APPENDIX

to

Heavy-Traffic Limit for the Initial Content Process

by

A. Korhan Aras, Yunan Liu and Ward Whitt

APPENDIX A: OVERVIEW

This appendix has extra materials to supplement the main paper. In §B we review a necessary and sufficient condition for tightness in space $\mathbb{D}_{\mathbb{D}}$. In §C we review useful results used in the proofs. In §D we give proofs omitted in the main paper. In §E we provide the first characterization of the steady-state distribution of the new and old content in the stationary $G/GI/\infty$ model. In §F we report simulation results for a challenging test case having non-Markov arrival process, non-exponential service-time distribution, and general initial conditions. In §G we provide additional simulation results.

APPENDIX B: TIGHTNESS IN $\mathbb{D}_{\mathbb{D}}$

We now review a necessary and sufficient condition for tightness of a stochastic process $\{X_n : n \geq 1\}$ in space $\mathbb{D}_{\mathbb{D}}$. Also see [4] for details.

LEMMA B.1. *A sequence of stochastic process $\{X_n : n \geq 1\}$ in $\mathbb{D}_{\mathbb{D}}$ is tight if and only if*

(*i*) *The sequence $\{X_n : n \geq 1\}$ is stochastically bounded in $\mathbb{D}_{\mathbb{D}}$, i.e., for all $\epsilon > 0$, there exists a compact subset $K \subset \mathbb{R}$ such that*

$$P(\|X_n\|_T \in K) > 1 - \epsilon, \qquad \textit{for all} \quad n \geq 1,$$

where $\|X_n\|_T = \sum_{s\in[0,T]} \sup_{t\in[0,T]} |X_n(s,t)|$; and any one of the following

(*ii*) *For all $\delta > 0$, and all uniformly bounded sequences $\{\tau_n : n \geq 1\}$ where for each n, τ_n is a stopping time with respect to the natural filtration $\mathbf{F}_n = \{\mathcal{F}_n(t), t \in [0,T]\}$ where $\mathcal{F}_n(t) = \sigma\{X_n(s,\cdot) : 0 \leq s \leq t\}$, there exists a constant $\beta > 0$ such that*

$$\lim_{\delta\downarrow 0} \limsup_{n\to\infty} \sup_{\tau_n} E\left[(1 \wedge d_{J_1}(X_n(\tau_n+\delta,0), X_n(\tau_n,\cdot)))^{\beta}\right] = 0;$$

or

(*ii'*) *For all $\delta > 0$, there exists a constant β and random variables $\gamma_n(\delta) \geq 0$ such that for each n, w.p.1.,*

$$E\left[(1 \wedge d_{J_1}(X_n(s+u,\cdot), X_n(s,\cdot)))^{\beta} | \mathcal{F}_n\right](1 \wedge d_{J_1}(X_n(s-v,\cdot), X_n(s,\cdot)))^{\beta} \leq E\left[\gamma_n(\delta)|\mathcal{F}_n\right],$$

for all $0 \leq s \leq t$, $0 \leq u \leq \delta$ and $0 \leq u \leq s \wedge \delta$, where $\mathbf{F}_n = \{\mathcal{F}_n(t) : t \in [0,T]\}$ with $\mathcal{F}_n(t) = \sigma\{X_n(s,\cdot) : 0 \leq s \leq t\}$ and

$$\lim_{\delta \downarrow 0} \limsup_{n\to\infty} E\left[\gamma_n(\delta)\right] = 0.$$

APPENDIX C: USEFUL INEQUALITIES

In this section we review two useful inequalities. Both are used to prove (4.22) in §4.

THEOREM C.1 (Lévy's inequality (symmetric case), Theorem 3.7.1 of Gut[9]). *Let X_1, $X_2, \ldots, X_n$ be independent real-valued symmetric random variables (satisfying $-X_i \stackrel{\mathrm{d}}{=} X_i$ for all $1 \leq i \leq n$) and let $S_n = \sum_{k=1}^n X_k$, $n \geq 1$ be the partial sums. Then, for any $x > 0$,*

$$\mathbb{P}\left(\max_{1\leq k\leq n} |S_k| > x\right) \leq 2\mathbb{P}(|S_n| > x). \tag{C.1}$$

THEOREM C.2 (Hoeffding's inequality, Theorem 3.1.3 of Gut[9]). *Let $X_1, X_2, \ldots, X_n$ be independent real-valued random variables such that $\mathbb{P}(a_k \leq X_k \leq b_k) = 1$ for $a_k, b_k \in \mathbb{R}$, $k = 1, \ldots, n$, and let $S_n = \sum_{k=1}^n X_k$, $n \geq 1$, denote the partial sums. Then*

$$\mathbb{P}(|S_n - \mathbb{E}[S_n]| > x) \leq 2\exp\left(-\frac{2x^2}{\sum_{k=1}^n (b_k - a_k)^2}\right). \tag{C.2}$$

APPENDIX D: ADDITIONAL PROOFS

We now provide proofs of Corollaries 3.4 and 5.1, and Lemma 4.2, which were omitted in the main paper.

Proof of Corollary 3.4. This follows from parts (ii) and (iii) of Theorem 3.2. We present the proof of the four-parameter covariance formulas in (i); the variance formulas in (i) and (ii) easily follow.

First, the covariances of $\hat{X}_1^{\nu}$ and $\hat{X}_2^{o}$ are

$$
\begin{aligned}
&\mathrm{Cov}(\hat{X}_1^{e,\nu}(t_1, y_1), \hat{X}_1^{e,\nu}(t_2, y_2)) \\
&= \mathbb{E}\left[\int_{(t_1-y_1)^+}^{t_1} G_\nu^c(t_1 - s)\, d\hat{N}(s) \times \int_{(t_2-y_2)^+}^{t_2} G_\nu^c(t_2 - s)\, d\hat{N}(s)\right] \\
&= c_\lambda^2 \int_{(t_1-y_1)^+\vee(t_2-y_2)^+}^{t_1\wedge t_2} G_\nu^c(t_1 - s) G_\nu^c(t_1 - s)\, d\Lambda(s),
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathrm{Cov}(\hat{X}_2^{e,o}(t_1, y_1), \hat{X}_2^{e,o}(t_2, y_2)) \\
&= \mathbb{E}\left[\int_0^{(y_1-t_1)^+} H_x^c(t_1)\, d\hat{X}_0(x) \times \int_0^{(y_2-t_2)^+} H_x^c(t_2)\, d\hat{X}_0(x)\right] \\
&= \int_0^{(y_1-t_1)^+\wedge(y_2-t_2)^+} H_x^c(t_1) H_x^c(t_2)\, d\Sigma_2^{e,o}(x).
\end{aligned}
$$

Second, the covariance of $\hat{X}_2^{\nu}$

$$
\begin{aligned}
&\mathrm{Cov}(\hat{X}_2^{e,\nu}(t_1,y_1),\hat{X}_2^{e,\nu}(t_2,y_2))\\
&= \mathbb{E}\left[\left(\int_{(t_1-y_1)^+}^{t_1}\int_0^\infty \mathbf{1}_{(x+s>t_1)}\, d\left(\hat{W}(\Lambda(s),G_\nu(x)) - G_\nu(x)\hat{W}(\Lambda(s),1)\right)\right)\right.\\
&\times \left.\left(\int_{(t_2-y_2)^+}^{t_2}\int_0^\infty \mathbf{1}_{(x+s>t_2)}\, d\left(\hat{W}(\Lambda(s),G_\nu(x)) - G_\nu(x)\hat{W}(\Lambda(s),1)\right)\right)\right]\\
&= \mathbb{E}\left[\int_{(t_1-y_1)^+}^{t_1}\int_0^\infty \mathbf{1}_{(x+s>t_1)}\, d\hat{W}(\Lambda(s),G_\nu(x)) \times \int_{(t_2-y_2)^+}^{t_2}\int_0^\infty \mathbf{1}_{(x+s>t_2)}\, d\hat{W}(\Lambda(s),G_\nu(x))\right]\\
&+ \mathbb{E}\left[\int_{(t_1-y_1)^+}^{t_1} G_\nu^c(t_1-s)\, d\hat{W}(\Lambda(s),1) \times \int_{(t_2-y_2)^+}^{t_2} G_\nu^c(t_2-s)\, d\hat{W}(\Lambda(s),1)\right]\\
&- \mathbb{E}\left[\int_{(t_1-y_1)^+}^{t_1}\int_0^\infty \mathbf{1}_{(x+s>t_1)}\, d\hat{W}(\Lambda(s),G_\nu(x)) \times \int_{(t_2-y_2)^+}^{t_2} G_\nu^c(t_2-s)\, d\hat{W}(\Lambda(s),1)\right]\\
&- \mathbb{E}\left[\int_{(t_1-y_1)^+}^{t_1} G_\nu^c(t_1-s)\, d\hat{W}(\Lambda(s),1) \times \int_{(t_2-y_2)^+}^{t_2}\int_0^\infty \mathbf{1}_{(x+s>t_2)}\, d\hat{W}(\Lambda(s),G_\nu(x))\right]\\
&= \int_{(t_1-y_1)^+\vee(t_2-y_2)^+}^{t_1\wedge t_2} G_\nu^c(t_1\vee t_2-s)d\Lambda(s) + \int_{(t_1-y_1)^+\vee(t_2-y_2)^+}^{t_1\wedge t_2} G_\nu^c(t_1-s)G_\nu^c(t_2-s)\, d\Lambda(s)\\
&- 2\int_{(t_1-y_1)^+\vee(t_2-y_2)^+}^{t_1\wedge t_2} G_\nu^c(t_1-s)G_\nu^c(t_2-s)\, d\Lambda(s)\\
&= \int_{(t_1-y_1)^+\vee(t_2-y_2)^+}^{t_1\wedge t_2} G_\nu^c(t_1\vee t_2-s)G_\nu(t_1\wedge t_2-s)\, d\Lambda(s). \quad \blacksquare
\end{aligned}
$$

Proof of Corollary 5.1. First, (5.1) and (5.2) easily follow Theorems 3.1 and 3.2 of [4] by considering the performance of a system at the end of interval $[0,t_0]$ (that is, at time t_0) with the system being initially empty (at time 0). Then, it suffices to apply a time shift, that is, shifting the interval to the left by t_0 so that the interval becomes $[-t_0,0]$.

When the system starts empty at $-t_0$, following (5.2) and Theorem 4.2 of [4], the variance function of $\hat{X}^e(0,y)$ is

(D.1)
$$\mathrm{Var}(\hat{X}^e(0,y)) = \int_0^y \left[(c_\lambda^2-1)G^c(s)^2 + G^c(s)\right]\lambda(-s)\, ds, \quad \text{for} \quad 0\le y\le t_0.$$

Now plugging (5.1) and (D.1) into (3.23) yields that

$$\begin{aligned}
\sigma^2_{\hat{X},o}(t) &= \int_0^{t_0} H_u(t)H_u^c(t)G^c(u)\lambda(-u)\,du \\
&+ \int_0^{t_0} (H_u^c(t))^2 \left[(c_\lambda^2-1)G^c(u)^2 + G^c(u)\right]\lambda(-u)\,du \\
&= \int_0^{t_0} H_u(t)G^c(t+u)\lambda(-u)\,du \\
&+ \int_0^{t_0} H_u^c(t)G^c(t+u)\left[(c_\lambda^2-1)G^c(u)+1\right]\lambda(-u)\,du \\
&= \int_0^{t_0} G^c(t+u)\left[(c_\lambda^2-1)G^c(t+u)+1\right]\lambda(-u)\,du \\
&= \int_{-t_0}^{0} G^c(t-s)\left[(c_\lambda^2-1)G^c(t-s)+1\right]\lambda(s)\,ds,
\end{aligned}$$

where the last equality holds by a change of variable. Summing the above equation with $\sigma^2_{\hat{X},\nu}(t)$ in (3.22) yields (5.4). ∎

Proof of Lemma 4.2. We mimic the proof of (A.14.15) in [33]. First for a deterministic $g \in \mathbb{D}_{\mathbb{D}}$, we have, for $t \in [0,T]$, $y \in [0, y^{\uparrow}]$

$$\mathbb{E}\|g - X\|_{T,y^{\uparrow}} \geq \mathbb{E}|g(t,y) - X(t,y)| \geq |g(t,y) - \mathbb{E}[X(t,y)]|,$$

where the second inequality holds by Jensen's inequality. Hence, we have

$$\mathbb{E}\|g - X\|_{T,y^{\uparrow}} \geq \sup_{(t,y)\in[0,T]\times[0,y^{\uparrow}]} |g(t,y) - \mathbb{E}[X(t,y)]| = \|g - \mathbb{E}[X]\|_{T,y^{\uparrow}}. \tag{D.2}$$

By conditioning on X, we have

$$\begin{aligned}
\mathbb{E}\left[\phi\left(\|X - X^*\|_{T,y^{\uparrow}}\right)\right] &= \mathbb{E}_X\left[\mathbb{E}_{X^*}\left[\phi\left(\|X - X^*\|_{T,y^{\uparrow}}\right)\middle| X\right]\right] \\
&\geq \mathbb{E}_X\left[\phi\left(\mathbb{E}_{X^*}\left[\|X - X^*\|_{T,y^{\uparrow}}\middle| X\right]\right)\right] \\
&\geq \mathbb{E}_X\left[\phi\left(\|X - \mathbb{E}\left[X^*\right]\|_{T,y^{\uparrow}}\right)\right] = \mathbb{E}\left[\phi\left(\|X - \mathbb{E}\left[X\right]\|_{T,y^{\uparrow}}\right)\right],
\end{aligned}$$

where the first inequality holds by Jensen's inequality and the second inequality holds by (D.2). ∎

APPENDIX E: STEADY-STATE APPROXIMATIONS

An important application of the results in this paper is generating useful approximations for the steady-state behavior of general stationary $G/GI/\infty$

IS models. Since we can apply Little's law to conclude that the steady-state mean number in system is $\rho \equiv \lambda E[S]$, where S is a service time, we assume that $E[S] < \infty$ in this section.

Finding general conditions for the existence of such steady-state distributions is complicated, even for the special case of the number in system with renewal (GI) arrival processes, as can be seen from Remark 2 of [37]. However, assuming that steady state for the process $\{\hat{X}_n^{e,\nu}(t,\cdot) : t \geq 0\}$ in $\mathbb{D}_{\mathbb{D}}$ for system n is well defined, it is natural to approximate by the steady-state stationary process associated with the limit process $\{\hat{X}^{e,\nu}(t,\cdot) : t \geq 0\}$ in $\mathbb{D}_{\mathbb{D}}$, which is itself an element of $\mathbb{D}$.

As observed by Glynn and Whitt [2], pp.193–195, the steady-state behavior of the limit process is relatively easy to analyze. For example, they observed, for service-time distributions with finite support in an interval $[0, y^*]$, that the number in system is in steady state after time y^*. We establish a generalization of that result, which holds because the driving processes $\mathcal{B}_a(s)$ and $\hat{K}(s,\cdot)$ have stationary increments in s.

Corollary E.1. (*Stationary version of the limiting IS age process*) *If $\hat{N}(\cdot) = c_a\sqrt{\lambda}\mathcal{B}_a(\cdot)$ and the system starts empty at time $-t_0 \leq 0$, then the limit processes (as $n \to \infty$) of the new input in the $G/GI/\infty$ model can be represented as*

(E.1)
$$(\hat{X}_1^{e,\nu}(t,y), \hat{X}_2^{e,\nu}(t,y)) = \left(\sqrt{\lambda c_a^2}\int_{(t-y)^+}^{t} G^c(t-s)\, d\mathcal{B}_a(s), \int_{(t-y)^+}^{t}\int_{t-s}^{\infty} d\hat{K}(\lambda s, x)\right),$$

(E.2)
$$(\hat{X}_1^{r,\nu}(t,y), \hat{X}_2^{r,\nu}(t,y)) = \left(\sqrt{\lambda c_a^2}\int_{-t_0}^{t} G^c(t+y-s)\, d\mathcal{B}_a(s), \int_{-t_0}^{t}\int_{t+y-s}^{\infty} d\hat{K}(\lambda s, x)\right).$$

(*a*) *If $G^c(y^*) = 0$, then the distribution of $(\hat{X}_1^{e,\nu}(t,\cdot), \hat{X}_2^{e,\nu}(t,\cdot))$ as a process in $\mathbb{D}^2$ is independent of t for $t + t_0 \geq y^*$ and thus reaches steady state at time y^* if $t_0 = 0$ (or is in steady state at time 0 if $t_0 \geq y^*$);*

(*b*) *As $-t_0 \downarrow -\infty$, corresponding to the system starting empty in the distant past, the processes in (E.1) and (E.2) converge w.p.1 to the associated* *stationary processes* (*as functions of* t)

(E.3)
$$(\hat{X}_1^{e,\nu}(t,y), \hat{X}_2^{e,\nu}(t,y)) = \left(\sqrt{\lambda c_a^2}\int_{t-y}^{t} G^c(t-s)\, d\mathcal{B}_a(s), \int_{t-y}^{t}\int_{t-s}^{\infty} d\hat{K}(\lambda s, x)\right),$$

(E.4)
$$(\hat{X}_1^{r,\nu}(t,y), \hat{X}_2^{r,\nu}(t,y)) = \left(\sqrt{\lambda c_a^2}\int_{-\infty}^{t} G^c(t+y-s)\, d\mathcal{B}_a(s), \int_{-\infty}^{t}\int_{t+y-s}^{\infty} d\hat{K}(\lambda s, x)\right),$$

whose marginal distribution as a function of y (*as a process in* $\mathbb{D}$) *can be seen by setting* $t = 0$*, yielding the steady-state processes*

(E.5)
$$(\hat{X}_1^{e,s}(y), \hat{X}_2^{e,s}(y)) \equiv \left(\sqrt{\lambda c_a^2}\int_{-y}^{0} G^c(-s)\, d\mathcal{B}_a(s),\ \int_{-y}^{0}\int_{-s}^{\infty} d\hat{K}(\lambda s, x)\right), \quad y \ge 0,$$

(E.6)
$$(\hat{X}_1^{r,s}(y), \hat{X}_2^{r,s}(y)) \equiv \left(\sqrt{\lambda c_a^2}\int_{-\infty}^{0} G^c(y-s)\, d\mathcal{B}_a(s),\ \int_{-\infty}^{0}\int_{y-s}^{\infty} d\hat{K}(\lambda s, u)\right), \quad y \ge 0,$$

with covariance formulas given by

$$\mathrm{Cov}(\hat{X}_1^{e,s}(y_1), \hat{X}_1^{e,s}(y_2)) = \lambda c_a^2 \int_0^{y_1\wedge y_2} G^c(s)^2\, ds,$$
$$\mathrm{Cov}(\hat{X}_2^{e,s}(y_1), \hat{X}_2^{e,s}(y_2)) = \lambda \int_0^{y_1\wedge y_2} G^c(s)G(s) ds,$$
$$\mathrm{Cov}(\hat{X}_1^{e,s}(y_1), \hat{X}_1^{e,s}(y_2)) = \lambda c_a^2 \int_0^{\infty} G^c(y_1+s)G^c(y_2+s)\, ds,$$

(E.7)
$$\mathrm{Cov}(\hat{X}_2^{e,s}(y_1), \hat{X}_2^{e,s}(y_2)) = \lambda \int_0^{\infty} (G^c((y_1 \vee y_2)+s) - G^c(y_1+s)G^c(y_2+s))\, ds.$$

Proof. We have already established (E.1) and (E.2). The other representations hold because both $\mathcal{B}_a(s)$ and $\hat{K}(\lambda s, \cdot)$ have stationary increments in s. The limit as $-t_0 \downarrow -\infty$ is relatively easy because the processes $\mathcal{B}_a(s)$ and $\hat{K}(\lambda s, \cdot)$ do not change over the interval $[-t_0, t]$ if we expand the interval on the left and consider the process over $(-\infty, t]$. The new contribution over the interval $(-\infty, -t_0]$ decreases as $-t_0 \downarrow -\infty$. This can be quantified through the variance of the zero-mean Gaussian random variable, which is asymptotically negligible. It thus remains to derive the covariance formulas in (E.7). We only prove the first two because the proofs of the others are similar. First, by the isometry of Brownian integrals,

(E.8)
$$\mathrm{Cov}(\hat{X}_1^{e,s}(y_1), \hat{X}_1^{e,s*}(y_2)) = \lambda \int_{-y_1\wedge y_2}^{0} G^c(-s)^2\, ds = \lambda \int_0^{y_1\wedge y_2} G^c(s)^2\, ds.$$

Next, exploiting the representation of the Kiefer process in terms of the Brownian sheet, i.e., $K(x,y) = W(x,y) - yW(x,1)$ (see the appendix of [4]), we have

$$\begin{aligned}
&\mathrm{Cov}(\hat{X}_2^{e,s}(y_1), \hat{X}_2^{e,s}(y_2)) \\
&= \mathbb{E}\left[\int_{-y_1}^{0}\int_{-s}^{\infty} d\left(W(\lambda s, G(x)) - G(x)W(\lambda s, 1)\right)\right. \\
&\qquad\qquad \left.\times \int_{-y_2}^{0}\int_{-s}^{\infty} d\left(W(\lambda s, G(x)) - G(x)W(\lambda s, 1)\right)\right]
\end{aligned}$$

(E.9)
$$= \lambda \int_{-y_1\wedge y_2}^{0} G^c(-s) ds + \lambda \int_{-y_1\wedge y_2}^{0} G^c(-s)^2 ds - 2\lambda \int_{-y_1\wedge y_2}^{0} G^c(-s)^2 ds,$$

which coincides with the second covariance formula in (E.7). ∎

Corollary E.1 adds to the insight about the stationary model provided by Corollaries 3.1, 4.1 and 4.2 of [4]. As indicated in Corollary 4.1 of [4], we see that the approximating stationary distribution depends on the arrival process beyond its constant arrival rate λ and the assumed FCLT only through the asymptotic variability parameter c_a^2. Thus this distribution is the same as for the $M/GI/\infty$ model if and only if $c_a^2 = 1$. It is thus instructive to also consider what can be established for the $M/GI/\infty$ model directly by exploiting its special structure. So now we consider the $M/GI/\infty$ model. It is well known that the steady-state number of customers $X_n(0) \equiv X_n^e(0,\infty)$ is a Poisson random variable with $E[X_n(0)] = Var(X_n(0)) = n\lambda E[S] = n\lambda \int_0^\infty G^c(u)du$ and, conditioned on that number, the ages (and the residual service times) are i.i.d. with the stationary-excess cdf $G_e(t) \equiv (1/E[S]) \int_0^t G^c(u)\, du$, $t \geq 0$ and ccdf $G_e^c(t) \equiv 1 - G_e(t)$; e.g., see [1, 3]. Thus we have the following corollary.

Corollary E.2. (*FCLT for the $M/GI/\infty$ model in steady state*) *Consider a sequence of $M/GI/\infty$ models in steady state at time 0, with service-time cdf G and constant arrival rate $n\lambda$. Then Assumption 1 holds with the FWLLN and FCLT limits for the initial age processes* (*and all $t \geq 0$*) *given* *by*

$$X^{e,s}(0,y) = \rho G_e(y) = \int_0^y a(y)dy = \int_0^y \lambda G^c(u)\,du \quad \textit{and} \tag{E.10}$$
$$\hat{X}^{e,s}(0,y) \stackrel{\mathrm{d}}{=} \hat{X}_1^{e,s*}(y) + \hat{X}_2^{e,s*}(y),$$

where $\rho \equiv \lambda E[S] = X^e(0,\infty) = X(0)$, *and* $\hat{X}_1^{e,s*}$ *and* $\hat{X}_1^{e,s*}$ *are independent processes with*

$$\hat{X}_1^{e,s*}(y) \equiv \hat{U}^*(\rho, G_e(y)) \stackrel{\mathrm{d}}{=} \sqrt{\rho}\,\mathcal{B}_s^*(G_e(y)) \quad \textit{and} \quad \hat{X}_2^{e,s*}(y) \equiv G_e(y)\hat{X}(0) \stackrel{\mathrm{d}}{=} \sqrt{\rho} G_e(y)\,\mathcal{Z}_0, \tag{E.11}$$

where $\hat{U}^*$ *is a standard Kiefer process associated with old customers,* $\mathcal{B}_s^*$ *is a standard Brownian bridge,* $\mathcal{Z}_0$ *is a standard Gaussian random variable, independent of* $\hat{U}^*$*. The steady-state version of the remaining-service-time process*

$$X^{r,s}(0,y) = \rho G_e^c(y) = \int_y^\infty \lambda G^c(u)\,du \quad \textit{and} \quad \hat{X}^{r,s}(0,y) \stackrel{\mathrm{d}}{=} \hat{X}_1^{r,s*}(y) + \hat{X}_2^{r,s*}(y), \quad t \geq 0, \tag{E.12}$$

where $\hat{X}_1^{r,s*}$ *and* $\hat{X}_2^{r,s*}$ *are independent zero-mean Gaussian processes, with*

$$\mathrm{Cov}\left(\hat{X}_1^{r,s*}(x_1), \hat{X}_1^{r,s*}(x_2)\right) = \int_0^\infty H_u(x_1 \wedge x_2) H_u^c(x_1 \vee x_2)\,dX^{e,s}(0,u),$$
$$\textit{and} \quad \hat{X}_2^{r,s*}(x) \stackrel{\mathrm{d}}{=} \sqrt{\rho} \int_0^\infty H_u^c(x)\,d\mathcal{B}_o^*(G_e(u)) + \sqrt{\rho} G_e^c(y)\,\mathcal{Z}_0. \tag{E.13}$$

The variance of $\hat{X}^{e,s*}(y)$ *is*

$$\mathrm{Var}(\hat{X}^{e,s*}(y)) = \int_0^y \lambda G^c(u)\,du, \quad y \geq 0. \tag{E.14}$$

As a consequence, the variance of the total content, as the sum of variances of the new content (that is (3.22)) and old content (that is (3.23)), is

$$\sigma_{\hat{X}}^2 = \sigma_{\hat{X},\nu}^2 + \sigma_{\hat{X},o}^2 = \lambda \int_0^t G^c(u)du + \lambda \int_0^\infty H_u^c(t) G^c(u)du = \lambda \int_0^\infty G^c(u)du = \rho. \tag{E.15}$$

Proof. Let $A_1, A_2, \ldots$ be the ages (e.g., elapsed time in service) of the customers in service at time 0. To prove the FWLLN in (E.10), we have

$$\bar{X}^e(0,y) = \frac{1}{n}\sum_{i=1}^{n\bar{X}_n(0)} \mathbf{1}(A_i \le y) \Rightarrow X(0)\, G_e(y) = \rho\, G_e(y) \quad \text{in } \mathbb{D}, \quad \text{as } n \to \infty,$$

where the convergence holds because the age A_i follows the equilibrium distribution G_e (see Theorem 1 of [1]). To prove the FCLT in (E.10), we have

$$\begin{aligned}
\text{(E.16)} \qquad \hat{X}^e(0,y) &= \frac{1}{\sqrt{n}}\left(\sum_{i=1}^{n\bar{X}_n(0)} \mathbf{1}(A_i \le y) - nX(0)\, G_e(y)\right) \\
&= \frac{1}{\sqrt{n}}\sum_{i=1}^{n\bar{X}_n(0)} \left(\mathbf{1}(A_i \le y) - G_e(y)\right) + \hat{X}_n(0)\, G_e(y) \\
&= \hat{K}_n\left(\bar{X}_n(0), G_e(y)\right) + \hat{X}_n(0)\, G_e(y) \\
&\Rightarrow \hat{U}^*\left(X(0), G_e(y)\right) + \hat{X}(0)\, G_e(y) \\
&\stackrel{\mathrm{d}}{=} \sqrt{X(0)}\mathcal{B}_s^*\left(G_e(y)\right) + \sqrt{X(0)}\, G_e(y)\, \mathcal{Z}_0 \quad \text{in } \mathbb{D},
\end{aligned}$$

as $n \to \infty$, where the second equality holds by adding and subtracting $G_e(y)$ in the sum, and the convergence holds by (i) the marginal convergence of each of the two terms in (E.16) (due to the Gaussian CLT limit for a Poisson random variable) and by (ii) Lemma 4.1 and the conditional independence of these two terms conditioning on $X_n(0)$ (with arguments similar to §4.1). The variance formulas (E.14) and (E.15) immediately follow from (E.12) and (3.23) with $G_\nu = G$. The proof of the FWLLN limit in (E.12) follows from (E.10) and Theorem 3.1, and the proof of (E.13) follows from (E.12), Theorem 3.2 and Corollary 3.3. ∎

COROLLARY E.3. (*Two equivalent decompositions*) *For the $M/GI/\infty$ model* (*with $c_a^2 = 1$*), *the two representations* (E.5) *and* (E.10) ((E.6) *and* (E.12)) *are equivalent independent decompositions, i.e.,*

$$\text{(E.17)} \qquad \hat{X}_1^{e,s*} + \hat{X}_2^{e,s*} \stackrel{\mathrm{d}}{=} \hat{X}_1^{e,s} + \hat{X}_2^{e,s} \quad \textit{and} \quad \hat{X}_1^{r,s*} + \hat{X}_2^{r,s*} \stackrel{\mathrm{d}}{=} \hat{X}_1^{r,s} + \hat{X}_2^{r,s}.$$

Proof. We only prove the first equality in (E.17) since the second equality follows similarly. Because all four terms in the first equality of (E.17) are zero-mean Gaussian processes, with $\hat{X}_1^{e,s*}$ independent of $\hat{X}_2^{e,s*}$ and $\hat{X}_1^{e,s}$

independent of $\hat{X}_2^{e,s}$, it suffices to show that

$$\sum_{k=1}^{2} \mathrm{Cov}(\hat{X}_k^{e,s*}(y_1), \hat{X}_k^{e,s*}(y_2)) = \sum_{k=1}^{2} \mathrm{Cov}(\hat{X}_k^{e,s}(y_1), \hat{X}_k^{e,s}(y_2)), \tag{E.18}$$

By (E.11), we have

$$\begin{aligned}
&\mathrm{Cov}(\hat{X}_1^{e,s*}(y_1), \hat{X}_1^{e,s*}(y_2)) = \rho\left[G_e(y_1) \wedge G_e(y_2) - G_e(y_1)G_e(y_2)\right], \\
&\mathrm{Cov}(\hat{X}_2^{e,s*}(y_1), \hat{X}_2^{e,s*}(y_2)) = \rho G_e(y_1)G_e(y_2).
\end{aligned}$$

so that the left-hand side of (E.18) is $\rho G_e(y_1) \wedge G_e(y_2) = \lambda \int_0^{y_1 \wedge y_2} G^c(u)\, du$, which coincides with the right-hand side of (E.18), according to (E.7) with $c_a = 1$. ∎

The corollaries above show that the evolution of new and old content after some time is somewhat complicated for this basic $M/GI/\infty$ model, even though the steady-state distribution at one time is remarkably simple. We now illustrate with an example.

EXAMPLE E.1. (*Simulation comparison for an $M/GI/\infty$ in steady state*) We consider an $M/H_2(1,4)/\infty$ model in $[0,T]$, having a Poisson arrival process with constant arrival rate $\lambda_n = n\lambda$ for $\lambda = 1$, an H_2 service distribution with balanced means,i.e., a mixture of two exponential r.v.'s with rates μ_1 and μ_2 with probability p. We set $\mu_1 = 2p\mu$, $\mu_2 = 2(1-p)\mu$, $\mu = 1$ and $p = 0.5(1-\sqrt{0.6})$ so that the mean $1/\mu = 1$ and $c_s^2 = 4$. We set $n = 100$.

Since the model is in steady state at time 0, we use a Poisson distribution with mean $n\lambda/\mu$ to generate the number of customers at time 0. To generate the elapsed times in service (e.g., ages) and the residual service times for these customers, we use the equilibrium version of the service distribution, which again follows an H_2 distribution, but with altered parameters, specifically with $p^* = p\mu_2/(p\mu_2 + (1-p)\mu_1)$, $\mu_1^* = \mu_1$ and $\mu_2^* = \mu$. See Theorem 1 of [1]. See Figure 2 for the simulation comparison, which is an analog of Figure 1.

APPENDIX F: A CHALLENGING TEST CASE

In order to more fully substantiate the theoretical results, we consider a $G_t/GI/\infty$ IS model with non-Markov arrival process, non-exponential service-time distribution and unconventional initial conditions. We let the (artificial) initial conditions be generated by a time changed renewal process.

COROLLARY F.1. (*Initial customers generated from a renewal process with a time change*) *In the* n^{th} $G_t/GI/\infty$ *system, if the initial age process*

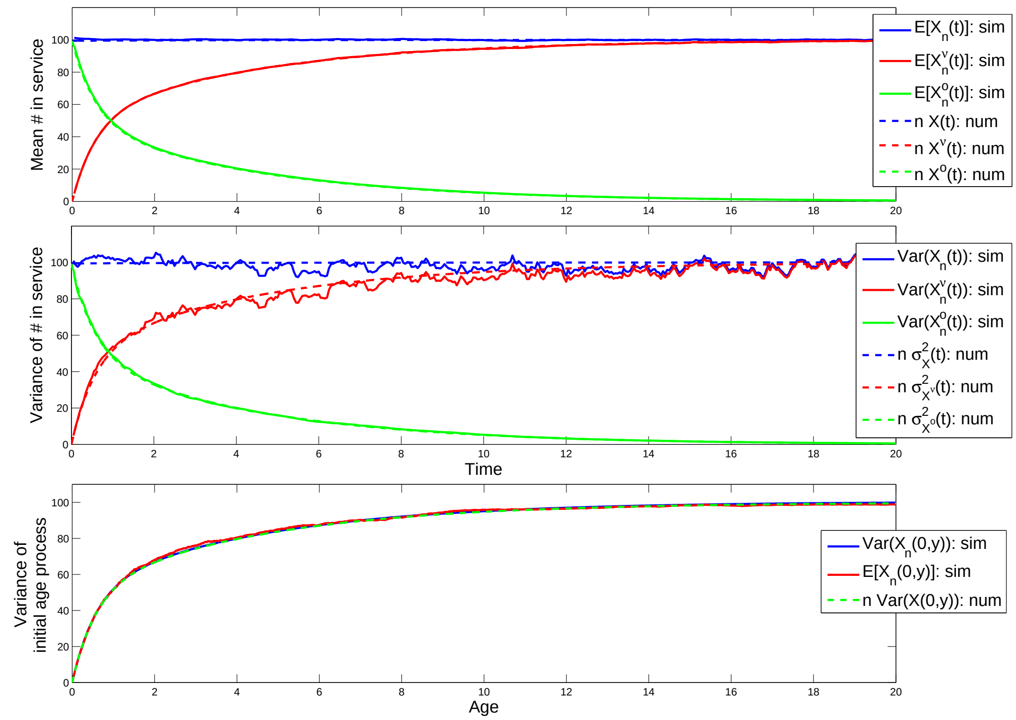

FIG 2. *Example 2: Simulation comparisons of the mean and variance for the number of new and old customers in service of an $M/H_2(1,4)/\infty$ model in steady state, with a constant arrival rate $n\lambda = 100$.*

$X_n^e(0,y) \equiv N^*(nX^*(0,y))$ where N^* is a rate-1 renewal process with inter-renewal SCV $\bar{c}_0^2$ and the deterministic function

$$X^*(y) \equiv \int_0^y b^*(y)dy < \infty, \tag{F.1}$$

then Assumption 2 is satisfied with $X(0,y) = X^*(y)$ and $\hat{X}(0,y) = \bar{c}_0 \mathcal{B}_0(X^*(y))$, for $y \geq 0$, where $\mathcal{B}_0$ is a standard BM. In addition, the variance of the ICP (that is (3.23)), is

$$\begin{aligned}
\sigma^2_{\hat{X},o}(t) &= \int_0^\infty H_u(t)H_u^c(t)\, dX^*(u) + \bar{c}_0^2 \int_0^\infty H_u^c(t)^2\, dX^*(u) \\
&= \int_0^\infty \left[\left(\bar{c}_0^2 - 1\right) H_u^c(t) + 1\right] H_u^c(t) b^*(u)\, du.
\end{aligned}$$

EXAMPLE F.1. (*Simulation comparison for an example of Corollary F.1*) In addition to the initial conditions specified above, we let the new input come from an $H_2^t(1,4)/LN(1,4)/\infty$ model in $[0,T]$, having a G_t arrival process $N_n(t) \equiv N^{(e)}(n\Lambda(t))$, where $N^{(e)}$ is a rate-1 equilibrium renewal process having an H_2 interrenewal distribution with balanced means and $c_\lambda^2 = 4$, while $\Lambda(t) \equiv \int_0^t \lambda(u)du$, where $\lambda(t)$ is the sinusoidal arrival rate with in (5.5) having parameters $a = c = 1$, $b = 0.6$, $\phi = 0$ and $n = 100$. This example has an LN service distribution with mean $1/\mu = 1$ and $c_s^2 = 4$. For the initial conditions, let N^* be a rate-1 Poisson process (so that $\bar{c}_0^2 = 1$) in Corollary F.1 and consider two density functions in (F.1)

$$b_1^*(u) = u\,\mathbf{1}_{(0\le u\le d)} + (2d-u)\,\mathbf{1}_{(d\le u\le 2d)} \quad \text{and} \quad b_2^*(u) = \frac{1}{3}u^2\,\mathbf{1}_{(0\le u\le 2d)}, \tag{F.2}$$

with $d = 1.5$, as shown in Figure 3.

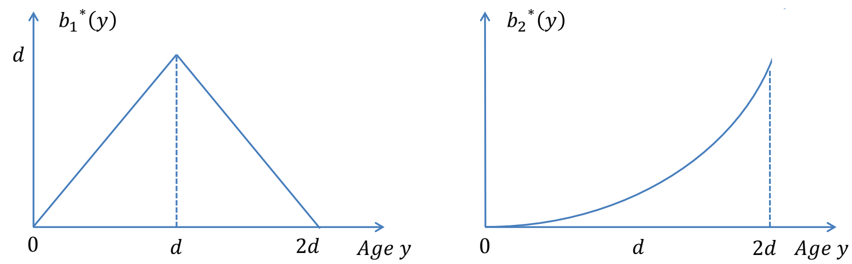

FIG 3. *Two choices of the initial-condition density $b^*(u)$ in (F.1).*

Comparisons with simulations are shown in Figure 4. The first three plots in Figure 4 are analogs of those in Figure 1. In the fourth and fifth plots we give simulation comparisons for the variances of the two-parameter process $\hat{X}^e(t,y)$. We provide the simulations for the case of b_2^* in the appendix.

APPENDIX G: AN ADDITIONAL EXAMPLE

EXAMPLE G.1. (*Simulation comparison for another example of Corollary F.1*) We now supplement Example 3 by considering that same $H_2^t(1,4)/LN(1,4)/\infty$ model with all parameters specified in Example 3, but with $b^* = b_2^*$ specified in (F.2). Comparisons with simulations are shown in Figure 5.

REFERENCES

[1] DUFFIELD, N. and WHITT, W. (1997). Control and recovery from rare congestion events in a large multi-server system. *Queueing Syst.* **26** 69–104. MR1480867

[2] GLYNN, P. W. and WHITT, W. (1991). A new view of the heavy-traffic limit theorem for the infinite-server queue. *Adv. in Appl. Probab.* **23**(1) 188–209. MR1091098

[3] GOLDBERG, D. and WHITT, W. (2008). The last departure time from an $M_t/G/\infty$ queue with a terminating arrival process. *Queueing Syst.* **58**(2) 77–104. MR2390269

[4] Pang, G. and Whitt, W. (2010). Two-parameter heavy-traffic limits for infinite-server queues. *Queueing Syst.* **65**(4) 325–364. MR2671058

Department of Industrial and Systems Engineering,
North Carolina State University Raleigh, North Carolina 27695-7906,
E-mail: yliu48@ncsu.edu

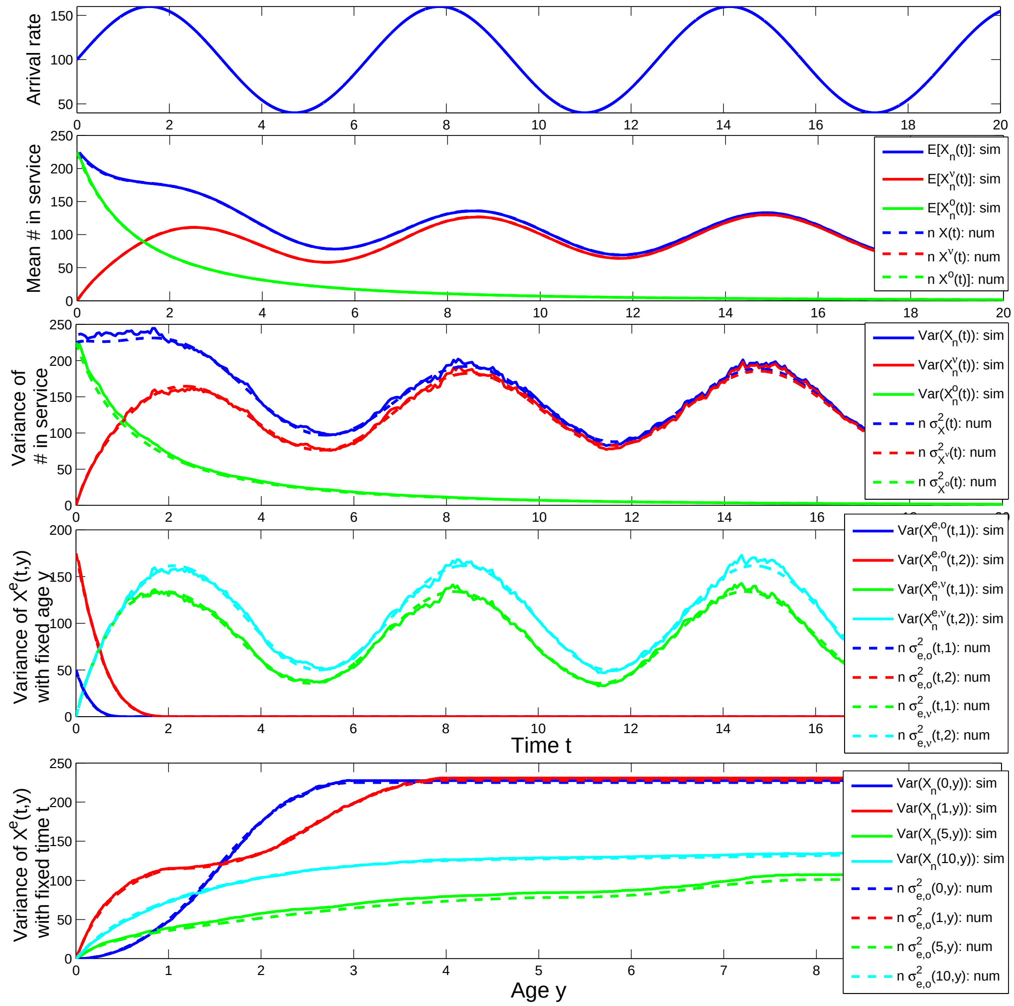

FIG 4. *Example 3 with* b_1^* *in* (F.2)*: Simulation comparisons of the mean and variance for the number of customers in service of an* $H_2^t(1,4)/LN(1,4)/\infty$ *model, with the sinusoidal arrival rate* (5.5) *having parameters* $a = c = 1$, $b = 0.6$, $\phi = 0$ *and* $n = 100$ *and general initial conditions.*

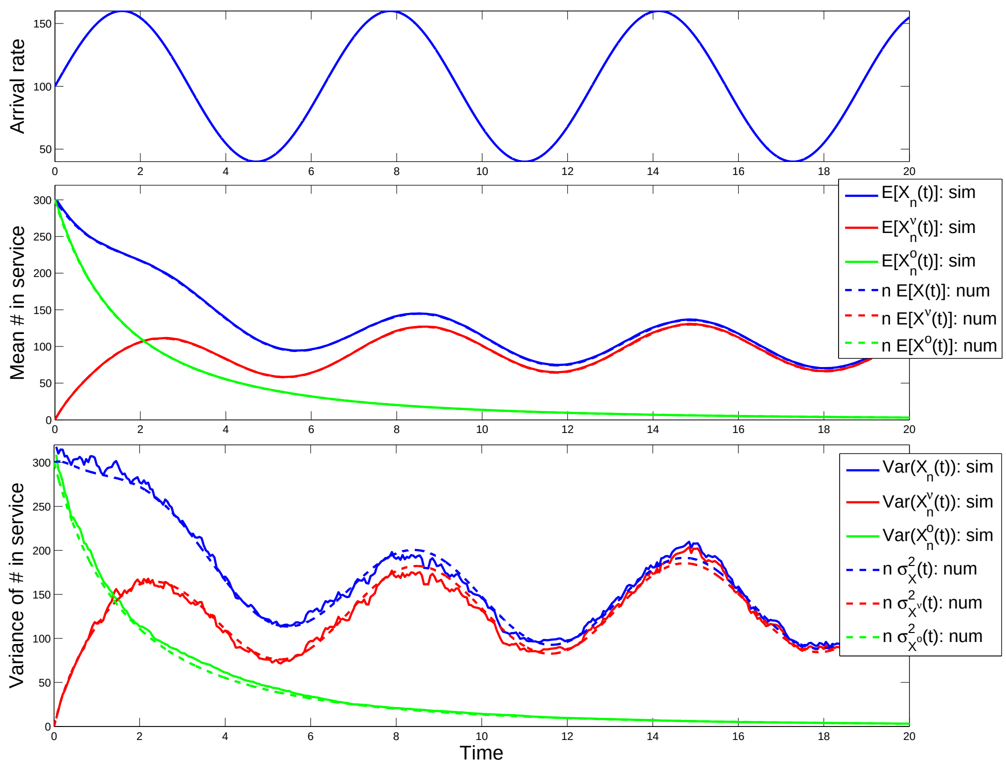

FIG 5. *Example 4 with* b_2^* *in* (F.2)*: Simulation comparisons of the mean and variance for the number of customers in service of an* $H_2^t(1,4)/LN(1,4)/\infty$ *model, with the sinusoidal arrival rate* (5.5) *having parameters* $a = c = 1$, $b = 0.6$, $\phi = 0$ *and* $n = 100$ *and general initial conditions.*