学习笔记

Last update: 2023 年 5 月 8 日

目录

Chapter 1

数学基础

1.1 泰勒展开

泰勒展开是一种思想!!!

泰勒级数

$$f(x+\Delta x)=f(x)+\frac{\partial f(x)}{\partial x}\Delta x+\frac{1}{2}\frac{\partial^2 f(x)}{\partial x^2}(\Delta x)^2+\dots \tag{1.1}$$

麦克劳林级数

$$f(x)=f(0)+\frac{\partial f(0)}{\partial x}x+\frac{1}{2}\frac{\partial^2 f(0)}{\partial x^2}x^2+\cdots=f(x)|_{x=0}+\left.\frac{\partial f(x)}{\partial x}\right|_{x=0}x+\frac{1}{2}\left.\frac{\partial^2 f(0)}{\partial x^2}\right|_{x=0}x^2+\dots$$

泰勒级数展开式使用时都有一个前提条件，不能随便用!

$(1+x)^\alpha$的泰勒展开使用条件：$|x|<1$

由此，有几个常用的当 $|x|<1$ 时

$$\frac{1}{1+x}=1-x+o(x),\quad \frac{1}{1-x}=1+x+o(x),\quad \sqrt{1+x}=1+\frac{1}{2}x+o(x)$$

e^x 的泰勒展开使用条件： $x\in\mathbb{R}$

$$e^x=1+x+...+\frac{x^n}{n!}+o(x^n)$$

欧拉公式（推荐电影：博士的爱情方程式）

$$e^{ix}=\cos x+i\sin x$$

所以

$$\cos x+i\sin x=e^{ix}=1+ix-\frac{x^2}{2!}-i\frac{x^3}{3!}+\frac{x^4}{4}+i\frac{x^5}{5!}+...+\frac{(ix)^n}{n!}+...$$

所以

$\cos x$ 的泰勒展开对应 e^{ix} 的泰勒展开的实部（奇数项1，3，5，7...）

$\sin x$ 的泰勒展开对应 e^{ix} 的泰勒展开的虚部（偶数项2，4，6，8...）

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}+\dots \tag{1.2}$$

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}+\dots \tag{1.3}$$

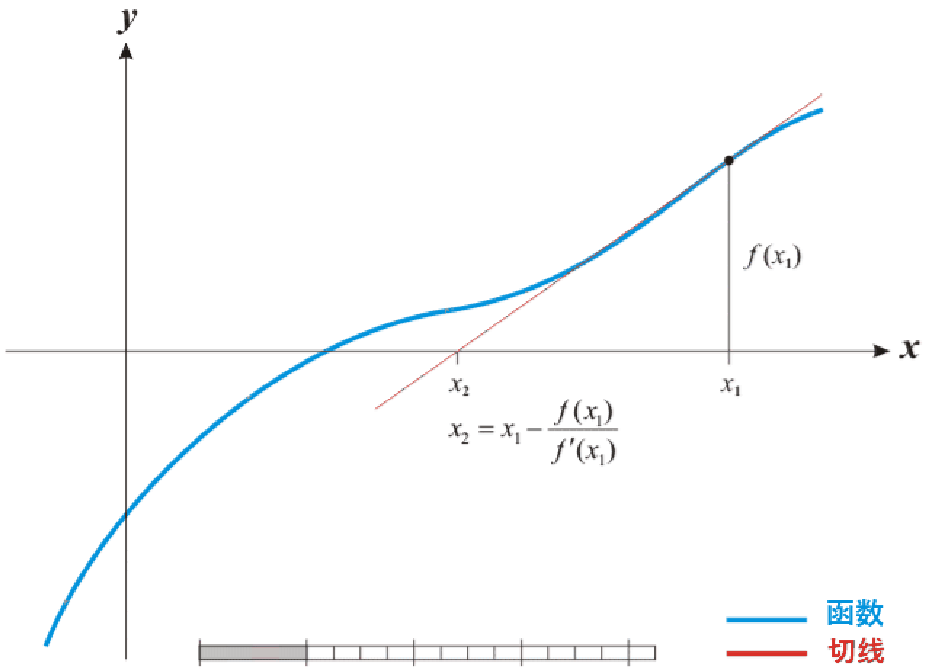
$$(1+x)^\alpha=1+\alpha x+\frac{\alpha(\alpha-1)}{2!}x^2+\frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3+... \tag{1.4}$$

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}+...+(-1)^{n-1}\frac{x^n}{n}+... \tag{1.5}$$

麦克劳林级数

$$\frac{1}{1+\varepsilon}=f(\varepsilon)=f(0)+f'(0)\varepsilon+\frac{1}{2}f''(0)\varepsilon^2=1+(-1)\varepsilon+o(\varepsilon)=1-\varepsilon$$

1.2 牛顿迭代法:求非线性方程的根



求 $f(x) = 0$ 的根（牛顿迭代法）

泰勒展开(仅使用一阶近似或说切线近似)

$$f(x) = f(x_0) + \frac{\partial f(x_0)}{\partial x}(x - x_0) + \cdots = 0 \tag{1.6}$$

则点x_0处切线方程的根为

$$x = -\frac{f(x_0)}{\frac{\partial f(x_0)}{\partial x}} + x_0 \tag{1.7}$$

写成迭代格式

$$x_{k+1} = -\frac{f(x_k)}{\frac{\partial f(x_k)}{\partial x}} + x_k \tag{1.8}$$

1.3 平面曲线的法向量—空间曲面的法向量

1.3.1 平面曲线的法向量

平面曲线方程为

$$f(x,y)=0 \tag{1.9}$$

两边全微分

$$\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy=0 \tag{1.10}$$

写成向量点积的形式

$$(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y})\cdot(dx,dy)=0 \tag{1.11}$$

因为 (dx,dy) 为切向量，所以 $(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y})$ 为法向量，即

$$\boldsymbol{n}=\nabla f=\frac{\partial f}{\partial \boldsymbol{x}}=\frac{\partial f}{\partial x_i}\boldsymbol{e}_i=(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}) \tag{1.12}$$

1.3.2 空间曲面的法向量

空间曲面方程为

$$f(x,y,z)=0 \tag{1.13}$$

两边全微分

$$\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\frac{\partial f}{\partial z}dz=0 \tag{1.14}$$

写成向量点积的形式

$$(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z})\cdot(dx,dy,dz)=0 \tag{1.15}$$

因为 (dx,dy,dz) 为切向量，所以 $(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z})$ 为法向量，即

$$\boldsymbol{n}=\nabla f=\frac{\partial f}{\partial \boldsymbol{x}}=\frac{\partial f}{\partial x_i}\boldsymbol{e}_i=(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}) \tag{1.16}$$

1.4 傅里叶展开

参考《数学物理方法》

1.4.1 以$2l$为周期的无穷区间上函数的傅里叶级数展开

法国数学家傅里叶(Fourier)发现，任何周期函数都可以用正弦函数和余弦函数构成的无穷级数来表示。这是一种特殊的三角级数

若函数$f(x)$为区间$[-l,l]$上可积，且满足$f(x+2l)=f(x)$的无穷区间上的周期函数,则该函数可展开为傅里叶级数

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l}) \tag{1.17}$$

其中

$$\begin{aligned}a_n&=\frac{1}{l}\int_{-\pi}^{\pi}f(x)\cos\frac{n\pi x}{l}dx\\ b_n&=\frac{1}{l}\int_{-\pi}^{\pi}f(x)\sin\frac{n\pi x}{l}dx\end{aligned} \tag{1.18}$$

奇函数的傅里叶级数只含正弦项，称为正弦级数。

偶函数的傅里叶级数只含余弦项，称为余弦级数。

1.4.2 定义在有限区间$[-l,l]$上的函数的傅里叶级数：一般延拓

若$f(x)$为定义在有限区间$[-l,l]$上的可积函数，令$F(x)=f(x),x\in[-l,l]$，且满足$F(x+2l)=F(x)$，则$F(x)$是以$T=2l$为周期的无穷区间上的周期函数。然后，可将$F(x)$展开为傅里叶级数

$$F(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l}) \tag{1.19}$$

其中

$$\begin{aligned}a_n&=\frac{1}{l}\int_{-l}^{l}F(x)\cos\frac{n\pi x}{l}dx\quad(n=0,1,2,3...)\\ b_n&=\frac{1}{l}\int_{-l}^{l}F(x)\sin\frac{n\pi x}{l}dx\quad(n=1,2,3...)\end{aligned} \tag{1.20}$$

当$x\in[-l,l]$时,$f(x)=F(x)$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l}) \tag{1.21}$$

其中

$$\begin{aligned}a_n&=\frac{1}{l}\int_{-l}^{l}f(x)\cos\frac{n\pi x}{l}dx\\ b_n&=\frac{1}{l}\int_{-l}^{l}f(x)\sin\frac{n\pi x}{l}dx\end{aligned} \tag{1.22}$$

1.4.3 定义在有限区间$[0,l]$上的函数的傅里叶级数：一般延拓

若$f(x)$为定义在有限区间$[0,l]$上的可积函数，令$F(x)=\begin{cases}f(x),x\in[0,l]\\ g(x),x\in(-l,0]\end{cases}$，且满足$F(x+2l)=F(x)$，则$F(x)$是以$T=2l$为周期的无穷区间上的周期函数。然后可将$F(x)$展开为傅里叶级数。当$x\in[0,l]$时，$f(x)=F(x)$。

为了方便，通常取$g(x)$使得$F(x)$为偶函数或奇函数，然后将$f(x)$展开为余弦级数或正弦级数。

1.4.4 定义在有限区间$[0,l]$上的函数的傅里叶级数：偶延拓

若$f(x)$为定义在有限区间$[0,l]$上的可积函数，令$F(x)=\begin{cases}f(x),x\in[0,l]\\ f(-x),x\in(-l,0]\end{cases}$，且满足$F(x+2l)=F(x)$，则$F(x)$是以$T=2l$为周期的无穷区间上的周期函数且为偶函数。然后可将$F(x)$展开为余弦傅里叶级数。

$$F(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{l} \tag{1.23}$$

其中

$$a_n = \frac{1}{l}\int_{-l}^{l} F(x)\cos\frac{n\pi x}{l}dx = \frac{2}{l}\int_{0}^{l} F(x)\cos\frac{n\pi x}{l}dx \quad (n=0,1,2,3...) \tag{1.24}$$

当$x\in[0,l]$时，$f(x)=F(x)$。

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} \tag{1.25}$$

其中

$$a_n = \frac{2}{l}\int_{0}^{l} f(x)\cos\frac{n\pi x}{l}dx \quad (n=0,1,2,3...) \tag{1.26}$$

余弦级数的和的导数在$x=0$和$x=l$处为零。

1.4.5 定义在有限区间$[0,l]$上的函数的傅里叶级数：奇延拓

若$f(x)$为定义在有限区间$[0,l]$上的可积函数，令$F(x)=\begin{cases} f(x), x\in[0,l] \\ -f(-x), x\in(-l,0] \end{cases}$，且满足$F(x+2l)=F(x)$，则$F(x)$是以$T=2l$为周期的无穷区间上的周期函数且为奇函数。然后可将$F(x)$展开为余弦傅里叶级数。

$$F(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \tag{1.27}$$

其中

$$b_n = \frac{1}{l}\int_{-l}^{l} F(x)\sin\frac{n\pi x}{l}dx = \frac{2}{l}\int_{0}^{l} F(x)\sin\frac{n\pi x}{l}dx \quad (n=1,2,3...) \tag{1.28}$$

当$x\in[0,l]$时，$f(x)=F(x)$。

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \tag{1.29}$$

其中

$$b_n = \frac{2}{l}\int_{0}^{l} f(x)\sin\frac{n\pi x}{l}dx \quad (n=1,2,3...) \tag{1.30}$$

正弦级数的和在$x=0$和$x=l$处为零。

1.4.6 定义在有限区间$[a,a+l]$上的函数的傅里叶级数：奇延拓

若$f(x)$为定义在有限区间$[a,a+l]$上的可积函数，令$F(x)=\begin{cases} f(x+a), x\in[0,l] \\ -f(-x+a), x\in(-l,0] \end{cases}$，且满足$F(x+2l)=F(x)$，则$F(x)$是以$T=2l$为周期的无穷区间上的周期函数且为奇函数。然后可将$F(x)$展开为余弦傅里叶级数。

$$F(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \tag{1.31}$$

其中

$$b_n = \frac{1}{l}\int_{-l}^{l} F(x)\sin\frac{n\pi x}{l}dx = \frac{2}{l}\int_{0}^{l} F(x)\sin\frac{n\pi x}{l}dx \quad (n=1,2,3...) \tag{1.32}$$

当$x\in[0,l]$时，$f(x+a)=F(x)$。

$$f(x+a) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \tag{1.33}$$

其中

$$b_n = \frac{2}{l}\int_{0}^{l} f(x+a)\sin\frac{n\pi x}{l}dx \quad (n=1,2,3...) \tag{1.34}$$

则$f(x)$为

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi(x-a)}{l} \tag{1.35}$$

其中

$$b_n = \frac{2}{l}\int_{a}^{a+l} f(x)\sin\frac{n\pi(x-a)}{l}dx \quad (n=1,2,3...) \tag{1.36}$$

1.5 一般实系数一元三次方程的解法

1.5.1 复数开n次方根

欧拉公式为

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{1.37}$$

实际上，复数域上的数可以根据欧拉公式简洁的写为

$$a + bi = R\left[\cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi)\right] = Re^{i(\theta+2k\pi)} = e^{\ln R + i(\theta+2k\pi)} \tag{1.38}$$

其中

$$R = \sqrt{a^2 + b^2}, \qquad \theta = \arctan\frac{b}{a} \tag{1.39}$$

从该式中已可以很轻易地看出，实际上，用等号左边的表示法表示的一个数，可以对应等号右边的表示法中的无穷多个数。这也恰恰是用等号左边的表示法表示数时，当进行需要开方的运算时，会出现多个答案的最重要的原因。不妨说，等号左边的表示法是不精确的。

$$\sqrt[n]{a + bi} = \sqrt[n]{Re^{i(\theta+2k\pi)}} = \sqrt[n]{R}e^{\frac{i(\theta+2k\pi)}{n}} \qquad k = 0, ..., n-1 \tag{1.40}$$

或者更为简单的算法

$$\sqrt[n]{a + bi} = \sqrt[n]{e^{\ln R + i(\theta+2k\pi)}} = e^{\frac{\ln R + i(\theta+2k\pi)}{n}} \qquad k = 0, ..., n-1 \tag{1.41}$$

这里，其实 k 不一定非取 n 个连续自然数，例如卡丹公式推导中 $k = 0, 1, 2$ 和 $k = 0, 1, -1$ 会得到相同的结果。

由于三角函数的周期性，开 n 次方会有 n 个不同的解。这就是复数域的开方运算的本质。

1.5.2 卡丹公式推导过程

一般实系数一元三次方程可写为

2-07-30}

$$ax^3 + bx^2 + cx + d = 0 (a \neq 0) \tag{1.42}$$

上式除以 a ，并设

2-07-30}

$$x = y - \frac{b}{3a} \tag{1.43}$$

则可化为如下形式

2-07-30}

$$y^3 + py + q = 0 \tag{1.44}$$

其中

$$p = \frac{3ac - b^2}{3a^2}, \qquad q = \frac{27a^2d - 9abc + 2b^3}{27a^3} \tag{1.45}$$

接下来，令

2-07-30}

$$y = u + v \tag{1.46}$$

然后代入公式 (1.44)，合并同类项，得

2-07-30}

$$u^3 + v^3 + (3uv + p)(u + v) + q = 0 \tag{1.47}$$

令

$$3uv + p = 0 \tag{1.48}$$

代入 (1.46)，得

2-07-30}

$$u^3 + v^3 = -q \tag{1.49}$$

再由公式 (1.47)，得

2-07-30}

$$uv = -\frac{p}{3} \tag{1.50}$$

所以

2-07-30}

$$u^3v^3 = -\frac{p^3}{27} \tag{1.51}$$

不妨令

$$z_1 = u^3, \qquad z_2 = v^3 \tag{1.52}$$

由公式 (1.49) 和 (1.51)

$$\begin{aligned} z_1+z_2&=-q \\ z_1z_2&=-\frac{p^3}{27} \end{aligned} \tag{1.53}$$

解得

$$\begin{aligned} z_1^2+qz_1-\frac{p^3}{27}=0 \\ z_2^2+qz_2-\frac{p^3}{27}=0 \end{aligned} \tag{1.54}$$

所以，z_1，z_2 为方程 $z^2+qz-\frac{p^3}{27}$ 的两个根。根据一元二次方程的根与系数的关系，得

$$\begin{aligned} z_1&=\frac{-b+\sqrt{b^2-4ac}}{2a}=-\frac{-q+\sqrt{q^2+\frac{4p^3}{27}}}{2}=-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}} \\ z_2&=\frac{-b-\sqrt{b^2-4ac}}{2a}=-\frac{-q-\sqrt{q^2+\frac{4p^3}{27}}}{2}=-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}} \end{aligned} \tag{1.55}$$

则

$$u^3=-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}},\qquad v^3=-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}} \tag{1.56}$$

故有

$$u=\begin{cases}\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}} \\ \sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}\cdot\omega \\ \sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}\cdot\omega^2\end{cases},\qquad v=\begin{cases}\sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}} \\ \sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}\cdot\omega \\ \sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}\cdot\omega^2\end{cases} \tag{1.57}$$

其中，ω 是1的三次单位根

$$\omega=-\frac{1}{2}+\frac{\sqrt{3}}{2}i \tag{1.58}$$

u，v 的取值必须满足公式 (1.47)，否则就有九种组合方式。据此选择合适的 u，v 值，再由公式 (1.46) 便可得方程 (1.44) 的解

$$\begin{aligned} y_1&=\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}+\sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}} \\ y_2&=\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}\cdot\omega+\sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}\cdot\omega^2 \\ y_3&=\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}\cdot\omega^2+\sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}\cdot\omega \end{aligned} \tag{1.59}$$

再由公式 (1.43)，可得方程 (1.42) 的解

$$\begin{aligned} x_1&=\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}+\sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}-\frac{b}{3a} \\ x_2&=\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}\cdot\omega+\sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}\cdot\omega^2-\frac{b}{3a} \\ x_3&=\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}\cdot\omega^2+\sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}\cdot\omega-\frac{b}{3a} \end{aligned} \tag{1.60}$$

此即为卡丹公式。

令

$$\Delta=\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3=\Delta_1+\Delta_2 \tag{1.61}$$

称为实系数一元三次方程的卡丹判别式，根的判别法则如下：

当 $\Delta>0$ 时，有一个实根和两个复根；

当 $\Delta=0$ 时，有三个实根，当 $p=q=0$ 时，有一个三重零根，$p,q\neq 0$ 时，三个实根中有两个相等；

当 $\Delta<0$ 时，有三个不等实根。

此外，当 $\Delta<0$ 时，u^3 和 v^3 变为

$$u^3=-\frac{q}{2}+i\sqrt{-\Delta},\qquad v^3=-\frac{q}{2}-i\sqrt{-\Delta} \tag{1.62}$$

$$R=\sqrt{\left(-\frac{q}{2}\right)^2+\left(\sqrt{-\Delta}\right)^2}=\sqrt{\frac{q^2}{4}-\Delta}=\sqrt{-\left(\frac{p}{3}\right)^3}=\sqrt{-\Delta_2},\qquad \theta_1=\arctan\frac{\sqrt{-\Delta}}{-q/2},\qquad \theta_2=\arctan\frac{-\sqrt{-\Delta}}{-q/2}=-\arctan\frac{\sqrt{-\Delta}}{-q/2}=-\theta_1 \tag{1.63}$$

根据欧拉公式，将复数写成复指数形式

$$e^{i\theta}=\cos\theta+i\sin\theta \tag{1.64}$$

可得

$$u^3=Re^{i(\theta_1+2k\pi)},\quad v^3=Re^{i(\theta_2+2k\pi)}=Re^{i(-\theta_1+2k\pi)} \tag{1.65}$$

所以

$$u=\sqrt[3]{R}e^{\frac{i(\theta_1+2k\pi)}{3}},\qquad v=\sqrt[3]{R}e^{\frac{i(-\theta_1+2k\pi)}{3}},\qquad k=-1,0,1\text{ or }k=0,1,2\text{ et al.} \tag{1.66}$$

不妨将 θ_1 写成 θ，再根据欧拉公式将其展开，得

$$u=\begin{cases}\sqrt[3]{R}\left(\cos\frac{\theta}{3}+i\sin\frac{\theta}{3}\right)\\ \sqrt[3]{R}\left(\cos\left(\frac{\theta}{3}+\frac{2\pi}{3}\right)+i\sin\left(\frac{\theta}{3}+\frac{2\pi}{3}\right)\right)\\ \sqrt[3]{R}\left(\cos\left(\frac{\theta}{3}-\frac{2\pi}{3}\right)+i\sin\left(\frac{\theta}{3}-\frac{2\pi}{3}\right)\right)\end{cases},\qquad v=\begin{cases}\sqrt[3]{R}\left(\cos\frac{\theta}{3}-i\sin\frac{\theta}{3}\right)\\ \sqrt[3]{R}\left(\cos\left(\frac{\theta}{3}+\frac{2\pi}{3}\right)-i\sin\left(\frac{\theta}{3}+\frac{2\pi}{3}\right)\right)\\ \sqrt[3]{R}\left(\cos\left(\frac{\theta}{3}-\frac{2\pi}{3}\right)-i\sin\left(\frac{\theta}{3}-\frac{2\pi}{3}\right)\right)\end{cases} \tag{1.67}$$

其中

$$\sqrt[3]{R}=\sqrt{-\frac{p}{3}} \tag{1.68}$$

因为

$$\theta=\theta_1=\arctan\frac{\sqrt{-\Delta}}{-q/2} \tag{1.69}$$

所以

$$\tan\theta=\frac{\sqrt{-\Delta}}{-q/2} \tag{1.70}$$

因为

$$\tan^2\theta+1=\frac{1}{\cos^2\theta} \tag{1.71}$$

所以

$$\cos^2\theta=\frac{1}{\tan^2\theta+1}=\frac{1}{\frac{-\Delta}{(q/2)^2}+1}=\frac{1}{\frac{(q/2)^2-\Delta}{(q/2)^2}}=\frac{1}{\frac{-(p/3)^3}{(q/2)^2}}=-\frac{(q/2)^2}{(p/3)^3} \tag{1.72}$$

这里，因为

$$\Delta=\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3<0 \tag{1.73}$$

所以

$$\left(\frac{p}{3}\right)^3<0 \tag{1.74}$$

所以

$$-\frac{(q/2)^2}{(p/3)^3}\geqslant 0 \tag{1.75}$$

所以

$$\cos\theta=\sqrt{-\frac{(q/2)^2}{(p/3)^3}} \tag{1.76}$$

这里，$\sqrt{(q/2)^2}$ 不一定等于 $q/2$.

所以

$$\theta=\arccos\sqrt{-\frac{(q/2)^2}{(p/3)^3}}=\arccos\sqrt{-\frac{\Delta_1}{\Delta_2}} \tag{1.77}$$

利用公式 (1.50) 选取合适的 u 和 v，再由公式 (1.46) 即可得方程 (1.44) 的求根公式

$$\begin{aligned}y_1&=2\sqrt[3]{R}\cos\frac{\theta}{3}\\ y_2&=2\sqrt[3]{R}\cos\left(\frac{\theta}{3}+\frac{2\pi}{3}\right)\\ y_3&=2\sqrt[3]{R}\cos\left(\frac{\theta}{3}-\frac{2\pi}{3}\right)\end{aligned} \tag{1.78}$$

利用公式 (1.43) 可得方程 (1.42) 的求根公式

$$
\begin{aligned}
x_1 &= 2\sqrt[3]{R}\cos\frac{\theta}{3} - \frac{b}{3a} \\
x_2 &= 2\sqrt[3]{R}\cos\left(\frac{\theta}{3} + \frac{2\pi}{3}\right) - \frac{b}{3a} \\
x_3 &= 2\sqrt[3]{R}\cos\left(\frac{\theta}{3} - \frac{2\pi}{3}\right) - \frac{b}{3a}
\end{aligned}
\tag{1.79}
$$

或表示为

$$
\begin{aligned}
x_1 &= 2\sqrt{-\frac{p}{3}}\cos\frac{\theta}{3} - \frac{b}{3a} \\
x_2 &= 2\sqrt{-\frac{p}{3}}\cos\left(\frac{\theta}{3} + \frac{2\pi}{3}\right) - \frac{b}{3a} \\
x_3 &= 2\sqrt{-\frac{p}{3}}\cos\left(\frac{\theta}{3} - \frac{2\pi}{3}\right) - \frac{b}{3a}
\end{aligned}
\tag{1.80}
$$

1.6 高阶常微分方程解法

高阶非齐次线性常微分方程一般形式为

$$f^{(n)} + c_1 f^{(n-1)} + ... + c_{n-1} f^{(1)} + c_n f = F(x)$$

若$F(x) = 0$，该方程称为齐次方程。求非齐次方程的步骤为：1.求出齐次方程通解;2.求出非齐次方程特解;3.叠加求非齐次方程通解。

1.6.1 齐次方程通解的求法：特征方程法

非齐次方程对应的齐次方程为$f^{(n)} + a_1 f^{(n-1)} + ... + a_{n-1} f^{(1)} + a_n f = 0$

特征方程为 $\lambda^n + a_1 \lambda^{n-1} + ... + a_{n-1}\lambda + a_n = 0$

若λ是特征方程的k重实数解($k = 1, 2, 3...$),则应该为它在通解中增加这样一项(共k小项)

$e^{\lambda x}(c_1 + c_2 x + ... + c_k x^{k-1})$

代入边界条件求系数。

若$\alpha + i\beta$是特征方程的一对k重共轭复数解($k = 1, 2, 3...$)，则应该为它在通解中增加这样一项(共2k小项)

$e^{\alpha x}[(c_1 + c_2 x + ... + c_k x^{k-1}) \cos(\beta x) + (d_1 + d_2 x + ... + d_k x^{k-1}) \sin(\beta x)]$

代入边界条件求系数。

1.6.2 非齐次方程特解的求法

若非齐次项形如$F(x) = e^{\alpha x} P_m(x) = e^{\alpha x}(s_0 + s_1 x + ... + s_m x^m)$，且$\alpha$是非齐次方程对应齐次方程的k重特征解($k = 0, 1, 2...$)，则非齐次方程特解可设为

$f^* = e^{\alpha x} x^k Q_m(x) = e^{\alpha x} x^k (b_0 + b_1 x + ... + b_m x^m)$

代回非齐次方程求系数。

若非齐次项形如$F(x) = e^{\alpha x}[s_1 \cos(\beta x) + s_2 \sin(\beta x)]$，且$\alpha + i\beta$是非齐次方程对应齐次方程的k重特征解($k = 0, 1, 2...$)，则非齐次方程特解可设为

$f^* = e^{\alpha x} x^k [b_1 \cos(\beta x) + b_2 \sin(\beta x)]$

代回非齐次方程求系数。

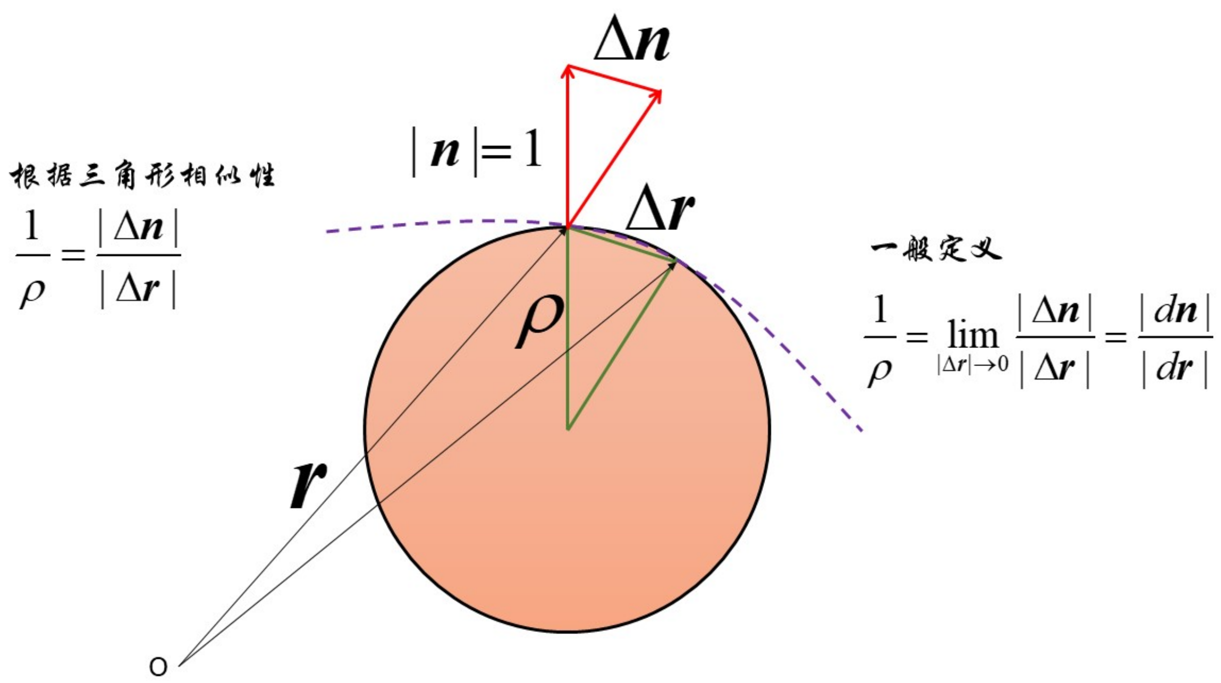
1.6.3 非齐次方程的通解

将非齐次方程对应的齐次方程的通解$\overline{f}$，和非齐次方程的特解f^*相加，即可得到非齐次方程的通解 $f(x) = \overline{f} + f^*$

Chapter 2

微分几何初步

2.1 平面曲线的曲率



问：为什么要引入曲率的概念？

答：刻画曲线的弯曲程度。

平面曲线的曲率的一般定义为

$$\kappa=\frac{1}{\rho}=|\frac{d\boldsymbol{N}}{ds}|=\lim_{|\Delta r|\to 0}\frac{|\Delta \boldsymbol{N}|}{|\Delta \boldsymbol{r}|}=\frac{|d\boldsymbol{N}|}{|d\boldsymbol{r}|}=\frac{\sqrt{d\boldsymbol{N}\cdot d\boldsymbol{N}}}{\sqrt{d\boldsymbol{r}\cdot d\boldsymbol{r}}}=\sqrt{\frac{d\boldsymbol{N}\cdot d\boldsymbol{N}}{d\boldsymbol{r}\cdot d\boldsymbol{r}}} \tag{2.1}$$

等效的定义

$$\kappa=|\frac{d\boldsymbol{T}}{ds}|=\sqrt{\frac{d\boldsymbol{T}\cdot d\boldsymbol{T}}{d\boldsymbol{r}\cdot d\boldsymbol{r}}} \tag{2.2}$$

等效的定义

$$ds=\rho d\theta \to \frac{1}{\rho}=\lim_{\Delta s\to 0}\frac{\Delta\theta}{\Delta s}=\frac{d\theta}{ds} \tag{2.3}$$

平面曲线的一般方程为

$$F(\boldsymbol{r})=F(x,y)=0,\quad F(\boldsymbol{r}(t))=F(x(t),y(t))=F(x(x),y(x))=0 \tag{2.4}$$

等号两边求全微分

$$dF(x,y)=\frac{\partial F}{\partial x}dx+\frac{\partial F}{\partial y}dy=(\frac{\partial F}{\partial x}\boldsymbol{e}_x+\frac{\partial F}{\partial y}\boldsymbol{e}_y)\cdot(dx\boldsymbol{e}_x+dy\boldsymbol{e}_y)=\nabla F\cdot d\boldsymbol{r}=0 \tag{2.5}$$

其中，$d\boldsymbol{r}$表示切矢量，$\frac{d\boldsymbol{r}}{ds}$表示单位切矢量，$\nabla F$表示法矢量，$\frac{\nabla F}{|\nabla F|}$表示单位法矢量。

平面曲线的单位法矢量为

$$\boldsymbol{N}=\frac{\nabla F}{|\nabla F|}=\frac{\nabla F}{\sqrt{\nabla F\cdot\nabla F}}=\frac{\frac{\partial F}{\partial x}\boldsymbol{e}_x+\frac{\partial F}{\partial y}\boldsymbol{e}_y}{\sqrt{(\frac{\partial F}{\partial x})^2+(\frac{\partial F}{\partial y})^2}} \tag{2.6}$$

在直角坐标系下

$$F(x,y)=y-f(x)=0 \tag{2.7}$$

单位法矢量为

$$\boldsymbol{n}=\frac{f'(x)\boldsymbol{e}_x+\boldsymbol{e}_y}{\sqrt{(f'(x))^2+1}}=\frac{f'\boldsymbol{e}_x+\boldsymbol{e}_y}{\sqrt{f'^2+1}}=(f'^2+1)^{-1/2}(f'\boldsymbol{e}_x+\boldsymbol{e}_y) \tag{2.8}$$

$$\begin{aligned}\frac{d}{dx}[(f'^2+1)^{-1/2}f'] &= [-\frac{1}{2}(f'^2+1)^{-3/2}2f'f'']f'+(f'^2+1)^{-1/2}f'' \\ &= -(f'^2+1)^{-3/2}f'^2f''+(f'^2+1)^{-1/2}f'' \\ &= -(f'^2+1)^{-3/2}f'^2f''+(f'^2+1)^{-3/2}(f''f'^2+f'') \\ &= (f'^2+1)^{-3/2}f''\end{aligned} \tag{2.9}$$

$$\frac{d}{dx}[(f'^2+1)^{-1/2}]=-\frac{1}{2}(f'^2+1)^{-3/2}2f'f''=-(f'^2+1)^{-3/2}f'f'' \tag{2.10}$$

对单位法矢量求微分

$$d\boldsymbol{N}=\frac{d\boldsymbol{N}}{dx}dx=(f'^2+1)^{-3/2}(f''\boldsymbol{e}_x-f'f''\boldsymbol{e}_y) \tag{2.11}$$

计算微分法矢量与其自身的内积

$$d\boldsymbol{N}\cdot d\boldsymbol{N}=(f'^2+1)^{-2}f''^2 \tag{2.12}$$

位置矢量为

$$\boldsymbol{r}=x\boldsymbol{e}_x+y\boldsymbol{e}_y=x\boldsymbol{e}_x+f(x)\boldsymbol{e}_y=x\boldsymbol{e}_x+f\boldsymbol{e}_y \tag{2.13}$$

对位置矢量求微分

$$d\boldsymbol{r}=\frac{d\boldsymbol{r}}{dx}dx=\boldsymbol{e}_x+f'\boldsymbol{e}_y \tag{2.14}$$

计算微分位置矢量与其自身的内积

$$d\boldsymbol{r}\cdot d\boldsymbol{r}=1+f'^2 \tag{2.15}$$

计算曲率

$$\kappa=\sqrt{\frac{d\boldsymbol{N}\cdot d\boldsymbol{N}}{d\boldsymbol{r}\cdot d\boldsymbol{r}}}=\sqrt{\frac{(f'^2+1)^{-2}f''^2}{1+f'^2}}=\sqrt{(1+f'^2)^{-3}f''^2}=(1+f'^2)^{-3/2}|f''|=\frac{|f''|}{(1+f'^2)^{3/2}} \tag{2.16}$$

2.2 空间曲线的曲率、挠率和Frenet标架：TNB标架

（弧坐标）自然坐标s，参数坐标t

切矢量，主法矢量，副法矢量

根据速度来定义切矢量:

若使t为时间，$\boldsymbol{r}$为位置矢量，曲线上一点的速度为

$$\frac{d\boldsymbol{r}}{dt}=\boldsymbol{v} \tag{2.17}$$

单位切矢量

$$\boldsymbol{T}=\frac{\boldsymbol{v}}{|\boldsymbol{v}|}=\frac{\frac{d\boldsymbol{r}}{dt}}{|\frac{ds}{dt}|}=\frac{\frac{d\boldsymbol{r}}{dt}}{\frac{ds}{dt}}=\frac{d\boldsymbol{r}}{ds} \tag{2.18}$$

单位长度上$\boldsymbol{T}$的转动率称为曲率，记作κ

$$\kappa=|\frac{d\boldsymbol{T}}{ds}| \tag{2.19}$$

因为

$$|\boldsymbol{T}|^2=\boldsymbol{T}\cdot\boldsymbol{T}=1 \tag{2.20}$$

两边对s求导

$$\frac{d\boldsymbol{T}}{ds}\cdot\boldsymbol{T}+\boldsymbol{T}\cdot\frac{d\boldsymbol{T}}{ds}=0\rightarrow\boldsymbol{T}\cdot\frac{d\boldsymbol{T}}{ds}=0 \tag{2.21}$$

所以

$$\boldsymbol{T}\perp\frac{d\boldsymbol{T}}{ds} \tag{2.22}$$

所以$\frac{d\boldsymbol{T}}{ds}$为法矢量，单位主法矢量$\boldsymbol{N}$为

$$\boldsymbol{N}=\frac{\frac{d\boldsymbol{T}}{ds}}{|\frac{d\boldsymbol{T}}{ds}|}=\frac{1}{\kappa}\frac{d\boldsymbol{T}}{ds} \tag{2.23}$$

亦有

$$\frac{d\boldsymbol{T}}{ds}=\kappa\boldsymbol{N} \tag{2.24}$$

单位副法矢量为

$$\boldsymbol{B} = \boldsymbol{T} \times \boldsymbol{N} \tag{2.25}$$

$$\begin{aligned} \frac{d\boldsymbol{B}}{ds} &= \frac{d\boldsymbol{T}}{ds} \times \boldsymbol{N} + \boldsymbol{T} \times \frac{d\boldsymbol{N}}{ds} \\ &= \frac{d\boldsymbol{T}}{ds} \times \frac{1}{\kappa}\frac{d\boldsymbol{T}}{ds} + \boldsymbol{T} \times \frac{d\boldsymbol{N}}{ds} \\ &= \boldsymbol{T} \times \frac{d\boldsymbol{N}}{ds} \end{aligned} \tag{2.26}$$

所以

$$\frac{d\boldsymbol{B}}{ds} \perp \boldsymbol{T} \tag{2.27}$$

因为（同$\boldsymbol{T}$）

$$\frac{d\boldsymbol{B}}{ds} \perp \boldsymbol{B} \tag{2.28}$$

所以

$$\frac{d\boldsymbol{B}}{ds} \parallel \boldsymbol{N} \tag{2.29}$$

令

$$\frac{d\boldsymbol{B}}{ds} = -\tau \boldsymbol{N} \tag{2.30}$$

两边同时右点乘$\boldsymbol{N}$

$$\frac{d\boldsymbol{B}}{ds} \cdot \boldsymbol{N} = -\tau \boldsymbol{N} \cdot \boldsymbol{N} \rightarrow \tau = -\frac{d\boldsymbol{B}}{ds} \cdot \boldsymbol{N} \tag{2.31}$$

光滑曲线的挠率定义为

$$\tau = -\frac{d\boldsymbol{B}}{ds} \cdot \boldsymbol{N} \tag{2.32}$$

它表示曲线偏离$\boldsymbol{TN}$平面的速率。

$$\begin{aligned} \frac{d}{ds}\boldsymbol{N} &= \frac{d}{ds}[\boldsymbol{B} \times \boldsymbol{T}] = \frac{d\boldsymbol{B}}{ds} \times \boldsymbol{T} + \boldsymbol{B} \times \frac{d\boldsymbol{T}}{ds} = -\tau \boldsymbol{N} \times \boldsymbol{T} + \boldsymbol{B} \times \kappa \boldsymbol{N} \\ &= \tau \boldsymbol{B} - \kappa \boldsymbol{T} \end{aligned} \tag{2.33}$$

以上三个与$\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}$相关的微分式子结合起来，就是Frenet公式

$$\frac{d\boldsymbol{T}}{ds} = \kappa \boldsymbol{N}, \quad \frac{d\boldsymbol{N}}{ds} = -\kappa \boldsymbol{T} + \tau \boldsymbol{B}, \quad \frac{d\boldsymbol{B}}{ds} = -\tau \boldsymbol{N} \tag{2.34}$$

或写成矩阵形式

$$\frac{d}{ds}\begin{bmatrix} \boldsymbol{T}(s) \\ \boldsymbol{N}(s) \\ \boldsymbol{B}(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{T}(s) \\ \boldsymbol{N}(s) \\ \boldsymbol{B}(s) \end{bmatrix} \tag{2.35}$$

Chapter 3

张量初步

3.1 张量代数（Tensor algebra）

3.1.1 指标记法

自由指标、哑标（单项中重复指标表示求和、爱因斯坦求和约定）

3.1.2 张量表示形式

实体形式（加粗体或下面加波浪线、不依赖于坐标系）、分量形式、矩阵形式

3.1.3 张量定义：线性映射（linear mapping）

一元函数$y=f(x)=x \to f(\alpha x)=\alpha x=\alpha f(x)$，标量可以提出去，这就是线性。

$y=f(x)=x^2 \to f(\alpha x)=\alpha^2 x^2=\alpha^2 f(x)$，即$f(\alpha x)\neq \alpha f(x)$，这就是非线性。

函数是映射的一种，输入一个自变量，通过某种对应法则，可以得到一个因变量。

那么对于一个二阶张量，是不是可以理解为输入一个矢量，通过某种作用(张量代数的定义与运算法则)，得到了另一个矢量。所以说张量也可以看作一种映射。

二阶张量(second-order tensor)就是从矢量(vector)到矢量的线性映射(linear mapping)。二阶张量作用于一个矢量得到另一个矢量。 $\boldsymbol{u},\boldsymbol{v}$ 为矢量，如果 $\boldsymbol{S}$ 是从 $\boldsymbol{u}$ 到 $\boldsymbol{v}$ 的线性映射，那么

$$\boldsymbol{v}=\boldsymbol{S}(\boldsymbol{u})$$

α,β 为标量, $\alpha,\beta\in\mathbb{R}$,映射的线性是指如下性质：

$$\boldsymbol{S}(\alpha\boldsymbol{u}+\beta\boldsymbol{v})=\alpha\boldsymbol{S}(\boldsymbol{u})+\beta\boldsymbol{S}(\boldsymbol{v})$$

♣ 映射：一一对应关系

♣ 线性：

对于标量函数

$$f(\alpha x+\beta y)=\alpha f(x)+\beta f(y)$$

对于张量函数

$$\boldsymbol{f}(\alpha\boldsymbol{a}+\beta\boldsymbol{b})=\alpha\boldsymbol{f}(\boldsymbol{a})+\beta\boldsymbol{f}(\boldsymbol{b})$$

3.1.4 Kronecker delta and Levi-Civita

♣ Kronecker delta

$$\boldsymbol{e}_i\otimes\boldsymbol{e}_j=\delta_{ij} \tag{3.1}$$

$$\boldsymbol{I}=\delta_{ij}\boldsymbol{e}_i\otimes\boldsymbol{e}_j \tag{3.2}$$

♣ Levi-Civita symbol

$$\boldsymbol{e}_i\otimes\boldsymbol{e}_j=e_{ijk}\boldsymbol{e}_k \tag{3.3}$$

$$\boldsymbol{e}=e_{ijk}\boldsymbol{e}_i\otimes\boldsymbol{e}_j\otimes\boldsymbol{e}_k \tag{3.4}$$

♣ $e-\delta$恒等式

$$e_{ijk}e_{imn}=\delta_{jm}\delta_{kn}-\delta_{jn}\delta_{km} \tag{3.5}$$

3.1.5 张量的分量形式（坐标系的引入、方便计算）

一阶张量：$\boldsymbol{u}=u_i\boldsymbol{e}_i$

二阶张量： $\boldsymbol{\varepsilon}=\varepsilon_{ij}\boldsymbol{e}_i\otimes\boldsymbol{e}_j$

三阶张量： $\boldsymbol{\eta}=\eta_{ijk}\boldsymbol{e}_i\otimes\boldsymbol{e}_j\otimes\boldsymbol{e}_k$

以此类推。

任意阶张量的分量形式定义如下

$$\boldsymbol{S}=S_{i_1i_2\ldots i_r}\boldsymbol{e}_{i_1}\otimes\boldsymbol{e}_{i_2}\otimes\ldots\otimes\boldsymbol{e}_{i_r}$$

$\boldsymbol{S}$ 为二阶张量，$\boldsymbol{e}_i$ 为正交坐标系[1](直，柱，球)的基矢量，二阶张量的分量定义为

$$S_{ij}=(\boldsymbol{S})_{ij}=\boldsymbol{e}_i\cdot\boldsymbol{S}\cdot\boldsymbol{e}_j$$

详细来源推导见补充资料(Supplemental materials)。

3.1.6 张量加减、相等

♣ 加减

$$\begin{aligned}&\boldsymbol{a}=\boldsymbol{b}+\boldsymbol{c}\\&\rightarrow a_i\boldsymbol{e}_i=b_j\boldsymbol{e}_j+c_k\boldsymbol{e}_k\\&\rightarrow a_i(\boldsymbol{e}_i\cdot\boldsymbol{e}_i)=b_j(\boldsymbol{e}_j\cdot\boldsymbol{e}_i)+c_k(\boldsymbol{e}_k\cdot\boldsymbol{e}_i)\\&\rightarrow a_i=b_i+c_i\end{aligned} \tag{3.6}$$

♣ 证明相等的两种方法：

（1）对应分量相等。

（2）对同一个张量的作用相同。例，两个二阶张量分别作用于同一个矢量得到另一个相同的矢量，说明这两个张量相等。

3.1.7 张量转置

$\boldsymbol{u},\boldsymbol{v}$ 为矢量，二阶张量 $\boldsymbol{S}$ 的转置(transpose)张量 $\boldsymbol{S}^T$ 定义为

$$\boldsymbol{v}\cdot\boldsymbol{S}\cdot\boldsymbol{u}=\boldsymbol{u}\cdot\boldsymbol{S}^T\cdot\boldsymbol{v}$$

$$\boldsymbol{S}=S_{ij}\boldsymbol{e}_i\otimes\boldsymbol{e}_j,\quad \boldsymbol{S}^T=S_{ij}\boldsymbol{e}_j\otimes\boldsymbol{e}_i=S_{ji}\boldsymbol{e}_i\otimes\boldsymbol{e}_j \tag{3.7}$$

性质(Properties): 1. $\boldsymbol{I}^T=\boldsymbol{I}$ 2. $(\boldsymbol{S}+\boldsymbol{T})^T=\boldsymbol{S}^T+\boldsymbol{T}^T$ 3. $(\boldsymbol{S}^T)^T=\boldsymbol{S}$

3.1.8 内积（缩并）

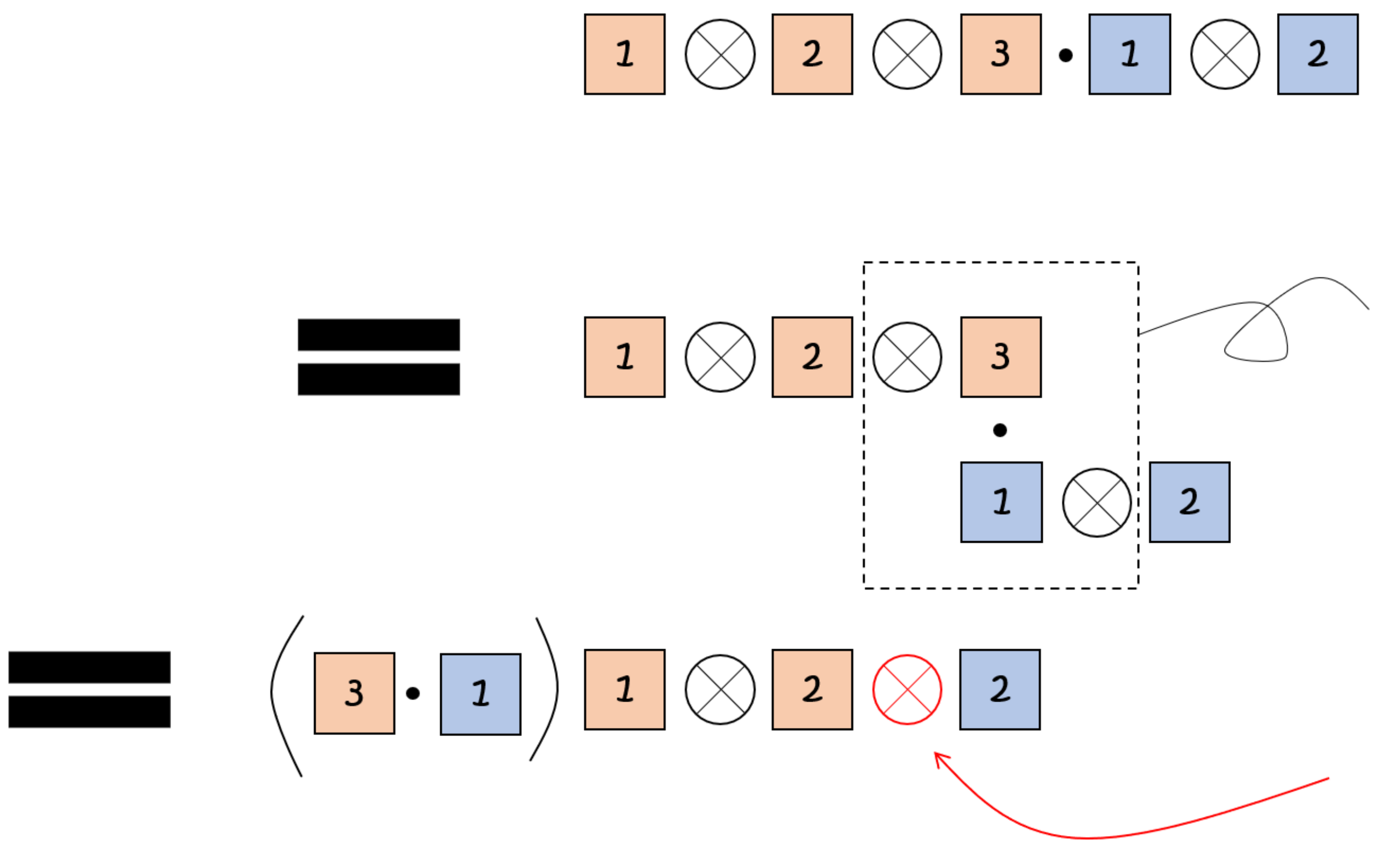
口诀：就近原则，错位相消，代数提前，其余相拼

惯用为最靠近的缩并（就近原则，前一个张量的尾段与后一个张量的首段按从左到右顺序进行缩并）。

张量内积满足结合律和分配律，但不满足交换律。(只有两个一阶张量的内乘满足交换律)

$$(\boldsymbol{S}\cdot\boldsymbol{T})\cdot\boldsymbol{u}=\boldsymbol{S}\cdot(\boldsymbol{T}\cdot\boldsymbol{u})$$

[1]对于正交坐标系$\boldsymbol{e}_i\cdot\boldsymbol{e}_j=\delta_{ij},\boldsymbol{e}_i\times\boldsymbol{e}_j=\epsilon_{ijk}\boldsymbol{e}_k$



$$
\begin{aligned}
&\underset{r阶}{\boldsymbol{A}} \underset{s}{\cdot} \underset{t阶}{\boldsymbol{B}} \\
&= (A_{i_1\dots i_{r-s} i_{r-s+1}\dots i_r} \boldsymbol{e}_{i_1} \otimes \dots \otimes \boldsymbol{e}_{i_{r-s}} \otimes \boldsymbol{e}_{i_{r-s+1}} \otimes \dots \otimes \boldsymbol{e}_{i_r}) \underset{s}{\cdot} (B_{j_1\dots j_s j_{s+1}\dots j_t} \boldsymbol{e}_{j_1} \otimes \dots \otimes \boldsymbol{e}_{j_s} \otimes \boldsymbol{e}_{j_{s+1}} \otimes \dots \otimes \boldsymbol{e}_{j_t}) \\
&= A_{i_1\dots i_{r-s} i_{r-s+1}\dots i_r} B_{j_1\dots j_s j_{s+1}\dots j_t} (\boldsymbol{e}_{i_1} \otimes \dots \otimes \boldsymbol{e}_{i_{r-s}} \otimes \boldsymbol{e}_{i_{r-s+1}} \otimes \dots \otimes \boldsymbol{e}_{i_r}) \underset{s}{\cdot} (\boldsymbol{e}_{j_1} \otimes \dots \otimes \boldsymbol{e}_{j_s} \otimes \boldsymbol{e}_{j_{s+1}} \otimes \dots \otimes \boldsymbol{e}_{j_t}) \\
&= A_{i_1\dots i_{r-s} i_{r-s+1}\dots i_r} B_{j_1\dots j_s j_{s+1}\dots j_t} (\boldsymbol{e}_{i_{r-s+1}} \cdot \boldsymbol{e}_{j_1}) \dots (\boldsymbol{e}_{i_r} \cdot \boldsymbol{e}_{j_s}) (\boldsymbol{e}_{i_1} \otimes \dots \otimes \boldsymbol{e}_{i_{r-s}}) \otimes (\boldsymbol{e}_{j_1} \otimes \dots \otimes \boldsymbol{e}_{j_s}) \\
&= A_{i_1\dots i_{r-s} i_{r-s+1}\dots i_r} B_{j_1\dots j_s j_{s+1}\dots j_t} \delta_{i_{r-s+1} j_1} \dots \delta_{i_r j_s} (\boldsymbol{e}_{i_1} \otimes \dots \otimes \boldsymbol{e}_{i_{r-s}}) \otimes (\boldsymbol{e}_{j_1} \otimes \dots \otimes \boldsymbol{e}_{j_s}) \\
&= A_{i_1\dots i_{r-s} j_1\dots j_s} B_{j_1\dots j_s j_{s+1}\dots j_t} (\boldsymbol{e}_{i_1} \otimes \dots \otimes \boldsymbol{e}_{i_{r-s}}) \otimes (\boldsymbol{e}_{j_{s+1}} \otimes \dots \otimes \boldsymbol{e}_{j_t}) \\
&= A_{i_1\dots i_{r-s} j_1\dots j_s} B_{j_1\dots j_s j_{s+1}\dots j_t} \boldsymbol{e}_{i_1} \otimes \dots \otimes \boldsymbol{e}_{i_{r-s}} \otimes \boldsymbol{e}_{j_{s+1}} \otimes \dots \otimes \boldsymbol{e}_{j_t}
\end{aligned} \tag{3.8}
$$

♠ 一次缩并(dot product)

$$
\boldsymbol{a} \cdot \boldsymbol{b} = a_i \boldsymbol{e}_i \cdot b_j \boldsymbol{e}_j = a_i b_j \boldsymbol{e}_i \cdot \boldsymbol{e}_j = a_i b_j \delta_{ij} = a_i b_i \tag{3.9}
$$

$$
\begin{aligned}
\boldsymbol{\sigma} \cdot \boldsymbol{n} &= (\sigma_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j) \cdot (n_k \boldsymbol{e}_k) \\
&= \sigma_{ij} n_k (\boldsymbol{e}_i \otimes \boldsymbol{e}_j) \cdot \boldsymbol{e}_k \\
&= \sigma_{ij} n_k (\boldsymbol{e}_j \cdot \boldsymbol{e}_k) \boldsymbol{e}_i \\
&= \sigma_{ij} n_k \delta_{jk} \boldsymbol{e}_i \\
&= \sigma_{ij} n_j \boldsymbol{e}_i
\end{aligned} \tag{3.10}
$$

$$
\begin{aligned}
\boldsymbol{n} \cdot \boldsymbol{\sigma} &= (n_i \boldsymbol{e}_i) \cdot (\sigma_{jk} \boldsymbol{e}_j \otimes \boldsymbol{e}_k) \\
&= n_i \sigma_{jk} (\boldsymbol{e}_i \cdot \boldsymbol{e}_j) \boldsymbol{e}_k \\
&= n_i \sigma_{jk} \delta_{ij} \boldsymbol{e}_k \\
&= n_i \sigma_{ik} \boldsymbol{e}_k
\end{aligned} \tag{3.11}
$$

♠ 二次缩并

$$
\begin{aligned}
\boldsymbol{A} : \boldsymbol{B} &= (A_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j) : (B_{kl} \boldsymbol{e}_k \otimes \boldsymbol{e}_l) \\
&= A_{ij} B_{kl} (\boldsymbol{e}_i \otimes \boldsymbol{e}_j) : (\boldsymbol{e}_k \otimes \boldsymbol{e}_l) \\
&= A_{ij} B_{kl} (\boldsymbol{e}_i \cdot \boldsymbol{e}_k)(\boldsymbol{e}_j \cdot \boldsymbol{e}_l) \\
&= A_{ij} B_{kl} \delta_{ik} \delta_{jl} \\
&= A_{ij} B_{ij}
\end{aligned} \tag{3.12}
$$

$$\begin{aligned}\boldsymbol{C}:\boldsymbol{\varepsilon}&=(C_{ijkl}\boldsymbol{e}_i\otimes\boldsymbol{e}_j\otimes\boldsymbol{e}_k\otimes\boldsymbol{e}_l):(\varepsilon_{mn}\boldsymbol{e}_m\otimes\boldsymbol{e}_n)\\&=C_{ijkl}\varepsilon_{mn}(\boldsymbol{e}_i\otimes\boldsymbol{e}_j\otimes\boldsymbol{e}_k\otimes\boldsymbol{e}_l):(\boldsymbol{e}_m\otimes\boldsymbol{e}_n)\\&=C_{ijkl}\varepsilon_{mn}(\boldsymbol{e}_k\cdot\boldsymbol{e}_m)(\boldsymbol{e}_l\cdot\boldsymbol{e}_n)\boldsymbol{e}_i\otimes\boldsymbol{e}_j\\&=C_{ijkl}\varepsilon_{mn}\delta_{km}\delta_{ln}\boldsymbol{e}_i\otimes\boldsymbol{e}_j\\&=C_{ijmn}\varepsilon_{mn}\boldsymbol{e}_i\otimes\boldsymbol{e}_j\end{aligned}\tag{3.13}$$

$$\begin{aligned}\boldsymbol{\varepsilon}:\boldsymbol{C}&=(\varepsilon_{mn}\boldsymbol{e}_m\otimes\boldsymbol{e}_n):(C_{ijkl}\boldsymbol{e}_i\otimes\boldsymbol{e}_j\otimes\boldsymbol{e}_k\otimes\boldsymbol{e}_l)\\&=\varepsilon_{mn}C_{ijkl}(\boldsymbol{e}_m\otimes\boldsymbol{e}_n):(\boldsymbol{e}_i\otimes\boldsymbol{e}_j\otimes\boldsymbol{e}_k\otimes\boldsymbol{e}_l)\\&=\varepsilon_{mn}C_{ijkl}(\boldsymbol{e}_m\cdot\boldsymbol{e}_i)(\boldsymbol{e}_n\cdot\boldsymbol{e}_j)\boldsymbol{e}_k\otimes\boldsymbol{e}_l\\&=\varepsilon_{mn}C_{ijkl}\delta_{mi}\delta_{nj}\boldsymbol{e}_k\otimes\boldsymbol{e}_l\\&=\varepsilon_{mn}C_{mnkl}\boldsymbol{e}_k\otimes\boldsymbol{e}_l\end{aligned}\tag{3.14}$$

♠ 更高阶的缩并以此类推

二次缩并（双点乘）

$$\ell^2(\boldsymbol{A}\otimes\boldsymbol{B})=\boldsymbol{A}:\boldsymbol{B}$$

$$\boldsymbol{S}:\boldsymbol{T}=S_{ij}\boldsymbol{e}_i\otimes\boldsymbol{e}_j:T_{kl}\boldsymbol{e}_k\otimes\boldsymbol{e}_l=S_{ij}T_{kl}\boldsymbol{e}_i\otimes\boldsymbol{e}_j:\boldsymbol{e}_k\otimes\boldsymbol{e}_l=S_{ij}T_{kl}\delta_{ik}\delta_{jl}=S_{ij}T_{ij}$$

$$\begin{aligned}\boldsymbol{S}:\boldsymbol{T}&=\mathrm{tr}(\boldsymbol{S}^T\cdot\boldsymbol{T})=\mathrm{tr}(S_{ij}\boldsymbol{e}_j\otimes\boldsymbol{e}_i\cdot T_{kl}\boldsymbol{e}_k\otimes\boldsymbol{e}_l)=\mathrm{tr}(S_{ij}T_{kl}\delta_{ik}\boldsymbol{e}_j\otimes\boldsymbol{e}_l)\\&=\mathrm{tr}(S_{kj}T_{kl}\boldsymbol{e}_j\otimes\boldsymbol{e}_l)=S_{kj}T_{kl}\mathrm{tr}(\boldsymbol{e}_j\otimes\boldsymbol{e}_l)=S_{kj}T_{kl}\boldsymbol{e}_j\cdot\boldsymbol{e}_l=S_{kj}T_{kj}=S_{ij}T_{ij}\\&=\mathrm{tr}(\boldsymbol{S}\cdot\boldsymbol{T}^T)\end{aligned}$$

性质(Properties): 1. $\boldsymbol{S}:\boldsymbol{T}=\boldsymbol{T}:\boldsymbol{S}$ 2. $\boldsymbol{I}:\boldsymbol{S}=\mathrm{tr}\boldsymbol{S}$

3.1.9 外积

$$\boldsymbol{a}\otimes\boldsymbol{b}=(a_i\boldsymbol{e}_i)\otimes(b_j\boldsymbol{e}_j)=a_ib_j\boldsymbol{e}_i\otimes\boldsymbol{e}_j\tag{3.15}$$

$$\boldsymbol{A}\otimes\boldsymbol{B}=(A_{ij}\boldsymbol{e}_i\otimes\boldsymbol{e}_j)\otimes(B_{kl}\boldsymbol{e}_k\otimes\boldsymbol{e}_l)=A_{ij}B_{kl}(\boldsymbol{e}_i\otimes\boldsymbol{e}_j)\otimes(\boldsymbol{e}_k\otimes\boldsymbol{e}_l)=A_{ij}B_{kl}\boldsymbol{e}_i\otimes\boldsymbol{e}_j\otimes\boldsymbol{e}_k\otimes\boldsymbol{e}_l\tag{3.16}$$

以此类推

3.1.10 二阶张量的迹

$\boldsymbol{u},\boldsymbol{v}$为矢量，任一个二阶张量可以表示为两个矢量的张量积(tensor product)，二阶张量的迹定义为

$$\mathrm{tr}(\boldsymbol{u}\otimes\boldsymbol{v})=\boldsymbol{u}\cdot\boldsymbol{v}$$

张量是线性算子，张量的迹也是线性算子，算子的线性表述如下

$$\mathrm{tr}(\alpha\boldsymbol{S}+\beta\boldsymbol{T})=\alpha\mathrm{tr}(\boldsymbol{S})+\beta\mathrm{tr}(\boldsymbol{T})$$

其中$\boldsymbol{S},\boldsymbol{T}$为二阶张量，$\alpha,\beta$为标量。

二阶张量的一次缩并也可以表示为**张量的迹**

$$tr(\boldsymbol{a}\otimes\boldsymbol{b})=tr(a_ib_j\boldsymbol{e}_i\otimes\boldsymbol{e}_j)=a_ib_jtr(\boldsymbol{e}_i\otimes\boldsymbol{e}_j)=a_ib_j\boldsymbol{e}_i\cdot\boldsymbol{e}_j=a_ib_j\delta_{ij}=a_ib_i\tag{3.17}$$

以此类推

$$\mathrm{tr}(\boldsymbol{u}\otimes\boldsymbol{v})=\boldsymbol{u}\cdot\boldsymbol{v},\qquad\mathrm{tr}(\boldsymbol{S})=\mathrm{tr}(S_{ij}\boldsymbol{e}_i\otimes\boldsymbol{e}_j)=S_{ij}\mathrm{tr}(\boldsymbol{e}_i\otimes\boldsymbol{e}_j)=S_{ij}\boldsymbol{e}_i\cdot\boldsymbol{e}_j=S_{ii}$$

$$\mathrm{tr}\boldsymbol{S}=\mathrm{tr}(S_{ij}\boldsymbol{e}_i\otimes\boldsymbol{e}_j)=S_{ij}\mathrm{tr}(\boldsymbol{e}_i\otimes\boldsymbol{e}_j)=S_{ij}(\boldsymbol{e}_i\cdot\boldsymbol{e}_j)=S_{ii}$$

性质：

$$(\boldsymbol{A}\cdot\boldsymbol{B})^T=\boldsymbol{B}^T\cdot\boldsymbol{A}^T$$

$$tr(\boldsymbol{A}\cdot\boldsymbol{B}) = tr(\boldsymbol{B}\cdot\boldsymbol{A}) = tr(\boldsymbol{A}^T\cdot\boldsymbol{B}^T) = tr(\boldsymbol{B}^T\cdot\boldsymbol{A}^T)$$

$$tr(\boldsymbol{A}\cdot\boldsymbol{B}^T) = tr(\boldsymbol{A}^T\cdot\boldsymbol{B})$$

证明：

$$\begin{aligned}(\boldsymbol{A}\cdot\boldsymbol{B})^T &= (A_{ij}\boldsymbol{e}_i\otimes\boldsymbol{e}_j\cdot B_{mn}\boldsymbol{e}_m\otimes\boldsymbol{e}_n)^T = (A_{ij}B_{mn}\delta_{jm}\boldsymbol{e}_i\otimes\boldsymbol{e}_n)^T = (A_{im}B_{mn}\boldsymbol{e}_i\otimes\boldsymbol{e}_n)^T\\ &= A_{nm}B_{mi}\boldsymbol{e}_i\otimes\boldsymbol{e}_n = A_{jm}B_{mi}\boldsymbol{e}_i\otimes\boldsymbol{e}_j\end{aligned}$$

$$\begin{aligned}\boldsymbol{B}^T\cdot\boldsymbol{A}^T &= B_{nm}\boldsymbol{e}_m\otimes\boldsymbol{e}_n\cdot A_{ji}\boldsymbol{e}_i\otimes\boldsymbol{e}_j = B_{nm}A_{ji}\delta_{ni}\boldsymbol{e}_m\otimes\boldsymbol{e}_j = B_{nm}A_{jn}\boldsymbol{e}_m\otimes\boldsymbol{e}_j\\ &= A_{jn}B_{nm}\boldsymbol{e}_m\otimes\boldsymbol{e}_j = A_{jn}B_{ni}\boldsymbol{e}_i\otimes\boldsymbol{e}_j = A_{jm}B_{mi}\boldsymbol{e}_i\otimes\boldsymbol{e}_j\end{aligned}$$

证明：

$$\begin{aligned}tr(\boldsymbol{A}\cdot\boldsymbol{B}) &= tr(A_{ij}\boldsymbol{e}_i\otimes\boldsymbol{e}_j\cdot B_{mn}\boldsymbol{e}_m\otimes\boldsymbol{e}_n) = tr(A_{ij}B_{mn}\delta_{jm}\boldsymbol{e}_i\otimes\boldsymbol{e}_n) = A_{ij}B_{mn}\delta_{jm}\delta_{in}\\ &= A_{ij}B_{ji}\end{aligned}$$

$$\begin{aligned}tr(\boldsymbol{B}\cdot\boldsymbol{A}) &= tr(B_{mn}\boldsymbol{e}_m\otimes\boldsymbol{e}_n\cdot A_{ij}\boldsymbol{e}_i\otimes\boldsymbol{e}_j) = tr(B_{mn}A_{ij}\delta_{ni}\boldsymbol{e}_m\otimes\boldsymbol{e}_j) = B_{mn}A_{ij}\delta_{ni}\delta_{mj}\\ &= B_{ji}A_{ij} = A_{ij}B_{ji}\end{aligned}$$

$$\begin{aligned}tr(\boldsymbol{A}^T\cdot\boldsymbol{B}^T) &= tr(A_{ji}\boldsymbol{e}_i\otimes\boldsymbol{e}_j\cdot B_{nm}\boldsymbol{e}_m\otimes\boldsymbol{e}_n) = tr(A_{ji}B_{nm}\delta_{jm}\boldsymbol{e}_i\otimes\boldsymbol{e}_n) = A_{ji}B_{nm}\delta_{jm}\delta_{in}\\ &= A_{ji}B_{ij} = A_{ij}B_{ji}\end{aligned}$$

$$\begin{aligned}tr(\boldsymbol{B}^T\cdot\boldsymbol{A}^T) &= tr(B_{nm}\boldsymbol{e}_m\otimes\boldsymbol{e}_n\cdot A_{ji}\boldsymbol{e}_i\otimes\boldsymbol{e}_j) = tr(B_{nm}A_{ji}\delta_{ni}\boldsymbol{e}_m\times\boldsymbol{e}_j) = B_{nm}A_{ji}\delta_{ni}\delta_{mj}\\ &= B_{ij}A_{ji} = A_{ji}B_{ij} = A_{ij}B_{ji}\end{aligned}$$

3.1.11 二阶张量的行列式

$$\det(\boldsymbol{F}) = e_{ijk}F_{i1}F_{j2}F_{k3} \tag{3.18}$$

$$\begin{aligned}
\det(\boldsymbol{F}) &= \begin{vmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{vmatrix} = F_{11}\begin{vmatrix} F_{22} & F_{23} \\ F_{32} & F_{33} \end{vmatrix} - F_{21}\begin{vmatrix} F_{12} & F_{13} \\ F_{32} & F_{33} \end{vmatrix} + F_{31}\begin{vmatrix} F_{12} & F_{13} \\ F_{22} & F_{23} \end{vmatrix} \\
&= F_{11}(e_{1jk}F_{j2}F_{k3}) - F_{21}(-e_{2jk}F_{j2}F_{k3}) + F_{31}(e_{3jk}F_{j2}F_{k3}) \\
&= e_{1jk}F_{11}F_{j2}F_{k3} + e_{2jk}F_{21}F_{j2}F_{k3} + e_{3jk}F_{31}F_{j2}F_{k3} \\
&= e_{ijk}F_{i1}F_{j2}F_{k3}
\end{aligned} \tag{3.19}$$

$$\det(\boldsymbol{F}) = e_{ijk}F_{1i}F_{2j}F_{3k} \tag{3.20}$$

$$\begin{aligned}
\det(\boldsymbol{F}) &= \begin{vmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{vmatrix} = F_{11}\begin{vmatrix} F_{22} & F_{23} \\ F_{32} & F_{33} \end{vmatrix} - F_{12}\begin{vmatrix} F_{21} & F_{23} \\ F_{31} & F_{33} \end{vmatrix} + F_{13}\begin{vmatrix} F_{21} & F_{22} \\ F_{31} & F_{32} \end{vmatrix} \\
&= F_{11}(e_{1jk}F_{2j}F_{3k}) - F_{12}(-e_{2jk}F_{2j}F_{3k}) + F_{13}(e_{3jk}F_{2j}F_{3k}) \\
&= e_{1jk}F_{11}F_{2j}F_{3k} + e_{2jk}F_{12}F_{2j}F_{3k} + e_{3jk}F_{13}F_{2j}F_{3k} \\
&= e_{ijk}F_{1i}F_{2j}F_{3k}
\end{aligned} \tag{3.21}$$

$$\begin{aligned}
e_{lmn}\det(\boldsymbol{F}) &= e_{lmn}e_{ijk}F_{i1}F_{j2}F_{k3} = e_{ijk}F_{il}F_{jm}F_{kn} \\
&= e_{lmn}e_{ijk}F_{1i}F_{2j}F_{3k} = e_{ijk}F_{li}F_{mj}F_{nk}
\end{aligned} \tag{3.22}$$

$$e_{lmn}e_{ijk} = \begin{vmatrix} \delta_{li} & \delta_{lj} & \delta_{lk} \\ \delta_{mi} & \delta_{mj} & \delta_{mk} \\ \delta_{ni} & \delta_{nj} & \delta_{nk} \end{vmatrix} \tag{3.23}$$

$$\begin{aligned}
&e_{lmn}\det(\boldsymbol{F}) \\
&= e_{lmn}e_{ijk}F_{i1}F_{j2}F_{k3} \\
&= F_{i1}F_{j2}F_{k3}[\delta_{li}\delta_{mj}\delta_{nk} + \delta_{lj}\delta_{mk}\delta_{ni} + \delta_{lk}\delta_{mi}\delta_{nj} - \delta_{lk}\delta_{mj}\delta_{ni} - \delta_{mk}\delta_{nj}\delta_{li} - \delta_{nk}\delta_{lj}\delta_{mi}] \\
&= [F_{l1}F_{m2}F_{n3} + F_{n1}F_{l2}F_{m3} + F_{m1}F_{n2}F_{l3} - F_{n1}F_{m2}F_{l3} - F_{l1}F_{n2}F_{m3} - F_{m1}F_{l2}F_{n3}] \\
&= F_{l1}(F_{m2}F_{n3} - F_{m3}F_{n2}) + F_{l2}(F_{m3}F_{n1} - F_{m1}F_{n3}) + F_{l3}(F_{m1}F_{n2} - F_{m2}F_{n1}) \\
&= F_{l1}(e_{1jk}F_{mj}F_{nk}) + F_{l2}(e_{2jk}F_{mj}F_{nk}) + F_{l3}(e_{3jk}F_{mj}F_{nk}) \\
&= e_{1jk}F_{l1}F_{mj}F_{nk} + e_{1jk}F_{l2}F_{mj}F_{nk} + e_{3jk}F_{l3}F_{mj}F_{nk} \\
&= e_{ijk}F_{li}F_{mj}F_{nk}
\end{aligned} \tag{3.24}$$

$$\begin{aligned}
&e_{lmn}\det(\boldsymbol{F}) \\
&= e_{lmn}e_{ijk}F_{1i}F_{2j}F_{3k} \\
&= F_{1i}F_{2j}F_{3k}[\delta_{li}\delta_{mj}\delta_{nk} + \delta_{lj}\delta_{mk}\delta_{ni} + \delta_{lk}\delta_{mi}\delta_{nj} - \delta_{lk}\delta_{mj}\delta_{ni} - \delta_{mk}\delta_{nj}\delta_{li} - \delta_{nk}\delta_{lj}\delta_{mi}] \\
&= [F_{1l}F_{2m}F_{3n} + F_{1n}F_{2l}F_{3m} + F_{1m}F_{2n}F_{3l} - F_{1n}F_{2m}F_{3l} - F_{1l}F_{2n}F_{3m} - F_{1m}F_{2l}F_{3n}] \\
&= F_{1l}(F_{2m}F_{3n} - F_{3m}F_{2n}) + F_{2l}(F_{3m}F_{1n} - F_{1m}F_{3n}) + F_{3l}(F_{1m}F_{2n} - F_{2m}F_{1n}) \\
&= F_{1l}(e_{1jk}F_{jm}F_{kn}) + F_{2l}(e_{2jk}F_{jm}F_{kn}) + F_{3l}(e_{3jk}F_{jm}F_{kn}) \\
&= e_{1jk}F_{1l}F_{jm}F_{kn} + e_{1jk}F_{2l}F_{jm}F_{kn} + e_{3jk}F_{3l}F_{jm}F_{kn} \\
&= e_{ijk}F_{il}F_{jm}F_{kn}
\end{aligned} \tag{3.25}$$

$$\begin{aligned}
e_{lmn}\det(\boldsymbol{F}) &= e_{lmn}e_{ijk}F_{1i}F_{2j}F_{3k} = e_{ijk}F_{il}F_{jm}F_{kn} \\
&= e_{lmn}e_{ijk}F_{i1}F_{j2}F_{k3} = e_{ijk}F_{li}F_{mj}F_{nk}
\end{aligned} \tag{3.26}$$

$$e_{lmn}e_{lmn}\det(\boldsymbol{F}) = e_{lmn}e_{ijk}F_{il}F_{jm}F_{kn} \tag{3.27}$$

$$e_{lmn}e_{lmn} = \delta_{mm}\delta_{nn} - \delta_{mn}\delta_{nm} = 9 - 3 = 6 \tag{3.28}$$

$$\det(\boldsymbol{F}) = \frac{1}{6}e_{ijk}e_{lmn}F_{il}F_{jm}F_{kn} \tag{3.29}$$

一些性质：

$$\det(\boldsymbol{F}) = \det(\boldsymbol{F}^T) \tag{3.30}$$

$$\det(\boldsymbol{A}\cdot\boldsymbol{B}) = \det(\boldsymbol{A})\det(\boldsymbol{B}) \tag{3.31}$$

$$\det(\alpha\boldsymbol{A}) = \alpha^3\det(\boldsymbol{A}) \tag{3.32}$$

3.1.12 张量的分解

加法分解

（1）对称反对称分解（2）静偏分解（3）其他

2.乘法分解

两种正交分解方法：

1. 对称反对称分解。例： $\varepsilon_{ij} = \varepsilon_{ij}^s + \varepsilon_{ij}^a = \frac{1}{2}(\varepsilon_{ij} + \varepsilon_{ji}) + \frac{1}{2}(\varepsilon_{ij} - \varepsilon_{ji})$
2. 静水偏量分解。例： $\varepsilon_{ij} = \varepsilon_{ij}^h + \varepsilon_{ij}^{'} = \frac{1}{3}\delta_{ij}\varepsilon_{kk}(\text{迹平均}) + \varepsilon_{ij}^{'}$

3.1.13 特殊张量及其性质

3.1.13.0.1 对称张量及反对称张量

二阶对称张量

$$\boldsymbol{S} = \boldsymbol{S}^T \qquad S_{ij} = S_{ji}$$

二阶反对称张量

$$\boldsymbol{A} = -\boldsymbol{A}^T \qquad A_{ij} = -A_{ji}$$

二阶对称张量与反对称张量的二次缩并（双点积）为零。

证明1：

$$\boldsymbol{S}:\boldsymbol{A} = tr(\boldsymbol{S}\boldsymbol{A}) = tr(\boldsymbol{A}^{\boldsymbol{T}}\boldsymbol{S}^{\boldsymbol{T}}) = tr(-\boldsymbol{A}\boldsymbol{S}) = -tr(\boldsymbol{A}\boldsymbol{S}) = -tr(\boldsymbol{S}\boldsymbol{A})$$

所以

$$tr(\boldsymbol{S}\boldsymbol{A}) = 0 \rightarrow \boldsymbol{S}:\boldsymbol{A} = 0$$

证明2：

$$\boldsymbol{S}:\boldsymbol{A} = tr(\boldsymbol{S}\boldsymbol{A}) = S_{ij}A_{ij} = -S_{ji}A_{ji} = -S_{ij}A_{ij}$$

所以，

$$S_{ij}A_{ij} = 0 \rightarrow \boldsymbol{S}:\boldsymbol{A} = 0$$

其中，最后一个等号用到了哑标的指标替换。

3.1.13.0.2 两个基本张量：零张量和单位张量

$\boldsymbol{0}$: 零张量. $\forall\boldsymbol{v}\in\mathbb{R}^3, \boldsymbol{0}\boldsymbol{v} = \boldsymbol{0}$. 零张量作用于矢量$\boldsymbol{v}$得到零矢量$\boldsymbol{0}$.

$\boldsymbol{I}$: 单位张量. $\forall\boldsymbol{v}\in\mathbb{R}^3, \boldsymbol{I}\boldsymbol{v} = \boldsymbol{v}$. 单位张量作用于矢量$\boldsymbol{v}$得到矢量$\boldsymbol{v}$本身.

3.2 张量微分（component form）

关键技术：链式法则、（利用哑标的可换性）凑指标、张量对称性

克罗内克符号和哑标：克罗内克符号可以将一个张量里的自由指标转化为哑标，反之，亦可以使一个张量中的哑标转化为自由指标。

$$\delta_{ij}\eta_{ijk} = \eta_{nnk}, \qquad \eta_{nnk} = \delta_{ij}\eta_{ijk}$$

应变梯度张量关于后两个指标对称

$$\eta_{ijk} = \eta_{ikj}$$

例子，因为偏导数下面是η_{ijk}，所以可以考虑在偏导数上面凑ijk，即将η_{iik}, η_{kjj}等凑成η_{ijk}，然后$\frac{\partial \eta_{ijk}}{\partial \eta_{ijk}} = 1$

$$\begin{aligned}
\frac{\partial \eta_{iik}\eta_{kjj}}{\partial \eta_{ijk}} &= \eta_{iik}\frac{\partial \eta_{kjj}}{\partial \eta_{ijk}} + \eta_{kjj}\frac{\partial \eta_{iik}}{\partial \eta_{ijk}} \\
&= \eta_{nnk}\frac{\partial \eta_{kjj}}{\partial \eta_{ijk}} + \eta_{kmm}\frac{\partial \eta_{iik}}{\partial \eta_{ijk}} = \eta_{nnk}\frac{\partial \delta_{ij}\eta_{kij}}{\partial \eta_{ijk}} + \eta_{kmm}\frac{\partial \delta_{ij}\eta_{ijk}}{\partial \eta_{ijk}} \\
&= \eta_{nnk}\frac{\delta_{ij}\partial \eta_{kij}}{\partial \eta_{ijk}} + \eta_{kmm}\frac{\delta_{ij}\partial \eta_{ijk}}{\partial \eta_{ijk}} = \eta_{nnr}\frac{\delta_{st}\partial \eta_{rst}}{\partial \eta_{ijk}} + \eta_{kmm}\frac{\delta_{ij}\partial \eta_{ijk}}{\partial \eta_{ijk}} \\
&= \eta_{nni}\frac{\delta_{jk}\partial \eta_{ijk}}{\partial \eta_{ijk}} + \eta_{kmm}\frac{\delta_{ij}\partial \eta_{ijk}}{\partial \eta_{ijk}} = \eta_{nni}\delta_{jk} + \eta_{kmm}\delta_{ij}
\end{aligned}$$

注意!!! 需要凑自变量的指标（这里是η_{ijk}），但是不用管自变量的指标，例如上面标蓝色这一项$\eta_{nni}\frac{\delta_{jk}\partial\eta_{ijk}}{\partial\eta_{ijk}}$，这样写本身并不合理，因为$i, j, k$分别出现了三次，按理来说不是哑标。但实际上，我们不用管自变量的指标，隐去自变量，其他的指标(该项的红色部分$\eta_{nni}\frac{\delta_{jk}\partial\eta_{ijk}}{\partial\eta_{ijk}}$)出现一次就按自由指标算，出现两次就按哑标算。

克罗内克符号和置换符号均可直接提到偏导数符号之外

$$\frac{\partial \delta_{ij}\eta_{ijk}}{\partial \eta_{ijk}} = \frac{\delta_{ij}\partial \eta_{ijk}}{\partial \eta_{ijk}} = \delta_{ij}\frac{\partial \eta_{ijk}}{\partial \eta_{ijk}} = \delta_{ij}$$

$$\frac{\partial e_{ijk}\eta_{ijk}}{\partial \eta_{ijk}} = \frac{e_{ijk}\partial \eta_{ijk}}{\partial \eta_{ijk}} = e_{ijk}\frac{\partial \eta_{ijk}}{\partial \eta_{ijk}} = e_{ijk}$$

广义胡克定律

$$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}$$

应变能密度函数

$$U = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} = \frac{1}{2}\lambda\varepsilon_{kk}\varepsilon_{nn} + \mu\varepsilon_{ij}\varepsilon_{ij}$$

满足

$$\sigma_{ij} = \frac{\partial U}{\partial \varepsilon_{ij}}$$

To see this:

$$\begin{aligned}
\frac{\partial U}{\partial \varepsilon_{ij}} &= \frac{1}{2}\lambda\frac{\partial \varepsilon_{kk}\varepsilon_{nn}}{\partial \varepsilon_{ij}} + \mu\frac{\partial \varepsilon_{ij}\varepsilon_{ij}}{\partial \varepsilon_{ij}} = \frac{1}{2}\lambda(\varepsilon_{kk}\frac{\partial \varepsilon_{nn}}{\partial \varepsilon_{ij}} + \varepsilon_{nn}\frac{\partial \varepsilon_{mm}}{\partial \varepsilon_{ij}}) + \mu(\varepsilon_{ij}\frac{\partial \varepsilon_{ij}}{\partial \varepsilon_{ij}} + \varepsilon_{ij}\frac{\partial \varepsilon_{ij}}{\partial \varepsilon_{ij}}) \\
&= \frac{1}{2}\lambda(\varepsilon_{kk}\frac{\partial \delta_{ij}\varepsilon_{ij}}{\partial \varepsilon_{ij}} + \varepsilon_{nn}\frac{\partial \delta_{ij}\varepsilon_{ij}}{\partial \varepsilon_{ij}}) + \mu(\varepsilon_{ij} + \varepsilon_{ij}) = \frac{1}{2}\lambda(\varepsilon_{kk}\delta_{ij} + \varepsilon_{nn}\delta_{ij}) + \mu(\varepsilon_{ij} + \varepsilon_{ij}) \\
&= \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} = \sigma_{ij}
\end{aligned}$$

也就是说，构建的应变能密度函数必须满足：通过功共轭关系，可以得到本构关系。

3.3 张量场论：梯度、散度、旋度

参考《Introduction to the Mechanics of a Continuous Medium by Lawrence E. Malvern》

3.3.1 Gradient

左梯度的基矢量在偏导符号的左边。

（左）梯度：在首端增加一个维度

$$\nabla \boldsymbol{u} = \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}} = \boldsymbol{e}_k \otimes \frac{\partial \boldsymbol{u}}{\partial x_k} = \boldsymbol{e}_k \otimes \frac{\partial u_i \boldsymbol{e}_i}{\partial x_k} = \frac{\partial u_i}{\partial x_k} \boldsymbol{e}_k \otimes \boldsymbol{e}_i = \partial_k u_i \boldsymbol{e}_k \otimes \boldsymbol{e}_i = u_{i,k} \boldsymbol{e}_k \otimes \boldsymbol{e}_i = u_{j,i} \boldsymbol{e}_i \otimes \boldsymbol{e}_j$$

右梯度的基矢量在偏导符号的右边。

（右）梯度:：在尾段增加一个维度 (**常用**)

$$\boldsymbol{u} \nabla = \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}} = \frac{\partial \boldsymbol{u}}{\partial x_k} \otimes \boldsymbol{e}_k = \frac{\partial u_i \boldsymbol{e}_i}{\partial x_k} \otimes \boldsymbol{e}_k = \frac{\partial u_i}{\partial x_k} \boldsymbol{e}_i \otimes \boldsymbol{e}_k = u_{i,k} \boldsymbol{e}_i \otimes \boldsymbol{e}_k = u_{i,j} \boldsymbol{e}_i \otimes \boldsymbol{e}_j$$

也有反着定义的。

3.3.2 Divergence

左散度的基矢量在偏导符号的右边。

（左）散度：在首段减小一个维度（**常用**）

$$\nabla \cdot \boldsymbol{\sigma} = \boldsymbol{e}_k \cdot \frac{\partial \boldsymbol{\sigma}}{\partial x_k} = \boldsymbol{e}_k \cdot \frac{\partial \sigma_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j}{\partial x_k} = \frac{\partial \sigma_{ij}}{\partial x_k} \boldsymbol{e}_k \cdot \boldsymbol{e}_i \otimes \boldsymbol{e}_j = \frac{\partial \sigma_{ij}}{\partial x_k} \delta_{ki} \boldsymbol{e}_j = \frac{\partial \sigma_{ij}}{\partial x_i} \boldsymbol{e}_j$$

右散度的基矢量在偏导符号的左边。

（右）散度：在尾段减小一个维度

$$\boldsymbol{\sigma} \cdot \nabla = \frac{\partial \boldsymbol{\sigma}}{\partial x_k} \cdot \boldsymbol{e}_k = \frac{\partial \sigma_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j}{\partial x_k} \cdot \boldsymbol{e}_k = \frac{\partial \sigma_{ij}}{\partial x_k} \boldsymbol{e}_i \otimes \boldsymbol{e}_j \cdot \boldsymbol{e}_k = \frac{\partial \sigma_{ij}}{\partial x_k} \delta_{jk} \boldsymbol{e}_i = \frac{\partial \sigma_{ij}}{\partial x_j} \boldsymbol{e}_i$$

高斯定理、散度定理（divergence theorem）

电通量、电位移矢量$\boldsymbol{D}$、散度

$$Q_f = \int_V \nabla \cdot \boldsymbol{D} dv = \int_{\partial V} d\boldsymbol{s} \cdot \boldsymbol{D} \to \boldsymbol{n} \cdot \boldsymbol{D} = q_f$$

其中，Q_f为自由电荷的电荷量，q_f为自由电荷的电荷面密度。

弹性力学平衡微分方程

由弹性体的静力平衡

$$\int_{\partial V} \boldsymbol{t} ds + \int_V \boldsymbol{f} dV = 0$$

其中$\boldsymbol{t}$为面力矢量，$\boldsymbol{f}$为体力矢量。

由柯西定理

$$\int_{\partial V} \boldsymbol{n} \cdot \boldsymbol{\sigma} ds + \int_V \boldsymbol{f} dV = 0$$

二阶张量高斯公式为

$$\int_V \nabla \cdot \boldsymbol{\sigma} dv = \int_{\partial V} d\boldsymbol{s} \cdot \boldsymbol{\sigma}$$

所以

$$\int_V \nabla \cdot \boldsymbol{\sigma} dv + \int_V \boldsymbol{f} dV = 0 \to \int_V \nabla \cdot \boldsymbol{\sigma} + \boldsymbol{f} = 0$$

因为积分区域可以任意选择，故要求被积函数为0，所以可得弹性力学平衡微分方程

$$\nabla \cdot \boldsymbol{\sigma} + \boldsymbol{f} = \boldsymbol{0}$$

3.3.3 Curl

$$\boldsymbol{e}_i \times \boldsymbol{e}_j = \epsilon_{ijk}\boldsymbol{e}_k$$

$$\boldsymbol{e}_{ijk}\boldsymbol{e}_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$$

左旋度的基矢量在偏导数的左边

左旋度:

$$\begin{aligned}\boldsymbol{w} &= \frac{1}{2}\nabla \times \boldsymbol{u} \\ &= \frac{1}{2}\boldsymbol{e}_k \times \frac{\partial u_i \boldsymbol{e}_i}{\partial x_k} = \frac{1}{2}\frac{\partial u_i}{\partial x_k}\boldsymbol{e}_k \times \boldsymbol{e}_i = \frac{1}{2}\frac{\partial u_i}{\partial x_k}\epsilon_{kij}\boldsymbol{e}_j = \frac{1}{2}\frac{\partial u_s}{\partial x_t}\epsilon_{tsi}\boldsymbol{e}_i = \frac{1}{2}\frac{\partial u_s}{\partial x_t}\epsilon_{its}\boldsymbol{e}_i \\ &= \frac{1}{2}\epsilon_{its}\partial_t u_s \boldsymbol{e}_i = \frac{1}{2}\epsilon_{its}u_{s,t}\boldsymbol{e}_i\end{aligned}$$

左旋度

$$\nabla \times \boldsymbol{\varepsilon} = \boldsymbol{e}_k \times \frac{\partial \boldsymbol{\varepsilon}}{\partial x_k} = \boldsymbol{e}_k \times \frac{\partial \varepsilon_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j}{\partial x_k} = \frac{\partial \varepsilon_{ij}}{\partial x_k}\boldsymbol{e}_k \times \boldsymbol{e}_i \otimes \boldsymbol{e}_j = \epsilon_{kim}\frac{\partial \varepsilon_{ij}}{\partial x_k}\boldsymbol{e}_m \otimes \boldsymbol{e}_j$$

右旋度的基矢量在偏导数的右边

右旋度:

$$\boldsymbol{\varepsilon} \times \nabla = \frac{\partial \boldsymbol{\varepsilon}}{\partial x_k} \times \boldsymbol{e}_k = \frac{\partial \varepsilon_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j}{\partial x_k} \times \boldsymbol{e}_k = \frac{\partial \varepsilon_{ij}}{\partial x_k}\boldsymbol{e}_i \otimes \boldsymbol{e}_j \times \boldsymbol{e}_k = \epsilon_{jkm}\frac{\partial \varepsilon_{ij}}{\partial x_k}\boldsymbol{e}_i \otimes \boldsymbol{e}_m$$

斯托克斯定理（Stokes theorem）[2]

tensor form:

$$\int_S \nabla \times \boldsymbol{u} \cdot d\boldsymbol{s} = \oint_L \boldsymbol{u} \cdot d\boldsymbol{l} \quad \text{or} \quad \int_S Curl\boldsymbol{u} \cdot d\boldsymbol{s} = \oint_L \boldsymbol{u} \cdot d\boldsymbol{l}$$

component form:

$$\nabla \times \boldsymbol{u} = \boldsymbol{e}_k \times \frac{\partial u_i \boldsymbol{e}_i}{\partial x_k} = \frac{\partial u_i}{\partial x_k}\boldsymbol{e}_k \times \boldsymbol{e}_i = \frac{\partial u_i}{\partial x_k}\boldsymbol{e}_j = e_{kij}\frac{\partial u_i}{\partial x_k}\boldsymbol{e}_j = e_{kij}\partial_k u_i \boldsymbol{e}_j$$

then

$$\begin{aligned}&\int_S e_{kij}\partial_k u_i \boldsymbol{e}_j \cdot \boldsymbol{n} ds = \oint_L \boldsymbol{u} \cdot \boldsymbol{t} dl \\ &\to \int_S e_{kij}(\partial_k u_i) n_j ds = \oint_L u_i t_i dl \to \int_S e_{kij} n_j \partial_k u_i ds = \oint_L u_i t_i dl\end{aligned}$$

that is

$$\int_S e_{kij}(\partial_k u_i) n_j ds = \oint_L u_i t_i dl \quad \text{or} \quad \int_S e_{kij} n_j \partial_k u_i ds = \oint_L u_i t_i dl$$

[2]参考《Introduction to Electrodynamics by David J. Griffiths》

Chapter 4

连续介质力学初步

机理为本，概念为先。

追本溯源，寻根究底。

抓住关键，分清主次。

4.1 应变分析

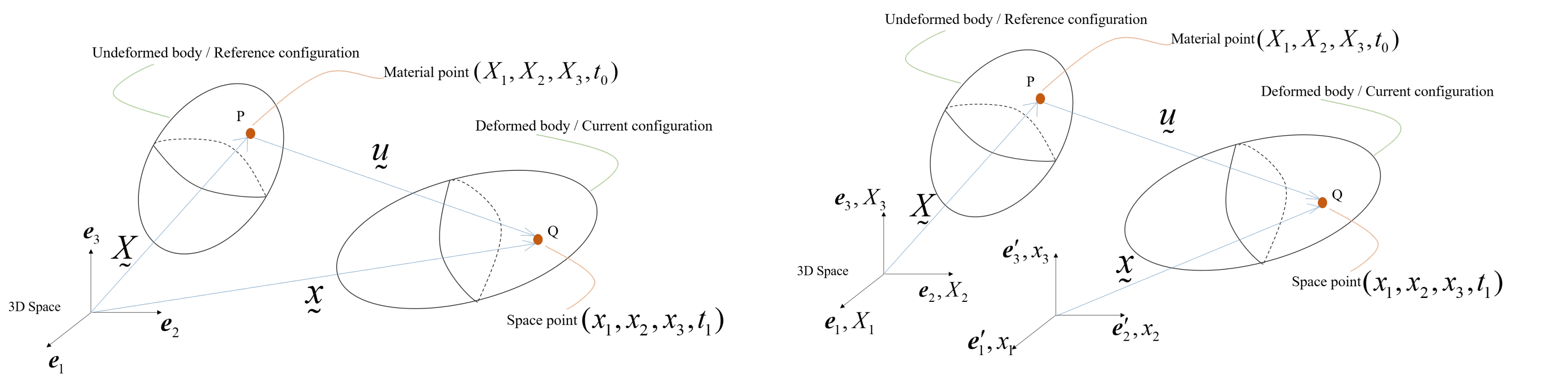
变形分析是研究所有固体力学问题的基础!!!

4.1.1 力的作用效果：物体运动及变形的根源

力的作用效果有两个：1.改变物体的运动状态。本质是物体速度的改变。物体可以离散成质点系，物体变形前的运动就是刚体运动。刚体运动包括平移和定轴转动。2.使物体发生变形。变形包括体积改变和形状改变。体积改变对应纯拉伸，形状改变对应纯剪切。所以，在力学中，力对物体的作用效果可以分解为：刚体运动+变形=平移+转动+纯拉伸+纯剪切。

4.1.2 变形映射(The deformation mapping)

假想将物体离散为质点系，**物质点**表示未变形的物体中的各质点所占据的空间点(**参考构型**)，**空间点**表示运动变形后的物体中各质点所占据的空间点(**当前构型**)，从而将对物质的研究转移到对空间点的研究上来，即考虑的是物质点系所占空间点系的变化，从而把真实存在的物质从问题中剥离出去了。物质点是特殊的空间点，只是它作为参考，称为物质点。



一个坐标系描述的构型变化(目前足够)

(a)

两个坐标系描述的构型变化(还需要考虑坐标系间的跃迁矢量)

(b)

参考构型和当前构型之间的空间点系是一一对应，形成映射。用 φ 表示这里的映射。

$$\boldsymbol{x} = \varphi(\boldsymbol{X}) = \boldsymbol{X} + \boldsymbol{u}(\boldsymbol{X}) = \boldsymbol{X} + \boldsymbol{u} \tag{4.1}$$

只要位移场不仅仅表示刚体运动，物体就会发生局部变形，可能是体积改变，也可能是形状改变。

4.1.3 局部变形描述(Description of local deformation)

泰勒展开一阶近似

$$\boldsymbol{x} + d\boldsymbol{x} = \varphi(\boldsymbol{X} + d\boldsymbol{X}) = \varphi(\boldsymbol{X}) + \frac{\partial \varphi(\boldsymbol{X})}{\partial \boldsymbol{X}} \cdot d\boldsymbol{X} \tag{4.2}$$

然后有

$$d\boldsymbol{x} = \frac{\partial \varphi(\boldsymbol{X})}{\partial \boldsymbol{X}} \cdot d\boldsymbol{X} = \boldsymbol{F} \cdot d\boldsymbol{X} \tag{4.3}$$

我们定义$\boldsymbol{F}$为变形梯度张量

$$\boldsymbol{F} = \frac{\partial \varphi(\boldsymbol{X})}{\partial \boldsymbol{X}} = \varphi\nabla = \boldsymbol{x}\nabla \tag{4.4}$$

变形梯度张量包含了材料局部转动和变形的完整信息。例如，未变形体中的小线段$d\boldsymbol{X}$如何旋转、拉伸，成为变形体中的线段$d\boldsymbol{x}$。现在，我们有变形前后的位置矢量，位移矢量，以及变形梯度张量。虽然，变形梯度张量包含了材料局部转动和变形的完整信息。然而，我们关注的是物体的变形信息。我们想通过某种方法将变形梯度张量中的转动信息和变形信息分离开来。我们可以通过对变形梯度张量的极分解实现这一目的。

4.1.3.1 变形梯度的极分解、以及应变张量的引出

极分解定理：任何二阶张量都可以分解为纯转动张量和对称张量的乘积。

极分解的主要目的：将物体的刚体转动和变形分开。

这里我们只考虑常用的变形体都张量右分解。变形梯度张量右分解后，作用于位置微元，可以使物体先发生变形，再发生刚体转动

$$\boldsymbol{F} = \boldsymbol{R} \cdot \boldsymbol{U} \qquad d\boldsymbol{x} = \boldsymbol{R} \cdot \boldsymbol{U} \cdot d\boldsymbol{X} = \boldsymbol{R} \cdot (\boldsymbol{U} \cdot d\boldsymbol{X}) \tag{4.5}$$

描述线段微元的变形，我们可以用其变形前后模的变化($d\boldsymbol{x} \sim d\boldsymbol{X}$)，也可以用其变形前后模的平方的变化($|d\boldsymbol{x}|^2 \sim |d\boldsymbol{X}|^2 = d\boldsymbol{x} \cdot d\boldsymbol{x} \sim d\boldsymbol{X} \cdot d\boldsymbol{X}$)。

我们想要我们可以在不确定旋转矩阵的情况下，也能计算与旋转无关的变形度量，那就要想办法消除转动张量$\boldsymbol{R}$。当我们用小线段模平方变化来表征变形度量时，可以达到这一目的。通过以下方式引出右柯西-格林应变张量$\boldsymbol{C}$的定义，可以描述物体在刚体转动之前的变形。

$$\begin{aligned} d\boldsymbol{x} \cdot d\boldsymbol{x} &= (\boldsymbol{F} \cdot d\boldsymbol{X}) \cdot (\boldsymbol{F} \cdot d\boldsymbol{X}) = d\boldsymbol{X} \cdot (\boldsymbol{F}^T \cdot \boldsymbol{F}) \cdot d\boldsymbol{X} \\ &= d\boldsymbol{X} \cdot [(\boldsymbol{R} \cdot \boldsymbol{U})^T \cdot (\boldsymbol{R} \cdot \boldsymbol{U})] \cdot d\boldsymbol{X} = d\boldsymbol{X} \cdot [\boldsymbol{U}^T \cdot \boldsymbol{R}^T \cdot \boldsymbol{R} \cdot \boldsymbol{U}] \cdot d\boldsymbol{X} = d\boldsymbol{X} \cdot \boldsymbol{U}^T \cdot \boldsymbol{U} \cdot d\boldsymbol{X} \\ &= d\boldsymbol{X} \cdot \boldsymbol{C} \cdot d\boldsymbol{X} \end{aligned} \tag{4.6}$$

对于未变形（仅作刚体运动）的物体，我们有$\boldsymbol{C} = \boldsymbol{I}$。进一步的，如果我们想要得到基于零的变形度量值，需要从$\boldsymbol{C}$中减去单位张量$\boldsymbol{I}$，从而引出格林-拉格朗日应变张量$\boldsymbol{E}$的定义

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{C} - \boldsymbol{I}) \tag{4.7}$$

该张量也可以描述物体在发生刚体转动之前产生的变形，但在未变形状态下的所有分量均为零。这里系数1/2的引入，只是为了保证各种应变定义在小变形情况下的一致性。

代入位移可得格林-拉格朗日应变和位移的关系

$$\begin{aligned} \boldsymbol{E} &= \frac{1}{2}[(\boldsymbol{I} + \boldsymbol{u}\nabla)^T \cdot (\boldsymbol{I} + \boldsymbol{u}\nabla) - \boldsymbol{I}] = \frac{1}{2}[(\boldsymbol{I}^T + (\boldsymbol{u}\nabla)^T) \cdot (\boldsymbol{I} + \boldsymbol{u}\nabla) - \boldsymbol{I}] \\ &= \frac{1}{2}[(\boldsymbol{I} + \boldsymbol{u}\nabla^T) \cdot (\boldsymbol{I} + \boldsymbol{u}\nabla) - \boldsymbol{I}] = \frac{1}{2}[\boldsymbol{I} + \boldsymbol{u}\nabla + \boldsymbol{u}\nabla^T + \boldsymbol{u}\nabla^T \cdot \boldsymbol{u}\nabla - \boldsymbol{I}] \\ &= \frac{1}{2}[\boldsymbol{u}\nabla + \boldsymbol{u}\nabla^T + \boldsymbol{u}\nabla^T \cdot \boldsymbol{u}\nabla] \\ &= \frac{1}{2}[\boldsymbol{u}\nabla + \boldsymbol{u}\nabla^T + \boldsymbol{u}\nabla^T\boldsymbol{u}\nabla] \end{aligned}$$

当应变和刚体转动的幅度都很小时，格林-拉格朗日应变张量中的二次项可以忽略不记。由此可得众所周知的小应变张量

$$\boldsymbol{\varepsilon} = \frac{1}{2}[\boldsymbol{u}\nabla + \boldsymbol{u}\nabla^T]$$

实际操作中，极分解的计算成本往往较高。因此，人们会尽量避免执行此类计算。但在理论思考方面，这一概念非常有用。

若只考虑x方向应变

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} \tag{4.8}$$

然而，由弹性力学(微元、泰勒展开)

$$u(x + dx) = u(x) + \frac{\partial u}{\partial x}dx \tag{4.9}$$

小应变定义为

$$\varepsilon_{xx} = \frac{\Delta L}{L} = \frac{u(x + dx) - u(x)}{dx} = \frac{\partial u}{\partial x} \tag{4.10}$$

由此，小应变中的$\frac{1}{2}$使得小应变定义和一维定义相合。

4.1.3.2 一维情形下应变的定义

变形度量：变形后的长度

$$L + \Delta L \tag{4.11}$$

无量纲化的变形度量：伸长率

$$\lambda = \frac{L + \Delta L}{L} = 1 + \frac{\Delta L}{L} \tag{4.12}$$

基于零的变形度量：应变

$$\varepsilon = \frac{L + \Delta L - L}{L} = \frac{\Delta L}{L} = \lambda - 1 \tag{4.13}$$

以上即依据线段长度定义应变。

对于小变形情形

$$\varepsilon = \frac{\Delta L}{L} \ll 1 \tag{4.14}$$

若用变形后的长度进行无量纲化

$$\varepsilon' = \frac{L + \Delta L - L}{L + \Delta L} = \frac{\Delta L}{L + \Delta L} = \frac{\varepsilon}{1 + \varepsilon} \doteq \varepsilon(1 - \varepsilon) = \varepsilon - \varepsilon^2 \doteq \varepsilon \tag{4.15}$$

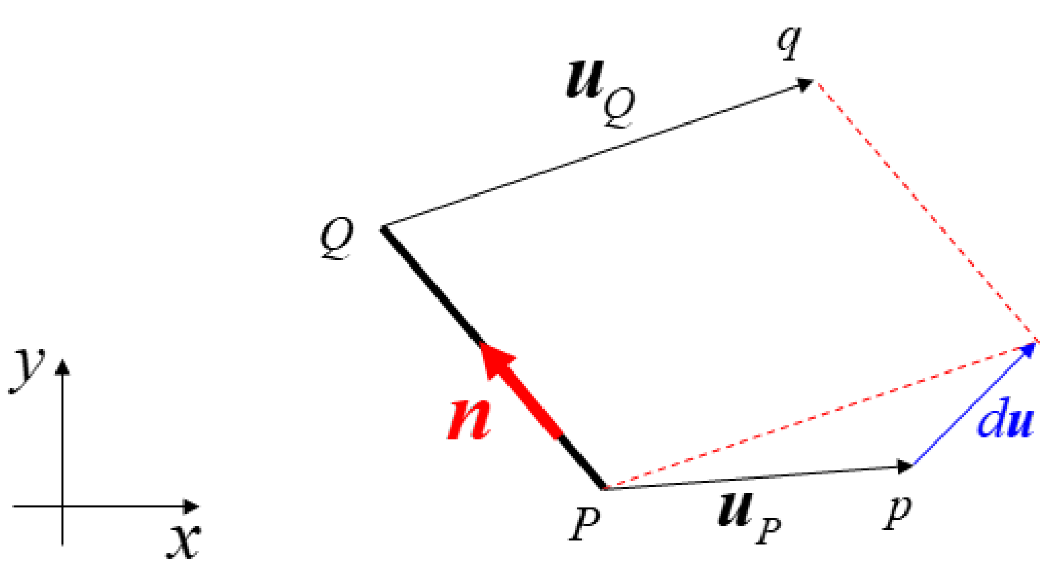
若依据线段长度的平方定义应变，在小变形情况下，为了与以上定义相合，需添加系数$\frac{1}{2}$，且小变形情况下可省略高阶项

$$\varepsilon = \frac{1}{2}\frac{(L+\delta L)^2 - L^2}{L^2} = \frac{1}{2}\frac{L^2 + 2L\Delta L + (\Delta L)^2 - L^2}{L^2} \doteq \frac{\Delta L}{L} \tag{4.16}$$

为什么分母上要加个2? 此处分母上加2的目的是为了保持小变形情况下应变的归一性，即使得各种应变的定义在小变形情况下保持相同。用别的数可以吗?5可以吗?自然是可以的，只是小变形情况下各种应变就不统一了(按理说，能用于大变形的应变定义应该也能用于小变形才是，你退化到小变形，和原来的定义不统一就不是很合适)。

4.1.4 位移梯度的分解：小应变张量和小转动张量

4.1.4.1 相对位移(Relative displacement)



相对位移(Relative displacement)

$$d\boldsymbol{u} = \boldsymbol{u}_Q - \boldsymbol{u}_P \tag{4.17}$$

单位相对位移(Unit relative displacement)

$$\frac{d\boldsymbol{u}}{dL} \tag{4.18}$$

引入直角坐标(Introduce the rectangular coordinates)

$$\boldsymbol{u} = u_x\boldsymbol{e}_x + u_y\boldsymbol{e}_y \tag{4.19}$$

方向矢量(direction vector)

$$\boldsymbol{n} = \frac{dx}{dL}\boldsymbol{e}_x + \frac{dy}{dL}\boldsymbol{e}_y \tag{4.20}$$

$$\begin{aligned}
\frac{d\boldsymbol{u}}{dL} &= \frac{du_x}{dL}\boldsymbol{e}_x + \frac{du_x}{dL}\boldsymbol{e}_y \\
&= \left[\frac{\partial u_x}{\partial x}\frac{dx}{dL} + \frac{\partial u_x}{\partial y}\frac{dy}{dL}\right]\boldsymbol{e}_x + \left[\frac{\partial u_y}{\partial x}\frac{dx}{dL} + \frac{\partial u_y}{\partial y}\frac{dy}{dL}\right]\boldsymbol{e}_y \\
&= \left[(\frac{\partial u_x}{\partial x}\boldsymbol{e}_x + \frac{\partial u_x}{\partial y}\boldsymbol{e}_y)\cdot(\frac{dx}{dL}\boldsymbol{e}_x + \frac{dy}{dL}\boldsymbol{e}_y)\right]\boldsymbol{e}_x + \left[(\frac{\partial u_y}{\partial x}\boldsymbol{e}_x + \frac{\partial u_y}{\partial y}\boldsymbol{e}_y)\cdot(\frac{dx}{dL}\boldsymbol{e}_x + \frac{dy}{dL}\boldsymbol{e}_y)\right]\boldsymbol{e}_y \\
&= \left[\boldsymbol{e}_x\otimes(\frac{\partial u_x}{\partial x}\boldsymbol{e}_x + \frac{\partial u_x}{\partial y}\boldsymbol{e}_y)\right]\cdot(\frac{dx}{dL}\boldsymbol{e}_x + \frac{dy}{dL}\boldsymbol{e}_y) + \left[\boldsymbol{e}_y\otimes(\frac{\partial u_y}{\partial x}\boldsymbol{e}_x + \frac{\partial u_y}{\partial y}\boldsymbol{e}_y)\right]\cdot(\frac{dx}{dL}\boldsymbol{e}_x + \frac{dy}{dL}\boldsymbol{e}_y) \\
&= \left[\boldsymbol{e}_x\otimes(\frac{\partial u_x}{\partial x}\boldsymbol{e}_x + \frac{\partial u_x}{\partial y}\boldsymbol{e}_y)\right]\cdot\boldsymbol{n} + \left[\boldsymbol{e}_y\otimes(\frac{\partial u_y}{\partial x}\boldsymbol{e}_x + \frac{\partial u_y}{\partial y}\boldsymbol{e}_y)\right]\cdot\boldsymbol{n} \\
&= \left[\frac{\partial u_x}{\partial x}\boldsymbol{e}_x\otimes\boldsymbol{e}_x + \frac{\partial u_x}{\partial y}\boldsymbol{e}_x\otimes\boldsymbol{e}_y + \frac{\partial u_y}{\partial x}\boldsymbol{e}_y\otimes\boldsymbol{e}_x + \frac{\partial u_y}{\partial y}\boldsymbol{e}_y\otimes\boldsymbol{e}_y\right]\cdot\boldsymbol{n} \\
&= \boldsymbol{u}\nabla\cdot\boldsymbol{n} = \boldsymbol{J}\cdot\boldsymbol{n}
\end{aligned} \tag{4.21}$$

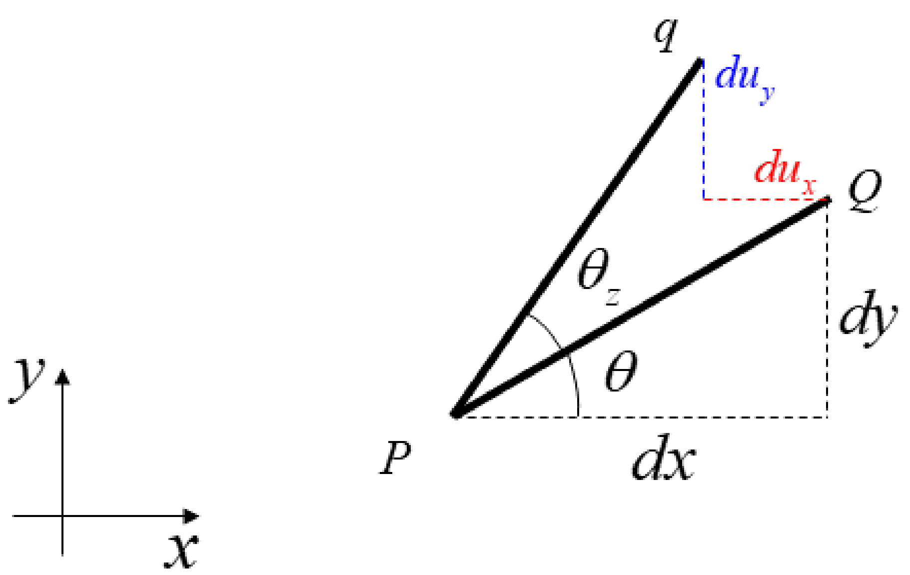
其中，$\boldsymbol{J} = \boldsymbol{u}\nabla$是雅可比矩阵(Jocabian)或单位相对位移矩阵(unit relative displacement matrix)。

$$\boldsymbol{u}\nabla = \frac{1}{2}(\boldsymbol{u}\nabla + \nabla\boldsymbol{u}) + \frac{1}{2}(\boldsymbol{u}\nabla - \nabla\boldsymbol{u}) = \boldsymbol{E} + \boldsymbol{\Omega} \tag{4.22}$$

其中，$\boldsymbol{E}$是小应变张量(small strain tensor), $\boldsymbol{\Omega}$是小转动张量(small rotation tensor)。

$$
\begin{aligned}
\boldsymbol{E} &= E_{xx}\boldsymbol{e}_x\otimes\boldsymbol{e}_x + E_{xy}\boldsymbol{e}_x\otimes\boldsymbol{e}_y + E_{yx}\boldsymbol{e}_y\otimes\boldsymbol{e}_x + E_{yy}\boldsymbol{e}_y\otimes\boldsymbol{e}_y\\
&= u_{x,x}\boldsymbol{e}_x\otimes\boldsymbol{e}_x + \frac{1}{2}(u_{x,y}+u_{y,x})\boldsymbol{e}_x\otimes\boldsymbol{e}_y + \frac{1}{2}(u_{y,x}+u_{x,y})\boldsymbol{e}_y\otimes\boldsymbol{e}_x + u_{y,y}\boldsymbol{e}_y\otimes\boldsymbol{e}_y\\
\boldsymbol{\Omega} &= \Omega_{xx}\boldsymbol{e}_x\otimes\boldsymbol{e}_x + \Omega_{xy}\boldsymbol{e}_x\otimes\boldsymbol{e}_y + \Omega_{yx}\boldsymbol{e}_y\otimes\boldsymbol{e}_x + \Omega_{yy}\boldsymbol{e}_y\otimes\boldsymbol{e}_y\\
&= \Omega_{xy}\boldsymbol{e}_x\otimes\boldsymbol{e}_y + \Omega_{yx}\boldsymbol{e}_y\otimes\boldsymbol{e}_x\\
&= \Omega_{xy}\boldsymbol{e}_x\otimes\boldsymbol{e}_y - \Omega_{xy}\boldsymbol{e}_y\otimes\boldsymbol{e}_x\\
&= \frac{1}{2}(u_{x,y}-u_{y,x})\boldsymbol{e}_x\otimes\boldsymbol{e}_y + \frac{1}{2}(u_{y,x}-u_{x,y})\boldsymbol{e}_y\otimes\boldsymbol{e}_x
\end{aligned}
\tag{4.23}
$$

4.1.4.2 纯转动(pure rotation)



$$
\begin{aligned}
d\boldsymbol{u} &= d\boldsymbol{u}_x + d\boldsymbol{u}_y = du_x\boldsymbol{e}_x + du_y\boldsymbol{e}_y\\
&= [dLcos(\theta_z+\theta) - dLcos\theta]\,\boldsymbol{e}_x + [dLsin(\theta_z+\theta) - dLsin\theta]\,\boldsymbol{e}_y\\
&= [dL(cos\theta_z cos\theta - sin\theta_z sin\theta) - dLcos\theta]\,\boldsymbol{e}_x + [dL(sin\theta_z cos\theta + cos\theta_z sin\theta) - dLsin\theta]\,\boldsymbol{e}_y\\
&= [dL(cos\theta_z - 1)cos\theta - dLsin\theta_z sin\theta]\,\boldsymbol{e}_x + [dLsin\theta_z cos\theta + dL(cos\theta_z-1)sin\theta]\,\boldsymbol{e}_y\\
&= dL\,[(cos\theta_z-1)cos\theta - sin\theta_z sin\theta]\,\boldsymbol{e}_x + dL\,[sin\theta_z cos\theta + (cos\theta_z-1)sin\theta]\,\boldsymbol{e}_y
\end{aligned}
\tag{4.24}
$$

$$
\begin{aligned}
\frac{d\boldsymbol{u}}{dL} &= [(cos\theta_z-1)cos\theta - sin\theta_z sin\theta]\,\boldsymbol{e}_x + [sin\theta_z cos\theta + (cos\theta_z-1)sin\theta]\,\boldsymbol{e}_y\\
&= ([(cos\theta_z-1)\boldsymbol{e}_x - sin\theta_z\boldsymbol{e}_y]\cdot[cos\theta\boldsymbol{e}_x + sin\theta\boldsymbol{e}_y])\,\boldsymbol{e}_x + ([sin\theta_z\boldsymbol{e}_x + (cos\theta_z-1)\boldsymbol{e}_y]\cdot[cos\theta\boldsymbol{e}_x + sin\theta\boldsymbol{e}_y])\,\boldsymbol{e}_y\\
&= ([(cos\theta_z-1)\boldsymbol{e}_x - sin\theta_z\boldsymbol{e}_y]\cdot\boldsymbol{n})\,\boldsymbol{e}_x + ([sin\theta_z\boldsymbol{e}_x + (cos\theta_z-1)\boldsymbol{e}_y]\cdot\boldsymbol{n})\,\boldsymbol{e}_y\\
&= (\boldsymbol{e}_x\otimes[(cos\theta_z-1)\boldsymbol{e}_x - sin\theta_z\boldsymbol{e}_y])\cdot\boldsymbol{n} + (\boldsymbol{e}_y\otimes[sin\theta_z\boldsymbol{e}_x + (cos\theta_z-1)\boldsymbol{e}_y])\cdot\boldsymbol{n}\\
&= [(cos\theta_z-1)\boldsymbol{e}_x\otimes\boldsymbol{e}_x - sin\theta_z\boldsymbol{e}_x\otimes\boldsymbol{e}_y + sin\theta_z\boldsymbol{e}_y\otimes\boldsymbol{e}_x + (cos\theta_z-1)\boldsymbol{e}_y\otimes\boldsymbol{e}_y]\cdot\boldsymbol{n}
\end{aligned}
\tag{4.25}
$$

Assume θ_z is small, $sin\theta_z \approx \theta_z, \quad cos\theta_z \approx 1$

$$
\frac{d\boldsymbol{u}}{dL} = [-\theta_z\boldsymbol{e}_x\otimes\boldsymbol{e}_y + \theta_z\boldsymbol{e}_y\otimes\boldsymbol{e}_x]\cdot\boldsymbol{n}
\tag{4.26}
$$

$$
\boldsymbol{u}\nabla = -\theta_z\boldsymbol{e}_x\otimes\boldsymbol{e}_y + \theta_z\boldsymbol{e}_y\otimes\boldsymbol{e}_x = \frac{1}{2}(\boldsymbol{u}\nabla - \nabla\boldsymbol{u}) = \boldsymbol{\theta} = \theta_{xy}\boldsymbol{e}_x\otimes\boldsymbol{e}_y - \theta_{xy}\boldsymbol{e}_y\otimes\boldsymbol{e}_x
\tag{4.27}
$$

So

$$
\theta_z = -\Omega_{xy} = \Omega_{yx}
\tag{4.28}
$$

拓展到三维

$$
\theta_z = -\Omega_{xy}, \quad \theta_x = -\Omega_{yz}, \quad \theta_y = -\Omega_{zx}
\tag{4.29}
$$

small rotation vector

$$
\boldsymbol{\theta} = \theta_1\boldsymbol{e}_1 + \theta_2\boldsymbol{e}_2 + \theta_3\boldsymbol{e}_3 = -\Omega_{23}\boldsymbol{e}_1 - \Omega_{31}\boldsymbol{e}_2 - \Omega_{12}\boldsymbol{e}_3 = -\frac{1}{2}\boldsymbol{e}:\boldsymbol{\Omega} = \frac{1}{2}\nabla\times\boldsymbol{u}
\tag{4.30}
$$

其中

$$\begin{aligned}\frac{1}{2}\boldsymbol{e}:\boldsymbol{\Omega} &= \frac{1}{2}e_{ijk}\Omega_{jk}\boldsymbol{e}_i = \frac{1}{2}e_{1jk}\Omega_{jk}\boldsymbol{e}_1 + \frac{1}{2}e_{2jk}\Omega_{jk}\boldsymbol{e}_2 + \frac{1}{2}e_{3jk}\Omega_{jk}\boldsymbol{e}_3 \\ &= \frac{1}{2}(e_{123}\Omega_{23} + e_{132}\Omega_{32})\boldsymbol{e}_1 + \frac{1}{2}(e_{231}\Omega_{31} + e_{213}\Omega_{13})\boldsymbol{e}_2 + \frac{1}{2}(e_{312}\Omega_{12} + e_{321}\Omega_{21})\boldsymbol{e}_3 \\ &= \frac{1}{2}(\Omega_{23} - \Omega_{32})\boldsymbol{e}_1 + \frac{1}{2}(\Omega_{31} - \Omega_{13})\boldsymbol{e}_2 + \frac{1}{2}(\Omega_{12} - \Omega_{21})\boldsymbol{e}_3 \\ &= \Omega_{23}\boldsymbol{e}_1 + \Omega_{31}\boldsymbol{e}_2 + \Omega_{12}\boldsymbol{e}_3\end{aligned} \tag{4.31}$$

$$\begin{aligned}&\left.\begin{aligned}\boldsymbol{e}:\boldsymbol{u}\nabla &= e_{ijk}u_{j,k}\boldsymbol{e}_i \\ \boldsymbol{u}\times\nabla &= \frac{\partial u_j\boldsymbol{e}_j}{\partial x_k}\times\boldsymbol{e}_k = u_{j,k}e_{jki}\boldsymbol{e}_i = e_{ijk}u_{j,k}\boldsymbol{e}_i\end{aligned}\right\} \Longrightarrow \boldsymbol{e}:\boldsymbol{u}\nabla = \boldsymbol{u}\times\nabla \\ &\left.\begin{aligned}\boldsymbol{e}:\nabla\boldsymbol{u} &= e_{ijk}u_{k,j}\boldsymbol{e}_i \\ \nabla\times\boldsymbol{u} &= \boldsymbol{e}_k\times\frac{\partial u_j\boldsymbol{e}_j}{\partial x_k} = u_{j,k}e_{kji}\boldsymbol{e}_i = e_{ikj}u_{j,k}\boldsymbol{e}_i = e_{ijk}u_{k,j}\boldsymbol{e}_i = -e_{ijk}u_{j,k}\boldsymbol{e}_i = -\boldsymbol{u}\times\nabla\end{aligned}\right\} \Longrightarrow \boldsymbol{e}:\nabla\boldsymbol{u} = \nabla\times\boldsymbol{u}\end{aligned} \tag{4.32}$$

4.2 应力分析

4.2.1 力

4.2.1.1 力的作用效果

力对物体的作用效果无非就是两个：1.使物体发生变形. 2.改变物体的运动状态.

4.2.1.2 力学中两架并行的马车：牛顿三大定律、热力学三大定律

牛顿第一运动定律：静力平衡。

牛顿第二运动定律：动力平衡

牛顿第三运动定律：作用力与反作用力

4.2.2 应力矢量

(应力矢量/表面力/表面应力)/ surface force (surface stress vector)

选取虚拟平面上的一个微小单元，其面积大小为a，作用于这个微元的合力为 $\boldsymbol{f}_a$，那么我们的应力矢量定义为：

$$\boldsymbol{t} = \lim_{a\to 0}\frac{\boldsymbol{f}_a}{a}$$

在这个定义中，力是矢量，面积却是标量。$\boldsymbol{t}$依赖于曲面外法向和微元在曲面的位置，即，$\boldsymbol{t} = (\boldsymbol{n}, \boldsymbol{x})$具体来讲，

$\boldsymbol{x} \rightarrow$微元位置(位置矢量)

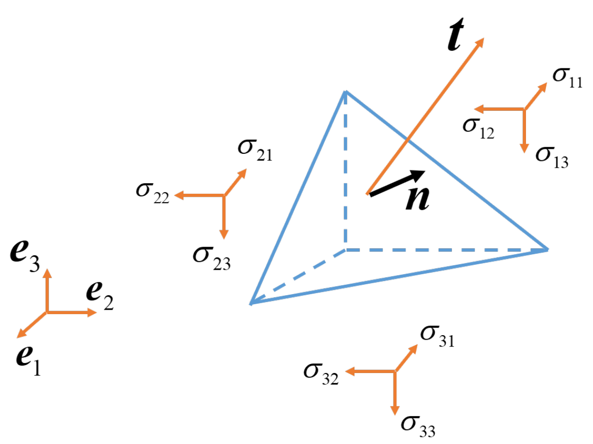
$\boldsymbol{n} \rightarrow \boldsymbol{x}$处曲面的法向矢量

$\boldsymbol{t}$称为表面力矢量或应力矢量。

为什么是用力矢量除以面元大小定义应力矢量?

面元是有方向的$d\boldsymbol{s} = ds\boldsymbol{n}$。因为没有矢量除法，所以不能直接这样$\boldsymbol{t} = \frac{\boldsymbol{f}}{d\boldsymbol{s}}$，定义应力矢量。而是用力矢量除以面元大小来定义应力矢量$\boldsymbol{t} = \frac{\boldsymbol{f}}{ds}$，由柯西定理联系应力张量和应力矢量$\boldsymbol{n}\cdot\boldsymbol{\sigma} = \boldsymbol{t}$.

4.2.3 柯西定理的来源–受力平衡



面积关系

$$A_j = A\boldsymbol{n}\cdot\boldsymbol{e}_j$$

考虑$\boldsymbol{e}_1$方向的受力平衡$\Sigma F_1=0$

$$\sigma_{11}A_1+\sigma_{21}A_2+\sigma_{31}A_3 = A\boldsymbol{t}\cdot\boldsymbol{e}_1$$

写成指标形式

$$\sigma_{j1}A_j = A\boldsymbol{t}\cdot\boldsymbol{e}_1$$

推广到3个方向

$$\sigma_{ji}A_j = A\boldsymbol{t}\cdot\boldsymbol{e}_i$$

代入面积关系

$$\sigma_{ji}A\boldsymbol{n}\cdot\boldsymbol{e}_j = A\boldsymbol{t}\cdot\boldsymbol{e}_i \quad \Rightarrow \sigma_{ji}\boldsymbol{n}\cdot\boldsymbol{e}_j = \boldsymbol{t}\cdot\boldsymbol{e}_i$$

下面对等式左边进行操作

$$\sigma_{ji}\boldsymbol{n}\cdot\boldsymbol{e}_j = \sigma_{ji}n_j = (\boldsymbol{e}_j\cdot\boldsymbol{\sigma}\boldsymbol{e}_i)n_j = (n_j\boldsymbol{e}_j)\cdot\boldsymbol{\sigma}\boldsymbol{e}_i = \boldsymbol{n}\cdot\boldsymbol{\sigma}\boldsymbol{e}_i = \boldsymbol{t}\cdot\boldsymbol{e}_i$$

$$\Rightarrow \boldsymbol{n}\cdot\boldsymbol{\sigma} = \boldsymbol{t}$$

所以，应力张量$\boldsymbol{\sigma}$只是受力平衡推导过程中引入的一个概念，其目的是为了方便计算。力是真实存在的，应力矢量和应力张量都不是真实存在。应力($\frac{N}{m^2}$ or $\frac{N}{m^3}$)只是力的强度的一种度量。

4.2.4 Cauchy应力—Cauchy定理建立了应力张量与应力矢量的联系（case1）

存在一个张量 $\boldsymbol{\sigma}$,使得

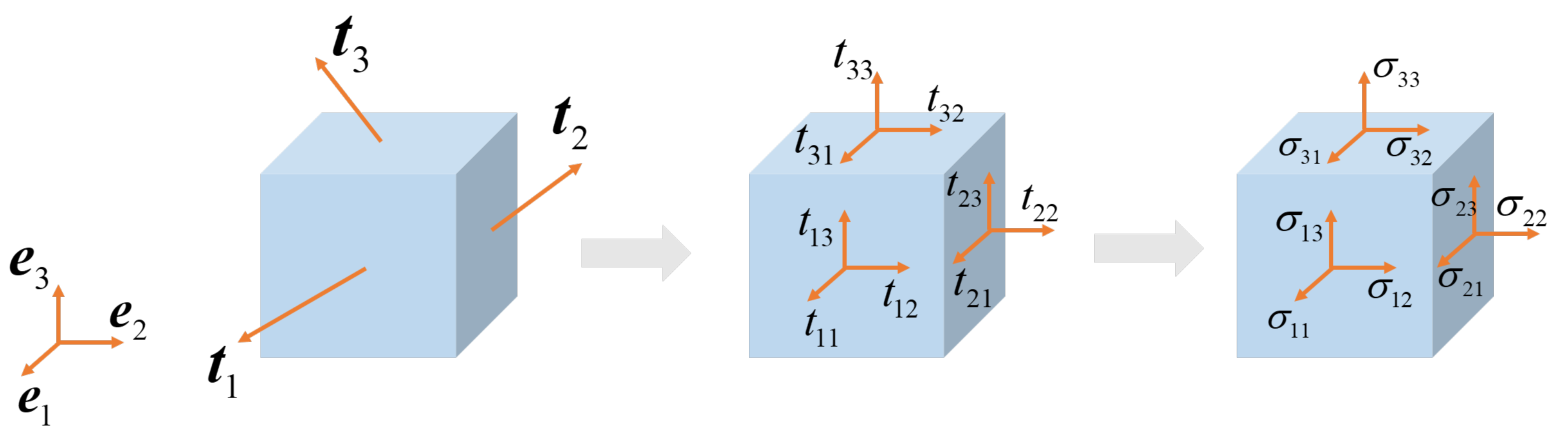
$$\boldsymbol{t}(\boldsymbol{n},\boldsymbol{x}) = \boldsymbol{n}\cdot\boldsymbol{\sigma}(\boldsymbol{x})$$

可以简写为

$$\boldsymbol{t} = \boldsymbol{n}\cdot\boldsymbol{\sigma}$$

$\boldsymbol{\sigma}$ 称为Cauchy应力张量。

4.2.5 微元体分析



σ_{ij}中i为面元指标，j为方向指标，指i面上沿j方向的应力矢量的大小。

已知在直角坐标系(Cartesian system)下，物体内一点处的应力张量为

$$\boldsymbol{\sigma} = \sigma_{ij}\boldsymbol{e}_i\otimes\boldsymbol{e}_j = \begin{bmatrix}\sigma_{xx} & \sigma_{xy} & \sigma_{xz}\\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz}\\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz}\end{bmatrix}$$

考虑x正面，其外法向矢量为$\boldsymbol{e}_1$，由柯西定理，x正面上的应力矢量为

$$\boldsymbol{t}_1 = \boldsymbol{e}_1\cdot\boldsymbol{\sigma} = \boldsymbol{e}_1\cdot\sigma_{ij}\boldsymbol{e}_i\otimes\boldsymbol{e}_j = \sigma_{ij}\delta_{1i}\boldsymbol{e}_j = \sigma_{1j}\boldsymbol{e}_j = \sigma_{11}\boldsymbol{e}_1+\sigma_{12}\boldsymbol{e}_2+\sigma_{13}\boldsymbol{e}_3$$

应力矢量在x正面，法向上(沿$\boldsymbol{e}_1$方向)的分量(正应力矢量)大小为

$$t_{11} = \boldsymbol{t}_1 \cdot \boldsymbol{e}_1 = (\boldsymbol{e}_1 \cdot \boldsymbol{\sigma}) \cdot \boldsymbol{e}_1 = (\sigma_{1j}\boldsymbol{e}_j) \cdot \boldsymbol{e}_1 = \sigma_{1j}\delta_{1j} = \sigma_{11}$$

应力矢量在x正面，沿$\boldsymbol{e}_2$方向的切向分量(切应力矢量)大小为

$$t_{12} = \boldsymbol{t}_1 \cdot \boldsymbol{e}_2 = (\boldsymbol{e}_1 \cdot \boldsymbol{\sigma}) \cdot \boldsymbol{e}_2 = (\sigma_{1j}\boldsymbol{e}_j) \cdot \boldsymbol{e}_2 = \sigma_{1j}\delta_{j2} = \sigma_{12} \tag{4.33}$$

应力矢量在x正面，沿$\boldsymbol{e}_3$方向的切向分量(切应力矢量)大小为

$$t_{13} = \boldsymbol{t}_1 \cdot \boldsymbol{e}_3 = (\boldsymbol{e}_1 \cdot \boldsymbol{\sigma}) \cdot \boldsymbol{e}_3 = (\sigma_{1j}\boldsymbol{e}_j) \cdot \boldsymbol{e}_3 = \sigma_{1j}\delta_{j3} = \sigma_{13} \tag{4.34}$$

其他5个面与上面分析类似，不再赘述。可以看出应力张量的分量就是坐标轴平面上应力矢量的分量。

4.2.6 变形前后面积微元的变化

$$\begin{aligned} d\boldsymbol{x} &= \boldsymbol{F} \cdot d\boldsymbol{X} = F_{iL} dX_L \boldsymbol{e}_i \\ d\boldsymbol{y} &= \boldsymbol{F} \cdot d\boldsymbol{Y} = F_{jM} dY_M \boldsymbol{e}_j \end{aligned} \tag{4.35}$$

$$dA\boldsymbol{N} = d\boldsymbol{X} \times d\boldsymbol{Y} = e_{LMN} dX_L dY_M \boldsymbol{e}_N \tag{4.36}$$

$$\begin{aligned} da\boldsymbol{n} &= d\boldsymbol{x} \times d\boldsymbol{y} \\ &= F_{iL} dX_L \boldsymbol{e}_i \times F_{jM} dY_M \boldsymbol{e}_j \\ &= F_{iL} F_{jM} dX_L dY_M \boldsymbol{e}_i \times \boldsymbol{e}_j \\ &= e_{ijk} F_{iL} F_{jM} dX_L dY_M \boldsymbol{e}_k \\ &= e_{ijk} F_{iL} F_{jM} dX_L dY_M \boldsymbol{e}_k \cdot (\boldsymbol{F} \cdot \boldsymbol{F}^{-1}) \\ &= e_{ijk} F_{iL} F_{jM} dX_L dY_M \boldsymbol{e}_k \cdot (F_{rN} F^{-1}_{Nt} \boldsymbol{e}_r \otimes \boldsymbol{e}_t) \\ &= e_{ijk} F_{iL} F_{jM} dX_L dY_M (F_{kN} F^{-1}_{Nt} \boldsymbol{e}_t) \\ &= e_{ijk} F_{iL} F_{jM} F_{kN} dX_L dY_M (F^{-1}_{Nt} \boldsymbol{e}_t) \\ &= e_{LMN} det(\boldsymbol{F}) dX_L dY_M (F^{-1}_{Nt} \boldsymbol{e}_t) \\ &= dA\boldsymbol{N} det(\boldsymbol{F}) (F^{-1}_{Nt} \boldsymbol{e}_t) \\ &= dA\boldsymbol{N} J F^{-1}_{Nt} \boldsymbol{e}_t \end{aligned} \tag{4.37}$$

4.2.7 变形前后线段、体积微元的变化[变形梯度的雅可比行列式(The Jacobian of the Deformation Gradient)]

$$\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{X}) = \boldsymbol{X} + \boldsymbol{u}(\boldsymbol{X}) \tag{4.38}$$

$$\begin{aligned} d\boldsymbol{x} &= \boldsymbol{F} \cdot d\boldsymbol{X} = F_{iL} dX_L \boldsymbol{e}_i \\ d\boldsymbol{y} &= \boldsymbol{F} \cdot d\boldsymbol{Y} = F_{jM} dY_M \boldsymbol{e}_j \\ d\boldsymbol{z} &= \boldsymbol{F} \cdot d\boldsymbol{Z} = F_{kN} dZ_N \boldsymbol{e}_k \end{aligned} \tag{4.39}$$

$$dV = (d\boldsymbol{X} \times d\boldsymbol{Y}) \cdot d\boldsymbol{Z} = e_{LMN} dX_L dY_M dZ_N \tag{4.40}$$

$$\begin{aligned}
dv &= (d\boldsymbol{x} \times d\boldsymbol{y}) \cdot d\boldsymbol{z} \\
&= (F_{iL} dX_L \boldsymbol{e}_i \times F_{jM} dY_M \boldsymbol{e}_j) \cdot F_{kN} dZ_N \boldsymbol{e}_k \\
&= F_{iL} F_{jM} F_{kN} dX_L dY_M dZ_N (\boldsymbol{e}_i \times \boldsymbol{e}_j) \cdot \boldsymbol{e}_k \\
&= e_{ijk} F_{iL} F_{jM} F_{kN} dX_L dY_M dZ_N \\
&= e_{LMN} det(\boldsymbol{F}) dX_L dY_M dZ_N \\
&= det(\boldsymbol{F}) dV
\end{aligned} \tag{4.41}$$

得出

$$J = det(\boldsymbol{F}) = \frac{dv}{dV} \tag{4.42}$$

Observe the following:

- For any physically admissible deformation, the volume of the deformed element must be positive (no matter how much you deform a solid, you can't make material disapper). Therefore, all physically admissible displacement fields must satisfy $J > 0$.
- If a material is incompressible, its volume remains constant. This requires $J = 1$.
- If the mass density of the material at a point in the undeformed solid is ρ_0, its mass density in the deformed solid is $\rho = \rho_0 / J$.

When working with constitutive equations, it is occasionally necessary to evaluate derivatives of J with respect to the components of $\boldsymbol{F}$. The following result (which can be proved by, for example, expanding the Jacobian using index notation) is extremely useful:

$$\frac{\partial J}{\partial F_{ij}} = J F_{ji}^{-1}$$

The proof of this equation is shown in the next page.

证明如下：

$$\boldsymbol{F} \cdot \boldsymbol{F}^{-1} = F_{ij} F_{jk}^{-1} = \boldsymbol{I}$$

$$tr(\boldsymbol{F} \cdot \boldsymbol{F}^{-1}) = F_{ij} F_{ji}^{-1} = \delta_{ii} = 3$$

$$\begin{aligned}
F_{ij} \frac{\partial J}{\partial F_{ij}} &= F_{ij} \frac{\partial \det(\boldsymbol{F})}{\partial F_{ij}} = F_{ij} \left[\frac{1}{6} \frac{\partial e_{rst} e_{lmn} F_{rl} F_{sm} F_{tn}}{\partial F_{ij}} \right] \\
&= F_{ij} \frac{1}{6} e_{rst} e_{lmn} \left[\frac{\partial F_{rl} F_{sm} F_{tn}}{\partial F_{ij}} \right] \\
&= F_{ij} \frac{1}{6} e_{rst} e_{lmn} \left[\frac{\partial F_{rl}}{\partial F_{ij}} F_{sm} F_{tn} + \frac{\partial F_{sm}}{\partial F_{ij}} F_{rl} F_{tn} + \frac{\partial F_{tn}}{\partial F_{ij}} F_{rl} F_{sm} \right] \\
&= F_{ij} \frac{1}{6} e_{rst} e_{lmn} \left[\delta_{ri} \delta_{lj} \frac{\partial F_{ij}}{\partial F_{ij}} F_{sm} F_{tn} + \delta_{si} \delta_{mj} \frac{\partial F_{ij}}{\partial F_{ij}} F_{rl} F_{tn} + \delta_{ti} \delta_{nj} \frac{\partial F_{ij}}{\partial F_{ij}} F_{rl} F_{sm} \right] \\
&= F_{ij} \frac{1}{6} e_{rst} e_{lmn} \left[\delta_{ri} \delta_{lj} F_{sm} F_{tn} + \delta_{si} \delta_{mj} F_{rl} F_{tn} + \delta_{ti} \delta_{nj} F_{rl} F_{sm} \right] \\
&= \frac{1}{6} e_{rst} e_{lmn} \left[F_{rl} F_{sm} F_{tn} + F_{sm} F_{rl} F_{tn} + F_{tn} F_{rl} F_{sm} \right] \\
&= \frac{1}{6} e_{rst} e_{lmn} \left[3 F_{rl} F_{sm} F_{tn} \right] \\
&= 3J
\end{aligned}$$

两边同乘F_{ji}^{-1}

$$F_{ij} F_{ji}^{-1} \frac{\partial J}{\partial F_{ij}} = 3 J F_{ji}^{-1} \rightarrow \frac{\partial J}{\partial F_{ij}} = J F_{ji}^{-1}$$

4.2.8 各种应力的定义

4.2.8.1 柯西应力

柯西应力$\boldsymbol{\sigma}$：作用于真实变形体上的每单位面积上的真实的力。

$$\boldsymbol{t} = \frac{d\boldsymbol{p}}{da} \quad \boldsymbol{n} \cdot \boldsymbol{\sigma} = \boldsymbol{t} \quad d\boldsymbol{p} = da \boldsymbol{n} \cdot \boldsymbol{\sigma} \tag{4.43}$$

4.2.8.2 基尔霍夫应力

基尔霍夫应力$\boldsymbol{\tau} = J\boldsymbol{\sigma}$

4.2.8.3 第一皮奥拉-基尔霍夫应力

第一皮奥拉-基尔霍夫应力(nominal stress)

$$
\begin{aligned}
d\boldsymbol{p} &= da\boldsymbol{t} = da\boldsymbol{n}\cdot\boldsymbol{\sigma} = dan_j\sigma_{jk}\boldsymbol{e}_k \\
&= dAN_N JF^{-1}_{Nj}\sigma_{jk}\boldsymbol{e}_k = dAN_N\boldsymbol{e}_N\cdot JF^{-1}_{Rj}\sigma_{jk}\boldsymbol{e}_R\otimes\boldsymbol{e}_k = dA\boldsymbol{N}\cdot\boldsymbol{S} = dAN_I S_{Ij}\boldsymbol{e}_j
\end{aligned} \tag{4.44}
$$

其中

$$\boldsymbol{S} = J\boldsymbol{F}^{-1}\cdot\boldsymbol{\sigma} \tag{4.45}$$

4.2.8.4 第二皮奥拉-基尔霍夫应力

第二皮奥拉-基尔霍夫应力(material stress)

$$d\boldsymbol{P} = \boldsymbol{F}^{-1}\cdot d\boldsymbol{p} = \boldsymbol{F}^{-1}\cdot(dA\boldsymbol{N}\cdot\boldsymbol{S}) = (dA\boldsymbol{N}\cdot\boldsymbol{S})\cdot\boldsymbol{F}^{-T} = dA\boldsymbol{N}\cdot(\boldsymbol{S}\cdot\boldsymbol{F}^{-T}) = dA\boldsymbol{N}\cdot\boldsymbol{\Sigma} = dAN_I\Sigma_{IJ}\boldsymbol{e}_J \tag{4.46}$$

4.2.9 斜截面上应力应变矢量的法向切向分解

柯西定理为

$$\boldsymbol{t} = \boldsymbol{\sigma}^{(n)} = \boldsymbol{n}\cdot\boldsymbol{\sigma} \tag{4.47}$$

应力矢量的分解

$$
\begin{aligned}
\boldsymbol{\sigma}_n^{(n)} &= \boldsymbol{n}\cdot\boldsymbol{\sigma}^{(n)} = [\boldsymbol{n}\cdot(\boldsymbol{n}\cdot\boldsymbol{\sigma})]\boldsymbol{n} \\
\boldsymbol{\sigma}_s^{(n)} &= \boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}_n^{(n)}
\end{aligned} \tag{4.48}
$$

类比柯西定理

$$\boldsymbol{t} = \boldsymbol{\varepsilon}^{(n)} = \boldsymbol{n}\cdot\boldsymbol{\varepsilon} \tag{4.49}$$

应变矢量的分解

$$
\begin{aligned}
\boldsymbol{\varepsilon}_n^{(n)} &= \boldsymbol{n}\cdot\boldsymbol{\varepsilon}^{(n)} = [\boldsymbol{n}\cdot(\boldsymbol{n}\cdot\boldsymbol{\varepsilon})]\boldsymbol{n} \\
\boldsymbol{\varepsilon}_s^{(n)} &= \boldsymbol{\varepsilon}^{(n)} - \boldsymbol{\varepsilon}_n^{(n)}
\end{aligned} \tag{4.50}
$$

物体内某一点处线段的伸长

$$\Delta L = \left|\boldsymbol{\varepsilon}_n^{(n)}\right|\cdot L \tag{4.51}$$

4.2.10 运动守恒方程

4.2.10.1 柯西应力表示的守恒方程

动量守恒

$$
\begin{aligned}
&\int_{\partial\Omega}\boldsymbol{t}da + \int_\Omega \rho\boldsymbol{b}dv - \frac{d}{dt}(\int_\Omega \rho\boldsymbol{v}dv) = \boldsymbol{0} \\
&\Downarrow \\
&\int_{\partial\Omega}\boldsymbol{n}\cdot\boldsymbol{\sigma}da + \int_\Omega \rho\boldsymbol{b}dv - \frac{d}{dt}(\int_\Omega \rho\boldsymbol{v}dv) = \boldsymbol{0} \\
&\Downarrow \\
&\int_\Omega \nabla\cdot\boldsymbol{\sigma}dv + \int_\Omega \rho\boldsymbol{b}dv - \int_\Omega \rho\boldsymbol{a}dv = \boldsymbol{0} \\
&\Downarrow \\
&\int_\Omega (\nabla\cdot\boldsymbol{\sigma} + \rho\boldsymbol{b} - \rho\boldsymbol{a})dv = \boldsymbol{0} \\
&\Downarrow \\
&\nabla\cdot\boldsymbol{\sigma} + \rho\boldsymbol{b} - \rho\boldsymbol{a} = \boldsymbol{0} \\
&\Downarrow \\
&\sigma_{ji,j} + \rho b_i - \rho a_i = 0
\end{aligned} \tag{4.52}
$$

$$
\begin{aligned}
&\int_{\partial\Omega} \boldsymbol{x}\times\boldsymbol{t}da=\int_{\partial\Omega} e_{ijk}x_i t_j da=\int_{\partial\Omega} e_{ijk}x_i n_m\sigma_{mj}da=\int_{\partial\Omega} n_m(e_{ijk}x_i\sigma_{mj})da=\int_{\Omega}\frac{\partial}{\partial x_m}(e_{ijk}x_i\sigma_{mj})dv=\int_{\Omega} e_{ijk}(\sigma_{ij}+x_i\frac{\partial\sigma_{mj}}{\partial x_m})dv\\
&\int_{\Omega} \boldsymbol{x}\times\rho\boldsymbol{b}dv=e_{ijk}x_i\rho b_j dv\\
&\int_{\Omega} \boldsymbol{x}\times\rho\boldsymbol{a}dv=\int_{\Omega} e_{ijk}x_i\rho a_j dv
\end{aligned}
\tag{4.53}
$$

动量矩守恒

$$
\begin{aligned}
&\int_{\partial\Omega} \boldsymbol{x}\times\boldsymbol{t}da+\int_{\Omega} \boldsymbol{x}\times\rho\boldsymbol{b}dv-\frac{d}{dt}(\int_{\Omega} \boldsymbol{x}\times\rho\boldsymbol{v}dv)=\boldsymbol{0}\\
&\quad\Downarrow\\
&\int_{\Omega} e_{ijk}[\sigma_{ij}+x_i(\sigma_{mj,m}+\rho b_j-\rho a_j)]dv=0\\
&\quad\Downarrow\\
&\int_{\Omega} e_{ijk}\sigma_{ij}dv=0\\
&\quad\Downarrow\\
&e_{ijk}\sigma_{ij}=0\\
&\quad\Downarrow\\
&\sigma_{ij}=\sigma_{ji}
\end{aligned}
\tag{4.54}
$$

其中

矢量为零的充分必要条件（或等价关系）

第一种：

$$\boldsymbol{a}=\boldsymbol{0}\Leftrightarrow a_1=a_2=a_3=0 \tag{4.55}$$

第二种：

$$\boldsymbol{a}=\boldsymbol{0}\Leftrightarrow \boldsymbol{a}\cdot\boldsymbol{e}=e_{imn}a_i=0 \tag{4.56}$$

$$e_{kmn}e_{ijk}\sigma_{ij}=e_{kmn}e_{kij}\sigma_{ij}=(\delta_{mi}\delta_{nj}-\delta_{mj}\delta_{ni})\sigma_{ij}=\sigma_{mn}-\sigma_{nm}=0 \tag{4.57}$$

$$e_{ijk}\sigma_{ij}=0\Leftrightarrow e_{kmn}e_{ijk}\sigma_{ij}=0\Leftrightarrow \sigma_{mn}-\sigma_{nm}=0 \tag{4.58}$$

需要注意，以下推导并不能成立

$$e_{ijk}\sigma_{ij}=e_{jik}\sigma_{ji}=-e_{ijk}\sigma_{ji}\Leftrightarrow e_{ijk}(\sigma_{ij}+\sigma_{ji})=0\not\Leftrightarrow \sigma_{ij}+\sigma_{ji}=0 \tag{4.59}$$

张量形式

应力张量中的行向量存储应力矢量信息。

平衡方程表示区域内任一点 (x,y,z) 的微元体的平衡条件。

微分体 $dv=dxdydz$，对于平面问题，dz设为1.

Force equilibrium and moment equilibrium

$$
\begin{aligned}
&\sum \boldsymbol{F}=0,\int_{\partial\Omega}\boldsymbol{t}ds+\int_{\Omega}\boldsymbol{f}dv=\boldsymbol{0}\\
&\sum \boldsymbol{M}=0,\int_{\partial\Omega}\boldsymbol{r}\times\boldsymbol{t}ds+\int_{\Omega}\boldsymbol{r}\times\boldsymbol{f}dv=\boldsymbol{0}
\end{aligned}
\tag{4.60}
$$

where $\boldsymbol{t}$ is surface force vector, $\boldsymbol{f}$ is body force vector, $\boldsymbol{r}$ is position vector.

Cauchy's theorem

$$\boldsymbol{t}=\boldsymbol{n}\cdot\boldsymbol{\sigma} \tag{4.61}$$

where $\boldsymbol{\sigma}$ is Cauchy stress tensor (symmetric).

Detailed derivation:

$$\begin{aligned}\int_{\partial\Omega}\boldsymbol{r}\times\boldsymbol{t}ds &= \int_{\partial\Omega}\boldsymbol{r}\times(\boldsymbol{n}\cdot\boldsymbol{\sigma})ds = -\int_{\partial\Omega}(\boldsymbol{n}\cdot\boldsymbol{\sigma})\times\boldsymbol{r}ds\\ &= -\int_{\partial\Omega}\boldsymbol{n}\cdot\boldsymbol{\sigma}\times\boldsymbol{r}ds = -\int_{\partial\Omega}\boldsymbol{n}\cdot(\boldsymbol{\sigma}\times\boldsymbol{r})ds\\ &= -\int_{\Omega}\nabla\cdot(\boldsymbol{\sigma}\times\boldsymbol{r})dv = \int_{\Omega}\boldsymbol{r}\times(\nabla\cdot\boldsymbol{\sigma})+\boldsymbol{e}\cdot\boldsymbol{\sigma}dv\end{aligned} \tag{4.62}$$

$$\begin{aligned}\nabla\cdot(\boldsymbol{\sigma}\times\boldsymbol{r}) &= \boldsymbol{e}_l\cdot\partial_l(\sigma_{ij}\boldsymbol{e}_i\otimes\boldsymbol{e}_j\times r_k\boldsymbol{e}_k) = \boldsymbol{e}_l\cdot\partial_l(\sigma_{ij}r_k)\boldsymbol{e}_i\otimes\boldsymbol{e}_m e_{jkm}\\ &= e_{jkm}\delta_{li}\partial_l(\sigma_{ij}r_k)\boldsymbol{e}_m = e_{jkm}\partial_i(\sigma_{ij}r_k)\boldsymbol{e}_m = e_{jkm}(\sigma_{ij,i}r_k+\sigma_{ij}r_{k,i})\boldsymbol{e}_m\\ &= e_{jkm}(\sigma_{ij,i}r_k+\sigma_{ij}\delta_{ki})\boldsymbol{e}_m = (e_{jkm}\sigma_{ij,i}r_k+e_{jkm}\sigma_{ij}\delta_{ki})\boldsymbol{e}_m\\ &= (e_{jkm}\sigma_{ij,i}r_k+e_{jim}\sigma_{ij})\boldsymbol{e}_m = -r_k\sigma_{ij,i}e_{kjm}\boldsymbol{e}_m - e_{mij}\sigma_{ij}\boldsymbol{e}_m\\ &= -\boldsymbol{r}\times(\nabla\cdot\boldsymbol{\sigma})-\boldsymbol{e}\cdot\boldsymbol{\sigma}\end{aligned} \tag{4.63}$$

By Cauchy's theorem and divergence theorem

ium-equ}

$$\begin{cases}\nabla\cdot\boldsymbol{\sigma}+\boldsymbol{f}=\boldsymbol{0}\\ \boldsymbol{r}\times(\nabla\cdot\boldsymbol{\sigma}+\boldsymbol{f})+\boldsymbol{e}\cdot\boldsymbol{\sigma}+\boldsymbol{p}=\boldsymbol{0}\end{cases}\to\begin{cases}\nabla\cdot\boldsymbol{\sigma}+\boldsymbol{f}=\boldsymbol{0}\\ \boldsymbol{e}\cdot\boldsymbol{\sigma}=\boldsymbol{0}\end{cases} \tag{4.64}$$

where $\boldsymbol{e}\cdot\boldsymbol{\sigma}=\boldsymbol{\sigma}\overset{\times}{\cdot}\boldsymbol{I}$.

$$\boldsymbol{e}:\boldsymbol{\sigma}=e_{ijk}\sigma_{jk}\boldsymbol{e}_i=-e_{ikj}\sigma_{jk}\boldsymbol{e}_i=-e_{ijk}\sigma_{kj}\boldsymbol{e}_i \tag{4.65}$$

then

$$e_{ijk}(\sigma_{jk}-\sigma_{kj})\boldsymbol{e}_i=0 \implies \sigma_{jk}=\sigma_{kj} \tag{4.66}$$

Thus, the tress tensor is symmetric.

4.2.10.2 名义应力表示的守恒方程

动量守恒

$$\begin{aligned}&\int_{\partial\Omega}n_j\sigma_{ji}da+\int_{\Omega}\rho b_i dv=0\\ &\quad\Downarrow\quad dan_t=dAN_NJF^{-1}_{Nt},\quad dv=JdV\\ &\int_{\partial\Omega_0}N_NJF^{-1}_{Nj}\sigma_{ji}dA+\int_{\Omega_0}\rho Jb_idV=0\\ &\quad\Downarrow\quad S_{Ni}=JF^{-1}\sigma_{ji},\quad \rho_0=J\rho\\ &\int_{\partial\Omega_0}N_NS_{Ni}dA+\int_{\Omega_0}\rho_0b_idV=0\\ &\quad\Downarrow\\ &\int_{\Omega_0}S_{Ni,N}dV+\int_{\Omega_0}\rho_0b_idV=0\\ &\quad\Downarrow\\ &\int_{\Omega_0}(S_{Ni,N}+\rho_0b_i)dV=0\\ &\quad\Downarrow\\ &S_{Ni,N}+\rho_0b_i=0\end{aligned} \tag{4.67}$$

动量矩守恒

$$\begin{aligned}&\boldsymbol{\sigma}=\boldsymbol{\sigma}^T\\ &\quad\Downarrow\boldsymbol{\sigma}=\frac{1}{J}\boldsymbol{F}\cdot\boldsymbol{S}\\ &\boldsymbol{F}\cdot\boldsymbol{S}=(\boldsymbol{F}\cdot\boldsymbol{S})^T\end{aligned} \tag{4.68}$$

4.2.10.3 物质应力表示的守恒方程

动量守恒

$$
\begin{aligned}
& S_{Ni,N} + \rho_0 b_i = 0 \\
& \qquad \Downarrow \quad S_{Ni} = \Sigma_{Nr} F_{ir} \\
& (E_{Nr} F_{ir})_{,N} + \rho_0 b_i = 0
\end{aligned}
\tag{4.69}
$$

动量矩守恒

$$
\begin{aligned}
& \boldsymbol{\sigma} = \boldsymbol{\sigma}^T \\
& \qquad \Downarrow \quad \boldsymbol{\sigma} = \frac{1}{J} \boldsymbol{F} \cdot \boldsymbol{\Sigma} \cdot \boldsymbol{F}^T \\
& \frac{1}{J} \boldsymbol{F} \cdot \boldsymbol{\Sigma} \cdot \boldsymbol{F}^T = \frac{1}{J} \boldsymbol{F} \cdot \boldsymbol{\Sigma}^T \cdot \boldsymbol{F}^T \\
& \qquad \Downarrow \\
& \boldsymbol{\Sigma} = \boldsymbol{\Sigma}^T
\end{aligned}
\tag{4.70}
$$

其中，转置算法

$$
(\boldsymbol{A} \cdot \boldsymbol{B}^T) = \boldsymbol{B} \cdot \boldsymbol{A}^T, \quad (\boldsymbol{F} \cdot \boldsymbol{\Sigma} \cdot \boldsymbol{F}^T)^T = \boldsymbol{F} \cdot (\boldsymbol{F} \cdot \boldsymbol{\Sigma})^T = \boldsymbol{F} \cdot \boldsymbol{\Sigma}^T \cdot \boldsymbol{F}^T \tag{4.71}
$$

4.3 本构关系

4.3.1 总起

Definition:

A material is elastic if the stress at a material point $\boldsymbol{x}$ at present time t depends only on the strain of the material point $\boldsymbol{x}$ at time t.

$$
\boldsymbol{\sigma}(\boldsymbol{x}, t) = \boldsymbol{f}(\varepsilon(\boldsymbol{x}, t), \boldsymbol{x}) \tag{4.72}
$$

为什么？自变量中多一个 $\boldsymbol{x}$? 因为弹性材料不一定是均匀的，比如功能梯度材料，应变和位置无关，但是应力和位置有关。

The material is homogeneous if the response function does not depend on the position vector $\boldsymbol{x}$.

$$
\boldsymbol{\sigma}(\boldsymbol{x}, t) = \boldsymbol{f}(\varepsilon(\boldsymbol{x}, t)) \tag{4.73}
$$

The material is linear elastic if the response function is linear in strain.

$$
\boldsymbol{\sigma}(\boldsymbol{x}, t) = \boldsymbol{C} \cdot (\varepsilon(\boldsymbol{x}, t)) \tag{4.74}
$$

where $\boldsymbol{C}$ is a fourth-order tensor also called elasticity tensor.

每放松一个要求，也是一个机遇。

For example, the constitutive equation for a homogeneous, isotropic, linear elastic material is

$$
\boldsymbol{\sigma} = \boldsymbol{C} \cdot \boldsymbol{\varepsilon} = \lambda tr(\boldsymbol{\varepsilon}) \boldsymbol{I} + 2\mu \boldsymbol{\varepsilon} \tag{4.75}
$$

where λ, μ are Lame' constants.

$$
\boldsymbol{C} = \mathbb{I} + \mathbb{I}^* \tag{4.76}
$$

where $\mathbb{I}$ is fourth-order unit tensor, $\mathbb{I}^*$ is the second fourth-order unit tensor, that is

$$
\boldsymbol{A} = \mathbb{I} \boldsymbol{A}, \qquad (tr\boldsymbol{A}) \boldsymbol{I} = \mathbb{I}^* \boldsymbol{A} \tag{4.77}
$$

4.3.2 (线性)本构关系是怎么建立起来的？

凡是现时应变唯一确定现时应力的材料，称为柯西弹性材料。

$$
\boldsymbol{\sigma} = \boldsymbol{f}(\varepsilon) \tag{4.78}
$$

一阶泰勒展开

$$\boldsymbol{\sigma} = \boldsymbol{f}(\boldsymbol{\varepsilon}) = \boldsymbol{f}(0) + \frac{\partial \boldsymbol{f}(0)}{\partial \boldsymbol{\varepsilon}} : \boldsymbol{\varepsilon} \tag{4.79}$$

无应变状态对应无应力状态，则 $\boldsymbol{f}(\mathbf{0}) = \mathbf{0}$，那么

$$\boldsymbol{\sigma}(\boldsymbol{\varepsilon}) = \frac{\partial \boldsymbol{f}(\mathbf{0})}{\partial \boldsymbol{\varepsilon}} : \boldsymbol{\varepsilon} = \boldsymbol{C} : \boldsymbol{\varepsilon} \tag{4.80}$$

其中

$$\boldsymbol{C} = \frac{\partial \boldsymbol{f}(\mathbf{0})}{\partial \boldsymbol{\varepsilon}} = \frac{\partial \boldsymbol{\sigma}(0)}{\partial \boldsymbol{\varepsilon}} \Longrightarrow C_{ijkl} = \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} \tag{4.81}$$

注意!!! 初应力为零 $\boldsymbol{\sigma}(0) = 0$，并不代表应力导数在0处为零

$$\boldsymbol{\sigma}(\mathbf{0}) = \mathbf{0} \not\Rightarrow \frac{\partial \boldsymbol{\sigma}(0)}{\partial \boldsymbol{\varepsilon}} = \left.\frac{\partial \boldsymbol{\sigma}(\varepsilon)}{\partial \boldsymbol{\varepsilon}}\right|_{\varepsilon=\mathbf{0}} = \mathbf{0} \tag{4.82}$$

4.3.3 你咋知道能量是应力乘以应变呢？

Consider the displacement $\boldsymbol{u}$ of a body that is in equilibrium subjected to body force $\boldsymbol{b}$ and surface force $\boldsymbol{t}$.

In a process in which the displacement has an increment $d\boldsymbol{u}$, the body force $\boldsymbol{b}$ and surface force $\boldsymbol{t}$ do work dW.

$$\begin{aligned}
dW &= \int_\Omega \boldsymbol{b}\cdot d\boldsymbol{u}dv + \int_{\partial\Omega_t} \boldsymbol{t}\cdot d\boldsymbol{u}da = \int_\Omega \boldsymbol{b}\cdot d\boldsymbol{u}dv + \int_{\partial\Omega_t} (\boldsymbol{n}\cdot\boldsymbol{\sigma})\cdot d\boldsymbol{u}da = \int_\Omega \boldsymbol{b}\cdot d\boldsymbol{u}dv + \int_{\partial\Omega_t} \boldsymbol{n}\cdot(\boldsymbol{\sigma}\cdot d\boldsymbol{u})da \\
&= \int_\Omega \boldsymbol{b}\cdot d\boldsymbol{u}dv + \int_\Omega \nabla\cdot(\boldsymbol{\sigma}\cdot d\boldsymbol{u})dv = \int_\Omega \boldsymbol{b}\cdot d\boldsymbol{u}dv + \int_\Omega [(\nabla\cdot\boldsymbol{\sigma})\cdot d\boldsymbol{u} + \boldsymbol{\sigma}\cdot(\nabla\otimes d\boldsymbol{u})]dv \\
&= \int_\Omega (\nabla\cdot\boldsymbol{\sigma} + \boldsymbol{b})\cdot d\boldsymbol{u}dv + \int_\Omega \boldsymbol{\sigma}\cdot(\nabla\otimes d\boldsymbol{u})dv = \int_\Omega \boldsymbol{\sigma}\cdot(\nabla\otimes d\boldsymbol{u})dv \\
&= \int_\Omega \boldsymbol{\sigma}\cdot\frac{1}{2}(\nabla\otimes d\boldsymbol{u} + (\nabla\otimes d\boldsymbol{u})^T)dv \\
&= \int_\Omega \boldsymbol{\sigma}\cdot d\boldsymbol{\varepsilon}dv
\end{aligned} \tag{4.83}$$

Then, the strain-energy function gives

$$dW^e = \frac{dW}{dv} = \boldsymbol{\sigma}\cdot d\boldsymbol{\varepsilon} \tag{4.84}$$

If

$$W^e = W^e(\boldsymbol{\varepsilon}) \tag{4.85}$$

Then

$$dW^e = W^e(\boldsymbol{\varepsilon} + d\boldsymbol{\varepsilon}) - W^e(\boldsymbol{\varepsilon}) = \frac{\partial W^e}{\partial \boldsymbol{\varepsilon}}\cdot d\boldsymbol{\varepsilon} + \dots \tag{4.86}$$

So

$$\boldsymbol{\sigma} = \frac{\partial W^e}{\partial \boldsymbol{\varepsilon}} \quad \text{or} \quad \sigma_{ij} = \frac{\partial W^e}{\partial \varepsilon_{ij}} \tag{4.87}$$

4.3.4 线弹性情形

有点问题，合不起来!!!

应变比能

$$W^e = \int_0^{\varepsilon_{ij}} \sigma_{ij} d\varepsilon_{ij} \tag{4.88}$$

引入应变能密度函数使

$$\sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}} \tag{4.89}$$

那么

$$W^e = \int_0^{\varepsilon_{ij}} \frac{\partial W}{\partial \varepsilon_{ij}} d\varepsilon_{ij} = \int_0^{\varepsilon_{ij}} dW = W(\varepsilon_{ij}) - W(0) \tag{4.90}$$

若 W 对 ε_{ij} 有二阶以上连续偏导数，由偏导的顺序无关性

$$\frac{\partial}{\partial \varepsilon_{kl}}\left(\frac{\partial W}{\partial \varepsilon_{ij}}\right) = \frac{\partial}{\partial \varepsilon_{ij}}\left(\frac{\partial W}{\partial \varepsilon_{kl}}\right) \Longrightarrow \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} = \frac{\partial \sigma_{kl}}{\partial \varepsilon_{ij}} \tag{4.91}$$

在无应变自然状态($\varepsilon_{ij} = 0$)附近把应变能函数$W(\varepsilon_{ij})$对应变分量展开成幂级数

$$W(\varepsilon_{ij}) = W(0) + \frac{\partial W(0)}{\partial \varepsilon_{ij}}\varepsilon_{ij} + \frac{1}{2}\frac{\partial^2 W(0)}{\partial \varepsilon_{ij}\partial \varepsilon_{kl}}\varepsilon_{ij}\varepsilon_{kl} + \dots = C_0 + C_{ij}\varepsilon_{ij} + \frac{1}{2}C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} + \dots \tag{4.92}$$

若取无应变状态为 $W=0$ 的参考状态，则 $C_0=0$; 若采用无初应力假设，则 $C_{ij}=0$; 对于小应变情形，略去高阶小项，那么

$$W=\frac{1}{2}C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} \tag{4.93}$$

其中

$$C_{ijkl}=\frac{\partial^2 W}{\partial\varepsilon_{ij}\partial\varepsilon_{kl}}=\frac{\partial^2 W}{\partial\varepsilon_{kl}\partial\varepsilon_{ij}}=C_{klij} \tag{4.94}$$

然后有

$$\begin{aligned}
\sigma_{mn}&=\frac{\partial W}{\partial\varepsilon_{mn}}=\frac{\partial}{\partial\varepsilon_{mn}}(\frac{1}{2}C_{ijkl}\varepsilon_{ij}\varepsilon_{kl})=\frac{1}{2}C_{ijkl}\frac{\partial}{\partial\varepsilon_{mn}}(\varepsilon_{ij}\varepsilon_{kl})=\frac{1}{2}C_{ijkl}(\frac{\partial\varepsilon_{ij}}{\partial\varepsilon_{mn}}\varepsilon_{kl}+\varepsilon_{ij}\frac{\partial\varepsilon_{kl}}{\partial\varepsilon_{mn}})=\frac{1}{2}C_{ijkl}(\delta_{im}\delta_{jn}\varepsilon_{kl}+\varepsilon_{ij}\delta_{km}\delta_{ln})\\
&=\frac{1}{2}C_{mnkl}\varepsilon_{kl}+\frac{1}{2}C_{ijmn}\varepsilon_{ij}=\frac{1}{2}C_{mnkl}\varepsilon_{kl}+\frac{1}{2}C_{mnij}\varepsilon_{ij}=C_{mnkl}\varepsilon_{kl}\\
&\Downarrow\\
\sigma_{ij}&=C_{ijkl}\varepsilon_{kl}
\end{aligned} \tag{4.95}$$

4.3.5 (一般各向异性)线弹性本构关系(21)

四阶弹性系数张量有81个系数。

$$(11,22,33,12,23,31,13,32,21)\times(11,22,33,12,23,31,13,32,21)=9\times 9=81 \tag{4.96}$$

4.3.5.1 STEP1: 小对称性(应力和应变的对称性)

$$\begin{aligned}
&\sigma_{ij}=C_{ijkl}\varepsilon_{kl}\\
&\sigma_{ji}=C_{jikl}\varepsilon_{kl}\implies C_{ijkl}=C_{jikl}\\
&\sigma_{ij}=\sigma_{ji}
\end{aligned} \tag{4.97}$$

$$\begin{aligned}
&\sigma_{ij}=C_{ijkl}\varepsilon_{kl}\\
&\sigma_{ij}=C_{ijlk}\varepsilon_{lk}\implies C_{ijkl}=C_{ijlk}\\
&\varepsilon_{kl}=\varepsilon_{lk}
\end{aligned} \tag{4.98}$$

$$(11,22,33,12,23,31)\times(11,22,33,12,23,31)=6\times 6=36 \tag{4.99}$$

4.3.5.2 STEP2: 大对称性(能量的对称性)

$$C_{ijkl}=C_{klij} \tag{4.100}$$

$$(11,22,33,12,23,31)\times(11,22,33,12,23,31)=6+5+4+3+2+1=21 \tag{4.101}$$

若令c的指标$1,2,3,4,5,6$分别对应C的双指标$11,22,33,12,23,31$

$$c_{11}=C_{1111},c_{12}=C_{1122}... \tag{4.102}$$

至此可得Voigt形式的本构关系

$$\begin{bmatrix}\sigma_{11}\\\sigma_{22}\\\sigma_{33}\\\sigma_{12}\\\sigma_{23}\\\sigma_{31}\end{bmatrix}=\begin{bmatrix}c_{11}&c_{12}&c_{13}&c_{14}&c_{15}&c_{16}\\&c_{22}&c_{23}&c_{24}&c_{25}&c_{26}\\&&c_{33}&c_{34}&c_{35}&c_{36}\\&&&c_{44}&c_{45}&c_{46}\\&syms&&&c_{55}&c_{56}\\&&&&&c_{66}\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\\varepsilon_{22}\\\varepsilon_{33}\\\varepsilon_{12}\\\varepsilon_{23}\\\varepsilon_{31}\end{bmatrix} \tag{4.103}$$

4.3.6 坐标与坐标转换

4.3.6.1 坐标

为了确切地描述物体的形状、位置以及发生在空间中的物理现象，人们经常借助于参考坐标系。最常用的是笛卡儿坐标系，又称直角坐标系，它的坐标轴是相互正交的三根直线。

设空间任意点P的矢径为$\boldsymbol{r}$。它在笛卡儿坐标系中的分解式为

$$\boldsymbol{r}(x_1,x_2,x_3)=x_1\boldsymbol{e}_1+x_2\boldsymbol{e}_2+x_3\boldsymbol{e}_3=x_i\boldsymbol{e}_i \tag{4.104}$$

其中，$x_i(i=1,2,3)$是P点的三个坐标。

当一个坐标任意变化另两个坐标保持不变时，空间点的轨迹称为坐标线。过每个空间点有三根坐标线，在笛卡儿坐标系中各坐标线都和相应的坐标轴平行。

当坐标变化时，矢径的变化为

$$d\boldsymbol{r}=\frac{\partial\boldsymbol{r}}{\partial x_1}dx_1+\frac{\partial\boldsymbol{r}}{\partial x_2}dx_2+\frac{\partial\boldsymbol{r}}{\partial x_3}dx_3=\frac{\partial\boldsymbol{r}}{\partial x_i}dx_i=dx_i\boldsymbol{g}_i \tag{4.105}$$

其中，用矢径对坐标的偏导数定义的三个矢量

$$\boldsymbol{g}_i=\frac{\partial\boldsymbol{r}}{\partial x_i}\quad(i=1,2,3) \tag{4.106}$$

称为基矢量。空间每点处有三个基矢量，它们组成一个参考架或坐标架。任何具有方向性的物理量都可以对其相应作用点处的参考架分解。例如P点处的矢量$\boldsymbol{u}$可以分解为

$$\boldsymbol{u}=u_i\boldsymbol{g}_i \tag{4.107}$$

其中u_i为矢量$\boldsymbol{u}$的分量。对笛卡儿坐标系，对矢径$\boldsymbol{r}$求导得

$$\boldsymbol{g}_1=\frac{\partial\boldsymbol{r}}{\partial x_1}=\boldsymbol{e}_1,\quad \boldsymbol{g}_2=\frac{\partial\boldsymbol{r}}{\partial x_2}=\boldsymbol{e}_2,\quad \boldsymbol{g}_3=\frac{\partial\boldsymbol{r}}{\partial x_3}=\boldsymbol{e}_3 \tag{4.108}$$

即笛卡儿坐标系得基矢量是三个互相正交得单位基矢量$\boldsymbol{e}_i$，它们构成正交标准化基。

欧几里得度量空间，即能在其中找到一个适用于全空间得笛卡儿坐标系得空间。一般而论，人们可以根据方便的原则任选三个独立参数作为三维欧氏空间中的参考坐标。上述关于坐标系基矢量、参考架以及分量的定义都仍然适用。所不同的是：现在的坐标线可能不再正交；不同点处的坐标线可能不再平行；基矢量的大小和方向都可能随点而异；各点处的参考架不再是正交标准化基。关于任意曲线坐标系的进一步论述可参考张量分析的一般理论。

4.3.6.2 坐标转换

张量分析要研究同一个物理规律在不同参考坐标系中描述时，其数学表达形式有什么联系和区别。为此先要弄清各坐标系之间的转换关系。先看笛卡儿坐标系。新、老两个坐标系 x_i 和 X_I 的正交标准化基 $\boldsymbol{e}_i$ 和 $\boldsymbol{E}_I$ 分别满足

$$\boldsymbol{e}_i\cdot\boldsymbol{e}_j=\delta_{ij},\quad \boldsymbol{E}_I\cdot\boldsymbol{E}_J=\delta_{IJ} \tag{4.109}$$

把新基 $\boldsymbol{e}_j$ 对老基 $\boldsymbol{E}_I$ 分解

$$\boldsymbol{e}_i=[\cos(\boldsymbol{e}_i,\boldsymbol{E}_1)]\boldsymbol{E}_1+[\cos(\boldsymbol{e}_i,\boldsymbol{E}_2)]\boldsymbol{E}_2+[\cos(\boldsymbol{e}_i,\boldsymbol{E}_3)]\boldsymbol{E}_3=[\cos(\boldsymbol{e}_i,\boldsymbol{E}_J)]\boldsymbol{E}_J=[\boldsymbol{e}_i\cdot\boldsymbol{E}_J]\boldsymbol{E}_J=\beta_{iJ}\boldsymbol{E}_J \tag{4.110}$$

其中 β_{iJ} 是新坐标轴 i 和老坐标轴 J 之间的夹角余弦，称为转换系数。反之，老基对新基的分解式为

$$\boldsymbol{E}_J=[\cos(\boldsymbol{e}_1,\boldsymbol{E}_J)]\boldsymbol{e}_1+[\cos(\boldsymbol{e}_2,\boldsymbol{E}_j)]\boldsymbol{e}_2+[\cos(\boldsymbol{e}_3,\boldsymbol{E}_j)]\boldsymbol{e}_3=[\cos(\boldsymbol{e}_i,\boldsymbol{E}_J)]\boldsymbol{e}_i=[\boldsymbol{e}_i\cdot\boldsymbol{E}_J]\boldsymbol{e}_i=\beta_{iJ}\boldsymbol{e}_i \tag{4.111}$$

某空间点P在新、老坐标系中的矢径为

$$\boldsymbol{x}=x_j\boldsymbol{e}_j,\quad \boldsymbol{X}=X_I\boldsymbol{E}_I \tag{4.112}$$

注意老原点在新坐标系中的矢径和新原点在老坐标系中的矢径分别为

$$\boldsymbol{c}=c_i\boldsymbol{e}_i,\quad \boldsymbol{C}=C_I\boldsymbol{E}_I \tag{4.113}$$

其中 c_i 是老原点的新坐标值，C_J 是新原点的老坐标值。

把矢量关系 $\boldsymbol{x}=\boldsymbol{c}+\boldsymbol{X}$ 向新坐标轴 $\boldsymbol{i}$ 投影，即用 $\boldsymbol{e}_i$ 点乘上式两边，则可得新坐标用老坐标表示的表达式

$$\boldsymbol{x}=\boldsymbol{c}+\boldsymbol{X} \implies \begin{matrix} \boldsymbol{x}\cdot\boldsymbol{e}_i = x_k\boldsymbol{e}_k\cdot\boldsymbol{e}_i = x_k\delta_{ki} = x_i \\ (\boldsymbol{c}+\boldsymbol{X})\cdot\boldsymbol{e}_i = c_k\boldsymbol{e}_k\cdot\boldsymbol{e}_i + X_J\boldsymbol{E}_J\cdot\boldsymbol{e}_i = c_i + X_J\beta_{iJ} \end{matrix} \implies x_i = c_i + \beta_{iJ}X_J \implies \boldsymbol{x}=\boldsymbol{c}+\boldsymbol{\beta}\cdot\boldsymbol{X} \tag{4.114}$$

把矢量关系 $\boldsymbol{X}=\boldsymbol{C}+\boldsymbol{x}$ 向老坐标轴 $\boldsymbol{J}$ 投影，即用 $\boldsymbol{E}_J$ 点乘上式两边，则类似地可得老坐标用新坐标表示的表达式

$$\boldsymbol{X}=\boldsymbol{C}+\boldsymbol{x} \implies \begin{matrix} \boldsymbol{X}\cdot\boldsymbol{E}_J = X_K\boldsymbol{E}_K\cdot\boldsymbol{E}_K = X_J \\ (\boldsymbol{C}+\boldsymbol{x})\cdot\boldsymbol{E}_J = C_I\boldsymbol{E}_I\cdot\boldsymbol{E}_J + x_k\boldsymbol{e}_k\cdot\boldsymbol{E}_J = C_J + x_k\beta_{kJ} \end{matrix} \implies X_J = C_J + x_i\beta_{iJ} \implies \boldsymbol{X}=\boldsymbol{C}+\boldsymbol{x}\cdot\boldsymbol{\beta} \tag{4.115}$$

将上两式求导可得转换系数为

$$\beta_{iJ} = \frac{\partial x_i}{\partial X_J} = \frac{\partial X_J}{\partial x_i} \tag{4.116}$$

若新、老坐标系原点重合

$$\begin{matrix} \boldsymbol{x}=\boldsymbol{\beta}\cdot\boldsymbol{X} \\ \boldsymbol{X}=\boldsymbol{\beta}^{-1}\cdot\boldsymbol{x} \end{matrix}, \quad \boldsymbol{X}=\boldsymbol{x}\cdot\boldsymbol{\beta}=\boldsymbol{\beta}^T\cdot\boldsymbol{x} \implies \boldsymbol{\beta}^T=\boldsymbol{\beta}^{-1} \tag{4.117}$$

其中 $\boldsymbol{\beta}$ 称为转换矩阵。两边左乘 $\boldsymbol{\beta}$ 后有

$$\boldsymbol{\beta}\cdot\boldsymbol{\beta}^T=\boldsymbol{I} \tag{4.118}$$

所以 $\boldsymbol{\beta}$ 是正交矩阵，其行列式的值为

$$\det(\boldsymbol{\beta})=1 \tag{4.119}$$

转换系数的互逆关系

$$\begin{matrix} \boldsymbol{e}_i\cdot\boldsymbol{e}_j=\delta_{ij} \\ \boldsymbol{e}_i\cdot\boldsymbol{e}_j=\beta_{iL}\boldsymbol{E}_L\cdot\beta_{jK}\boldsymbol{E}_K=\beta_{iL}\beta_{jK}\delta_{LK}=\beta_{iK}\beta_{jK} \end{matrix} \implies \beta_{iK}\beta_{jK}=\delta_{ij} \tag{4.120}$$

同理

$$\begin{matrix} \boldsymbol{E}_I\cdot\boldsymbol{E}_J=\delta_{IJ} \\ \boldsymbol{E}_I\cdot\boldsymbol{E}_J=\beta_{lI}\boldsymbol{e}_l\cdot\beta_{kJ}\boldsymbol{e}_k=\beta_{lI}\beta_{kJ}\boldsymbol{e}_l\cdot\boldsymbol{e}_k=\beta_{kI}\beta_{kJ} \end{matrix} \implies \beta_{kI}\beta_{kJ}=\delta_{IJ} \tag{4.121}$$

坐标转换的一般定义是：设在三维欧氏空间中任选两个新、老曲线坐标系，x_i 和 X_J 是同一空间点 P 的新、老坐标值，则方程组

$$x_i=x_i(X_J) \quad (i,J=1,2,3) \tag{4.122}$$

定义了由老坐标到新坐标的坐标转换，称为正转换。对上式微分

$$dx_i=\frac{\partial x_i}{\partial X_J}dX_J \tag{4.123}$$

在每个空间点的无限小邻域内导数 $\partial x_i/\partial X_J$ 可看作常数，所以上式给出了由老坐标微分 dX_J 确定新坐标微分 dx_i 的线性变换。若其系数行列式(称为雅可比行列式)

$$J=\det(\frac{\partial x_i}{\partial X_J}) \tag{4.124}$$

处处不为零，则存在相应的逆变换，即可反过来用 dx_i 唯一地确定 dX_J，即

$$X_J=X_J(x_i) \quad (i,J=1,2,3) \tag{4.125}$$

由单值、一阶偏导数连续、且雅可比行列式不为零的转换函数所实现的坐标转换称为容许转换，它一定是可逆的。雅可比 J 处处为正的转换把一个右手坐标系转换成另一个右手坐标系，称为正常转换。J 处处为负的转换把一个右手系转换成左手系，称为反常转换。

4.3.7 具有材料对称性的线弹性本构关系

4.3.7.1 横观(具有一个弹性对称面的)各向异性弹性体(13)

4.3.7.1.1 关于 $x-y$ 平面对称

若把 z 轴反向作为新坐标系，转换矩阵为

$$\boldsymbol{x}=\boldsymbol{\beta}\cdot\boldsymbol{X}\implies\boldsymbol{\beta}=\boldsymbol{e}_i\cdot\boldsymbol{E}_J=\begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix}\tag{4.126}$$

根据二阶张量的转换规律

$$\sigma_{ij}=\beta_{iM}\beta_{jN}\Sigma_{MN}\implies\boldsymbol{\sigma}=\boldsymbol{\beta}\cdot\boldsymbol{\Sigma}\cdot\boldsymbol{\beta}^T\tag{4.127}$$

新、老坐标系中的应力关系和应变关系分别为

$$\begin{bmatrix}\sigma'_{11}&\sigma'_{12}&\sigma'_{13}\\\sigma'_{21}&\sigma'_{22}&\sigma'_{23}\\\sigma'_{31}&\sigma'_{32}&\sigma'_{33}\end{bmatrix}=\begin{bmatrix}\sigma_{11}&\sigma_{12}&-\sigma_{13}\\\sigma_{21}&\sigma_{22}&-\sigma_{23}\\-\sigma_{31}&-\sigma_{32}&\sigma_{33}\end{bmatrix},\qquad\begin{bmatrix}\varepsilon'_{11}&\varepsilon'_{12}&\varepsilon'_{13}\\\varepsilon'_{21}&\varepsilon'_{22}&\varepsilon'_{23}\\\varepsilon'_{31}&\varepsilon'_{32}&\varepsilon'_{33}\end{bmatrix}=\begin{bmatrix}\varepsilon_{11}&\varepsilon_{12}&-\varepsilon_{13}\\\varepsilon_{21}&\varepsilon_{22}&-\varepsilon_{23}\\-\varepsilon_{31}&-\varepsilon_{32}&\varepsilon_{33}\end{bmatrix}\tag{4.128}$$

写成Voigt形式，并代入本构关系

$$\left.\begin{aligned}&[\boldsymbol{\sigma}']=[\boldsymbol{\beta}]\cdot[\boldsymbol{\sigma}]\implies[\boldsymbol{\sigma}]=[\boldsymbol{\beta}]^{-1}\cdot[\boldsymbol{\sigma}']\\&[\boldsymbol{\varepsilon}']=[\boldsymbol{\beta}]\cdot[\boldsymbol{\varepsilon}]\implies[\boldsymbol{\varepsilon}]=[\boldsymbol{\beta}]^{-1}\cdot[\boldsymbol{\varepsilon}']\end{aligned}\right.\implies[\boldsymbol{\sigma}]=[\boldsymbol{C}]\cdot[\boldsymbol{\varepsilon}]\implies[\boldsymbol{\sigma}']=([\boldsymbol{\beta}]\cdot[\boldsymbol{C}]\cdot[\boldsymbol{\beta}]^{-1})\cdot[\boldsymbol{\varepsilon}']\tag{4.129}$$

新坐标系中的本构关系为

$$\begin{bmatrix}\sigma'_{11}\\\sigma'_{22}\\\sigma'_{33}\\\sigma'_{12}\\-\sigma'_{23}\\-\sigma'_{31}\end{bmatrix}=\begin{bmatrix}c_{11}&c_{12}&c_{13}&c_{14}&c_{15}&c_{16}\\&c_{22}&c_{23}&c_{24}&c_{25}&c_{26}\\&&c_{33}&c_{34}&c_{35}&c_{36}\\&&&c_{44}&c_{45}&c_{46}\\&syms&&&c_{55}&c_{56}\\&&&&&c_{66}\end{bmatrix}\begin{bmatrix}\varepsilon'_{11}\\\varepsilon'_{22}\\\varepsilon'_{33}\\\varepsilon'_{12}\\-\varepsilon'_{23}\\-\varepsilon'_{31}\end{bmatrix}\implies\begin{bmatrix}\sigma'_{11}\\\sigma'_{22}\\\sigma'_{33}\\\sigma'_{12}\\\sigma'_{23}\\\sigma'_{31}\end{bmatrix}=\begin{bmatrix}c_{11}&c_{12}&c_{13}&c_{14}&-c_{15}&-c_{16}\\&c_{22}&c_{23}&c_{24}&-c_{25}&-c_{26}\\&&c_{33}&c_{34}&-c_{35}&-c_{36}\\&&&c_{44}&-c_{45}&-c_{46}\\&syms&&&c_{55}&c_{56}\\&&&&&c_{66}\end{bmatrix}\begin{bmatrix}\varepsilon'_{11}\\\varepsilon'_{22}\\\varepsilon'_{33}\\\varepsilon'_{12}\\\varepsilon'_{23}\\\varepsilon'_{31}\end{bmatrix}\tag{4.130}$$

由于 z 轴正负两个方向的弹性相同，因此在上述坐标变换前后的本构关系应该相同，故必有

$$C_{15}=C_{25}=C_{35}=C_{45}=C_{16}=C_{26}=C_{36}=C_{46}=0\tag{4.131}$$

于是独立的弹性常数减小到13个。

$$\begin{bmatrix}\sigma_{11}\\\sigma_{22}\\\sigma_{33}\\\sigma_{12}\\\sigma_{23}\\\sigma_{31}\end{bmatrix}=\begin{bmatrix}c_{11}&c_{12}&c_{13}&c_{14}&0&0\\&c_{22}&c_{23}&c_{24}&0&0\\&&c_{33}&c_{34}&0&0\\&&&c_{44}&0&0\\&syms&&&c_{55}&c_{56}\\&&&&&c_{66}\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\\varepsilon_{22}\\\varepsilon_{33}\\\varepsilon_{12}\\\varepsilon_{23}\\\varepsilon_{31}\end{bmatrix}\tag{4.132}$$

4.3.7.2 正交各向异性线弹性体(9)

4.3.7.2.1 关于 $y-z$ 平面对称

若把 x 轴反向作为新坐标系，新、老坐标系中的应力和应变关系分别为

$$\begin{bmatrix}\sigma'_{11}&\sigma'_{12}&\sigma'_{13}\\\sigma'_{21}&\sigma'_{22}&\sigma'_{23}\\\sigma'_{31}&\sigma'_{32}&\sigma'_{33}\end{bmatrix}=\begin{bmatrix}\sigma_{11}&-\sigma_{12}&-\sigma_{13}\\-\sigma_{21}&\sigma_{22}&\sigma_{23}\\-\sigma_{31}&\sigma_{32}&\sigma_{33}\end{bmatrix},\qquad\begin{bmatrix}\varepsilon'_{11}&\varepsilon'_{12}&\varepsilon'_{13}\\\varepsilon'_{21}&\varepsilon'_{22}&\varepsilon'_{23}\\\varepsilon'_{31}&\varepsilon'_{32}&\varepsilon'_{33}\end{bmatrix}=\begin{bmatrix}\varepsilon_{11}&-\varepsilon_{12}&-\varepsilon_{13}\\-\varepsilon_{21}&\varepsilon_{22}&\varepsilon_{23}\\-\varepsilon_{31}&\varepsilon_{32}&\varepsilon_{33}\end{bmatrix}\tag{4.133}$$

写成Voigt形式，并代入本构关系

$$\begin{array}{l}[\boldsymbol{\sigma}'] = [\boldsymbol{\beta}]\cdot[\boldsymbol{\sigma}] \implies [\boldsymbol{\sigma}] = [\boldsymbol{\beta}]^{-1}\cdot[\boldsymbol{\sigma}'] \\ [\boldsymbol{\varepsilon}'] = [\boldsymbol{\beta}]\cdot[\boldsymbol{\varepsilon}] \implies [\boldsymbol{\varepsilon}] = [\boldsymbol{\beta}]^{-1}\cdot[\boldsymbol{\varepsilon}']\end{array} \implies [\boldsymbol{\sigma}] = [\boldsymbol{C}]\cdot[\boldsymbol{\varepsilon}] \implies [\boldsymbol{\sigma}'] = ([\boldsymbol{\beta}]\cdot[\boldsymbol{C}]\cdot[\boldsymbol{\beta}]^{-1})\cdot[\boldsymbol{\varepsilon}'] \tag{4.134}$$

新坐标系中的本构关系为

$$\begin{bmatrix}\sigma'_{11}\\ \sigma'_{22}\\ \sigma'_{33}\\ -\sigma'_{12}\\ \sigma'_{23}\\ -\sigma'_{31}\end{bmatrix} = \begin{bmatrix}c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16}\\ & c_{22} & c_{23} & c_{24} & c_{25} & c_{26}\\ & & c_{33} & c_{34} & c_{35} & c_{36}\\ & & & c_{44} & c_{45} & c_{46}\\ & syms & & & c_{55} & c_{56}\\ & & & & & c_{66}\end{bmatrix}\begin{bmatrix}\varepsilon'_{11}\\ \varepsilon'_{22}\\ \varepsilon'_{33}\\ -\varepsilon'_{12}\\ \varepsilon'_{23}\\ -\varepsilon'_{31}\end{bmatrix} \implies \begin{bmatrix}\sigma'_{11}\\ \sigma'_{22}\\ \sigma'_{33}\\ \sigma'_{12}\\ \sigma'_{23}\\ \sigma'_{31}\end{bmatrix} = \begin{bmatrix}c_{11} & c_{12} & c_{13} & -c_{14} & c_{15} & -c_{16}\\ & c_{22} & c_{23} & -c_{24} & c_{25} & -c_{26}\\ & & c_{33} & -c_{34} & c_{35} & -c_{36}\\ & & & c_{44} & -c_{45} & c_{46}\\ & syms & & & c_{55} & -c_{56}\\ & & & & & c_{66}\end{bmatrix}\begin{bmatrix}\varepsilon'_{11}\\ \varepsilon'_{22}\\ \varepsilon'_{33}\\ \varepsilon'_{12}\\ \varepsilon'_{23}\\ \varepsilon'_{31}\end{bmatrix} \tag{4.135}$$

4.3.7.2.2 关于 $z-x$ 平面对称

若把 y 轴反向作为新坐标系，新、老坐标系中的应力和应变关系分别为

$$\begin{bmatrix}\sigma'_{11} & \sigma'_{12} & \sigma'_{13}\\ \sigma'_{21} & \sigma'_{22} & \sigma'_{23}\\ \sigma'_{31} & \sigma'_{32} & \sigma'_{33}\end{bmatrix} = \begin{bmatrix}\sigma_{11} & -\sigma_{12} & \sigma_{13}\\ -\sigma_{21} & \sigma_{22} & -\sigma_{23}\\ \sigma_{31} & -\sigma_{32} & \sigma_{33}\end{bmatrix}, \quad \begin{bmatrix}\varepsilon'_{11} & \varepsilon'_{12} & \varepsilon'_{13}\\ \varepsilon'_{21} & \varepsilon'_{22} & \varepsilon'_{23}\\ \varepsilon'_{31} & \varepsilon'_{32} & \varepsilon'_{33}\end{bmatrix} = \begin{bmatrix}\varepsilon_{11} & -\varepsilon_{12} & \varepsilon_{13}\\ -\varepsilon_{21} & \varepsilon_{22} & -\varepsilon_{23}\\ \varepsilon_{31} & -\varepsilon_{32} & \varepsilon_{33}\end{bmatrix} \tag{4.136}$$

写成Voigt形式，并代入本构关系

$$\begin{array}{l}[\boldsymbol{\sigma}'] = [\boldsymbol{\beta}]\cdot[\boldsymbol{\sigma}] \implies [\boldsymbol{\sigma}] = [\boldsymbol{\beta}]^{-1}\cdot[\boldsymbol{\sigma}'] \\ [\boldsymbol{\varepsilon}'] = [\boldsymbol{\beta}]\cdot[\boldsymbol{\varepsilon}] \implies [\boldsymbol{\varepsilon}] = [\boldsymbol{\beta}]^{-1}\cdot[\boldsymbol{\varepsilon}']\end{array} \implies [\boldsymbol{\sigma}] = [\boldsymbol{C}]\cdot[\boldsymbol{\varepsilon}] \implies [\boldsymbol{\sigma}'] = ([\boldsymbol{\beta}]\cdot[\boldsymbol{C}]\cdot[\boldsymbol{\beta}]^{-1})\cdot[\boldsymbol{\varepsilon}'] \tag{4.137}$$

新坐标系中的本构关系为

$$\begin{bmatrix}\sigma'_{11}\\ \sigma'_{22}\\ \sigma'_{33}\\ -\sigma'_{12}\\ -\sigma'_{23}\\ \sigma'_{31}\end{bmatrix} = \begin{bmatrix}c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16}\\ & c_{22} & c_{23} & c_{24} & c_{25} & c_{26}\\ & & c_{33} & c_{34} & c_{35} & c_{36}\\ & & & c_{44} & c_{45} & c_{46}\\ & syms & & & c_{55} & c_{56}\\ & & & & & c_{66}\end{bmatrix}\begin{bmatrix}\varepsilon'_{11}\\ \varepsilon'_{22}\\ \varepsilon'_{33}\\ -\varepsilon'_{12}\\ -\varepsilon'_{23}\\ \varepsilon'_{31}\end{bmatrix} \implies \begin{bmatrix}\sigma'_{11}\\ \sigma'_{22}\\ \sigma'_{33}\\ \sigma'_{12}\\ \sigma'_{23}\\ \sigma'_{31}\end{bmatrix} = \begin{bmatrix}c_{11} & c_{12} & c_{13} & -c_{14} & -c_{15} & c_{16}\\ & c_{22} & c_{23} & -c_{24} & -c_{25} & c_{26}\\ & & c_{33} & -c_{34} & -c_{35} & c_{36}\\ & & & c_{44} & c_{45} & -c_{46}\\ & syms & & & c_{55} & -c_{56}\\ & & & & & c_{66}\end{bmatrix}\begin{bmatrix}\varepsilon'_{11}\\ \varepsilon'_{22}\\ \varepsilon'_{33}\\ \varepsilon'_{12}\\ \varepsilon'_{23}\\ \varepsilon'_{31}\end{bmatrix} \tag{4.138}$$

由于 x, y 轴正负两个方向的弹性相同，因此在上述坐标变换前后的本构关系应该相同，故必有

$$C_{14} = C_{24} = C_{34} = C_{56} = 0 \tag{4.139}$$

此时，独立的弹性系数只有9个。

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{12}\\ \sigma_{23}\\ \sigma_{31}\end{bmatrix} = \begin{bmatrix}c_{11} & c_{12} & c_{13} & 0 & 0 & 0\\ & c_{22} & c_{23} & 0 & 0 & 0\\ & & c_{33} & 0 & 0 & 0\\ & & & c_{44} & 0 & 0\\ & syms & & & c_{55} & 0\\ & & & & & c_{66}\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ \varepsilon_{12}\\ \varepsilon_{23}\\ \varepsilon_{31}\end{bmatrix} \tag{4.140}$$

4.3.7.3 横观各向同性线弹性体(5)

设体内每一点存在一轴（z轴），在与此轴垂直的平面（Oxy）内，所有射线方向的弹性性质均相同。称该平面为各向同性面。

具有各向同性面，且各各向同性面相互平行（或具有弹性对称轴）的物体，称为横观各向同性材料。

设 xy 平面绕 z 轴旋转任意角度θ，旋转前后应力应变关系不变，比较其弹性常数可得

取两个特殊的变换：

将 x, y 轴互换时，材料弹性关系不变 $c_{11} = c_{22}, c_{13} = c_{23}, c_{55} = c_{66}$。

将坐标系绕 z 轴旋转45°，剪切应力应变关系不变，得 $c_{44} = c_{11} - c_{12}$

4.3.7.3.1 将 x, y 轴互换作为新坐标系

若把 x, y 轴互换作为新坐标系，转换矩阵为

$$\boldsymbol{x} = \boldsymbol{\beta} \cdot \boldsymbol{X} \implies \boldsymbol{\beta} = \boldsymbol{e}_i \cdot \boldsymbol{E}_J = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.141}$$

根据二阶张量的转换规律

$$\sigma_{ij} = \beta_{iM}\beta_{jN}\Sigma_{MN} \implies \boldsymbol{\sigma} = \boldsymbol{\beta} \cdot \boldsymbol{\Sigma} \cdot \boldsymbol{\beta}^T \tag{4.142}$$

新、老坐标系中的应力和应变关系分别为

$$\begin{bmatrix} \sigma'_{11} & \sigma'_{12} & \sigma'_{13} \\ \sigma'_{21} & \sigma'_{22} & \sigma'_{23} \\ \sigma'_{31} & \sigma'_{32} & \sigma'_{33} \end{bmatrix} = \begin{bmatrix} \sigma_{22} & \sigma_{21} & \sigma_{23} \\ \sigma_{12} & \sigma_{11} & \sigma_{13} \\ \sigma_{32} & \sigma_{31} & \sigma_{33} \end{bmatrix}, \qquad \begin{bmatrix} \varepsilon'_{11} & \varepsilon'_{12} & \varepsilon'_{13} \\ \varepsilon'_{21} & \varepsilon'_{22} & \varepsilon'_{23} \\ \varepsilon'_{31} & \varepsilon'_{32} & \varepsilon'_{33} \end{bmatrix} = \begin{bmatrix} \varepsilon_{22} & \varepsilon_{21} & \varepsilon_{23} \\ \varepsilon_{12} & \varepsilon_{11} & \varepsilon_{13} \\ \varepsilon_{32} & \varepsilon_{31} & \varepsilon_{33} \end{bmatrix} \tag{4.143}$$

写成Voigt形式，并代入本构关系

$$\begin{matrix} [\boldsymbol{\sigma}'] = [\boldsymbol{\beta}] \cdot [\boldsymbol{\sigma}] \implies [\boldsymbol{\sigma}] = [\boldsymbol{\beta}]^{-1} \cdot [\boldsymbol{\sigma}'] \\ [\boldsymbol{\varepsilon}'] = [\boldsymbol{\beta}] \cdot [\boldsymbol{\varepsilon}] \implies [\boldsymbol{\varepsilon}] = [\boldsymbol{\beta}]^{-1} \cdot [\boldsymbol{\varepsilon}'] \end{matrix} \implies [\boldsymbol{\sigma}] = [\boldsymbol{C}] \cdot [\boldsymbol{\varepsilon}] \implies [\boldsymbol{\sigma}'] = ([\boldsymbol{\beta}] \cdot [\boldsymbol{C}] \cdot [\boldsymbol{\beta}]^{-1}) \cdot [\boldsymbol{\varepsilon}'] \tag{4.144}$$

新坐标系中的本构关系为

$$\begin{bmatrix} \sigma'_{22} \\ \sigma'_{11} \\ \sigma'_{33} \\ \sigma'_{12} \\ \sigma'_{31} \\ \sigma'_{23} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{13} & c_{23} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix} \begin{bmatrix} \varepsilon'_{22} \\ \varepsilon'_{11} \\ \varepsilon'_{33} \\ \varepsilon'_{12} \\ \varepsilon'_{31} \\ \varepsilon'_{23} \end{bmatrix} \implies \begin{bmatrix} \sigma'_{11} \\ \sigma'_{22} \\ \sigma'_{33} \\ \sigma'_{12} \\ \sigma'_{23} \\ \sigma'_{31} \end{bmatrix} = \begin{bmatrix} c_{22} & c_{12} & c_{23} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{23} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{66} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{55} \end{bmatrix} \begin{bmatrix} \varepsilon'_{11} \\ \varepsilon'_{22} \\ \varepsilon'_{33} \\ \varepsilon'_{12} \\ \varepsilon'_{23} \\ \varepsilon'_{31} \end{bmatrix} \tag{4.145}$$

由于横观各向同性，因此在上述坐标变换前后的本构关系应该相同，故必有

$$c_{11} = c_{22}, c_{13} = c_{23}, c_{55} = c_{66} \tag{4.146}$$

此时

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ & c_{11} & c_{13} & 0 & 0 & 0 \\ & & c_{33} & 0 & 0 & 0 \\ & & & c_{44} & 0 & 0 \\ & syms & & & c_{55} & 0 \\ & & & & & c_{55} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{bmatrix} \tag{4.147}$$

4.3.7.3.2 将坐标系绕 z 轴旋转45°

若坐标系绕 z 轴旋转45°，转换矩阵为

$$\boldsymbol{x} = \boldsymbol{\beta} \cdot \boldsymbol{X} \implies \boldsymbol{\beta} = \boldsymbol{e}_i \cdot \boldsymbol{E}_J = \begin{bmatrix} \cos\frac{\pi}{4} & \cos(\frac{\pi}{2} - \frac{\pi}{4}) & 0 \\ \cos(\frac{\pi}{2} + \frac{\pi}{4}) & \cos\frac{\pi}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.148}$$

根据二阶张量的转换规律

$$\sigma_{ij} = \beta_{iM}\beta_{jN}\Sigma_{MN} \implies \boldsymbol{\sigma} = \boldsymbol{\beta} \cdot \boldsymbol{\Sigma} \cdot \boldsymbol{\beta}^T \tag{4.149}$$

新、老坐标系中的应力和应变关系分别为

$$\begin{bmatrix} \sigma'_{11} & \sigma'_{12} & \sigma'_{13} \\ \sigma'_{21} & \sigma'_{22} & \sigma'_{23} \\ \sigma'_{31} & \sigma'_{32} & \sigma'_{33} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(\sigma_{11}+2\sigma_{12}+\sigma_{22}) & \frac{1}{2}(-\sigma_{11}+\sigma_{22}) & \frac{1}{\sqrt{2}}(\sigma_{23}+\sigma_{31}) \\ \frac{1}{2}(-\sigma_{11}+\sigma_{22}) & \frac{1}{2}(\sigma_{11}-2\sigma_{12}+\sigma_{22}) & \frac{1}{\sqrt{2}}(\sigma_{23}-\sigma_{31}) \\ \frac{1}{\sqrt{2}}(\sigma_{23}+\sigma_{31}) & \frac{1}{\sqrt{2}}(\sigma_{23}-\sigma_{31}) & \sigma_{33} \end{bmatrix} \tag{4.150}$$

和

$$\begin{bmatrix} \varepsilon'_{11} & \varepsilon'_{12} & \varepsilon'_{13} \\ \varepsilon'_{21} & \varepsilon'_{22} & \varepsilon'_{23} \\ \varepsilon'_{31} & \varepsilon'_{32} & \varepsilon'_{33} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(\varepsilon_{11}+2\varepsilon_{12}+\varepsilon_{22}) & \frac{1}{2}(-\varepsilon_{11}+\varepsilon_{22}) & \frac{1}{\sqrt{2}}(\varepsilon_{23}+\varepsilon_{31}) \\ \frac{1}{2}(-\varepsilon_{11}+\varepsilon_{22}) & \frac{1}{2}(\varepsilon_{11}-2\varepsilon_{12}+\varepsilon_{22}) & \frac{1}{\sqrt{2}}(\varepsilon_{23}-\varepsilon_{31}) \\ \frac{1}{\sqrt{2}}(\varepsilon_{23}+\varepsilon_{31}) & \frac{1}{\sqrt{2}}(\varepsilon_{23}-\varepsilon_{31}) & \varepsilon_{33} \end{bmatrix} \tag{4.151}$$

那么写成Voigt形式

$$\begin{bmatrix} \sigma'_{11} \\ \sigma'_{22} \\ \sigma'_{33} \\ \sigma'_{12} \\ \sigma'_{23} \\ \sigma'_{31} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} \Longrightarrow [\boldsymbol{\sigma}'] = [\boldsymbol{\beta}] \cdot [\boldsymbol{\sigma}] \Longrightarrow [\boldsymbol{\sigma}] = [\boldsymbol{\beta}]^{-1} \cdot [\boldsymbol{\sigma}'] \tag{4.152}$$

和

$$\begin{bmatrix} \varepsilon'_{11} \\ \varepsilon'_{22} \\ \varepsilon'_{33} \\ \varepsilon'_{12} \\ \varepsilon'_{23} \\ \varepsilon'_{31} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{bmatrix} \Longrightarrow [\boldsymbol{\varepsilon}'] = [\boldsymbol{\beta}] \cdot [\boldsymbol{\varepsilon}] \Longrightarrow [\boldsymbol{\varepsilon}] = [\boldsymbol{\beta}]^{-1} \cdot [\boldsymbol{\varepsilon}'] \tag{4.153}$$

Voigt形式的本构关系为

$$[\boldsymbol{\sigma}] = [\boldsymbol{C}] \cdot [\boldsymbol{\varepsilon}] \tag{4.154}$$

用上上上式和上上式代入

$$[\boldsymbol{\beta}]^{-1} \cdot [\boldsymbol{\sigma}'] = [\boldsymbol{C}] \cdot ([\boldsymbol{\beta}]^{-1} \cdot [\boldsymbol{\varepsilon}']) \Longrightarrow [\boldsymbol{\sigma}'] = [\boldsymbol{\beta}] \cdot \{[\boldsymbol{C}] \cdot ([\boldsymbol{\beta}]^{-1} \cdot [\boldsymbol{\varepsilon}'])\} = ([\boldsymbol{\beta}] \cdot [\boldsymbol{C}] \cdot [\boldsymbol{\beta}]^{-1}) \cdot [\boldsymbol{\varepsilon}'] \tag{4.155}$$

用mathematica求得新坐标系中的弹性系数矩阵

```
In[1]:= Simplify[(1/2  1/2  0  1   0       0
                  1/2  1/2  0  -1  0       0
                  0    0    1  0   0       0
                  -1/2 1/2  0  0   0       0
                  0    0    0  0   1/√2    -1/√2
                  0    0    0  0   1/√2    1/√2).(c11 c12 c13 0   0   0
                                                   c12 c11 c13 0   0   0
                                                   c13 c13 c33 0   0   0
                                                   0   0   0   c44 0   0
                                                   0   0   0   0   c55 0
                                                   0   0   0   0   0   c55).Inverse[(1/2  1/2  0  1   0     0
                                                                                     1/2  1/2  0  -1  0     0
                                                                                     0    0    1  0   0     0
                                                                                     -1/2 1/2  0  0   0     0
                                                                                     0    0    0  0   1/√2  -1/√2
                                                                                     0    0    0  0   1/√2  1/√2)]] // MatrixForm

Out[1]//MatrixForm=
(1/2 (c11 + c12 + c44)   1/2 (c11 + c12 - c44)   c13   0           0     0
 1/2 (c11 + c12 - c44)   1/2 (c11 + c12 + c44)   c13   0           0     0
 c13                     c13                     c33   0           0     0
 0                       0                       0     c11 - c12   0     0
 0                       0                       0     0           c55   0
 0                       0                       0     0           0     c55)
```

代入

$$\begin{bmatrix} \sigma'_{11} \\ \sigma'_{22} \\ \sigma'_{33} \\ \sigma'_{12} \\ \sigma'_{23} \\ \sigma'_{31} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(c_{11}+c_{12}+c_{44}) & \frac{1}{2}(c_{11}+c_{12}-c_{44}) & c_{13} & 0 & 0 & 0 \\ \frac{1}{2}(c_{11}-c_{12}+c_{44}) & \frac{1}{2}(c_{11}+c_{12}+c_{44}) & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{11}-c_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{55} \end{bmatrix} \begin{bmatrix} \varepsilon'_{11} \\ \varepsilon'_{22} \\ \varepsilon'_{33} \\ \varepsilon'_{12} \\ \varepsilon'_{23} \\ \varepsilon'_{31} \end{bmatrix} \tag{4.156}$$

所以可得

$$c_{44} = c_{11} - c_{12} \tag{4.157}$$

最终独立的弹性常数减少到5个。

$$\begin{bmatrix}\sigma_{11}\\\sigma_{22}\\\sigma_{33}\\\sigma_{12}\\\sigma_{23}\\\sigma_{31}\end{bmatrix}=\begin{bmatrix}c_{11}&c_{12}&c_{13}&0&0&0\\&c_{11}&c_{13}&0&0&0\\&&c_{33}&0&0&0\\&&&c_{11}-c_{12}&0&0\\&syms&&&c_{55}&0\\&&&&&c_{55}\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\\varepsilon_{22}\\\varepsilon_{33}\\\varepsilon_{12}\\\varepsilon_{23}\\\varepsilon_{31}\end{bmatrix}\tag{4.158}$$

工程上一般用两个弹性模量 E_{xy}, E_z，两个泊松比 μ_{xy}, μ_z 和一个切变模量 G 表示。地层、层状岩体、复合板材等可简化为横观各向同性弹性材料。

4.3.7.4 各向同性线弹性体(2)

在横观各向同性材料的基础上，将 x 轴与 z 轴互换，或将 y 轴与 z 轴互换时，材料弹性关系不变，这种材料称为各向同性材料。

4.3.7.4.1 将 x 轴与 z 轴互换

若将 x 轴与 z 轴互换作为新坐标系，转换矩阵为

$$\boldsymbol{x}=\boldsymbol{\beta}\cdot\boldsymbol{X}\implies\boldsymbol{\beta}=\boldsymbol{e}_i\cdot\boldsymbol{E}_J=\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}\tag{4.159}$$

根据二阶张量的转换规律

$$\sigma_{ij}=\beta_{iM}\beta_{jN}\Sigma_{MN}\implies\boldsymbol{\sigma}=\boldsymbol{\beta}\cdot\boldsymbol{\Sigma}\cdot\boldsymbol{\beta}^T\tag{4.160}$$

新、老坐标系中的应力和应变关系分别为

$$\begin{bmatrix}\sigma'_{11}&\sigma'_{12}&\sigma'_{13}\\\sigma'_{21}&\sigma'_{22}&\sigma'_{23}\\\sigma'_{31}&\sigma'_{32}&\sigma'_{33}\end{bmatrix}=\begin{bmatrix}\sigma_{33}&\sigma_{32}&\sigma_{31}\\\sigma_{23}&\sigma_{22}&\sigma_{21}\\\sigma_{13}&\sigma_{12}&\sigma_{11}\end{bmatrix},\qquad\begin{bmatrix}\varepsilon'_{11}&\varepsilon'_{12}&\varepsilon'_{13}\\\varepsilon'_{21}&\varepsilon'_{22}&\varepsilon'_{23}\\\varepsilon'_{31}&\varepsilon'_{32}&\varepsilon'_{33}\end{bmatrix}=\begin{bmatrix}\varepsilon_{33}&\varepsilon_{32}&\varepsilon_{31}\\\varepsilon_{23}&\varepsilon_{22}&\varepsilon_{21}\\\varepsilon_{13}&\varepsilon_{12}&\varepsilon_{11}\end{bmatrix}\tag{4.161}$$

写成Voigt形式，并代入本构关系

$$\begin{matrix}[\boldsymbol{\sigma}']=[\boldsymbol{\beta}]\cdot[\boldsymbol{\sigma}]\implies[\boldsymbol{\sigma}]=[\boldsymbol{\beta}]^{-1}\cdot[\boldsymbol{\sigma}']\\{}[\boldsymbol{\varepsilon}']=[\boldsymbol{\beta}]\cdot[\boldsymbol{\varepsilon}]\implies[\boldsymbol{\varepsilon}]=[\boldsymbol{\beta}]^{-1}\cdot[\boldsymbol{\varepsilon}']\end{matrix}\implies[\boldsymbol{\sigma}]=[\boldsymbol{C}]\cdot[\boldsymbol{\varepsilon}]\implies[\boldsymbol{\sigma}']=([\boldsymbol{\beta}]\cdot[\boldsymbol{C}]\cdot[\boldsymbol{\beta}]^{-1})\cdot[\boldsymbol{\varepsilon}']\tag{4.162}$$

新坐标系中的本构关系为

$$\begin{bmatrix}\sigma'_{33}\\\sigma'_{22}\\\sigma'_{11}\\\sigma'_{23}\\\sigma'_{12}\\\sigma'_{31}\end{bmatrix}=\begin{bmatrix}c_{11}&c_{12}&c_{13}&0&0&0\\&c_{11}&c_{13}&0&0&0\\&&c_{33}&0&0&0\\&&&c_{11}-c_{12}&0&0\\&syms&&&c_{55}&0\\&&&&&c_{55}\end{bmatrix}\begin{bmatrix}\varepsilon'_{33}\\\varepsilon'_{22}\\\varepsilon'_{11}\\\varepsilon'_{23}\\\varepsilon'_{12}\\\varepsilon'_{31}\end{bmatrix}\implies\begin{bmatrix}\sigma'_{11}\\\sigma'_{22}\\\sigma'_{33}\\\sigma'_{12}\\\sigma'_{23}\\\sigma'_{31}\end{bmatrix}=\begin{bmatrix}c_{33}&c_{13}&c_{13}&0&0&0\\c_{13}&c_{11}&c_{12}&0&0&0\\c_{13}&c_{12}&c_{11}&0&0&0\\0&0&0&c_{55}&0&0\\0&0&0&0&c_{11}-c_{12}&0\\0&0&0&0&0&c_{55}\end{bmatrix}\begin{bmatrix}\varepsilon'_{11}\\\varepsilon'_{22}\\\varepsilon'_{33}\\\varepsilon'_{12}\\\varepsilon'_{23}\\\varepsilon'_{31}\end{bmatrix}\tag{4.163}$$

所以

$$c_{11}=c_{33},c_{12}=c_{13},c_{55}=c_{11}-c_{22}\tag{4.164}$$

最终

$$\begin{bmatrix}\sigma_{11}\\\sigma_{22}\\\sigma_{33}\\\sigma_{12}\\\sigma_{23}\\\sigma_{31}\end{bmatrix}=\begin{bmatrix}c_{11}&c_{12}&c_{12}&0&0&0\\&c_{11}&c_{12}&0&0&0\\&&c_{11}&0&0&0\\&&&c_{11}-c_{12}&0&0\\&syms&&&c_{11}-c_{12}&0\\&&&&&c_{11}-c_{12}\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\\varepsilon_{22}\\\varepsilon_{33}\\\varepsilon_{12}\\\varepsilon_{23}\\\varepsilon_{31}\end{bmatrix}\tag{4.165}$$

4.3.7.4.2 将 y 轴与 z 轴互换

若将 y 轴与 z 轴互换作为新坐标系，转换矩阵为

$$\boldsymbol{x}=\boldsymbol{\beta}\cdot\boldsymbol{X}\implies\boldsymbol{\beta}=\boldsymbol{e}_i\cdot\boldsymbol{E}_J=\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}\tag{4.166}$$

根据二阶张量的转换规律

$$\sigma_{ij}=\beta_{iM}\beta_{jN}\Sigma_{MN}\implies\boldsymbol{\sigma}=\boldsymbol{\beta}\cdot\boldsymbol{\Sigma}\cdot\boldsymbol{\beta}^T\tag{4.167}$$

新、老坐标系中的应力和应变关系分别为

$$\begin{bmatrix}\sigma'_{11}&\sigma'_{12}&\sigma'_{13}\\\sigma'_{21}&\sigma'_{22}&\sigma'_{23}\\\sigma'_{31}&\sigma'_{32}&\sigma'_{33}\end{bmatrix}=\begin{bmatrix}\sigma_{11}&\sigma_{13}&\sigma_{12}\\\sigma_{31}&\sigma_{33}&\sigma_{32}\\\sigma_{21}&\sigma_{23}&\sigma_{22}\end{bmatrix},\qquad\begin{bmatrix}\varepsilon'_{11}&\varepsilon'_{12}&\varepsilon'_{13}\\\varepsilon'_{21}&\varepsilon'_{22}&\varepsilon'_{23}\\\varepsilon'_{31}&\varepsilon'_{32}&\varepsilon'_{33}\end{bmatrix}=\begin{bmatrix}\varepsilon_{11}&\varepsilon_{13}&\varepsilon_{12}\\\varepsilon_{31}&\varepsilon_{33}&\varepsilon_{32}\\\varepsilon_{21}&\varepsilon_{23}&\varepsilon_{22}\end{bmatrix}\tag{4.168}$$

写成Voigt形式，并代入本构关系

$$\begin{matrix}[\boldsymbol{\sigma}']=[\boldsymbol{\beta}]\cdot[\boldsymbol{\sigma}]\implies[\boldsymbol{\sigma}]=[\boldsymbol{\beta}]^{-1}\cdot[\boldsymbol{\sigma}']\\ [\boldsymbol{\varepsilon}']=[\boldsymbol{\beta}]\cdot[\boldsymbol{\varepsilon}]\implies[\boldsymbol{\varepsilon}]=[\boldsymbol{\beta}]^{-1}\cdot[\boldsymbol{\varepsilon}']\end{matrix}\implies[\boldsymbol{\sigma}]=[\boldsymbol{C}]\cdot[\boldsymbol{\varepsilon}]\implies[\boldsymbol{\sigma}']=([\boldsymbol{\beta}]\cdot[\boldsymbol{C}]\cdot[\boldsymbol{\beta}]^{-1})\cdot[\boldsymbol{\varepsilon}']\tag{4.169}$$

新坐标系中的本构关系为

$$\begin{bmatrix}\sigma'_{11}\\\sigma'_{33}\\\sigma'_{22}\\\sigma'_{13}\\\sigma'_{23}\\\sigma'_{12}\end{bmatrix}=\begin{bmatrix}c_{11}&c_{12}&c_{12}&0&0&0\\&c_{11}&c_{12}&0&0&0\\&&c_{11}&0&0&0\\&&&c_{11}-c_{12}&0&0\\&syms&&&c_{11}-c_{12}&0\\&&&&&c_{11}-c_{12}\end{bmatrix}\begin{bmatrix}\varepsilon'_{11}\\\varepsilon'_{33}\\\varepsilon'_{22}\\\varepsilon'_{13}\\\varepsilon'_{23}\\\varepsilon'_{12}\end{bmatrix}\implies\begin{bmatrix}\sigma'_{11}\\\sigma'_{22}\\\sigma'_{33}\\\sigma'_{12}\\\sigma'_{23}\\\sigma'_{31}\end{bmatrix}=\begin{bmatrix}c_{11}&c_{12}&c_{12}&0&0&0\\c_{12}&c_{11}&c_{12}&0&0&0\\c_{12}&c_{12}&c_{11}&0&0&0\\0&0&0&c_{11}-c_{12}&0&0\\0&0&0&0&c_{11}-c_{12}&0\\0&0&0&0&0&c_{11}-c_{12}\end{bmatrix}\begin{bmatrix}\varepsilon'_{11}\\\varepsilon'_{22}\\\varepsilon'_{33}\\\varepsilon'_{12}\\\varepsilon'_{23}\\\varepsilon'_{31}\end{bmatrix}\tag{4.170}$$

最终

$$\begin{bmatrix}\sigma_{11}\\\sigma_{22}\\\sigma_{33}\\\sigma_{12}\\\sigma_{23}\\\sigma_{31}\end{bmatrix}=\begin{bmatrix}c_{11}&c_{12}&c_{12}&0&0&0\\&c_{11}&c_{12}&0&0&0\\&&c_{11}&0&0&0\\&&&c_{11}-c_{12}&0&0\\&syms&&&c_{11}-c_{12}&0\\&&&&&c_{11}-c_{12}\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\\varepsilon_{22}\\\varepsilon_{33}\\\varepsilon_{12}\\\varepsilon_{23}\\\varepsilon_{31}\end{bmatrix}\tag{4.171}$$

可见弹性系数矩阵没有变化。最终，独立的弹性系数减少到2个。令

$$c_{11}=\lambda+2\mu,\quad c_{12}=\lambda\tag{4.172}$$

即可转换成常见形式

$$\begin{bmatrix}\sigma_{11}\\\sigma_{22}\\\sigma_{33}\\\sigma_{12}\\\sigma_{23}\\\sigma_{31}\end{bmatrix}=\begin{bmatrix}\lambda+2\mu&\lambda&\lambda&0&0&0\\\lambda&\lambda+2\mu&\lambda&0&0&0\\\lambda&\lambda&\lambda+2\mu&0&0&0\\0&0&0&2\mu&0&0\\0&0&0&0&2\mu&0\\0&0&0&0&0&2\mu\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\\varepsilon_{22}\\\varepsilon_{33}\\\varepsilon_{12}\\\varepsilon_{23}\\\varepsilon_{31}\end{bmatrix}\tag{4.173}$$

4.3.8 拉梅常数和弹性模量、泊松比的关系

$$\lambda=\frac{E\nu}{(1+\nu)(1-2\nu)},\quad\mu=\frac{E}{2(1+\nu)}\tag{4.174}$$

4.4 实功

4.4.1 柯西应力表示的实功

$$
\begin{aligned}
\dot{U} &= \int_{\partial\Omega} t_i v_i da + \int_{\Omega} \rho b_i v_i dv = \int_{\partial\Omega} n_j \sigma_{ji} v_i da + \int_{\Omega} \rho b_i v_i dv = \int_{\Omega} \frac{\partial}{\partial x_j}(\sigma_{ji} v_i) dv + \int_{\Omega} \rho b_i v_i dv \\
&= \int_{\Omega} (\frac{\partial \sigma_{ji}}{\partial x_j} v_i + \sigma_{ji} \frac{\partial v_i}{\partial x_j}) dv + \int_{\Omega} \rho b_i v_i dv = \int_{\Omega} [\sigma_{ji} \frac{\partial v_i}{\partial x_j} + v_i (\frac{\partial \sigma_{ji}}{\partial x_j} + \rho b_i)] dv = \int_{\Omega} (\sigma_{ji} \frac{\partial v_i}{\partial x_j} + v_i \rho a_i) dv \\
&= \int_{\Omega} (\frac{1}{2} \sigma_{ji} \frac{\partial v_i}{\partial x_j} + \frac{1}{2} \sigma_{ji} \frac{\partial v_i}{\partial x_j} + v_i \rho a_i) dv \\
&\quad \Downarrow \quad \sigma_{ij} = \sigma_{ji} \\
&= \int_{\Omega} (\frac{1}{2} \sigma_{ij} \frac{\partial v_j}{\partial x_i} + \frac{1}{2} \sigma_{ij} \frac{\partial v_i}{\partial x_j} + v_i \rho a_i) dv = \int_{\Omega} \sigma_{ij} [\frac{1}{2} (\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j})] dv + \frac{d}{dt} \int_{\Omega} \frac{1}{2} \rho v_i v_i dv = \int_{\Omega} \sigma_{ij} \dot{\varepsilon}_{ij} dv + \frac{d}{dt} \int_{\Omega} \frac{1}{2} \rho v_i v_i dv
\end{aligned}
\tag{4.175}
$$

4.4.2 名义应力表示的实功

$$
\begin{aligned}
\dot{U} &= \int_{\partial\Omega} t_i v_i da + \int_{\Omega} \rho b_i v_i dv = \int_{\partial\Omega} n_j \sigma_{ji} v_i da + \int_{\Omega} \rho b_i v_i dv = \int_{\partial\Omega_0} N_J S_{Ji} v_i dA + \int_{\Omega_0} \rho_0 b_i v_i dV \\
&= \int_{\Omega_0} \frac{\partial}{\partial X_J} (S_{Ji} v_i) dV + \int_{\Omega_0} \rho_0 b_i v_i dV = \int_{\Omega_0} (\frac{\partial S_{Ji}}{\partial X_J} v_i + S_{Ji} \frac{\partial v_i}{\partial X_J}) dV + \int_{\Omega_0} \rho_0 b_i v_i dV = \int_{\Omega_0} [S_{Ji} \frac{\partial v_i}{\partial X_J} + (\frac{\partial S_{Ji}}{\partial X_J} + \rho_0 b_i) v_i] dV \\
&= \int_{\Omega_0} [S_{Ji} \frac{\partial v_i}{\partial X_J} + \rho_0 a_i v_i] dV \\
&\quad \Downarrow \quad F_{iJ} = \delta_{iJ} + \frac{\partial u_i}{\partial X_J} \Rightarrow \dot{F}_{iJ} = \frac{\partial \dot{u}_i}{\partial X_J} = \frac{\partial v_i}{\partial X_J} \\
&= \int_{\Omega_0} S_{Ji} \dot{F}_{iJ} dV + \frac{d}{dt} \int_{\Omega_0} \frac{1}{2} \rho_0 v_i v_i dV
\end{aligned}
\tag{4.176}
$$

4.4.3 物质应力表示的实功

$$
\begin{aligned}
S_{Ji} \dot{F}_{iJ} &= \Sigma_{JR} F_{iR} \dot{F}_{iJ} = \frac{1}{2} (\Sigma_{JR} + \Sigma_{RJ}) F_{iR} \dot{F}_{iJ} = \frac{1}{2} \Sigma_{JR} F_{iR} \dot{F}_{iJ} + \frac{1}{2} \Sigma_{RJ} F_{iR} \dot{F}_{iJ} \\
&= \frac{1}{2} \Sigma_{JR} F_{iR} \dot{F}_{iJ} + \frac{1}{2} \Sigma_{JR} F_{iJ} \dot{F}_{iR} = \Sigma_{JR} [\frac{1}{2} (F_{iR} \dot{F}_{iJ} + F_{iJ} \dot{F}_{iR})] = \Sigma_{JR} \dot{E}_{RJ} = \Sigma_{RJ} \dot{E}_{RJ}
\end{aligned}
\tag{4.177}
$$

其中

$$
\begin{aligned}
&\boldsymbol{F}^T \cdot \boldsymbol{F} = F_{iL} \boldsymbol{e}_L \otimes \boldsymbol{e}_i \cdot F_{jM} \boldsymbol{e}_j \otimes \boldsymbol{e}_M = F_{iL} F_{jM} \delta_{ij} \boldsymbol{e}_L \otimes \boldsymbol{e}_M = F_{iL} F_{iM} \boldsymbol{e}_L \otimes \boldsymbol{e}_M \\
&\qquad \Downarrow \\
&\qquad \boldsymbol{E} = \frac{1}{2} (\boldsymbol{F}^T \cdot \boldsymbol{F} - \boldsymbol{I}) = \frac{1}{2} (F_{mI} F_{mJ} \boldsymbol{e}_I \otimes \boldsymbol{e}_J - \delta_{IJ} \boldsymbol{e}_I \otimes \boldsymbol{e}_J) = \frac{1}{2} (F_{mI} F_{mJ} - \delta_{IJ}) \boldsymbol{e}_I \otimes \boldsymbol{e}_J \\
&\qquad \Downarrow \\
&\qquad \dot{\boldsymbol{E}} = \frac{1}{2} (F_{mI} \dot{F}_{mJ} + \dot{F}_{mI} F_{mJ}) \boldsymbol{e}_I \otimes \boldsymbol{e}_J
\end{aligned}
\tag{4.178}
$$

4.5 虚功原理

计算准备：

$$
\sigma_{ij} \delta D_{ij} = \sigma_{ij} [\frac{1}{2} (\frac{\partial \delta v_i}{\partial x_j} + \frac{\partial \delta v_j}{\partial x_i})] = \frac{1}{2} (\sigma_{ij} \frac{\partial \delta v_i}{\partial x_j} + \sigma_{ij} \frac{\partial \delta v_j}{\partial x_i}) = \frac{1}{2} (\sigma_{ji} \frac{\partial \delta v_i}{\partial x_j} + \sigma_{ji} \frac{\partial \delta v_i}{\partial x_j}) = \sigma_{ji} \frac{\partial \delta v_i}{\partial x_j} = \frac{\partial}{\partial x_j} (\sigma_{ji} \delta v_i) - \frac{\partial \sigma_{ji}}{\partial x_j} \delta v_i \tag{4.179}
$$

$$
\int_{\Omega} \sigma_{ij} \delta D_{ij} dv = \int_{\Omega} [\frac{\partial}{\partial x_j} (\sigma_{ji} \delta v_i) - \frac{\partial \sigma_{ji}}{\partial x_j} \delta v_i] dv = \int_{\partial\Omega_t} n_j \sigma_{ji} \delta v_i dv - \int_{\Omega} \frac{\partial \sigma_{ji}}{\partial x_j} \delta v_i dv \tag{4.180}
$$

$$
\begin{aligned}
&\int_{\Omega} \sigma_{ij} \delta D_{ij} dv + \int_{\Omega} \rho a_i \delta v_i dv - \int_{\Omega} \rho b_i \delta v_i dv - \int_{\partial_t\Omega} t_i \delta v_i da = 0 \\
&\qquad \Downarrow \\
&- \int_{\Omega} (\frac{\partial \sigma_{ji}}{\partial x_j} + \rho b_i - \rho a_i) \delta v_i dv + \int_{\partial\Omega_t} (n_j \sigma_{ji} - t_i) \delta v_i da = 0
\end{aligned}
\tag{4.181}
$$

由δv_i任意性，必有

$$\begin{cases} \dfrac{\partial \sigma_{ji}}{\partial x_j} + \rho b_i - \rho a_i = 0 \qquad \text{in}\Omega \\ n_j\sigma_{ji} - t_i = 0 \qquad \text{on}\partial\Omega \end{cases} \tag{4.182}$$

若应力满足虚功方程，必满足控制方程与边界条件，由实功推导可直接写出虚功形式!!!

4.5.1 能量函数的构造

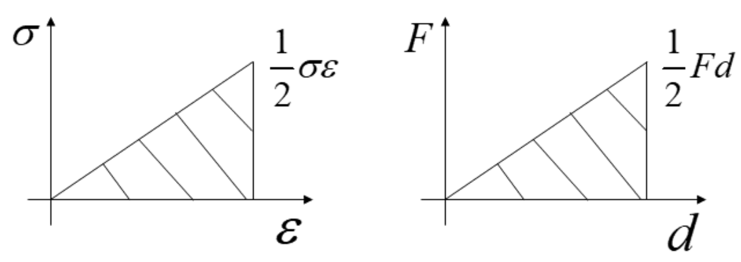
由本构关系和功共轭关系构造能量函数

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} = \frac{\partial U}{\partial \varepsilon_{ij}} = \frac{1}{2}C_{ijkl}(\varepsilon_{kl} + \varepsilon_{kl}) = \frac{1}{2}C_{ijkl}(\frac{\partial \varepsilon_{ij}}{\partial \varepsilon_{ij}}\varepsilon_{kl} + \varepsilon_{ij}\frac{\partial \varepsilon_{kl}}{\partial \varepsilon_{ij}}) = \frac{1}{2}C_{ijkl}\frac{\partial \varepsilon_{ij}\varepsilon_{kl}}{\partial \varepsilon_{ij}} = \frac{\partial \frac{1}{2}C_{ijkl}\varepsilon_{ij}\varepsilon_{kl}}{\partial \varepsilon_{ij}} \tag{4.183}$$

所以，应变能密度函数定义为

$$U = \frac{1}{2}C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} \tag{4.184}$$

所以，对于线性本构关系，其能量密度就是以上形式。



4.6 应力极值问题

4.6.1 实对称张量的特征值与特征矢量

实对称张量必定存在实数特征值和相互正交的实特征矢量。若张量不对称，则既不能保证存在实数的特征值，也不可能通过旋转坐标系将其简化为对角形式。因此对称性是一个很重要的优点。

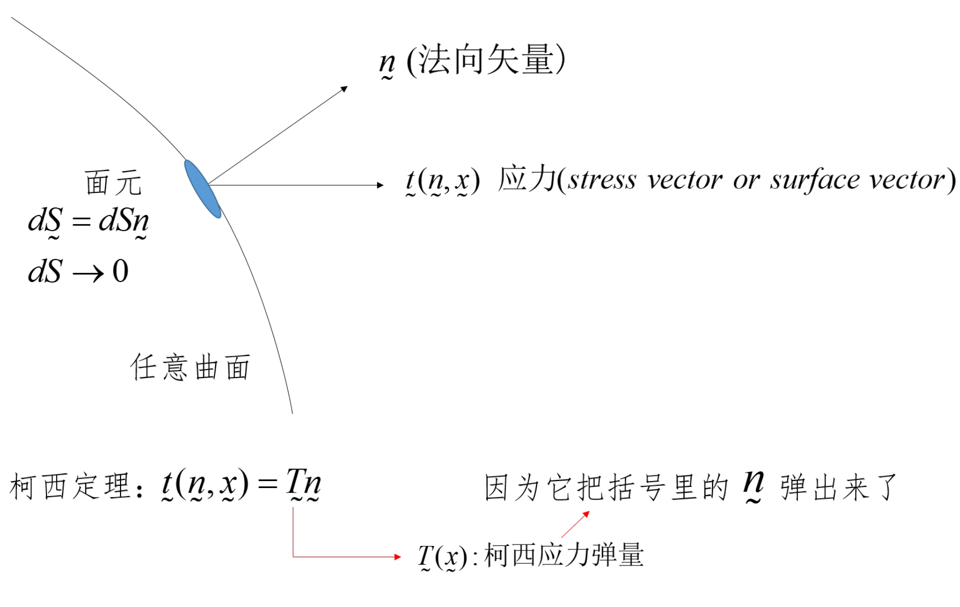
不论你怎样选取坐标系，张量唯一对应一组特征值与一组特征矢量，即特征值与特征矢量与坐标系无关?。

应力张量的特征矢量组成的坐标系的坐标轴称为主轴，垂直于主轴的平面的称为主平面，主平面上的正应力称为主应力，在主平面上应力矢量垂直于该平面且没有切应力分量。应力张量的特征值对应主应力的大小。

已知过一个点的某个平面的法向矢量为$\boldsymbol{n}$，应力矢量为$\boldsymbol{t} = \sigma\boldsymbol{n}$，由柯西定理$\boldsymbol{t} = \boldsymbol{\sigma n}$，则 $\boldsymbol{\sigma n} = \sigma\boldsymbol{n} \Rightarrow \boldsymbol{\sigma n} - \sigma\boldsymbol{n} = \boldsymbol{0} \Rightarrow \boldsymbol{\sigma n} - \sigma\boldsymbol{In} = \boldsymbol{0} \Rightarrow (\boldsymbol{\sigma} - \sigma\boldsymbol{I})\boldsymbol{n} = \boldsymbol{0}$，形成应力张量的特征值问题。

4.6.2 应力极值问题：最大/最小正应力和切应力

4.6.2.1 正应力的极值问题



$\boldsymbol{t}$是应力矢量(stress vector)，$\boldsymbol{n}$是曲面的单位法向矢量(the normal to the plane)。应力矢量在法线上的投影为正应力分量，即$\boldsymbol{n}\cdot\boldsymbol{t}=\boldsymbol{n}\cdot(\boldsymbol{n}\cdot\boldsymbol{T})$。对于一个应力张量$T$，定义其正应力分量(normal component of the stress tensor)为

$$T_n(\boldsymbol{n})=\boldsymbol{n}\cdot(\boldsymbol{n}\cdot\boldsymbol{T})$$

令

$$f(\boldsymbol{n},\lambda)=T_n-\lambda(\boldsymbol{n}\cdot\boldsymbol{n}-1)=\boldsymbol{n}\cdot(\boldsymbol{n}\cdot\boldsymbol{T})-\lambda(\boldsymbol{n}\cdot\boldsymbol{n}-1)=n_in_jT_{ij}-\lambda(n_kn_k-1)$$

极值条件为：

$$\frac{\partial f}{\partial \boldsymbol{n}}=0,\quad \frac{\partial f}{\partial \lambda}=0$$

其中

$$\begin{aligned}\frac{\partial f}{\partial \boldsymbol{n}}&=\frac{\partial}{\partial n_m}[n_in_jT_{ij}-\lambda(n_kn_k-1)]=\frac{\partial n_i}{\partial n_m}n_jT_{ij}+n_i\frac{\partial n_j}{\partial n_m}T_{ij}-\lambda(\frac{\partial n_k}{\partial n_m}n_k+n_k\frac{\partial n_k}{\partial n_m})\\&=\delta_{im}n_jT_{ij}+n_i\delta_{jm}T_{ij}-\lambda(\delta_{km}n_k+n_k\delta_{km})=n_jT_{mj}+n_iT_{im}-\lambda(n_m+n_m)=n_jT_{mj}+n_jT_{mj}-\lambda(2n_m)\\&=2(n_jT_{mj}-\lambda n_m)=2(\boldsymbol{n}\cdot\boldsymbol{T}-\lambda\boldsymbol{n})=2(\boldsymbol{n}\cdot\boldsymbol{T}-\boldsymbol{n}\cdot\lambda\boldsymbol{I})=2\boldsymbol{n}\cdot(\boldsymbol{T}-\lambda\boldsymbol{I})=0\\\frac{\partial f}{\partial \lambda}&=\boldsymbol{n}\cdot\boldsymbol{n}-1=0\end{aligned}\tag{4.185}$$

总结：

最大（或最小）正应力发生在一个平面，这个平面的单位法向矢量是应力张量T的特征矢量。

正应力的极值大小等于应力张量特征值的大小 $T_n|_{max}=\boldsymbol{n}\cdot(\boldsymbol{n}\cdot\boldsymbol{T})=\boldsymbol{n}\cdot(\boldsymbol{n}\cdot\lambda)=\lambda$。

特征值问题若想有解，要求系数矩阵的行列式为零

$$det(\boldsymbol{T}-\lambda\boldsymbol{I})=\begin{vmatrix}T_{11}-\lambda & T_{12} & T_{13}\\T_{21} & T_{22}-\lambda & T_{23}\\T_{31} & T_{32} & T_{33}-\lambda\end{vmatrix}=-\lambda^3+I_1\lambda^2-I_2\lambda+I_3=0\tag{4.186}$$

其中

$$I_1=T_{11}+T_{22}+T_{33},\quad I_2=\begin{vmatrix}T_{22}&T_{23}\\T_{32}&T_{33}\end{vmatrix}+\begin{vmatrix}T_{33}&T_{31}\\T_{13}&T_{11}\end{vmatrix}+\begin{vmatrix}T_{11}&T_{12}\\T_{21}&T_{22}\end{vmatrix},\quad I_3=\begin{vmatrix}T_{11}&T_{12}&T_{13}\\T_{21}&T_{22}&T_{23}\\T_{31}&T_{32}&T_{33}\end{vmatrix}\tag{4.187}$$

若$\lambda_1,\lambda_2,\lambda_3$是方程的根，则可以写为

$$(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)=0\tag{4.188}$$

展开(可以用mathematica)

$$\lambda^3+(-\lambda_1-\lambda_2-\lambda_3)\lambda^2+(\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3)\lambda-\lambda_1\lambda_2\lambda_3=0\tag{4.189}$$

可以看出，根与系数之间存在如下关系

$$I_1=\lambda_1+\lambda_2+\lambda_3,\quad I_2=\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3,\quad I_3=\lambda_1\lambda_2\lambda_3\tag{4.190}$$

4.6.2.2 切应力的极值问题

$$\begin{aligned}T_n&=\boldsymbol{n}\cdot\boldsymbol{t}=\boldsymbol{n}\cdot(\boldsymbol{n}\cdot\boldsymbol{T})\\T_s&=|\boldsymbol{t}-T_n\boldsymbol{n}|=|\boldsymbol{t}-(\boldsymbol{n}\cdot\boldsymbol{t})\boldsymbol{n}|\end{aligned}\tag{4.191}$$

问题表述为：在条件$n\cdot n=1$，寻找T_s的极大值和极小值。为了方便分析，我们考虑T_s^2在条件$\boldsymbol{n}\cdot\boldsymbol{n}=1$下的极值.

$$\begin{aligned}T_s^2&=[\boldsymbol{t}-(\boldsymbol{n}\cdot\boldsymbol{t})\boldsymbol{n}]\cdot[\boldsymbol{t}-(\boldsymbol{n}\cdot\boldsymbol{t})\boldsymbol{n}]=\boldsymbol{t}\cdot\boldsymbol{t}-(\boldsymbol{n}\cdot\boldsymbol{t})^2-(\boldsymbol{n}\cdot\boldsymbol{t})^2+(\boldsymbol{n}\cdot\boldsymbol{t})^2=\boldsymbol{t}\cdot\boldsymbol{t}-(\boldsymbol{n}\cdot\boldsymbol{t})^2\\&=(\boldsymbol{n}\cdot\boldsymbol{T})\cdot(\boldsymbol{n}\cdot\boldsymbol{T})-[\boldsymbol{n}\cdot(\boldsymbol{n}\cdot\boldsymbol{T})]^2\end{aligned}\tag{4.192}$$

令

$$g(\boldsymbol{n},\mu)=(\boldsymbol{n}\cdot\boldsymbol{T})\cdot(\boldsymbol{n}\cdot\boldsymbol{T})-[\boldsymbol{n}\cdot(\boldsymbol{n}\cdot\boldsymbol{T})]^2-\mu(\boldsymbol{n}\cdot\boldsymbol{n}-1)=n_in_pT_{ij}T_{pj}-(n_in_jT_{ij})(n_pn_qT_{pq})-\mu(n_kn_k-1)\tag{4.193}$$

极值条件为

$$\frac{\partial g}{\partial \boldsymbol{n}}=0,\quad \frac{\partial g}{\partial \mu}=0 \tag{4.194}$$

其中

$$\begin{aligned}
\frac{\partial}{\partial n_m}(n_i n_p T_{ij} T_{pj}) &= \frac{\partial n_i}{\partial n_m}(n_p T_{ij} T_{pj}) + n_i \frac{\partial n_p}{\partial n_m}(T_{ij} T_{pj}) = n_p T_{mj} T_{pj} + n_i T_{ij} T_{mj} = 2 n_i T_{ij} T_{mj} = 2(\boldsymbol{n}\cdot\boldsymbol{T})\cdot\boldsymbol{T}^T = 2\boldsymbol{T}\cdot(\boldsymbol{n}\cdot\boldsymbol{T}) \\
\frac{\partial}{\partial n_m}[(n_i n_j T_{ij})(n_p n_q T_{pq})] &= [\frac{\partial}{\partial n_m}(n_i n_j T_{ij})](n_p n_q T_{pq}) + (n_i n_j T_{ij})[\frac{\partial}{\partial n_m}(n_p n_q T_{pq})] \\
&= [n_j T_{mj} + n_i T_{im}](n_p n_q T_{pq}) + (n_i n_j T_{ij})[n_q T_{mq} + n_p T_{pm}] \\
&= [n_q T_{mq} + n_p T_{pm}](n_i n_j T_{ij}) + (n_i n_j T_{ij})[n_q T_{mq} + n_p T_{pm}] \\
&= 2(n_i n_j T_{ij})[n_q T_{mq} + n_p T_{pm}] \\
&= 4(n_i n_j T_{ij})(n_p T_{pm}) \\
&= 4[\boldsymbol{n}\cdot(\boldsymbol{n}\cdot\boldsymbol{T})](\boldsymbol{n}\cdot\boldsymbol{T})
\end{aligned} \tag{4.195}$$

极值条件具体为

$$\begin{aligned}
\frac{\partial g}{\partial \boldsymbol{n}} &= 2 n_i T_{ij} T_{mj} - 4(n_i n_j T_{ij})(n_p T_{pm}) - 2\mu n_m = 2\boldsymbol{T}\cdot(\boldsymbol{n}\cdot\boldsymbol{T}) - 4[\boldsymbol{n}\cdot(\boldsymbol{n}\cdot\boldsymbol{T})](\boldsymbol{n}\cdot\boldsymbol{T}) - 2\mu\boldsymbol{n} = 0 \\
\frac{\partial g}{\partial \mu} &= \boldsymbol{n}\cdot\boldsymbol{n} = 1
\end{aligned} \tag{4.196}$$

假设$\boldsymbol{n}=\frac{1}{\sqrt{2}}(\boldsymbol{n}_1+\boldsymbol{n}_2), \boldsymbol{n}_1\cdot\boldsymbol{n}_1=1, \boldsymbol{n}_2\cdot\boldsymbol{n}_2=1, \boldsymbol{n}_1\cdot\boldsymbol{n}_2=0$,然后有

$$\begin{aligned}
\boldsymbol{n}\cdot\boldsymbol{T} &= \frac{1}{\sqrt{2}}(\boldsymbol{n}_1+\boldsymbol{n}_2)\cdot\boldsymbol{T} = \frac{1}{\sqrt{2}}(\boldsymbol{n}_1\cdot\boldsymbol{T}+\boldsymbol{n}_2\cdot\boldsymbol{T}) = \frac{1}{\sqrt{2}}(\lambda_1\boldsymbol{n}_1+\lambda_2\boldsymbol{n}_2) \\
&\Downarrow \\
\boldsymbol{n}\cdot(\boldsymbol{n}\cdot\boldsymbol{T}) &= \boldsymbol{n}\cdot\frac{1}{\sqrt{2}}(\lambda_1\boldsymbol{n}_1+\lambda_2\boldsymbol{n}_2) = \frac{1}{\sqrt{2}}(\boldsymbol{n}_1+\boldsymbol{n}_2)\cdot\frac{1}{\sqrt{2}}(\lambda_1\boldsymbol{n}_1+\lambda_2\boldsymbol{n}_2) = \boxed{\frac{\lambda_1+\lambda_2}{2}} \\
&\Downarrow \\
\frac{\partial g}{\partial \boldsymbol{n}} &= \boldsymbol{T}\cdot(\boldsymbol{n}\cdot\boldsymbol{T}) - 2[\boldsymbol{n}\cdot(\boldsymbol{n}\cdot\boldsymbol{T})](\boldsymbol{n}\cdot\boldsymbol{T}) - \mu\boldsymbol{n} \\
&= \boldsymbol{T}\cdot\frac{1}{\sqrt{2}}(\lambda_1\boldsymbol{n}_1+\lambda_2\boldsymbol{n}_2) - 2\frac{\lambda_1+\lambda_2}{2}\frac{1}{\sqrt{2}}(\lambda_1\boldsymbol{n}_1+\lambda_2\boldsymbol{n}_2) - \mu\boldsymbol{n} \\
&= \frac{1}{\sqrt{2}}(\lambda_1^2\boldsymbol{n}_1+\lambda_2^2\boldsymbol{n}_2) - \frac{1}{\sqrt{2}}(\lambda_1^2\boldsymbol{n}_1+\lambda_1\lambda_2\boldsymbol{n}_1+\lambda_1\lambda_2\boldsymbol{n}_2+\lambda_2^2\boldsymbol{n}_2) - \mu\boldsymbol{n} \\
&= -\frac{1}{\sqrt{2}}\lambda_1\lambda_2(\boldsymbol{n}_1+\boldsymbol{n}_2) - \mu\boldsymbol{n} = -\lambda_1\lambda_2\boldsymbol{n} - \mu\boldsymbol{n} = 0
\end{aligned} \tag{4.197}$$

所以，$\mu=-\lambda_1\lambda_2$

此时，

$$\begin{aligned}
T_s &= \sqrt{(\boldsymbol{n}\cdot\boldsymbol{T})\cdot(\boldsymbol{n}\cdot\boldsymbol{T}) - [\boldsymbol{n}\cdot(\boldsymbol{n}\cdot\boldsymbol{T})]^2} \\
&= \sqrt{\boldsymbol{n}\cdot[\boldsymbol{T}\cdot(\boldsymbol{n}\cdot\boldsymbol{T})] - [\boldsymbol{n}\cdot(\boldsymbol{n}\cdot\boldsymbol{T})]^2} \\
&\Downarrow \text{代入}\frac{\partial g}{\partial \boldsymbol{n}}=0 \\
&= \sqrt{\boldsymbol{n}\cdot\{2[\boldsymbol{n}\cdot(\boldsymbol{n}\cdot\boldsymbol{T})](\boldsymbol{n}\cdot\boldsymbol{T}) + \mu\boldsymbol{n}\} - [\boldsymbol{n}\cdot(\boldsymbol{n}\cdot\boldsymbol{T})]^2} \\
&= \sqrt{\{2[\boldsymbol{n}\cdot(\boldsymbol{n}\cdot\boldsymbol{T})][\boldsymbol{n}\cdot(\boldsymbol{n}\cdot\boldsymbol{T})] + \mu\boldsymbol{n}\cdot\boldsymbol{n}\} - [\boldsymbol{n}\cdot(\boldsymbol{n}\cdot\boldsymbol{T})]^2} \\
&= \sqrt{\{2[\boldsymbol{n}\cdot(\boldsymbol{n}\cdot\boldsymbol{T})]^2 + \mu\boldsymbol{n}\cdot\boldsymbol{n}\} - [\boldsymbol{n}\cdot(\boldsymbol{n}\cdot\boldsymbol{T})]^2} \\
&= \sqrt{[\boldsymbol{n}\cdot(\boldsymbol{n}\cdot\boldsymbol{T})]^2 + \mu\boldsymbol{n}\cdot\boldsymbol{n}} \\
&= \sqrt{[\boldsymbol{n}\cdot(\boldsymbol{n}\cdot\boldsymbol{T})]^2 + \mu} \\
&\Downarrow \text{代入}\boldsymbol{n}\cdot(\boldsymbol{n}\cdot\boldsymbol{T}) = \boxed{\frac{\lambda_1+\lambda_2}{2}} \\
&= \sqrt{(\frac{\lambda_1+\lambda_2}{2})^2 - \lambda_1\lambda_2} \\
&= \frac{|\lambda_1-\lambda_2|}{2}
\end{aligned} \tag{4.198}$$

当$\boldsymbol{n}=\frac{1}{2}(\boldsymbol{n}_1+\boldsymbol{n}_2)$, $\mu=-\lambda_1\lambda_2$，切应力取得极值。此时，切应力大小为最大最小正应力之差的一半，切应力所在平面法向量与最大和最小主应力方向间的夹角为45°。

4.6.2.3 应力张量的主应力，主应力方向以及三个不变量

切应力为零的平面称为主平面。主平面上的正应力称为主应力。

解应力张量的特征值问题得到应力张量的主应力，主应力方向以及三个不变量。

4.6.3 已知应力张量求解主应力

4.6.3.1 解析法-卡丹公式法求解一元三次方程

一般的一元三次方程可写为

$$ax^3+bx^2+cx+d=0(a\neq 0) \tag{4.199}$$

上式除以 a ，并设 $x=y-\frac{b}{3a}$，则可化为如下形式

$$y^3+py+q=0 \tag{4.200}$$

其中

$$p=\frac{3ac-b^2}{3a^2},\quad q=\frac{27a^2d-9abc+2b^3}{27a^3} \tag{4.201}$$

判别式为

$$\Delta=\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3=\Delta_1+\Delta_2 \tag{4.202}$$

当 $\Delta>0$ 时，有一个实根和两个复根；当 $\Delta=0$ 时，有三个实根，当 $p=q=0$ 时，有一个三重零根，$p,q\neq 0$ 时，三个实根中有两个相等；$\Delta<0$ 时，有三个不等实根。

特殊方程的解为

$$\begin{aligned}
y_1&=\sqrt[3]{-\frac{q}{2}+\sqrt{\Delta}}+\sqrt[3]{-\frac{q}{2}-\sqrt{\Delta}}\\
y_2&=\omega\sqrt[3]{-\frac{q}{2}+\sqrt{\Delta}}+\omega^2\sqrt[3]{-\frac{q}{2}-\sqrt{\Delta}}\\
y_3&=\omega^2\sqrt[3]{-\frac{q}{2}+\sqrt{\Delta}}+\omega\sqrt[3]{-\frac{q}{2}-\sqrt{\Delta}}
\end{aligned} \tag{4.203}$$

其中

$$\omega=\frac{-1+\sqrt{3}i}{2} \tag{4.204}$$

最终，可求得

$$x_1=y_1-\frac{b}{3a},\quad x_2=y_2-\frac{b}{3a},\quad x_3=y_3-\frac{b}{3a} \tag{4.205}$$

仅当 $\Delta<0$ 时，三个根可以表示为三角函数表达式

$$\begin{aligned}
y_1&=2\sqrt[3]{R}\cos\theta\\
y_2&=2\sqrt[3]{R}\cos(\theta+\frac{2}{3}\pi)\\
y_3&=2\sqrt[3]{R}\cos(\theta+\frac{4}{3}\pi)
\end{aligned} \tag{4.206}$$

其中

$$R=\sqrt{-\Delta_2},\quad \theta=\frac{1}{3}\arccos\sqrt{-\frac{\Delta_1}{\Delta_2}} \tag{4.207}$$

4.6.3.2 解析法-利用卡丹公式求解主应力

以下过程使用matlab编程即可。

对于主应力的特征值问题

$$\sigma^3-I_1\sigma^2+I_2\sigma-I_3=0 \tag{4.208}$$

三个应力不变量分别为(参考《弹性理论》陆明万等 [1])

$$\begin{aligned} I_1 &= \sigma_{11}+\sigma_{22}+\sigma_{33} \\ I_2 &= \sigma_{11}\sigma_{22}+\sigma_{22}\sigma_{33}+\sigma_{33}\sigma_{11}-\sigma_{12}^2-\sigma_{23}^2-\sigma_{31}^2 \\ I_3 &= \sigma_{11}\sigma_{22}\sigma_{33}+2\sigma_{12}\sigma_{23}\sigma_{31}-\sigma_{11}\sigma_{23}^2-\sigma_{22}\sigma_{31}^2-\sigma_{33}\sigma_{12}^2 \end{aligned} \tag{4.209}$$

一元三次方程的系数为

$$a=1,\quad b=-I_1,\quad c=I_2,\quad d=-I_3,\quad \frac{b}{3a}=-\frac{I_1}{3} \tag{4.210}$$

$$p=\frac{3ac-b^2}{3a^2}=\frac{1}{3}(-I_1^2+3I_2)=-3\left(\left(\frac{I_1}{3}\right)^2-\frac{I_2}{3}\right),\quad q=\frac{27a^2d-9abc+2b^3}{27a^3}=\frac{1}{27}(-2I_1^3+9I_1I_2-27I_3) \tag{4.211}$$

$$\Delta_1=\left(\frac{q}{2}\right)^2,\qquad \Delta_2=\left(\frac{p}{3}\right)^3 \tag{4.212}$$

$$R=\sqrt{-\Delta_2},\quad \theta=\frac{1}{3}\arccos\sqrt{-\frac{\Delta_1}{\Delta_2}} \tag{4.213}$$

$$\begin{aligned} \sigma_1 &= 2\sqrt[3]{R}\cos\theta+\frac{I_1}{3} \\ \sigma_2 &= 2\sqrt[3]{R}\cos(\theta+\frac{2}{3}\pi)+\frac{I_1}{3} \\ \sigma_3 &= 2\sqrt[3]{R}\cos(\theta+\frac{4}{3}\pi)+\frac{I_1}{3} \end{aligned} \tag{4.214}$$

路面力学计算中主应力的求法即为卡丹公式的变形。

4.6.3.3 数值法-求二阶张量的特征值与特征向量

对于特征值问题

$$\boldsymbol{A}\boldsymbol{x}=\lambda\boldsymbol{x} \tag{4.215}$$

其中，$\boldsymbol{A}$ 为二维方阵，$\boldsymbol{x}$ 为列向量。

假设三个特征值分别为 $\lambda_1,\lambda_2,\lambda_3$，三个特征向量分别为

$$\boldsymbol{x}_1=[x_{11}\ x_{21}\ x_{31}]^T=\begin{bmatrix}x_{11}\\x_{21}\\x_{31}\end{bmatrix},\quad \boldsymbol{x}_2=[x_{12}\ x_{22}\ x_{32}]^T=\begin{bmatrix}x_{12}\\x_{22}\\x_{32}\end{bmatrix},\quad \boldsymbol{x}_3=[x_{13}\ x_{23}\ x_{33}]^T=\begin{bmatrix}x_{13}\\x_{23}\\x_{33}\end{bmatrix} \tag{4.216}$$

则

$$\boldsymbol{A}\boldsymbol{x}_1=\lambda_1\boldsymbol{x}_1,\quad \boldsymbol{A}\boldsymbol{x}_2=\lambda_2\boldsymbol{x}_2,\quad \boldsymbol{A}\boldsymbol{x}_3=\lambda_3\boldsymbol{x}_3 \tag{4.217}$$

将三个方程合并

$$\boldsymbol{A}[\boldsymbol{x}_1\ \boldsymbol{x}_2\ \boldsymbol{x}_3]=[\lambda_1\boldsymbol{x}_1\ \lambda_2\boldsymbol{x}_2\ \lambda_3\boldsymbol{x}_3] \tag{4.218}$$

对于等式右端

$$[\lambda_1\boldsymbol{x}_1\ \lambda_2\boldsymbol{x}_2\ \lambda_3\boldsymbol{x}_3]=\begin{bmatrix}\lambda_1x_{11} & \lambda_2x_{12} & \lambda_3x_{13}\\ \lambda_1x_{21} & \lambda_2x_{22} & \lambda_3x_{23}\\ \lambda_1x_{31} & \lambda_2x_{32} & \lambda_3x_{33}\end{bmatrix}=\begin{bmatrix}x_{11} & x_{12} & x_{13}\\ x_{21} & x_{22} & x_{23}\\ x_{31} & x_{32} & x_{33}\end{bmatrix}\begin{bmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{bmatrix} \tag{4.219}$$

这里可以看出矩阵运算的一个性质：一个方阵右乘一个对角矩阵，相当于对角矩阵的对角元素按顺序乘到方阵的列上。那么，一个方阵左乘一个对角矩阵，相当于对角矩阵的对角元素按顺序乘到方阵的行上。

令

$$\boldsymbol{X}=[\boldsymbol{x}_1\ \boldsymbol{x}_2\ \boldsymbol{x}_3]=\begin{bmatrix}x_{11} & x_{12} & x_{13}\\ x_{21} & x_{22} & x_{23}\\ x_{31} & x_{32} & x_{33}\end{bmatrix} \tag{4.220}$$

则

$$\boldsymbol{A}\boldsymbol{X}=\boldsymbol{X}\mathrm{diag}(\lambda_1,\lambda_2,\lambda_3) \tag{4.221}$$

等式两边同时左乘 $\boldsymbol{X}^{-1}$，得

$$\boldsymbol{X}^{-1}\boldsymbol{A}\boldsymbol{X} = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3) \tag{4.222}$$

在线性代数中，此即为矩阵 $\boldsymbol{A}$ 的相似变换，此处的对角矩阵即为矩阵 $\boldsymbol{A}$ 的相似矩阵（前提是矩阵 $\boldsymbol{X}$ 可逆）。

若等式右边同时右乘 $\boldsymbol{X}^{-1}$，则

$$\boldsymbol{A} = \boldsymbol{X}\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)\boldsymbol{X}^{-1} \tag{4.223}$$

在matlab中，可以使用eig()函数求矩阵的特征值与特征向量，具体使用方法为

$$[\mathrm{eigenvectors}, \mathrm{eigenvalues}] = \mathrm{eig}(\boldsymbol{\sigma}) \tag{4.224}$$

其中

$$\mathrm{eigenvectors} = \boldsymbol{X}, \quad \mathrm{eigenvalues} = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3) \tag{4.225}$$

在matlab中，使用inv()函数求矩阵的逆，具体用法为

$$\boldsymbol{A}^{-1} = \mathrm{inv}(\boldsymbol{A}) \tag{4.226}$$

在matlab中，使用sort(a,'descend')函数对向量进行降序排列，使用sort(a)对向量进行升序排列。

一个小问题：若 $\boldsymbol{A}$ 为对称矩阵，$\boldsymbol{X}^{-1}\boldsymbol{A}\boldsymbol{X}$ 和 $\boldsymbol{X}\boldsymbol{A}\boldsymbol{X}^{-1}$ 相同吗？

$$\begin{aligned}
\boldsymbol{X}^{-1}\boldsymbol{A}\boldsymbol{X} &= X_{ij}^{-1}\boldsymbol{e}_i \otimes \boldsymbol{e}_j \cdot A_{kl}\boldsymbol{e}_k \otimes \boldsymbol{e}_l \cdot X_{mn}\boldsymbol{e}_m \otimes \boldsymbol{e}_n \\
&= X_{ij}^{-1}A_{kl}X_{mn}\boldsymbol{e}_i \otimes \boldsymbol{e}_j \cdot \boldsymbol{e}_k \otimes \boldsymbol{e}_l \cdot \boldsymbol{e}_m \otimes \boldsymbol{e}_n \\
&= X_{ij}^{-1}A_{kl}X_{mn}\delta_{jk}\delta_{lm}\boldsymbol{e}_i \otimes \boldsymbol{e}_n \\
&= X_{ik}^{-1}A_{km}X_{mn}\boldsymbol{e}_i \otimes \boldsymbol{e}_n
\end{aligned} \tag{4.227}$$

$$\begin{aligned}
\boldsymbol{X}\boldsymbol{A}\boldsymbol{X}^{-1} &= X_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j \cdot A_{kl}\boldsymbol{e}_k \otimes \boldsymbol{e}_l \cdot X_{mn}^{-1}\boldsymbol{e}_m \otimes \boldsymbol{e}_n \\
&= X_{ij}A_{kl}X_{mn}^{-1}\boldsymbol{e}_i \otimes \boldsymbol{e}_j \cdot \boldsymbol{e}_k \otimes \boldsymbol{e}_l \cdot \boldsymbol{e}_m \otimes \boldsymbol{e}_n \\
&= X_{ij}A_{kl}X_{mn}^{-1}\delta_{jk}\delta_{lm}\boldsymbol{e}_i \otimes \boldsymbol{e}_n \\
&= X_{ik}A_{km}X_{mn}^{-1}\boldsymbol{e}_i \otimes \boldsymbol{e}_n
\end{aligned} \tag{4.228}$$

若 $\boldsymbol{X}$ 同时为正交矩阵且为对称矩阵，即 $\boldsymbol{X}^{-1} = \boldsymbol{X}^T$ 和 $\boldsymbol{X}^T = \boldsymbol{X}$，那么有

$$X_{ik}A_{km}X_{mn}^{-1} = X_{ik}A_{km}X_{mn}^T = X_{ki}^TA_{km}X_{nm} = X_{ki}^{-1}A_{km}X_{nm} = X_{ik}^{-1}A_{km}X_{mn} \tag{4.229}$$

其中，还用到了矩阵的一个性质：对称矩阵的逆矩阵也是对称矩阵？

故，对于对称矩阵 $\boldsymbol{A}$，只有当 $\boldsymbol{X}$ 为对称正交矩阵时，才有

$$\boldsymbol{X}^{-1}\boldsymbol{A}\boldsymbol{X} = \boldsymbol{X}\boldsymbol{A}\boldsymbol{X}^{-1} \tag{4.230}$$

4.6.3.4 数值法-求应力张量的特征值即为主应力

在matlab中，可以使用eig()函数求矩阵的特征值与特征向量，具体使用方法为

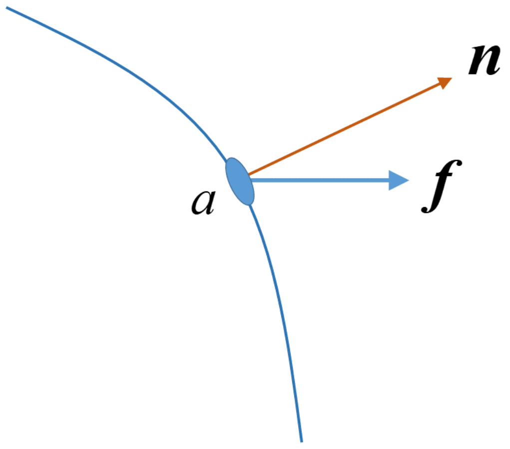
$$\mathrm{eigenvalues} = \mathrm{eig}(\boldsymbol{\sigma}) \tag{4.231}$$

再用sort()函数对其降序排列即可。

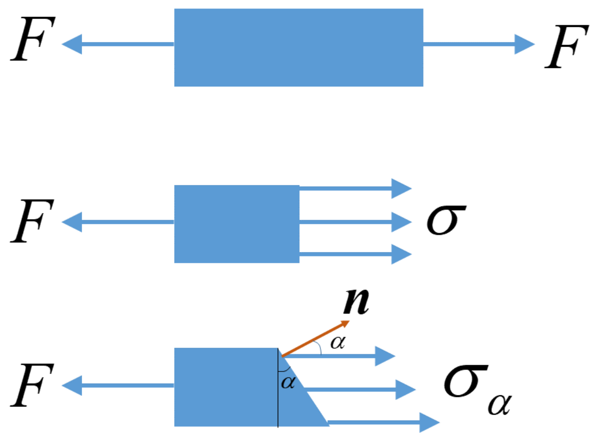
Chapter 5

一维情形：杆梁理论

5.1 单轴拉伸



(a)



(b)

应力矢量的定义

$$\boldsymbol{t} = \lim_{\Delta a \to 0} \frac{\Delta \boldsymbol{f}}{\Delta a} = \frac{d\boldsymbol{f}}{da} \tag{5.1}$$

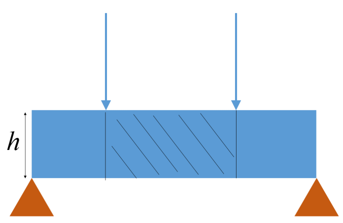
单轴拉伸杆件所受外力与横截面上内力平衡方程

$$F = \int_A \sigma dA \tag{5.2}$$

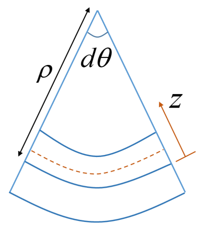
斜截面上的应力

$$F = \int_A \sigma_\alpha dA = \sigma_\alpha \int_A dA = \sigma_\alpha A_\alpha = \sigma_\alpha \frac{A}{cos\alpha} \to \sigma_\alpha = \frac{Fcos\alpha}{A} \tag{5.3}$$

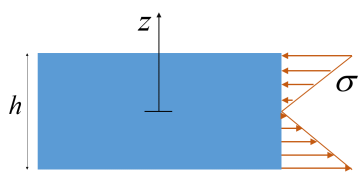
5.2 纯弯曲



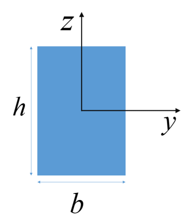
(c)



(d)



(e)



(f)

纯弯曲：横截面上只有弯矩，没有剪力。可以通过四点弯曲来实现纯弯曲。

几何方程

$$\varepsilon = \frac{\rho d\theta - (\rho - z)d\theta}{\rho d\theta} = \frac{z}{\rho} \tag{5.4}$$

物理方程

$$\sigma = E\varepsilon \tag{5.5}$$

平衡方程

$$\begin{aligned} F &= \int_A \sigma dA = \int_A E\varepsilon dA = \int_A E\frac{z}{\rho} dA = \frac{E}{\rho}\int_A z dA = \frac{E}{\rho} S_y \\ M &= \int_A \sigma z dA = \int_A E\varepsilon z dA = \int_A E\frac{z}{\rho} z dA = \frac{E}{\rho}\int_A z^2 dA = \frac{E}{\rho} I_y \end{aligned} \tag{5.6}$$

可得

$$\frac{1}{\rho} = \frac{M}{EI_y} \tag{5.7}$$

接下来建立曲率与挠度的关系。梁弯曲的挠度函数的一阶导为

$$w' = \tan\theta \tag{5.8}$$

梁弯曲的挠度函数的二阶导为

$$w'' = \frac{d\tan\theta}{dx} = \sec^2\theta\frac{d\theta}{dx} = (1+\tan^2\theta)\frac{d\theta}{dx} = (1+w'^2)\frac{d\theta}{dx} \to \frac{d\theta}{dx} = \frac{w''}{(1+w'^2)} \tag{5.9}$$

弧微分

$$\frac{ds}{dx} = \frac{\sqrt{(dx)^2+(dy)^2}}{dx} = \sqrt{1+(\frac{dy}{dx})^2} = \sqrt{1+(\frac{dw}{dx})^2} = \sqrt{1+w'^2} \tag{5.10}$$

曲率

$$\rho d\theta = ds \to \frac{1}{\rho} = \frac{d\theta}{ds} = \frac{d\theta}{dx}\frac{dx}{ds} = \frac{w''}{(1+w'^2)}\frac{1}{\sqrt{1+w'^2}} = \frac{w''}{(1+w'2)^{3/2}} \tag{5.11}$$

因为小挠度情形下，w'^2 与 1 相比是个小量，所以可以省略，

$$\frac{1}{\rho} = w'' \to w'' = \frac{M}{EI_y} \tag{5.12}$$

$$\theta \approx \tan\theta = w' = \int \frac{M}{EI_y}dx + c_1 \tag{5.13}$$

$$w = \int\int \frac{M}{EI_y}dxdx + c_1x + c_2 \tag{5.14}$$

注意：这里是不定积分，而且需要分段计算。

$$M = F_{NL}x = \frac{Fx}{2} \tag{5.15}$$

$$w = \int\int \frac{1}{EI_y}\frac{Fx}{2}dxdx + c_1x + c_2 = \frac{Fx^3}{12EI} + c_1x + c_2 \tag{5.16}$$

边界条件为

$$w(0) = 0, \quad w'(\frac{L}{2}) = 0 \tag{5.17}$$

$$w = \frac{Fx^3}{12EI} - \frac{FL^2x}{16EI} = -\frac{FL^3}{48EI} \tag{5.18}$$

反正，因为挠度函数是个分段函数，所以需要分段算。右段的弯矩函数和左段的不同，所以右段的挠度函数需要再单独算!!! 算一段的函数时，只能用这一段内的边界条件。比如说$0 < x < L$段，只能用$0 < x < L$的边界条件。

5.3 能量法：卡氏第二定理

以简支梁中间作用集中载荷为例

内力势能为

$$U_I = \int_\theta \frac{1}{2}Md\theta = \int_x \frac{1}{2}M\frac{1}{\rho}dx = \int_x \frac{M^2}{2EI}dx \tag{5.19}$$

外力势能为

$$U_F = -Fd \tag{5.20}$$

总势能为

$$U = U_I + U_F = \int_x \frac{M^2}{2EI}dx - Fd \tag{5.21}$$

把U看成F的函数（在“输入-系统-输出”中，F为独立的输入变量），由最小势能原理

$$\delta U(F)=0=\frac{\partial U}{\partial F}\delta F=\int_x \frac{M}{EI}\frac{\partial M}{\partial F}dx-d \tag{5.22}$$

故，控制方程为

$$d=\int_x \frac{M}{EI}\frac{\partial M}{\partial F}dx=\int_0^L \frac{M}{EI}\frac{\partial M}{\partial F}dx=2\int_0^{L/2} \frac{M}{EI}\frac{\partial M}{\partial F}dx \tag{5.23}$$

因为挠度是个分段函数，所以需要分段计算，对于本问题，左右侧积分相同，所以直接2倍。

$$M=F_{NL}x=\frac{Fx}{2} \tag{5.24}$$

故

$$d=2\int_0^{L/2} \frac{1}{EI}\frac{Fx}{2}\frac{x}{2}dx=\int_0^{L/2} \frac{Fx^2}{2EI}dx=\frac{FL^3}{48EI} \tag{5.25}$$

故，给定力边界条件F后，即可确定中间的挠度。

5.4 欧拉梁-弹性理论

欧拉梁位移场为

$$\boldsymbol{u} = u_i\boldsymbol{e}_i = -x_3\frac{\partial u_3}{\partial x_1}\boldsymbol{e}_1 + u_3(x_1)\boldsymbol{e}_3 \tag{5.26}$$

应变张量为

$$\boldsymbol{\varepsilon} = \frac{1}{2}(\boldsymbol{u}\nabla + \nabla\boldsymbol{u}) = \frac{1}{2}(u_{i,j} + u_{j,i})\boldsymbol{e}_i \otimes \boldsymbol{e_j} = -x_3\frac{\partial^2 u_3}{\partial x_1^2}\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 \tag{5.27}$$

应力张量为

$$\begin{aligned}\boldsymbol{\sigma} &= \lambda tr(\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon} = (\lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij})\boldsymbol{e}_i \otimes \boldsymbol{e}_j \\ &= -(\lambda+2\mu)x_3\frac{\partial^2 u_3}{\partial x_1^2}\boldsymbol{e}_1\otimes\boldsymbol{e}_1 - \lambda x_3\frac{\partial^2 u_3}{\partial x_1^2}\boldsymbol{e}_2\otimes\boldsymbol{e}_2 - \lambda x_3\frac{\partial^2 u_3}{\partial x_1^2}\boldsymbol{e}_3\otimes\boldsymbol{e}_3\end{aligned} \tag{5.28}$$

其中$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$， $\mu = \frac{E}{2(1+\nu)}$，若$\nu = 0$, $\lambda + 2\mu = E$.

总势能为

$$\begin{aligned}U &= \int_\Omega \frac{1}{2}\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}dv - \int_{\partial\Omega}\boldsymbol{t}\cdot\boldsymbol{u}ds \\ &= \int_\Omega \frac{1}{2}(\lambda+2\mu)x_3^2(\frac{\partial^2 u_3}{\partial x_1^2})^2 dv - \int_{\partial\Omega}\boldsymbol{t}\cdot\boldsymbol{u}ds \\ &= \frac{1}{2}(\lambda+2\mu)\int_\Omega x_3^2(\frac{\partial^2 u_3}{\partial x_1^2})^2dv - \int_{\partial\Omega}t_iu_ids \\ &= \int_0^L\int_{-\frac{b}{2}}^{\frac{b}{2}}\int_{-\frac{h}{2}}^{\frac{h}{2}}(\lambda+2\mu)x_3^2(\frac{\partial^2 u_3}{\partial x_1^2})^2dx_1dx_2dx_3 - \int_{\partial\Omega}t_iu_ids\end{aligned} \tag{5.29}$$

最小势能原理

$$\delta U = 0, \quad \delta f(x) = \frac{\partial f}{\partial x}\delta x, \quad \int_a^b u\frac{\partial^2 v}{\partial x^2}dx = u\frac{\partial v}{\partial x}|_a^b - \frac{\partial u}{\partial x}v|_a^b + \int_a^b\frac{\partial^2 u}{\partial x^2}vdx \tag{5.30}$$

$$\begin{aligned}\delta U = 0 &= \frac{1}{2}(\lambda+2\mu)\delta\int_\Omega x_3^2(\frac{\partial^2 u_3}{\partial x_1^2})^2dv - \delta\int_{\partial\Omega}t_iu_ids = \frac{1}{2}(\lambda+2\mu)\int_\Omega x_3^2(2\frac{\partial^2 u_3}{\partial x_1^2})\delta(\frac{\partial^2 u_3}{\partial x_1^2})dv - \int_{\partial\Omega}t_i\delta u_ids \\ &= (\lambda+2\mu)\int_\Omega x_3^2\frac{\partial^2 u_3}{\partial x_1^2}\delta(\frac{\partial^2 u_3}{\partial x_1^2})dv - \int_{\partial\Omega}t_i\delta u_ids = (\lambda+2\mu)\int_\Omega x_3^2\frac{\partial^2 u_3}{\partial x_1^2}(\frac{\partial^2\delta u_3}{\partial x_1^2})dv - \int_{\partial\Omega}t_i\delta u_ids \\ &= (\lambda+2\mu)\int_\Omega x_3^2\frac{\partial^2 u_3}{\partial x_1^2}(\frac{\partial^2\delta u_3}{\partial x_1^2})dv - \int_{\partial\Omega}t_i\delta u_ids \\ &= (\lambda+2\mu)\int_{\partial\Omega}x_3^2\frac{\partial^2 u_3}{\partial x_1^2}(\frac{\partial\delta u_3}{\partial x_1})|_0^Lds - (\lambda+2\mu)\int_{\partial\Omega}x_3^2\frac{\partial^3 u_3}{\partial x_1^3}\delta u_3|_0^Lds \\ &+ (\lambda+2\mu)\int_\Omega x_3^2\frac{\partial^4 u_3}{\partial x_1^4}\delta u_3dv - \int_{\partial\Omega}t_i\delta u_ids \\ &= (\lambda+2\mu)\int_{\partial\Omega}x_3^2\frac{\partial^2 u_3}{\partial x_1^2}(\frac{\partial\delta u_3}{\partial x_1})|_0^Ldx_2dx_3 - (\lambda+2\mu)\int_{\partial\Omega}x_3^2\frac{\partial^3 u_3}{\partial x_1^3}\delta u_3|_0^Ldx_2dx_3 \\ &+ (\lambda+2\mu)\int_\Omega x_3^2\frac{\partial^4 u_3}{\partial x_1^4}\delta u_3dv - \int_{\partial\Omega}t_3\delta u_3|_{x_3=-\frac{h}{2}}dx_1dx_2 = 0 \\ &= (\lambda+2\mu)(\int_{-\frac{h}{2}}^{\frac{h}{2}}x_3^2dx_3\int_{-\frac{b}{2}}^{\frac{b}{2}}dx_2)\frac{\partial^2 u_3}{\partial x_1^2}(\frac{\partial\delta u_3}{\partial x_1})|_0^L - (\lambda+2\mu)(\int_{-\frac{h}{2}}^{\frac{h}{2}}x_3^2dx_3\int_{-\frac{b}{2}}^{\frac{b}{2}}dx_2)\frac{\partial^3 u_3}{\partial x_1^3}\delta u_3|_0^L \\ &+ (\lambda+2\mu)(\int_{-\frac{h}{2}}^{\frac{h}{2}}x_3^2dx_3\int_{-\frac{b}{2}}^{\frac{b}{2}}dx_2)\int_0^L\frac{\partial^4 u_3}{\partial x_1^4}\delta u_3dx_1 - \int_{\frac{b}{2}}^{\frac{b}{2}}dx_2\int_0^Lt_3\delta u_3|_{x_3=-\frac{h}{2}}dx_1 \\ &= (\lambda+2\mu)I\frac{\partial^2 u_3}{\partial x_1^2}(\frac{\partial\delta u_3}{\partial x_1})|_0^L - (\lambda+2\mu)I\frac{\partial^3 u_3}{\partial x_1^3}\delta u_3|_0^L \\ &+ (\lambda+2\mu)I\int_0^L\frac{\partial^4 u_3}{\partial x_1^4}\delta u_3dx_1 - \int_0^Lbt_3\delta u_3|_{x_3=-\frac{h}{2}}dx_1 \\ &= (\lambda+2\mu)I\frac{\partial^2 u_3}{\partial x_1^2}(\frac{\partial\delta u_3}{\partial x_1})|_0^L - (\lambda+2\mu)I\frac{\partial^3 u_3}{\partial x_1^3}\delta u_3|_0^L \\ &+ (\lambda+2\mu)I\int_0^L\frac{\partial^4 u_3}{\partial x_1^4}\delta u_3dx_1 - \int_0^Lq\delta u_3|_{x_3=-\frac{h}{2}}dx_1\end{aligned} \tag{5.31}$$

其中t_3是面载荷，q是线载荷。

得控制方程

$$(\lambda+2\mu)I\frac{\partial^4 u_3}{\partial x_1^4} - bt_3 = (\lambda+2\mu)I\frac{\partial^4 u_3}{\partial x_1^4} - q = 0 \tag{5.32}$$

边界条件(运动边界条件 or 自然边界条件) (形式一：可以写成这样)

$$\begin{aligned}&\frac{\partial u_3}{\partial x_1}=\text{给定}\quad \text{or}\quad (\lambda+2\mu)I\frac{\partial^2 u_3}{\partial x_1^2}=0\quad \text{at}\quad x=0,L\\&u_3=\text{给定}\quad \text{or}\quad (\lambda+2\mu)I\frac{\partial^3 u_3}{\partial x_1^3}=0\quad \text{at}\quad x=0,L\end{aligned}\tag{5.33}$$

5.5 弹性欧拉梁理论-算例

5.5.1 简支梁受均布载荷

控制方程为

$$Su_3^{(4)}(x_1)-q=0\tag{5.34}$$

边界条件为

$$\begin{aligned}&u_3(0)=0,u_3(L)=0\\&u_3^{(2)}(0)=0,u_3^{(2)}(L)=0\end{aligned}\tag{5.35}$$

控制方程为非齐次方程，求解步骤：1.先求对应齐次方程的通解；2求非齐次方程的特解。

求齐次方程通解的方法：特征根法

齐次方程为

$$Su_3^{(4)}(x_1)=0\tag{5.36}$$

特征方程为

$$S\lambda^4=0\tag{5.37}$$

特征根为

$$\lambda=0\quad (4\text{重实数特征根})\tag{5.38}$$

故齐次方程的通解为

$$\overline{u}_3(x_1)=c_1+c_2x+c3x^2+c4x^3\tag{5.39}$$

设非齐次方程的特解为

$$u_3^*=x^4b_0\tag{5.40}$$

代入非齐次方程得

$$b_0=\frac{q}{24S}\tag{5.41}$$

综上，非齐次方程的解为

$$u_3(x)=c_1+c_2x+c3x^2+c4x^3+\frac{q}{24S}x^4\tag{5.42}$$

代入边界条件即可求出未知系数。可以考虑使用matlab或mathematica解方程组。

5.5.2 简支梁中间受集中载荷

控制方程为

$$Su_3^{(4)}(x_1)-q(x)=0\tag{5.43}$$

边界条件为

$$\begin{aligned}&u_3(0)=0,u_3(L)=0\\&u_3^{(2)}(0)=0,u_3^{(2)}(L)=0\end{aligned}\tag{5.44}$$

控制方程为非齐次方程，求解方法：级数解法

将非齐次方程的解展开为傅里叶级数

$$u_3(x)=\sum_{n=1}^{\infty}A_n\sin\frac{n\pi x}{L}\tag{5.45}$$

当外载荷为集中载荷时，分布载荷可以表示为

$$q(x)=F\delta(x-\frac{L}{2})\tag{5.46}$$

将分布载荷展开为傅里叶级数

$$q(x) = \sum_{n=1}^{\infty} Q_n \sin\frac{n\pi x}{L} \tag{5.47}$$

傅里叶系数为

$$Q_n = \frac{2}{L}\int_0^L q(x)\sin\frac{n\pi x}{L} = \frac{2}{L}\int_0^L F\delta(x-\frac{L}{2})\sin\frac{n\pi x}{L}dx = \frac{2F}{L}\sin\frac{n\pi}{2} \tag{5.48}$$

将位移级数形式和载荷级数形式代入控制方程

$$\begin{aligned} &\sum_{n=1}^{\infty} SA_n(\sin\frac{n\pi x}{L})^{(4)} = \sum_{n=1}^{\infty} Q_n \sin\frac{n\pi x}{L} \\ \Longrightarrow &\sum_{n=1}^{\infty} SA_n(\frac{n\pi x}{L})^4 \sin\frac{n\pi x}{L} = \sum_{n=1}^{\infty} Q_n \sin\frac{n\pi x}{L} \\ \Longrightarrow &A_n = \frac{Q_n}{S(\frac{n\pi x}{L})^4} \end{aligned} \tag{5.49}$$

实际上，级数只需要取有限项即可满足精度要求。

最终，

$$\begin{aligned} A_n = \frac{Q_n}{S(\frac{n\pi x}{L})^4} \rightarrow u_3(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} \\ Q_n = \frac{2F}{L}\sin\frac{n\pi}{2} \rightarrow q(x) = \sum_{n=1}^{\infty} Q_n \sin\frac{n\pi x}{L} \end{aligned} \tag{5.50}$$

5.5.3 悬臂梁受均布载荷

控制方程为

$$Su_3^{(4)}(x_1) - q0 = 0 \tag{5.51}$$

边界条件为

$$\begin{aligned} u_3(0) = 0, u_3^{(1)}(0) = 0 \\ u_3^{(2)}(L) = 0, u_3^{(3)}(L) = 0 \end{aligned} \tag{5.52}$$

控制方程为非齐次方程，求解步骤：1.先求对应齐次方程的通解；2求非齐次方程的特解。

求齐次方程通解的方法：特征根法

齐次方程为

$$Su_3^{(4)}(x_1) = 0 \tag{5.53}$$

特征方程为

$$S\lambda^4 = 0 \tag{5.54}$$

特征根为

$$\lambda = 0 \quad (4\text{重实数特征根}) \tag{5.55}$$

故齐次方程的通解为

$$\overline{u}_3(x_1) = c_1 + c_2 x + c3x^2 + c4x^3 \tag{5.56}$$

设非齐次方程的特解为

$$u_3^* = x^4 b_0 \tag{5.57}$$

代入非齐次方程得

$$b_0 = \frac{q}{24S} \tag{5.58}$$

综上，非齐次方程的解为

$$u_3(x) = c_1 + c_2 x + c3x^2 + c4x^3 + \frac{q}{24S}x^4 \tag{5.59}$$

代入边界条件即可求出未知系数。可以考虑使用matlab或mathematica解方程组。

5.5.4 悬臂梁末端受集中载荷

若采用边界条件

$$\begin{aligned} &u_3(0)=0, u_3^{(1)}(0)=0 \\ &u_3^{(2)}(L)=0, u_3^{(3)}(L)=F \end{aligned} \tag{5.60}$$

可用一般解法。

若用级数解法，边界条件可能略有不同。

控制方程为

$$S u_3^{(4)}(x_1)-q(x)=0 \tag{5.61}$$

边界条件为

$$\begin{aligned} &u_3(0)=0, u_3^{(1)}(0)=0 \\ &u_3^{(2)}(L)=0, u_3^{(3)}(L)=0 \end{aligned} \tag{5.62}$$

控制方程为非齐次方程，求解方法：级数解法

将非齐次方程的解展开为傅里叶级数

$$u_3(x)=\sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} \tag{5.63}$$

当外载荷为集中载荷时，分布载荷可以表示为

$$q(x)=F\delta(x-\frac{L}{2}) \tag{5.64}$$

将分布载荷展开为傅里叶级数

$$q(x)=\sum_{n=1}^{\infty} Q_n \sin \frac{n\pi x}{L} \tag{5.65}$$

傅里叶系数为

$$Q_n=\frac{2}{L}\int_0^L q(x)\sin\frac{n\pi x}{L}=\frac{2}{L}\int_0^L F\delta(x-\frac{L}{2})\sin\frac{n\pi x}{L}dx=\frac{2F}{L}\sin\frac{n\pi}{2} \tag{5.66}$$

将位移级数形式和载荷级数形式代入控制方程

$$\begin{aligned} &\sum_{n=1}^{\infty} SA_n(\sin\frac{n\pi x}{L})^{(4)}=\sum_{n=1}^{\infty} Q_n\sin\frac{n\pi x}{L} \\ \Longrightarrow\ &\sum_{n=1}^{\infty} SA_n(\frac{n\pi x}{L})^4\sin\frac{n\pi x}{L}=\sum_{n=1}^{\infty} Q_n\sin\frac{n\pi x}{L} \\ \Longrightarrow\ &A_n=\frac{Q_n}{S(\frac{n\pi x}{L})^4} \end{aligned} \tag{5.67}$$

实际上，级数只需要取有限项即可满足精度要求。

最终，

$$\begin{aligned} &A_n=\frac{Q_n}{S(\frac{n\pi x}{L})^4}\to u_3(x)=\sum_{n=1}^{\infty} A_n\sin\frac{n\pi x}{L} \\ &Q_n=\frac{2F}{L}\sin\frac{n\pi}{2}\to q(x)=\sum_{n=1}^{\infty} Q_n\sin\frac{n\pi x}{L} \end{aligned} \tag{5.68}$$

5.6 欧拉梁-对称偶应力理论（Yang）

欧拉梁位移场为

$$\boldsymbol{u} = u_i \boldsymbol{e}_i = -x_3 \frac{\partial u_3}{\partial x_1} \boldsymbol{e}_1 + u_3(x_1) \boldsymbol{e}_3 \tag{5.69}$$

转动场为

$$\boldsymbol{\theta} = \frac{1}{2} \nabla \times \boldsymbol{u} = \frac{1}{2} \partial_i u_j e_{ijk} \boldsymbol{e}_k = -\frac{\partial u_3}{\partial x_1} \boldsymbol{e}_2 \tag{5.70}$$

应变张量和曲率张量为

$$\boldsymbol{\varepsilon} = \frac{1}{2}(\boldsymbol{u}\nabla + \nabla \boldsymbol{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}) \boldsymbol{e}_i \otimes \boldsymbol{e}_j = -x_3 \frac{\partial^2 u_3}{\partial x_1^2} \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 \tag{5.71}$$

$$\boldsymbol{\chi} = \frac{1}{2}(\boldsymbol{\theta}\nabla + \nabla \boldsymbol{\theta}) = \frac{1}{2}(\theta_{i,j} + \theta_{j,i}) \boldsymbol{e}_i \otimes \boldsymbol{e}_j = -\frac{1}{2} \frac{\partial^2 u_3}{\partial x_1^2} \boldsymbol{e}_1 \otimes \boldsymbol{e}_2 \tag{5.72}$$

应力张量和偶应力张量为

$$\begin{aligned} \boldsymbol{\sigma} &= \lambda tr(\boldsymbol{\varepsilon}) \boldsymbol{I} + 2\mu \boldsymbol{\varepsilon} = (\lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}) \boldsymbol{e}_i \otimes \boldsymbol{e}_j \\ &= -(\lambda + 2\mu) x_3 \frac{\partial^2 u_3}{\partial x_1^2} \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 - \lambda x_3 \frac{\partial^2 u_3}{\partial x_1^2} \boldsymbol{e}_2 \otimes \boldsymbol{e}_2 - \lambda x_3 \frac{\partial^2 u_3}{\partial x_1^2} \boldsymbol{e}_3 \otimes \boldsymbol{e}_3 \end{aligned} \tag{5.73}$$

其中$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$, $\quad \mu = \frac{E}{2(1+\nu)}$, 若$\nu = 0$, $\lambda + 2\mu = E$.

$$\boldsymbol{m} = 2\mu l_0^2 \boldsymbol{\chi} = 2\mu l_0^2 \chi_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j = -\mu l_0^2 \frac{\partial^2 u_3}{\partial x_1^2} \boldsymbol{e}_1 \otimes \boldsymbol{e}_2 \tag{5.74}$$

总势能为

$$\begin{aligned} U &= \int_\Omega \frac{1}{2} \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon} + \frac{1}{2} \boldsymbol{m} \cdot \boldsymbol{\chi} dv - \int_{\partial\Omega} \boldsymbol{t} \cdot \boldsymbol{u} ds = \int_\Omega \frac{1}{2} \sigma_{ij} \varepsilon_{ij} + \frac{1}{2} m_{ij} \chi_{ij} dv - \int_{\partial\Omega} t_i u_i ds \\ &= \int_\Omega \frac{1}{2}(\lambda + 2\mu) x_3^2 (\frac{\partial^2 u_3}{\partial x_1^2})^2 + \frac{1}{4} \mu l_0^2 (\frac{\partial^2 u_3}{\partial x_1^2})^2 dv - \int_{\partial\Omega} t_i u_i ds \\ &= \int_\Omega [\frac{1}{2}(\lambda + 2\mu) x_3^2 + \frac{1}{4} \mu l_0^2] (\frac{\partial^2 u_3}{\partial x_1^2})^2 dv - \int_{\partial\Omega} t_i u_i ds \end{aligned} \tag{5.75}$$

最小势能原理

$$\delta U = 0, \quad \delta f(x) = \frac{\partial f}{\partial x} \delta x, \quad \int_a^b u \frac{\partial^2 v}{\partial x^2} dx = u \frac{\partial v}{\partial x}|_a^b - \frac{\partial u}{\partial x} v|_a^b + \int_a^b \frac{\partial^2 u}{\partial x^2} v dx \tag{5.76}$$

$$\begin{aligned} \delta U = 0 &= \int_\Omega [\frac{1}{2}(\lambda + 2\mu) x_3^2 + \frac{1}{4} \mu l_0^2] (2 \frac{\partial^2 u_3}{\partial x_1^2} \delta \frac{\partial^2 u_3}{\partial x_1^2}) dv - \int_{\partial\Omega} t_i \delta u_i ds \\ &= \int_\Omega [(\lambda + 2\mu) x_3^2 + \frac{1}{2} \mu l_0^2] (\frac{\partial^2 u_3}{\partial x_1^2} \delta \frac{\partial^2 u_3}{\partial x_1^2}) dv - \int_{\partial\Omega} t_i \delta u_i ds \\ &= \int_{\partial\Omega} \int_0^L [(\lambda + 2\mu) x_3^2 + \frac{1}{2} \mu l_0^2] (\frac{\partial^2 u_3}{\partial x_1^2} \delta \frac{\partial^2 u_3}{\partial x_1^2}) dx_1 ds - \int_{\partial\Omega} t_i \delta u_i ds \\ &= \int_{\partial\Omega} [(\lambda + 2\mu) x_3^2 + \frac{1}{2} \mu l_0^2] (\frac{\partial^2 u_3}{\partial x_1^2} \delta \frac{\partial u_3}{\partial x_1})|_0^L ds - \int_{\partial\Omega} [(\lambda + 2\mu) x_3^2 + \frac{1}{2} \mu l_0^2] (\frac{\partial^3 u_3}{\partial x_1^3} \delta u_3)|_0^L ds \\ &\quad + \int_\Omega [(\lambda + 2\mu) x_3^2 + \frac{1}{2} \mu l_0^2] (\frac{\partial^4 u_3}{\partial x_1^4} \delta u_3) dv - \int_{\partial\Omega} t_i \delta u_i ds \\ &= (\int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{-\frac{b}{2}}^{\frac{b}{2}} [(\lambda + 2\mu) x_3^2 + \frac{1}{2} \mu l_0^2] dx_2 dx_3) (\frac{\partial^2 u_3}{\partial x_1^2} \delta \frac{\partial u_3}{\partial x_1})|_0^L \\ &\quad - (\int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{-\frac{b}{2}}^{\frac{b}{2}} [(\lambda + 2\mu) x_3^2 + \frac{1}{2} \mu l_0^2] dx_2 dx_3) (\frac{\partial^3 u_3}{\partial x_1^3} \delta u_3)|_0^L \\ &\quad + (\int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{-\frac{b}{2}}^{\frac{b}{2}} [(\lambda + 2\mu) x_3^2 + \frac{1}{2} \mu l_0^2] dx_2 dx_3) (\int_0^L \frac{\partial^4 u_3}{\partial x_1^4} \delta u_3 dx_1) - \int_0^L q \delta u_3 ds \\ &= [(\lambda + 2\mu) I + \frac{1}{2} \mu l_0^2 bh] (\frac{\partial^2 u_3}{\partial x_1^2} \delta \frac{\partial u_3}{\partial x_1})|_0^L \\ &\quad - [(\lambda + 2\mu) I + \frac{1}{2} \mu l_0^2 bh] (\frac{\partial^3 u_3}{\partial x_1^3} \delta u_3)|_0^L \\ &\quad + [(\lambda + 2\mu) I + \frac{1}{2} \mu l_0^2 bh] (\int_0^L \frac{\partial^4 u_3}{\partial x_1^4} \delta u_3 dx_1) - \int_0^L q \delta u_3 ds \end{aligned} \tag{5.77}$$

得控制方程

$$[(\lambda + 2\mu) I + \frac{1}{2} \mu l_0^2 bh] \frac{\partial^4 u_3}{\partial x_1^4} - q = 0 \tag{5.78}$$

边界条件 (形式二：也可以写成这样)

$$
\begin{aligned}
&[(\lambda+2\mu)I+\frac{1}{2}\mu l_0^2bh](\frac{\partial^2 u_3}{\partial x_1^2}\delta\frac{\partial u_3}{\partial x_1})|_0^L=0\\
&[(\lambda+2\mu)I+\frac{1}{2}\mu l_0^2bh](\frac{\partial^3 u_3}{\partial x_1^3}\delta u_3)|_0^L=0
\end{aligned}
\tag{5.79}
$$

Chapter 6

二维情形：板壳理论

6.1 基尔霍夫板理论:位移场的来龙去脉

6.1.1 Assumption❶ and Assumption❷

Assumption❶:平面应力假设（Plane stress）

$$\sigma_{zz} = 0, \quad \sigma_{zx} = \sigma_{zy} = 0$$

Tip:

平面应力假设是一个非常强的假设，因为切应力为零，所以该假设认为材料不抗剪。

现在，只剩三个应力分量，且$\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$和$x, y, z$有关，即

$$\sigma_{xx} = \sigma_{xx}(x, y, z), \quad \sigma_{yy} = \sigma_{yy}(x, y, z), \quad \sigma_{zz} = \sigma_{zz}(x, y, z) \tag{6.1}$$

解耦不是空穴来风!!! 阳老师的思路是：基于泰勒展开!!!（基尔霍夫没说泰勒展开吧）

引申：$f(x, y, z) \quad |z| \ll |x|, |y|$, z的dimension远远小于x, y的。

$$f(x, y, z) = f(x, y, 0) + \frac{\partial f}{\partial z}(x, y, 0)z + \frac{1}{2}\frac{\partial^2 f}{\partial z^2}(x, y, 0)z^2 + o(z^2) = f_1(x, y) + zf_2(x, y)$$

Assumption❷:应力沿厚度方向线性变化(The stresses vary linearly throughout the thickness.)

$$\sigma_{xx} = a_1(x, y) + zb_1(x, y)$$
$$\sigma_{yy} = a_2(x, y) + zb_2(x, y)$$
$$\sigma_{zz} = a_3(x, y) + zb_3(x, y)$$

阳老师：我喜欢从假设出发，这样，即时错了也是假设错了而不是我错了。

6.1.2 Constitutive law

本构方程为(The general Hooke's law)

$$\varepsilon_{xx} = \frac{1}{E}[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})]$$
$$\varepsilon_{yy} = \frac{1}{E}[\sigma_{yy} - \nu(\sigma_{zz} + \sigma_{xx})]$$
$$\varepsilon_{zz} = \frac{1}{E}[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})]$$
$$\varepsilon_{xy} = \frac{1}{2G}\sigma_{xy}, \quad \varepsilon_{yz} = \frac{1}{2G}\sigma_{yz}, \quad \varepsilon_{zx} = \frac{1}{2G}\sigma_{zx}$$

将assumption 1代入本构方程中

$$\varepsilon_{xx} = \frac{1}{E}(\sigma_{xx} - \nu\sigma_{yy})$$
$$\varepsilon_{yy} = \frac{1}{E}(\sigma_{yy} - \nu\sigma_{xx})$$
$$\varepsilon_{zz} = \frac{-\nu}{E}(\sigma_{xx} + \sigma_{yy})$$
$$\varepsilon_{xy} = \frac{1}{2G}\sigma_{xy}, \quad \varepsilon_{yz} = 0, \quad \varepsilon_{zx} = 0$$

对于小变形(small deformation)，几何方程为

$$\varepsilon_{ij}=\frac{1}{2}(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i})$$

将本构方程代入几何方程中

$$\varepsilon_{zz}=\frac{\partial u_z}{\partial z}=\frac{-\nu}{E}(\sigma_{xx}+\sigma_{yy})$$

将假设2代入上式

$$\varepsilon_{zz}=\frac{\partial u_z}{\partial z}=\frac{-\nu}{E}\left\{[a_1(x,y)+a_2(x,y)]+[b_1(x,y)+b_2(x,y)]z\right\}$$

积分求u_z

$$u_z=\frac{-\nu}{E}\left\{[a_1(x,y)+a_2(x,y)]z+[b_1(x,y)+b_2(x,y)]\frac{z^2}{2}\right\}+w(x,y)$$

Tips:

①通过平面应力假设，将独立应力分量减少为3个，但是应变分量还是6个

②$\sigma_{yz}=0,\sigma_{zx}=0$使得应变减少为4个

③$\sigma_{zz}=0$使得$\varepsilon_{xx},\varepsilon_{yy},\varepsilon_{zz},\varepsilon_{xy}$都为$(\sigma_{xx},\sigma_{yy},\sigma_{xy})$的函数。

6.1.3 Assumption❸

Assumption❸: $|w(x,y)|>>|z|$

$$u_z=w(x,y)$$

将本构方程代入几何方程中

$$\varepsilon_{yz}=\frac{1}{2}(\frac{\partial u_y}{\partial z}+\frac{\partial u_z}{\partial y})=0\implies\frac{\partial u_x}{\partial z}=-\frac{\partial u_z}{\partial x}=-\frac{\partial w}{\partial x}$$
$$\varepsilon_{zx}=\frac{1}{2}(\frac{\partial u_z}{\partial x}+\frac{\partial u_x}{\partial z})=0\implies\frac{\partial u_y}{\partial z}=-\frac{\partial u_z}{\partial y}=-\frac{\partial w}{\partial y}$$

积分求u_x,u_y

$$u_x=u(x,y)-z\frac{\partial w}{\partial x}$$
$$u_y=v(x,y)-z\frac{\partial w}{\partial y}$$

其中，$u(x,y)$和$v(x,y)$为积分出来的与z无关的常函数。

6.1.4 三个假设得到板壳理论最著名的方程之一

最终，基尔霍夫板的位移场如下

$$u_z=w(x,y)$$
$$u_x=u(x,y)-z\frac{\partial w}{\partial x}$$
$$u_y=v(x,y)-z\frac{\partial w}{\partial y}$$

6.1.5 上述过程中的一个bug

论文中常见的情形是：先给出位移场，然后求应变，然后求应力，再用能量法求控制方程。但是，结合上述分析过程，将会出现一个bug，而避免这个bug的方式是，不考虑bug存在的那些地方。

根据基尔霍夫板假设，位移场为

$$u_z=w(x,y)$$
$$u_x=u(x,y)-z\frac{\partial w}{\partial x}$$
$$u_y=v(x,y)-z\frac{\partial w}{\partial y}$$

几何方程为

$$\varepsilon_{ij}=\frac{1}{2}(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i})$$

将位移场代入几何方程

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} = \frac{\partial u}{\partial x} - z\frac{\partial^2 w}{\partial x^2}$$
$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y} = \frac{\partial v}{\partial y} - z\frac{\partial^2 w}{\partial y^2}$$
$$\varepsilon_{zz} = \frac{\partial u_z}{\partial z} = 0$$
$$\varepsilon_{xy} = \frac{1}{2}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) - z\frac{\partial^2 w}{\partial x \partial y}$$
$$\varepsilon_{yz} = \varepsilon_{zx} = 0$$

Bug就在于此，因为前面的推导过程中ε_{zz}是一个关于z的线性函数，而这里$\varepsilon_{zz} = 0$，矛盾。但实际上，ε_{zz}应该是一个小量，属于次要矛盾，可以忽略。

绕开这个问题的一个小技巧(trick):
用能量法的话，因为平面应力假设($\sigma_{zz} = 0, \sigma_{zx} = \sigma_{zy} = 0$)，所以没必要讨论$\varepsilon_{zz}, \varepsilon_{zx}, \varepsilon_{zy}$。

$$U = \int_\Omega \frac{1}{2}\left[\sigma_{xx}\varepsilon_{xx} + \sigma_{yy}\varepsilon_{yy} + 2\sigma_{xy}\varepsilon_{xy}\right] dv \tag{6.2}$$

本构方程为

$$\sigma_{xx} = \frac{E}{1-v^2}(\varepsilon_{xx} + \nu\varepsilon_{yy})$$
$$\sigma_{yy} = \frac{E}{1-v^2}(\varepsilon_{yy} + \nu\varepsilon_{xx})$$
$$\sigma_{xy} = 2\mu\varepsilon_{xy}$$

将几何方程代入本构方程

$$\sigma_{xx} = -\left[\frac{E}{1-\nu^2}(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2})\right] z$$
$$\sigma_{yy} = -\left[\frac{E}{1-\nu^2}(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2})\right] z$$
$$\sigma_{xy} = -\left[2\mu\frac{\partial^2 w}{\partial x \partial y}\right] z$$

考虑次要应力

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} = 0$$
$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} = 0$$
$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = 0$$

$$\begin{aligned}
\frac{\partial \sigma_{zx}}{\partial z} = -[\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y}] &= \left[\frac{E}{1-\nu^2}(\frac{\partial^3 w}{\partial x^3} + \nu\frac{\partial^3 w}{\partial x \partial y^2}) + 2\mu\frac{\partial^3 w}{\partial x \partial y^2}\right] z \\
&= \left[\frac{E}{1-\nu^2}(\frac{\partial^3 w}{\partial x^3} + \nu\frac{\partial^3 w}{\partial x \partial y^2}) + \frac{E}{1+\nu}\frac{\partial^3 w}{\partial x \partial y^2}\right] z \\
&= \left[\frac{E}{1-\nu^2}(\frac{\partial^3 w}{\partial x^3} + \nu\frac{\partial^3 w}{\partial x \partial y^2}) + \frac{E}{1-\nu^2}(1-\nu)\frac{\partial^3 w}{\partial x \partial y^2}\right] z \\
&= \left[\frac{E}{1-\nu^2}(\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2})\right] z
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \sigma_{zy}}{\partial z} = -[\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y}] &= \left[\frac{E}{1-\nu^2}(\frac{\partial^3 w}{\partial y^3} + \nu\frac{\partial^3 w}{\partial x^2 \partial y}) + 2\mu\frac{\partial^3 w}{\partial x^2 \partial y}\right] z \\
&= \left[\frac{E}{1-\nu^2}(\frac{\partial^3 w}{\partial y^3} + \nu\frac{\partial^3 w}{\partial x^2 \partial y}) + \frac{E}{1+\nu}\frac{\partial^3 w}{\partial x^2 \partial y}\right] z \\
&= \left[\frac{E}{1-\nu^2}(\frac{\partial^3 w}{\partial y^3} + \nu\frac{\partial^3 w}{\partial x^2 \partial y}) + \frac{E}{1-\nu^2}(1-\nu)\frac{\partial^3 w}{\partial x^2 \partial y}\right] z \\
&= \left[\frac{E}{1-\nu^2}(\frac{\partial^3 w}{\partial y^3} + \frac{\partial^3 w}{\partial x^2 \partial y})\right] z
\end{aligned}$$

积分得

$$\sigma_{zx} = \int \frac{\partial \sigma_{zx}}{\partial z} dz + f_1(x,y) = \left[\frac{E}{1-\nu^2}(\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2})\right] \frac{z^2}{2} + f_1(x,y)$$
$$\sigma_{zy} = \int \frac{\partial \sigma_{zy}}{\partial z} dz + f_2(x,y) = \left[\frac{E}{1-\nu^2}(\frac{\partial^3 w}{\partial y^3} + \frac{\partial^3 w}{\partial x^2 \partial y})\right] \frac{z^2}{2} + f_2(x,y)$$

$$\begin{aligned}\frac{\partial\sigma_{zz}}{\partial z}&=-\left[\frac{\partial\sigma_{xz}}{\partial x}+\frac{\partial\sigma_{yz}}{\partial y}\right]\\&=-\left\{\left[\frac{E}{1-\nu^2}(\frac{\partial^4 w}{\partial x^4}+\frac{\partial^4 w}{\partial x^2\partial y^2})\right]\frac{z^2}{2}+\frac{\partial f_1}{\partial x}\right\}\\&\quad-\left\{\left[\frac{E}{1-\nu^2}(\frac{\partial^4 w}{\partial y^4}+\frac{\partial^4 w}{\partial x^2\partial y^2})\right]\frac{z^2}{2}+\frac{\partial f_2}{\partial y}\right\}\\&=-\left\{\left[\frac{E}{1-\nu^2}(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4})\right]\frac{z^2}{2}+\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}\right\}\end{aligned}$$

积分得

$$\begin{aligned}\sigma_{zz}&=\int\frac{\partial\sigma_{zz}}{\partial z}dz+f_3(x,y)\\&=-\left\{\left[\frac{E}{1-\nu^2}(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4})\right]\frac{z^3}{6}+\left(\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}\right)z+f_3(x,y)\right\}\end{aligned}$$

由边界条件确定积分常数

$$\begin{aligned}\sigma_{zx}(\pm\frac{h}{2})=0\to f_1(x,y)&=-\left[\frac{E}{1-\nu^2}(\frac{\partial^3 w}{\partial x^3}+\frac{\partial^3 w}{\partial x\partial y^2})\right]\frac{h^2}{8}\\\sigma_{zy}(\pm\frac{h}{2}=0)\to f_2(x,y)&=-\left[\frac{E}{1-\nu^2}(\frac{\partial^3 w}{\partial y^3}+\frac{\partial^3 w}{\partial x^2\partial y})\right]\frac{h^2}{8}\end{aligned}$$

$$\begin{aligned}\sigma_{zx}&=\int\frac{\partial\sigma_{zx}}{\partial z}dz+f_1(x,y)=\left[\frac{E}{1-\nu^2}(\frac{\partial^3 w}{\partial x^3}+\frac{\partial^3 w}{\partial x\partial y^2})\right](\frac{z^2}{2}-\frac{h^2}{8})\\\sigma_{zy}&=\int\frac{\partial\sigma_{zy}}{\partial z}dz+f_2(x,y)=\left[\frac{E}{1-\nu^2}(\frac{\partial^3 w}{\partial y^3}+\frac{\partial^3 w}{\partial x^2\partial y})\right](\frac{z^2}{2}-\frac{h^2}{8})\end{aligned}$$

$$\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}=-\left[\frac{E}{1-\nu^2}\left(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}\right)\right]\frac{h^2}{8}$$

故

$$\sigma_{zz}=-\left\{\left[\frac{E}{1-\nu^2}(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4})\right](\frac{z^3}{6}-\frac{h^2 z}{8})+f_3(x,y)\right\}$$

由边界条件确定积分常数

$$\sigma_{zz}(\frac{h}{2})=0\to f_3(x,y)=\left[\frac{E}{1-\nu^2}(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4})\right]\frac{h^3}{24}$$

$$\sigma_{zz}=-\left\{\left[\frac{E}{1-\nu^2}(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4})\right](\frac{z^3}{6}-\frac{h^2 z}{8}+\frac{h^3}{24})\right\}$$

$$\sigma_{zz}(-\frac{h}{2})=-q\to\left[\frac{E}{1-\nu^2}(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4})\right]\frac{h^3}{12}=q$$

若是z坐标朝上，

$$\left[\frac{E}{1-\nu^2}(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4})\right]\frac{h^3}{12}=-q$$

所以坐标朝下的好处是，可以使等号右边没有负号。

6.2 基尔霍夫板

6.2.1 力矩表示的平衡微分方程

弹性力学三维平衡微分方程为

$$\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \sigma_{yx}}{\partial y}+\frac{\partial \sigma_{zx}}{\partial z}=0$$
$$\frac{\partial \sigma_{xy}}{\partial x}+\frac{\partial \sigma_{yy}}{\partial y}+\frac{\partial \sigma_{zy}}{\partial z}=0$$
$$\frac{\partial \sigma_{xz}}{\partial x}+\frac{\partial \sigma_{yz}}{\partial y}+\frac{\partial \sigma_{zz}}{\partial z}=0$$

对z积分

$$\frac{\partial}{\partial x}\left[\int_{-h/2}^{h/2}\sigma_{xx}dz\right]+\frac{\partial}{\partial y}\left[\int_{-h/2}^{h/2}\sigma_{yx}dz\right]+\sigma_{zx}|_{-h/2}^{h/2}=0$$
$$\frac{\partial}{\partial x}\left[\int_{-h/2}^{h/2}\sigma_{xx}dz\right]+\frac{\partial}{\partial y}\left[\int_{-h/2}^{h/2}\sigma_{yx}dz\right]+\sigma_{zy}|_{-h/2}^{h/2}=0$$
$$\frac{\partial}{\partial x}\left[\int_{-h/2}^{h/2}\sigma_{xz}dz\right]+\frac{\partial}{\partial y}\left[\int_{-h/2}^{h/2}\sigma_{yz}dz\right]+\sigma_{zz}|_{-h/2}^{h/2}=0$$

定义薄板面内张力N和薄板面外剪力Q

$$\frac{\partial N_{xx}}{\partial x}+\frac{\partial N_{yz}}{\partial y}+\sigma_{zx}|_{-h/2}^{h/2}=0$$
$$\frac{\partial N_{xy}}{\partial x}+\frac{\partial N_{yy}}{\partial y}+\sigma_{zy}|_{-h/2}^{h/2}=0$$
$$\frac{\partial Q_{xz}}{\partial x}+\frac{\partial Q_{yz}}{\partial y}+\sigma_{zz}|_{-h/2}^{h/2}=0$$

其中，N_{xx},N_{yy},N_{xy}为薄板张力的分量，Q_{xz},Q_{yz}为薄板剪力的分量。

只在上表面作用外力，边界条件为

$$\sigma_{zx}(h/2)=\sigma_{zx}(-h/2)=0$$
$$\sigma_{zy}(h/2)=\sigma_{zy}(-h/2)=0$$
$$\sigma_{zz}(\frac{h}{2})=0,\quad \sigma_{zz}(-\frac{h}{2})=-q(x,y)$$

将边界条件代入前式得

$$\frac{\partial N_{xx}}{\partial x}+\frac{\partial N_{yz}}{\partial y}=0$$
$$\frac{\partial N_{xy}}{\partial x}+\frac{\partial N_{yy}}{\partial y}=0$$
$$\frac{\partial Q_{xz}}{\partial x}+\frac{\partial Q_{yz}}{\partial y}+q(x,y)=0$$

弹性力学三维平衡微分方程前两式乘z

$$z\frac{\partial \sigma_{xx}}{\partial x}+z\frac{\partial \sigma_{yx}}{\partial y}+z\frac{\partial \sigma_{zx}}{\partial z}=0$$
$$z\frac{\partial \sigma_{xy}}{\partial x}+z\frac{\partial \sigma_{yy}}{\partial y}+z\frac{\partial \sigma_{zy}}{\partial z}=0$$

对z积分

$$\frac{\partial}{\partial x}\left[\int_{-h/2}^{h/2}z\sigma_{xx}dz\right]+\frac{\partial}{\partial y}\left[\int_{-h/2}^{h/2}z\sigma_{yx}dz\right]+\int_{-h/2}^{h/2}z\frac{\partial \sigma_{zx}}{\partial z}dz=0$$
$$\frac{\partial}{\partial x}\left[\int_{-h/2}^{h/2}z\sigma_{xy}dz\right]+\frac{\partial}{\partial y}\left[\int_{-h/2}^{h/2}z\sigma_{yy}dz\right]+\int_{-h/2}^{h/2}z\frac{\partial \sigma_{zy}}{\partial z}d=0$$

定义弯矩和扭矩M

$$\frac{\partial M_{xx}}{\partial x}+\frac{\partial M_{yx}}{\partial y}+\int_{-h/2}^{h/2}z\frac{\partial \sigma_{zx}}{\partial z}dz=0$$
$$\frac{\partial M_{xy}}{\partial x}+\frac{\partial M_{yy}}{\partial y}+\int_{-h/2}^{h/2}z\frac{\partial \sigma_{zy}}{\partial z}d=0$$

式中积分项结合边界条件为

$$\int_{-h/2}^{h/2}z\frac{\partial \sigma_{zx}}{\partial z}dz=\int_{-h/2}^{h/2}zd\sigma_{zx}=\int_{-h/2}^{h/2}[d(z\sigma_{zx})-\sigma_{zx}dz]=z\sigma_{zx}|_{-h/2}^{h/2}-\int_{-h/2}^{h/2}\sigma_{zx}dz=-Q_{zx}$$
$$\int_{-h/2}^{h/2}z\frac{\partial \sigma_{zy}}{\partial z}dz=\int_{-h/2}^{h/2}zd\sigma_{zy}=\int_{-h/2}^{h/2}[d(z\sigma_{zy})-\sigma_{zy}dz]=z\sigma_{zy}|_{-h/2}^{h/2}-\int_{-h/2}^{h/2}\sigma_{zy}dz=-Q_{zy}$$

故

$$\frac{\partial M_{xx}}{\partial x}+\frac{\partial M_{yx}}{\partial y}-Q_{zx}=0$$
$$\frac{\partial M_{xy}}{\partial x}+\frac{\partial M_{yy}}{\partial y}-Q_{zy}=0$$

由

$$Q_{zx}=Q_{xz},\quad Q_{zy}=Q_{yz},\quad M_{yx}=M_{xy}$$

代入得

$$\frac{\partial^2 M_{xx}}{\partial x^2}+2\frac{\partial^2 M_{xy}}{\partial x\partial y}+\frac{\partial^2 M_{yy}}{\partial y^2}+q=0$$

6.2.2 对几何方程进行假设及简化得位移场

基尔霍夫的主要假设：薄板的厚度比薄板其他尺寸小，薄板的挠度比薄板的厚度小。

$$h\ll l,\quad u_z\ll h$$

u_z泰勒展开一阶近似

$$u_z(x,y,z+\Delta z)=u_z(x,y,z)+\frac{\partial u_z(x,y,z)}{\partial z}\Delta z+o(\Delta z)$$

假设1：平面应变假设

$$\varepsilon_{zz}=\varepsilon_{zx}=\varepsilon_{zy}=0$$

由

$$\varepsilon_{zz}=\frac{\partial u_z(x,y,z)}{\partial z}=0$$

得

$$u_z(x,y,z)=w(x,y)\quad \text{and}\quad u_z(x,y,z+\Delta z)=u_z(x,y,z)$$

由

$$\varepsilon_{zx}=\frac{1}{2}(\frac{\partial u_z}{\partial x}+\frac{\partial u_x}{\partial z})=0$$
$$\varepsilon_{zy}=\frac{1}{2}(\frac{\partial u_z}{\partial y}+\frac{\partial u_y}{\partial z})=0$$

移项后积分得

$$u_x=-\int\frac{\partial u_z}{\partial x}dz+f_1(x,y)=-z\frac{\partial w}{\partial x}+f_1(x,y)$$
$$u_y=-\int\frac{\partial u_z}{\partial y}dz+f_2(x,y)=-z\frac{\partial w}{\partial y}+f_2(x,y)$$

假设2：中面无面内位移

$$u_x(x,y,0)=u_y(x,y,0)=0$$

故

$$f_1(x,y)=f_2(x,y)=0$$

所以薄板问题位移场为

$$u_x=-z\frac{\partial w}{\partial x}$$
$$u_y=-z\frac{\partial w}{\partial y}$$
$$u_z=w(x,y)$$

应变场为

$$\varepsilon_{xx}=\frac{\partial u_x}{\partial x}=-z\frac{\partial^2 w}{\partial x^2}$$
$$\varepsilon_{yy}=\frac{\partial u_y}{\partial y}=-z\frac{\partial^2 w}{\partial y^2}$$
$$\varepsilon_{xy}=\frac{1}{2}(\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x})=-z\frac{\partial^2 w}{\partial x\partial y}$$
$$\varepsilon_{zz}=\varepsilon_{yz}=\varepsilon_{zx}=0$$

6.2.3 利用简化的本构方程求应力场

应力场为

$$\sigma_{xx} = \frac{E}{1-\nu^2}(\varepsilon_{xx} + \nu\varepsilon_{yy}) = -\left[\frac{E}{1-\nu^2}(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2})\right] z$$
$$\sigma_{yy} = \frac{E}{1-\nu^2}(\varepsilon_{yy} + \nu\varepsilon_{xx}) = -\left[\frac{E}{1-\nu^2}(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2})\right] z$$
$$\sigma_{xy} = \frac{E}{1+\nu}\varepsilon_{xy} = -\left[\frac{E}{1+\nu}\frac{\partial^2 w}{\partial x \partial y}\right] z$$

这里满足薄板张力平衡方程。在薄板假设之下，需要舍弃与z有关的本构关系，否则会出现bug。

6.2.4 最后得用挠度表示的平衡微分方程

力矩为

$$M_{xx} = \int_{-h/2}^{h/2} z\sigma_{xx} dz = -\left[\frac{E}{1-\nu^2}(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2})\right] \frac{z^3}{3}|_{-h/2}^{h/2}$$
$$M_{yy} = \int_{-h/2}^{h/2} z\sigma_{yy} dz = -\left[\frac{E}{1-\nu^2}(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2})\right] \frac{z^3}{3}|_{-h/2}^{h/2}$$
$$M_{xy} = -\left[\frac{E}{1+\nu}\frac{\partial^2 w}{\partial x \partial y}\right] \frac{z^3}{3}|_{-h/2}^{h/2}$$

代入力矩平衡微分方程得挠度表示的平衡微分方程

$$-\left[\frac{E}{1-\nu^2}(\frac{\partial^4 w}{\partial x^4} + \nu\frac{\partial^4 w}{\partial x^2 \partial y^2})\right] \frac{h^3}{12} - \left[\frac{E}{1-\nu^2}(\frac{\partial^4 w}{\partial y^4} + \nu\frac{\partial^4 w}{\partial x^2 \partial y^2})\right] \frac{h^3}{12} - 2\left[\frac{E}{1+\nu}\frac{\partial^4 w}{\partial x^2 \partial y^2}\right] \frac{h^3}{12} + q = 0$$

$$\frac{E}{1-\nu^2}\frac{h^3}{12}\left[(\frac{\partial^4 w}{\partial x^4} + 2\nu\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} + 2(1-v)\frac{\partial^4 w}{\partial x^2 \partial y^2})\right] = q$$

$$\frac{E}{1-\nu^2}\frac{h^3}{12}\left[(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4})\right] = q$$

6.3 FvK model

$$\boldsymbol{x} \equiv x_\alpha \boldsymbol{e}_\alpha \tag{6.3}$$

The deformation of the sheet is denoted by

$$\boldsymbol{y} \equiv \boldsymbol{x} + \boldsymbol{u} + w\boldsymbol{e}_3 \tag{6.4}$$

The deformation gradient is given by

$$\boldsymbol{F} \equiv \boldsymbol{y}\nabla = \boldsymbol{I} + \boldsymbol{u}\nabla + \nabla w \otimes \boldsymbol{e}_3 \tag{6.5}$$

The Green-Lagrange strain $\boldsymbol{E}$ is

$$\boldsymbol{E} \equiv \frac{1}{2}(\boldsymbol{F}^T \cdot \boldsymbol{F} - \boldsymbol{I}) = \frac{1}{2}(\boldsymbol{u}\nabla + \boldsymbol{u}\nabla^T + \boldsymbol{u}\nabla \cdot \boldsymbol{u}\nabla^T + \nabla w \otimes \nabla w) \tag{6.6}$$

The linearized curvature tensor $\boldsymbol{\kappa}$ is

$$\boldsymbol{\kappa} \equiv -\nabla\nabla w \tag{6.7}$$

In the FvK model the elastic energy is assumed as the following quadratic form

$$U = \frac{1}{2}\int_\Omega \mathbb{A}\boldsymbol{E} \cdot \boldsymbol{E} + \frac{1}{2}\int_\Omega \mathbb{D}\boldsymbol{\kappa} \cdot \boldsymbol{\kappa} \tag{6.8}$$

splitted as the sum of the membrane energy U_m and the bending energy U_b. On limiting our attention to linear elastic isotropic materials, the membrane stiffness $\mathbb{A}$ and the bending stiffness $\mathbb{D}$ are proportional and read

$$\mathbb{D} = \frac{h^2}{12}\mathbb{A}, \qquad \mathbb{A}\boldsymbol{E} \equiv \frac{Eh}{1-\nu^2}((1-\nu)\boldsymbol{E} + \nu(tr\boldsymbol{E})\boldsymbol{I}) \tag{6.9}$$

Chapter 7

三维情形：笛卡尔直角坐标

7.1 广义胡克定律

广义胡克定律为

$$\begin{aligned}
\varepsilon_{11} &= \frac{1}{E}\left[\sigma_{11}-\nu(\sigma_{22}+\sigma_{33})\right] = \frac{1}{E}\left[(1+\nu)\sigma_{11}-\nu\sigma_{kk}\right] \\
\varepsilon_{22} &= \frac{1}{E}\left[\sigma_{22}-\nu(\sigma_{33}+\sigma_{11})\right] = \frac{1}{E}\left[(1+\nu)\sigma_{22}-\nu\sigma_{kk}\right] \\
\varepsilon_{33} &= \frac{1}{E}\left[\sigma_{33}-\nu(\sigma_{11}+\sigma_{22})\right] = \frac{1}{E}\left[(1+\nu)\sigma_{33}-\nu\sigma_{kk}\right] \\
\varepsilon_{12} &= \frac{1}{2\mu}\sigma_{12} = \frac{1+\nu}{E}\sigma_{12} \\
\varepsilon_{23} &= \frac{1}{2\mu}\sigma_{23} = \frac{1+\nu}{E}\sigma_{23} \\
\varepsilon_{31} &= \frac{1}{2\mu}\sigma_{31} = \frac{1+\nu}{E}\sigma_{31}
\end{aligned} \tag{7.1}$$

可以写成统一的形式

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} \tag{7.2}$$

上上式的前三式相加

$$\varepsilon_{kk} = \frac{1}{E}\left[(1+\nu)\sigma_{kk} - 3\nu\sigma_{kk}\right] = \frac{1-2\nu}{E}\sigma_{kk} \tag{7.3}$$

将上式代入上上式

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{1-2\nu}\varepsilon_{kk}\delta_{ij} \tag{7.4}$$

移项

$$\frac{1+\nu}{E}\sigma_{ij} = \varepsilon_{ij} + \frac{\nu}{1-2\nu}\varepsilon_{kk}\delta_{ij} \tag{7.5}$$

化简

$$\sigma_{ij} = \frac{E\nu}{(1+\nu)(1-2\nu)}\varepsilon_{kk}\delta_{ij} + \frac{E}{1+\nu}\varepsilon_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} \tag{7.6}$$

矩阵化表示：

$$\begin{aligned}
\sigma_{11} &= \lambda(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}) + 2\mu\varepsilon_{11} = (\lambda+2\mu)\varepsilon_{11} + \lambda\varepsilon_{22} + \lambda\varepsilon_{33} \\
\sigma_{22} &= \lambda(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}) + 2\mu\varepsilon_{22} = \lambda\varepsilon_{11} + (\lambda+2\mu)\varepsilon_{22} + \lambda\varepsilon_{33} \\
\sigma_{33} &= \lambda(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}) + 2\mu\varepsilon_{33} = \lambda\varepsilon_{11} + \lambda\varepsilon_{22} + (\lambda+2\mu)\varepsilon_{33} \\
\sigma_{12} &= 2\mu\varepsilon_{12} \\
\sigma_{23} &= 2\mu\varepsilon_{23} \\
\sigma_{31} &= 2\mu\varepsilon_{31}
\end{aligned} \tag{7.7}$$

那么

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{12}\\ \sigma_{23}\\ \sigma_{31}\end{bmatrix} = \begin{bmatrix}\lambda+2\mu & \lambda & \lambda & 0 & 0 & 0\\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0\\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0\\ 0 & 0 & 0 & 2\mu & 0 & 0\\ 0 & 0 & 0 & 0 & 2\mu & 0\\ 0 & 0 & 0 & 0 & 0 & 2\mu\end{bmatrix} = \begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ \varepsilon_{12}\\ \varepsilon_{23}\\ \varepsilon_{31}\end{bmatrix} \tag{7.8}$$

7.2 相容方程

小应变张量的分量形式

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{7.9}$$

STEP1:求二阶偏导

20331-1}

$$\varepsilon_{ij,kl} = \frac{1}{2}(u_{i,jkl} + u_{j,ikl}) \tag{7.10}$$

STEP2:为了建立不同应变分量之间的关系，把两个分量指标和两个偏导指标双双对换

$$i \longleftrightarrow k, \quad j \longleftrightarrow l \tag{7.11}$$

然后有

20331-2}

$$\varepsilon_{kl,ij} = \frac{1}{2}(u_{k,lij} + u_{l,kij}) \tag{7.12}$$

此时，公式(7.10)和(7.12)右端出现位移分量u_i, u_j, u_k, u_l的四个三阶偏导数。

STEP3:把某个分量指标和某个导数指标对换(例如$j \longleftrightarrow k$)

20331-3}

$$\varepsilon_{ik,jl} = \frac{1}{2}(u_{i,kjl} + u_{k,ijl}) \tag{7.13}$$

STEP4:再双双对换分量指标和偏导指标

$$i \longleftrightarrow k, \quad j \longleftrightarrow l \tag{7.14}$$

然后有

20331-4}

$$\varepsilon_{jl,ik} = \frac{1}{2}(u_{j,lik} + u_{l,jik}) \tag{7.15}$$

此时，公式(7.13)和(7.15)右端出现位移分量u_i, u_j, u_k, u_l的四个三阶偏导数。

若位移场单值连续且存在三阶以上连续偏导数，根据偏导数与求导顺序无关

20331-5}

$$\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{ik,jl} - \varepsilon_{jl,ik} = 0 \tag{7.16}$$

此为存在单值连续位移场的充要条件。由于推导过程中只用了连续函数的求导顺序无关性，所以其本质是变形连续条件。在数学上，上式是能由几何方程积分出单值连续位移场的充要条件，简称可积条件。

对于任一张量A_{ij}，$A_{ij} - A_{ji}$为反对称张量，指标i和j反对称。$A_{ij} - A_{ji} = 0$，2个自由指标，共9个方程。实际上，其中有3个等于零的恒等方程和3对等价方程。所以，$A_{ij} - A_{ji} = 0$共包含3个独立方程。可以采用排列符号将其缩写为

$$e_{mij}A_{ij} = 0 \tag{7.17}$$

公式(7.16)中含有4个自由指标，共81个方程，但其中有不少是恒等方程或等价方程。公式(7.16)左端关于指标j和k反对称，故可以采用排列符号将其缩写为

$$e_{mjk}\varepsilon_{ij,kl} - e_{mjk}\varepsilon_{jl,ik} = 0 \tag{7.18}$$

化简为

20331-6}

$$e_{mjk}(\varepsilon_{ij,kl} - \varepsilon_{jl,ik}) = 0 \tag{7.19}$$

公式(7.19)关于指标i和l反对称，故可以继续采用排列符号将其缩写为

20331-7}

$$e_{mjk}e_{nil}\varepsilon_{ij,kl} = 0 \tag{7.20}$$

实际上，公式(7.20)关于指标m和n对称，所以最终变形连续方程只有6个独立方程。证明如下：

$$
\begin{aligned}
e_{mjk}e_{nil}\varepsilon_{ij,kl} &= \begin{vmatrix} \delta_{mn} & \delta_{mi} & \delta_{ml} \\ \delta_{jn} & \delta_{ji} & \delta_{jl} \\ \delta_{kn} & \delta_{ki} & \delta_{kl} \end{vmatrix} \varepsilon_{ij,kl} \\
&= [(\delta_{mn}\delta_{ji}\delta_{kl} - \delta_{mn}\delta_{ki}\delta_{jl}) + (-\delta_{mi}\delta_{jn}\delta_{kl} - \delta_{mi}\delta_{kn}\delta_{jl}) + (\delta_{ml}\delta_{jn}\delta_{ki} - \delta_{ml}\delta_{kn}\delta_{ji})]\,\varepsilon_{ij,kl} \\
&= (\delta_{mn}\varepsilon_{ii,kk} - \delta_{mn}\varepsilon_{kl,kl}) + (-\varepsilon_{mn,kk} + \varepsilon_{ml,nl}) + (\varepsilon_{kn,km} - \varepsilon_{jj,nm})
\end{aligned} \tag{7.21}
$$

$$
\begin{aligned}
e_{njk}e_{mil}\varepsilon_{ij,kl} &= \begin{vmatrix} \delta_{nm} & \delta_{ni} & \delta_{nl} \\ \delta_{jm} & \delta_{ji} & \delta_{jl} \\ \delta_{km} & \delta_{ki} & \delta_{kl} \end{vmatrix} \varepsilon_{ij,kl} \\
&= [(\delta_{nm}\delta_{ji}\delta_{kl} - \delta_{nm}\delta_{ki}\delta_{jl}) + (-\delta_{ni}\delta_{jm}\delta_{kl} + \delta_{ni}\delta_{km}\delta_{jl}) + (\delta_{nl}\delta_{jm}\delta_{ki} - \delta_{nl}\delta_{km}\delta_{ji})]\,\varepsilon_{ij,kl} \\
&= (\delta_{nm}\varepsilon_{ii,kk} - \delta_{nm}\varepsilon_{kl,kl}) + (-\varepsilon_{nm,kk} + \varepsilon_{nl,ml}) + (\varepsilon_{km,kn} - \varepsilon_{jj,mn}) \\
&= (\delta_{mn}\varepsilon_{ii,kk} - \delta_{mn}\varepsilon_{kl,kl}) + (-\varepsilon_{mn,kk} + \varepsilon_{kn,km}) + (\varepsilon_{ml,nl} - \varepsilon_{jj,mn})
\end{aligned} \tag{7.22}
$$

其中最后用到了应变的对称性和偏导指标的可换性。所以

$$
e_{mjk}e_{nil}\varepsilon_{ij,kl} = e_{njk}e_{mil}\varepsilon_{ij,kl} \tag{7.23}
$$

指标m和n对称得证。

(m,n)分别取$(1,1),(2,2),(3,3),(1,2),(2,3),(3,1)$

$$
\begin{aligned}
e_{1jk}e_{1il}\varepsilon_{ij,kl} &= e_{123}e_{1il}\varepsilon_{i2,3l} + e_{132}e_{1il}\varepsilon_{i3,2l} = e_{123}(e_{123}\varepsilon_{22,33} + e_{132}\varepsilon_{32,32}) + e_{132}(e_{123}\varepsilon_{23,23} + e_{132}\varepsilon_{33,22}) \\
&= \varepsilon_{22,33} - \varepsilon_{32,32} - \varepsilon_{23,23} + \varepsilon_{33,22} = \varepsilon_{22,33} - 2\varepsilon_{23,23} + \varepsilon_{33,22} = 0
\end{aligned} \tag{7.24}
$$

$$
\begin{aligned}
e_{2jk}e_{2il}\varepsilon_{ij,kl} &= e_{231}e_{2il}\varepsilon_{i3,1l} + e_{213}e_{2il}\varepsilon_{i1,3l} = e_{231}(e_{231}\varepsilon_{33,11} + e_{213}\varepsilon_{13,13}) + e_{213}(e_{231}\varepsilon_{31,31} + e_{213}\varepsilon_{11,33}) \\
&= \varepsilon_{33,11} - \varepsilon_{13,13} - \varepsilon_{31,31} + \varepsilon_{11,33} = \varepsilon_{33,11} - 2\varepsilon_{13,13} + \varepsilon_{11,33} = 0
\end{aligned} \tag{7.25}
$$

$$
\begin{aligned}
e_{3jk}e_{3il}\varepsilon_{ij,kl} &= e_{312}e_{3il}\varepsilon_{i1,2l} + e_{321}e_{3il}\varepsilon_{i2,1l} = e_{312}(e_{312}\varepsilon_{11,22} + e_{321}\varepsilon_{21,21}) + e_{321}(e_{312}\varepsilon_{12,12} + e_{321}\varepsilon_{22,11}) \\
&= \varepsilon_{11,22} - \varepsilon_{21,21} - \varepsilon_{12,12} + \varepsilon_{22,11} = \varepsilon_{11,22} - 2\varepsilon_{12,12} + \varepsilon_{22,11} = 0
\end{aligned} \tag{7.26}
$$

$$
\begin{aligned}
e_{1jk}e_{2il}\varepsilon_{ij,kl} &= e_{123}e_{2il}\varepsilon_{i2,3l} + e_{132}e_{2il}\varepsilon_{i3,2l} = e_{123}(e_{231}\varepsilon_{32,31} + e_{213}\varepsilon_{12,33}) + e_{132}(e_{231}\varepsilon_{33,21} + e_{213}\varepsilon_{13,23}) \\
&= \varepsilon_{32,31} - \varepsilon_{12,33} - \varepsilon_{33,21} + \varepsilon_{13,23} = 0
\end{aligned} \tag{7.27}
$$

$$
\begin{aligned}
e_{2jk}e_{3il}\varepsilon_{ij,kl} &= e_{213}e_{3il}\varepsilon_{i1,3l} + e_{231}e_{3il}\varepsilon_{i3,1l} = e_{213}(e_{312}\varepsilon_{11,32} + e_{321}\varepsilon_{21,31}) + e_{231}(e_{312}\varepsilon_{13,12} + e_{321}\varepsilon_{23,11}) \\
&= \varepsilon_{13,12} - \varepsilon_{23,11} - \varepsilon_{11,32} + \varepsilon_{21,31} = 0
\end{aligned} \tag{7.28}
$$

$$
\begin{aligned}
e_{1jk}e_{3il}\varepsilon_{ij,kl} &= e_{123}e_{3il}\varepsilon_{i2,3l} + e_{132}e_{3il}\varepsilon_{i3,2l} = e_{123}(e_{312}\varepsilon_{12,32} + e_{321}\varepsilon_{22,31}) + e_{132}(e_{312}\varepsilon_{13,22} + e_{321}\varepsilon_{23,21}) \\
&= \varepsilon_{12,32} - \varepsilon_{22,31} - \varepsilon_{13,22} + \varepsilon_{23,21} = 0
\end{aligned} \tag{7.29}
$$

Chapter 8

三维情形：柱坐标

8.1 数学知识

8.1.1 柱坐标拉普拉斯算子

标量函数的梯度为

$$\nabla f = \frac{1}{h_k} e_k \frac{\partial f}{\partial x_k} \tag{8.1}$$

其中，h_k为拉梅系数，对于柱坐标$h_k = (1, \rho, 1)$。

标量函数的拉普拉斯运算为

8.2 控制方程

8.2.1 几何方程

$$
\begin{aligned}
\boldsymbol{\varepsilon} &= \frac{1}{2}[\boldsymbol{u}\nabla + \boldsymbol{u}\nabla^T] \\
&= \begin{bmatrix}
\frac{\partial u_\rho}{\partial x_\rho} & \frac{1}{2}\left[\frac{\partial u_\theta}{\partial x_\rho} + \frac{1}{\rho}\left(\frac{\partial u_\rho}{\partial x_\theta} - u_\theta\right)\right] & \frac{1}{2}\left(\frac{\partial u_z}{\partial x_\rho} + \frac{\partial u_\rho}{\partial x_z}\right) \\
\frac{1}{2}\left[\frac{\partial u_\theta}{\partial x_\rho} + \frac{1}{\rho}\left(\frac{\partial u_\rho}{\partial x_\theta} - u_\theta\right)\right] & \frac{1}{\rho}\left(\frac{\partial u_\theta}{\partial x_\theta} + u_\rho\right) & \frac{1}{2}\left(\frac{1}{\rho}\frac{\partial u_z}{\partial x_\theta} + \frac{\partial u_\theta}{\partial x_z}\right) \\
\frac{1}{2}\left(\frac{\partial u_z}{\partial x_\rho} + \frac{\partial u_\rho}{\partial x_z}\right) & \frac{1}{2}\left(\frac{1}{\rho}\frac{\partial u_z}{\partial x_\theta} + \frac{\partial u_\theta}{\partial x_z}\right) & \frac{\partial u_z}{\partial x_z}
\end{bmatrix}
\end{aligned} \tag{8.2}
$$

其中

$$
\begin{aligned}
u\nabla &= \frac{1}{h_k}\frac{\partial u}{\partial x_k} \otimes e_k \\
&= \frac{1}{h_k}\frac{\partial u_i e_i}{\partial x_k} \otimes e_k \\
&= \frac{1}{h_k}\left[\frac{\partial u_i}{\partial x_k} e_i + u_i \frac{\partial e_i}{\partial x_k}\right] \otimes e_k \\
&= \frac{1}{h_k}\frac{\partial u_i}{\partial x_k} e_i \otimes e_k + \frac{1}{h_k} u_i \frac{\partial e_i}{\partial x_k} \otimes e_k \\
&= \frac{1}{h_k}\frac{\partial u_i}{\partial x_k} e_i \otimes e_k + \frac{1}{h_\theta} u_i \frac{\partial e_i}{\partial x_\theta} \otimes e_\theta \\
&= \frac{1}{h_k}\frac{\partial u_i}{\partial x_k} e_i \otimes e_k + \frac{1}{h_\theta} u_\rho \frac{\partial e_\rho}{\partial x_\theta} \otimes e_\theta + \frac{1}{h_\theta} u_\theta \frac{\partial e_\theta}{\partial x_\theta} \otimes e_\theta \\
&= \frac{1}{h_k}\frac{\partial u_i}{\partial x_k} e_i \otimes e_k + \frac{1}{h_\theta} u_\rho e_\theta \otimes e_\theta - \frac{1}{h_\theta} u_\theta e_\rho \otimes e_\theta
\end{aligned} \tag{8.3}
$$

$$
\begin{aligned}
\nabla u &= \frac{1}{h_k} e_k \otimes \frac{\partial u}{\partial x_k} \\
&= \frac{1}{h_k} e_k \otimes \frac{\partial u_i e_i}{\partial x_k} \\
&= \frac{1}{h_k} e_k \otimes \left[\frac{\partial u_i}{\partial x_k} e_i + u_i \frac{\partial e_i}{\partial x_k}\right] \\
&= \frac{1}{h_k} e_k \otimes \frac{\partial u_i}{\partial x_k} e_i + \frac{1}{h_k} e_k \otimes u_i \frac{\partial e_i}{\partial x_k} \\
&= \frac{1}{h_k}\frac{\partial u_i}{\partial x_k} e_k \otimes e_i + \frac{1}{h_k} u_i e_k \otimes \frac{\partial e_i}{\partial x_k} \\
&= \frac{1}{h_k}\frac{\partial u_i}{\partial x_k} e_k \otimes e_i + \frac{1}{h_\theta} u_i e_\theta \otimes \frac{\partial e_i}{\partial x_\theta} \\
&= \frac{1}{h_k}\frac{\partial u_i}{\partial x_k} e_k \otimes e_i + \frac{1}{h_\theta} u_\rho e_\theta \otimes \frac{\partial e_\rho}{\partial x_\theta} + \frac{1}{h_\theta} u_\theta e_\theta \otimes \frac{\partial e_\theta}{\partial x_\theta} \\
&= \frac{1}{h_k}\frac{\partial u_i}{\partial x_k} e_k \otimes e_i + \frac{1}{h_\theta} u_\rho e_\theta \otimes e_\theta - \frac{1}{h_\theta} u_\theta e_\theta \otimes e_\rho
\end{aligned} \tag{8.4}
$$

$$
\begin{aligned}
&\frac{1}{2}[u\nabla + \nabla u] \\
=&\frac{1}{2}\left[\frac{1}{h_k}\frac{\partial u_i}{\partial x_k} e_i \otimes e_k + \frac{1}{h_\theta} u_\rho e_\theta \otimes e_\theta - \frac{1}{h_\theta} u_\theta e_\rho \otimes e_\theta + \frac{1}{h_k}\frac{\partial u_i}{\partial x_k} e_k \otimes e_i + \frac{1}{h_\theta} u_\rho e_\theta \otimes e_\theta - \frac{1}{h_\theta} u_\theta e_\theta \otimes e_\rho\right] \\
=&\frac{1}{2}\left[\frac{1}{h_k}\frac{\partial u_i}{\partial x_k} e_i \otimes e_k + \frac{1}{h_\theta} u_\rho e_\theta \otimes e_\theta - \frac{1}{h_\theta} u_\theta e_\rho \otimes e_\theta + \frac{1}{h_i}\frac{\partial u_k}{\partial x_i} e_i \otimes e_k + \frac{1}{h_\theta} u_\rho e_\theta \otimes e_\theta - \frac{1}{h_\theta} u_\theta e_\theta \otimes e_\rho\right] \\
=&\frac{1}{2}\left[\left(\frac{1}{h_k}\frac{\partial u_i}{\partial x_k} + \frac{1}{h_i}\frac{\partial u_k}{\partial x_i}\right) e_i \otimes e_k + \frac{2}{h_\theta} u_\rho e_\theta \otimes e_\theta - \frac{1}{h_\theta} u_\theta e_\rho \otimes e_\theta - \frac{1}{h_\theta} u_\theta e_\theta \otimes e_\rho\right] \\
=&\begin{bmatrix}
\frac{\partial u_\rho}{\partial x_\rho} & \frac{1}{2}\left[\frac{\partial u_\theta}{\partial x_\rho} + \frac{1}{\rho}\left(\frac{\partial u_\rho}{\partial x_\theta} - u_\theta\right)\right] & \frac{1}{2}\left(\frac{\partial u_z}{\partial x_\rho} + \frac{\partial u_\rho}{\partial x_z}\right) \\
\frac{1}{2}\left[\frac{\partial u_\theta}{\partial x_\rho} + \frac{1}{\rho}\left(\frac{\partial u_\rho}{\partial x_\theta} - u_\theta\right)\right] & \frac{1}{\rho}\left(\frac{\partial u_\theta}{\partial x_\theta} + u_\rho\right) & \frac{1}{2}\left(\frac{1}{\rho}\frac{\partial u_z}{\partial x_\theta} + \frac{\partial u_\theta}{\partial x_z}\right) \\
\frac{1}{2}\left(\frac{\partial u_z}{\partial x_\rho} + \frac{\partial u_\rho}{\partial x_z}\right) & \frac{1}{2}\left(\frac{1}{\rho}\frac{\partial u_z}{\partial x_\theta} + \frac{\partial u_\theta}{\partial x_z}\right) & \frac{\partial u_z}{\partial x_z}
\end{bmatrix}
\end{aligned} \tag{8.5}
$$

工程剪应变和小应变剪切分量存在如下关系

$$
\gamma_{ij} = 2\varepsilon_{ij} \quad (i \neq j) \tag{8.6}
$$

8.2.2 物理方程：广义胡克定律

$$
\begin{aligned}
\boldsymbol{\sigma} &= \lambda tr(\boldsymbol{\varepsilon})I + 2\mu\boldsymbol{\varepsilon} \\
&= \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} \\
&= \begin{bmatrix} \lambda\varepsilon_{kk} + 2\mu\varepsilon_{\rho\rho} & 2\mu\varepsilon_{\rho\theta} & 2\mu\varepsilon_{\rho z} \\ 2\mu\varepsilon_{\rho\theta} & \lambda\varepsilon_{kk} + 2\mu\varepsilon_{\theta\theta} & 2\mu\varepsilon_{\theta z} \\ 2\mu\varepsilon_{\rho z} & 2\mu\varepsilon_{\theta z} & \lambda\varepsilon_{kk} + 2\mu\varepsilon_{zz} \end{bmatrix}
\end{aligned} \tag{8.7}
$$

8.2.3 平衡方程：动量守恒

$$
\begin{aligned}
\nabla \cdot \boldsymbol{\sigma} &= \mathbf{0} \\
&= \begin{bmatrix} \frac{\partial\sigma_{\rho\rho}}{\partial x_\rho} + \frac{\partial\sigma_{z\rho}}{\partial x_z} + \frac{1}{\rho}\left(\frac{\partial\sigma_{\theta\rho}}{\partial x_\theta} + \sigma_{\rho\rho} - \sigma_{\theta\theta}\right) \\ \frac{\partial\sigma_{\rho\theta}}{\partial x_\rho} + \frac{\partial\sigma_{z\theta}}{\partial x_z} + \frac{1}{\rho}\left(\frac{\partial\sigma_{\theta\theta}}{\partial x_\theta} + \sigma_{\rho\theta} + \sigma_{\theta\rho}\right) \\ \frac{\partial\sigma_{\rho z}}{\partial x_\rho} + \frac{\partial\sigma_{zz}}{\partial x_z} + \frac{1}{\rho}\left(\frac{\partial\sigma_{\theta z}}{\partial x_\theta} + \sigma_{\rho z}\right) \end{bmatrix}
\end{aligned} \tag{8.8}
$$

其中

$$
\begin{aligned}
&\nabla \cdot \sigma \\
&= \frac{1}{h_k} e_k \cdot \frac{\partial\sigma}{\partial x_k} \\
&= \frac{1}{h_k} e_k \cdot \frac{\partial\sigma_{ij} e_i \otimes e_j}{\partial x_k} \\
&= \frac{1}{h_k} e_k \cdot \left[\frac{\partial\sigma_{ij}}{\partial x_k} e_i \otimes e_j + \sigma_{ij}\frac{\partial e_i}{\partial x_k} \otimes e_j + \sigma_{ij} e_i \otimes \frac{\partial e_j}{\partial x_k}\right] \\
&= \frac{1}{h_k} e_k \cdot \frac{\partial\sigma_{ij}}{\partial x_k} e_i \otimes e_j + \frac{1}{h_k} e_k \cdot \sigma_{ij}\frac{\partial e_i}{\partial x_k} \otimes e_j + \frac{1}{h_k} e_k \cdot \sigma_{ij} e_i \otimes \frac{\partial e_j}{\partial x_k} \\
&= \frac{1}{h_k}\delta_{ki}\frac{\partial\sigma_{ij}}{\partial x_k} e_j + \frac{1}{h_\theta} e_\theta \cdot \sigma_{ij}\frac{\partial e_i}{\partial x_\theta} \otimes e_j + \frac{1}{h_\theta} e_\theta \cdot \sigma_{ij} e_i \otimes \frac{\partial e_j}{\partial x_\theta} \\
&= \frac{1}{h_k}\delta_{ki}\frac{\partial\sigma_{ij}}{\partial x_k} e_j + \left[\frac{1}{h_\theta} e_\theta \cdot \sigma_{\rho j}\frac{\partial e_\rho}{\partial x_\theta} \otimes e_j + \frac{1}{h_\theta} e_\theta \cdot \sigma_{\theta j}\frac{\partial e_\theta}{\partial x_\theta} \otimes e_j\right] + \left[\frac{1}{h_\theta} e_\theta \cdot \sigma_{i\rho} e_i \otimes \frac{\partial e_\rho}{\partial x_\theta} + \frac{1}{h_\theta} e_\theta \cdot \sigma_{i\theta} e_i \otimes \frac{\partial e_\theta}{\partial x_\theta}\right] \\
&= \frac{1}{h_k}\delta_{ki}\frac{\partial\sigma_{ij}}{\partial x_k} e_j + \left[\frac{1}{h_\theta} e_\theta \cdot \sigma_{\rho j} e_\theta \otimes e_j - \frac{1}{h_\theta} e_\theta \cdot \sigma_{\theta j} e_\rho \otimes e_j\right] + \left[\frac{1}{h_\theta} e_\theta \cdot \sigma_{i\rho} e_i \otimes e_\theta - \frac{1}{h_\theta} e_\theta \cdot \sigma_{i\theta} e_i \otimes e_\rho\right] \\
&= \frac{1}{h_k}\delta_{ki}\frac{\partial\sigma_{ij}}{\partial x_k} e_j + \left[\frac{1}{h_\theta}\sigma_{\rho j} e_j\right] + \left[\frac{1}{h_\theta}\delta_{\theta i}\sigma_{i\rho} e_\theta - \frac{1}{h_\theta}\delta_{\theta i}\sigma_{i\theta} e_\rho\right] \\
&= \frac{1}{h_k}\frac{\partial\sigma_{kj}}{\partial x_k} e_j + \frac{1}{h_\theta}\sigma_{\rho j} e_j + \frac{1}{h_\theta}\sigma_{\theta\rho} e_\theta - \frac{1}{h_\theta}\sigma_{\theta\theta} e_\rho
\end{aligned} \tag{8.9}
$$

8.2.4 相容方程：变形连续方程、应变协调方程

应变表示的相容方程为

$$
\begin{aligned}
\nabla\times\varepsilon\times\nabla \mid (e_\rho \otimes e_\rho) &= -\frac{1}{\rho^2}\frac{\partial^2\varepsilon_{zz}}{\partial x_\theta \partial x_\theta} + \frac{1}{\rho}\left[2\frac{\partial^2\varepsilon_{z\theta}}{\partial x_\theta \partial x_z} + 2\frac{\partial\varepsilon_{z\rho}}{\partial x_z} - \frac{\partial\varepsilon_{zz}}{\partial x_\rho}\right] - \frac{\partial^2\varepsilon_{\theta\theta}}{\partial x_z \partial x_z} = 0 \\
\nabla\times\varepsilon\times\nabla \mid (e_\theta \otimes e_\theta) &= -\frac{\partial^2\varepsilon_{zz}}{\partial x_\rho \partial x_\rho} + 2\frac{\partial^2\varepsilon_{z\rho}}{\partial x_\rho \partial x_z} - \frac{\partial^2\varepsilon_{\rho\rho}}{\partial x_z \partial x_z} = 0 \\
\nabla\times\varepsilon\times\nabla \mid (e_z \otimes e_z) &= \frac{1}{\rho^2}\left[-\frac{\partial^2\varepsilon_{\rho\rho}}{\partial x_\theta \partial x_\theta} + 2\frac{\partial\varepsilon_{\rho\theta}}{\partial x_\theta}\right] + \frac{1}{\rho}\left[2\frac{\partial^2\varepsilon_{\theta\rho}}{\partial x_\rho \partial x_\theta} - 2\frac{\partial\varepsilon_{\theta\theta}}{\partial x_\rho} + \frac{\partial\varepsilon_{\rho\rho}}{\partial x_\rho}\right] - \frac{\partial^2\varepsilon_{\theta\theta}}{\partial x_\rho \partial x_\rho} = 0 \\
\nabla\times\varepsilon\times\nabla \mid (e_\rho \otimes e_\theta) &= \frac{-1}{\rho^2}\left(\frac{\partial\varepsilon_{zz}}{\partial x_\theta}\right) + \frac{1}{\rho}\left[\frac{\partial^2\varepsilon_{zz}}{\partial x_\theta \partial x_\rho} - \frac{\partial^2\varepsilon_{z\rho}}{\partial x_\theta \partial x_z} + \frac{\partial\varepsilon_{z\theta}}{\partial x_z}\right] - \frac{\partial^2\varepsilon_{\theta z}}{\partial x_z \partial x_\rho} + \frac{\partial^2\varepsilon_{\theta\rho}}{\partial x_z \partial x_z} = 0 \\
\nabla\times\varepsilon\times\nabla \mid (e_\rho \otimes e_z) &= \frac{1}{\rho^2}\left[\frac{\partial^2\varepsilon_{z\rho}}{\partial x_\theta \partial x_\theta} - \frac{\partial\varepsilon_{z\theta}}{\partial x_\theta}\right] + \frac{1}{\rho}\left[-\frac{\partial^2\varepsilon_{z\theta}}{\partial x_\theta \partial x_\rho} - \frac{\partial^2\varepsilon_{\theta\rho}}{\partial x_z \partial x_\theta} - \frac{\partial\varepsilon_{\rho\rho}}{\partial x_z} + \frac{\partial\varepsilon_{\theta\theta}}{\partial x_z}\right] + \frac{\partial^2\varepsilon_{\theta\theta}}{\partial x_z \partial x_\rho} = 0 \\
\nabla\times\varepsilon\times\nabla \mid (e_\theta \otimes e_z) &= \frac{1}{\rho^2}\left[\frac{\partial\varepsilon_{z\rho}}{\partial x_\theta} - \varepsilon_{z\theta}\right] + \frac{1}{\rho}\left[-\frac{\partial^2\varepsilon_{z\rho}}{\partial x_\rho \partial x_\theta} + \frac{\partial\varepsilon_{z\theta}}{\partial x_\rho} + \frac{\partial^2\varepsilon_{\rho\rho}}{\partial x_z \partial x_\theta} - 2\frac{\partial\varepsilon_{\theta\rho}}{\partial x_z}\right] + \frac{\partial^2\varepsilon_{z\theta}}{\partial x_\rho \partial x_\rho} - \frac{\partial^2\varepsilon_{\rho\theta}}{\partial x_z \partial x_\rho} = 0
\end{aligned} \tag{8.10}
$$

应力表示的相容方程为

$$
\begin{aligned}
&\nabla\cdot\nabla\sigma+\frac{1}{1+v}\nabla\nabla\Theta \mid (e_\rho\otimes e_\rho)=\nabla^2\sigma_{\rho\rho}-\frac{1}{\rho^2}\left[4\frac{\partial\sigma_{\theta\rho}}{\partial x_\theta}+2\sigma_{\rho\rho}-2\sigma_{\theta\theta}\right]+\frac{1}{1+v}\left(\frac{\partial}{\partial x_\rho}\frac{\partial\Theta}{\partial x_\rho}\right)\\
&\nabla\cdot\nabla\sigma+\frac{1}{1+v}\nabla\nabla\Theta \mid (e_\theta\otimes e_\theta)=\nabla^2\sigma_{\theta\theta}+\frac{1}{\rho^2}\left[4\frac{\partial\sigma_{\rho\theta}}{\partial x_\theta}+2\sigma_{\rho\rho}-2\sigma_{\theta\theta}\right]+\frac{1}{1+v}\left[\frac{1}{\rho}\frac{\partial\Theta}{\partial x_\rho}+\frac{1}{\rho^2}\left(\frac{\partial}{\partial x_\theta}\frac{\partial\Theta}{\partial x_\theta}\right)\right]\\
&\nabla\cdot\nabla\sigma+\frac{1}{1+v}\nabla\nabla\Theta \mid (e_z\otimes e_z)=\nabla^2\sigma_{zz}+\frac{1}{1+v}\left[\frac{\partial}{\partial x_z}\frac{\partial\Theta}{\partial x_z}\right]\\
&\nabla\cdot\nabla\sigma+\frac{1}{1+v}\nabla\nabla\Theta \mid (e_\rho\otimes e_\theta)=\nabla^2\sigma_{\rho\theta}+\frac{1}{\rho^2}\left[-2\frac{\partial\sigma_{\theta\theta}}{\partial x_\theta}+2\frac{\partial\sigma_{\rho\rho}}{\partial x_\theta}-4\sigma_{\rho\theta}\right]+\frac{1}{1+v}\left[-\frac{1}{\rho^2}\frac{\partial\Theta}{\partial x_\theta}+\frac{1}{\rho}\left(\frac{\partial}{\partial x_\rho}\frac{\partial\Theta}{\partial x_\theta}\right)\right]\\
&\nabla\cdot\nabla\sigma+\frac{1}{1+v}\nabla\nabla\Theta \mid (e_\rho\otimes e_z)=\nabla^2\sigma_{\rho z}+\frac{1}{\rho^2}\left[-2\frac{\partial\sigma_{\theta z}}{\partial x_\theta}-\sigma_{\rho z}\right]+\frac{1}{1+v}\left[\frac{\partial}{\partial x_\rho}\frac{\partial\Theta}{\partial x_z}\right]\\
&\nabla\cdot\nabla\sigma+\frac{1}{1+v}\nabla\nabla\Theta \mid (e_\theta\otimes e_z)=\nabla^2\sigma_{\theta z}+\frac{1}{\rho^2}\left[2\frac{\partial\sigma_{\rho z}}{\partial x_\theta}-\sigma_{\theta z}\right]+\frac{1}{1+v}\left[\frac{1}{\rho}\left(\frac{\partial}{\partial x_\theta}\frac{\partial\Theta}{\partial x_z}\right)\right]
\end{aligned}
\tag{8.11}
$$

8.2.5 解法：应力函数法

假设由应力函数表示的位移形式如下

$$
\begin{aligned}
u_\rho&=-\frac{1+v}{E}\left[\frac{\partial^2\Pi}{\partial\rho\partial z}-\frac{2}{\rho}\frac{\partial\Lambda}{\partial\theta}\right]\\
u_\theta&=-\frac{1+v}{E}\left[\frac{1}{\rho}\frac{\partial^2\Pi}{\partial\theta\partial z}+2\frac{\partial\Lambda}{\partial\rho}\right]\\
u_z&=\frac{1+v}{E}\left[2(1-v)\nabla^2\Pi-\frac{\partial^2\Pi}{\partial z^2}\right]
\end{aligned}
\tag{8.12}
$$

由几何方程求得应变为

$$
\begin{aligned}
\varepsilon_{\rho\rho}&=\frac{1+v}{E}\left[-\frac{\partial^3\Pi}{\partial\rho^2\partial z}-\frac{2}{\rho^2}\frac{\partial\Lambda}{\partial\theta}+\frac{2}{\rho}\frac{\partial^2\Lambda}{\partial\rho\partial\theta}\right]\\
\varepsilon_{\theta\theta}&=\frac{1+v}{E}\left[-\frac{1}{\rho^2}\frac{\partial^3\Pi}{\partial\theta^2\partial z}-\frac{2}{\rho}\frac{\partial^2\Lambda}{\partial\rho\partial\theta}-\frac{1}{\rho}\frac{\partial^2\Pi}{\partial\rho\partial z}+\frac{2}{\rho^2}\frac{\partial\Lambda}{\partial\theta}\right]\\
\varepsilon_{zz}&=\frac{1+v}{E}\left[2(1-v)\frac{\partial}{\partial z}\nabla^2\Pi-\frac{\partial^3\Pi}{\partial z^3}\right]\\
\varepsilon_{\rho\theta}&=\frac{1+v}{E}\frac{1}{2}\left[\frac{2}{\rho^2}\frac{\partial^2\Pi}{\partial\theta\partial z}-\frac{2}{\rho}\frac{\partial^3\Pi}{\partial\rho\partial\theta\partial z}-2\frac{\partial^2\Lambda}{\partial\rho^2}+\frac{2}{\rho^2}\frac{\partial^2\Lambda}{\partial\theta^2}+\frac{2}{\rho}\frac{\partial\Lambda}{\partial\rho}\right]\\
\varepsilon_{\rho z}&=\frac{1+v}{E}\frac{1}{2}\left[2(1-v)\frac{\partial}{\partial\rho}\nabla^2\Pi-2\frac{\partial^3\Pi}{\partial\rho\partial z^2}+\frac{2}{\rho}\frac{\partial^2\Lambda}{\partial\theta\partial z}\right]\\
\varepsilon_{\theta z}&=\frac{1+v}{E}\frac{1}{2}\left[2(1-v)\frac{1}{\rho}\frac{\partial}{\partial\theta}\nabla^2\Pi-\frac{2}{\rho}\frac{\partial^3\Pi}{\partial\theta\partial z^2}-2\frac{\partial^2\Lambda}{\partial\rho\partial z}\right]
\end{aligned}
\tag{8.13}
$$

第一应变不变量为

$$
\varepsilon_{\rho\rho}+\varepsilon_{\theta\theta}+\varepsilon_{zz}=(1-2v)\frac{\partial}{\partial z}\nabla^2\Pi
\tag{8.14}
$$

由物理方程求得应力为

$$
\begin{aligned}
\sigma_{\rho\rho}&=\frac{\partial}{\partial z}\left(v\nabla^2\Pi-\frac{\partial^2\Pi}{\partial\rho^2}\right)+\frac{2}{\rho}\left(\frac{\partial^2\Lambda}{\partial\rho\partial\theta}-\frac{1}{\rho}\frac{\partial\Lambda}{\partial\theta}\right)\\
\sigma_{\theta\theta}&=\frac{\partial}{\partial z}\left(v\nabla^2\Pi-\frac{1}{\rho}\frac{\partial\Pi}{\partial\rho}-\frac{1}{\rho^2}\frac{\partial^2\Pi}{\partial\theta^2}\right)+\frac{2}{\rho}\left(\frac{1}{\rho}\frac{\partial\Lambda}{\partial\theta}-\frac{\partial^2\Lambda}{\partial\rho\partial\theta}\right)\\
\sigma_{zz}&=(2-v)\frac{\partial}{\partial z}\left(\nabla^2\Pi-\frac{\partial^2\Pi}{\partial z^2}\right)\\
\sigma_{\rho\theta}&=\frac{\partial}{\partial z}\left(\frac{1}{\rho^2}\frac{\partial\Pi}{\partial\theta}-\frac{1}{\rho}\frac{\partial^2\Pi}{\partial\rho\partial\theta}\right)-2\frac{\partial^2\Lambda}{\partial\rho^2}-\frac{\partial^2\Lambda}{\partial z^2}\\
\sigma_{\rho z}&=\frac{\partial}{\partial\rho}\left((1-v)\nabla^2\Pi-\frac{\partial^2\Pi}{\partial z^2}\right)+\frac{1}{\rho}\frac{\partial^2\Lambda}{\partial\theta\partial z}\\
\sigma_{\theta z}&=\frac{\partial}{\partial\theta}\left((1-v)\frac{1}{\rho}\nabla^2\Pi-\frac{1}{\rho}\frac{\partial^2\Pi}{\partial z^2}\right)-\frac{\partial^2\Lambda}{\partial\rho\partial z}
\end{aligned}
\tag{8.15}
$$

此时，相容方程将转换为重调和方程和调和方程

$$
\begin{aligned}
\nabla^2\nabla^2\Pi&=0\\
\nabla^2\Lambda&=0
\end{aligned}
\tag{8.16}
$$

一般采用Hankel积分变换法解此偏微分方程。首先将应力函数分解为级数形式

$$
\begin{aligned}
\Pi(\rho,\theta,z)&=\sum_{k=0}^{\infty}\Pi_k(\rho,z)\cos k\theta\\
\Lambda(\rho,\theta,z)&=\sum_{k=0}^{\infty}\Lambda_k(\rho,z)\sin k\theta
\end{aligned}
\tag{8.17}
$$

根据Hankel积分变换理论，对Π_k和Λ_k使用k阶Hankel积分变换

$$\begin{aligned}\Pi_k(\rho,z)&=\int_0^\infty \bar{\Pi}_k(\xi,z)J_k(\xi\rho)\xi d\xi\\ \Lambda_k(\rho,z)&=\int_0^\infty \bar{\Lambda}_k(\xi,z)J_k(\xi\rho)\xi d\xi\end{aligned}\tag{8.18}$$

此时，应力函数可以分别表示为

$$\begin{aligned}\Pi(\rho,\theta,z)&=\sum_0^\infty \Pi_k(\rho,z)\cos k\theta=\sum_0^\infty\int_0^\infty \bar{\Pi}_k J_k(\xi\rho)\xi d\xi\cos k\theta\\ \Lambda(\rho,\theta,z)&=\sum_0^\infty \Lambda_k(\rho,z)\sin k\theta=\sum_0^\infty\int_0^\infty \bar{\Lambda}_k J_k(\xi\rho)\xi d\xi\sin k\theta\end{aligned}\tag{8.19}$$

对上式进行拉普拉斯运算得

$$\begin{aligned}\nabla^4\Pi&=\sum_0^\infty\left[\nabla_k^2\nabla_k^2\Pi_k(\rho,z)\right]\cos k\theta=\sum_0^\infty\int_0^\infty\left[\frac{\partial^2}{\partial z^2}-\xi^2\right]^2\bar{\Pi}_k\xi J_k(\xi\rho)\cos k\theta d\xi\\ \nabla^2\Lambda&=\sum_0^\infty\left[\nabla_k^2\Lambda_k(\rho,z)\right]\sin k\theta=\sum_0^\infty\int_0^\infty\left[\frac{\partial^2}{\partial z^2}-\xi^2\right]\xi J_k(\xi\rho)\sin k\theta d\xi\end{aligned}\tag{8.20}$$

所以

$$\begin{aligned}\nabla_k^2\nabla_k^2\Pi_k(\rho,z)&=\int_0^\infty\left[\frac{\partial^2}{\partial z^2}-\xi^2\right]^2\bar{\Pi}_k(\xi,z)\xi J_k(\xi\rho)d\xi\\ \nabla_k^2\Lambda_k(\rho,z)&=\int_0^\infty\left[\frac{\partial^2}{\partial z^2}-\xi^2\right]\bar{\Lambda}_k(\xi,z)\xi J_k(\xi\rho)d\xi\end{aligned}\tag{8.21}$$

对上式进行Hankel逆变换

$$\begin{aligned}\textstyle\int_0^\infty \nabla_k^2\nabla_k^2\Pi_k(\rho,z)J_k(\xi\rho)\rho d\rho&=\left[\tfrac{\partial^2}{\partial z^2}-\xi^2\right]^2\bar{\Pi}_k(\xi,z)\\ \textstyle\int_0^\infty \nabla_k^2\Lambda_k(\rho,z)J_k(\xi\rho)\rho d\rho&=\left[\tfrac{\partial^2}{\partial z^2}-\xi^2\right]\bar{\Lambda}_k(\xi,z)\end{aligned}\tag{8.22}$$

因为等式左边被积函数为零，所以

$$\begin{aligned}\left[\frac{\partial^2}{\partial z^2}-\xi^2\right]^2\bar{\Pi}_k(\xi,z)=0\\ \left[\frac{\partial^2}{\partial z^2}-\xi^2\right]\bar{\Lambda}_k(\xi,z)=0\end{aligned}\tag{8.23}$$

至此，通过Hankel积分变换将以(ρ,θ,z)为变量的偏微分方程转换为以z为变量的常微分方程。根据常微分方程理论，很容易得到上式通解

$$\begin{aligned}\bar{\Pi}_k&=(\bar{A}+\bar{B}z)e^{-\xi z}+(\bar{C}+\bar{D}z)e^{\xi z}\\ \bar{\Lambda}_k&=\bar{E}e^{-\xi z}+\bar{F}e^{\xi z}\end{aligned}\tag{8.24}$$

回代得

$$\begin{aligned}\Pi_k&=\int_0^\infty\bar{\Pi}_kJ_k(\xi\rho)\xi d\xi=\int_0^\infty\left[(\bar{A}+\bar{B}z)e^{-\xi z}+(\bar{C}+\bar{D}z)e^{\xi z}\right]J_k(\xi\rho)\xi d\xi\\ \Lambda_k&=\int_0^\infty\bar{\Lambda}_kJ_k(\xi\rho)\xi d\xi=\int_0^\infty\left[\bar{E}e^{-\xi z}+\bar{F}e^{\xi z}\right]J_k(\xi\rho)\xi d\xi\end{aligned}\tag{8.25}$$

最终，问题一般解（应力函数）为

$$\begin{aligned}\Pi(\rho,\theta,z)&=\sum_0^\infty\int_0^\infty\left[(\bar{A}+\bar{B}z)e^{-\xi z}+(\bar{C}+\bar{D}z)e^{\xi z}\right]J_k(\xi\rho)\xi d\xi\cos k\theta\\ \Lambda(\rho,\theta,z)&=\sum_0^\infty\int_0^\infty\left[\bar{E}e^{-\xi z}+\bar{F}e^{\xi z}\right]J_k(\xi\rho)\xi d\xi\sin k\theta\end{aligned}\tag{8.26}$$

回代得应力和位移的一般解

$$
\begin{aligned}
u_\rho &= -\frac{1+v}{2E}\sum_0^\infty\int_0^\infty \left\langle \frac{2\rho}{k+1}U_{k+1} - \frac{2\rho}{k-1}U_{k-1}\right\rangle d\xi\cos k\theta \\
u_\theta &= -\frac{1+v}{2E}\sum_0^\infty\int_0^\infty \left\langle \frac{2\rho}{k+1}U_{k+1} + \frac{2\rho}{k-1}U_{k-1}\right\rangle d\xi\sin k\theta \\
u_z &= \frac{1+v}{E}\sum_0^\infty\int_0^\infty \left\langle \left[-\xi^3\bar{A} + (-2+4v-\xi z)\xi^2\bar{B}\right]e^{-\xi z} + \left[-\xi^3\bar{C} + (2-4v-\xi z)\xi^2\bar{D}\right]e^{\xi z}\right\rangle J_k(\xi\rho)d\xi\cos k\theta \\
\sigma_{\rho\rho} &= \sum_0^\infty\int_0^\infty \left\langle\left\{\left[-\xi^3\bar{A} + (1+2v-\xi z)\xi^2\bar{B}\right]\xi e^{-\xi z} + \left[\xi^3\bar{C} + (1+2v+\xi z)\xi^2\bar{D}\right]\xi e^{\xi z}\right\} J_k(\xi\rho) + U_{k+1} + U_{k-1}\right\rangle d\xi\cos k\theta \\
\sigma_{\theta\theta} &= \sum_0^\infty\int_0^\infty \left[\left(2v\xi^2\bar{B}e^{-\xi z} + 2v\xi^2\bar{D}e^{\xi z}\right)\xi J_k(\xi\rho) - U_{k+1} - U_{k-1}\right] d\xi\cos k\theta \\
\sigma_{zz} &= \sum_0^\infty\int_0^\infty \left\langle \left[\xi^3\bar{A} + (1-2v+\xi z)\xi^2\bar{B}\right]\xi e^{-\xi z} + \left[-\xi^3\bar{C} + (1-2v-\xi z)\xi^2\bar{D}\right]\xi e^{\xi z}\right\rangle J_k(\xi\rho)d\xi\cos k\theta \\
\sigma_{\rho\theta} &= \sum_0^\infty\int_0^\infty \left\langle\left[\left(\xi^2\bar{E}e^{-\xi z} + \xi^2\bar{F}e^{\xi z}\right)\right]\xi J_k(\xi\rho) + U_{k+1} - U_{k-1}\right\rangle d\xi\sin k\theta \\
\sigma_{\rho z} &= \sum_0^\infty\int_0^\infty \frac{1}{2}\left\langle H_{k+1} - H_{k-1}\right\rangle d\xi\cos k\theta \\
\sigma_{\theta z} &= \sum_0^\infty\int_0^\infty \frac{1}{2}\left\langle H_{k+1} + H_{k-1}\right\rangle d\xi\sin k\theta \\
U_{k+1} &= \frac{k+1}{2\rho}J_{k+1}(\xi\rho)\left\{\left[\xi^3\bar{A} + (-1+\xi z)\xi^2\bar{B} - 2\xi^2\bar{E}\right]e^{-\xi z} + \left[-\xi^3\bar{C} + (-1-\xi z)\xi^2\bar{D} - 2\xi^2\bar{F}\right]e^{\xi z}\right\} \\
U_{k-1} &= \frac{k-1}{2\rho}J_{k-1}(\xi\rho)\left\{\left[\xi^3\bar{A} + (-1+\xi z)\xi^2\bar{B} + 2\xi^2\bar{E}\right]e^{-\xi z} + \left[-\xi^3\bar{C} + (-1-\xi z)\xi^2\bar{D} + 2\xi^2\bar{F}\right]e^{\xi z}\right\} \\
H_{k+1} &= J_{k+1}(\xi\rho)\left\{\left[\xi^3\bar{A} + (-2v+\xi z)\xi^2\bar{B} - \xi^2\bar{E}\right]\xi e^{-\xi z} + \left[\xi^3\bar{C} + (2v+\xi z)\xi^2\bar{D} + \xi^2\bar{F}\right]\xi e^{\xi z}\right\} \\
H_{k-1} &= J_{k-1}(\xi\rho)\left\{\left[\xi^3\bar{A} + (-2v+\xi z)\xi^2\bar{B} + \xi^2\bar{E}\right]\xi e^{-\xi z} + \left[\xi^3\bar{C} + (2v+\xi z)\xi^2\bar{D} - \xi^2\bar{F}\right]\xi e^{\xi z}\right\}
\end{aligned}
\tag{8.27}
$$

现做如下变换以求简化

$$
A = \xi^3\bar{A},\quad B = \xi^2\bar{B},\quad C = \xi^3\bar{C},\quad D = \xi^2\bar{D},\quad E = \xi^2\bar{E},\quad F = \xi^2\bar{F} \tag{8.28}
$$

$$
\begin{aligned}
u_\rho &= -\frac{1+v}{2E}\sum_0^\infty\int_0^\infty \left\langle \frac{2\rho}{k+1}U_{k+1} - \frac{2\rho}{k-1}U_{k-1}\right\rangle d\xi\cos k\theta \\
u_\theta &= -\frac{1+v}{2E}\sum_0^\infty\int_0^\infty \left\langle \frac{2\rho}{k+1}U_{k+1} + \frac{2\rho}{k-1}U_{k-1}\right\rangle d\xi\sin k\theta \\
u_z &= \frac{1+v}{E}\sum_0^\infty\int_0^\infty \left\langle [-A + (-2+4v-\xi z)B]e^{-\xi z} + [-C + (2-4v-\xi z)D]e^{\xi z}\right\rangle J_k(\xi\rho)d\xi\cos k\theta \\
\sigma_{\rho\rho} &= \sum_0^\infty\int_0^\infty \left\langle\left\{[-A + (1+2v-\xi z)B]\xi e^{-\xi z} + [C + (1+2v+\xi z)D]\xi e^{\xi z}\right\} J_k(\xi\rho) + U_{k+1} + U_{k-1}\right\rangle d\xi\cos k\theta \\
\sigma_{\theta\theta} &= \sum_0^\infty\int_0^\infty \left[\left(2vBe^{-\xi z} + 2vDe^{\xi z}\right)\xi J_k(\xi\rho) - U_{k+1} - U_{k-1}\right] d\xi\cos k\theta \\
\sigma_{zz} &= \sum_0^\infty\int_0^\infty \left\langle [A + (1-2v+\xi z)B]\xi e^{-\xi z} + [-C + (1-2v-\xi z)D]\xi e^{\xi z}\right\rangle J_k(\xi\rho)d\xi\cos k\theta \\
\sigma_{\rho\theta} &= \sum_0^\infty\int_0^\infty \left\langle\left[\left(Ee^{-\xi z} + Fe^{\xi z}\right)\right]\xi J_k(\xi\rho) + U_{k+1} - U_{k-1}\right\rangle d\xi\sin k\theta \\
\sigma_{\rho z} &= \sum_0^\infty\int_0^\infty \frac{1}{2}\left\langle H_{k+1} - H_{k-1}\right\rangle d\xi\cos k\theta \\
\sigma_{\theta z} &= \sum_0^\infty\int_0^\infty \frac{1}{2}\left\langle H_{k+1} + H_{k-1}\right\rangle d\xi\sin k\theta \\
U_{k+1} &= \frac{k+1}{2\rho}J_{k+1}(\xi\rho)\left\{[A + (-1+\xi z)B - 2E]e^{-\xi z} + [-C + (-1-\xi z)D - 2F]e^{\xi z}\right\} \\
U_{k-1} &= \frac{k-1}{2\rho}J_{k-1}(\xi\rho)\left\{[A + (-1+\xi z)B + 2E]e^{-\xi z} + [-C + (-1-\xi z)D + 2F]e^{\xi z}\right\} \\
H_{k+1} &= J_{k+1}(\xi\rho)\left\{[A + (-2v+\xi z)B - E]\xi e^{-\xi z} + [C + (2v+\xi z)D + F]\xi e^{\xi z}\right\} \\
H_{k-1} &= J_{k-1}(\xi\rho)\left\{[A + (-2v+\xi z)B + E]\xi e^{-\xi z} + [C + (2v+\xi z)D - F]\xi e^{\xi z}\right\}
\end{aligned}
\tag{8.29}
$$

8.3 边界条件

为了求得待定系数，先考虑无穷远处边界条件

$$
\begin{aligned}
&\lim_{\rho\to+\infty}(\sigma_{\rho\rho},\sigma_{\theta\theta},\sigma_{zz},\sigma_{\rho\theta},\sigma_{\rho z},\sigma_{\theta z},u_\rho,u_\theta,u_z)=0\\
&\lim_{z\to+\infty}(\sigma_{\rho\rho},\sigma_{\theta\theta},\sigma_{zz},\sigma_{\rho\theta},\sigma_{\rho z},\sigma_{\theta z},u_\rho,u_\theta,u_z)=0
\end{aligned}
\tag{8.30}
$$

观察发现，只有当正次幂指数函数项为零时，才能满足无穷远边界条件，所以

$$
C=D=F=0 \tag{8.31}
$$

其次，考虑表面边界条件，并将其展开为级数

$$
\begin{aligned}
\sigma_{zz}|_{z=0}&=-p(\rho,\theta)=-\sum_0^\infty p_k(\rho)\cos k\theta\\
\sigma_{z\rho}|_{z=0}&=-g(\rho,\theta)=-\sum_0^\infty g_k(\rho)\cos k\theta\\
\sigma_{z\theta}|_{z=0}&=-t(\rho,\theta)=-\sum_0^\infty t_k(\rho)\sin k\theta
\end{aligned}
\tag{8.32}
$$

将应力代入得

$$
\begin{aligned}
A+(1-2v)B&=-\bar{p}_k^k(\xi)\\
A-2vB-E&=-\bar{g}_k^{k+1}(\xi)-\bar{t}_k^{k+1}(\xi)\\
A-2vB+E&=\bar{g}_k^{k-1}(\xi)-\bar{t}_k^{k-1}(\xi)
\end{aligned}
\tag{8.33}
$$

解方程组得

$$
\begin{aligned}
A&=-2v\bar{p}_k^k-(1-2v)\left[\frac{1}{2}\left(\bar{t}_k^{k+1}+\bar{t}_k^{k-1}+\bar{g}_k^{k+1}-\bar{g}_k^{k-1}\right)\right]\\
B&=-\bar{p}_k^k+\frac{1}{2}\left(\bar{t}_k^{k+1}+\bar{t}_k^{k-1}+\bar{g}_k^{k+1}-\bar{g}_k^{k-1}\right)\\
E&=\frac{1}{2}\left(\bar{t}_k^{k+1}-\bar{t}_k^{k-1}+\bar{g}_k^{k+1}+\bar{g}_k^{k-1}\right)
\end{aligned}
\tag{8.34}
$$

至此，已完成半无界非轴对称空间问题的解答。

8.3.1 水平均布圆载荷

表面仅作用圆面积均布水平载荷时，可列出三个边界条件

$$
\begin{aligned}
&\sigma_{zz}|_{z=0}=0\\
&\sigma_{z\rho}|_{z=0}=-g(\rho,\theta)=-g_1(\rho)\cos\theta=q(\rho)\cos\theta=\begin{cases}q\cos\theta & \rho<a\\ 0 & \rho>a\end{cases}\\
&\sigma_{z\theta}|_{z=0}=-t(\rho,\theta)=-t_1(\rho)\sin\theta=-q(\rho)\sin\theta=\begin{cases}-q\sin\theta & \rho<a\\ 0 & \rho>a\end{cases}
\end{aligned}
\tag{8.35}
$$

可知只有$k=1$的项。

$$
\begin{aligned}
&A+(1-2\nu)B=-\bar{p}_1(\xi)=0\\
&A-2\nu B-E=-\bar{g}_1^2(\xi)-\bar{t}_1^2(\xi)=-\int_0^\infty\rho g_1(\rho)J_2(\xi\rho)d\rho-\int_0^\infty\rho t_1(\rho)J_2(\xi\rho)d\rho=\int_0^\infty\rho q(\rho)J_2(\xi\rho)d\rho-\int_0^\infty\rho q(\rho)J_2(\xi\rho)d\rho=0\\
&A-2\nu B+E=\bar{g}_1^0(\xi)-\bar{t}_1^0(\xi)=\int_0^\infty\rho g_1(\rho)J_0(\xi\rho)d\rho-\int_0^\infty\rho t_1(\rho)J_0(\xi\rho)d\rho=-\int_0^\infty\rho q(\rho)J_0(\xi\rho)d\rho-\int_0^\infty\rho q(\rho)J_0(\xi\rho)d\rho=-2\bar{q}(\xi)
\end{aligned}
\tag{8.36}
$$

用matlab解方程组得

$$
A=-(1-2v)\bar{q}(\xi),\quad B=\bar{q}(\xi),\quad E=-\bar{q}(\xi),\quad C=D=F=0 \tag{8.37}
$$

其中

$$
\bar{q}(\xi)=\int_0^\infty\rho q(\rho)J_0(\xi\rho)d\rho=q\int_0^a\rho J_0(\xi\rho)d\rho=q\left[\frac{\rho J_1(\xi\rho)}{\xi}\right]_0^a=q\frac{aJ_1(\xi a)}{\xi} \tag{8.38}
$$

其中高亮部分是因为

$$\frac{d}{d\rho}\left[\rho\frac{J_1(\xi\rho)}{\xi}\right]=\rho J_0(\xi\rho) \tag{8.39}$$

参考《路面力学计算》(2-60)。

8.3.2 垂直均布圆载荷

当表面仅作用有圆面积均布垂直载荷时，表面边界条件为

$$\begin{aligned}
&\sigma_{zz}|_{z=0}=-p(\rho,\theta)=-p_0(\rho)=-p(\rho)=\begin{cases}-p & \rho<a\\ 0 & \rho>a\end{cases}\\
&\sigma_{z\rho}|_{z=0}=0\\
&\sigma_{z\theta}|_{z=0}=0
\end{aligned} \tag{8.40}$$

可知只有$k=0$的项。

边界条件为

$$\begin{aligned}
&A+(1-2v)B=-\overline{p}_0^0(\xi)=-\int_0^\infty \rho p_0(\rho)J_0(\xi\rho)d\rho=-\int_0^a \rho p(\rho)J_0(\xi\rho)d\rho=-p\int_0^a \rho J_0(\xi\rho)d\rho=-p\frac{aJ_1(\xi a)}{\xi}\\
&A-2vB-E=0\\
&A-2vB+E=0
\end{aligned} \tag{8.41}$$

其中，高亮部分同前。

8.3.3 一些处理

$$\overline{u}_\rho=\frac{u_\rho}{p},\overline{u}_\theta=\frac{u_\theta}{p},\overline{u}_z=\frac{u_z}{p},\overline{\sigma}_{\rho\rho}=\frac{\sigma_{\rho\rho}}{p},\overline{\sigma}_{\theta\theta}=\frac{\sigma_{\theta\theta}}{p},\overline{\sigma}_{zz}=\frac{\sigma_{zz}}{p},\overline{\sigma}_{\rho\theta}=\frac{\sigma_{\rho\theta}}{p},\overline{\sigma}_{\rho z}=\frac{\sigma_{\rho z}}{p},\overline{\sigma}_{\theta z}=\frac{\sigma_{\theta z}}{p} \tag{8.42}$$

$$\overline{A}=\frac{A}{p},\overline{B}=\frac{B}{p},\overline{C}=\frac{C}{p},\overline{D}=\frac{D}{p},\overline{E}=\frac{E}{p},\overline{F}=\frac{F}{p} \tag{8.43}$$

$$\begin{aligned}
&\overline{u}_\rho=-\frac{1+v}{2E}\sum_0^\infty\int_0^\infty\left\langle\frac{2\rho}{k+1}U_{k+1}-\frac{2\rho}{k-1}U_{k-1}\right\rangle d\xi\cos k\theta\\
&\overline{u}_\theta=-\frac{1+v}{2E}\sum_0^\infty\int_0^\infty\left\langle\frac{2\rho}{k+1}U_{k+1}+\frac{2\rho}{k-1}U_{k-1}\right\rangle d\xi\sin k\theta\\
&\overline{u}_z=\frac{1+v}{E}\sum_0^\infty\int_0^\infty\left\langle[-\overline{A}+(-2+4v-\xi z)\overline{B}]e^{-\xi z}+[-\overline{C}+(2-4v-\xi z)\overline{D}]e^{\xi z}\right\rangle J_k(\xi\rho)d\xi\cos k\theta\\
&\overline{\sigma}_{\rho\rho}=\sum_0^\infty\int_0^\infty\left\langle\left\{[-\overline{A}+(1+2v-\xi z)\overline{B}]\xi e^{-\xi z}+[\overline{C}+(1+2v+\xi z)\overline{D}]\xi e^{\xi z}\right\}J_k(\xi\rho)+U_{k+1}+U_{k-1}\right\rangle d\xi\cos k\theta\\
&\overline{\sigma}_{\theta\theta}=\sum_0^\infty\int_0^\infty\left[\left(2v\overline{B}e^{-\xi z}+2v\overline{D}e^{\xi z}\right)\xi J_k(\xi\rho)-U_{k+1}-U_{k-1}\right]d\xi\cos k\theta\\
&\overline{\sigma}_{zz}=\sum_0^\infty\int_0^\infty\left\langle[\overline{A}+(1-2v+\xi z)\overline{B}]\xi e^{-\xi z}+[-\overline{C}+(1-2v-\xi z)\overline{D}]\xi e^{\xi z}\right\rangle J_k(\xi\rho)d\xi\cos k\theta\\
&\overline{\sigma}_{\rho\theta}=\sum_0^\infty\int_0^\infty\left\langle\left[\left(\overline{E}e^{-\xi z}+\overline{F}e^{\xi z}\right)\right]\xi J_k(\xi\rho)+U_{k+1}-U_{k-1}\right\rangle d\xi\sin k\theta\\
&\overline{\sigma}_{\rho z}=\sum_0^\infty\int_0^\infty\frac{1}{2}\left\langle H_{k+1}-H_{k-1}\right\rangle d\xi\cos k\theta\\
&\overline{\sigma}_{\theta z}=\sum_0^\infty\int_0^\infty\frac{1}{2}\left\langle H_{k+1}+H_{k-1}\right\rangle d\xi\sin k\theta\\
&U_{k+1}=\frac{k+1}{2\rho}J_{k+1}(\xi\rho)\left\{[\overline{A}+(-1+\xi z)\overline{B}-2\overline{E}]e^{-\xi z}+[-\overline{C}+(-1-\xi z)\overline{D}-2\overline{F}]e^{\xi z}\right\}\\
&U_{k-1}=\frac{k-1}{2\rho}J_{k-1}(\xi\rho)\left\{[\overline{A}+(-1+\xi z)\overline{B}+2\overline{E}]e^{-\xi z}+[-\overline{C}+(-1-\xi z)\overline{D}+2\overline{F}]e^{\xi z}\right\}\\
&H_{k+1}=J_{k+1}(\xi\rho)\left\{[\overline{A}+(-2v+\xi z)\overline{B}-\overline{E}]\xi e^{-\xi z}+[\overline{C}+(2v+\xi z)\overline{D}+\overline{F}]\xi e^{\xi z}\right\}\\
&H_{k-1}=J_{k-1}(\xi\rho)\left\{[\overline{A}+(-2v+\xi z)\overline{B}+\overline{E}]\xi e^{-\xi z}+[\overline{C}+(2v+\xi z)\overline{D}-\overline{F}]\xi e^{\xi z}\right\}
\end{aligned} \tag{8.44}$$

$$
\begin{aligned}
&A+(1-2v)B=-p\frac{aJ_1(\xi a)}{\xi} & & \overline{A}+(1-2v)\overline{B}=-\frac{aJ_1(\xi a)}{\xi} \\
&A-2vB-E=0 & \Longrightarrow \quad & \overline{A}-2v\overline{B}-\overline{E}=0 \\
&A-2vB+E=0 & & \overline{A}-2v\overline{B}+\overline{E}=0
\end{aligned}
\tag{8.45}
$$

$$
\begin{gathered}
t=\xi a \longrightarrow \xi=\frac{t}{a} \longrightarrow d\xi=\frac{dt}{a} \\
\xi z=\frac{t}{a}z=t\overline{z}, \quad \xi\rho=\frac{t}{a}\rho=t\overline{\rho}
\end{gathered}
\tag{8.46}
$$

$$
\begin{aligned}
&\overline{u}_\rho=-\frac{1+v}{2E}\sum_0^\infty\int_0^\infty\left\langle\frac{2a\overline{\rho}}{k+1}U_{k+1}-\frac{2a\overline{\rho}}{k-1}U_{k-1}\right\rangle d\frac{t}{a}\cos k\theta \\
&\overline{u}_\theta=-\frac{1+v}{2E}\sum_0^\infty\int_0^\infty\left\langle\frac{2a\overline{\rho}}{k+1}U_{k+1}+\frac{2a\overline{\rho}}{k-1}U_{k-1}\right\rangle d\frac{t}{a}\sin k\theta \\
&\overline{u}_z=\frac{1+v}{E}\sum_0^\infty\int_0^\infty\langle[-\overline{A}+(-2+4v-t\overline{z})\overline{B}]e^{-t\overline{z}}+[-\overline{C}+(2-4v-t\overline{z})\overline{D}]e^{t\overline{z}}\rangle J_k(t\overline{\rho})d\frac{t}{a}\cos k\theta \\
&\overline{\sigma}_{\rho\rho}=\sum_0^\infty\int_0^\infty\left\langle\{[-\overline{A}+(1+2v-t\overline{z})\overline{B}]e^{-t\overline{z}}+[\overline{C}+(1+2v+t\overline{z})\overline{D}]e^{t\overline{z}}\}\frac{t}{a}J_k(t\overline{\rho})+U_{k+1}+U_{k-1}\right\rangle d\frac{t}{a}\cos k\theta \\
&\overline{\sigma}_{\theta\theta}=\sum_0^\infty\int_0^\infty\left[\left(2v\overline{B}e^{-t\overline{z}}+2v\overline{D}e^{t\overline{z}}\right)\frac{t}{a}J_k(t\overline{\rho})-U_{k+1}-U_{k-1}\right]d\frac{t}{a}\cos k\theta \\
&\overline{\sigma}_{zz}=\sum_0^\infty\int_0^\infty\langle[\overline{A}+(1-2v+t\overline{z})\overline{B}]e^{-t\overline{z}}+[-\overline{C}+(1-2v-t\overline{z})\overline{D}]e^{t\overline{z}}\rangle\frac{t}{a}J_k(t\overline{\rho})d\frac{t}{a}\cos k\theta \\
&\overline{\sigma}_{\rho\theta}=\sum_0^\infty\int_0^\infty\left\langle\left[\left(\overline{E}e^{-t\overline{z}}+\overline{F}e^{t\overline{z}}\right)\right]\frac{t}{a}J_k(t\overline{\rho})+U_{k+1}-U_{k-1}\right\rangle d\frac{t}{a}\sin k\theta \\
&\overline{\sigma}_{\rho z}=\sum_0^\infty\int_0^\infty\frac{1}{2}\langle H_{k+1}-H_{k-1}\rangle d\frac{t}{a}\cos k\theta \\
&\overline{\sigma}_{\theta z}=\sum_0^\infty\int_0^\infty\frac{1}{2}\langle H_{k+1}+H_{k-1}\rangle d\frac{t}{a}\sin k\theta \\
&U_{k+1}=\frac{k+1}{2a\overline{\rho}}J_{k+1}(t\overline{\rho})\{[\overline{A}+(-1+t\overline{z})\overline{B}-2\overline{E}]e^{-t\overline{z}}+[-\overline{C}+(-1-t\overline{z})\overline{D}-2\overline{F}]e^{t\overline{z}}\} \\
&U_{k-1}=\frac{k-1}{2a\overline{\rho}}J_{k-1}(t\overline{\rho})\{[\overline{A}+(-1+t\overline{z})\overline{B}+2\overline{E}]e^{-t\overline{z}}+[-\overline{C}+(-1-t\overline{z})\overline{D}+2\overline{F}]e^{t\overline{z}}\} \\
&H_{k+1}=J_{k+1}(t\overline{\rho})\left\{[\overline{A}+(-2v+t\overline{z})\overline{B}-\overline{E}]\frac{t}{a}e^{-t\overline{z}}+[\overline{C}+(2v+t\overline{z})\overline{D}+\overline{F}]\frac{t}{a}e^{t\overline{z}}\right\} \\
&H_{k-1}=J_{k-1}(t\overline{\rho})\left\{[\overline{A}+(-2v+t\overline{z})\overline{B}+\overline{E}]\frac{t}{a}e^{-t\overline{z}}+[\overline{C}+(2v+t\overline{z})\overline{D}-\overline{F}]\frac{t}{a}e^{t\overline{z}}\right\}
\end{aligned}
\tag{8.47}
$$

Chapter 9

高阶连续介质力学理论

转动矢量（rotation vector）

$$\boldsymbol{\theta} = \frac{1}{2}\nabla \times \boldsymbol{u}$$

转动梯度张量（曲率张量curvature tensor）

$$\boldsymbol{\chi} = \frac{1}{2}(\nabla\boldsymbol{\theta} + \nabla\boldsymbol{\theta}^T)$$

位移梯度

$$\nabla\boldsymbol{u}$$

柯西应变（位移梯度的对称部分）

$$\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T)$$

9.1 Couple stress theory for solids-Hadjesfandiari

9.1.1 equilibrium

Force equilibrium and moment equilibrium

$$\begin{aligned}
&\int_{\partial\Omega} \boldsymbol{t} ds + \int_{\Omega} \boldsymbol{f} dv = \boldsymbol{0} \\
&\int_{\partial\Omega} \boldsymbol{r}\times\boldsymbol{t} + \boldsymbol{q} ds + \int_{\Omega} \boldsymbol{r}\times\boldsymbol{f} + \boldsymbol{p} dv = \boldsymbol{0}
\end{aligned} \tag{9.1}$$

where $\boldsymbol{t}$ is surface force vector, $\boldsymbol{f}$ is body force vector, $\boldsymbol{q}$ is surface moment vector, $\boldsymbol{p}$ is body moment vector, $\boldsymbol{r}$ is position vector.

Cauchy's theorem and similar form of couple stress

$$\boldsymbol{t} = \boldsymbol{n}\cdot\boldsymbol{\sigma}, \qquad \boldsymbol{q} = \boldsymbol{n}\cdot\boldsymbol{\mu} \tag{9.2}$$

where $\boldsymbol{\sigma}$ is Cosserat stress tensor (non-symmetric).

Detailed derivation:

$$\begin{aligned}
\int_{\partial\Omega} \boldsymbol{r}\times\boldsymbol{t} ds &= \int_{\partial\Omega} \boldsymbol{r}\times(\boldsymbol{n}\cdot\boldsymbol{\sigma}) ds = -\int_{\partial\Omega} (\boldsymbol{n}\cdot\boldsymbol{\sigma})\times\boldsymbol{r} ds \\
&= -\int_{\partial\Omega} \boldsymbol{n}\cdot\boldsymbol{\sigma}\times\boldsymbol{r} ds = -\int_{\partial\Omega} \boldsymbol{n}\cdot(\boldsymbol{\sigma}\times\boldsymbol{r}) ds \\
&= -\int_{\Omega} \nabla\cdot(\boldsymbol{\sigma}\times\boldsymbol{r}) dv = \int_{\Omega} \boldsymbol{r}\times(\nabla\cdot\boldsymbol{\sigma}) + \boldsymbol{e}\cdot\boldsymbol{\sigma} dv
\end{aligned} \tag{9.3}$$

$$\begin{aligned}
\nabla\cdot(\boldsymbol{\sigma}\times\boldsymbol{r}) &= \boldsymbol{e}_l\cdot\partial_l(\sigma_{ij}\boldsymbol{e}_i\otimes\boldsymbol{e}_j\times r_k\boldsymbol{e}_k) = \boldsymbol{e}_l\cdot\partial_l(\sigma_{ij}r_k)\boldsymbol{e}_i\otimes\boldsymbol{e}_m e_{jkm} \\
&= e_{jkm}\delta_{li}\partial_l(\sigma_{ij}r_k)\boldsymbol{e}_m = e_{jkm}\partial_i(\sigma_{ij}r_k)\boldsymbol{e}_m = e_{jkm}(\sigma_{ij,i}r_k + \sigma_{ij}r_{k,i})\boldsymbol{e}_m \\
&= e_{jkm}(\sigma_{ij,i}r_k + \sigma_{ij}\delta_{ki})\boldsymbol{e}_m = (e_{jkm}\sigma_{ij,i}r_k + e_{jkm}\sigma_{ij}\delta_{ki})\boldsymbol{e}_m \\
&= (e_{jkm}\sigma_{ij,i}r_k + e_{jim}\sigma_{ij})\boldsymbol{e}_m = -r_k\sigma_{ij,i}e_{kjm}\boldsymbol{e}_m - e_{mij}\sigma_{ij}\boldsymbol{e}_m \\
&= -\boldsymbol{r}\times(\nabla\cdot\boldsymbol{\sigma}) - \boldsymbol{e}\cdot\boldsymbol{\sigma}
\end{aligned} \tag{9.4}$$

By Cauchy's theorem and divergence theorem

ium-equ}

$$\begin{cases} \nabla\cdot\boldsymbol{\sigma} + \boldsymbol{f} = \boldsymbol{0} \\ \boldsymbol{r}\times(\nabla\cdot\boldsymbol{\sigma} + \boldsymbol{f}) + \boldsymbol{e}\cdot\boldsymbol{\sigma} + \nabla\cdot\boldsymbol{\mu} + \boldsymbol{p} = \boldsymbol{0} \end{cases} \rightarrow \begin{cases} \nabla\cdot\boldsymbol{\sigma} + \boldsymbol{f} = \boldsymbol{0} \\ \boldsymbol{e}\cdot\boldsymbol{\sigma} + \nabla\cdot\boldsymbol{\mu} + \boldsymbol{p} = \boldsymbol{0} \end{cases} \tag{9.5}$$

where $\boldsymbol{e}\cdot\boldsymbol{\sigma} = \boldsymbol{\sigma}\overset{\times}{\cdot}\boldsymbol{I}$.

9.1.2 kinematics

Relative displacement

$$du_i = u_{i,j}dx_j, \qquad d\boldsymbol{u} = \boldsymbol{u}\nabla\cdot d\boldsymbol{x} \tag{9.6}$$

Gradient of displacement vector

$$u_{i,j} = \frac{1}{2}(u_{i,j} + u_{j,i}) + \frac{1}{2}(u_{i,j} - u_{j,i}) = u_{(i,j)} + u_{[i,j]} = \varepsilon_{ij} + \omega_{ij} \tag{9.7}$$

where ε_{ij} and ω_{ij} are small deformation strain and rotation tensor.

Decompose the relative displacement

$$du_i = (\varepsilon_{ij} + \omega_{ij})dx_j = \varepsilon_{ij}dx_j + \omega_{ij}dx_j = du_i^{(1)} + du_i^{(2)} \tag{9.8}$$

$$du_i^{(2)}dx_i = \omega_{ij}dx_jdx_i = 0 \tag{9.9}$$

Rotation vecor is defined by

$$\theta_i = \frac{1}{2}e_{ijk}\omega_{kj} = \frac{1}{2}e_{ijk}u_{k,j}, \qquad \boldsymbol{\theta} = \frac{1}{2}\nabla\times\boldsymbol{u} = \frac{1}{2}\boldsymbol{e}\cdot\nabla\boldsymbol{u} \tag{9.10}$$

$$\omega_{ji} = e_{ijk}\theta_k = -e_{jik}\theta_k, \qquad \boldsymbol{\omega} = -\boldsymbol{e}\cdot\boldsymbol{\theta} \tag{9.11}$$

Relative rotation

$$d\theta_i = \theta_{i,j} dx_j, \qquad d\boldsymbol{\theta} = \boldsymbol{\theta}\nabla \cdot d\boldsymbol{x} \tag{9.12}$$

Gradient of rotation vector

$$\theta_{i,j} = \frac{1}{2}(\theta_{i,j} + \theta_{j,i}) + \frac{1}{2}(\theta_{i,j} - \theta_{j,i}) = \theta_{(i,j)} + \theta_{[i,j]} = \chi_{ij} + \kappa_{ij} \tag{9.13}$$

where χ_{ij} and κ_{ij} are the small deformation twist and curvature tensor.

Decompose the relative rotation

$$d\theta_i = (\chi_{ij} + \kappa_{ij})dx_j = \chi_{ij}dx_j + \kappa_{ij}dx_j = d\theta_i^{(1)} + d\theta_i^{(2)} \tag{9.14}$$

$$d\theta_i^{(2)} dx_i = \kappa_{ij} dx_j dx_i = 0 \tag{9.15}$$

The curvature vector dual to the curvature tensor is defined by

$$\phi_i = \frac{1}{2} e_{ijk}\kappa_{kj} = \frac{1}{2} e_{ijk}\theta_{k,j}, \qquad \boldsymbol{\phi} = \frac{1}{2}\nabla \times \boldsymbol{\theta} = \frac{1}{4}\nabla \times (\nabla \times \boldsymbol{u}) \tag{9.16}$$

$$\kappa_{ji} = e_{ijk}\phi_k \tag{9.17}$$

$$\phi_i = \frac{1}{2} e_{ijk}\theta_{k,j} = (\frac{1}{2} e_{ijk}\theta_k)_{,j} = \omega_{ji,j} \tag{9.18}$$

Consider the material continuum occupying a volume Ω bounded by a surface $\partial\Omega$. The standard form of the equilibrium equations for this medium was given in (9.5). Let us multiply (9.5)a by a virtual displacement δu_i and integrate over the volume and also multiply (9.5)b by the corresponding virtual rotation $\delta\theta_i$ and integrate this over the volume as well. Therefore, we have

$$\begin{cases} \int_\Omega (\nabla \cdot \boldsymbol{\sigma} + \boldsymbol{f}) \cdot \delta \boldsymbol{u} dv = 0 \\ \int_\Omega (\boldsymbol{e} \cdot \boldsymbol{\sigma} + \nabla \cdot \boldsymbol{\mu} + \boldsymbol{p}) \cdot \delta\boldsymbol{\theta} dv = 0 \end{cases} \rightarrow \begin{cases} \int_\Omega (\sigma_{ji,j} + f_i)\delta u_i dv = 0 \\ \int_\Omega (e_{ijk}\sigma_{jk} + \mu_{ji,j} + p_i)\delta\theta_i dv = 0 \end{cases} \tag{9.19}$$

$$(\nabla \cdot \boldsymbol{\sigma}) \cdot \delta \boldsymbol{u} = \nabla \cdot (\boldsymbol{\sigma} \cdot \boldsymbol{u}) - \boldsymbol{\sigma} \cdot \nabla \boldsymbol{u} \rightarrow \sigma_{ji,j}\delta u_i = (\sigma_{ji}\delta u_i)_{,j} - \sigma_{ji}\delta u_{i,j} \tag{9.20}$$

By the chain rule and the divergence theorem,

$$\begin{aligned} \int_\Omega (\nabla \cdot \boldsymbol{\sigma} + \boldsymbol{f}) \cdot \delta \boldsymbol{u} dv &= \int_\Omega (\nabla \cdot \boldsymbol{\sigma}) \cdot \delta\delta \boldsymbol{u} + \boldsymbol{f} \cdot \delta \boldsymbol{u} dv = \int_\Omega \nabla \cdot (\boldsymbol{\sigma} \cdot \boldsymbol{u}) - \boldsymbol{\sigma} \cdot \nabla \delta \boldsymbol{u} + \boldsymbol{f} \cdot \delta \boldsymbol{u} dv \\ &= \int_\Omega \nabla \cdot (\boldsymbol{\sigma} \cdot \delta \boldsymbol{u}) dv + \int_\Omega -\boldsymbol{\sigma} \cdot \nabla \delta \boldsymbol{u} dv + \int_\Omega \boldsymbol{f} \cdot \delta \boldsymbol{u} dv \\ &= \int_{\partial\Omega} \boldsymbol{n} \cdot (\boldsymbol{\sigma} \cdot \delta \boldsymbol{u}) dv + \int_\Omega -\boldsymbol{\sigma} \cdot \nabla \delta \boldsymbol{u} dv + \int_\Omega \boldsymbol{f} \cdot \delta \boldsymbol{u} dv \\ &= \int_{\partial\Omega} (\boldsymbol{n} \cdot \boldsymbol{\sigma}) \cdot \delta \boldsymbol{u} dv + \int_\Omega -\boldsymbol{\sigma} \cdot \nabla \delta \boldsymbol{u} dv + \int_\Omega \boldsymbol{f} \cdot \delta \boldsymbol{u} dv = 0 \\ &= \int_{\partial\Omega} \boldsymbol{t} \cdot \delta \boldsymbol{u} dv + \int_\Omega -\boldsymbol{\sigma} \cdot \nabla \delta \boldsymbol{u} dv + \int_\Omega \boldsymbol{f} \cdot \delta \boldsymbol{u} dv = 0 \end{aligned} \tag{9.21}$$

Then

$$\int_\Omega \boldsymbol{\sigma} \cdot \nabla \delta \boldsymbol{u} dv = \int_{\partial\Omega} \boldsymbol{t} \cdot \delta \boldsymbol{u} dv + \int_\Omega \boldsymbol{f} \cdot \delta \boldsymbol{u} dv \tag{9.22}$$ {hh-44}

The chain rule gives

$$(\nabla \cdot \boldsymbol{\mu}) \cdot \delta\boldsymbol{\theta} + (\boldsymbol{e} \cdot \boldsymbol{\sigma}) \cdot \delta\boldsymbol{\theta} = \nabla \cdot (\boldsymbol{\mu} \cdot \delta\boldsymbol{\theta}) - \boldsymbol{\mu} \cdot \nabla\delta\boldsymbol{\theta} - \boldsymbol{\sigma} \cdot \delta\boldsymbol{\omega} \tag{9.23}$$

where

$$\begin{aligned} &\boldsymbol{\omega} = -\boldsymbol{e} \cdot \boldsymbol{\theta} \\ &(\boldsymbol{e} \cdot \boldsymbol{\sigma}) \cdot \delta\boldsymbol{\theta} = \delta\boldsymbol{\theta} \cdot \boldsymbol{e} \cdot \boldsymbol{\sigma} = \boldsymbol{\sigma} \cdot \boldsymbol{e} \cdot \delta\boldsymbol{\theta} = \sigma \cdot (\boldsymbol{e} \cdot \delta\boldsymbol{\theta}) = -\boldsymbol{\sigma} \cdot \delta\boldsymbol{\omega} \\ &(\nabla \cdot \boldsymbol{\mu}) \cdot \delta\boldsymbol{\theta} = \nabla \cdot (\boldsymbol{\mu} \cdot \delta\boldsymbol{\theta}) - \boldsymbol{\mu} \cdot \nabla\delta\boldsymbol{\theta} \end{aligned} \tag{9.24}$$

By the divergence theorem

$$\begin{aligned}
&\int_\Omega (\boldsymbol{e}\cdot\boldsymbol{\sigma}+\nabla\cdot\boldsymbol{\mu}+\boldsymbol{p})\cdot\delta\boldsymbol{\theta}dv \\
&=\int_\Omega (\boldsymbol{e}\cdot\boldsymbol{\sigma})\cdot\delta\boldsymbol{\theta}+(\nabla\cdot\boldsymbol{\mu})\cdot\delta\boldsymbol{\theta}+\boldsymbol{p}\cdot\delta\boldsymbol{\theta}dv \\
&=\int_\Omega \nabla\cdot(\boldsymbol{\mu}\cdot\delta\boldsymbol{\theta})-\boldsymbol{\mu}\cdot\nabla\delta\boldsymbol{\theta}-\boldsymbol{\sigma}\cdot\delta\boldsymbol{\omega}+\boldsymbol{p}\cdot\delta\boldsymbol{\theta}dv \\
&=\int_{\partial\Omega}(\boldsymbol{n}\cdot\boldsymbol{\mu})\cdot\delta\boldsymbol{\theta}ds+\int_\Omega -\boldsymbol{\mu}\cdot\nabla\delta\boldsymbol{\theta}-\boldsymbol{\sigma}\cdot\delta\boldsymbol{\omega}+\boldsymbol{p}\cdot\delta\boldsymbol{\theta}dv \\
&=\int_{\partial\Omega}\boldsymbol{q}\cdot\delta\boldsymbol{\theta}ds+\int_\Omega -\boldsymbol{\mu}\cdot\nabla\delta\boldsymbol{\theta}-\boldsymbol{\sigma}\cdot\delta\boldsymbol{\omega}+\boldsymbol{p}\cdot\delta\boldsymbol{\theta}dv=0
\end{aligned} \tag{9.25}$$

Then

{hh-46}
$$\int_\Omega \boldsymbol{\mu}\cdot\nabla\delta\boldsymbol{\theta}+\boldsymbol{\sigma}\cdot\delta\boldsymbol{\omega}dv=\int_{\partial\Omega}\boldsymbol{q}\cdot\delta\boldsymbol{\theta}ds+\int_\Omega\boldsymbol{p}\cdot\delta\boldsymbol{\theta}dv \tag{9.26}$$

By adding (9.22) and (9.26), we obtain

$$\int_\Omega \boldsymbol{\sigma}\cdot(\nabla\delta\boldsymbol{u}+\delta\boldsymbol{\omega})+\boldsymbol{\mu}\cdot\nabla\delta\boldsymbol{\theta}dv=\int_{\partial\Omega}\boldsymbol{t}\cdot\delta\boldsymbol{u}+\boldsymbol{q}\cdot\delta\boldsymbol{\theta}ds+\int_\Omega\boldsymbol{f}\cdot\delta\boldsymbol{u}+\boldsymbol{p}\cdot\delta\boldsymbol{\theta}dv \tag{9.27}$$

$$\boldsymbol{u}\nabla=\boldsymbol{\varepsilon}+\boldsymbol{\omega}\longrightarrow\delta\boldsymbol{u}\nabla=\delta\boldsymbol{\varepsilon}+\delta\boldsymbol{\omega}\longrightarrow\delta\boldsymbol{\varepsilon}=\delta\boldsymbol{u}\nabla-\delta\boldsymbol{\omega} \tag{9.28}$$

$$\boldsymbol{\sigma}\cdot(\nabla\delta\boldsymbol{u}+\delta\boldsymbol{\omega})=\boldsymbol{\sigma}\cdot[(\nabla\delta\boldsymbol{u})^T+(\delta\boldsymbol{\omega})^T]^T=\boldsymbol{\sigma}\cdot(\delta\boldsymbol{u}\nabla-\delta\boldsymbol{\omega})^T=\boldsymbol{\sigma}\cdot(\delta\boldsymbol{\varepsilon})^T=\boldsymbol{\sigma}\cdot\delta\boldsymbol{\varepsilon} \tag{9.29}$$

Then

$$\int_\Omega \boldsymbol{\sigma}\cdot\delta\boldsymbol{\varepsilon}+\boldsymbol{\mu}\cdot\nabla\delta\boldsymbol{\theta}dv=\int_{\partial\Omega}\boldsymbol{t}\cdot\delta\boldsymbol{u}+\boldsymbol{q}\cdot\delta\boldsymbol{\theta}ds+\int_\Omega\boldsymbol{f}\cdot\delta\boldsymbol{u}+\boldsymbol{p}\cdot\delta\boldsymbol{\theta}dv \tag{9.30}$$

Since $\delta\varepsilon_{ij}$ is symmetric, we also have

$$\sigma_{ij}\delta\varepsilon_{ij}=\sigma_{(ij)}\delta\varepsilon_{ij}\qquad \boldsymbol{\sigma}\cdot\delta\boldsymbol{\varepsilon}=\boldsymbol{\sigma}^s\cdot\boldsymbol{\varepsilon} \tag{9.31}$$

Thus, the principle of virtual work can be written as

{51}
$$\int_\Omega \boldsymbol{\sigma}^s\cdot\delta\boldsymbol{\varepsilon}+\boldsymbol{\mu}\cdot\nabla\delta\boldsymbol{\theta}dv=\int_{\partial\Omega}\boldsymbol{t}\cdot\delta\boldsymbol{u}+\boldsymbol{q}\cdot\delta\boldsymbol{\theta}ds+\int_\Omega\boldsymbol{f}\cdot\delta\boldsymbol{u}+\boldsymbol{p}\cdot\delta\boldsymbol{\theta}dv \tag{9.32}$$

The right hand side of (9.32) shows that the boundary conditions on the surface of the body can be either vectors $\boldsymbol{u}$ and $\boldsymbol{\theta}$ as essential (geometrical) boundary conditions, or $\boldsymbol{t}$ and $\boldsymbol{q}$ as natural (mechanical) boundary conditions. The left hand side of (9.32) shows that $\boldsymbol{\sigma}^s$ and $\boldsymbol{\varepsilon}$ are energy conjugate tensors, and the skew symmetric part of Cosserat stress tensor $\boldsymbol{\sigma}^a$ has no contribution to internal virtual work. At this point, it is also seen that $\boldsymbol{\mu}$ and $\boldsymbol{\nabla\theta}$ are energy conjugate tensors. Therefore, the compatible curvature tensor must be developed from $\nabla\boldsymbol{\theta}$. The virtual work principle (9.32) shows that there is no room for strain gradients as fundamental measures of deformation in a consistent couple stress theory. Interestingly, (9.32) can reveal more insight about the structure of this consistent couple stress theory.

$$\boldsymbol{e}\cdot[\nabla(\boldsymbol{p}\otimes\boldsymbol{u})]=\boldsymbol{e}\cdot(\nabla\boldsymbol{p}\otimes\boldsymbol{u})-\boldsymbol{e}\cdot(\boldsymbol{p}\otimes\nabla\boldsymbol{u}),\quad \text{can not remove the Livi-Civita (nice!)} \tag{9.33}$$

$$\begin{aligned}
\boldsymbol{p}\cdot\delta\boldsymbol{\theta}&=\boldsymbol{p}\cdot(\frac{1}{2}\nabla\times\delta\boldsymbol{u})=\frac{1}{2}\boldsymbol{p}\cdot(\nabla\times\delta\boldsymbol{u})=\frac{1}{2}\boldsymbol{e}\cdot(\boldsymbol{p}\otimes\nabla\delta\boldsymbol{u}) \\
&=\frac{1}{2}\boldsymbol{e}\cdot(\nabla\boldsymbol{p}\otimes\delta\boldsymbol{u})-\frac{1}{2}\boldsymbol{e}\cdot\nabla(\boldsymbol{p}\otimes\delta\boldsymbol{u}) \\
&=\frac{1}{2}\boldsymbol{e}\cdot(\nabla\boldsymbol{p}\otimes\delta\boldsymbol{u})-\frac{1}{2}\nabla\cdot[\boldsymbol{e}\cdot(\boldsymbol{p}\otimes\delta\boldsymbol{u})]
\end{aligned} \tag{9.34}$$

$$\begin{aligned}
\int_\Omega\boldsymbol{p}\cdot\delta\boldsymbol{\theta}dv&=\int_\Omega\frac{1}{2}\boldsymbol{e}\cdot(\nabla\boldsymbol{p}\otimes\delta\boldsymbol{u})-\frac{1}{2}\nabla\cdot[\boldsymbol{e}\cdot(\boldsymbol{p}\otimes\delta\boldsymbol{u})]dv \\
&=\int_\Omega\frac{1}{2}\boldsymbol{e}\cdot(\nabla\boldsymbol{p}\otimes\delta\boldsymbol{u})dv-\int_{\partial\Omega}\frac{1}{2}\boldsymbol{n}\cdot[\boldsymbol{e}\cdot(\boldsymbol{p}\otimes\delta\boldsymbol{u})]ds \\
&=\int_\Omega\frac{1}{2}\nabla\boldsymbol{p}\cdot\boldsymbol{e}\cdot\delta\boldsymbol{u}dv-\int_{\partial\Omega}\frac{1}{2}\boldsymbol{p}\cdot(\boldsymbol{n}\cdot\boldsymbol{e})\cdot\delta\boldsymbol{u}ds \\
&=\int_\Omega\frac{1}{2}(\nabla\boldsymbol{p}\cdot\boldsymbol{e})\cdot\delta\boldsymbol{u}dv-\int_{\partial\Omega}\frac{1}{2}[\boldsymbol{p}\cdot(\boldsymbol{n}\cdot\boldsymbol{e})]\cdot\delta\boldsymbol{u}ds \\
&=\int_\Omega\frac{1}{2}(\nabla\boldsymbol{p}\cdot\boldsymbol{e})\cdot\delta\boldsymbol{u}dv+\int_{\partial\Omega}\frac{1}{2}[(\boldsymbol{e}\cdot\boldsymbol{n})\cdot\boldsymbol{p}]\cdot\delta\boldsymbol{u}ds \\
&=\int_\Omega\frac{1}{2}e_{ijk}(\partial_ip_j)\delta u_kdv-\int_{\partial\Omega}\frac{1}{2}n_ie_{ijk}p_j\delta u_kds \\
&=\int_\Omega\frac{1}{2}e_{ijk}(\partial_jp_k)\delta u_idv+\int_{\partial\Omega}\frac{1}{2}e_{ijk}p_jn_k\delta u_ids
\end{aligned} \tag{9.35}$$

which means that the body couple $\boldsymbol{p}$ transforms into an equivalent body force $\frac{1}{2}(\nabla\boldsymbol{p}\cdot\boldsymbol{e})$ in the volume and a surface force $\frac{1}{2}[(\boldsymbol{e}\cdot\boldsymbol{n})\cdot\boldsymbol{p}]$ on the bounding surface. This shows that in a continuum theory of materials, the body couple is not distinguishable from the body force. Therefore, in couple stress theory, we must only consider body forces. This is analogous to the impossibility of distinguishing a distributed moment load in Euler-Bernouli beam theory, in which the moment must be replaced by the equivalent distributed force and end concentrated loads. Therefore, for a proper couple stress theory, the equilibrium equations become

$$\boldsymbol{e}\cdot\nabla\boldsymbol{p}=(\nabla\boldsymbol{p})\cdot\boldsymbol{e}=\nabla\times\boldsymbol{p},\qquad(\boldsymbol{e}\cdot\boldsymbol{n})\cdot\boldsymbol{p}=\boldsymbol{p}\times\boldsymbol{n}\qquad\text{magic!}\tag{9.36}$$

$$\begin{cases}\boldsymbol{F}=\boldsymbol{f}+\dfrac{1}{2}(\nabla\boldsymbol{p}\cdot\boldsymbol{e})=\boldsymbol{f}+\dfrac{1}{2}\nabla\times\boldsymbol{p}\\ \boldsymbol{T}=\boldsymbol{t}+\dfrac{1}{2}[(\boldsymbol{e}\cdot\boldsymbol{n})\cdot\boldsymbol{p}]=\boldsymbol{t}+\dfrac{1}{2}\boldsymbol{p}\times\boldsymbol{n}\end{cases}\tag{9.37}$$

体力偶可以转化为等效的体力和面力，所以只考虑体力而不考虑体力偶。刚开始时（面力、体力、面力偶、体力偶），现在（等效面力、等效体力、面力偶）。

$$\begin{cases}\nabla\cdot\boldsymbol{\sigma}+\boldsymbol{F}=\boldsymbol{0}\\ \boldsymbol{e}\cdot\boldsymbol{\sigma}+\nabla\cdot\boldsymbol{\mu}=\boldsymbol{0}\end{cases}\tag{9.38}$$

The virtual work theorem (there is no body moment) reduces to

$$\int_{\Omega}\boldsymbol{\sigma}^{s}\cdot\delta\boldsymbol{\varepsilon}+\boldsymbol{\mu}\cdot\nabla\delta\boldsymbol{\theta}dv=\int_{\partial\Omega}\boldsymbol{T}\cdot\delta\boldsymbol{u}+\boldsymbol{q}\cdot\delta\boldsymbol{\theta}ds+\int_{\Omega}\boldsymbol{F}\cdot\delta\boldsymbol{u}dv\tag{9.39}$$ {59}

In particular, if $\boldsymbol{u}$ is specified on the boundary surface, then the normal component of the rotation $\boldsymbol{\theta}$ corresponding to twisting cannot be prescribed independent.

$$\boldsymbol{\theta}^{\perp}=(\boldsymbol{n}\otimes\boldsymbol{n})\cdot\boldsymbol{\theta}=\boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{\theta})\tag{9.40}$$

However, the tangential component of rotation θ corresponding to bending may be specified in addition,

$$\boldsymbol{\theta}^{\parallel}=\boldsymbol{\theta}-\boldsymbol{\theta}^{\perp}=(\boldsymbol{I}-\boldsymbol{n}\otimes\boldsymbol{n})\cdot\boldsymbol{\theta}\tag{9.41}$$

and the number of geometric or essential boundary conditions that can be specified is therefore five.

Next, we let $\boldsymbol{q}^{\perp}$ and $\boldsymbol{q}^{\parallel}$ represent the normal and tangential components of the surface moment vector $\boldsymbol{q}$, respectively, where $\boldsymbol{q}^{\perp}$ causes twisting, while $\boldsymbol{q}^{\parallel}$ is responsible for bending.

$$\boldsymbol{q}^{\perp}=(\boldsymbol{n}\otimes\boldsymbol{n})\cdot\boldsymbol{q}=\boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{q})=\boldsymbol{n}[\boldsymbol{n}\cdot(\boldsymbol{n}\cdot\boldsymbol{\mu})],\qquad\boldsymbol{q}^{\parallel}=\boldsymbol{q}-\boldsymbol{q}^{\perp}\tag{9.42}$$

For kinematics, since $\boldsymbol{n}\cdot\boldsymbol{\theta}$ is not an independent generalized degree of freedom, its apparent corresponding generalized force must be zero. Thus, for the normal component of the surface moment vector $\boldsymbol{q}^{\perp}$, we must enforce the condition

$$\boldsymbol{q}^{\perp}=\boldsymbol{0}\tag{9.43}$$

Furthermore, the boundary moment surface virtual work becomes

$$\int_{\partial\Omega}\boldsymbol{q}\cdot\delta\boldsymbol{\theta}ds=\int_{\partial\Omega}\boldsymbol{q}^{\parallel}\cdot\delta\boldsymbol{\theta}ds=\int_{\partial\Omega}\boldsymbol{q}^{\parallel}\cdot\delta\boldsymbol{\theta}^{\parallel}ds\tag{9.44}$$

First, we notice that the virtual work theorem can be written for every arbitrary volume Ω_a with surface $\partial\Omega_a$ within the body Ω. Thus

$$\int_{\Omega_a}\boldsymbol{\sigma}^{s}\cdot\delta\boldsymbol{\varepsilon}+\boldsymbol{\mu}\cdot\nabla\delta\boldsymbol{\theta}dv=\int_{\partial\Omega_a}\boldsymbol{T}\cdot\delta\boldsymbol{u}+\boldsymbol{q}\cdot\delta\boldsymbol{\theta}ds+\int_{\Omega_a}\boldsymbol{F}\cdot\delta\boldsymbol{u}dv\tag{9.45}$$

For any point on this arbitrary surface with unit normal $\boldsymbol{n}$, we must have

$$\boldsymbol{q}^{\perp}=\boldsymbol{n}[\boldsymbol{n}\cdot(\boldsymbol{n}\cdot\boldsymbol{\mu})]=\boldsymbol{n}[(\boldsymbol{n}\otimes\boldsymbol{n})\cdot\boldsymbol{\mu}]=\boldsymbol{0}\quad\text{in}\quad\Omega\tag{9.46}$$

Since $\boldsymbol{n}\otimes\boldsymbol{n}$ is symmetric and arbitrary, $\boldsymbol{\mu}$ must be skew-symmetric. Thus

$$\boldsymbol{\mu}=-\boldsymbol{\mu}^{T}\tag{9.47}$$

9.2 一些分解

两种正交分解方法:

1. 对称反对称分解。例: $\varepsilon_{ij} = \varepsilon_{ij}^s + \varepsilon_{ij}^a = \frac{1}{2}(\varepsilon_{ij} + \varepsilon_{ji}) + \frac{1}{2}(\varepsilon_{ij} - \varepsilon_{ji})$
2. 静水偏量分解。例: $\varepsilon_{ij} = \varepsilon_{ij}^h + \varepsilon_{ij}^{'} = \frac{1}{3}\delta_{ij}\varepsilon_{kk}(\text{迹平均}) + \varepsilon_{ij}^{'}$

$$\varepsilon_{ij} = \varepsilon_{ij}^s + \varepsilon_{ij}^a = \frac{1}{2}(\varepsilon_{ij} + \varepsilon_{ji}) + \frac{1}{2}(\varepsilon_{ij} - \varepsilon_{ji}) \tag{9.48}$$

$$\sigma_{ij} = \sigma_{ij}^s + \sigma_{ij}^a = \frac{1}{2}(\sigma_{ij} + \sigma_{ji}) + \frac{1}{2}(\sigma_{ij} - \sigma_{ji}) \tag{9.49}$$

$$\begin{aligned} \eta_{ijk} &= \eta_{ijk}^{s0} + \eta_{ijk}^{s1} + \eta_{ijk}^{as} + \eta_{ijk}^{aa} \\ &= \eta_{ijk}^{h} + \eta_{ijk}^{'s0} + \eta_{ijk}^{'s1} + \eta_{ijk}^{'as} + \eta_{ijk}^{'aa} \quad = \eta_{ijk}^{h} + \eta_{ijk}^{'s1} + \eta_{ijk}^{'as} + \eta_{ijk}^{'2} \end{aligned} \tag{9.50}$$

9.3 关于全应变梯度理论到反对称偶应力理论的一些退化推导

$$\theta_i = \frac{1}{2}e_{ijk}u_{k,j} \quad \boldsymbol{\theta} = \frac{1}{2}\nabla\times\boldsymbol{u} = \frac{1}{2}\boldsymbol{e}\cdot\nabla\boldsymbol{u} \tag{9.51}$$

$$\chi_{ij} = \theta_{i,j} = \frac{1}{2}e_{imn}u_{n,mj} \tag{9.52}$$

$$\chi_{ij} = \chi_{ij}^s + \chi_{ij}^a = \frac{1}{2}(\chi_{ij}+\chi_{ji}) + \frac{1}{2}(\chi_{ij}-\chi_{ji}) \tag{9.53}$$

$$\chi_{ij}^s = \frac{1}{2}(\chi_{ij}+\chi_{ji}) = \frac{1}{2}(\frac{1}{2}e_{imn}u_{n,mj} + \frac{1}{2}e_{jmn}u_{n,mi}) \tag{9.54}$$

$$\chi'_{ij} = e_{imn}\eta'_{mnj}, \quad \eta'_{ijk} = \varepsilon^{s'}_{jk,i} = \varepsilon^s_{jk,i} - \frac{1}{3}\delta_{jk}\varepsilon^s_{nn,i} \tag{9.55}$$

$$\begin{aligned}
\chi'^s_{ij} &= \frac{1}{2}(e_{imn}\eta'_{mnj} + e_{jmn}\eta'_{mni}) \\
&= \frac{1}{2}[e_{imn}(\varepsilon^s_{nj,m} - \frac{1}{3}\delta_{nj}\varepsilon^s_{rr,m}) + e_{jmn}(\varepsilon^s_{ni,m} - \frac{1}{3}\delta_{ni}\varepsilon^s_{rr,m})] \\
&= \frac{1}{2}[e_{imn}\varepsilon^s_{nj,m} - \frac{1}{3}e_{imj}\varepsilon^s_{rr,m} + e_{jmn}\varepsilon^s_{ni,m} - \frac{1}{3}e_{jmi}\varepsilon^s_{rr,m}] \\
&= \frac{1}{2}[e_{imn}\frac{1}{2}(u_{n,jm}+u_{j,nm}) - \frac{1}{3}e_{imj}u_{r,rm} + e_{jmn}\frac{1}{2}(u_{n,im}+u_{i,nm}) - \frac{1}{3}e_{jmi}u_{r,rm}] \\
&= \frac{1}{2}[\frac{1}{2}e_{imn}u_{n,jm} + \frac{1}{2}e_{imn}u_{j,nm} - \frac{1}{3}e_{imj}u_{r,rm} + \frac{1}{2}e_{jmn}u_{n,im} + \frac{1}{2}e_{jmn}u_{i,nm} - \frac{1}{3}e_{jmi}u_{r,rm}] \\
&= \frac{1}{2}[\frac{1}{2}e_{imn}u_{n,jm} + \frac{1}{2}e_{jmn}u_{n,im}] = \frac{1}{2}[\frac{1}{2}e_{imn}u_{n,mj} + \frac{1}{2}e_{jmn}u_{n,mi}] \\
&= \chi^s_{ij}
\end{aligned} \tag{9.56}$$

Thus(Therefore、For this reason、Hence、Accordingly...)

$$\chi^s_{ij} = \chi'^s_{ij} \tag{9.57}$$

9.4 反对称

$$\gamma_i = \varepsilon^s_{mm,i} = u_{m,mi} \tag{9.58}$$

$$\chi^a_{ij} = \frac{1}{2}(\chi_{ij}-\chi_{ji}) = \frac{1}{2}(\frac{1}{2}e_{imn}u_{n,mj} - \frac{1}{2}e_{jmn}u_{n,mi}) \tag{9.59}$$

$$\begin{aligned}
\chi'^a_{ij} &= \frac{1}{2}(e_{imn}\eta'_{mnj} - e_{jmn}\eta'_{mni}) \\
&= \frac{1}{2}[e_{imn}(\varepsilon^s_{nj,m} - \frac{1}{3}\delta_{nj}\varepsilon^s_{rr,m}) - e_{jmn}(\varepsilon^s_{ni,m} - \frac{1}{3}\delta_{ni}\varepsilon^s_{rr,m})] \\
&= \frac{1}{2}[e_{imn}\varepsilon^s_{nj,m} - \frac{1}{3}e_{imj}\varepsilon^s_{rr,m} - e_{jmn}\varepsilon^s_{ni,m} + \frac{1}{3}e_{jmi}\varepsilon^s_{rr,m}] \\
&= \frac{1}{2}[e_{imn}\frac{1}{2}(u_{n,jm}+u_{j,nm}) - \frac{1}{3}e_{imj}u_{r,rm} - e_{jmn}\frac{1}{2}(u_{n,im}+u_{i,nm}) + \frac{1}{3}e_{jmi}u_{r,rm}] \\
&= \frac{1}{2}[\frac{1}{2}e_{imn}u_{n,jm} + \frac{1}{2}e_{imn}u_{j,nm} - \frac{1}{3}e_{imj}u_{r,rm} - \frac{1}{2}e_{jmn}u_{n,im} - \frac{1}{2}e_{jmn}u_{i,nm} + \frac{1}{3}e_{jmi}u_{r,rm}] \\
&= \frac{1}{2}[\frac{1}{2}e_{imn}u_{n,jm} - \frac{1}{2}e_{jmn}u_{n,im} - \frac{2}{3}e_{imj}u_{r,rm}] \\
&= \frac{1}{2}[\frac{1}{2}e_{imn}u_{n,mj} - \frac{1}{2}e_{jmn}u_{n,mi}] - \frac{1}{3}e_{imj}u_{r,rm} \\
&= \chi^a_{ij} - \frac{1}{3}e_{imj}\gamma_m
\end{aligned} \tag{9.60}$$

9.4.1 变形分析示例：刚体平面转动

参考构型和当前构型中的位置矢量分别为

$$\boldsymbol{X} = X_I \boldsymbol{E}_I, \qquad \boldsymbol{x} = x_i \boldsymbol{e}_i \tag{9.61}$$

两个位置矢量一一对应，形成映射

$$\boldsymbol{x} = \varphi(\boldsymbol{X}) \tag{9.62}$$

那么，映射的具体形式是什么呢？

首先，将空间坐标系的基矢量用物质坐标系的基矢量来表示

$$\boldsymbol{e}_i = (\boldsymbol{e}_i \cdot \boldsymbol{E}_J)\boldsymbol{E}_J \tag{9.63}$$

其次，将空间坐标用物质坐标来表示

$$x_i = X_I \tag{9.64}$$

然后，将上述表示代入当前构型中的位置矢量

$$\begin{aligned}
\boldsymbol{x} &= x_i\boldsymbol{e}_i = X_I(\boldsymbol{e}_i \cdot \boldsymbol{E}_J)\boldsymbol{E}_J = X_I(\boldsymbol{e}_i \cdot \boldsymbol{E}_1)\boldsymbol{E}_1 + X_I(\boldsymbol{e}_i \cdot \boldsymbol{E}_2)\boldsymbol{E}_2 + X_I(\boldsymbol{e}_i \cdot \boldsymbol{E}_3)\boldsymbol{E}_3 \\
&= [X_1(\boldsymbol{e}_1 \cdot \boldsymbol{E}_1) + X_2(\boldsymbol{e}_2 \cdot \boldsymbol{E}_1) + X_3(\boldsymbol{e}_3 \cdot \boldsymbol{E}_1)]\boldsymbol{E}_1 \\
&+ [X_1(\boldsymbol{e}_1 \cdot \boldsymbol{E}_2) + X_2(\boldsymbol{e}_2 \cdot \boldsymbol{E}_2) + X_3(\boldsymbol{e}_3 \cdot \boldsymbol{E}_2)]\boldsymbol{E}_2 \\
&+ [X_1(\boldsymbol{e}_1 \cdot \boldsymbol{E}_3) + X_2(\boldsymbol{e}_2 \cdot \boldsymbol{E}_3) + X_3(\boldsymbol{e}_3 \cdot \boldsymbol{E}_3)]\boldsymbol{E}_3 \\
&= [X_1\cos\theta + X_2\cos(\frac{\pi}{2}+\theta)]\boldsymbol{E}_1 + [X_1\cos(\frac{\pi}{2}-\theta) + X_2\cos\theta]\boldsymbol{E}_2 + X_3\boldsymbol{E}_3 \\
&= [X_1\cos\theta - X_2\sin\theta]\boldsymbol{E}_1 + [X_1\sin\theta + X_2\cos\theta]\boldsymbol{E}_2 + X_3\boldsymbol{E}_3
\end{aligned} \tag{9.65}$$

变形梯度张量为

$$\begin{aligned}
\boldsymbol{F} &= \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} = \boldsymbol{x}\nabla = \frac{\partial \boldsymbol{x}}{\partial X_I} \otimes \boldsymbol{E}_I \\
&= \frac{\partial \boldsymbol{x}}{\partial X_1} \otimes \boldsymbol{E}_1 + \frac{\partial \boldsymbol{x}}{\partial X_2} \otimes \boldsymbol{E}_2 + \frac{\partial \boldsymbol{x}}{\partial X_3} \otimes \boldsymbol{E}_3 \\
&= [\cos\theta\boldsymbol{E}_1 + \sin\theta\boldsymbol{E}_2] \otimes \boldsymbol{E}_1 + [-\sin\theta\boldsymbol{E}_1 + \cos\theta\boldsymbol{E}_2] \otimes \boldsymbol{E}_2 + \boldsymbol{E}_3 \otimes \boldsymbol{E}_3 \\
&= \cos\theta\boldsymbol{E}_1 \otimes \boldsymbol{E}_1 - \sin\theta\boldsymbol{E}_1 \otimes \boldsymbol{E}_2 + \sin\theta\boldsymbol{E}_2 \otimes \boldsymbol{E}_1 + \cos\theta\boldsymbol{E}_2 \otimes \boldsymbol{E}_2 + \boldsymbol{E}_3 \otimes \boldsymbol{E}_3 \\
&= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned} \tag{9.66}$$

右柯西-格林应变张量$\boldsymbol{C}$为

$$\begin{aligned}
\boldsymbol{C} =& \boldsymbol{F}^T \cdot \boldsymbol{F} \\
=& [\cos\theta\boldsymbol{E}_1 \otimes \boldsymbol{E}_1 + \sin\theta\boldsymbol{E}_1 \otimes \boldsymbol{E}_2 - \sin\theta\boldsymbol{E}_2 \otimes \boldsymbol{E}_1 + \cos\theta\boldsymbol{E}_2 \otimes \boldsymbol{E}_2 + \boldsymbol{E}_3 \otimes \boldsymbol{E}_3] \\
& \cdot \\
& [\cos\theta\boldsymbol{E}_1 \otimes \boldsymbol{E}_1 - \sin\theta\boldsymbol{E}_1 \otimes \boldsymbol{E}_2 + \sin\theta\boldsymbol{E}_2 \otimes \boldsymbol{E}_1 + \cos\theta\boldsymbol{E}_2 \otimes \boldsymbol{E}_2 + \boldsymbol{E}_3 \otimes \boldsymbol{E}_3] \\
=& \cos^2\theta\boldsymbol{E}_1 \otimes \boldsymbol{E}_1 - \cos\theta\sin\theta\boldsymbol{E}_1 \otimes \boldsymbol{E}_2 + \sin^2\theta\boldsymbol{E}_1 \otimes \boldsymbol{E}_1 + \sin\theta\cos\theta\boldsymbol{E}_1 \otimes \boldsymbol{E}_2 \\
& - \sin\theta\cos\theta\boldsymbol{E}_2 \otimes \boldsymbol{E}_1 + \sin^2\theta\boldsymbol{E}_2 \otimes \boldsymbol{E}_2 + \cos\theta\sin\theta\boldsymbol{E}_2 \otimes \boldsymbol{E}_1 + \cos^2\theta\boldsymbol{E}_2 \otimes \boldsymbol{E}_2 \\
& + \boldsymbol{E}_3 \otimes \boldsymbol{E}_3 \\
=& \boldsymbol{E}_1 \otimes \boldsymbol{E}_1 + \boldsymbol{E}_2 \otimes \boldsymbol{E}_2 + \boldsymbol{E}_3 \otimes \boldsymbol{E}_3 \\
=& \boldsymbol{E}_I \otimes \boldsymbol{E}_I = \delta_{IJ}\boldsymbol{E}_I \otimes \boldsymbol{E}_J = \boldsymbol{I}
\end{aligned} \tag{9.67}$$

结论：当物体发生刚体转动时，右柯西-格林应变张量$\boldsymbol{C} = \boldsymbol{I}$，主伸长率（特征值）都为1.

位移矢量为

$$\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{X} = [X_1(\cos\theta - 1) - X_2\sin\theta]\boldsymbol{E}_1 + [X_1\sin\theta + X_2(\cos\theta - 1)]\boldsymbol{E}_2 \tag{9.68}$$

位移梯度为

$$\begin{aligned}\boldsymbol{u}\nabla &= \frac{\partial \boldsymbol{u}}{\partial X_J}\otimes \boldsymbol{E}_J = \frac{\partial \boldsymbol{u}}{\partial X_1}\otimes \boldsymbol{E}_1 + \frac{\partial \boldsymbol{u}}{\partial X_2}\otimes \boldsymbol{E}_2 + \frac{\partial \boldsymbol{u}}{\partial X_3}\otimes \boldsymbol{E}_3 \\ &= (\cos\theta - 1)\boldsymbol{E}_1\otimes\boldsymbol{E}_1 + \sin\theta\boldsymbol{E}_2\otimes\boldsymbol{E}_1 - \sin\theta\boldsymbol{E}_1\otimes\boldsymbol{E}_2 + (\cos\theta-1)\boldsymbol{E}_2\otimes\boldsymbol{E}_2 \\ &= (\cos\theta - 1)\boldsymbol{E}_1\otimes\boldsymbol{E}_1 - \sin\theta\boldsymbol{E}_1\otimes\boldsymbol{E}_2 + \sin\theta\boldsymbol{E}_2\otimes\boldsymbol{E}_1 + (\cos\theta-1)\boldsymbol{E}_2\otimes\boldsymbol{E}_2\end{aligned} \tag{9.69}$$

小应变张量为

$$\boldsymbol{\varepsilon} = \frac{1}{2}[\boldsymbol{u}\nabla + \nabla\boldsymbol{u}] = (\cos\theta-1)\boldsymbol{E}_1\otimes\boldsymbol{E}_1 + (\cos\theta-1)\boldsymbol{E}_2\otimes\boldsymbol{E}_2 = \begin{bmatrix}\cos\theta-1 & 0 & 0\\ 0 & \cos\theta-1 & 0\\ 0 & 0 & 0\end{bmatrix} \tag{9.70}$$

对于小角度转动，使用泰勒展开一阶近似可得

$$\boldsymbol{\varepsilon} = \begin{bmatrix}-\frac{\theta^2}{2} & 0 & 0\\ 0 & -\frac{\theta^2}{2} & 0\\ 0 & 0 & 0\end{bmatrix} \tag{9.71}$$

初看上去，上式误差很小。然而，即使是很小的数值，已经预示着较大的应变。对于金属(钢的杨氏模量为$200GPa$)来说，通常应变阶数为10^{-4}时便会产生实质性的应力。这意味着，使用小应变张量将引起明显的误差，即时刚体转动$1°$时也是如此。

刚体转动不应该产生应变和应力，而使用小应变时会产生应变，误差就在于此。

9.4.2 变形分析示例-动能计算

质量守恒

$$dm = dM \tag{9.72}$$

质量和体积的关系为

$$dM = \rho_0 d\Omega_r, \quad dm = \rho d\Omega_c \tag{9.73}$$

其中，$\Omega_r = \Omega_{reference}$表示参考构型，$\Omega_c = \Omega_{current}$表示当前构型。

所以

$$\rho d\Omega_c = \rho_0 d\Omega_r \tag{9.74}$$

其中，ρ_0为物体变形前的密度，ρ为物体变形后的密度。

因为

$$J = \frac{d\Omega_c}{d\Omega_r} \tag{9.75}$$

所以

$$\rho J = \rho_0 \tag{9.76}$$

对于不可压缩情形$(J = 1)$

$$\rho = \rho_0 \tag{9.77}$$

动能

$$T = \int_{\Omega_c} \frac{1}{2}(dm)\|\mathbf{v}\|^2 = \int_{\Omega_c} \frac{1}{2}(\rho d\Omega_c)\|\mathbf{v}\|^2 = \int_{\Omega_c} \frac{1}{2}\rho\|\mathbf{v}\|^2 d\Omega_c = \int_{\Omega_r} \frac{1}{2}\rho_0\|\mathbf{v}\|^2 d\Omega_r \tag{9.78}$$

其中，积分区域应该跟着积分微元变化!!! 最后一个等号是利用质量守恒实现微元体及积分区域的转换。

对于不可压缩情形

$$T = \int_{\Omega_c} \frac{1}{2}\rho\|\mathbf{v}\|^2 d\Omega_c = \int_{\Omega_r} \frac{1}{2}\rho\|\mathbf{v}\|^2 d\Omega_r \tag{9.79}$$

引入直角坐标系，假设物体运动为

$$x = \frac{1}{\sqrt{\lambda(t)}}X, \quad y = \frac{1}{\sqrt{\lambda(t)}}Y, \quad z = \lambda(t)Z \tag{9.80}$$

也可以写为逆关系

$$X = \sqrt{\lambda(t)}x, \quad Y = \sqrt{\lambda(t)}y, \quad Z = \frac{1}{\lambda(t)}z \tag{9.81}$$

物体速度为

$$\dot{x} = \left(-\frac{1}{2}\lambda^{-\frac{3}{2}}\dot{\lambda}\right)X, \quad \dot{y} = \left(-\frac{1}{2}\lambda^{-\frac{3}{2}}\dot{\lambda}\right)Y, \quad \dot{z} = \dot{\lambda}Z \tag{9.82}$$

若带入逆关系

$$\dot{x} = \left(-\frac{1}{2}\frac{\dot{\lambda}}{\lambda}\right)x, \quad \dot{y} = \left(-\frac{1}{2}\frac{\dot{\lambda}}{\lambda}\right)y, \quad \dot{z} = \frac{\dot{\lambda}}{\lambda}z \tag{9.83}$$

所以速度既可以表示为物质坐标的函数也可以表示为空间坐标的函数。

物体动能为

$$\begin{aligned} T &= \int_{\Omega_r} \frac{1}{2}\rho\|\mathbf{v}\|^2 d\Omega_r = \int_{\Omega_c} \frac{1}{2}\rho\|\mathbf{v}\|^2 d\Omega_c \\ &= \int_{\Omega_r} \frac{1}{2}\rho\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) d\Omega_r = \int_{\Omega_c} \frac{1}{2}\rho\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) d\Omega_c \end{aligned} \tag{9.84}$$

计算准备:

因为

$$\frac{\partial}{\partial x}\left(\frac{\dot{x}^3}{3}\right) = \dot{x}^2\frac{\partial \dot{x}}{\partial x} = \dot{x}^2\left(-\frac{1}{2}\frac{\dot{\lambda}}{\lambda}\right) \tag{9.85}$$

可见，$\dot{x}^2$的原函数并非$\frac{\dot{x}^3}{3}$，对z亦是如此。

所以

$$\dot{x}^2 = -2\frac{\lambda}{\dot{\lambda}}\frac{\partial}{\partial x}\left(\frac{\dot{x}^3}{3}\right) \tag{9.86}$$

所以

$$
\begin{aligned}
\int_{\Omega_c} \dot{x}^2 d\Omega_c &= \int_{\Omega_c} -2\frac{\lambda}{\dot{\lambda}}\frac{\partial}{\partial x}\left(\frac{\dot{x}^3}{3}\right) d\Omega_c \\
&= -2\frac{\lambda}{\dot{\lambda}}\int_{\Omega_c}\frac{\partial}{\partial x}\left(\frac{\dot{x}^3}{3}\right) d\Omega_c \\
&= -2\frac{\lambda}{\dot{\lambda}}\int_{-\lambda H}^{\lambda H}\int_{-L/\sqrt{\lambda}}^{L/\sqrt{\lambda}}\int_{-L/\sqrt{\lambda}}^{L/\sqrt{\lambda}}\frac{\partial}{\partial x}\left(\frac{\dot{x}^3}{3}\right) dxdydz \\
&= -2\frac{\lambda}{\dot{\lambda}}\int_{-\lambda H}^{\lambda H}\int_{-L/\sqrt{\lambda}}^{L/\sqrt{\lambda}} dydz\int_{-L/\sqrt{\lambda}}^{L/\sqrt{\lambda}}\frac{\partial}{\partial x}\left(\frac{\dot{x}^3}{3}\right) dx \\
&= -2\frac{\lambda}{\dot{\lambda}}\cdot 2\lambda H\cdot 2L/\sqrt{\lambda}\cdot\left.\frac{\dot{x}^3}{3}\right|_{-L/\sqrt{\lambda}}^{L/\sqrt{\lambda}} \\
&= -2\frac{\lambda}{\dot{\lambda}}\cdot 2\lambda H\cdot 2L/\sqrt{\lambda}\cdot\left\langle\frac{1}{3}\left[\frac{d}{dt}(L/\sqrt{\lambda})\right]^3-\frac{1}{3}\left[\frac{d}{dt}(-L/\sqrt{\lambda})\right]^3\right\rangle \\
&= -2\frac{\lambda}{\dot{\lambda}}\cdot 2\lambda H\cdot 2L/\sqrt{\lambda}\cdot\frac{2}{3}\left[\frac{d}{dt}(L/\sqrt{\lambda})\right]^3 \\
&= -2\frac{\lambda}{\dot{\lambda}}\cdot 2\lambda H\cdot 2L/\sqrt{\lambda}\cdot\frac{2}{3}\left[-\frac{1}{2}\lambda^{-\frac{3}{2}}\dot{\lambda}L\right]^3 \\
&= -2\frac{\lambda}{\dot{\lambda}}\cdot 2\lambda H\cdot 2L/\sqrt{\lambda}\cdot\frac{2}{3}\left[-\frac{1}{8}\lambda^{-\frac{9}{2}}\dot{\lambda}^3L^3\right] \\
&= \frac{2}{3}HL^4\frac{\dot{\lambda}^2}{\lambda^3}
\end{aligned}
\tag{9.87}
$$

因为

$$
\frac{\partial}{\partial z}\left(\frac{\dot{z}^3}{3}\right)=\dot{z}^2\frac{\partial\dot{z}}{\partial z}=\dot{z}^2\frac{\dot{\lambda}}{\lambda}
\tag{9.88}
$$

所以

$$
\dot{z}^2=\frac{\lambda}{\dot{\lambda}}\frac{\partial}{\partial z}\left(\frac{\dot{z}^3}{3}\right)
\tag{9.89}
$$

$$
\begin{aligned}
\int_{\Omega_c}\dot{z}^2 d\Omega_c &= \int_{\Omega_c}\frac{\lambda}{\dot{\lambda}}\frac{\partial}{\partial z}\left(\frac{\dot{z}^3}{3}\right) d\Omega_c \\
&= \frac{\lambda}{\dot{\lambda}}\int_{\Omega_c}\frac{\partial}{\partial z}\left(\frac{\dot{z}^3}{3}\right) d\Omega_c \\
&= \frac{\lambda}{\dot{\lambda}}\int_{-\lambda H}^{\lambda H}\int_{-L/\sqrt{\lambda}}^{L/\sqrt{\lambda}}\int_{-L/\sqrt{\lambda}}^{L/\sqrt{\lambda}}\frac{\partial}{\partial x}\left(\frac{\dot{z}^3}{3}\right) dxdydz \\
&= \frac{\lambda}{\dot{\lambda}}\int_{-L/\sqrt{\lambda}}^{L/\sqrt{\lambda}}\int_{-L/\sqrt{\lambda}}^{L/\sqrt{\lambda}} dxdy\int_{-\lambda H}^{\lambda H}\frac{\partial}{\partial x}\left(\frac{\dot{z}^3}{3}\right) dz \\
&= \frac{\lambda}{\dot{\lambda}}2L/\sqrt{\lambda}\cdot 2L/\sqrt{\lambda}\cdot\left.\frac{\dot{z}^3}{3}\right|_{-\lambda H}^{\lambda H} \\
&= \frac{\lambda}{\dot{\lambda}}2L/\sqrt{\lambda}\cdot 2L/\sqrt{\lambda}\cdot\frac{2}{3}\left[\frac{d}{dt}(\lambda H)\right]^3 \\
&= \frac{\lambda}{\dot{\lambda}}2L/\sqrt{\lambda}\cdot 2L/\sqrt{\lambda}\cdot\frac{2}{3}\left(\dot{\lambda}H\right)^3 \\
&= \frac{\lambda}{\dot{\lambda}}2L/\sqrt{\lambda}\cdot 2L/\sqrt{\lambda}\cdot\frac{2}{3}\dot{\lambda}^3H^3 \\
&= \frac{8}{3}L^2H^3\dot{\lambda}^2
\end{aligned}
\tag{9.90}
$$

若在当前构型(当前物体所占空间区域)中计算动能

$$
\begin{aligned}
&\int_{\Omega_c}\frac{1}{2}\rho\dot{x}^2 d\Omega_c=\frac{1}{2}\rho\int_{\Omega_c}\dot{x}^2 d\Omega_c=\frac{1}{2}\rho\frac{2}{3}HL^4\frac{\dot{\lambda}^2}{\lambda^3}=\frac{1}{3}\rho HL^4\frac{\dot{\lambda}^2}{\lambda^3} \\
&\int_{\Omega_c}\frac{1}{2}\rho\dot{y}^2 d\Omega_c=\int_{\Omega_c}\frac{1}{2}\rho\dot{x}^2 d\Omega_c=\frac{1}{3}\rho HL^4\frac{\dot{\lambda}^2}{\lambda^3} \\
&\int_{\Omega_c}\frac{1}{2}\rho\dot{z}^2 d\Omega_c=\frac{1}{2}\rho\int_{\Omega_c}\dot{z}^2 d\Omega_c=\frac{1}{2}\rho\cdot\frac{8}{3}L^2H^3\dot{\lambda}^2=\frac{4}{3}\rho L^2H^3\dot{\lambda}^2
\end{aligned}
\tag{9.91}
$$

计算准备:

$$
\begin{aligned}
&\dot{x}^2=\left(-\frac{1}{2}\lambda^{-\frac{3}{2}}\dot{\lambda}\right)^2X^2=\left(\frac{1}{4}\lambda^{-3}\dot{\lambda}^2\right)X^2 \\
&\dot{z}^2=(\dot{\lambda}Z)^2=\dot{\lambda}^2Z^2
\end{aligned}
\tag{9.92}
$$

$$
\begin{aligned}
\int_{\Omega_r} \dot{x}^2 d\Omega_r &= \int_{-H}^{H}\int_{-L}^{L}\int_{-L}^{L}\left(\frac{1}{4}\lambda^{-3}\dot{\lambda}^2\right)X^2 dX dY dZ \\
&= \left(\frac{1}{4}\lambda^{-3}\dot{\lambda}^2\right)\int_{-H}^{H}\int_{-L}^{L} dY dZ \int_{-L}^{L} X^2 dX \\
&= \left(\frac{1}{4}\lambda^{-3}\dot{\lambda}^2\right)\int_{-H}^{H}\int_{-L}^{L} dY dZ \int_{-L}^{L} \frac{\partial}{\partial X}\left(\frac{X^3}{3}\right) dX \\
&= \left(\frac{1}{4}\lambda^{-3}\dot{\lambda}^2\right)\cdot 2H \cdot 2L \cdot \left.\frac{X^3}{3}\right|_{-L}^{L} \\
&= \left(\frac{1}{4}\lambda^{-3}\dot{\lambda}^2\right)\cdot 2H \cdot 2L \cdot \frac{2}{3}L^3 \\
&= \frac{2}{3}HL^4\frac{\dot{\lambda}^2}{\lambda^3}
\end{aligned}
\tag{9.93}
$$

$$
\begin{aligned}
\int_{\Omega_r} \dot{z}^2 d\Omega_r &= \int_{-H}^{H}\int_{-L}^{L}\int_{-L}^{L} \dot{\lambda}^2 Z^2 dX dY dZ \\
&= \dot{\lambda}^2 \int_{-L}^{L}\int_{-L}^{L} dX dY \int_{-H}^{H} Z^2 dZ \\
&= \dot{\lambda}^2 \int_{-L}^{L}\int_{-L}^{L} dX dY \int_{-H}^{H} \frac{\partial}{\partial Z}\left(\frac{Z^3}{3}\right) dZ \\
&= \dot{\lambda}^2 \cdot 2L \cdot 2L \cdot \left.\frac{Z^3}{3}\right|_{-H}^{H} \\
&= \dot{\lambda}^2 \cdot 2L \cdot 2L \cdot \frac{2}{3}H^3 \\
&= \dot{\lambda}^2 \cdot 2L \cdot 2L \cdot \frac{2}{3}H^3 \\
&= \frac{8}{3}H^3L^2\dot{\lambda}^2
\end{aligned}
\tag{9.94}
$$

若在参考构型(初始物体所占空间区域)中计算动能

$$
\begin{aligned}
&\int_{\Omega_c} \frac{1}{2}\rho\dot{x}^2 d\Omega_c = \frac{1}{2}\rho\int_{\Omega_c} \dot{x}^2 d\Omega_c = \frac{1}{2}\rho\frac{2}{3}HL^4\frac{\dot{\lambda}^2}{\lambda^3} = \frac{1}{3}\rho HL^4\frac{\dot{\lambda}^2}{\lambda^3} \\
&\int_{\Omega_c} \frac{1}{2}\rho\dot{y}^2 d\Omega_c = \int_{\Omega_c} \frac{1}{2}\rho\dot{x}^2 d\Omega_c = \frac{1}{3}\rho HL^4\frac{\dot{\lambda}^2}{\lambda^3} \\
&\int_{\Omega_c} \frac{1}{2}\rho\dot{z}^2 d\Omega_c = \frac{1}{2}\rho\int_{\Omega_c} \dot{z}^2 d\Omega_c = \frac{1}{2}\rho\cdot\frac{8}{3}L^2H^3\dot{\lambda}^2 = \frac{4}{3}\rho L^2H^3\dot{\lambda}^2
\end{aligned}
\tag{9.95}
$$

可见，动能虽然是在当前构型中定义的，但是可以通过质量守恒把积分区域转换到参考构型中去!!!

Chapter 10

有限元法初步

10.1 微分方程及其等效积分形式

微分方程

$$\boldsymbol{A}(\boldsymbol{u}) = \boldsymbol{0} \qquad \text{in } \Omega \qquad \bigcirc \qquad \boldsymbol{B}(\boldsymbol{u}) = \boldsymbol{0} \qquad \text{on } \overline{\Omega} \tag{10.1}$$

等效积分形式

$$\int_{\Omega} \boldsymbol{v} \cdot \boldsymbol{A}(\boldsymbol{u}) d\Omega + \int_{\overline{\Omega}} \overline{\boldsymbol{v}} \cdot \boldsymbol{B}(\boldsymbol{u}) d\overline{\Omega} = \boldsymbol{0} \tag{10.2}$$

等效积分形式的弱形式

$$\int_{\Omega} \boldsymbol{C}(\boldsymbol{v}) \cdot \boldsymbol{D}(\boldsymbol{u}) d\Omega + \int_{\overline{\Omega}} \boldsymbol{E}(\overline{\boldsymbol{v}}) \cdot \boldsymbol{F}(\boldsymbol{u}) d\overline{\Omega} = \boldsymbol{0} \tag{10.3}$$

10.2 加权余量法

设近似解

$$\boldsymbol{u} \approx \boldsymbol{N} \cdot \boldsymbol{a} \tag{10.4}$$

若近似解不完全等于精确解，则微分方程和边界条件会产生余量，余量 $\boldsymbol{R}$ 和 $\overline{\boldsymbol{R}}$ 为

$$\boldsymbol{R} = \boldsymbol{A}(\boldsymbol{N} \cdot \boldsymbol{a}), \qquad \overline{\boldsymbol{R}} = \boldsymbol{B}(\boldsymbol{N} \cdot \boldsymbol{a}) \tag{10.5}$$

令余量的加权积分为零

$$\int_{\Omega} \boldsymbol{w} \cdot \boldsymbol{A}(\boldsymbol{N} \cdot \boldsymbol{a}) d\Omega + \int_{\overline{\Omega}} \overline{\boldsymbol{w}} \cdot \boldsymbol{B}(\boldsymbol{N} \cdot \boldsymbol{a}) d\overline{\Omega} = 0 \tag{10.6}$$

其弱形式为

$$\int_{\Omega} \boldsymbol{C}(\boldsymbol{w}) \cdot \boldsymbol{D}(\boldsymbol{N} \cdot \boldsymbol{a}) d\Omega + \int_{\overline{\Omega}} \boldsymbol{E}(\overline{\boldsymbol{w}}) \cdot \boldsymbol{F}(\boldsymbol{N} \cdot \boldsymbol{a}) d\overline{\Omega} = \boldsymbol{0} \tag{10.7}$$

10.3 伽辽金加权余量法

令

$$\boldsymbol{w} = \boldsymbol{N}, \quad \overline{\boldsymbol{w}} = -\boldsymbol{w} = -\boldsymbol{N}$$

伽辽金加权余量法的形式

$$\int_{\Omega} \boldsymbol{N} \cdot \boldsymbol{A}(\boldsymbol{N} \cdot \boldsymbol{a}) d\Omega - \int_{\Gamma} \boldsymbol{N} \cdot \boldsymbol{B}(\boldsymbol{N} \cdot \boldsymbol{a}) d\Gamma = 0 \tag{10.8}$$

伽辽金加权余量法的等效积分形式的弱形式

$$\int_{\Omega} \boldsymbol{C}(\boldsymbol{N}) \cdot \boldsymbol{D}(\boldsymbol{N} \cdot \boldsymbol{a}) d\Omega - \int_{\Gamma} \boldsymbol{E}(N) \cdot \boldsymbol{F}(\boldsymbol{N} \cdot \boldsymbol{a}) d\Gamma = \boldsymbol{0} \tag{10.9}$$

近似解的变分

$$\delta \boldsymbol{u} = \boldsymbol{N} \cdot \delta \boldsymbol{a}$$

可以更简洁的表示为

$$\int_{\Omega} \delta \boldsymbol{u} \cdot \boldsymbol{A}(\boldsymbol{u}) d\Omega - \int_{\Gamma} \delta \boldsymbol{u} \cdot \boldsymbol{B}(\boldsymbol{u}) d\Gamma = 0 \tag{10.10}$$

弱形式

$$\int_\Omega \boldsymbol{C}(\delta\boldsymbol{u})\cdot\boldsymbol{D}(\boldsymbol{u})d\Omega-\int_\Gamma \boldsymbol{E}(\delta\boldsymbol{u})\cdot\boldsymbol{F}(\boldsymbol{u})d\Gamma=\mathbf{0} \tag{10.11}$$

10.4 里兹变分原理

L是线性、自伴随的微分算子

$$\int_\Omega \boldsymbol{L}(\boldsymbol{u})\boldsymbol{v}d\Omega=\int_\Omega \boldsymbol{u}\boldsymbol{L}(\boldsymbol{v})d\Omega+b.t.(\boldsymbol{u},\boldsymbol{v}) \tag{10.12}$$

其中，$b.t.(\boldsymbol{u},\boldsymbol{v})$ 表示在 Ω 的边界 Γ 上由 $\boldsymbol{u}$ 和 $\boldsymbol{v}$ 及其导数组成的积分项。

原问题的微分方程和边界条件表达如下

$$\begin{aligned}&\boldsymbol{A}(\boldsymbol{u})=\boldsymbol{L}(\boldsymbol{u})+\boldsymbol{f}=\mathbf{0}\qquad \text{in }\Omega\\&\boldsymbol{B}(\boldsymbol{u})=\mathbf{0}\qquad \text{on }\Gamma\end{aligned} \tag{10.13}$$

和以上微分方程及边界条件等效的伽辽金提法可表示如下

$$\int_\Omega \delta\boldsymbol{u}\cdot[\boldsymbol{L}(\boldsymbol{u})+\boldsymbol{f}]d\Omega-\int_\Gamma \delta\boldsymbol{u}\cdot\boldsymbol{B}(\boldsymbol{u})d\Gamma=\mathbf{0} \tag{10.14}$$

分部积分

$$\begin{aligned}\int_\Omega \delta\boldsymbol{u}\cdot\boldsymbol{L}(\boldsymbol{u})d\Omega&=\int_\Omega \frac{1}{2}\delta\boldsymbol{u}\cdot\boldsymbol{L}(\boldsymbol{u})+\frac{1}{2}\delta\boldsymbol{u}\cdot\boldsymbol{L}(\boldsymbol{u})d\Omega\\&=\int_\Omega \frac{1}{2}\delta\boldsymbol{u}\cdot\boldsymbol{L}(\boldsymbol{u})+\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{L}(\boldsymbol{\delta u})d\Omega+b.t.(\delta\boldsymbol{u},\boldsymbol{u})\\&=\int_\Omega \frac{1}{2}\delta\boldsymbol{u}\cdot\boldsymbol{L}(\boldsymbol{u})+\frac{1}{2}\boldsymbol{u}\cdot\delta\boldsymbol{L}(\boldsymbol{u})d\Omega+b.t.(\delta\boldsymbol{u},\boldsymbol{u})\\&=\int_\Omega \frac{1}{2}\delta(\boldsymbol{u}\cdot\boldsymbol{L}(\boldsymbol{u}))d\Omega+b.t.(\delta\boldsymbol{u},\boldsymbol{u})\\&=\delta\int_\Omega \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{L}(\boldsymbol{u})d\Omega+b.t.(\delta\boldsymbol{u},\boldsymbol{u})\end{aligned} \tag{10.15}$$

令

$$\begin{aligned}\delta\Pi(\boldsymbol{u})&=\int_\Omega \delta\boldsymbol{u}\cdot[\boldsymbol{L}(\boldsymbol{u})+\boldsymbol{f}]d\Omega-\int_\Gamma \delta\boldsymbol{u}\cdot\boldsymbol{B}(\boldsymbol{u})d\Gamma\\&=\int_\Omega \delta\boldsymbol{u}\cdot\boldsymbol{L}(\boldsymbol{u})d\Omega+\int_\Omega \delta\boldsymbol{u}\cdot\boldsymbol{f}d\Omega-\int_\Gamma \delta\boldsymbol{u}\cdot\boldsymbol{B}(\boldsymbol{u})d\Gamma\\&=\delta\int_\Omega \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{L}(\boldsymbol{u})d\Omega+b.t.(\delta\boldsymbol{u},\boldsymbol{u})+\delta\int_\Omega \boldsymbol{u}\cdot\boldsymbol{f}d\Omega-\int_\Gamma \delta\boldsymbol{u}\cdot\boldsymbol{B}(\boldsymbol{u})d\Gamma\\&=\delta\int_\Omega \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{L}(\boldsymbol{u})+\boldsymbol{u}\cdot\boldsymbol{f}d\Omega+\delta b.t.(\boldsymbol{u})\\&=\delta\left[\int_\Omega \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{L}(\boldsymbol{u})+\boldsymbol{u}\cdot\boldsymbol{f}d\Omega+b.t.(\boldsymbol{u})\right]\\&=0\end{aligned} \tag{10.16}$$

所以原问题的泛函为

$$\Pi(\boldsymbol{u})=\int_\Omega \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{L}(\boldsymbol{u})+\boldsymbol{u}\cdot\boldsymbol{f}d\Omega+b.t.(\boldsymbol{u}) \tag{10.17}$$

若$\boldsymbol{L}$为$2m$阶线性自伴随的算子,可将微分阶数降为m阶

$$\begin{aligned}&\int_\Omega (-1)^m\boldsymbol{C}(\delta\boldsymbol{u})\cdot\boldsymbol{C}(\boldsymbol{u})d\Omega+b.t.(\delta\boldsymbol{u},\boldsymbol{u})=\mathbf{0}\\&\int_\Omega (-1)^m\delta\boldsymbol{C}(\boldsymbol{u})\cdot\boldsymbol{C}(\boldsymbol{u})d\Omega+b.t.(\delta\boldsymbol{u},\boldsymbol{u})=\mathbf{0}\\&\int_\Omega (-1)^m\delta[\frac{1}{2}\boldsymbol{C}(\boldsymbol{u})\cdot\boldsymbol{C}(\boldsymbol{u})]d\Omega+b.t.(\delta\boldsymbol{u},\boldsymbol{u})=\mathbf{0}\\&\delta\int_\Omega (-1)^m[\frac{1}{2}\boldsymbol{C}(\boldsymbol{u})\cdot\boldsymbol{C}(\boldsymbol{u})]d\Omega+b.t.(\delta\boldsymbol{u},\boldsymbol{u})=\mathbf{0}\end{aligned} \tag{10.18}$$

从而可得泛函

10.5 例子

$$\frac{d^2u}{dx^2}+u+x=0\qquad x\in[0,1] \tag{10.19}$$

$$u|_{x=0}=0$$
$$u|_{x=1}=0 \tag{10.20}$$

设满足边界条件的近似解为

$$\begin{aligned} u&=x(1-x)(a_1+a_2x+...)\\ &=a_1x(1-x) \qquad \text{取一项}\end{aligned} \tag{10.21}$$

余量为

$$R=A(\overline{u})=-2a_1+a_1(x-x^2)+x=a_1(-2+x-x^2)+x \tag{10.22}$$

采用伽辽金法，取近似函数作为权函数

$$w=x(1-x) \tag{10.23}$$

$$\int_0^1 wA(\overline{u})dx=\int_0^1 wRdx=0 \tag{10.24}$$

解得

$$a_1=\frac{5}{18} \tag{10.25}$$

所以，近似解为

$$u=\frac{5}{18}x(1-x) \tag{10.26}$$

10.6 变分原理与里兹法

L是线性、自伴随的微分算子

$$\int_\Omega \boldsymbol{L}(\boldsymbol{u})\boldsymbol{v}d\Omega=\int_\Omega \boldsymbol{u}\boldsymbol{L}(\boldsymbol{v})d\Omega+b.t.(\boldsymbol{u},\boldsymbol{v}) \tag{10.27}$$

其中，$b.t.(\boldsymbol{u},\boldsymbol{v})$ 表示在 Ω 的边界 $\varGamma$ 上由 $\boldsymbol{u}$ 和 $\boldsymbol{v}$ 及其导数组成的积分项。

原问题的微分方程和边界条件表达如下

$$\begin{aligned} &\boldsymbol{A}(\boldsymbol{u})=\boldsymbol{L}(\boldsymbol{u})+\boldsymbol{f}=\boldsymbol{0} \qquad \text{in } \Omega\\ &\boldsymbol{B}(\boldsymbol{u})=\boldsymbol{0} \qquad \text{on } \varGamma\end{aligned} \tag{10.28}$$

和以上微分方程及边界条件等效的伽辽金提法可表示如下

$$\int_\Omega \delta\boldsymbol{u}\cdot[\boldsymbol{L}(\boldsymbol{u})+\boldsymbol{f}]d\Omega-\int_\varGamma \delta\boldsymbol{u}\cdot\boldsymbol{B}(\boldsymbol{u})d\varGamma=\boldsymbol{0} \tag{10.29}$$

分部积分

$$\begin{aligned} \int_\Omega \delta\boldsymbol{u}\cdot\boldsymbol{L}(\boldsymbol{u})d\Omega&=\int_\Omega \frac{1}{2}\delta\boldsymbol{u}\cdot\boldsymbol{L}(\boldsymbol{u})+\frac{1}{2}\delta\boldsymbol{u}\cdot\boldsymbol{L}(\boldsymbol{u})d\Omega\\ &=\int_\Omega \frac{1}{2}\delta\boldsymbol{u}\cdot\boldsymbol{L}(\boldsymbol{u})+\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{L}(\boldsymbol{\delta u})d\Omega+b.t.(\delta\boldsymbol{u},\boldsymbol{u})\\ &=\int_\Omega \frac{1}{2}\delta\boldsymbol{u}\cdot\boldsymbol{L}(\boldsymbol{u})+\frac{1}{2}\boldsymbol{u}\cdot\delta\boldsymbol{L}(\boldsymbol{u})d\Omega+b.t.(\delta\boldsymbol{u},\boldsymbol{u})\\ &=\int_\Omega \frac{1}{2}\delta(\boldsymbol{u}\cdot\boldsymbol{L}(\boldsymbol{u}))d\Omega+b.t.(\delta\boldsymbol{u},\boldsymbol{u})\\ &=\delta\int_\Omega \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{L}(\boldsymbol{u})d\Omega+b.t.(\delta\boldsymbol{u},\boldsymbol{u})\end{aligned} \tag{10.30}$$

令

$$\begin{aligned} \delta\Pi(\boldsymbol{u})&=\int_\Omega \delta\boldsymbol{u}\cdot[\boldsymbol{L}(\boldsymbol{u})+\boldsymbol{f}]d\Omega-\int_\varGamma \delta\boldsymbol{u}\cdot\boldsymbol{B}(\boldsymbol{u})d\varGamma\\ &=\int_\Omega \delta\boldsymbol{u}\cdot\boldsymbol{L}(\boldsymbol{u})d\Omega+\int_\Omega \delta\boldsymbol{u}\cdot\boldsymbol{f}d\Omega-\int_\varGamma \delta\boldsymbol{u}\cdot\boldsymbol{B}(\boldsymbol{u})d\varGamma\\ &=\delta\int_\Omega \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{L}(\boldsymbol{u})d\Omega+b.t.(\delta\boldsymbol{u},\boldsymbol{u})+\delta\int_\Omega \boldsymbol{u}\cdot\boldsymbol{f}d\Omega-\int_\varGamma \delta\boldsymbol{u}\cdot\boldsymbol{B}(\boldsymbol{u})d\varGamma\\ &=\delta\int_\Omega \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{L}(\boldsymbol{u})+\boldsymbol{u}\cdot\boldsymbol{f}d\Omega+\delta b.t.(\boldsymbol{u})\\ &=\delta\left[\int_\Omega \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{L}(\boldsymbol{u})+\boldsymbol{u}\cdot\boldsymbol{f}d\Omega+b.t.(\boldsymbol{u})\right]\\ &=0\end{aligned} \tag{10.31}$$

所以原问题的泛函为

$$\Pi(\boldsymbol{u})=\int_\Omega \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{L}(\boldsymbol{u})+\boldsymbol{u}\cdot\boldsymbol{f}d\Omega+b.t.(\boldsymbol{u}) \tag{10.32}$$

10.7 拉格朗日乘子法

考虑离散结构模型稳态分析：[1]

$\boldsymbol{u}$ 为位移矢量，$\boldsymbol{K}$ 为二阶对称刚度张量，$\boldsymbol{R}$ 为载荷矢量，$\boldsymbol{B}$ 为二阶系数张量，$\boldsymbol{\lambda}$ 为乘子矢量。

$$\prod^{*}=\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{K}\cdot\boldsymbol{u}-\boldsymbol{u}\cdot\boldsymbol{R}+\boldsymbol{\lambda}\cdot(\boldsymbol{B}\cdot\boldsymbol{u}-\boldsymbol{u}^{*}) \tag{10.33}$$

求一阶变分得控制方程

$$\begin{aligned}\delta\prod^{*}&=\delta\boldsymbol{u}\cdot\boldsymbol{K}\cdot\boldsymbol{u}-\delta\boldsymbol{u}\cdot\boldsymbol{R}+\delta\boldsymbol{\lambda}\cdot(\boldsymbol{B}\cdot\boldsymbol{u}-\boldsymbol{u}^{*})+\boldsymbol{\lambda}\cdot\boldsymbol{B}\cdot\delta\boldsymbol{u}\\&=\delta\boldsymbol{u}\cdot\boldsymbol{K}\cdot\boldsymbol{u}-\delta\boldsymbol{u}\cdot\boldsymbol{R}+\delta\boldsymbol{\lambda}\cdot(\boldsymbol{B}\cdot\boldsymbol{u}-\boldsymbol{u}^{*})+\delta\boldsymbol{u}\cdot\boldsymbol{B}^{T}\cdot\boldsymbol{\lambda}\\&=\delta\boldsymbol{u}\cdot(\boldsymbol{K}\cdot\boldsymbol{u}-\boldsymbol{R}+\boldsymbol{B}^{T}\cdot\boldsymbol{\lambda})+\delta\boldsymbol{\lambda}\cdot(\boldsymbol{B}\cdot\boldsymbol{u}-\boldsymbol{u}^{*})\\&=0\end{aligned} \tag{10.34}$$

由变分的任意性

$$\begin{cases}\boldsymbol{K}\cdot\boldsymbol{u}+\boldsymbol{B}^{T}\cdot\boldsymbol{\lambda}=\boldsymbol{R}\\\boldsymbol{B}\cdot\boldsymbol{u}-\boldsymbol{u}^{*}=\boldsymbol{0}\end{cases}\rightarrow\begin{bmatrix}\boldsymbol{K}&\boldsymbol{B}^{T}\\\boldsymbol{B}&\boldsymbol{0}\end{bmatrix}\begin{bmatrix}\boldsymbol{u}\\\boldsymbol{\lambda}\end{bmatrix}=\begin{bmatrix}\boldsymbol{R}\\\boldsymbol{u}^{*}\end{bmatrix} \tag{10.35}$$

若给定位移约束 $\boldsymbol{u}=\boldsymbol{u}^{*}$，则 $\boldsymbol{R}=\boldsymbol{0}$，求得的 $\boldsymbol{\lambda}=-\boldsymbol{R}$ 为节点处施加位移约束所应的反作用力。

10.8 罚函数法

$$\prod^{**}=\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{K}\cdot\boldsymbol{u}-\boldsymbol{u}\cdot\boldsymbol{R}+\frac{\alpha}{2}(\boldsymbol{B}\cdot\boldsymbol{u}-\boldsymbol{u}^{*})\cdot(\boldsymbol{B}\cdot\boldsymbol{u}-\boldsymbol{u}^{*}) \tag{10.36}$$

$$\begin{aligned}\delta\prod^{**}&=\delta\boldsymbol{u}\cdot\boldsymbol{K}\cdot\boldsymbol{u}-\delta\boldsymbol{u}\cdot\boldsymbol{R}+\frac{\alpha}{2}(\boldsymbol{B}\cdot\delta\boldsymbol{u})\cdot(\boldsymbol{B}\cdot\boldsymbol{u}-\boldsymbol{u}^{*})+\frac{\alpha}{2}(\boldsymbol{B}\cdot\boldsymbol{u}-\boldsymbol{u}^{*})\cdot(\boldsymbol{B}\cdot\delta\boldsymbol{u})\\&=\delta\boldsymbol{u}\cdot\boldsymbol{K}\cdot\boldsymbol{u}-\delta\boldsymbol{u}\cdot\boldsymbol{R}+\frac{\alpha}{2}\left[(\boldsymbol{B}\cdot\boldsymbol{u}-\boldsymbol{u}^{*})\cdot\boldsymbol{B}\cdot\delta\boldsymbol{u}+(\boldsymbol{B}\cdot\boldsymbol{u}-\boldsymbol{u}^{*})\cdot\boldsymbol{B}\cdot\delta\boldsymbol{u}\right]\\&=\delta\boldsymbol{u}\cdot\boldsymbol{K}\cdot\boldsymbol{u}-\delta\boldsymbol{u}\cdot\boldsymbol{R}+\alpha(\boldsymbol{B}\cdot\boldsymbol{u}-\boldsymbol{u}^{*})\cdot\boldsymbol{B}\cdot\delta\boldsymbol{u}\\&=\delta\boldsymbol{u}\cdot\boldsymbol{K}\cdot\boldsymbol{u}-\delta\boldsymbol{u}\cdot\boldsymbol{R}+\alpha\delta\boldsymbol{u}\cdot\boldsymbol{B}^{T}\cdot(\boldsymbol{B}\cdot\boldsymbol{u}-\boldsymbol{u}^{*})\\&=\delta\boldsymbol{u}\cdot\boldsymbol{K}\cdot\boldsymbol{u}-\delta\boldsymbol{u}\cdot\boldsymbol{R}+\alpha\delta\boldsymbol{u}\cdot(\boldsymbol{B}^{T}\cdot\boldsymbol{B}\cdot\boldsymbol{u}-\boldsymbol{B}^{T}\cdot\boldsymbol{u}^{*})\\&=\delta\boldsymbol{u}\cdot(\boldsymbol{K}\cdot\boldsymbol{u}-\boldsymbol{R}+\alpha\boldsymbol{B}^{T}\cdot\boldsymbol{B}\cdot\boldsymbol{u}-\alpha\boldsymbol{B}^{T}\cdot\boldsymbol{u}^{*})\\&=0\end{aligned} \tag{10.37}$$

$$(\boldsymbol{K}+\alpha\boldsymbol{B}^{T}\cdot\boldsymbol{B})\cdot\boldsymbol{u}=\boldsymbol{R}+\alpha\boldsymbol{B}^{T}\cdot\boldsymbol{u}^{*} \tag{10.38}$$

[1]参考有限元法：理论、格式与求解方法. Finite Element Procedures. Klaus- Jurgen Bathe 著，轩建平译 3.4节

$$\boldsymbol{K}\cdot\boldsymbol{u}=K_{ij}\boldsymbol{e}_i\otimes\boldsymbol{j}\cdot u_k\boldsymbol{e}_k=K_{ij}u_k\delta_{jk}\boldsymbol{e}_i=K_{ij}u_j\boldsymbol{e}_i \tag{10.39}$$

$$\boldsymbol{u}\cdot\boldsymbol{K}=u_k\boldsymbol{e}_k\cdot K_{ij}\boldsymbol{e}_i\otimes\boldsymbol{e}_j=u_kK_{ij}\delta_{ki}\boldsymbol{e}_j=K_{ij}u_i\boldsymbol{e}_j \tag{10.40}$$

$$\boldsymbol{u}\cdot\boldsymbol{K}\cdot\boldsymbol{u}=\boldsymbol{u}\cdot K_{ij}\boldsymbol{e}_i\otimes\boldsymbol{e}_j\cdot u_k\boldsymbol{e}_k=\boldsymbol{u}\cdot K_{ij}u_k\delta_{jk}\boldsymbol{e}_i=\boldsymbol{u}\cdot K_{ij}u_j\boldsymbol{e}_i=u_k\boldsymbol{e}_k\cdot K_{ij}u_j\boldsymbol{e}_i=u_kK_{ij}u_j\delta_{ki}=u_iK_{ij}u_j \tag{10.41}$$

张量内乘表达式对矢量求导

$$\begin{aligned}\frac{\partial\boldsymbol{u}\cdot\boldsymbol{K}\cdot\boldsymbol{u}}{\partial\boldsymbol{u}}&=\frac{\partial u_iK_{ij}u_j}{\partial\boldsymbol{u}}=\frac{\partial u_iK_{ij}u_j}{\partial u_k}\boldsymbol{e}_k=K_{ij}\frac{\partial u_iu_j}{\partial u_k}\boldsymbol{e}_k=K_{ij}\left(u_i\frac{\partial u_j}{\partial u_k}+u_j\frac{\partial u_i}{\partial u_k}\right)\boldsymbol{e}_k\\&=K_{ij}\left(u_i\delta_{jk}+u_j\delta_{ik}\right)\boldsymbol{e}_k=K_{ij}u_i\boldsymbol{e}_j+K_{ij}u_j\boldsymbol{e}_i=K_{ji}u_j\boldsymbol{e}_i+K_{ij}u_j\boldsymbol{e}_i\\&=\boldsymbol{u}\cdot\boldsymbol{K}+\boldsymbol{K}\cdot\boldsymbol{u}=\boldsymbol{K}^T\cdot\boldsymbol{u}+\boldsymbol{K}\cdot\boldsymbol{u}=(\boldsymbol{K}^T+\boldsymbol{K})\cdot\boldsymbol{u}\\&=2K_{ij}u_j\boldsymbol{e}_i=2\boldsymbol{K}\cdot\boldsymbol{u}\qquad\text{if }\boldsymbol{K}\text{ is symmetric tensor}\\&=2K_{ij}u_i\boldsymbol{e}_j=2\boldsymbol{u}\cdot\boldsymbol{K}\qquad\text{if }\boldsymbol{K}\text{ is symmetric tensor}\end{aligned} \tag{10.42}$$

$$\boldsymbol{A}\cdot\boldsymbol{B}=A_{ij}\boldsymbol{e}_i\otimes\boldsymbol{e}_j\cdot B_{kl}\boldsymbol{e}_k\otimes\boldsymbol{e}_l=A_{ij}B_{kl}\delta_{jk}\boldsymbol{e}_i\otimes\boldsymbol{e}_l=A_{ik}B_{kl}\boldsymbol{e}_i\otimes\boldsymbol{e}_l=A_{ik}B_{kj}\boldsymbol{e}_i\otimes\boldsymbol{e}_j \tag{10.43}$$

$$\boldsymbol{B}\cdot\boldsymbol{A}=B_{kl}\boldsymbol{e}_k\otimes\boldsymbol{e}_l\cdot A_{ij}\boldsymbol{e}_i\otimes\boldsymbol{e}_j=A_{ij}B_{kl}\delta_{li}\boldsymbol{e}_k\otimes\boldsymbol{e}_j=A_{ij}B_{ki}\boldsymbol{e}_k\otimes\boldsymbol{e}_j=A_{kj}B_{ik}\boldsymbol{e}_i\otimes\boldsymbol{e}_j \tag{10.44}$$

$$\boldsymbol{B}\cdot\boldsymbol{A}^T=B_{kl}\boldsymbol{e}_k\otimes\boldsymbol{e}_l\cdot A_{ji}\boldsymbol{e}_i\otimes\boldsymbol{e}_j=A_{ji}B_{kl}\delta_{li}\boldsymbol{e}_k\otimes\boldsymbol{e}_j=A_{ji}B_{ki}\boldsymbol{e}_k\otimes\boldsymbol{e}_j=A_{jk}B_{ik}\boldsymbol{e}_i\otimes\boldsymbol{e}_j \tag{10.45}$$

$$\boldsymbol{B}^T\cdot\boldsymbol{A}=B_{lk}\boldsymbol{e}_k\otimes\boldsymbol{e}_l\cdot A_{ij}\boldsymbol{e}_i\otimes\boldsymbol{e}_j=A_{ij}B_{lk}\delta_{li}\boldsymbol{e}_k\otimes\boldsymbol{e}_j=A_{ij}B_{ik}\boldsymbol{e}_k\otimes\boldsymbol{e}_j=A_{kj}B_{ki}\boldsymbol{e}_i\otimes\boldsymbol{e}_j \tag{10.46}$$

$$\boldsymbol{B}^T\cdot\boldsymbol{A}^T=B_{lk}\boldsymbol{e}_k\otimes\boldsymbol{e}_l\cdot A_{ji}\boldsymbol{e}_i\otimes\boldsymbol{e}_j=A_{ji}B_{lk}\delta_{li}\boldsymbol{e}_k\otimes\boldsymbol{e}_j=A_{ji}B_{ik}\boldsymbol{e}_k\otimes\boldsymbol{e}_j=A_{jk}B_{ki}\boldsymbol{e}_i\otimes\boldsymbol{e}_j \tag{10.47}$$

结果：两个二阶张量的内乘不满足交换律，且

$$(\boldsymbol{A}\cdot\boldsymbol{B})^T=\boldsymbol{B}^T\cdot\boldsymbol{A}^T \tag{10.48}$$

Chapter 11

哈密顿原理推导

11.1 哈密顿原理推导–我喜欢的表示方法

$\Box^m$ denotes a m-order tensor.

$\Box \cdot \Box$ denotes contraction of two any-order tensors.

$\nabla \cdot (\Box^m \cdot \Box^{m-1}) = (\nabla \cdot \Box^m) \cdot \Box^{m-1} + \Box^m \cdot (\nabla \otimes \Box^{m-1})$ denates the chain rule.

$\nabla \otimes \Box^1 = (\boldsymbol{n} \otimes \boldsymbol{n}) \cdot \nabla \otimes \Box^1 + (\boldsymbol{I} - \boldsymbol{n} \otimes \boldsymbol{n}) \cdot \nabla \otimes \Box^1 = \boldsymbol{n} \otimes \nabla^{\perp}\Box^1 + \nabla^{\parallel} \otimes \Box^1$ denotes the surface gradient decomposition, where $\nabla^{\perp}$ is a normal gradient scalar operator, $\nabla^{\parallel}$ is the tangential gradient vecotr operator and the chain rule of ∇ is applicable to $\boldsymbol{n} \otimes \nabla^{\perp}$ and $\nabla^{\parallel}$. The "$\otimes$" behind the gradient vecotr operator could be omitted.

$\int_v \nabla \cdot \Box = \int_s \boldsymbol{n} \cdot \Box$ denotes the divergence theorem.

$\int_s \nabla^{\parallel} \cdot \Box^1 = \int_s (\nabla^{\parallel} \cdot \boldsymbol{n})\boldsymbol{n} \cdot \Box^1$ denotes the surface divergence theorem for a smooth closed surface.

预备推导一：

$$\begin{aligned} \Box^3 \cdot \nabla\nabla\Box^1 &= \nabla \cdot (\Box^3 \cdot \nabla\Box^1) - (\nabla \cdot \Box^3) \cdot \nabla\Box^1 \\ &= \nabla \cdot (\Box^3 \cdot \nabla\Box^1) - \nabla \cdot \left[(\nabla \cdot \Box^3) \cdot \Box^1\right] + \left[\nabla \cdot (\nabla \cdot \Box^3)\right] \cdot \Box^1 \end{aligned} \tag{11.1}$$

Let $\Box^1 \to \delta\boldsymbol{u}$

$$\Box^3 \cdot \nabla\nabla\delta\boldsymbol{u} = \nabla \cdot (\Box^3 \cdot \nabla\delta\boldsymbol{u}) - \nabla \cdot \left[(\nabla \cdot \Box^3) \cdot \delta\boldsymbol{u}\right] + \left[\nabla \cdot (\nabla \cdot \Box^3)\right] \cdot \delta\boldsymbol{u} \tag{11.2}$$

预备推导二：

$$\begin{aligned} \int_s \Box^2 \cdot \nabla\Box^1 &= \int_s \Box^2 \cdot (\boldsymbol{n} \otimes \nabla^{\perp}\Box^1 + \nabla^{\parallel}\Box^1) \\ &= \int_s \boldsymbol{n} \cdot \Box^2 \cdot \nabla^{\perp}\Box^1 + \Box^2 \cdot \nabla^{\parallel}\Box^1 \\ &= \int_s \boldsymbol{n} \cdot \Box^2 \cdot \nabla^{\perp}\Box^1 + \nabla^{\parallel} \cdot (\Box^2 \cdot \Box^1) - (\nabla^{\parallel} \cdot \Box^2) \cdot \Box^1 \\ &\text{表面散度定理(光滑闭合表面)} \\ &= \int_s \boldsymbol{n} \cdot \Box^2 \cdot \nabla^{\perp}\Box^1 + (\nabla^{\parallel} \cdot \boldsymbol{n})\boldsymbol{n} \cdot (\Box^2 \cdot \Box^1) - (\nabla^{\parallel} \cdot \Box^2) \cdot \Box^1 \\ &= \int_s \boldsymbol{n} \cdot \Box^2 \cdot \nabla^{\perp}\Box^1 + (\nabla^{\parallel} \cdot \boldsymbol{n})\boldsymbol{n} \cdot \Box^2 \cdot \Box^1 - (\nabla^{\parallel} \cdot \Box^2) \cdot \Box^1 \\ &= \int_s \boldsymbol{n} \cdot \Box^2 \cdot \nabla^{\perp}\Box^1 + \left[(\nabla^{\parallel} \cdot \boldsymbol{n})\boldsymbol{n} \cdot \Box^2 - \nabla^{\parallel} \cdot \Box^2\right] \cdot \Box^1 \end{aligned} \tag{11.3}$$

Let $\Box^1 \to \delta\boldsymbol{u}, \Box^2 \to \boldsymbol{n} \cdot \boldsymbol{\mu}$

$$\int_s (\boldsymbol{n} \cdot \boldsymbol{\mu}) \cdot \nabla\boldsymbol{u} = \int_s \boldsymbol{n} \cdot (\boldsymbol{n} \cdot \boldsymbol{\mu}) \cdot \nabla^{\perp}\boldsymbol{u} + \left[(\nabla^{\parallel} \cdot \boldsymbol{n})\boldsymbol{n} \cdot (\boldsymbol{n} \cdot \boldsymbol{\mu}) - \nabla^{\parallel} \cdot (\boldsymbol{n} \cdot \boldsymbol{\mu})\right] \cdot \boldsymbol{u} \tag{11.4}$$

哈密顿原理变分推导：

$$
\begin{aligned}
\delta w = \delta w(\nabla \boldsymbol{u}, \nabla\nabla \boldsymbol{u}) &= \int_v \boldsymbol{\sigma} \cdot \nabla \delta \boldsymbol{u} + \boldsymbol{\mu} \cdot \nabla\nabla \delta \boldsymbol{u} \\
&\quad \text{微分链式法则} \\
&= \int_v \nabla \cdot (\boldsymbol{\sigma} \cdot \delta \boldsymbol{u}) - (\nabla \cdot \boldsymbol{\sigma}) \cdot \delta \boldsymbol{u} + \nabla \cdot (\boldsymbol{\mu} \cdot \nabla \delta \boldsymbol{u}) - \nabla \cdot [(\nabla \cdot \boldsymbol{\mu}) \cdot \delta \boldsymbol{u}] + [\nabla \cdot (\nabla \cdot \boldsymbol{\mu})] \cdot \delta \boldsymbol{u} \\
&\quad \text{合并同类项+散度定理} \\
&= \int_v \nabla \cdot (\nabla \cdot \boldsymbol{\mu} - \boldsymbol{\sigma}) \cdot \delta \boldsymbol{u} + \int_s \boldsymbol{n} \cdot (\boldsymbol{\sigma} \cdot \delta \boldsymbol{u}) + \boldsymbol{n} \cdot (\boldsymbol{\mu} \cdot \nabla \delta \boldsymbol{u}) - \boldsymbol{n} \cdot [(\nabla \cdot \boldsymbol{\mu}) \cdot \delta \boldsymbol{u}] \\
&= \int_v \nabla \cdot (\nabla \cdot \boldsymbol{\mu} - \boldsymbol{\sigma}) \cdot \delta \boldsymbol{u} + \int_s \boldsymbol{n} \cdot \boldsymbol{\sigma} \cdot \delta \boldsymbol{u} + \boldsymbol{n} \cdot \boldsymbol{\mu} \cdot \nabla \delta \boldsymbol{u} - \boldsymbol{n} \cdot (\nabla \cdot \boldsymbol{\mu}) \cdot \delta \boldsymbol{u} \\
&\quad \text{合并同类项} \\
&= \int_v \nabla \cdot (\nabla \cdot \boldsymbol{\mu} - \boldsymbol{\sigma}) \cdot \delta \boldsymbol{u} + \int_s \boldsymbol{n} \cdot (\boldsymbol{\sigma} - \nabla \cdot \boldsymbol{\mu}) \cdot \delta \boldsymbol{u} + \boldsymbol{n} \cdot \boldsymbol{\mu} \cdot \nabla \delta \boldsymbol{u}
\end{aligned}
\tag{11.5}
$$

处理方法一(表面梯度分解+表面散度定理)：

$$
\int_s \boldsymbol{n} \cdot \boldsymbol{\mu} \cdot \nabla \delta \boldsymbol{u} = \int_s \boldsymbol{n} \cdot (\boldsymbol{n} \cdot \boldsymbol{\mu}) \cdot \nabla^{\perp} \delta \boldsymbol{u} + \left[(\nabla^{\parallel} \cdot \boldsymbol{n}) \boldsymbol{n} \cdot (\boldsymbol{n} \cdot \boldsymbol{\mu}) - \nabla^{\parallel} \cdot (\boldsymbol{n} \cdot \boldsymbol{\mu}) \right] \cdot \delta \boldsymbol{u} \tag{11.6}
$$

然后，

$$
\begin{aligned}
\delta w = &\int_v \nabla \cdot (\nabla \cdot \boldsymbol{\mu} - \boldsymbol{\sigma}) \cdot \delta \boldsymbol{u} \\
&+ \int_s \left[\boldsymbol{n} \cdot (\boldsymbol{\sigma} - \nabla \cdot \boldsymbol{\mu}) + (\nabla^{\parallel} \cdot \boldsymbol{n}) \boldsymbol{n} \cdot (\boldsymbol{n} \cdot \boldsymbol{\mu}) - \nabla^{\parallel} \cdot (\boldsymbol{n} \cdot \boldsymbol{\mu}) \right] \cdot \delta \boldsymbol{u} + \boldsymbol{n} \cdot (\boldsymbol{n} \cdot \boldsymbol{\mu}) \cdot \nabla^{\perp} \delta \boldsymbol{u}
\end{aligned}
\tag{11.7}
$$

处理方法二(链式法则)：

$$
\int_s \boldsymbol{n} \cdot \boldsymbol{\mu} \cdot \nabla \delta \boldsymbol{u} = \int_s \nabla \cdot (\boldsymbol{n} \cdot \boldsymbol{\mu} \cdot \delta \boldsymbol{u}) - [\nabla \cdot (\boldsymbol{n} \cdot \boldsymbol{\mu})] \cdot \delta \boldsymbol{u} \tag{11.8}
$$

然后，

$$
\begin{aligned}
\delta w = &\int_v \nabla \cdot (\nabla \cdot \boldsymbol{\mu} - \boldsymbol{\sigma}) \cdot \delta \boldsymbol{u} \\
&+ \int_s [\boldsymbol{n} \cdot (\boldsymbol{\sigma} - \nabla \cdot \boldsymbol{\mu}) - \nabla \cdot (\boldsymbol{n} \cdot \boldsymbol{\mu})] \cdot \delta \boldsymbol{u} + \int_s \nabla \cdot (\boldsymbol{n} \cdot \boldsymbol{\mu} \cdot \delta \boldsymbol{u})
\end{aligned}
\tag{11.9}
$$

11.2 哈密顿原理推导–常规的表示方法

$\square^m$ denotes a m-order tensor.

$\nabla\cdot(\square^m\cdot\square^{m-1})=(\nabla\cdot\square^m)\cdot\square^{m-1}+\square^m\cdot(\nabla\otimes\square^{m-1})$ denates the chain rule.

$\nabla\otimes\square^1=(\boldsymbol{n}\otimes\boldsymbol{n})\cdot\nabla\otimes\square^1+(\boldsymbol{I}-\boldsymbol{n}\otimes\boldsymbol{n})\cdot\nabla\otimes\square^1=\boldsymbol{n}\otimes\nabla^{\perp}\square^1+\nabla^{\parallel}\otimes\square^1$ denotes the surface gradient decomposition, where $\nabla^{\perp}$ is a normal gradient scalar operator, $\nabla^{\parallel}$ is the tangential gradient vecotr operator and the chain rule of ∇ is applicable to $\boldsymbol{n}\otimes\nabla^{\perp}$ and $\nabla^{\parallel}$. The "$\otimes$" behind the gradient vecotr operator could be omitted.

$\int_v\nabla\cdot\square=\int_s\boldsymbol{n}\cdot\square$ denotes the divergence theorem.

$\int_s\nabla^{\parallel}\cdot\square^1=\int_s(\nabla^{\parallel}\cdot\boldsymbol{n})\boldsymbol{n}\cdot\square^1$ denotes the surface divergence theorem for a smooth closed surface.

$\boldsymbol{a}$ is a vecotr, $\boldsymbol{A}$ is a second-order tensor, $\boldsymbol{\kappa}$ is third-order tensor.

预备推导一:

$$
\begin{aligned}
\boldsymbol{\kappa}\,\vdots\,\nabla\nabla\boldsymbol{a}&=\nabla\cdot(\boldsymbol{\kappa}:\nabla\boldsymbol{a})-(\nabla\cdot\boldsymbol{\kappa}):\nabla\boldsymbol{a}\\
&=\nabla\cdot(\boldsymbol{\kappa}:\nabla\boldsymbol{a})-\nabla\cdot[(\nabla\cdot\boldsymbol{\kappa})\cdot\boldsymbol{a}]+[\nabla\cdot(\nabla\cdot\boldsymbol{\kappa})]\cdot\boldsymbol{a}
\end{aligned}
\tag{11.10}
$$

Let $\boldsymbol{a}\to\delta\boldsymbol{u},\boldsymbol{\kappa}\to\boldsymbol{\mu}$

$$
\boldsymbol{\mu}\,\vdots\,\nabla\nabla\delta\boldsymbol{u}=\nabla\cdot(\boldsymbol{\mu}:\nabla\delta\boldsymbol{u})-\nabla\cdot[(\nabla\cdot\boldsymbol{\mu})\cdot\delta\boldsymbol{u}]+[\nabla\cdot(\nabla\cdot\boldsymbol{\mu})]\cdot\delta\boldsymbol{u}
\tag{11.11}
$$

预备推导二:

$$
\begin{aligned}
\int_s\boldsymbol{A}:\nabla\boldsymbol{a}&=\int_s\boldsymbol{A}:(\boldsymbol{n}\otimes\nabla^{\perp}\boldsymbol{a}+\nabla^{\parallel}\boldsymbol{a})\\
&=\int_s\boldsymbol{n}\cdot\boldsymbol{A}\cdot\nabla^{\perp}\boldsymbol{a}+\boldsymbol{A}:\nabla^{\parallel}\boldsymbol{a}\\
&=\int_s\boldsymbol{n}\cdot\boldsymbol{A}\cdot\nabla^{\perp}\boldsymbol{a}+\nabla^{\parallel}\cdot(\boldsymbol{A}\cdot\boldsymbol{a})-(\nabla^{\parallel}\cdot\square^2)\cdot\boldsymbol{a}\\
&\text{表面散度定理(光滑闭合表面)}\\
&=\int_s\boldsymbol{n}\cdot\boldsymbol{A}\cdot\nabla^{\perp}\boldsymbol{a}+(\nabla^{\parallel}\cdot\boldsymbol{n})\boldsymbol{n}\cdot(\boldsymbol{A}\cdot\boldsymbol{a})-(\nabla^{\parallel}\cdot\boldsymbol{A})\cdot\boldsymbol{a}\\
&=\int_s\boldsymbol{n}\cdot\boldsymbol{A}\cdot\nabla^{\perp}\boldsymbol{a}+(\nabla^{\parallel}\cdot\boldsymbol{n})\boldsymbol{n}\cdot\boldsymbol{A}\cdot\boldsymbol{a}-(\nabla^{\parallel}\cdot\boldsymbol{A})\cdot\boldsymbol{a}\\
&=\int_s\boldsymbol{n}\cdot\boldsymbol{A}\cdot\nabla^{\perp}\boldsymbol{a}+\left[(\nabla^{\parallel}\cdot\boldsymbol{n})\boldsymbol{n}\cdot\boldsymbol{A}-\nabla^{\parallel}\cdot\boldsymbol{A}\right]\cdot\boldsymbol{a}
\end{aligned}
\tag{11.12}
$$

Let $\boldsymbol{a}\to\delta\boldsymbol{u},\boldsymbol{A}\to\boldsymbol{n}\cdot\boldsymbol{\mu}$

$$
\int_s(\boldsymbol{n}\cdot\boldsymbol{\mu}):\nabla\delta\boldsymbol{u}=\int_s\boldsymbol{n}\cdot(\boldsymbol{n}\cdot\boldsymbol{\mu})\cdot\nabla^{\perp}\delta\boldsymbol{u}+\left[(\nabla^{\parallel}\cdot\boldsymbol{n})\boldsymbol{n}\cdot(\boldsymbol{n}\cdot\boldsymbol{\mu})-\nabla^{\parallel}\cdot(\boldsymbol{n}\cdot\boldsymbol{\mu})\right]\cdot\delta\boldsymbol{u}
\tag{11.13}
$$

哈密顿原理变分推导:

$$
\begin{aligned}
\delta w=\delta w(\nabla\boldsymbol{u},\nabla\nabla\boldsymbol{u})&=\int_v\boldsymbol{\sigma}:\nabla\delta\boldsymbol{u}+\boldsymbol{\mu}\,\vdots\,\nabla\nabla\delta\boldsymbol{u}\\
&\text{微分链式法则}\\
&=\int_v\nabla\cdot(\boldsymbol{\sigma}\cdot\delta\boldsymbol{u})-(\nabla\cdot\boldsymbol{\sigma})\cdot\delta\boldsymbol{u}+\nabla\cdot(\boldsymbol{\mu}:\nabla\delta\boldsymbol{u})-\nabla\cdot[(\nabla\cdot\boldsymbol{\mu})\cdot\delta\boldsymbol{u}]+[\nabla\cdot(\nabla\cdot\boldsymbol{\mu})]\cdot\delta\boldsymbol{u}\\
&\text{合并同类项+散度定理}\\
&=\int_v[\nabla\cdot(\nabla\cdot\boldsymbol{\mu}-\boldsymbol{\sigma})]\cdot\delta\boldsymbol{u}+\int_s\boldsymbol{n}\cdot(\boldsymbol{\sigma}\cdot\delta\boldsymbol{u})+\boldsymbol{n}\cdot(\boldsymbol{\mu}:\nabla\delta\boldsymbol{u})-\boldsymbol{n}\cdot[(\nabla\cdot\boldsymbol{\mu})\cdot\delta\boldsymbol{u}]\\
&=\int_v[\nabla\cdot(\nabla\cdot\boldsymbol{\mu}-\boldsymbol{\sigma})]\cdot\delta\boldsymbol{u}+\int_s\boldsymbol{n}\cdot\boldsymbol{\sigma}\cdot\delta\boldsymbol{u}+\boldsymbol{n}\cdot\boldsymbol{\mu}:\nabla\delta\boldsymbol{u}-\boldsymbol{n}\cdot(\nabla\cdot\boldsymbol{\mu})\cdot\delta\boldsymbol{u}\\
&\text{合并同类项}\\
&=\int_v[\nabla\cdot(\nabla\cdot\boldsymbol{\mu}-\boldsymbol{\sigma})]\cdot\delta\boldsymbol{u}+\int_s[\boldsymbol{n}\cdot(\boldsymbol{\sigma}-\nabla\cdot\boldsymbol{\mu})]\cdot\delta\boldsymbol{u}+(\boldsymbol{n}\cdot\boldsymbol{\mu}):\nabla\delta\boldsymbol{u}
\end{aligned}
\tag{11.14}
$$

处理方法一(表面梯度分解+表面散度定理):

$$
\int_s(\boldsymbol{n}\cdot\boldsymbol{\mu}):\nabla\delta\boldsymbol{u}=\int_s\boldsymbol{n}\cdot(\boldsymbol{n}\cdot\boldsymbol{\mu})\cdot\nabla^{\perp}\delta\boldsymbol{u}+\left[(\nabla^{\parallel}\cdot\boldsymbol{n})\boldsymbol{n}\cdot(\boldsymbol{n}\cdot\boldsymbol{\mu})-\nabla^{\parallel}\cdot(\boldsymbol{n}\cdot\boldsymbol{\mu})\right]\cdot\delta\boldsymbol{u}
\tag{11.15}
$$

然后，

$$\begin{aligned}\delta w = &\int_v \nabla\cdot(\nabla\cdot\boldsymbol{\mu}-\boldsymbol{\sigma})\cdot\delta\boldsymbol{u}\\ &+\int_s \left[\boldsymbol{n}\cdot(\boldsymbol{\sigma}-\nabla\cdot\boldsymbol{\mu})+(\nabla^{\parallel}\cdot\boldsymbol{n})\boldsymbol{n}\cdot(\boldsymbol{n}\cdot\boldsymbol{\mu})-\nabla^{\parallel}\cdot(\boldsymbol{n}\cdot\boldsymbol{\mu})\right]\cdot\delta\boldsymbol{u}+\boldsymbol{n}\cdot(\boldsymbol{n}\cdot\boldsymbol{\mu})\cdot\nabla^{\perp}\delta\boldsymbol{u}\end{aligned}\tag{11.16}$$

处理方法二(链式法则)：

$$\int_s(\boldsymbol{n}\cdot\boldsymbol{\mu})\cdot\nabla\delta\boldsymbol{u}=\int_s\nabla\cdot[(\boldsymbol{n}\cdot\boldsymbol{\mu})\cdot\delta\boldsymbol{u}]-[\nabla\cdot(\boldsymbol{n}\cdot\boldsymbol{\mu})]\cdot\delta\boldsymbol{u}\tag{11.17}$$

然后，

$$\begin{aligned}\delta w = &\int_v \nabla\cdot(\nabla\cdot\boldsymbol{\mu}-\boldsymbol{\sigma})\cdot\delta\boldsymbol{u}\\ &+\int_s[\boldsymbol{n}\cdot(\boldsymbol{\sigma}-\nabla\cdot\boldsymbol{\mu})-\nabla\cdot(\boldsymbol{n}\cdot\boldsymbol{\mu})]\cdot\delta\boldsymbol{u}+\int_s\nabla\cdot[(\boldsymbol{n}\cdot\boldsymbol{\mu})\cdot\delta\boldsymbol{u}]\end{aligned}\tag{11.18}$$

11.3 surface divergence theorem

stokes theorem

$$\int_S\boldsymbol{n}\cdot(\nabla\times\boldsymbol{G})=\oint_{\partial S}\boldsymbol{t}\cdot\boldsymbol{G}\tag{11.19}$$

Let

$$\boldsymbol{G}=\boldsymbol{n}\times\boldsymbol{F}\tag{11.20}$$

then

$$\int_S\boldsymbol{n}\cdot[\nabla\times(\boldsymbol{n}\times\boldsymbol{F})]=\oint_{\partial S}\boldsymbol{t}\cdot(\boldsymbol{n}\times\boldsymbol{F})=\oint_{\partial S}(\boldsymbol{t}\times\boldsymbol{n})\cdot\boldsymbol{F}\tag{11.21}$$

$$\begin{aligned}\boldsymbol{n}\cdot[\nabla\times(\boldsymbol{n}\times\boldsymbol{F})]&=\boldsymbol{n}\cdot[\nabla\times(n_iF_je_{ijk}\boldsymbol{e}_k)]\\&=\boldsymbol{n}\cdot[\boldsymbol{e}_m\times\partial_m(n_iF_je_{ijk})\boldsymbol{e}_k]\\&=\boldsymbol{n}\cdot e_{mkn}\partial_m(n_iF_je_{ijk})\boldsymbol{e}_n\\&=e_{mkn}n_n\partial_m(n_iF_je_{ijk})\\&=e_{knm}e_{kij}n_n\partial_m(n_iF_j)\\&=(\delta_{ni}\delta_{mj}-\delta_{nj}\delta_{mi})n_n\partial_m(n_iF_j)\\&=n_i\partial_j(n_iF_j)-n_j\partial_i(n_iF_j)\\&=\partial_jF_j+n_iF_j\partial_jn_i-n_in_j\partial_iF_j-n_jF_j\partial_in_i\\&=\partial_jF_j-n_in_j\partial_iF_j-n_jF_j\partial_in_i\\&=\nabla\cdot\boldsymbol{F}-(\nabla\cdot\boldsymbol{n})\boldsymbol{n}\cdot\boldsymbol{F}-(\boldsymbol{n}\cdot\nabla\boldsymbol{F})\cdot\boldsymbol{n}\\ \partial_j&=n_jn_k\partial_k+\nabla_j^{\parallel}\quad\text{or}\quad\nabla=(\boldsymbol{n}\otimes\boldsymbol{n})\cdot\nabla+(\boldsymbol{I}-\boldsymbol{n}\otimes\boldsymbol{n})\cdot\nabla\\&=n_jn_k\partial_kF_j+\nabla_j^{\parallel}F_j-n_in_jn_in_k\partial_kF_j-n_in_j\nabla_i^{\parallel}F_j-n_jF_jn_in_k\partial_kn_i-n_jF_j\nabla_i^{\parallel}n_i\\&=n_jn_k\partial_kF_j+\nabla_j^{\parallel}F_j-n_jn_k\partial_kF_j-n_in_j\nabla_i^{\parallel}F_j-n_jF_jn_in_k\partial_kn_i-n_jF_j\nabla_i^{\parallel}n_i\\&=\nabla_j^{\parallel}F_j-n_jF_j\nabla_i^{\parallel}n_i\\&=\nabla^{\parallel}\cdot\boldsymbol{F}-(\nabla^{\parallel}\cdot\boldsymbol{n})\boldsymbol{n}\cdot\boldsymbol{F}\end{aligned}\tag{11.22}$$

$n_in_j\nabla_i^{\parallel}F_j$可以被证明。

$$\begin{aligned}n_in_j\nabla_i^{\parallel}F_j&=(\boldsymbol{n}\otimes\boldsymbol{n}):\nabla^{\parallel}\boldsymbol{F}\\&=\boldsymbol{n}\cdot\nabla^{\parallel}\boldsymbol{F}\cdot\boldsymbol{n}\\&=\boldsymbol{n}\cdot[(\boldsymbol{I}-\boldsymbol{n}\otimes\boldsymbol{n})\cdot(\nabla\boldsymbol{F})]\cdot\boldsymbol{n}\\&=[\boldsymbol{n}\cdot(\boldsymbol{I}-\boldsymbol{n}\otimes\boldsymbol{n})]\cdot(\nabla\boldsymbol{F})\cdot\boldsymbol{n}\\&=\boldsymbol{0}\end{aligned}\tag{11.23}$$

$n_iF_j\partial_jn_i=0, n_kn_jF_jn_i\partial_kn_i=0$. 因为$\boldsymbol{F}$是任意的，为什么$n_i\partial_jn_i=0$ or $(\nabla\boldsymbol{n})\cdot\boldsymbol{n}=0$

then

$$\int_S\nabla^{\parallel}\cdot\boldsymbol{F}-(\nabla^{\parallel}\cdot\boldsymbol{n})\boldsymbol{n}\cdot\boldsymbol{F}=\oint_{\partial S}(\boldsymbol{t}\times\boldsymbol{n})\cdot\boldsymbol{F}\tag{11.24}$$

for smooth closed surface

$$\int_S\nabla^{\parallel}\cdot\boldsymbol{F}-(\nabla^{\parallel}\cdot\boldsymbol{n})\boldsymbol{n}\cdot\boldsymbol{F}=0\tag{11.25}$$

that is

$$\nabla^{\parallel} \cdot \boldsymbol{F} - (\nabla^{\parallel} \cdot \boldsymbol{n})\boldsymbol{n} \cdot \boldsymbol{F} = 0 \tag{11.26}$$

Chapter 12

Vector and tensor analysis

12.1 Alternative Form of Transformation

If we replace $\boldsymbol{f}$ by $\boldsymbol{ng}$ in (100.4), where $\boldsymbol{g}$ is any continuously differentiable tensor, we have, from (97.13),

$$\int Grad\boldsymbol{g} - (Div\boldsymbol{n})\boldsymbol{ng}dS = \oint \boldsymbol{T} \times \boldsymbol{ng}ds \tag{12.1}$$

12.2 reciprocal bases

u^i is contravariant component.

u_i is covariant components.

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_i^j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{12.2}$$

If $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ and $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ form reciprocal sets,

$$\boldsymbol{a} \otimes \boldsymbol{u} + \boldsymbol{b} \otimes \boldsymbol{v} + \boldsymbol{c} \otimes \boldsymbol{w} = \boldsymbol{I} \tag{12.3}$$

some reciprocal sets

$$\begin{aligned} &\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{n} && \boldsymbol{r}_u, \boldsymbol{r}_v, \boldsymbol{n} \\ &\nabla u, \nabla v, \nabla w && \boldsymbol{r}_u, \boldsymbol{r}_v, \boldsymbol{r}_w \\ &\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3 && \boldsymbol{e}^1, \boldsymbol{e}^2, \boldsymbol{e}^3 \end{aligned} \tag{12.4}$$

12.3 Gradient of a tensor

$$\frac{d\boldsymbol{f}}{ds} = \lim_{s\to 0}\frac{\boldsymbol{f}(\boldsymbol{r}_1 + s\boldsymbol{e}) - \boldsymbol{f}(\boldsymbol{r}_1)}{s} \qquad \text{definition} \tag{12.5}$$

If $\boldsymbol{f}(\boldsymbol{r})$ is given as a function $\boldsymbol{f}(x,y,z)$ of rectangular coordinates,

$$\frac{d\boldsymbol{f}}{ds} = \frac{\partial \boldsymbol{f}}{\partial x}\frac{dx}{ds} + \frac{\partial \boldsymbol{f}}{\partial y}\frac{dy}{ds} + \frac{\partial \boldsymbol{f}}{\partial z}\frac{dz}{ds} = \boldsymbol{f}_x\frac{dx}{ds} + \boldsymbol{f}_y\frac{dy}{ds} + \boldsymbol{f}_z\frac{dz}{ds} \qquad \text{Chain rule} \tag{12.6}$$

Since

$$\frac{dx}{ds} = \boldsymbol{e}\cdot\boldsymbol{e}_x, \quad \frac{dy}{ds} = \boldsymbol{e}\cdot\boldsymbol{e}_y, \quad \frac{dz}{ds} = \boldsymbol{e}\cdot\boldsymbol{e}_z \qquad \text{geometric projection} \tag{12.7}$$

then

$$\begin{aligned}\frac{d\boldsymbol{f}}{ds} &= \frac{\partial \boldsymbol{f}}{\partial x}(\boldsymbol{e}\cdot\boldsymbol{e}_x) + \frac{\partial \boldsymbol{f}}{\partial y}(\boldsymbol{e}\cdot\boldsymbol{e}_y) + \frac{\partial \boldsymbol{f}}{\partial z}(\boldsymbol{e}\cdot\boldsymbol{e}_z)\\ &= \boldsymbol{e}\cdot(\boldsymbol{e}_x\otimes\frac{\partial \boldsymbol{f}}{\partial x}) + \boldsymbol{e}\cdot(\boldsymbol{e}_y\otimes\frac{\partial \boldsymbol{f}}{\partial y}) + \boldsymbol{e}\cdot(\boldsymbol{e}_z\otimes\frac{\partial \boldsymbol{f}}{\partial z})\\ &= \boldsymbol{e}\cdot\left[\boldsymbol{e}_x\otimes\frac{\partial \boldsymbol{f}}{\partial x} + \boldsymbol{e}_y\otimes\frac{\partial \boldsymbol{f}}{\partial y} + \boldsymbol{e}_z\otimes\frac{\partial \boldsymbol{f}}{\partial z}\right]\\ &= \boldsymbol{e}\cdot\nabla\boldsymbol{f}\end{aligned} \tag{12.8}$$

then

$$\frac{d\boldsymbol{f}}{ds} = \boldsymbol{e}\cdot\nabla\boldsymbol{f} \tag{12.9}$$

which is a general form independent of the choice of coordinates. At any point, $\nabla\boldsymbol{f}$ is completely determined when $\frac{d\boldsymbol{f}}{ds_i}$ is given for three non-coplanar direction $s_i\boldsymbol{e}_i$

$$\frac{d\boldsymbol{f}}{ds_1} = \boldsymbol{e}_1\cdot\nabla\boldsymbol{f}, \quad \frac{d\boldsymbol{f}}{ds_2} = \boldsymbol{e}_2\cdot\nabla\boldsymbol{f}, \quad \frac{d\boldsymbol{f}}{ds_3} = \boldsymbol{e}_3\cdot\nabla\boldsymbol{f} \tag{12.10}$$

add the equations

$$\begin{aligned}&\boldsymbol{e}^1\otimes\frac{d\boldsymbol{f}}{ds_1} + \boldsymbol{e}^2\otimes\frac{d\boldsymbol{f}}{ds_2} + \boldsymbol{e}^3\otimes\frac{d\boldsymbol{f}}{ds_3}\\ &= \boldsymbol{e}^1\otimes(\boldsymbol{e}_1\cdot\nabla\boldsymbol{f}) + \boldsymbol{e}^2\otimes(\boldsymbol{e}_2\cdot\nabla\boldsymbol{f}) + \boldsymbol{e}^3\otimes(\boldsymbol{e}_3\cdot\nabla\boldsymbol{f})\\ &= (\boldsymbol{e}^1\otimes\boldsymbol{e}_1)\cdot\nabla\boldsymbol{f} + (\boldsymbol{e}^2\otimes\boldsymbol{e}_2)\cdot\nabla\boldsymbol{f} + (\boldsymbol{e}^3\otimes\boldsymbol{e}_3)\cdot\nabla\boldsymbol{f}\\ &= (\boldsymbol{e}^1\otimes\boldsymbol{e}_1 + \boldsymbol{e}^2\otimes\boldsymbol{e}_2 + \boldsymbol{e}^3\otimes\boldsymbol{e}_3)\cdot\nabla\boldsymbol{f}\\ &= \boldsymbol{I}\cdot\nabla\boldsymbol{f}\\ &= \nabla\boldsymbol{f}\end{aligned} \tag{12.11}$$

then

$$\nabla\boldsymbol{f} = \boldsymbol{e}^1\otimes\frac{d\boldsymbol{f}}{ds_1} + \boldsymbol{e}^2\otimes\frac{d\boldsymbol{f}}{ds_2} + \boldsymbol{e}^3\otimes\frac{d\boldsymbol{f}}{ds_3} \tag{12.12}$$

If $\boldsymbol{f}(\boldsymbol{r})$ is given as a function $\boldsymbol{f}(u,v,w)$ of curvilinear coordinates,

$$\frac{d\boldsymbol{f}}{ds} = \boldsymbol{e}\cdot\nabla\boldsymbol{f} = \frac{\partial \boldsymbol{f}}{\partial u}\frac{du}{ds} + \frac{\partial \boldsymbol{f}}{\partial v}\frac{dv}{ds} + \frac{\partial \boldsymbol{f}}{\partial w}\frac{dw}{ds} \tag{12.13}$$

$$\frac{du}{ds} = \boldsymbol{e}\cdot\nabla u, \quad \frac{dv}{ds} = \boldsymbol{e}\cdot\nabla v, \quad \frac{dw}{ds} = \boldsymbol{e}\cdot\nabla w \tag{12.14}$$

then

$$\begin{aligned}\frac{d\boldsymbol{f}}{ds} &= \boldsymbol{e}\cdot\nabla\boldsymbol{f}\\ &= \frac{\partial \boldsymbol{f}}{\partial u}(\boldsymbol{e}\cdot\nabla u) + \frac{\partial \boldsymbol{f}}{\partial v}(\boldsymbol{e}\cdot\nabla v) + \frac{\partial \boldsymbol{f}}{\partial w}(\boldsymbol{e}\cdot\nabla w)\\ &= \boldsymbol{e}\cdot\left[\nabla u\otimes\frac{\partial \boldsymbol{f}}{\partial u} + \nabla v\otimes\frac{\partial \boldsymbol{f}}{\partial v} + \nabla w\otimes\frac{\partial \boldsymbol{f}}{\partial w}\right]\end{aligned} \tag{12.15}$$

then

$$\nabla\boldsymbol{f} = \nabla u\otimes\frac{\partial \boldsymbol{f}}{\partial u} + \nabla v\otimes\frac{\partial \boldsymbol{f}}{\partial v} + \nabla w\otimes\frac{\partial \boldsymbol{f}}{\partial w} \tag{12.16}$$

for rectangular coordinates

$$\boldsymbol{r} = x_i\boldsymbol{e}_i \tag{12.17}$$

$$\nabla \boldsymbol{r} = \boldsymbol{e}_k \otimes \frac{\partial x_i \boldsymbol{e}_i}{\partial x_k} = \frac{\partial x_i}{\partial x_k} \boldsymbol{e}_k \otimes \boldsymbol{e}_i = \delta_{ik} \boldsymbol{e}_k \otimes \boldsymbol{e}_i = \delta_{ki} \boldsymbol{e}_k \otimes \boldsymbol{e}_i = \boldsymbol{I} \tag{12.18}$$

Here $\nabla \boldsymbol{r} = \boldsymbol{I}$ is a general expression independent of coordinates. For curvilinear coordinates

$$\nabla \boldsymbol{r} = \nabla u \otimes \frac{\partial \boldsymbol{r}}{\partial u} + \nabla v \otimes \frac{\partial \boldsymbol{r}}{\partial v} + \nabla w \otimes \frac{\partial \boldsymbol{r}}{\partial w} = \boldsymbol{I} \tag{12.19}$$

then $\nabla u, \nabla v, \nabla w$ is reciprocal sets of $\boldsymbol{r}_u, \boldsymbol{r}_v, \boldsymbol{r}_w$

For the set $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{n}$ is reciprocal to $\boldsymbol{r}_u, \boldsymbol{r}_v, \boldsymbol{n}$

$$\boldsymbol{I} = \boldsymbol{a} \otimes \boldsymbol{r}_u + \boldsymbol{b} \otimes \boldsymbol{r}_v + \boldsymbol{n} \otimes \boldsymbol{n} \tag{12.20}$$

For the position vector $\boldsymbol{r}$ to the surface, we have

$$\nabla^{\parallel} \boldsymbol{r} = \boldsymbol{a} \otimes \boldsymbol{r}_u + \boldsymbol{b} \otimes \boldsymbol{r}_v = \boldsymbol{I} - \boldsymbol{n} \otimes \boldsymbol{n} \tag{12.21}$$

12.4 Surface

a curve could be represented by a position vector with one parameter

$$\boldsymbol{r} = \boldsymbol{r}(t) \tag{12.22}$$

a surface could be represented by a position vector with two parameters

$$\boldsymbol{r} = \boldsymbol{r}(u, v) \tag{12.23}$$

vector element of area

$$d\boldsymbol{S} = \boldsymbol{r}_u \times \boldsymbol{r}_v du dv, \quad d\boldsymbol{S} = \boldsymbol{n} dS \tag{12.24}$$

scalar element of area

$$dS = |\boldsymbol{r}_u \times \boldsymbol{r}_v| \, du dv \tag{12.25}$$

a normal to the surface

$$\boldsymbol{r}_u \times \boldsymbol{r}_v \tag{12.26}$$

the unit normal $\boldsymbol{n}$ to the surface

$$\boldsymbol{n} = \frac{\boldsymbol{r}_u \times \boldsymbol{r}_v}{|\boldsymbol{r}_u \times \boldsymbol{r}_v|} = \frac{\boldsymbol{r}_u \times \boldsymbol{r}_v}{\sqrt{EG - F^2}} = \frac{\boldsymbol{r}_u \times \boldsymbol{r}_v}{H} \tag{12.27}$$

where

$$E = \boldsymbol{r}_u \cdot \boldsymbol{r}_u, \quad F = \boldsymbol{r}_u \cdot \boldsymbol{r}_v, \quad G = \boldsymbol{r}_v \cdot \boldsymbol{r}_v \tag{12.28}$$

$$(\boldsymbol{r}_u \times \boldsymbol{r}_v) \cdot (\boldsymbol{r}_u \times \boldsymbol{r}_v) = \begin{vmatrix} \boldsymbol{r}_u \cdot \boldsymbol{r}_u & \boldsymbol{r}_u \cdot \boldsymbol{r}_v \\ \boldsymbol{r}_v \cdot \boldsymbol{r}_u & \boldsymbol{r}_v \cdot \boldsymbol{r}_v \end{vmatrix} = \begin{vmatrix} E & F \\ F & G \end{vmatrix} = EG - F^2 \tag{12.29}$$

because

$$(\boldsymbol{a} \times \boldsymbol{b}) \cdot (\boldsymbol{c} \times \boldsymbol{d}) = \begin{vmatrix} \boldsymbol{a} \cdot \boldsymbol{c} & \boldsymbol{a} \cdot \boldsymbol{d} \\ \boldsymbol{b} \cdot \boldsymbol{c} & \boldsymbol{b} \cdot \boldsymbol{d} \end{vmatrix} \tag{12.30}$$

a tangential vector $s\boldsymbol{e}$ with a unit tangential vector $\boldsymbol{e}$ of the surface in some direction

$$\boldsymbol{e} = \frac{d\boldsymbol{r}}{ds} = \frac{\partial \boldsymbol{r}}{\partial u}\frac{du}{ds} + \frac{\partial \boldsymbol{r}}{\partial v}\frac{dv}{ds} = \boldsymbol{r}_u \frac{du}{ds} + \boldsymbol{r}_v \frac{dv}{ds} \quad \text{Chain rule} \tag{12.31}$$

set $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ reciprocal to $\boldsymbol{r}_u, \boldsymbol{r}_v, \boldsymbol{n}$, i.e.,

$$\boldsymbol{a} \cdot \boldsymbol{r}_u = 1, \quad \boldsymbol{b} \cdot \boldsymbol{r}_v = 1, \quad \boldsymbol{c} \cdot \boldsymbol{n} = 1, \quad ... \tag{12.32}$$

let

$$\boldsymbol{a} = \frac{\boldsymbol{r}_v \times \boldsymbol{n}}{H}, \quad \boldsymbol{b} = \frac{\boldsymbol{n} \times \boldsymbol{r}_u}{H}, \quad \boldsymbol{c} = \frac{\boldsymbol{r}_u \times \boldsymbol{r}_v}{H} = \boldsymbol{n} \tag{12.33}$$

then

$$\boldsymbol{a}\cdot\boldsymbol{e}=\frac{du}{ds},\quad \boldsymbol{b}\cdot\boldsymbol{e}=\frac{dv}{ds} \tag{12.34}$$

for

$$\frac{d\boldsymbol{f}}{ds}=\frac{\partial\boldsymbol{f}}{\partial u}\frac{du}{ds}+\frac{\partial\boldsymbol{f}}{\partial v}\frac{dv}{ds}=\boldsymbol{f}_u\frac{du}{ds}+\boldsymbol{f}_v\frac{dv}{ds}\quad \text{Chain rule} \tag{12.35}$$

then

$$\begin{aligned}\frac{d\boldsymbol{f}}{ds}&=\boldsymbol{f}_u(\boldsymbol{a}\cdot\boldsymbol{e})+\boldsymbol{f}_v(\boldsymbol{b}\cdot\boldsymbol{e})\\&=\boldsymbol{f}_u(\boldsymbol{e}\cdot\boldsymbol{a})+\boldsymbol{f}_v(\boldsymbol{e}\cdot\boldsymbol{a})=\boldsymbol{e}\cdot[\boldsymbol{a}\otimes\boldsymbol{f}_u+\boldsymbol{b}\otimes\boldsymbol{f}_v]\\&=\boldsymbol{e}\cdot[\boldsymbol{a}\boldsymbol{f}_u+\boldsymbol{b}\boldsymbol{f}_v]\end{aligned} \tag{12.36}$$

Then define the surface gradient of $\boldsymbol{f}$

$$\nabla^{\parallel}\boldsymbol{f}=\boldsymbol{a}\otimes\boldsymbol{f}_u+\boldsymbol{b}\otimes\boldsymbol{f}_v \tag{12.37}$$

give two non-collinear directions $s_1\boldsymbol{e}_1,\ s_2\boldsymbol{e}_2$ in the tangential plane, $n\boldsymbol{n}$ is the normal.

$$\nabla\boldsymbol{f}=\boldsymbol{e}^1\otimes\frac{d\boldsymbol{f}}{ds_1}+\boldsymbol{e}^2\otimes\frac{d\boldsymbol{f}}{ds_2}+\boldsymbol{e}^3\otimes\frac{d\boldsymbol{f}}{ds_3}=\boldsymbol{e}^1\otimes\frac{d\boldsymbol{f}}{ds_1}+\boldsymbol{e}^2\otimes\frac{d\boldsymbol{f}}{ds_2}+\boldsymbol{n}\otimes\frac{d\boldsymbol{f}}{dn} \tag{12.38}$$

$$\nabla^{\parallel}\boldsymbol{f}=\boldsymbol{e}^1\otimes\frac{d\boldsymbol{f}}{ds_1}+\boldsymbol{e}^2\otimes\frac{d\boldsymbol{f}}{ds_2} \tag{12.39}$$

set $\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{n}$ reciprocal to $\boldsymbol{e}^1,\boldsymbol{e}^2,\boldsymbol{n}$, then

$$\boldsymbol{I}=\boldsymbol{e}^1\otimes\boldsymbol{e}_1+\boldsymbol{e}^2\otimes\boldsymbol{e}_2+\boldsymbol{n}\otimes\boldsymbol{n}=\boldsymbol{e}_1\otimes\boldsymbol{e}^1+\boldsymbol{e}_2\otimes\boldsymbol{e}^2+\boldsymbol{n}\otimes\boldsymbol{n} \tag{12.40}$$

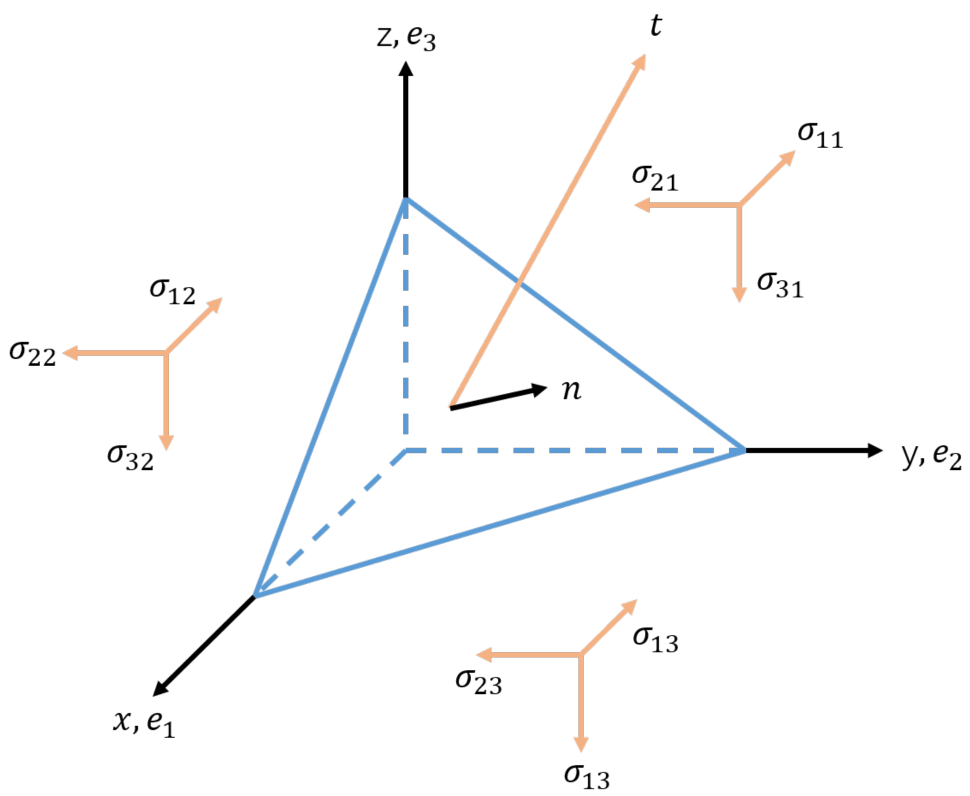
$$(\boldsymbol{n}\otimes\boldsymbol{n})\cdot\nabla\boldsymbol{f}=\boldsymbol{n}\otimes[\boldsymbol{n}\cdot\boldsymbol{n}\otimes\frac{d\boldsymbol{f}}{dn}]=\boldsymbol{n}\otimes[(\boldsymbol{n}\cdot\boldsymbol{n})\frac{d\boldsymbol{f}}{dn}]=\boldsymbol{n}\otimes\frac{d\boldsymbol{f}}{dn}=\nabla^{\perp}\boldsymbol{f} \tag{12.41}$$

$$(\boldsymbol{I}-\boldsymbol{n}\otimes\boldsymbol{n})\cdot\nabla\boldsymbol{f}=(\boldsymbol{e}^1\otimes\boldsymbol{e}_1+\boldsymbol{e}^2\otimes\boldsymbol{e}_2)\cdot\nabla\otimes\boldsymbol{f}=\boldsymbol{e}^1\otimes\frac{d\boldsymbol{f}}{ds_1}+\boldsymbol{e}^2\otimes\frac{d\boldsymbol{f}}{ds_2}=\nabla^{\parallel}\boldsymbol{f} \tag{12.42}$$

Chapter 13

Appendix

13.1 （其他的张量表示）–柯西定理的来源（case2）



面积关系

$$A_j = A\boldsymbol{n} \cdot \boldsymbol{e}_j$$

考虑$\boldsymbol{e}_1$方向的受力平衡$\Sigma F_1 = 0$

$$\sigma_{11}A_1 + \sigma_{12}A_2 + \sigma_{13}A_3 = A\boldsymbol{t} \cdot \boldsymbol{e}_1$$

写成指标形式

$$\sigma_{1j}A_j = A\boldsymbol{t} \cdot \boldsymbol{e}_1$$

推广到3个方向

$$\sigma_{ij}A_j = A\boldsymbol{t} \cdot \boldsymbol{e}_i$$

代入面积关系

$$\sigma_{ij}A\boldsymbol{n} \cdot \boldsymbol{e}_j = A\boldsymbol{t} \cdot \boldsymbol{e}_i \quad \Rightarrow \sigma_{ij}\boldsymbol{n} \cdot \boldsymbol{e}_j = \boldsymbol{t} \cdot \boldsymbol{e}_i$$

下面对等式左边进行操作

$$\sigma_{ij}\boldsymbol{n} \cdot \boldsymbol{e}_j = \sigma_{ij}n_j = \boldsymbol{e}_i \cdot \boldsymbol{\sigma}\boldsymbol{e}_j n_j = \boldsymbol{e}_i \cdot \boldsymbol{\sigma}(n_j\boldsymbol{e}_j) = \boldsymbol{e}_i \cdot \boldsymbol{\sigma}\boldsymbol{n} = \boldsymbol{e}_i \cdot \boldsymbol{t}$$

$$\Rightarrow \boldsymbol{\sigma}\boldsymbol{n} = \boldsymbol{t}$$

所以，应力张量$\boldsymbol{\sigma}$只是受力平衡推导过程中引入的一个概念，其目的是为了方便计算。力是真实存在的，应力矢量和应力张量都不是真实存在。应力($\frac{N}{m^2}$ or $\frac{N}{m^3}$)只是力的强度的一种度量。

13.1.1 柯西应力—柯西定理建立了应力张量与应力矢量的联系（case2）

存在一个张量 $\boldsymbol{\sigma}$,使得

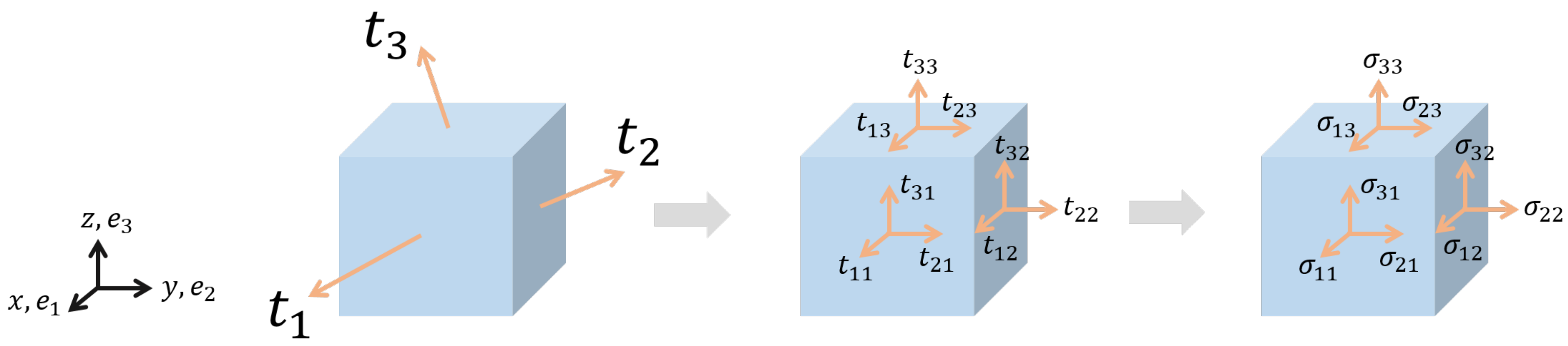
$$\boldsymbol{t}(\boldsymbol{n}, \boldsymbol{x}) = \boldsymbol{\sigma}(\boldsymbol{x})\boldsymbol{n}$$

可以简写为

$$t = \sigma n$$

σ 称为柯西应力张量。

13.1.2 微元体分析（case2）



[1]

σ_{ij}中j为面元指标，i为方向指标，指j面上沿i方向的应力矢量的大小。

已知在直角坐标系(Cartesian system)下，物体内一点处的应力张量为

$$\boldsymbol{\sigma} = \sigma_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$

考虑x正面，其外法向矢量为$\boldsymbol{e}_1$，由柯西定理，x正面上的应力矢量为

$$\boldsymbol{t}_1 = \boldsymbol{\sigma}\boldsymbol{e}_1 = \sigma_{ij}(\boldsymbol{e}_i \otimes \boldsymbol{e}_j)\boldsymbol{e}_1 = \sigma_{ij}(\boldsymbol{e}_j \cdot \boldsymbol{e}_1)\boldsymbol{e}_i = \sigma_{ij}\delta_{1j}\boldsymbol{e}_i = \sigma_{i1}\boldsymbol{e}_i = \sigma_{11}\boldsymbol{e}_1 + \sigma_{21}\boldsymbol{e}_2 + \sigma_{31}\boldsymbol{e}_3$$

应力矢量在x正面，法向上(沿$\boldsymbol{e}_1$方向)的分量(正应力矢量)大小为

$$t_{11} = \boldsymbol{t}_1 \cdot \boldsymbol{e}_1 = (\boldsymbol{\sigma}\boldsymbol{e}_1) \cdot \boldsymbol{e}_1 = (\sigma_{i1}\boldsymbol{e}_i) \cdot \boldsymbol{e}_1 = \sigma_{i1}\delta_{i1} = \sigma_{11}$$

应力矢量在x正面，沿$\boldsymbol{e}_2$方向的切向分量(切应力矢量)大小为

$$t_{21} = \boldsymbol{t}_1 \cdot \boldsymbol{e}_2 = (\boldsymbol{\sigma}\boldsymbol{e}_1) \cdot \boldsymbol{e}_2 = (\sigma_{i1}\boldsymbol{e}_i) \cdot \boldsymbol{e}_2 = \sigma_{i1}\delta_{i2} = \sigma_{21}$$

应力矢量在x正面，沿$\boldsymbol{e}_3$方向的切向分量(切应力矢量)大小为

$$t_{31} = \boldsymbol{t}_1 \cdot \boldsymbol{e}_3 = (\boldsymbol{\sigma}\boldsymbol{e}_1) \cdot \boldsymbol{e}_3 = (\sigma_{i1}\boldsymbol{e}_i) \cdot \boldsymbol{e}_3 = \sigma_{i1}\delta_{i3} = \sigma_{31}$$

其他5个面与上面分析类似，不再赘述。

可以看出应力张量的分量就是坐标轴平面上应力矢量的分量。

最大切应力的方向与两个主方向都成45度。若两个主方向为$\boldsymbol{n}_1, \boldsymbol{n}_2$，最大切应力的方向为$\boldsymbol{n} = \frac{\boldsymbol{n}_1+\boldsymbol{n}_2}{|\boldsymbol{n}_1+\boldsymbol{n}_2|}$

13.1.3 平衡方程（case2）

建立微元体的差异会导致以下算法不同, case2是应力张量中的列向量存储应力矢量信息。

散度在尾段去掉维度。

$$\boldsymbol{\sigma} \cdot \nabla = \frac{\partial \boldsymbol{\sigma}}{\partial x_k} \cdot \boldsymbol{e}_k = \frac{\partial \sigma_{ij}}{\partial x_k}\boldsymbol{e}_i \otimes \boldsymbol{e}_j \cdot \boldsymbol{e}_k = \frac{\partial \sigma_{ij}}{\partial x_k}\delta_{jk}\boldsymbol{e}_i = \frac{\partial \sigma_{ij}}{\partial x_j}\boldsymbol{e}_i = \sigma_{ij,j}\boldsymbol{e}_i = \boldsymbol{0}$$

[1]力是真实存在的，应力张量并不是真实存在的，只是一个中间的度量，只是为了求应力矢量或者应变的一个中间概念，数学或力学概念。

Chapter 14

电学

14.1 直角坐标和球坐标中的微元的映射关系

$$\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{X}) \tag{14.1}$$

$$d\boldsymbol{x} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{X}} \cdot d\boldsymbol{X} = (\boldsymbol{f}\nabla) \cdot d\boldsymbol{X} = \boldsymbol{J} \cdot d\boldsymbol{X} \tag{14.2}$$

$$\boldsymbol{x} = (x, y, z), \quad \boldsymbol{X} = (r, \phi, \theta), \quad \begin{aligned} x &= r\sin\phi\cos\theta \\ y &= r\sin\phi\sin\theta \\ z &= r\cos\phi \end{aligned} \tag{14.3}$$

$$\boldsymbol{J} = \frac{\partial x_i}{\partial X_j}\boldsymbol{e}_i \otimes \boldsymbol{e}_j = \begin{bmatrix} \sin\phi\cos\theta & -r\sin\phi\sin\theta & r\cos\phi\cos\theta \\ \sin\phi\sin\theta & r\sin\phi\cos\theta & r\cos\phi\sin\theta \\ \cos\phi & 0 & -r\sin\phi \end{bmatrix} \tag{14.4}$$

$$\det(\boldsymbol{J}) = \frac{dv}{dV} = r^2\sin\phi \tag{14.5}$$

14.2 球体积的计算方法

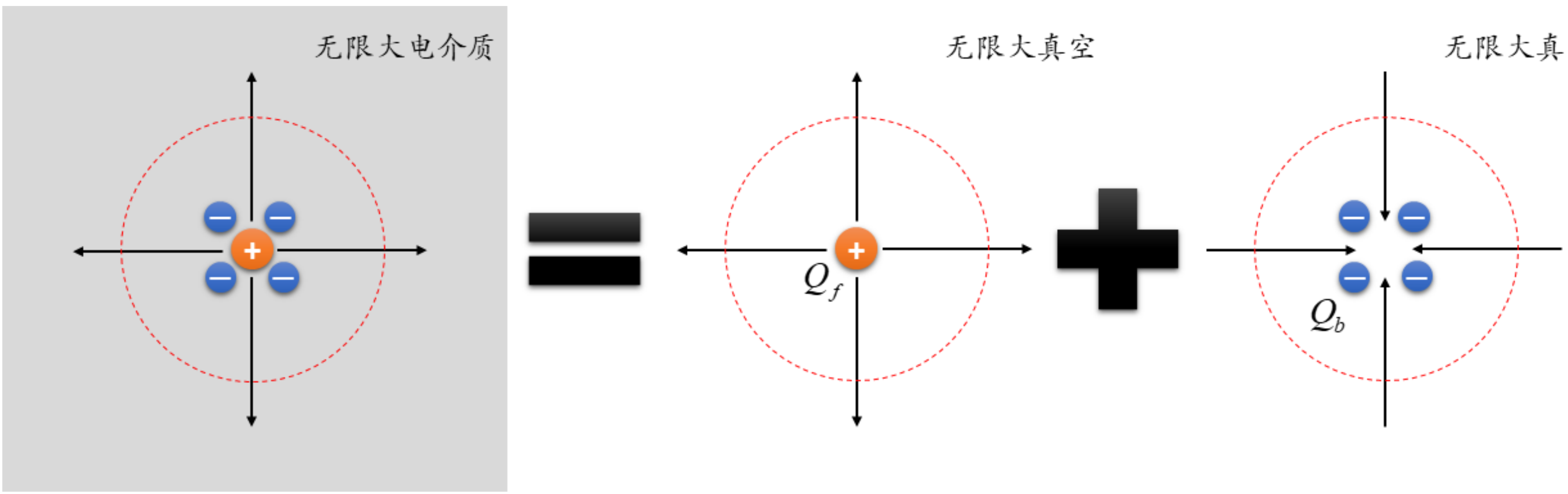
球的体积为

$$\begin{aligned} \oint_{\Omega_c} dv = \oint_{\Omega_r} r^2\sin\phi dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} r^2\sin\phi dr d\phi d\theta = \frac{4}{3}\pi r^3 &= \frac{1}{3}r(4\pi r^2) \\ &= \oint_{\partial\Omega}\frac{1}{3}r da = \frac{1}{3}r\oint_{\partial\Omega} da \end{aligned} \tag{14.6}$$

所以

$$4\pi r^2 = \oint_{\partial\Omega} da = \oint_{\partial\Omega} \boldsymbol{r}\cdot d\boldsymbol{a} \implies 4\pi = \oint_{\partial\Omega}\frac{\boldsymbol{r}}{r^2}\cdot d\boldsymbol{a} \tag{14.7}$$

14.3 库仑定律和高斯定理



库仑定律(真空中点电荷产生的电场)

$$\boldsymbol{E}_f = \frac{Q_f}{4\pi\epsilon_0}\frac{\boldsymbol{r}}{r^2} \tag{14.8}$$

极化电荷一但形成，其产生的电场和在真空中产生的电场等效

$$\boldsymbol{E}_b = \frac{Q_b}{4\pi\epsilon_0}\frac{\boldsymbol{r}}{r^2} \tag{14.9}$$

若取球形闭合曲面，可得高斯定理(电场通过闭合曲面的通量等于闭合曲面内部的电荷量)

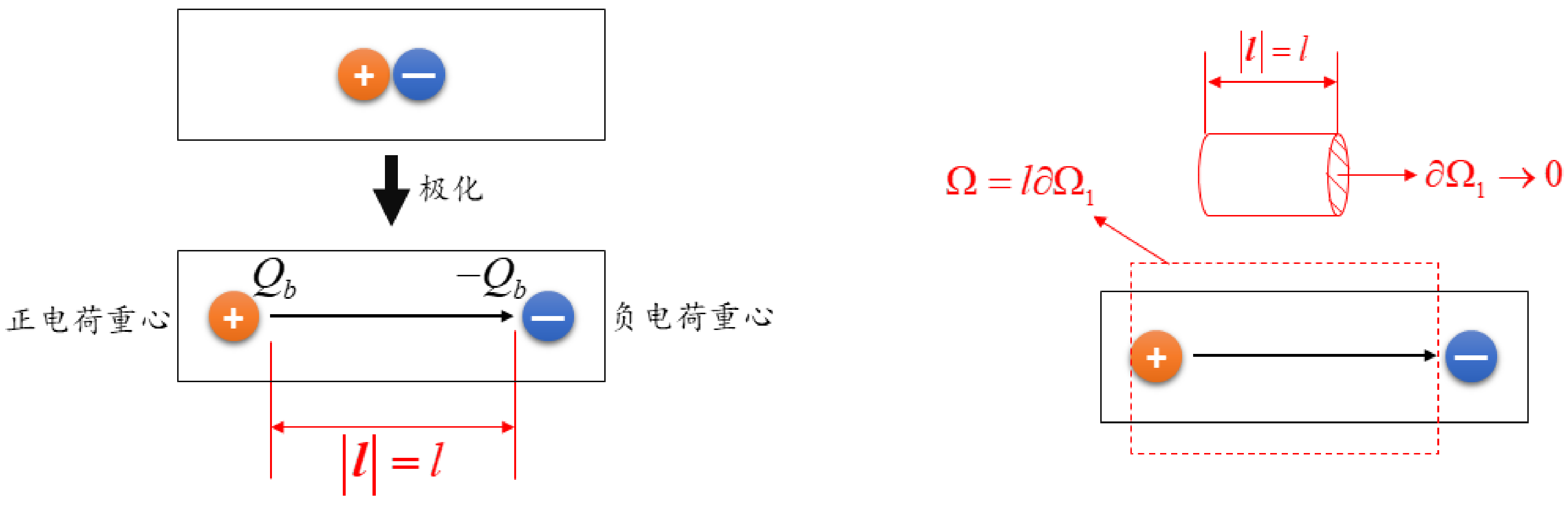
$$\Phi_f = \oint_{\partial\Omega} \boldsymbol{E}_f \cdot d\boldsymbol{a} = \oint_{\partial\Omega} \frac{Q_f}{4\pi\epsilon_0}\frac{\boldsymbol{r}}{r^2} \cdot d\boldsymbol{a} = \frac{Q_f}{\epsilon_0} \tag{14.10}$$

$$\Phi_b = \oint_{\partial\Omega} \boldsymbol{E}_b \cdot d\boldsymbol{a} = \oint_{\partial\Omega} \frac{Q_b}{4\pi\epsilon_0}\frac{\boldsymbol{r}}{r^2} \cdot d\boldsymbol{a} = \frac{Q_b}{\epsilon_0} \tag{14.11}$$

根据电场叠加原理

$$\Phi = \oint_{\partial\Omega} \boldsymbol{E} \cdot d\boldsymbol{a} = \oint_{\partial\Omega} \boldsymbol{E}_f \cdot d\boldsymbol{a} + \oint_{\partial\Omega} \boldsymbol{E}_b \cdot d\boldsymbol{a} = \frac{Q_f}{\epsilon_0} + \frac{Q_b}{\epsilon_0} = \frac{Q_f + Q_b}{\epsilon_0} \tag{14.12}$$

14.4 电介质



(a) (b)

分子电矩定义为($\boldsymbol{l}$ 从正电荷指向负电荷，负号使得极化电场能与外电场相抵消)

$$\boldsymbol{p}_m = Q_b l \boldsymbol{n}_m \tag{14.13}$$

极化强度矢量和极化电场强度矢量方向相反...只有电子可以移动

电极化强度矢量定义为(单位体积内的电矩矢量和)

$$\boldsymbol{p} = \lim_{\Delta\Omega\to 0} \frac{\Delta \boldsymbol{p}_m}{\Delta\Omega} = \frac{Q_b l \boldsymbol{n}_m}{\Omega_m} \tag{14.14}$$

取圆柱形闭合曲面

$$\oint_{\partial\Omega} \boldsymbol{p} \cdot d\boldsymbol{a} = \oint_{\partial\Omega} \frac{Q_b l \boldsymbol{n}_m}{\Omega_m} \cdot d\boldsymbol{a} = \frac{Q_b l(-A_1)}{\Omega_m} = -Q_b \tag{14.15}$$

若电介质既包括自由电荷又包括束缚电荷，由高斯定理

$$\oint_{\partial\Omega} \boldsymbol{E} \cdot d\boldsymbol{a} = \frac{1}{\epsilon_0}(Q_f + Q_b) \implies \oint_{\partial\Omega} \epsilon_0 \boldsymbol{E} \cdot d\boldsymbol{a} = Q_f + Q_b = Q_f - \oint_{\partial\Omega} \boldsymbol{p} \cdot d\boldsymbol{a} \tag{14.16}$$

$$\oint_{\partial\Omega} \boldsymbol{E} \cdot d\boldsymbol{a} = \oint_{\partial\Omega} (\boldsymbol{E_f} - \boldsymbol{E}_b) \cdot d\boldsymbol{a} = \oint_{\partial\Omega} \boldsymbol{E_f} \cdot d\boldsymbol{a} - \oint_{\partial\Omega} \boldsymbol{E}_b \cdot d\boldsymbol{a} = \frac{Q_f}{\epsilon_0} - \frac{Q_b}{\epsilon_0} \tag{14.17}$$

其中，$\boldsymbol{E}$ 实际上为自由电荷产生的电场和束缚电荷产生的电场(极化电场)的叠加，也就是说 $\boldsymbol{E} = \boldsymbol{E}_f - \boldsymbol{E}_b$。对于各向同性电介质，极化强度和电场强度存在线性关系 $\boldsymbol{p} = \chi\varepsilon_0 \boldsymbol{E}_b$。

所以

$$\oint_{\partial\Omega} (\epsilon_0 \boldsymbol{E} + \boldsymbol{p}) d\boldsymbol{a} = Q_f \tag{14.18}$$

定义电位移矢量为

$$\boldsymbol{D} = \epsilon_0 \boldsymbol{E} + \boldsymbol{p} \tag{14.19}$$

那么

$$\oint_{\partial\Omega} \boldsymbol{D} \cdot d\boldsymbol{a} = Q_f \tag{14.20}$$

由散度定理

$$\oint_{\Omega} \nabla \cdot \boldsymbol{D} d\boldsymbol{v} = \oint_{\partial\Omega} \boldsymbol{D} \cdot d\boldsymbol{a} \tag{14.21}$$

最终

$$\oint_{\Omega} \nabla \cdot \boldsymbol{D} d\boldsymbol{v} = \oint_{\partial\Omega} \boldsymbol{D} \cdot d\boldsymbol{a} = Q_f \tag{14.22}$$

Chapter 15

信号分析

周期和频率的关系

周期为 T 的正弦函数和余弦函数

$$\sin(\omega x) = \sin(2\pi f x) = \sin(\frac{2\pi}{T}x), \qquad \cos(\omega x) = \cos(2\pi f x) = \cos(\frac{2\pi}{T}x) \tag{15.1}$$

周期为 T 的复指数

$$e^{i\omega t} = \cos(\omega t) + i\sin(\omega t) \tag{15.2}$$

傅里叶变换的物理意义

周期函数的傅里叶级数实质上是将函数 $x(t)$ 分解为无数个不同频率、不同幅值的正、余弦信号，而这些信号的频率都是基频 ω_0 的整数倍（即 $k\omega_0$ ）。换言之，我们是在用无数个这样不同频率，不同幅值的正、余弦信号来逼近周期函数 $x(t)$ 。分解的过程中，对于每一个 $k\omega_0$，我们都得到了对应的幅值，这样就组成了一个函数关系（自变量为 $k\omega_0$ ，因变量为幅值，即相应频率信号的强度）？我们称之为频谱函数。

而对于非周期函数，傅里叶变换则是求频谱密度函数，该函数的自变量是 ω ，因变量是信号幅值在频域中的分布密度，即单位频率信号的强度。(如果你学过概率论，可以将频谱函数和频谱密度函数类比为离散概率分布和概率密度函数)

周期函数的傅里叶级数和傅里叶变换

$x(t)$ 是周期为 $T = \frac{\omega_0}{2\pi}$ 的周期函数，则可以用傅里叶级数近似表示为

$$\begin{aligned} x(t) &= \sum_{k=-\infty}^{k=+\infty} \hat{X}[k] e^{jk\omega_0 t} \\ \hat{X}[k] &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) e^{-jk\omega_0 t} dt \end{aligned} \tag{15.3}$$

其中，基频或者频率间隔为 ω_0。然而，对于更为常见的非周期函数，傅里叶级数无法对其进行逼近操作。

非周期函数的傅里叶变换-连续傅里叶变换

那么对于非周期函数，我们把它的周期看作无穷大，基频 $\omega_0 = \frac{2\pi}{T}$ 则趋近于无穷小

$$T \to \infty \implies \omega_0 \to 0 \tag{15.4}$$

又因为基频相当于周期函数的傅里叶级数中两个相邻频率的差值 $(k+1)\omega_0 - k\omega_0$ ，我们可以把它记作 $\Delta\omega$ 或者微分 $d\omega$

$$\Delta\omega = (k+1)\omega_0 - k\omega_0 = \omega_0 \tag{15.5}$$

$k\omega_0$ 则相当于连续变量 ω

$$\omega = k\omega_0 \tag{15.6}$$

这样就得到了针对非周期函数的频谱函数如下

$$\hat{X}[k]=\lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}x(t)e^{-jk\omega_0 t}dt=\lim_{T\to\infty}\frac{\omega_0}{2\pi}\int_{-\frac{T}{2}}^{\frac{T}{2}}x(t)e^{-jk\omega_0 t}dt=\frac{\Delta\omega}{2\pi}\int_{-\infty}^{+\infty}x(t)e^{-jwt}dt \tag{15.7}$$

可以看出，这里 $\hat{X}[k]\to 0$。傅里叶变换的函数值就都是无穷小，这显然对我们没有任何帮助。

代入傅里叶级数中

$$x(t)=\sum_{k=-\infty}^{k=+\infty}\hat{X}[k]e^{jk\omega_0 t}=\sum_{k=-\infty}^{k=+\infty}\left[\frac{\Delta\omega}{2\pi}\int_{-\infty}^{+\infty}x(t)e^{-jwt}dt\right]e^{jk\omega_0 t}=\int_{-\infty}^{+\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{+\infty}x(t)e^{-jwt}dt\right]e^{jk\omega_0 t}d\omega \tag{15.8}$$

则非周期函数的傅里叶变换定义为

$$\hat{x}(\omega)=\int_{-\infty}^{+\infty}x(t)e^{-jwt}dt \tag{15.9}$$

我们可以发现

$$\hat{X}[k]=\hat{x}(\omega)\frac{d\omega}{2\pi}=\hat{x}(f)df \tag{15.10}$$

也就是说，我们选取信号幅值在频域中的分布密度 $\hat{x}(\omega)$ 来表示傅里叶变换，而不是相应频率的信号幅值大小 $\hat{X}[k]$ 。

我们一般也用频率 f 来进行傅里叶变换

$$x(t)=\int_{-\infty}^{\infty}\hat{x}(f)e^{i2\pi ft}df \tag{15.11}$$

所以我们可以说，傅里叶变换的目的就是将信号转化为无数个不同频率的正弦信号的叠加，然后揭示这些正弦信号的强度和频率的关系。

综上所述，非周期函数的傅里叶变换对为

$$\begin{aligned}\hat{x}(f)&=\int_{-\infty}^{\infty}x(t)e^{-i2\pi ft}dt\\ x(t)&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{x}(f)e^{i\omega t}d\omega=\int_{-\infty}^{\infty}\hat{x}(f)e^{i2\pi ft}df\end{aligned} \tag{15.12}$$

离散傅里叶变换

我们说过，傅里叶变换的目的就是得到信号的频谱密度函数（自变量是 ω ，因变量是信号幅值在频域中的分布密度，即单位频率信号的强度），它实际上揭示了信号的强度和频率的关系。对于傅里叶变换

$$\hat{x}(f)=\int_{-\infty}^{\infty}x(t)e^{-i2\pi ft}dt \tag{15.13}$$

我们做数学题时碰到的 $x(t)$ 大多数是在 t 上连续的，但由于计算机采集的信号在时域中是离散的，故实际应用中的 $x(t)$ 都是其经采样处理后得到的 $x_s(t)$ 。同时，计算机也只可能计算出有限个频率上对应的幅值密度，即 f 也是离散的。DFT 就是 t 和 f 都为离散值的傅里叶变换。

采样

采样的具体操作是什么？我们首先引入冲激函数（也叫Dirac函数）的概念。冲激函数定义为

$$\delta(t)=\begin{cases}\infty & t=0\\ 0 & t\neq 0\end{cases}\qquad \int_{-\infty}^{\infty}\delta(t)dt=1 \tag{15.14}$$

根据它的定义，我们可知

$$\int_{-\infty}^{\infty}x(t)\delta(t-t_0)dt=x(t_0) \tag{15.15}$$

这是Dirac函数的重要性质（筛选性质），容易看出，它可以筛选出 $x(t)$ 在 t_0 处的函数值，即起到采样的作用。但是Dirac函数一次只能选取一个函数值，所以我们将很多个采样点不同的Dirac函数叠加起来，就可以实现时域上的采样了。这样叠加的函数被称为梳状函数，表达式为

$$\delta_s(t)=\sum_{n=-\infty}^{\infty}\delta(t-n\Delta t) \tag{15.16}$$

其中 Δt 为采样周期。

将时域上的连续信号 $x(t)$ 与它相乘，即可得到采样信号 $x_s(t)$

$$x_s(t)=x(t)\delta_s(t)=\sum_{n=-\infty}^{\infty}x(t)\delta(t-n\Delta t) \tag{15.17}$$

时域离散化计算

对上述采样得到的采样信号进行傅里叶变换。

$$\hat{x}(\omega)=\int_{-\infty}^{\infty} x_s(t)e^{-i\omega t}dt=\int_{-\infty}^{\infty}\left[\sum_{n=-\infty}^{\infty} x(t)\delta(t-n\Delta t)\right]e^{-i\omega t}dt=\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}\left[x(t)\delta(t-n\Delta t)\right]e^{-i\omega t}dt=\sum_{n=-\infty}^{\infty} x(n\Delta t)e^{-j\omega n\Delta t} \tag{15.18}$$

这样就完成了我们的时域离散化计算。

频域离散化计算

时域离散化得到的结果 $\hat{x}(\omega)$ 在频域上仍是连续的，而计算机只能求取有限个 ω 对应的频谱密度。此外，$\hat{x}(\omega)$ 中的时域采样次数N为无穷大，实际应用中显然不会进行无穷多次时域采样。

我们首先解决 N 为无穷大的问题。对于连续信号 $x(t)$ 进行 N 次（N 为有限值）采样，采样时间间隔为 Δt 。然后对采样得到的信号进行时域上的周期延拓（延拓至正负无穷），这样我们就得到了一个周期为 $T=N\Delta t$ 的函数。对于周期函数而言，它的频谱密度函数是离散化的，这样我们就成功把频域也进行了离散化。具体计算方法如下：

在一个周期内，离散信号的表达式为

$$x_s(t)=\sum_{n=0}^{N-1} x(t)\delta\left(t-n\Delta t\right) \tag{15.19}$$

离散信号的傅里叶变换

$$\begin{aligned}
\hat{X}\left(k\omega_0\right)&=\frac{1}{T}\int_0^T x_s(t)e^{-jkw_0t}dt=\frac{1}{T}\int_0^T\left(\sum_{n=0}^{N-1} x(t)\delta\left(t-n\Delta t\right)\right)e^{-jkw_0t}dt=\frac{1}{T}\sum_{n=0}^{N-1}\int_0^T x(t)\delta\left(t-n\Delta t\right)e^{-jk\omega_0 t}dt=\frac{1}{T}\sum_{n=0}^{N-1} x(n\Delta t)e^{-jk\omega_0 n\Delta t}\\
&=\frac{1}{N\Delta t}\sum_{n=0}^{N-1} x(n\Delta t)e^{-j\frac{2\pi}{N\Delta t}kn\Delta t}=\frac{1}{N\Delta t}\sum_{n=0}^{N-1} x(n\Delta t)e^{-j\frac{2\pi}{N}kn}=\Delta f\sum_{n=0}^{N-1} x(n\Delta t)e^{-j\frac{2\pi}{N}kn}\\
&\Longrightarrow\ \hat{x}(k\omega_0)=\frac{\hat{X}(k\omega_0)}{\Delta f}=\sum_{n=0}^{N-1} x(n\Delta t)e^{-j\frac{2\pi}{N}kn}\\
&\Longrightarrow\ \hat{x}[k]=\sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn}
\end{aligned} \tag{15.20}$$

离散傅里叶逆变换可通过如下过程求得

$$\begin{aligned}
x[n]=x(n\Delta t)&\approx\lim_{\Delta t\to 0} x_s(t)\Delta t=\lim_{\Delta t\to 0}\left[\sum_{k=-\infty}^{\infty}\hat{X}[k]e^{jk\omega_0 t}\right]\Delta t=\lim_{\Delta t\to 0}\sum_{k=-\infty}^{\infty}\hat{X}[k]e^{jk\omega_0 t}\Delta t\\
&\approx\lim_{\Delta t\to 0}\sum_{k=0}^{N-1}\hat{X}[k]e^{jk\omega_0 n\Delta t}\Delta t=\lim_{\Delta t\to 0}\sum_{k=0}^{N-1}\frac{\hat{x}[k]}{N\Delta t}e^{jk\omega_0 n\Delta t}\Delta t=\frac{1}{N}\sum_{n=0}^{N-1}\hat{x}[k]e^{jk\omega_0 n\Delta t}
\end{aligned} \tag{15.21}$$

综上所述，离散傅里叶变换对为

$$\begin{aligned}
\hat{x}[k]&=\sum_{n=0}^{N-1} x(n)e^{-i2\pi k\frac{n}{N}}\\
x[n]&=\frac{1}{N}\sum_{k=0}^{N-1}\hat{x}_k e^{i2\pi k\frac{n}{N}}
\end{aligned} \tag{15.22}$$

Tips: 连续傅里叶变换的黎曼近似和上述过程的关系？（好像无法用黎曼近似进行离散化）

$$\begin{aligned}
\hat{x}(f)&=\int_{-\infty}^{\infty} x(t)e^{-i2\pi ft}dt=\sum_{n=0}^{N-1} x(t)e^{-i2\pi ft}\Delta t=\sum_{n=0}^{N-1} x(n\Delta t)e^{-i2\pi fn\Delta t}\Delta t=\sum_{n=0}^{N-1} x(n\Delta t)e^{-i2\pi(k\frac{f_s}{N})n\Delta t}\Delta t=\sum_{n=0}^{N-1} x(n\Delta t)e^{-i2\pi k\frac{n}{N}}\Delta t\\
&=\frac{1}{f_s}\sum_{n=0}^{N-1} x(n\Delta t)e^{-i2\pi k\frac{n}{N}}\\
&\Longrightarrow\ \hat{x}(k)=\frac{1}{f_s}\sum_{n=0}^{N-1} x(n)e^{-i2\pi k\frac{n}{N}}
\end{aligned} \tag{15.23}$$

如何调整行列间距

$$
\begin{array}{ccccccccc}
m & 0 & 1 & ... & \frac{N}{2}-1 & \frac{N}{2} & \frac{N}{2}+1 & ... & N-1 \\
t & 0 & \Delta t & ... & (\frac{N}{2}-1)\Delta t & \frac{N}{2}\Delta t & (\frac{N}{2}+1)\Delta t & ... & (N-1)\Delta t \\
x & x_0 & x_1 & ... & x_(\frac{N}{2}-1) & x_{\frac{N}{2}} & x_{\frac{N}{2}+1} & ... & x_{N-1} \\
\hat{x} & \hat{x}_0 & \hat{x}_1 & ... & \hat{x}_{\frac{N}{2}-1} & \hat{x}_{\frac{N}{2}} & \hat{x}_{\frac{N}{2}+1} & ... & \hat{x}_{N-1} \\
f & 0 & \frac{f_s}{N} & ... & (\frac{N}{2}-1)\frac{f_s}{N} & \frac{f_s}{2} & & & \\
|\hat{x}| & |\hat{x}_0| & |\hat{x}_1| & ... & \left|\hat{x}_{\frac{N}{2}-1}\right| & \left|\hat{x}_{\frac{N}{2}}\right| & \left|\hat{x}_{\frac{N}{2}+1}\right| & ... & |\hat{x}_{N-1}| \\
|\hat{x}|^2 & |\hat{x}_0|^2 & |\hat{x}_1|^2 & ... & \left|\hat{x}_{\frac{N}{2}-1}\right|^2 & \left|\hat{x}_{\frac{N}{2}}\right|^2 & \left|\hat{x}_{\frac{N}{2}+1}\right|^2 & ... & |\hat{x}_{N-1}|^2 \\
\frac{|\hat{x}|^2}{N} & \frac{|\hat{x}_0|^2}{N} & \frac{|\hat{x}_1|^2}{N} & ... & \frac{\left|\hat{x}_{\frac{N}{2}-1}\right|^2}{N} & \frac{\left|\hat{x}_{\frac{N}{2}}\right|^2}{N} & \frac{\left|\hat{x}_{\frac{N}{2}+1}\right|^2}{N} & ... & \frac{|\hat{x}_{N-1}|^2}{N} \\
\frac{|\hat{x}|^2}{N\cdot f_s} & \frac{|\hat{x}_0|^2}{N\cdot f_s} & \frac{|\hat{x}_1|^2}{N\cdot f_s} & ... & \frac{\left|\hat{x}_{\frac{N}{2}-1}\right|^2}{N\cdot f_s} & \frac{\left|\hat{x}_{\frac{N}{2}}\right|^2}{N\cdot f_s} & \frac{\left|\hat{x}_{\frac{N}{2}+1}\right|^2}{N\cdot f_s} & ... & \frac{|\hat{x}_{N-1}|^2}{N\cdot f_s} \\
\frac{2|\hat{x}|^2}{N\cdot f_s} & \frac{|\hat{x}_0|^2}{N\cdot f_s} & \frac{2|\hat{x}_1|^2}{N\cdot f_s} & ... & \frac{2\left|\hat{x}_{\frac{N}{2}-1}\right|^2}{N\cdot f_s} & \frac{\left|\hat{x}_{\frac{N}{2}}\right|^2}{N\cdot f_s} & & &
\end{array}
\tag{15.24}
$$

Chapter 16

abaqus-umat用户子程序二次开发

16.1 塑性力学

16.1.1 von mises应力/effective stress

参考《塑性力学基础》王仁 [2]

由主应力计算：

$$\sigma = \sqrt{\frac{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}{2}} \tag{16.1}$$

$$\sigma = \sqrt{\sigma_1^2+\sigma_2^2+\sigma_3^2-\sigma_1\sigma_2-\sigma_2\sigma_3-\sigma_3\sigma_1} \tag{16.2}$$

$$\begin{aligned}\sigma &= \sqrt{\frac{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}{2}}\\ &= \sqrt{\frac{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2+\sigma_2^2+\sigma_3^2-2\sigma_2\sigma_3+\sigma_3^2+\sigma_1^2-2\sigma_3\sigma_1}{2}}\\ &= \sqrt{\frac{2\sigma_1^2+2\sigma_2^2+2\sigma_3^2-2\sigma_1\sigma_2-2\sigma_2\sigma_3-2\sigma_3\sigma_1}{2}}\\ &= \sqrt{\sigma_1^2+\sigma_2^2+\sigma_3^2-\sigma_1\sigma_2-\sigma_2\sigma_3-\sigma_3\sigma_1}\end{aligned} \tag{16.3}$$

由应力分量计算：

$$\sigma = \sqrt{\frac{(\sigma_x-\sigma_y)^2+(\sigma_y-\sigma_z)^2+(\sigma_z-\sigma_x)^2+6(\tau_{xy}^2+\tau_{yz}^2+\tau_{zx}^2)}{2}} \tag{16.4}$$

由应力偏量计算：

$$\sigma = \sqrt{\frac{3}{2}\boldsymbol{S}:\boldsymbol{S}} \tag{16.5}$$

其中,$\boldsymbol{S}$ 为偏应力张量。

$$\sigma = \sqrt{\frac{3}{2}\left[S_x^2+S_y^2+S_z^2+2(S_{xy}^2+S_{yz}^2+S_{zx}^2)\right]} \tag{16.6}$$

$$J_1' = s_x+s_y+s_z = (\sigma_x+\sigma_y+\sigma_z)-3\sigma = 0 \tag{16.7}$$

$$J_2' = -(s_xs_y+s_ys_z+s_zs_x)+s_{xy}^2+s_{yz}^2+s_{zx}^2 = -(s_1s_2+s_2s_3+s_3s_1) \tag{16.8}$$

$$J_3' = \begin{bmatrix} s_x & s_{xy} & s_{xz}\\ s_{xy} & s_y & s_{yz}\\ s_{xz} & s_{yz} & s_z\end{bmatrix} = s_1s_2s_3 \tag{16.9}$$

$$(J_1')^2 = (s_x+s_y+s_z)^2 = s_x^2+s_y^2+s_z^2+2(s_xs_y+s_ys_z+s_zs_x) = 0 \tag{16.10}$$

因而有

$$-(s_xs_y+s_ys_z+s_zs_x) = \frac{1}{2}(s_x^2+s_y^2+s_z^2) \Longleftrightarrow -2(s_xs_y+s_ys_z+s_zs_x) = (s_x^2+s_y^2+s_z^2) \tag{16.11}$$

所以

$$J_2' = \frac{1}{2}(s_x^2+s_y^2+s_z^2)+s_{xy}^2+s_{yz}^2+s_{zx}^2 = \frac{1}{2}(s_x^2+s_y^2+s_z^2+2s_{xy}^2+2s_{yz}^2+2s_{zx}^2) = \frac{1}{2}s_{ij}s_{ij} = \frac{1}{2}\boldsymbol{s}:\boldsymbol{s} = \frac{1}{2}(s_1^2+s_2^2+s_3^2) \tag{16.12}$$

又因(直接分解)

$$\begin{aligned} s_x^2+s_y^2+s_z^2 &= \frac{2}{3}(s_x^2+s_y^2+s_z^2)+\frac{1}{3}(s_x^2+s_y^2+s_z^2) = \frac{2}{3}(s_x^2+s_y^2+s_z^2)-\frac{2}{3}(s_xs_y+s_ys_z+s_zs_x) \\ &= \frac{2}{3}(s_x^2+s_y^2+s_z^2-s_xs_y-s_ys_z-s_zs_x) = \frac{1}{3}\left[(s_x-s_y)^2+(s_y-s_z)^2+(s_z-s_x)^2\right] \end{aligned} \tag{16.13}$$

~~$a+b=c$~~

由此得

$$\begin{aligned} J_2' &= \frac{1}{6}\left[(s_x-s_y)^2+(s_y-s_z)^2+(s_z-s_x)^2\right]+s_{xy}^2+s_{yz}^2+s_{zx}^2 \\ &= \frac{1}{6}\left[(s_x-s_y)^2+(s_y-s_z)^2+(s_z-s_x)^2+6s_{xy}^2+6s_{yz}^2+6s_{zx}^2\right] \\ &= \frac{1}{6}\left[2s_x^2+2s_y^2+2s_z^2-2s_xs_y-2s_ys_z-2s_zs_x+6s_{xy}^2+6s_{yz}^2+6s_{zx}^2\right] \\ &= \frac{1}{6}\left[2s_x^2+2s_y^2+2s_z^2+4s_{xy}^2+4s_{yz}^2+4s_{zx}^2\right] \\ &= \frac{1}{3}\left[s_x^2+s_y^2+s_z^2+2s_{xy}^2+2s_{yz}^2+2s_{zx}^2\right] \end{aligned} \tag{16.14}$$

因为

$$\sigma = \frac{1}{3}(\sigma_x+\sigma_y+\sigma_z) = \frac{1}{3}J_1, \qquad s_{ij} = \sigma_{ij}-\sigma\delta_{ij} \tag{16.15}$$

$$s_x = \sigma_x-\sigma, \quad s_y = \sigma_y-\sigma, \quad s_z = \sigma_z-\sigma, \quad s_{xy} = \sigma_{xy}, \quad s_{yz} = \sigma_{yz}, \quad s_{zx} = \sigma_{zx} \tag{16.16}$$

$$\begin{aligned} s_x-s_y &= \sigma_x-\sigma-\sigma_y+\sigma = \sigma_x-\sigma_y \\ s_y-s_z &= \sigma_y-\sigma-\sigma_z+\sigma = \sigma_y-\sigma_z \\ s_z-s_x &= \sigma_z-\sigma-\sigma_x+\sigma = \sigma_z-\sigma_x \\ s_{xy}-s_{yz} &= \sigma_{xy}-\sigma_{yz} \\ s_{yz}-s_{zx} &= \sigma_{yz}-\sigma_{zx} \\ s_{zx}-s_{xy} &= \sigma_{zx}-\sigma_{xy} \end{aligned} \tag{16.17}$$

所以

$$J_2' = \frac{1}{6}\left[(\sigma_x-\sigma_y)^2+(\sigma_y-\sigma_z)^2+(\sigma_z-\sigma_x)^2\right]+\sigma_{xy}^2+\sigma_{yz}^2+\sigma_{zx}^2 \tag{16.18}$$

又因不论在何种坐标系下(主应力空间坐标系或者普通直角坐标系)，J_2' 的不变性, 由此得

$$J_2' = \frac{1}{6}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right] \tag{16.19}$$

展开得

$$J_2' = \frac{1}{3}\left[\sigma_1^2+\sigma_2^2+\sigma_3^2-\sigma_1\sigma_2-\sigma_2\sigma_3-\sigma_3\sigma_1\right] \tag{16.20}$$

下面是定义式，或者说根据三轴拉伸最初的公式，其他形式都是导出式!!!

$$\sigma = \sqrt{\frac{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}{2}} = \sqrt{3J_2'} \tag{16.21}$$

16.1.2 effective strain

为什么等效应力的系数是$\frac{3}{2}$，而等效应变的系数是$\frac{2}{3}$???

参考《塑性力学引论》王仁等

$$\varepsilon = \sqrt{\frac{4}{3}I_2'} \tag{16.22}$$

$$\overline{\sigma} = \sqrt{\frac{3}{2}s_{ij}s_{ij}} \qquad s_{ij} = \sigma_{ij} - \hat{\sigma}\delta_{ij} \qquad \hat{\sigma} = \frac{1}{3}\sigma_{ii} \tag{16.23}$$

$$\overline{\varepsilon} = \sqrt{\frac{2}{3}e_{ij}e_{ij}} \qquad e_{ij} = \varepsilon_{ij} - \hat{\varepsilon}\delta_{ij} \qquad \hat{\varepsilon} = \frac{1}{3}\varepsilon_{ii} \tag{16.24}$$

能简单验证的，就是单轴的拉伸压缩，不论什么情况的定义，这样如此简单的情形都是应该满足的，否则无法generalize。

既然你已经知道了等效应力是和单轴的应力去比较。那么应变也是一样的，也是和单轴的进行比较，但是还要满足一定的本构关系，如果将两者完全视为一样的话（二阶张量），形式就不应该存在不同啦。如果那样等效应力和等效应变就不存在联系了。至于前面的系数是为了满足单轴的情况。

等效应变里的系数2/3，是为了保证本构关系从三维情况下退化到一维情况时，本构关系定义的一致性，即能实现$\overline{\sigma} = E\overline{\varepsilon} \Longrightarrow \sigma = E\varepsilon$。也就是说，对于单轴拉伸情形（不可压条件下，泊松比为0.5），使用等效应力和等效应变定义式计算得到的值应各自等于单轴应力和单轴应变 $\overline{\sigma} = \sigma$, $\overline{\varepsilon} = \varepsilon$。具体可参考《塑性力学引论》王仁等p81。

在简单拉伸时，如果材料不可压缩，考虑到三维情形应变的泊松比效应，由

$$\varepsilon_{11} = \varepsilon > 0, \quad \varepsilon_{22} = \varepsilon_{33} = -\nu\varepsilon_{11} = -\frac{1}{2}\varepsilon \quad \text{and} \quad \varepsilon_{\rm ij} = 0 \text{ when } \rm i \neq j \tag{16.25}$$

可得各偏应变分量

$$\hat{\varepsilon} = \frac{1}{3}(\varepsilon - \frac{1}{2}\varepsilon - \frac{1}{2}\varepsilon) = 0, \quad e_{11} = \varepsilon_{11} = \varepsilon, \quad e_{22} = \varepsilon_{22} = -\frac{1}{2}\varepsilon, \quad e_{33} = \varepsilon_{33} = -\frac{1}{2}\varepsilon, \quad e_{12} = e_{23} = e_{31} = 0 \tag{16.26}$$

代入定义式得

$$\overline{\varepsilon} = \sqrt{\frac{2}{3}(e_{11}^2 + e_{22}^2 + e_{33}^2 + 2e_{12}^2 + 2e_{23}^2 + 2e_{31}^2)} = \sqrt{\frac{2}{3}(\varepsilon^2 + \frac{1}{4}\varepsilon^2 + \frac{1}{4}\varepsilon^2)} = \sqrt{\frac{2}{3}(\frac{3}{2}\varepsilon^2)} = \varepsilon \tag{16.27}$$

由

$$\sigma_{11} = \sigma > 0, \quad \sigma_{22} = \sigma_{33} = 0, \quad \text{and} \quad \sigma_{\rm ij} = 0 \text{ when } \rm i \neq j \tag{16.28}$$

可得各偏应力分量

$$\hat{\sigma} = \frac{1}{3}\sigma_{11} = \frac{1}{3}\sigma, \quad s_{11} = \sigma_{11} - \hat{\sigma} = \frac{2}{3}\sigma, \quad s_{22} = \sigma_{22} - \hat{\sigma} = -\frac{1}{3}\sigma, \quad s_{33} = \sigma_{33} - \hat{\sigma} = -\frac{1}{3}\sigma, \quad s_{12} = s_{23} = s_{31} = 0 \tag{16.29}$$

代入定义式得

$$\overline{\sigma} = \sqrt{\frac{3}{2}(s_{11}^2 + s_{22}^2 + s_{33}^2 + 2s_{12}^2 + 2s_{23}^2 + 2s_{31}^2)} = \sqrt{\frac{3}{2}(\frac{4}{9}\sigma^2 + \frac{1}{9}\sigma^2 + \frac{1}{9}\sigma^2)} = \sqrt{\frac{3}{2}(\frac{2}{3}\sigma^2)} = \sigma \tag{16.30}$$

此时，本构方程相合

$$\overline{\sigma} = E\overline{\varepsilon} \Longleftrightarrow \sigma = E\varepsilon \tag{16.31}$$

16.1.3 各向同性mises塑性模型

$$\boldsymbol{\sigma} = \lambda tr(\boldsymbol{\varepsilon}^e)\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon}^e \tag{16.32}$$

$$\boldsymbol{\varepsilon}^e = \boldsymbol{\varepsilon}_t^e + \Delta\boldsymbol{\varepsilon}^e = \boldsymbol{\varepsilon}_t^e + \Delta\boldsymbol{\varepsilon} - \Delta\boldsymbol{\varepsilon}^p \tag{16.33}$$

$$\begin{aligned}\boldsymbol{\sigma} &= \lambda tr(\boldsymbol{\varepsilon}_t^e + \Delta\boldsymbol{\varepsilon} - \Delta\boldsymbol{\varepsilon}^p)\boldsymbol{I} + 2\mu(\boldsymbol{\varepsilon}_t^e + \Delta\boldsymbol{\varepsilon} - \Delta\boldsymbol{\varepsilon}^p) \\ &= \lambda tr(\boldsymbol{\varepsilon}_t^e + \Delta\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu(\boldsymbol{\varepsilon}_t^e + \Delta\boldsymbol{\varepsilon}) - 2\mu\Delta\boldsymbol{\varepsilon}^p \\ &= \boldsymbol{\sigma}^{try} - 2\mu\Delta\boldsymbol{\varepsilon}^p \\ &= \boldsymbol{\sigma}^{try} - 2\mu\Delta\overline{\varepsilon}^p\boldsymbol{n} \\ &= \boldsymbol{\sigma}^{try} - 2\mu\Delta\overline{\varepsilon}^p\frac{3}{2}\frac{\boldsymbol{\sigma}'}{\overline{\sigma}}\end{aligned} \tag{16.34}$$

应力的静偏分解

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}' + \frac{1}{3}tr(\boldsymbol{\sigma})\boldsymbol{I} \tag{16.35}$$

所以

$$\boldsymbol{\sigma}' + \frac{1}{3}tr(\boldsymbol{\sigma})\boldsymbol{I} = \boldsymbol{\sigma}^{try} - 2\mu\Delta\overline{\varepsilon}^p\frac{3}{2}\frac{\boldsymbol{\sigma}'}{\overline{\sigma}} \tag{16.36}$$

移项、合并同类项

$$(1 + 3\mu\frac{\Delta\overline{\varepsilon}^p}{\overline{\sigma}})\boldsymbol{\sigma}' = \boldsymbol{\sigma}^{try} - \frac{1}{3}tr(\boldsymbol{\sigma})\boldsymbol{I} \tag{16.37}$$

对于等式右端

$$\begin{aligned}\boldsymbol{\sigma}^{try} - \frac{1}{3}tr(\boldsymbol{\sigma})\boldsymbol{I} &= \lambda tr(\boldsymbol{\varepsilon}_t^e + \Delta\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu(\boldsymbol{\varepsilon}_t^e + \Delta\boldsymbol{\varepsilon}) - Ktr(\boldsymbol{\varepsilon}^e)\boldsymbol{I} \\ &= \lambda tr(\boldsymbol{\varepsilon}_t^e + \Delta\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu(\boldsymbol{\varepsilon}_t^e + \Delta\boldsymbol{\varepsilon}) - Ktr(\boldsymbol{\varepsilon}_t^e + \Delta\boldsymbol{\varepsilon}^e)\boldsymbol{I} \\ &= \lambda tr(\boldsymbol{\varepsilon}_t^e + \Delta\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu(\boldsymbol{\varepsilon}_t^e + \Delta\boldsymbol{\varepsilon}) - Ktr(\boldsymbol{\varepsilon}_t^e + \Delta\boldsymbol{\varepsilon} - \Delta\boldsymbol{\varepsilon}^p)\boldsymbol{I} \\ &= \lambda tr(\boldsymbol{\varepsilon}_t^e + \Delta\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu(\boldsymbol{\varepsilon}_t^e + \Delta\boldsymbol{\varepsilon}) - Ktr(\boldsymbol{\varepsilon}_t^e + \Delta\boldsymbol{\varepsilon})\boldsymbol{I} \\ &= (\lambda - K)tr(\boldsymbol{\varepsilon}_t^e + \Delta\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu(\boldsymbol{\varepsilon}_t^e + \Delta\boldsymbol{\varepsilon}) \\ &= \boldsymbol{\sigma}^{try'}\end{aligned} \tag{16.38}$$

所以

$$(1 + 3\mu\frac{\Delta\overline{\varepsilon}^p}{\overline{\sigma}})\boldsymbol{\sigma}' = \boldsymbol{\sigma}^{try'} \tag{16.39}$$

等式两边同时作自身的内积

$$(1 + 3\mu\frac{\Delta\overline{\varepsilon}^p}{\overline{\sigma}})^2\boldsymbol{\sigma}' : \boldsymbol{\sigma}' = \boldsymbol{\sigma}^{try'} : \boldsymbol{\sigma}^{try'} \tag{16.40}$$

因为等效应力和等效尝试应力

$$\overline{\sigma} = (\frac{3}{2}\boldsymbol{\sigma}' : \boldsymbol{\sigma}')^{\frac{1}{2}}, \qquad \overline{\sigma}^{try} = (\frac{3}{2}\boldsymbol{\sigma}^{try} : \boldsymbol{\sigma}^{try})^{\frac{1}{2}} \tag{16.41}$$

所以

$$(1 + 3\mu\frac{\Delta\overline{\varepsilon}^p}{\overline{\sigma}})\overline{\sigma} = \overline{\sigma}^{try} \tag{16.42}$$

化简

$$\overline{\sigma} + 3\mu\Delta\overline{\varepsilon}^p = \overline{\sigma}^{try} \tag{16.43}$$

Mises屈服条件为

$$f = \overline{\sigma} - \sigma_{yield} = \overline{\sigma}^{try} - 3\mu\Delta\overline{\varepsilon}^p - \sigma_{yield} = 0 \tag{16.44}$$

其中，$f = \overline{\sigma} - \sigma_{yield}$为屈服函数。这里，$f(\Delta\overline{\varepsilon}^p)$ 为关于 $\Delta\overline{\varepsilon}^p$ 的函数，该方程为关于 $\Delta\overline{\varepsilon}^p$ 的非线性方程，可通过牛顿迭代法求解。

通过泰勒展开将非线性方程线性化

$$f + \frac{\partial f}{\partial\Delta\overline{\varepsilon}^p}d\Delta\overline{\varepsilon}^p = 0 \tag{16.45}$$

其中，

$$\frac{\partial f}{\partial\Delta\overline{\varepsilon}^p} = -3\mu \tag{16.46}$$

将 f 及其导数代入线性化屈服方程

$$\overline{\sigma}^{try} - 3\mu\Delta\overline{\varepsilon}^p - \sigma_{yield} - 3\mu d\Delta\overline{\varepsilon}^p = 0 \tag{16.47}$$

化简

$$d\Delta\overline{\varepsilon}^p = \frac{\overline{\sigma}^{try} - 3\mu\Delta\overline{\varepsilon}^p - \sigma_{yield}}{3\mu} \tag{16.48}$$

写成增量形式

$$
\begin{aligned}
d\Delta\bar{\varepsilon}^p &= \frac{\bar{\sigma}^{try} - 3\mu\Delta\bar{\varepsilon}^{p(k)} - \sigma_{yield}}{3\mu} \\
\Delta\bar{\varepsilon}^p &= \Delta\bar{\varepsilon}^{p(k)} + d\Delta\bar{\varepsilon}^p
\end{aligned}
\tag{16.49}
$$

Chapter 17

应用固体力学

8.1.5 有限元的简单一维实现

参考文献

[1] 陆明万, 弹性理论基础. 2021.

[2] 王仁, 塑性力学基础. 1982.