/\mu_k$, $\bar{T}_k(t) = \rho_k t$, $\bar{Q}_k(t) = 0$ and $\bar{X}_k(t) = (\mu_{k'_l} - \mu_k)t < 0$ for all $t \geq 0$. Since $Q^*_k = 0$ w.p.1 for all $k = k'_l + 1, k'_l + 2, \ldots, k'_{l+1} - 1$. This, together with (4.2), gives that $Z^*_k = 0$ w.p.1 for all $k = k'_l + 1, k'_l + 2, \ldots, k'_{l+1} - 1$.

For LILs of $I_k(t)$ and $T_k(t)$, $k = k'_l + 1, k'_l + 2, \ldots, k'_{l+1} - 1$, by (2.10) and (4.2), as (4.47) we have

$$\widetilde{X}_k(t) - \bar{X}_k(t) = -\mu_{k'_l}\widetilde{I}_{k'_l}(t) - \sum_{i=k'_l+1}^{k-1} \widetilde{Q}_i(t) + \mu^{1/2}_{k'_l} c_{k'_l} W_{k'_l}(\bar{T}_{k'_l}(t)) - \mu^{1/2}_k c_k W_k(\bar{T}_k(t)). \tag{4.11}$$

So,

$$\begin{aligned}
\widetilde{I}_k(t) - \bar{I}_k(t) &= \frac{1}{\mu_k}\left[\widetilde{Y}_k(t) - \bar{Y}_k(t)\right] = \frac{1}{\mu_k}\widetilde{Q}_k(t) - \frac{1}{\mu_k}\left[\widetilde{X}_k(t) - \bar{X}_k(t)\right] \\
&= \frac{\mu_{k'_l}}{\mu_k}\widetilde{I}_{k'_l}(t) + \frac{1}{\mu_k}\sum_{i=k'_l+1}^{k} \widetilde{Q}_i(t) - \frac{1}{\mu_k}\left[\mu^{1/2}_{k'_l} c_{k'_l} W_{k'_l}(\bar{T}_{k'_l}(t)) - \mu^{1/2}_k c_k W_k(\bar{T}_k(t))\right].
\end{aligned}$$

Since $I^*_k = Q^*_k = 0$ for all $k = k'_l + 1, k'_l + 2, \ldots, k'_{l+1} - 1$, w.p.1, we have $I^*_k = \sqrt{\mu_{k'_l}(c^2_{k'_l} + c^2_k)}/\mu_k$ for all $k = k'_l + 1, k'_l + 2, \ldots, k'_{l+1} - 1$, w.p.1, because $\mu^{1/2}_{k'_l} c_{k'_l} W_{k'_l}(\bar{T}_{k'_l}(t)) - \mu^{1/2}_k c_k W_k(\bar{T}_k(t))$ is a driftless Brownian motion with variance parameter $\mu_{k'_l}(c^2_{k'_l} + c^2_k)$.

For $T_k(t)$, $k = k'_l + 1, k'_l + 2, \ldots, k'_{l+1} - 1$, it follows from (4.2) that $\widetilde{T}_k(t) - \bar{T}_k(t) = -\left[\widetilde{I}_k(t) - \bar{I}_k(t)\right]$, so we have $T^*_k = \sqrt{\mu_{k'_l}(c^2_{k'_l} + c^2_k)}/\mu_k$ for all $k = k'_l + 1, k'_l + 2, \ldots, k'_{l+1} - 1$, w.p.1.

For $D_k(t)$, $k = k'_l+1, k'_l+2, \ldots, k'_{l+1}-1$, by (4.2) and (4.11),

$$\begin{aligned}
\widetilde{D}_k(t) - \bar{D}_k(t) &= \mu_k\left[\widetilde{T}_k(t) - \bar{T}_k(t)\right] + \mu_k^{1/2} c_k W_k(\bar{T}_k(t)) \\
&= -\left[\widetilde{Y}_k(t) - \bar{Y}_k(t)\right] + \mu_k^{1/2} c_k W_k(\bar{T}_k(t)) \\
&= -\widetilde{Q}_k(t) + \left[\widetilde{X}_k(t) - \bar{X}_k(t)\right] + \mu_k^{1/2} c_k W_k(\bar{T}_k(t)) \\
&= -\mu_{k'_l}\widetilde{I}_{k'_l}(t) - \sum_{i=k'_l+1}^{k} \widetilde{Q}_i(t) + \mu_{k'_l}^{1/2} c_{k'_l} W_{k'_l}(\bar{T}_{k'_l}(t)),
\end{aligned}$$

where $\bar{T}_{k'_l}(t) = t$ for all $t \geq 0$. Notice that $I^*_{k'_l} = Q^*_k = 0$ w.p.1 for all $k = k'_l+1, k'_l+2, \ldots, k'_{l+1}-1$, we have $D^*_k = \mu_{k'_l}^{1/2} c_{k'_l}$ w.p.1 for all $k = k'_l+1, k'_l+2, \ldots, k'_{l+1}-1$.

All LILs for class $k'_{l_0+1} \equiv K$ can be obtained similarly with $k = k'_l+1, k'_l+2, \ldots, k'_{l+1}-1$. So far, we get all LILs in (3.6).

Hence, all LILs are obtained. □

4.3 Proof of Theorem 3.2

We firstly prove **Case 1** and then **Case 2**.

Case 1. We firstly note that $\mu_0 = \mu_{k_0} < \mu_k$, $k = 1, 2, \ldots, k_0-1$ and $\mu_{k_0} < \mu_l$, $l = k_0+1, k_0+2, \ldots, K$ under condition $\rho_{k_0} = 1$ and $\rho_k < 1$ for all $k \neq k_0$. The corresponding fluid solution:

$$\bar{\mathbb{X}}_k(t) = (\eta, \eta, \rho_k t, (1-\rho_k)t, \mu_0 t) \quad \text{with} \quad \rho_k = \begin{cases} \frac{\mu_k}{\mu_0}, & k = 1, 2, \ldots, k_0-1, \\ 1, & k = k_0, \\ \frac{\mu_k}{\mu_{k_0}}, & k = k_0+1, \ldots, K. \end{cases} \tag{4.12}$$

Since the up-stream stages are unaffected by the down-stream, then the LILs for classes 1 to k_0-1 satisfy (3.3) when $\rho_k < 1$ for all $k \neq k_0$. Next, we proceed to find LILs for stage k_0, and then for stages $k_0+1, k_0+2, \ldots, K$.

By Lemma 4.3, we firstly note that $Q^*_k = Z^*_k = 0$ for all $k \neq k_0$ because $\rho_k < 1$ for all $k \neq k_0$, w.p.1. Next, we consider the LILs for stage k_0 in **Case 1** and then its following stages $k > k_0$.

Case 1: LILs for stage k_0. For stage k_0, under $\rho_k < 1$ for all $k = 1, 2, \ldots, k_0-1$ and $\rho_{k_0} = 1$, it follows that $\mu_{k_0} = \mu_0 < \mu_k, k = 1, 2, \ldots, k_0-1$. By (4.2) and Lemma 2.1, as (4.46) we have

$$\begin{aligned}
\widetilde{X}_{k_0}(t) &= \mu_{k_0-1}\widetilde{T}_{k_0-1}(t) - \mu_{k_0}t + \mu_{k_0-1}^{1/2} c_{k_0-1} W_{k_0-1}(\bar{T}_{k_0-1}(t)) - \mu_{k_0}^{1/2} c_{k_0} W_{k_0}(\bar{T}_{k_0}(t)) \\
&= \widetilde{X}_1(t) - \sum_{i=1}^{k_0-1} \widetilde{Q}_i(t) + (\mu_1 - \mu_{k_0})t + \mu_1^{1/2} c_1 W_1(\bar{T}_1(t)) - \mu_{k_0}^{1/2} c_{k_0} W_{k_0}(\bar{T}_{k_0}(t)) \\
&= -\sum_{i=1}^{k_0-1} \widetilde{Q}_i(t) + \mu_0^{1/2} c_0 W_0(t) - \mu_{k_0}^{1/2} c_{k_0} W_{k_0}(\bar{T}_{k_0}(t)).
\end{aligned} \tag{4.13}$$

For I_{k_0} and T_{k_0}, since $\rho_{k_0} = 1$, we have $\bar{X}_{k_0}(t) = \bar{I}_{k_0}(t) = 0$ and $\bar{T}_{k_0}(t) = t$. This, together with $Q_k^* = 0$ for all $k = 1, 2, \ldots, k_0 - 1$, implies that the norm-LIL

$$
\begin{aligned}
I_{k_0}^* &= \limsup_{L\to\infty} \frac{\sup_{0\le t\le L}\left|\widetilde{I}_{k_0}(t)\right|}{\varphi(L)} = \frac{1}{\mu_{k_0}} \limsup_{L\to\infty} \frac{\sup_{0\le t\le L}\left|\widetilde{Y}_{k_0}(t)\right|}{\varphi(L)} = \frac{1}{\mu_{k_0}} \limsup_{L\to\infty} \frac{\sup_{0\le t\le L}\left[\widetilde{Y}_{k_0}(t)\right]}{\varphi(L)} \\
&= \frac{1}{\mu_{k_0}} \limsup_{L\to\infty} \frac{\sup_{0\le t\le L}\left\{\sup_{0\le s\le t}\left[-\widetilde{X}_{k_0}(s)\right]\right\}}{\varphi(L)} = \frac{1}{\mu_{k_0}} \limsup_{L\to\infty} \frac{\sup_{0\le t\le L}\left[-\widetilde{X}_{k_0}(t)\right]}{\varphi(L)} \\
&= \frac{1}{\mu_{k_0}} \limsup_{L\to\infty} \frac{\sup_{0\le t\le L}\left[-\mu_0^{1/2}c_0W_0(t) + \mu_{k_0}^{1/2}c_{k_0}W_{k_0}(t)\right]}{\varphi(L)} \\
&= \frac{1}{\mu_{k_0}}\sqrt{\mu_0(c_0^2 + c_{k_0}^2)} \equiv \frac{C_{0,k_0}}{\sqrt{\mu_0}}, \quad \text{w.p.1,}
\end{aligned} \tag{4.14}
$$

where the fourth equality holds because $\widetilde{Y}(t) = \sup_{0\le s\le t}\left[-\widetilde{X}_{k_0}(s)\right]^+$ and $\widetilde{X}_{k_0}(0) = 0$, the seventh equality is from (4.4). Since $\widetilde{T}_{k_0}(t) - \bar{T}_{k_0}(t) = -\widetilde{I}_{k_0}(t)$ for all $t \ge 0$, we have the LIL $T_{k_0}^* = \sqrt{\mu_0(c_0^2 + c_{k_0}^2)/\mu_{k_0}} \equiv C_{0,k_0}/\sqrt{\mu_0}$ w.p.1.

For Q_{k_0} and Z_{k_0}, we firstly note that $\widetilde{Q}_{k_0}(t) = \Phi(\widetilde{X}_{k_0})(t)$ from (4.2). Since Φ is Lipschitz continuous under uniform topology, similarly with (4.13), for $L > 0$,

$$
\begin{aligned}
&\sup_{0\le t\le L}\left|\Phi(\widetilde{X}_{k_0})(t) - \Phi(\mu_0^{1/2}c_0W_0 - \mu_{k_0}^{1/2}c_{k_0}W_{k_0})(t)\right| \\
\le\ &\sup_{0\le t\le L}\left|\Psi(\widetilde{X}_{k_0})(t) - \Psi(\mu_0^{1/2}c_0W_0 - \mu_{k_0}^{1/2}c_{k_0}W_{k_0})(t)\right| \\
&+ \sup_{0\le t\le L}\left|\widetilde{X}_{k_0}(t) - (\mu_0^{1/2}c_0W_0(t) - \mu_{k_0}^{1/2}c_{k_0}W_{k_0}(t))\right| \\
\le\ &2\sup_{0\le t\le L}\left|\widetilde{X}_{k_0}(t) - (\mu_0^{1/2}c_0W_0(t) - \mu_{k_0}^{1/2}c_{k_0}W_{k_0}(t))\right| = 2\sup_{0\le t\le L}\sum_{i=1}^{k_0-1}\widetilde{Q}_i(t),
\end{aligned}
$$

where the second inequality is from (2.13). Notice that, for all $k = 1, 2, \ldots, k_0 - 1$, $Q_k^* = 0$ w.p.1, and $\bar{Q}_k(t) = 0$ for all $t \ge 0$, we have

$$
\begin{aligned}
&\limsup_{L\to\infty} \frac{\sup_{0\le t\le L}\left|\Phi(\widetilde{X}_{k_0})(t) - \Phi(\mu_0^{1/2}c_0W_0(t) - \mu_{k_0}^{1/2}c_{k_0}W_{k_0}(t))\right|}{\varphi(L)} \\
&\le 2\limsup_{L\to\infty} \frac{\sup_{0\le t\le L}\sum_{i=1}^{k_0-1}\widetilde{Q}_i(t)}{\varphi(L)} = 0, \quad \text{w.p.1.}
\end{aligned} \tag{4.15}
$$

So,

$$
\begin{aligned}
Q^*_{k_0} &= \limsup_{L\to\infty} \frac{\sup_{0\le t\le L}\left|\Phi(\widetilde{X}_{k_0})(t)\right|}{\varphi(L)} \\
&= \limsup_{L\to\infty} \frac{\sup_{0\le t\le L}\left| \begin{array}{r} \left[\Phi(\widetilde{X}_{k_0})(t) - \Phi(\mu_0^{1/2}c_0W_0(t) - \mu_{k_0}^{1/2}c_{k_0}W_{k_0}(t))\right] \\ +\Phi(\mu_0^{1/2}c_0W_0(t) - \mu_{k_0}^{1/2}c_{k_0}W_{k_0}(t)) \end{array}\right|}{\varphi(L)} \\
&= \limsup_{L\to\infty} \frac{\sup_{0\le t\le L}\left|\Phi(\mu_0^{1/2}c_0W_0(t) - \mu_{k_0}^{1/2}c_{k_0}W_{k_0}(t))\right|}{\varphi(L)} \\
&= \limsup_{L\to\infty} \frac{\sup_{0\le t\le L}\left|\mu_0^{1/2}c_0W_0(t) - \mu_{k_0}^{1/2}c_{k_0}W_{k_0}(t)\right|}{\varphi(L)} \\
&= \sqrt{\mu_0(c_0^2+c_{k_0}^2)}, \quad \text{w.p.1}, \qquad (4.16)
\end{aligned}
$$

where, in (4.16), the third equality is from (4.15), the fourth equality holds because $\Phi(\mu_0^{1/2}c_0W_0 - \mu_{k_0}^{1/2}c_{k_0}W_{k_0})(t)$ is a reflected BM, and is identically distributed as the absolute value $\left|\mu_0^{1/2}c_0W_0(t) - \mu_{k_0}^{1/2}c_{k_0}W_{k_0}(t)\right|$, the fifth equality holds because $\mu_0^{1/2}c_0W_0(t) - \mu_{k_0}^{1/2}c_{k_0}W_{k_0}(t)$ is a BM with variance parameter $\mu_0(c_0^2+c_{k_0}^2)$. By (2.10) and (4.2), we have $\bar{Z}_{k_0}(t) = \bar{Q}_{k_0}(t)/\mu_{k_0} = 0$ and $\widetilde{Z}_{k_0}(t) = \widetilde{Q}_{k_0}(t)/\mu_{k_0}$. This follows $Z^*_{k_0} = \sqrt{\mu_0(c_0^2+c_{k_0}^2)}/\mu_{k_0}$ w.p.1. So, the LILs for Q_{k_0} and Z_{k_0} are obtained.

For D_{k_0}, by (2.10), (4.2) and (4.13), we note that

$$
\begin{aligned}
& \widetilde{D}_{k_0}(t) - \bar{D}_{k_0}(t) \qquad (4.17) \\
=\ & \mu_{k_0}\left[\widetilde{T}_{k_0}(t) - \bar{T}_{k_0}(t)\right] + \mu_{k_0}^{1/2}c_{k_0}W_{k_0}(\bar{T}_{k_0}(t)) \\
=\ & -\widetilde{Y}_{k_0}(t) + \mu_{k_0}^{1/2}c_{k_0}W_{k_0}(t) = -\sup_{0\le s\le t}\left[-\widetilde{X}_{k_0}(s)\right] + \mu_{k_0}^{1/2}c_{k_0}W_{k_0}(t) \\
=\ & -\sup_{0\le s\le t}\left[\sum_{i=1}^{k_0-1}\widetilde{Q}_i(s) - \mu_0^{1/2}c_0W_0(s) + \mu_{k_0}^{1/2}c_{k_0}W_{k_0}(s)\right] + \mu_{k_0}^{1/2}c_{k_0}W_{k_0}(t) \\
\stackrel{\text{d}}{=}\ & -\sup_{0\le s\le t}\left[\sum_{i=1}^{k_0-1}\widetilde{Q}_i(s) - \mu_0^{1/2}c_0W_0(s) - \mu_{k_0}^{1/2}c_{k_0}W_{k_0}(t-s)\right]. \qquad (4.18)
\end{aligned}
$$

Because $Q^*_k = 0$ w.p.1 for all $k = 1, 2, \ldots, k_0 - 1$, we have, w.p.1,

$$
D^*_{k_0} = \limsup_{L\to\infty} \frac{\sup_{0\le t\le L}\sup_{0\le s\le t}\left|\mu_0^{1/2}c_0W_0(s) + \mu_{k_0}^{1/2}c_{k_0}W_{k_0}(t-s)\right|}{\varphi(L)} = \sqrt{\mu_0}C_{0,k_0}, \qquad (4.19)
$$

where the last equality is from Lemma 4 in [16].

So, we get all the norm-LILs and functional LILs for stage k_0, that is, (3.7) holds.

Case 1: LILs for stages $k > k_0$. For $l = 1, 2, \ldots, K - k_0$, by (2.10) and (4.2),

$$
\begin{aligned}
& \widetilde{X}_{k_0+l}(t) - \bar{X}_{k_0+l}(t) \\
= \; & \mu_{k_0+l-1}\left[\widetilde{T}_{k_0+l-1}(t) - \bar{T}_{k_0+l-1}(t)\right] \\
& + \mu_{k_0+l-1}^{1/2} c_{k_0+l-1} W_{k_0+l-1}(\bar{T}_{k_0+l-1}(t)) - \mu_{k_0+l}^{1/2} c_{k_0+l} W_{k_0+l}(\bar{T}_{k_0+l}(t)) \\
= \; & -\mu_{k_0+l-1}\left[\widetilde{I}_{k_0+l-1}(t) - \bar{I}_{k_0+l-1}(t)\right] \\
& + \mu_{k_0+l-1}^{1/2} c_{k_0+l-1} W_{k_0+l-1}(\bar{T}_{k_0+l-1}(t)) - \mu_{k_0+l}^{1/2} c_{k_0+l} W_{k_0+l}(\bar{T}_{k_0+l}(t)) \\
= \; & \left[\widetilde{X}_{k_0+l-1}(t) - \bar{X}_{k_0+l-1}(t)\right] - \left[\widetilde{Q}_{k_0+l-1}(t) - \bar{Q}_{k_0+l-1}(t)\right] \\
& + \mu_{k_0+l-1}^{1/2} c_{k_0+l-1} W_{k_0+l-1}(\bar{T}_{k_0+l-1}(t)) - \mu_{k_0+l}^{1/2} c_{k_0+l} W_{k_0+l}(\bar{T}_{k_0+l}(t)) \\
= \; & \left[\widetilde{X}_{k_0+l-2}(t) - \bar{X}_{k_0+l-2}(t)\right] - \left[\widetilde{Q}_{k_0+l-2}(t) - \bar{Q}_{k_0+l-2}(t)\right] - \left[\widetilde{Q}_{k_0+l-1}(t) - \bar{Q}_{k_0+l-1}(t)\right] \\
& + \mu_{k_0+l-2}^{1/2} c_{k_0+l-2} W_{k_0+l-2}(\bar{T}_{k_0+l-2}(t)) - \mu_{k_0+l}^{1/2} c_{k_0+l} W_{k_0+l}(\bar{T}_{k_0+l}(t)) \\
= \; & \cdots \\
= \; & \left[\widetilde{X}_{k_0+1}(t) - \bar{X}_{k_0+1}(t)\right] - \sum_{i=1}^{l-1}\left[\widetilde{Q}_{k_0+i}(t) - \bar{Q}_{k_0+i}(t)\right] \\
& + \mu_{k_0+1}^{1/2} c_{k_0+1} W_{k_0+1}(\bar{T}_{k_0+1}(t)) - \mu_{k_0+l}^{1/2} c_{k_0+l} W_{k_0+l}(\bar{T}_{k_0+l}(t)) \\
= \; & \mu_{k_0}\left[\widetilde{T}_{k_0}(t) - \bar{T}_{k_0}(t)\right] - \sum_{i=1}^{l-1}\left[\widetilde{Q}_{k_0+i}(t) - \bar{Q}_{k_0+i}(t)\right] \\
& + \mu_{k_0}^{1/2} c_{k_0} W_{k_0}(\bar{T}_{k_0}(t)) - \mu_{k_0+l}^{1/2} c_{k_0+l} W_{k_0+l}(\bar{T}_{k_0+l}(t)) \\
= \; & -\mu_{k_0}\widetilde{I}_{k_0}(t) - \sum_{i=1}^{l-1}\left[\widetilde{Q}_{k_0+i}(t) - \bar{Q}_{k_0+i}(t)\right] + \mu_{k_0}^{1/2} c_{k_0} W_{k_0}(\bar{T}_{k_0}(t)) - \mu_{k_0+l}^{1/2} c_{k_0+l} W_{k_0+l}(\bar{T}_{k_0+l}(t)).
\end{aligned}
$$

Together with (4.13), we have

$$
\begin{aligned}
& \widetilde{X}_{k_0+l}(t) - \bar{X}_{k_0+l}(t) \\
= \; & -\sup_{0\le u\le t}\left\{\sum_{i=1}^{k_0-1}\left[\widetilde{Q}_i(u) - \bar{Q}_i(u)\right] - \mu_0^{1/2} c_0 W_0(u) + \mu_{k_0}^{1/2} c_{k_0} W_{k_0}(\bar{T}_{k_0}(u))\right\} \\
& - \sum_{i=1}^{l-1}\left[\widetilde{Q}_{k_0+i}(t) - \bar{Q}_{k_0+i}(t)\right] + \mu_{k_0}^{1/2} c_{k_0} W_{k_0}(\bar{T}_{k_0}(t)) - \mu_{k_0+l}^{1/2} c_{k_0+l} W_{k_0+l}(\bar{T}_{k_0+l}(t)) \\
= \; & \inf_{0\le u\le t}\left\{\sum_{i=1}^{k_0-1}\left[-\left(\widetilde{Q}_i(u) + \bar{Q}_i(u)\right)\right] + \mu_0^{1/2} c_0 W_0(u) - \mu_{k_0}^{1/2} c_{k_0} W_{k_0}(\bar{T}_{k_0}(u))\right\} \qquad (4.20) \\
& - \sum_{i=1}^{l-1}\left[\widetilde{Q}_{k_0+i}(t) - \bar{Q}_{k_0+i}(t)\right] + \mu_{k_0}^{1/2} c_{k_0} W_{k_0}(\bar{T}_{k_0}(t)) - \mu_{k_0+l}^{1/2} c_{k_0+l} W_{k_0+l}(\bar{T}_{k_0+l}(t)).
\end{aligned}
$$

Since, by (4.2) and (2.10),

$$\begin{aligned}
\widetilde{I}_{k_0+l}(t) - \bar{I}_{k_0+l}(t) &= \frac{1}{\mu_{k_0+l}}\left[\widetilde{Q}_{k_0+l}(t) - \widetilde{X}_{k_0+l}(t)\right] - \left(t - \bar{T}_{k_0+l}(t)\right) \\
&= \frac{1}{\mu_{k_0+l}}\widetilde{Q}_{k_0+l}(t) - \frac{1}{\mu_{k_0+l}}\left[\widetilde{X}_{k_0+l}(t) - \mu_{k_0+l}\left(\bar{T}_{k_0+l}(t) - t\right)\right] \\
&= \frac{1}{\mu_{k_0+l}}\widetilde{Q}_{k_0+l}(t) - \frac{1}{\mu_{k_0+l}}\left[\widetilde{X}_{k_0+l}(t) - \bar{X}_{k_0+l}(t)\right]. \qquad (4.21)
\end{aligned}$$

So, with Lemma 2.1, (4.2), (4.20) and (4.21), we have, w.p.1, $I^*_{k_0+l}$

$$\begin{aligned}
I^*_{k_0+l} &= \limsup_{L\to\infty} \frac{\left\|\widetilde{I}_{k_0+l} - \bar{I}_{k_0+l}\right\|_L}{\varphi(L)} \\
&= \limsup_{L\to\infty} \frac{\left\|\widetilde{Q}_{k_0+l} - \left(\widetilde{X}_{k_0+l} - \bar{X}_{k_0+l}\right)\right\|_L}{\mu_{k_0+l}\varphi(L)} = \frac{1}{\mu_{k_0+l}} \limsup_{L\to\infty} \frac{\left\|\widetilde{X}_{k_0+l} - \bar{X}_{k_0+l}\right\|_L}{\varphi(L)} \\
&= \frac{1}{\mu_{k_0+l}} \limsup_{L\to\infty} \frac{\sup\limits_{0\le t\le L}\left| \begin{array}{l} -\sup\limits_{0\le u\le t}\left\{\sum\limits_{i=1}^{k_0-1}\left[\widetilde{Q}_i(u) - \bar{Q}_i(u)\right] - \mu_0^{1/2}c_0W_0(u)\right. \\ \left. +\mu_{k_0}^{1/2}c_{k_0}W_{k_0}(T_{k_0}(u))\right\} + \mu_{k_0}^{1/2}c_{k_0}W_{k_0}(T_{k_0}(t)) \\ -\sum\limits_{i=1}^{l-1}\left[\widetilde{Q}_{k_0+i}(t) - \bar{Q}_{k_0+i}(t)\right] - \mu_{k_0+l}^{1/2}c_{k_0+l}W_{k_0+l}(T_{k_0+l}(t)) \end{array}\right|}{\varphi(L)} \\
&= \frac{1}{\mu_{k_0+l}} \limsup_{L\to\infty} \frac{\sup\limits_{0\le t\le L}\left| \begin{array}{l} -\sup\limits_{0\le u\le t}\left\{-\mu_0^{1/2}c_0W_0(u) + \mu_{k_0}^{1/2}c_{k_0}W_{k_0}(T_{k_0}(u))\right\} \\ +\mu_{k_0}^{1/2}c_{k_0}W_{k_0}(T_{k_0}(t)) - \mu_{k_0+l}^{1/2}c_{k_0+l}W_{k_0+l}(T_{k_0+1}(t)) \end{array}\right|}{\varphi(L)} \qquad (4.22) \\
&= \frac{1}{\mu_{k_0+l}} \limsup_{L\to\infty} \frac{\sup\limits_{0\le t\le L}\left| \begin{array}{l} \sup\limits_{0\le u\le t}\left\{-\mu_0^{1/2}c_0W_0(u) + \mu_{k_0}^{1/2}c_{k_0}W_{k_0}(T_{k_0}(u))\right\} \\ -\mu_{k_0}^{1/2}c_{k_0}W_{k_0}(T_{k_0}(t)) + \mu_{k_0+l}^{1/2}c_{k_0+l}W_{k_0+l}(T_{k_0+1}(t)) \end{array}\right|}{\varphi(L)} \\
&= \frac{1}{\mu_{k_0+l}} \limsup_{L\to\infty} \frac{\sup\limits_{0\le t\le L}\left| \begin{array}{l} \sup\limits_{0\le u\le t}\left\{-\mu_0^{1/2}c_0W_0(u) + \mu_{k_0}^{1/2}c_{k_0}W_{k_0}(T_{k_0}(t-u))\right\} \\ +\mu_{k_0+l}^{1/2}c_{k_0+l}W_{k_0+l}(T_{k_0+l}(t)) \end{array}\right|}{\varphi(L)} \\
&= \limsup_{L\to\infty} \frac{\sup\limits_{0\le t\le L}\left| \sup\limits_{0\le u\le t}\left\{-\mu_0^{1/2}c_0W_0(u) + \mu_{k_0}^{1/2}c_{k_0}W_{k_0}(t-u)\right\} + \mu_{k_0}^{1/2}c_{k_0+l}W_{k_0+l}(t) \right|}{\mu_{k_0+l}\varphi(L)},
\end{aligned}$$

where the third equality holds because $Q^*_{k_0+l} = 0$, the fifth equality holds because $Q^*_{k_0+l} = 0$. The numerator in the last quality in (4.22) is

$$\begin{aligned}
&= \sup_{0\le t\le L}\left| \sup_{0\le\theta\le 1}\left\{-\mu_0^{1/2}c_0W_0(\theta t) + \mu_{k_0}^{1/2}c_{k_0}W_{k_0}((1-\theta)t) + \mu_{k_0}^{1/2}c_{k_0+l}W_{k_0+l}(t)\right\}\right| \\
&\stackrel{\mathrm{d}}{=} \sup_{0\le t\le L}\left| \sup_{0\le\theta\le 1}\sqrt{\mu_0c_0^2\theta + \mu_{k_0}c_{k_0}^2(1-\theta) + \mu_{k_0}c_{k_0+l}^2}\,W(t)\right|,
\end{aligned}$$

where $W(t)$ is a one-dimensional standard BM, the last equality in distribution holds because W_0, W_{k_0} and $W_{k_0+l}(t)$ are mutually independent BMs. So, this, together with (4.22), implies that

$$I^*_{k_0+l} = \frac{\sup_{0\le\theta\le 1}\sqrt{\mu_0 c_0^2\theta + \mu_{k_0}c_{k_0}^2(1-\theta) + \mu_{k_0}c_{k_0+l}^2}}{\mu_{k_0+l}} = \frac{\sqrt{\mu_0((c_0^2\vee c_{k_0}^2) + c_{k_0+l}^2)}}{\mu_{k_0+l}}, \quad \text{w.p.1.}$$

where the last equality holds because $\mu_0 = \mu_{k_0}$. Note that $\widetilde{T}_{k_0+l}(t) - \bar{T}_{k_0+l}(t) = \bar{I}_{k_0+l}(t) - \widetilde{I}_{k_0+l}(t)$ for all $t > 0$, we have $T^*_{k_0+l} = \frac{1}{\mu_{k_0+l}}\sqrt{\mu_0(c_0^2\vee c_{k_0}^2 + c_{k_0+l}^2)}$, w.p.1.

For departure D_{k_0+l}, we have

$$\begin{aligned}
& \widetilde{D}_{k_0+l}(t) - \bar{D}_{k_0+l}(t) \\
= \ & \mu_{k_0+l}\left[\widetilde{T}_{k_0+l}(t) - \bar{T}_{\mu_{k_0+l}}(t)\right] + \mu_{k_0+l}^{1/2}c_{k_0+l}W_{k_0+l}(\bar{T}_{k_0+l}(t)) \\
= \ & -\mu_{k_0+l}\left[\widetilde{I}_{k_0+l}(t) - \bar{I}_{k_0+l}(t)\right] + \mu_{k_0+l}^{1/2}c_{k_0+l}W_{k_0+l}(\bar{T}_{k_0+l}(t)) \\
= \ & -\widetilde{Q}_{k_0+l}(t) + \left[\widetilde{X}_{k_0+l}(t) - \bar{X}_{k_0+l}(t)\right] + \mu_{k_0+l}^{1/2}c_{k_0+l}W_{k_0+l}(\bar{T}_{k_0+l}(t)) \\
= \ & -\sup_{0\le u\le t}\left\{\sum_{i=1}^{k_0-1}\left[\widetilde{Q}_i(u) - \bar{Q}_i(u)\right] - \mu_0^{1/2}c_0W_0(u) + \mu_{k_0}^{1/2}c_{k_0}W_{k_0}(\bar{T}_{k_0}(u))\right\} \\
& -\sum_{i=1}^{l}\left[\widetilde{Q}_{k_0+i}(t) - \bar{Q}_{k_0+i}(t)\right] + \mu_{k_0}^{1/2}c_{k_0}W_{k_0}(\bar{T}_{k_0}(t)) \\
= \ & \inf_{0\le u\le t}\left\{-\sum_{i=1}^{k_0-1}\left[\widetilde{Q}_i(u) - \bar{Q}_i(u)\right] + \mu_0^{1/2}c_0W_0(u) - \mu_{k_0}^{1/2}c_{k_0}W_{k_0}(\bar{T}_{k_0}(u))\right\} \\
& -\sum_{i=1}^{l}\left[\widetilde{Q}_{k_0+i}(t) - \bar{Q}_{k_0+i}(t)\right] + \mu_{k_0}^{1/2}c_{k_0}W_{k_0}(\bar{T}_{k_0}(t)),
\end{aligned} \tag{4.23}$$

where the forth equality is from (4.20) and the fluid $\bar{Q}_{k_0+l}(t) = 0$ for all t.

For the LIL $D^*_{k_0+l}$, by (4.23), it follows from similar analysis for I^*_{k+l} in (4.22) that

$$\begin{aligned}
& D^*_{k_0+l} \\
= \ & \limsup_{L\to\infty}\frac{\left|\left|\widetilde{D}_{k_0+l} - \bar{D}_{k_0+l}\right|\right|_L}{\varphi(L)} \\
= \ & \limsup_{L\to\infty}\frac{\sup_{0\le t\le L}\left|\sup_{0\le u\le t}\left\{-\mu_0^{1/2}c_0W_0(u) + \mu_{k_0}^{1/2}c_{k_0}W_{k_0}(T_{k_0}(u))\right\} - \mu_{k_0}^{1/2}c_{k_0}W_{k_0}(T_{k_0}(t))\right|}{\varphi(L)} \\
= \ & \limsup_{L\to\infty}\frac{\sup_{0\le t\le L}\left|\sup_{0\le u\le t}\left\{-\mu_0^{1/2}c_0W_0(u) - \mu_{k_0}^{1/2}c_{k_0}W_{k_0}(t-u)\right\}\right|}{\varphi(L)} \\
= \ & \mu_0^{1/2}(c_0\vee c_{k_0}), \quad \text{w.p.1,}
\end{aligned}$$

where the forth equality is from Lemma 4 in [16] and $\mu_0 = \mu_{k_0}$.

So, we get all the LILs in (3.8).

Case 2. Since $\rho_i > 1$ for all $i \neq k_0$, then the first $k_0 - 1$ stages constitute an independent tandem queue as **Case 2** in Theorem 3.1. As a result, all limits $\mathcal{X}_k^*$ are given by (3.4) for $k = 1, 2, \ldots, k_0 - 1$.

Next, we proceed to find LILs for stage k_0, and then for stages $k_0+1, k_0+2, \ldots, K$. At first, we note that, if $\rho_{k_0} = 1$ and $\rho_k > 1$ for all $k \neq k_0$, then $\mu_0 > \mu_1 > \cdots > \mu_{k_0-1} = \mu_{k_0} > \mu_{k_0+1} > \cdots > \mu_K$ and $\rho_k = \mu_{k-1}/\mu_k$, the corresponding fluid solution is

$$\bar{\mathbb{X}}_k(t) = ((\mu_{k-1} - \mu_k)t, (\rho_k - 1)t, t, \eta, \mu_k t) \quad k = 1, 2, \ldots, K. \tag{4.24}$$

Case 2: LILs for stage k_0. For stage k_0, since $\rho_k > 1$ for all $k = 1, 2, \ldots, k_0 - 1$ and $\rho_{k_0} = 1$, we have $\mu_{k_0} = \mu_{k_0-1} > \mu_k, k = 1, 2, \ldots, k_0 - 1$. It follows from (4.2) and (4.24) that

$$\begin{aligned}
\widetilde{X}_{k_0}(t) &= \mu_{k_0-1}\widetilde{T}_{k_0-1}(t) - \mu_{k_0}t + \mu_{k_0-1}^{1/2}c_{k_0-1}W_{k_0-1}(t) - \mu_{k_0}^{1/2}c_{k_0}W_{k_0}(t) \\
&= -\widetilde{Y}_{k_0-1}(t) + (\mu_{k_0-1} - \mu_{k_0})t + \mu_{k_0-1}^{1/2}c_{k_0-1}W_{k_0-1}(t) - \mu_{k_0}^{1/2}c_{k_0}W_{k_0}(t).
\end{aligned}$$

Because $\bar{X}_{k_0}(t) = (\mu_{k_0-1} - \mu_{k_0})t = 0$ by (4.24), this gives us

$$\widetilde{X}_{k_0}(t) - \bar{X}_{k_0}(t) = -\widetilde{Y}_{k_0-1}(t) + \mu_{k_0-1}^{1/2}c_{k_0-1}W_{k_0-1}(t) - \mu_{k_0}^{1/2}c_{k_0}W_{k_0}(t). \tag{4.25}$$

So, for the queue length Q_{k_0}, by (4.25),

$$\begin{aligned}
\widetilde{Q}_{k_0}^* &= \limsup_{L\to\infty} \frac{\left|\left|\Phi(\widetilde{X}_{k_0})\right|\right|_L}{\varphi(L)} = \limsup_{L\to\infty} \frac{\left|\left|\Phi(-\widetilde{Y}_{k_0-1} + \mu_{k_0-1}^{1/2}c_{k_0-1}W_{k_0-1} - \mu_{k_0}^{1/2}c_{k_0}W_{k_0})\right|\right|_L}{\varphi(L)} \\
&= \limsup_{L\to\infty} \frac{\left|\left|\Phi(\mu_{k_0-1}^{1/2}c_{k_0-1}W_{k_0-1} - \mu_{k_0}^{1/2}c_{k_0}W_{k_0})\right|\right|_L}{\varphi(L)} \\
&= \limsup_{L\to\infty} \frac{\left|\left|\mu_{k_0-1}^{1/2}c_{k_0-1}W_{k_0-1} - \mu_{k_0}^{1/2}c_{k_0}W_{k_0}\right|\right|_L}{\varphi(L)} \\
&= \sqrt{\mu_{k_0}}C_{k_0-1,k_0}, \quad \text{w.p.1},
\end{aligned}$$

where $C_{k_0-1,k_0} = \sqrt{c_{k_0-1}^2 + c_{k_0}^2}$, the third equality is from $I_{k_0-1}^* = 0$ and the forth equality holds because

$$\Phi(\mu_{k_0-1}^{1/2}c_{k_0-1}W_{k_0-1} - \mu_{k_0}^{1/2}c_{k_0}W_{k_0})(t) \stackrel{\mathrm{d}}{=} \left|\mu_{k_0-1}^{1/2}c_{k_0-1}W_{k_0-1}(t) - \mu_{k_0}^{1/2}c_{k_0}W_{k_0}(t)\right|$$

for all $t \geq 0$. As a result, $Z^*_{k_0} = C_{k_0-1,k_0}/\sqrt{\mu_{k_0}}$ w.p.1. For the LIL of I_{k_0},

$$
\begin{aligned}
\widetilde{I}^*_{k_0} &= \frac{1}{\mu_{k_0}} \limsup_{L\to\infty} \frac{\left|\left|\Psi(\widetilde{X}_{k_0})\right|\right|_L}{\varphi(L)} \\
&= \frac{1}{\mu_{k_0}} \limsup_{L\to\infty} \frac{\left|\left|\Psi(-\widetilde{Y}_{k_0-1} + \mu_{k_0-1}^{1/2} c_{k_0-1} W_{k_0-1} - \mu_{k_0}^{1/2} c_{k_0} W_{k_0})\right|\right|_L}{\varphi(L)} \\
&= \frac{1}{\mu_{k_0}} \limsup_{L\to\infty} \frac{\left|\left|\Psi(\mu_{k_0-1}^{1/2} c_{k_0-1} W_{k_0-1} - \mu_{k_0}^{1/2} c_{k_0} W_{k_0})\right|\right|_L}{\varphi(L)} \\
&= \frac{1}{\mu_{k_0}} \limsup_{L\to\infty} \frac{\left|\left|\mu_{k_0-1}^{1/2} c_{k_0-1} W_{k_0-1} - \mu_{k_0}^{1/2} c_{k_0} W_{k_0}\right|\right|_L}{\varphi(L)} \\
&= \frac{C_{k_0-1,k_0}}{\sqrt{\mu_{k_0}}}, \quad \text{w.p.1},
\end{aligned}
$$

where the last equality similarly holds with $\widetilde{Q}^*_{k_0}$ above. This follows $T^*_{k_0} = C_{k_0-1,k_0}/\sqrt{\mu_{k_0}}$ w.p.1 because $\widetilde{T}_{k_0}(t) - \bar{T}_{k_0}(t) = \bar{I}_{k_0}(t) - \widetilde{I}_{k_0}(t)$ for all $t \geq 0$.

For D_{k_0}, by (4.2), Lemma 2.1 and (4.25),

$$
\begin{aligned}
& \widetilde{D}_{k_0}(t) - \bar{D}_{k_0}(t) \\
= \; & \mu_{k_0}\left[\widetilde{T}_{k_0}(t) - \bar{T}_{k_0}(t)\right] + \mu_{k_0}^{1/2} c_{k_0} W_{k_0}(\bar{T}_{k_0}(t)) \\
= \; & -\mu_{k_0}\left[\widetilde{I}_{k_0}(t) - \bar{I}_{k_0}(t)\right] + \mu_{k_0}^{1/2} c_{k_0} W_{k_0}(\bar{T}_{k_0}(t)) \\
= \; & -\widetilde{Y}_{k_0}(t) + \mu_{k_0}^{1/2} c_{k_0} W_{k_0}(t) \qquad (4.26) \\
= \; & -\sup_{0\leq u\leq t}\left[-\widetilde{Y}_{k_0-1}(u) - \mu_{k_0-1}^{1/2} c_{k_0-1} W_{k_0-1}(u) + \mu_{k_0}^{1/2} c_{k_0} W_{k_0}(u)\right] + \mu_{k_0}^{1/2} c_{k_0} W_{k_0}(t) \\
= \; & \inf_{0\leq u\leq t}\left[-\mu_{k_0-1}\widetilde{I}_{k_0-1}(u) + \mu_{k_0-1}^{1/2} c_{k_0-1} W_{k_0-1}(u) - \mu_{k_0}^{1/2} c_{k_0} W_{k_0}(u)\right] + \mu_{k_0}^{1/2} c_{k_0} W_{k_0}(t).
\end{aligned}
$$

For the norm-LIL of D_{k_0}, by (4.26) we have, with similar analysis for $I^*_{k_0}$ in (4.22),

$$
\begin{aligned}
\widetilde{D}^*_{k_0} &= \limsup_{L\to\infty} \frac{\left|\left|\widetilde{D}_{k_0} - \bar{D}_{k_0}\right|\right|_L}{\varphi(L)} \\
&= \limsup_{L\to\infty} \frac{\sup_{0\leq t\leq L}\left| \begin{array}{l} \sup_{0\leq u\leq t}\left[-\widetilde{Y}_{k_0-1}(u) - \mu_{k_0-1}^{1/2} c_{k_0-1} W_{k_0-1}(u) + \mu_{k_0}^{1/2} c_{k_0} W_{k_0}(u)\right] \\ -\mu_{k_0}^{1/2} c_{k_0} W_{k_0}(t) \end{array} \right|}{\varphi(L)}
\end{aligned}
$$

$$
\begin{aligned}
&= \limsup_{L\to\infty} \frac{\sup\limits_{0\le t\le L}\left|\sup\limits_{0\le u\le t}\left[-\mu_{k_0-1}^{1/2}c_{k_0-1}W_{k_0-1}(u)+\mu_{k_0}^{1/2}c_{k_0}W_{k_0}(u)\right]-\mu_{k_0}^{1/2}c_{k_0}W_{k_0}(t)\right|}{\varphi(L)}\\
&= \limsup_{L\to\infty} \frac{\sup\limits_{0\le t\le L}\left|\sup\limits_{0\le u\le t}\left[-\mu_{k_0-1}^{1/2}c_{k_0-1}W_{k_0-1}(u)-\mu_{k_0}^{1/2}c_{k_0}W_{k_0}(t-u)\right]\right|}{\varphi(L)}\\
&= \mu_{k_0}^{1/2}(c_{k_0-1}\vee c_{k_0}), \quad \text{w.p.1},
\end{aligned}
$$

where the third equality holds because $I^*_{k_0-1}=0$ and the fifth and sixth equalities come from Lemma 4 in [16] and $\mu_{k_0-1}=\mu_{k_0}$, respectively.

So, we get all the LILs in (3.9).

Case 2: LILs for stages $k>k_0$. By Lemma 4.4, it follows that $I^*_{k_0+l}=0$ for $l=1,2,\ldots,K-k_0$, w.p.1. As a result, $T^*_{k_0+l}=0$ for $l=1,2,\ldots,K-k_0$, w.p.1. For Q_{k_0+l}, $l=1,2,\ldots,K-k_0$, by Lemma 2.1, (4.2) and (4.24),

$$
\begin{aligned}
\widetilde{Q}_{k_0+l}(t)-\bar{Q}_{k_0+l}(t) &= \mu_{k_0+l-1}\left[\widetilde{T}_{k_0+l-1}(t)-t\right]+\mu_{k_0+l-1}^{1/2}c_{k_0+l-1}W_{k_0+l-1}(\bar{T}_{k_0+l-1}(t))\\
&\quad -\mu_{k_0+l}^{1/2}c_{k_0+l}W_{k_0+l}(\bar{T}_{k_0+l}(t))+\mu_{k_0+l}\widetilde{I}_{k_0+l}(t)\\
&= -\widetilde{Y}_{k_0+l-1}(t)+\mu_{k_0+l-1}^{1/2}c_{k_0+l-1}W_{k_0+l-1}(\bar{T}_{k_0+l-1}(t))\\
&\quad -\mu_{k_0+l}^{1/2}c_{k_0+l}W_{k_0+l}(\bar{T}_{k_0+l}(t))+\mu_{k_0+l}\widetilde{I}_{k_0+l}(t). \qquad (4.27)
\end{aligned}
$$

When $l=1$, by (4.2), we have

$$
\begin{aligned}
&\widetilde{Q}_{k_0+1}(t)-\bar{Q}_{k_0+1}(t)\\
&= -\widetilde{Y}_{k_0}(t)+\mu_{k_0}^{1/2}c_{k_0}W_{k_0}(\bar{T}_{k_0}(t))-\mu_{k_0+1}^{1/2}c_{k_0+1}W_{k_0+1}(\bar{T}_{k_0+1}(t))+\mu_{k_0+1}\widetilde{I}_{k_0+1}(t).\\
&= -\Psi(\widetilde{X}_{k_0})(t)+\mu_{k_0}^{1/2}c_{k_0}W_{k_0}(\bar{T}_{k_0}(t))-\mu_{k_0+1}^{1/2}c_{k_0+1}W_{k_0+1}(\bar{T}_{k_0+1}(t))+\mu_{k_0+1}\widetilde{I}_{k_0+1}(t).\\
&= -\sup_{0\le u\le t}\left[-\mu_{k_0-1}\widetilde{T}_{k_0-1}(u)+\mu_{k_0}u-\mu_{k_0-1}^{1/2}c_{k_0-1}W_{k_0-1}(\bar{T}_{k_0-1}(u))+\mu_{k_0}^{1/2}c_{k_0}W_{k_0}(\bar{T}_{k_0}(u))\right]\\
&\quad +\mu_{k_0}^{1/2}c_{k_0}W_{k_0}(\bar{T}_{k_0}(t))-\mu_{k_0+1}^{1/2}c_{k_0+1}W_{k_0+1}(\bar{T}_{k_0+1}(t))+\mu_{k_0+1}\widetilde{I}_{k_0+1}(t)\\
&= -\sup_{0\le u\le t}\left[\mu_{k_0-1}\widetilde{I}_{k_0-1}(u)-\mu_{k_0-1}^{1/2}c_{k_0-1}W_{k_0-1}(u)+\mu_{k_0}^{1/2}c_{k_0}W_{k_0}(u)\right]\\
&\quad +\mu_{k_0}^{1/2}c_{k_0}W_{k_0}(t)-\mu_{k_0+1}^{1/2}c_{k_0+1}W_{k_0+1}(t)+\mu_{k_0+1}\widetilde{I}_{k_0+1}(t)\\
&= -\sup_{0\le u\le t}\left[\mu_{k_0-1}\widetilde{I}_{k_0-1}(u)-\mu_{k_0-1}^{1/2}c_{k_0-1}W_{k_0-1}(u)-\mu_{k_0}^{1/2}c_{k_0}W_{k_0}(t-u)\right]\\
&\quad -\mu_{k_0+1}^{1/2}c_{k_0+1}W_{k_0+1}(t)+\mu_{k_0+1}\widetilde{I}_{k_0+1}(t). \qquad (4.28)
\end{aligned}
$$

For the LIL of Q_{k_0+1}, by (4.28),

$$
\begin{aligned}
& Q^*_{k_0+1} \\
= \ & \limsup_{L\to\infty} \frac{\left|\left|\widetilde{Q}_{k_0+1} - \bar{Q}_{k_0+1}\right|\right|_L}{\varphi(L)} \\
= \ & \limsup_{n\to\infty} \frac{\sup\limits_{0\le t\le L}\left| \begin{array}{c} \sup\limits_{0\le u\le t}\left[\mu_{k_0-1}\widetilde{I}_{k_0-1}(u) - \mu^{1/2}_{k_0-1}c_{k_0-1}W_{k_0-1}(u) - \mu^{1/2}_{k_0}c_{k_0}W_{k_0}(t-u)\right] \\ +\mu^{1/2}_{k_0+1}c_{k_0+1}W_{k_0+1}(t) - \mu_{k_0+1}\widetilde{I}_{k_0+1}(t) \end{array}\right|}{\varphi(L)} \\
= \ & \limsup_{L\to\infty} \frac{\sup\limits_{0\le t\le L}\left| \begin{array}{c} \sup\limits_{0\le u\le t}\left[-\mu^{1/2}_{k_0-1}c_{k_0-1}W_{k_0-1}(u) - \mu^{1/2}_{k_0}c_{k_0}W_{k_0}(t-u)\right] \\ +\mu^{1/2}_{k_0+1}c_{k_0+1}W_{k_0+1}(t) \end{array}\right|}{\varphi(L)} \\
= \ & \sqrt{\mu_{k_0}(c^2_{k_0-1}\vee c^2_{k_0}) + \mu_{k_0+1}c^2_{k_0+1}}, \quad \text{w.p.1,}
\end{aligned}
\tag{4.29}
$$

where the third equality holds because $I^*_{k_0-1} = I^*_{k_0+1} = 0$, the forth equality is from Lemma 4.3 in [**?**] and $\mu_{k_0-1} = \mu_{k_0}$.

For $Q_{k_0+l}, l = 2, 3, \ldots, K-k_0$, by (4.27) we have,

$$
\begin{aligned}
Q^*_{k_0+l} &= \limsup_{L\to\infty} \frac{\sup\limits_{0\le t\le L}\left| \begin{array}{l} -\mu_{k_0+l-1}\widetilde{I}_{k_0+l-1}(t) + \mu_{k_0+l}\widetilde{I}_{k_0+l}(t) \\ +\mu^{1/2}_{k_0+l-1}c_{k_0+l-1}W_{k_0+l-1}(t) - \mu^{1/2}_{k_0+l}c_{k_0+l}W_{k_0+l}(t) \end{array}\right|}{\varphi(L)} \\
&= \limsup_{L\to\infty} \frac{\sup\limits_{0\le t\le L}\left|\mu^{1/2}_{k_0+l-1}c_{k_0+l-1}W_{k_0+l-1}(t) - \mu^{1/2}_{k_0+l}c_{k_0+l}W_{k_0+l}(t)\right|}{\varphi(L)} \\
&= \sqrt{\mu_{k_0+l-1}c^2_{k_0+l-1} + \mu_{k_0+l}c^2_{k_0+l}}, \quad \text{w.p.1.}
\end{aligned}
\tag{4.30}
$$

For $Z_{k_0+l}, l = 1, 2, \ldots, K-k_0$, by Lemma 2.1 and (4.2),

$$
\begin{aligned}
& \widetilde{Z}_{k_0+l}(t) - \bar{Z}_{k_0+l}(t) \\
= \ & \frac{1}{\mu_{k_0+l}}\left[\widetilde{Q}_{k_0+l}(t) - \bar{Q}_{k_0+l}(t) + \mu^{1/2}_{k_0+l}c_{k_0+l}(W_{k_0+l}(\bar{T}_{k_0+l}(t)) - W_{k_0+l}(\rho_{k_0+l}t))\right] \\
= \ & \frac{1}{\mu_{k_0+l}}\Big[-\mu_{k_0+l-1}\widetilde{I}_{k_0+l-1}(t) + \mu_{k_0+l}\widetilde{I}_{k_0+l}(t) \\
& +\mu^{1/2}_{k_0+l-1}c_{k_0+l-1}W_{k_0+l-1}(\bar{T}_{k_0+l-1}(t)) - \mu^{1/2}_{k_0+l}c_{k_0+l}W_{k_0+l}(\rho_{k_0+l}t)\Big] \qquad (4.31) \\
= \ & \frac{1}{\mu_{k_0+l}}\Big[-\mu_{k_0+l-1}\widetilde{I}_{k_0+l-1}(t) + \mu_{k_0+l}\widetilde{I}_{k_0+l}(t) \\
& +\mu^{1/2}_{k_0+l-1}(c_{k_0+l-1}W_{k_0+l-1}(t) - c_{k_0+l}W_{k_0+l}(t))\Big].
\end{aligned}
$$

When $l=1$, if follows from (4.31) that

$$
\begin{aligned}
& \widetilde{Z}_{k_0+1}(t)-\bar{Z}_{k_0+1}(t) \\
= & \frac{1}{\mu_{k_0+1}}\left[-\widetilde{Y}_{k_0}(t)+\mu_{k_0+1}\widetilde{I}_{k_0+1}(t)+\mu_{k_0}^{1/2}c_{k_0}(W_{k_0}(t)-W_{k_0+1}(t))\right] \\
= & -\frac{\sup\limits_{0\le u\le t}\left[-\mu_{k_0-1}\widetilde{T}_{k_0-1}(u)+\mu_{k_0}u-\mu_{k_0-1}^{1/2}c_{k_0-1}W_{k_0-1}(\bar{T}_{k_0-1}(u))+\mu_{k_0}^{1/2}c_{k_0}W_{k_0}(\bar{T}_{k_0}(u))\right]}{\mu_{k_0+1}} \\
& +\frac{1}{\mu_{k_0+1}}\left[\mu_{k_0+1}\widetilde{I}_{k_0+1}(t)+\mu_{k_0}^{1/2}(c_{k_0}W_{k_0}(t)-c_{k_0+1}W_{k_0+1}(t))\right] \\
= & -\frac{1}{\mu_{k_0+1}}\sup_{0\le u\le t}\left[\mu_{k_0-1}\widetilde{I}_{k_0-1}(u)-\mu_{k_0-1}^{1/2}c_{k_0-1}W_{k_0-1}(u)+\mu_{k_0}^{1/2}c_{k_0}W_{k_0}(u)\right] \\
& +\frac{1}{\mu_{k_0+1}}\left[\mu_{k_0+1}\widetilde{I}_{k_0+1}(t)+\mu_{k_0}^{1/2}(c_{k_0}W_{k_0}(t)-c_{k_0+1}W_{k_0+1}(t))\right] \\
= & -\frac{1}{\mu_{k_0+1}}\sup_{0\le u\le t}\left[\mu_{k_0-1}\widetilde{I}_{k_0-1}(u)-\mu_{k_0-1}^{1/2}c_{k_0-1}W_{k_0-1}(u)-\mu_{k_0}^{1/2}c_{k_0}W_{k_0}(t-u)\right] \\
& +\frac{1}{\mu_{k_0+1}}\left[\mu_{k_0+1}\widetilde{I}_{k_0+1}(t)-\mu_{k_0}^{1/2}c_{k_0+1}W_{k_0+1}(t)\right], \qquad (4.32)
\end{aligned}
$$

For the LIL of Z_{k_0+1}, by (4.32),

$$
\begin{aligned}
& Z^*_{k_0+1} \\
= & \limsup_{L\to\infty}\frac{\left\|\widetilde{Z}_{k_0+1}-\bar{Z}_{k_0+1}\right\|_L}{\mu_{k_0+1}\varphi(L)} \\
= & \limsup_{n\to\infty}\frac{\sup\limits_{0\le t\le L}\left|\begin{array}{l}\sup\limits_{0\le u\le t}\left[\mu_{k_0-1}\widetilde{I}_{k_0-1}(u)-\mu_{k_0-1}^{1/2}c_{k_0-1}W_{k_0-1}(u)-\mu_{k_0}^{1/2}c_{k_0}W_{k_0}(t-u)\right] \\ +\mu_{k_0}^{1/2}c_{k_0+1}W_{k_0+1}(t)-\mu_{k_0+1}\widetilde{I}_{k_0+1}(t)\end{array}\right|}{\mu_{k_0+1}\varphi(L)} \\
= & \frac{1}{\mu_{k_0+1}}\sqrt{\mu_{k_0}((c^2_{k_0-1}\vee c^2_{k_0})+c^2_{k_0+1})}, \quad \text{w.p.1},
\end{aligned}
$$

where the last equality holds similarly with (4.29).

For $Z_{k_0+l}, l=2,3,\dots,K-k_0$, similar with (4.30), with (4.31) we have,

$$
\begin{aligned}
Z^*_{k_0+l} &= \limsup_{L\to\infty}\frac{\sup\limits_{0\le t\le L}\left|\begin{array}{l}-\mu_{k_0+l-1}\widetilde{I}_{k_0+l-1}(t)+\mu_{k_0+l}\widetilde{I}_{k_0+l}(t) \\ +\mu_{k_0+l-1}^{1/2}\left[c_{k_0+l-1}W_{k_0+l-1}(t)-c_{k_0+l}W_{k_0+l}(t)\right]\end{array}\right|}{\varphi(L)} \\
&= \limsup_{L\to\infty}\frac{\sup\limits_{0\le t\le L}\left|\mu_{k_0+l-1}^{1/2}\left[c_{k_0+l-1}W_{k_0+l-1}(t)-c_{k_0+l}W_{k_0+l}(t)\right]\right|}{\varphi(L)} \\
&= \sqrt{\mu_{k_0+l-1}(c^2_{k_0+l-1}+c^2_{k_0+l})}, \quad \text{w.p.1}. \qquad (4.33)
\end{aligned}
$$

For $D_{k_0+l}, l = 1, 2, \ldots, K - k_0$, by Lemma 2.1 and (4.2),

$$\begin{aligned}
\widetilde{D}_{k_0+l}(t) - \bar{D}_{k_0+l}(t) &= \mu_{k_0+l}\left[\widetilde{T}_{k_0+l}(t) - \bar{T}_{k_0+l}(t)\right] + \mu_{k_0+l}^{1/2} c_{k_0+l} W_{k_0+l}(\bar{T}_{k_0+l}(t)) \\
&= -\mu_{k_0+l}\left[\widetilde{I}_{k_0+l}(t) - \bar{I}_{k_0+l}(t)\right] + \mu_{k_0+l}^{1/2} c_{k_0+l} W_{k_0+l}(t),
\end{aligned}$$

this, together with the LIL $I^*_{k_0+l} = 0$ and functional LIL $\mathcal{K}_{I_{k_0+l}} = \{0\}$, implies that, w.p.1,

$$\begin{aligned}
D^*_{k_0+l} &= \limsup_{L\to\infty} \frac{\left\|-\mu_{k_0+l}\widetilde{I}_{k_0+l} + \mu_{k_0+l}^{1/2} c_{k_0+l} W_{k_0+l}\right\|_L}{\varphi(L)} \\
&= \limsup_{L\to\infty} \frac{\left\|\mu_{k_0+l}^{1/2} c_{k_0+l} W_{k_0+l}\right\|_L}{\varphi(L)} = \mu_{k_0+l}^{1/2} c_{k_0+l}.
\end{aligned}$$

So, we get all the LILs in (3.10) and (3.11).

Hence, we get all the LILs and complete the proof. □

4.4 Proof of Lemma 4.3

We prove Lemma 4.3 based on Lemmas 4.5 and 4.6, which are presented in advance and are proved after Lemma 4.3. Lemmas 4.5 and 4.6 show us the queue length in the underloaded regime is almost stochastically bounded in probability.

Lemma 4.5. *If $\rho_k < 1$ for all $k = 1, 2, \ldots, K$, then*

$$\mathsf{P}\left\{\sup_{0\le t\le T} \widetilde{Q}_k(t) \ge z\right\} \le k \exp\left\{-\gamma_k z\right\}, \tag{4.34}$$

where

$$\gamma_1 = \frac{\mu_1 - \mu_0}{\mu_0(c_0^2 + c_1^2)}, \quad \gamma_k = \min\left\{\frac{\mu_k - \mu_0}{\mu_0(c_0^2 + c_k^2)}, \frac{\gamma_1}{2(k-1)}, \frac{\gamma_2}{2(k-1)}, \ldots, \frac{\gamma_{k-1}}{2(k-1)}\right\} \tag{4.35}$$

for all $k = 2, 3, \ldots, K$.

Lemma 4.6. *If there exist k_0 and l_0 such that $\rho_{k_0} \ge 1$, $\rho_{k_0+l} < 1$ for all $l = 1, 2, \ldots, l_0$ and $\rho_{k_0+l_0+1} \ge 1$ if $k_0 + l_0 + 1 \le K$, then for all $k \ge k_0$ such that $\rho_k < 1$, we have*

$$\mathsf{P}\left\{\sup_{0\le t\le T} \widetilde{Q}_{k_0+l}(t) \ge z\right\} \le l \exp\left\{-\gamma'_l z\right\}, \tag{4.36}$$

where

$$\gamma'_1 = \frac{\mu_{k_0+1} - \mu_{k_0}}{\mu_{k_0}(c_{k_0}^2 + c_{k_0+1}^2)}, \tag{4.37}$$

$$\gamma'_l = \min\left\{\frac{\mu_{k_0+1} - \mu_{k_0}}{\mu_{k_0}(c_{k_0}^2 + c_{k_0+1}^2)}, \frac{\gamma'_1}{2(l-1)}, \frac{\gamma'_2}{2(l-1)}, \ldots, \frac{\gamma'_{l-1}}{2(l-1)}\right\} \tag{4.38}$$

for all $l = 2, 3, \ldots, l_0$.

Lemmas 4.5 and 4.6 show us that the queue length in the underloaded regime is s-tochastically bounded in probability regardless of its front stages, no matter what stage they are.

Proof of Lemma 4.3 Given k, we prove the result in two cases: (i) $\rho_i < 1$ for all $i = 1, 2, \ldots, k$. By Lemma 4.5, letting $z = 2\log T/\gamma_k$ in (4.34) yields

$$\mathsf{P}\left\{\sup_{0\le t\le T} \widetilde{Q}_k(t)/\log T \ge \frac{2}{\gamma_k}\right\} = k\frac{1}{T^2}.$$

This and Borel-Cantelli Lemma (see Ross [34]) imply that $\sup_{0\le t\le T} \widetilde{Q}_k(t) = O(\log T)$. (ii) there exists a $k' < k$ such that $\rho_{k'} \ge 1$ and $\rho_i < 1$ for all $i = k'+1, k'+2, \ldots, k$. For this case, by Lemma 4.6, the proof is similar with case (1) and is omitted.

As a result, $Q_k^* = 0$ w.p.1 when $\rho_k < 1$. It follows that $Z_k^* = 0$ w.p.1 under $\rho_k < 1$ because $\mu_k \widetilde{Z}_k = \widetilde{Q}_k$ and $\mu_k \bar{Z}_k = \bar{Q}_k$ by (2.10) and (4.2). □

4.4.1 Proof of Lemma 4.5

Since $\rho_k < 1$ for all $k = 1, 2, \ldots, K$, we have $\rho_k = \mu_0/\mu_k$ and $\mu_0 < \mu_k$ for all $k = 1, 2, \ldots, K$.

We inductively prove the result. Firstly, we prove (4.34) holds with $k = 1$. By (4.2),

$$\widetilde{X}_1(t) \;=\; (\mu_0 - \mu_1)t + \mu_0^{1/2} c_0 W_0(t) - \mu_1^{1/2} c_1 W_1(\rho_1 t)$$

is a reflected Brownian motion with negative drift $(\mu_0 - \mu_1)$ and variance parameter $\mu_0(c_0^2 + c_1^2)$, then it follows from Theorem 6.3 in [8] that

$$\mathsf{P}\left\{\sup_{0\le t\le T} \widetilde{Q}_1(t) \ge z\right\} \le \exp\left\{-\frac{2(\mu_1 - \mu_0)}{\mu_0(c_0^2 + c_1^2)} z\right\} = N_1 \exp\left\{-\gamma_1 z\right\}, \tag{4.39}$$

that is, (4.34) holds with $k = 1$.

We next prove (4.34) holds with $k > 1$. For any $k > 1$, by (4.2),

$$\begin{aligned}
\mathsf{P}\left\{\sup_{0\le t\le T} \widetilde{Q}_k(t) \ge z\right\} &= \mathsf{P}\left\{\sup_{0\le t\le T} \left[\widetilde{X}_k(t) + \widetilde{Y}_k(t)\right] \ge z\right\} \\
&= \mathsf{P}\left\{\sup_{0\le t\le T} \left[\widetilde{X}_k(t) + \sup_{0\le s\le t}\left[-\widetilde{X}_k(s)\right]\right] \ge z\right\} \\
&= \mathsf{P}\left\{\sup_{0\le t\le T}\sup_{0\le s\le t} \left[\widetilde{X}_k(t) - \widetilde{X}_k(s)\right] \ge z\right\}.
\end{aligned} \tag{4.40}$$

Since, for all $0 \le s \le t$,

$$\begin{aligned}
\widetilde{X}_k(t) - \widetilde{X}_k(s) &= \mu_{k-1}\left[\widetilde{T}_{k-1}(t) - \widetilde{T}_{k-1}(s)\right] - \mu_k(t-s) \\
&\quad + \mu_{k-1}^{1/2} c_{k-1}\left[W_{k-1}(\bar{T}_{k-1}(t)) - W_{k-1}(\bar{T}_{k-1}(s))\right] \\
&\quad - \mu_k^{1/2} c_k\left[W_k(\bar{T}_k(t)) - W_k(\bar{T}_k(s))\right],
\end{aligned} \tag{4.41}$$

where

$$\begin{aligned}
\mu_{k-1}\left[\widetilde{T}_{k-1}(t) - \widetilde{T}_{k-1}(s)\right] &= \mu_{k-1}(t-s) - \left[\widetilde{Y}_{k-1}(t) - \widetilde{Y}_{k-1}(s)\right] \\
&= \mu_{k-1}(t-s) - \left[\widetilde{Q}_{k-1}(t) - \widetilde{Q}_{k-1}(s)\right] + \left[\widetilde{X}_{k-1}(t) - \widetilde{X}_{k-1}(s)\right],
\end{aligned}$$

we have

$$\begin{aligned}
\widetilde{X}_k(t)-\widetilde{X}_k(s) &= \mu_{k-1}(t-s)-\mu_k(t-s)-\left[\widetilde{Q}_{k-1}(t)-\widetilde{Q}_{k-1}(s)\right] \\
&\quad +\mu_{k-1}^{1/2}c_{k-1}\left[W_{k-1}(\bar{T}_{k-1}(t))-W_{k-1}(\bar{T}_{k-1}(s))\right] \\
&\quad -\mu_k^{1/2}c_k\left[W_k(\bar{T}_k(t))-W_k(\bar{T}_k(s))\right] \\
&\quad +\left[\widetilde{X}_{k-1}(t)-\widetilde{X}_{k-1}(s)\right]. \qquad (4.42)
\end{aligned}$$

So, by iteration, we have, for all $0 \le s \le t$,

$$\begin{aligned}
\widetilde{X}_k(t)-\widetilde{X}_k(s) &= \mu_1(t-s)-\mu_k(t-s)-\sum_{l=1}^{k-1}\left[\widetilde{Q}_l(t)-\widetilde{Q}_l(s)\right]+\left[\widetilde{X}_1(t)-\widetilde{X}_1(s)\right] \\
&\quad +\mu_1^{1/2}c_1\left[W_1(\widetilde{T}_1(t))-W_1(\widetilde{T}_1(s))\right]-\mu_k^{1/2}c_k\left[W_k(\bar{T}_k(t))-W_k(\bar{T}_k(s))\right] \\
&= (\mu_0-\mu_k)(t-s)-\sum_{l=1}^{k-1}\left[\widetilde{Q}_l(t)-\widetilde{Q}_l(s)\right] \\
&\quad +\mu_0^{1/2}c_0\left[W_0(t)-W_0(s)\right]-\mu_k^{1/2}c_k\left[W_k(\bar{T}_k(t))-W_k(\bar{T}_k(s))\right] \\
&\overset{\mathrm{d}}{=} (\mu_0-\mu_k)(t-s)-\sum_{l=1}^{k-1}\left[\widetilde{Q}_l(t)-\widetilde{Q}_l(s)\right] \\
&\quad +\mu_0^{1/2}c_0\left[W_0(t)-W_0(s)\right]-\mu_0^{1/2}c_k\left[W_k(t)-W_k(s)\right] \\
&\le (\mu_0-\mu_k)(t-s)+\sum_{l=1}^{k-1}\widetilde{Q}_l(s)+\mu_0^{1/2}c_0W_0(t-s)-\mu_0^{1/2}c_kW_k(t-s).
\end{aligned}$$

By (4.40),

$$\begin{aligned}
&\mathsf{P}\left\{\sup_{0\le t\le T}\widetilde{Q}_k(t)\ge z\right\} \\
\le\ &\mathsf{P}\left\{\sup_{0\le t\le T}\sup_{0\le s\le t}\left[(\mu_0-\mu_k)(t-s)+\mu_0^{1/2}c_0W_0(t-s)-\mu_0^{1/2}c_kW_k(t-s)\right]\ge\frac{z}{2}\right\} \\
&+\mathsf{P}\left\{\sup_{0\le t\le T}\sup_{0\le s\le t}\sum_{l=1}^{k-1}\widetilde{Q}_l(s)\ge\frac{z}{2}\right\} \\
=\ &\mathsf{P}\left\{\sup_{0\le t\le T}\left[(\mu_0-\mu_k)t+\mu_0^{1/2}c_0W_0(t)-\mu_0^{1/2}c_kW_k(t)\right]\ge\frac{z}{2}\right\} \qquad (4.43) \\
&+\mathsf{P}\left\{\sup_{0\le t\le T}\sum_{l=1}^{k-1}\widetilde{Q}_l(t)\ge\frac{z}{2}\right\} \\
\le\ &\exp\left\{-\frac{2(\mu_k-\mu_0)}{\mu_0(c_0^2+c_k^2)}\frac{z}{2}\right\}+\mathsf{P}\left\{\sum_{l=1}^{k-1}\sup_{0\le t\le T}\widetilde{Q}_l(t)\ge\frac{z}{2}\right\} \\
\le\ &\exp\left\{-\frac{2(\mu_k-\mu_0)}{\mu_0(c_0^2+c_k^2)}\frac{z}{2}\right\}+\sum_{l=1}^{k-1}\mathsf{P}\left\{\sup_{0\le t\le T}\widetilde{Q}_l(t)\ge\frac{z}{2(k-1)}\right\},
\end{aligned}$$

where the second inequality holds because $(\mu_0 - \mu_k)t + \mu_0^{1/2} c_0 W_0(t) - \mu_0^{1/2} c_k W_k(t)$ is a Brownian motion with negative drift $(\mu_0 - \mu_k)$ and variance parameter $\mu_0(c_0^2 + c_k^2)$.

Next, we inductively prove (4.34) holds for $k = 2, 3, \ldots, K$. We first prove (4.34) holds for $k = 2$. By (4.43),

$$\begin{aligned}
\mathsf{P}\left\{\sup_{0\le t\le T} \widetilde{Q}_2(t) \ge z\right\} &\le \exp\left\{-\frac{2(\mu_2 - \mu_0)}{\mu_0(c_0^2 + c_2^2)}\frac{z}{2}\right\} + \mathsf{P}\left\{\sup_{0\le t\le T} \widetilde{Q}_1(t) \ge \frac{z}{2}\right\} \\
&\le \exp\left\{-\frac{2(\mu_2 - \mu_0)}{\mu_0(c_0^2 + c_2^2)}\frac{z}{2}\right\} + \exp\left\{-\frac{\gamma_1}{2}z\right\} \\
&\le 2\exp\left\{-\gamma_2 t\right\},
\end{aligned}$$

where γ_2 is defined in (4.35).

Suppose that (4.34) holds for all $2, 3, \ldots, k-1$, we next show that (4.34) holds for all k.

By the inductive hypothesis and (4.43),

$$\begin{aligned}
\mathsf{P}\left\{\sup_{0\le t\le T} \widetilde{Q}_k(t) \ge z\right\} &\le \exp\left\{-\frac{2(\mu_k - \mu_0)}{\mu_0(c_0^2 + c_k^2)}\frac{z}{2}\right\} + \sum_{l=1}^{k-1} \mathsf{P}\left\{\sup_{0\le t\le T} \widetilde{Q}_l(t) \ge \frac{z}{2(k-1)}\right\} \\
&\le \exp\left\{-\frac{2(\mu_k - \mu_0)}{\mu_0(c_0^2 + c_k^2)}\frac{z}{2}\right\} + \sum_{l=1}^{k-1} \exp\left\{-\frac{\gamma_l}{2(k-1)}z\right\} \\
&\le k\exp\left\{-\gamma_k t\right\},
\end{aligned}$$

where γ_k is defined in (4.35). So, (4.34) holds for $k = 2, 3, \ldots, K$.

Hence, (4.34) holds inductively. □

4.4.2 Proof of Lemma 4.6

We inductively prove the result. We first prove (4.36) holds for $k = k_0 + 1$. By (4.41),

$$\begin{aligned}
\widetilde{X}_{k_0+1}(t) - \widetilde{X}_{k_0+1}(s) &= (\mu_{k_0} - \mu_{k_0+1})(t-s) - \left[\widetilde{Y}_{k_0}(t) - \widetilde{Y}_{k_0}(s)\right] \\
&\quad + \mu_{k_0}^{1/2} c_{k_0}\left[W_{k_0}(\bar{T}_{k_0}(t)) - W_{k_0}(\bar{T}_{k_0}(s))\right] \\
&\quad - \mu_{k_0+1}^{1/2} c_{k_0+1}\left[W_{k_0+1}(\bar{T}_{k_0+1}(t)) - W_{k_0+1}(\bar{T}_{k_0+1}(s))\right].
\end{aligned}$$

With the definition of traffic intensity ρ, we have

$$\rho_{k_0} = \frac{\min\{\mu_0, \mu_1, \ldots, \mu_{k_0-1}\}}{\mu_{k_0}} \ge 1,$$

then $\min\{\mu_0, \mu_1, \ldots, \mu_{k_0-1}\} \ge \mu_{k_0}$. This, and

$$\rho_{k_0+1} = \frac{\min\{\mu_0, \mu_1, \ldots, \mu_{k_0}\}}{\mu_{k_0+1}} = \frac{\mu_{k_0}}{\mu_{k_0+1}} < 1,$$

implies that $\mu_{k_0} < \mu_{k_0+1}$. In addition, since $\widetilde{Y}_k(t) = \sup_{0\le s\le t}\left[-\widetilde{X}_k(s)\right]^+$ in (2.12), we have, for all $0 \le s \le t$, $\widetilde{Y}_{k-1}(t) - \widetilde{Y}_{k-1}(s) \ge 0$ for all $k = 1, 2, \ldots, K$. Notice that $\bar{T}_{k_0}(t) = t$ and $\bar{T}_{k_0+1}(t) = \rho_{k_0+1}t$ for all $t \ge 0$, then

$$
\begin{aligned}
& \widetilde{X}_{k_0+1}(t) - \widetilde{X}_{k_0+1}(s) \\
\le\ & (\mu_{k_0} - \mu_{k_0+1})(t-s) + \mu_{k_0}^{1/2} c_{k_0}\left[W_{k_0}(t) - W_{k_0}(s)\right] \\
& -\mu_{k_0+1}^{1/2} c_{k_0+1}\left[W_{k_0+1}(\rho_{k_0+1}(t)) - W_{k_0+1}(\rho_{k_0+1}(s))\right] \\
\stackrel{\mathrm{d}}{=}\ & (\mu_{k_0} - \mu_{k_0+1})(t-s) + \mu_{k_0}^{1/2} c_{k_0} W_{k_0}(t-s) - \mu_{k_0}^{1/2} c_{k_0+1} W_{k_0+1}(t-s).
\end{aligned}
$$

Similarly with (4.43),

$$
\begin{aligned}
& \mathsf{P}\left\{\sup_{0\le t\le T} \widetilde{Q}_{k_0+1}(t) \ge z\right\} \\
\le\ & \mathsf{P}\left\{\sup_{0\le t\le T}\left[(\mu_{k_0} - \mu_{k_0+1})t + \mu_{k_0}^{1/2} c_{k_0} W_{k_0}(t) - \mu_{k_0}^{1/2} c_{k_0+1} W_{k_0+1}(t)\right] \ge z\right\} \\
\le\ & \exp\left\{-\frac{\mu_{k_0+1} - \mu_{k_0}}{\mu_{k_0}(c_{k_0}^2 + c_{k_0+1}^2)} z\right\} \\
=\ & \exp\left\{-\gamma_1' z\right\}
\end{aligned}
$$

where γ_1 is defined in (4.37), the second inequality is from Theorem 6.3 in [8] because $(\mu_{k_0} - \mu_{k_0+1})t + \mu_{k_0}^{1/2} c_{k_0} W_{k_0}(t) - \mu_{k_0}^{1/2} c_{k_0+1} W_{k_0+1}(t)$ is a Brownian motion with negative drift $(\mu_{k_0} - \mu_{k_0+1})$ and variance parameter $\mu_{k_0}(c_{k_0}^2 + c_{k_0+1}^2)$.

Next we consider (4.41) with $k = k_0+2, \ldots, k_0+l (< k_0+l_0)$. By (4.42), for $l : 1 < l \le l_0$,

$$
\begin{aligned}
& \widetilde{X}_{k_0+l}(t) - \widetilde{X}_{k_0+l}(s) \\
=\ & \mu_{k_0+l-1}(t-s) - \mu_{k_0+l}(t-s) - \left[\widetilde{Q}_{k_0+l-1}(t) - \widetilde{Q}_{k_0+l-1}(s)\right] \\
& +\mu_{k_0+l-1}^{1/2} c_{k_0+l-1}\left[W_{k_0+l-1}(\bar{T}_{k_0+l-1}(t)) - W_{k_0+l-1}(\bar{T}_{k_0+l-1}(s))\right] \\
& -\mu_{k_0+l}^{1/2} c_{k_0+l}\left[W_{k_0+l}(\bar{T}_{k_0+l}(t)) - W_{k_0+l}(\bar{T}_{k_0+l}(s))\right] \\
& +\left[\widetilde{X}_{k_0+l-1}(t) - \widetilde{X}_{k_0+l-1}(s)\right] \\
=\ & \mu_{k_0+1}(t-s) - \mu_{k_0+l}(t-s) - \sum_{l=k_0+1}^{k_0+l-1}\left[\widetilde{Q}_i(t) - \widetilde{Q}_i(s)\right] \\
& +\mu_{k_0+1}^{1/2} c_{k_0+1}\left[W_{k_0+1}(\bar{T}_{k_0+1}(t)) - W_{k_0+1}(\bar{T}_{k_0+1}(s))\right] \\
& -\mu_{k_0+l}^{1/2} c_{k_0+l}\left[W_{k_0+l}(\bar{T}_{k_0+l}(t)) - W_{k_0+l}(\bar{T}_{k_0+l}(s))\right] \\
& +\left[\widetilde{X}_{k_0+1}(t) - \widetilde{X}_{k_0+1}(s)\right],
\end{aligned}
$$

where, by (4.2),

$$\begin{aligned}
\widetilde{X}_{k_0+1}(t) - \widetilde{X}_{k_0+1}(s) &= \mu_{k_0}\left[\widetilde{T}_{k_0}(t) - \widetilde{T}_{k_0}(s)\right] - \mu_{k_0+1}(t-s) \\
&\quad + \mu_{k_0}^{1/2} c_{k_0}\left[W_{k_0}(\bar{T}_{k_0}(t)) - W_{k_0}(\bar{T}_{k_0}(s))\right] \\
&\quad - \mu_{k_0+1}^{1/2} c_{k_0+1}\left[W_{k_0+1}(\bar{T}_{k_0+1}(t)) - W_{k_0+1}(\bar{T}_{k_0+1}(s))\right] \\
&= \mu_{k_0}(t-s) - \mu_{k_0+1}(t-s) - \left[\widetilde{Y}_{k_0}(t) - \widetilde{Y}_{k_0}(s)\right] \\
&\quad + \mu_{k_0}^{1/2} c_{k_0}\left[W_{k_0}(\bar{T}_{k_0}(t)) - W_{k_0}(\bar{T}_{k_0}(s))\right] \\
&\quad - \mu_{k_0+1}^{1/2} c_{k_0+1}\left[W_{k_0+1}(\bar{T}_{k_0+1}(t)) - W_{k_0+1}(\bar{T}_{k_0+1}(s))\right].
\end{aligned}$$

Since $\widetilde{Y}_{k_0}(t) - \widetilde{Y}_{k_0}(s) \geq 0$ and $\widetilde{Q}_k(t) \geq 0$ for all class k and $0 \leq s \leq t$, we have

$$\begin{aligned}
\widetilde{X}_{k_0+l}(t) - \widetilde{X}_{k_0+l}(s) &\leq (\mu_{k_0} - \mu_{k_0+l})(t-s) + \sum_{l=k_0+1}^{k_0+l-1} \widetilde{Q}_i(s) \\
&\quad + \mu_{k_0}^{1/2} c_{k_0}\left[W_{k_0}(\bar{T}_{k_0}(t)) - W_{k_0}(\bar{T}_{k_0}(s))\right] \\
&\quad - \mu_{k_0+l}^{1/2} c_{k_0+l}\left[W_{k_0+l}(\bar{T}_{k_0+l}(t)) - W_{k_0+l}(\bar{T}_{k_0+l}(s))\right] \\
&\stackrel{\mathrm{d}}{=} (\mu_{k_0} - \mu_{k_0+l})(t-s) + \sum_{l=k_0+1}^{k_0+l-1} \widetilde{Q}_i(s) \\
&\quad + \mu_{k_0}^{1/2} c_{k_0} W_{k_0}(t-s) - \mu_{k_0}^{1/2} c_{k_0+l} W_{k_0+l}(t-s),
\end{aligned}$$

where the equality in distribution holds because $\bar{T}_{k_0}(t) = t$ and $\bar{T}_{k_0+l}(t) = \rho_{k_0+l} t = (\mu_{k_0}/\mu_{k_0+l})t$ for all $t \geq 0$.

With (4.40),

$$\begin{aligned}
&\mathsf{P}\left\{\sup_{0\leq t\leq T} \widetilde{Q}_{k_0+l}(t) \geq z\right\} \\
\leq\ & \mathsf{P}\left\{\sup_{0\leq t\leq T}\sup_{0\leq s\leq t}\left[(\mu_{k_0} - \mu_{k_0+l})(t-s) + \mu_{k_0}^{1/2} c_{k_0} W_{k_0}(t-s)\right.\right. \\
&\qquad\qquad \left.\left. - \mu_{k_0}^{1/2} c_{k_0+l} W_{k_0+l}(t-s)\right] \geq \frac{z}{2}\right\} \\
&+ \mathsf{P}\left\{\sup_{0\leq t\leq T} \sum_{l=k_0+1}^{k_0+l-1} \widetilde{Q}_i(t) \geq \frac{z}{2}\right\} \\
=\ & \mathsf{P}\left\{\sup_{0\leq t\leq T}\left[(\mu_{k_0} - \mu_{k_0+l})t + \mu_{k_0}^{1/2} c_{k_0} W_{k_0}(t) - \mu_{k_0}^{1/2} c_{k_0+l} W_{k_0+l}(t)\right] \geq \frac{z}{2}\right\} \\
&+ \sum_{l=k_0+1}^{k_0+l-1} \mathsf{P}\left\{\sup_{0\leq t\leq T} \widetilde{Q}_i(t) \geq \frac{z}{2(l-1)}\right\} \\
\leq\ & \exp\left\{-\frac{2(\mu_{k_0+l} - \mu_{k_0})}{\mu_{k_0}(c_{k_0}^2 + c_{k_0+l}^2)}\frac{z}{2}\right\} + \sum_{l=k_0+1}^{k_0+l-1} \mathsf{P}\left\{\sup_{0\leq t\leq T} \widetilde{Q}_i(t) \geq \frac{z}{2(l-1)}\right\}. \qquad (4.44)
\end{aligned}$$

Next we inductively prove (4.41) holds for all $k_0+2, k_0+3, \ldots, k_0+l(<k_0+l_0)$. We first prove (4.41) holds for $k=k_0+2$. For $k=k_0+2$,

$$
\begin{aligned}
\mathsf{P}\left\{\sup_{0\le t\le T} \widetilde{Q}_{k_0+2}(t) \ge z\right\} &\le \exp\left\{-\frac{2(\mu_{k_0+l}-\mu_{k_0})}{\mu_{k_0}(c_{k_0}^2+c_{k_0+l}^2)}\frac{z}{2}\right\} + \mathsf{P}\left\{\sup_{0\le t\le T} \widetilde{Q}_{k_0+1}(t) \ge \frac{z}{2}\right\} \\
&\le \exp\left\{-\frac{\mu_{k_0+l}-\mu_{k_0}}{\mu_{k_0}(c_{k_0}^2+c_{k_0+l}^2)}z\right\} + \exp\left\{-\frac{\gamma_1'}{2}z\right\} \\
&\le 2\exp\left\{-\gamma_2' z\right\},
\end{aligned}
$$

where γ_2' is defined in (4.37).

Suppose that (4.41) holds for all $k_0+2, k_0+3, \ldots, k_0+l-1$, we next show that (4.34) holds for all k_0+l. By the inductive hypothesis and (4.44),

$$
\begin{aligned}
\mathsf{P}\left\{\sup_{0\le t\le T} \widetilde{Q}_{k_0+l}(t) \ge z\right\} &\le \exp\left\{-\frac{\mu_{k_0+l}-\mu_{k_0}}{\mu_{k_0}(c_{k_0}^2+c_{k_0+l}^2)}z\right\} + \sum_{l=k_0+1}^{k_0+l-1} \exp\left\{-\frac{\gamma_1'}{2(l-1)}z\right\} \\
&\le l\exp\left\{-\gamma_l' z\right\},
\end{aligned}
$$

where γ_l' is defined in (4.37). So, (4.41) holds for all $k_0+2, k_0+3, \ldots, k_0+l(<k_0+l_0)$.

Hence, (4.36) holds. □

4.5 Proof of Lemma 4.4

At first, we note that the stages with $\rho_k>1$ are in Cases 2 and 3 in Theorem 3.1, Case 2 in Theorem 3.2. For the above three cases, we note that it suffices to prove that

$$
I_k^* = 0, \quad \text{w.p.1}, \tag{4.45}
$$

for $k:\rho_k>1$, because, together with $\bar{T}_k(t)=t-\bar{I}_k(t)$ by (2.10) and $\widetilde{T}_k(t)=t-\widetilde{I}_k(t)$ by (4.2), (4.45) implies that $T_k^*=0$, w.p.1.

Case 2 in Theorem 3.1. Accordingly, we have $\rho_k>1$ for all $k=1,2,\ldots,K$. We next prove that (4.45) for all $k=1,2,\ldots,K$ by induction.

For $k=1$, by (4.2), $\lim_{t\to\infty}\widetilde{X}_1(t)=+\infty$ w.p.1, because $\widetilde{X}_1(t)$ is a BM with positive drift $\mu_0-\mu_1>0$. Since $\widetilde{I}_1(t)=\widetilde{Y}_1(t)/\mu_1=\sup_{0\le s\le t}\left[-\widetilde{X}_1(s)\right]^+/\mu_1$, we have $\sup_{t\ge 0}\widetilde{I}_1(t)<+\infty$ w.p.1. Since $\widetilde{I}_1(t)=\widetilde{Y}_1(t)/\mu_1=\sup_{0\le s\le t}\left[-\widetilde{X}_1(s)\right]^+/\mu_1$, we have $\sup_{t\ge 0}\widetilde{I}_1(t)<+\infty$ w.p.1. So, (4.45) holds for $k=1$.

Suppose that (4.45) holds for stages $1,2,\ldots,k$. Next we prove (4.45) holds for stage $k+1$. By (2.10) and (4.2),

$$
\widetilde{X}_{k+1}(t) = -\mu_k\widetilde{I}_k(t) + (\mu_k-\mu_{k+1})t + \mu_k^{1/2}c_kW_k(t) - \mu_{k+1}^{1/2}c_{k+1}W_{k+1}(t).
$$

Since $\rho_{k+1}>1$ and $I_k^*=0$ by inductive hypothesis, we have $\lim_{t\to\infty}\widetilde{X}_{k+1}(t)=\infty$, w.p.1. Similarly we have (4.45) holds for stage $k+1$. Hence, (4.45) holds for all $k=1,2,\ldots,K$.

Case 3 in Theorem 3.1. Accordingly, $\rho_k > 1$ for all $k = k'_1, k'_2, \ldots, k'_{l_0}$ and $\rho_k < 1$ for all $k \neq k'_1, k'_2, \ldots, k'_{l_0}$. At first, we note that by (2.10) and (4.2), for all $t \geq 0$, $\bar{T}_k(t) = t$ for all $k = k'_1, k'_2, \ldots, k'_{l_0}$, $\bar{T}_k(t) = \rho_k t$ for all $k \neq k'_1, k'_2, \ldots, k'_{l_0}$. Next, we inductively prove that (4.45) holds for all $k = k'_1, k'_2, \ldots, k'_{l_0}$.

If $k'_1 = 1$, then it follows from (3.4) that (4.45) holds. If $k'_1 > 1$, by (2.10) and (4.2), then

$$\begin{aligned}
\widetilde{X}_{k'_1}(t) &= \mu_{k'_1-1}\widetilde{T}_{k'_1-1}(t) - \mu_{k'_1}t + \mu_{k'_1-1}^{1/2}c_{k'_1-1}W_{k'_1-1}(\bar{T}_{k'_1-1}(t)) - \mu_{k'_1}^{1/2}c_{k'_1}W_{k'_1}(\bar{T}_{k'_1}(t)) \\
&= -\widetilde{Y}_{k'_1-1}(t) + (\mu_{k'_1-1} - \mu_{k'_1})t + \mu_{k'_1-1}^{1/2}c_{k'_1-1}W_{k'_1-1}(\bar{T}_{k'_1-1}(t)) - \mu_{k'_1}^{1/2}c_{k'_1}W_{k'_1}(\bar{T}_{k'_1}(t)) \\
&= \widetilde{X}_{k'_1-1}(t) - \widetilde{Q}_{k'_1-1}(t) + (\mu_{k'_1-1} - \mu_{k'_1})t \\
&\quad + \mu_{k'_1-1}^{1/2}c_{k'_1-1}W_{k'_1-1}(\bar{T}_{k'_1-1}(t)) - \mu_{k'_1}^{1/2}c_{k'_1}W_{k'_1}(\bar{T}_{k'_1}(t)) \\
&= \widetilde{X}_1(t) - \sum_{i=1}^{k'_1-1}\widetilde{Q}_i(t) + (\mu_1 - \mu_{k'_1})t + \mu_1^{1/2}c_1W_1(\bar{T}_1(t)) - \mu_{k'_1}^{1/2}c_{k'_1}W_{k'_1}(\bar{T}_{k'_1}(t)) \\
&= -\sum_{i=1}^{k'_1-1}\widetilde{Q}_i(t) + (\mu_0 - \mu_{k'_1})t + \mu_0^{1/2}c_0W_0(t) - \mu_{k'_1}^{1/2}c_{k'_1}W_{k'_1}(\bar{T}_{k'_1}(t)).
\end{aligned} \tag{4.46}$$

By Lemma 4.3, for all $k = 1, 2, \ldots, k'_1 - 1$, $Q_k^* = 0$ w.p.1 because $\rho_k < 1$. This, together with $\mu_0 > \mu_{k'_1}$ from (4.9), implies that $\lim_{t\to\infty}\widetilde{X}_{k'_1}(t) = +\infty$ w.p.1. By the reflected continuous mapping Ψ, $\sup_{0\leq t\leq\infty}\widetilde{Y}_{k'_1}(t) < \infty$ w.p.1. So, (4.45) holds for $k = k'_1$.

Suppose that (4.45) holds for all $k = k'_1, k'_2, \ldots, k'_l$, we next prove that (4.45) holds for $k = k'_{l+1}$. By (2.10) and (4.2), we have

$$\begin{aligned}
\widetilde{X}_{k'_{l+1}}(t) &= \mu_{k'_{l+1}-1}\widetilde{T}_{k'_{l+1}-1}(t) - \mu_{k'_{l+1}}t + \mu_{k'_{l+1}-1}^{1/2}c_{k'_{l+1}-1}W_{k'_{l+1}-1}(\bar{T}_{k'_{l+1}-1}(t)) \\
&\quad - \mu_{k'_{l+1}}^{1/2}c_{k'_{l+1}}W_{k'_{l+1}}(\bar{T}_{k'_{l+1}}(t)) \\
&= -\widetilde{Y}_{k'_{l+1}-1}(t) + (\mu_{k'_{l+1}-1} - \mu_{k'_{l+1}})t + \mu_{k'_{l+1}-1}^{1/2}c_{k'_{l+1}-1}W_{k'_{l+1}-1}(\bar{T}_{k'_{l+1}-1}(t)) \\
&\quad - \mu_{k'_{l+1}}^{1/2}c_{k'_{l+1}}W_{k'_{l+1}}(\bar{T}_{k'_{l+1}}(t)) \\
&= \widetilde{X}_{k'_{l+1}-1}(t) - \widetilde{Q}_{k'_{l+1}-1}(t) + (\mu_{k'_{l+1}-1} - \mu_{k'_{l+1}})t \\
&\quad + \mu_{k'_{l+1}-1}^{1/2}c_{k'_{l+1}-1}W_{k'_{l+1}-1}(\bar{T}_{k'_{l+1}-1}(t)) - \mu_{k'_{l+1}}^{1/2}c_{k'_{l+1}}W_{k'_{l+1}}(\bar{T}_{k'_{l+1}}(t)) \\
&= -\mu_{k'_l}\widetilde{I}_{k'_l}(t) - \sum_{i=k'_l+1}^{k'_{l+1}-1}\widetilde{Q}_i(t) + (\mu_{k'_l} - \mu_{k'_{l+1}})t \\
&\quad + \mu_{k'_l}^{1/2}c_{k'_l}W_{k'_l}(\bar{T}_{k'_l}(t)) - \mu_{k'_{l+1}}^{1/2}c_{k'_{l+1}}W_{k'_{l+1}}(\bar{T}_{k'_{l+1}}(t)).
\end{aligned} \tag{4.47}$$

By Lemma 4.3, $Q_k^* = 0$ for all $k \neq k'_1, k'_2, \ldots, k'_{l_0}$, w.p.1, and $\bar{T}_{k'_l}(t) = \bar{T}_{k'_{l+1}}(t) = t$ for all $t \geq 0$. Since $\mu_{k'_l} - \mu_{k'_{l+1}} > 0$ by (4.9), we have $\lim_{t\to\infty}\widetilde{X}_{k'_{l+1}}(t) = +\infty$ w.p.1, and further $\sup_{t\geq 0}\widetilde{I}_{k'_{l+1}}(t) < \infty$ w.p.1 by the definition of the reflected continuous mapping Ψ. So, (4.45) holds for $k = k'_{l+1}$.

Hence, (4.45) holds for all $k = k'_1, k'_2, \ldots, k'_{l_0}$.

Case 2 in Theorem 3.2. Accordingly, $\rho_{k_0} = 1$ and $\rho_{k_0} > 1$ for all $k \neq k_0$. Because the first $k_0 - 1$ stages independently form a tandem queueing system. As **Case 2 in Theorem 3.1** above, (4.45) holds for all $k = 1, 2, \ldots, k_0 - 1$. At first, we note that, for stage $k_0 + l$, $l = 1, 2, \ldots, K - k_0$, by (4.2),

$$\begin{aligned}\widetilde{X}_{k_0+l}(t) &= -\mu_{k_0+l-1}\widetilde{I}_{k_0+l-1}(t) + (\mu_{k_0+l-1} - \mu_{k_0+l})t + \mu_{k_0+l-1}^{1/2} c_{k_0+l-1} W_{k_0+l-1}(t) \\ &\quad -\mu_{k_0+l}^{1/2} c_{k_0+l} W_{k_0+l}(t). \end{aligned} \tag{4.48}$$

Next, we inductively prove that (4.45) holds for $k = k_0 + l$ with $l = 1, 2, \ldots, K - k_0$.

For $l = 1$, notice that $\mu_{k_0-1} = \mu_{k_0}$ because $\rho_{k_0} = 1$, it follows from (4.48) that

$$\widetilde{X}_{k_0+1}(t) = -\Psi(\widetilde{X}_{k_0})(t) + (\mu_{k_0} - \mu_{k_0+1})t + \mu_{k_0}^{1/2} c_{k_0} W_{k_0}(t) - \mu_{k_0+1}^{1/2} c_{k_0+1} W_{k_0+1}(t),$$

with

$$\widetilde{X}_{k_0}(t) = -\mu\widetilde{I}_{k_0-1}(t) + \mu_{k_0-1}^{1/2} c_{k_0-1} W_{k_0-1}(t) - \mu_{k_0}^{1/2} c_{k_0} W_{k_0}(t).$$

Notice that $I^*_{k_0-1} = 0$ w.p.1, both $\mu_{k_0-1}^{1/2} c_{k_0-1} W_{k_0-1}(t) - \mu_{k_0}^{1/2} c_{k_0} W_{k_0}(t)$ and $\mu_{k_0}^{1/2} c_{k_0} W_{k_0}(t) - \mu_{k_0+1}^{1/2} c_{k_0+1} W_{k_0+1}(t)$ are two driftless BMs, this, together with $\mu_{k_0} - \mu_{k_0+1} > 0$, implies that $\lim_{t\to+\infty} \widetilde{X}_{k_0+1}(t) = +\infty$, w.p.1. As a result, $\sup_{t\geq 0} \widetilde{I}_{k_0+1}(t) < +\infty$ w.p.1 since $\widetilde{I}_{k_0+1}(t) = \sup_{0\leq s\leq t} \left[-\widetilde{X}_{k_0+1}(s)\right]^+ / \mu_{k_0+1}$. So, (4.45) holds for $k = k_0 + 1$.

Suppose that (4.45) holds for stages $k = k_0 + 1, k_0 + 2, \ldots, k_0 + l_0$. Next we prove (4.45) holds for stage $k = k_0 + l_0 + 1$. For (4.48) with $l = l_0 + 1$, that is,

$$\begin{aligned}\widetilde{X}_{k_0+l_0+1}(t) &= -\mu_{k_0+l_0}\widetilde{I}_{k_0+l_0}(t) + (\mu_{k_0+l_0} - \mu_{k_0+l_0+1})t + \mu_{k_0+l_0}^{1/2} c_{k_0+l_0} W_{k_0+l_0}(t) \\ &\quad -\mu_{k_0+l_0+1}^{1/2} c_{k_0+l_0+1} W_{k_0+l_0+1}(t).\end{aligned}$$

Since $I^*_{k_0+l_0} = 0$ w.p.1 by inductive hypothesis, we have $\lim_{t\to+\infty} \widetilde{X}_{k_0+l_0+1}(t) = +\infty$ w.p.1 because $\mu_{k_0+l_0} > \mu_{k_0+l_0+1}$ and

$$\mu_{k_0+l_0}^{1/2} c_{k_0+l_0} W_{k_0+l_0}(\bar{T}_{k_0+l_0}(t)) - \mu_{k_0+l_0+1}^{1/2} c_{k_0+l_0+1} W_{k_0+l_0+1}(\bar{T}_{k_0+l_0+1}(t))$$

is a driftless Brownian motion. As a result, (4.45) holds for $k = k_0 + l_0 + 1$.

Hence, (4.45) holds for $k = k_0 + 1, k_0 + 2, \ldots, K$. □

5 Conclusion

In this paper, we study a type of the LIL for a multi-stage tandem queue. For a given stochastic process, for example, renewal process, the LIL is generally used to characterize its variability around the mean values, that is, the fluid approximations. In queues, some stochastic processes associated with the renewal process are usually used to describe the dynamic performance equations. We characterize the variability of performance measures of tandem queue by the LIL. In order to describe the variability better, the tandem queue is assumed to be in two cases with one or without critically loaded stages. For the case without critically loaded stages, we consider three subcases: all stages are underloaded, or overloaded, or both, whose corresponding results are in Theorem 3.1. We get all the LILs for the following performance measures: the queue length, workload, busy time, idle time and departure processes, which show us some interesting or counter-intuitive insights too, see Remarks 3.1, 3.2 and 3.3. For the case with one critically loaded stage, that is, only one stage is critically loaded, and others are underloaded or overloaded, we get all the LILs for the five performance measures, the corresponding results and its insight are in Theorem 3.2 and Remark 3.4 and 3.5, respectively. We prove all the results by a strong approximation method, which is established on the fluid and strong approximations of the multi-stage tandem queue. The strong approximation method transforms the original LIL into one associated with BM, and this is why we always deal with the BM in the proof.

For the multi-stage tandem queue considered, our main result only covers two special cases without critically loaded stages and with only one such stage. For these two cases, all the LILs can be obtained by the strong approximation method. For other cases with more than one critically loaded stage beyond this paper, the strong approximation method can foresee all the LILs associated with underloaded and overloaded stages, and could not predict the LILs of the critically loaded stages. We are looking forward to seeing some effective methods (existing now or developed in future) for more general cases, and further for generalized Jackson networks.

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