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ROBUST LINEAR EXTRAPOLATIONS OF STATIONARY PROCESSES IN L^P

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We consider minimax extrapolation problems in line with the seminal work by Huber, who introduced the minimax variance to the field of statistics. Extrapolation problem, as known as the extremal problem, can be regarded as a linear approximation on the unit disk in the complex plane. Although robust one-step ahead predictor and interpolator has already been considered separately in the previous literature, we give two conditions, as sufficient condition, to find the minimax extrapolator in the general framework from both the point of regarding the observation set as any subset of the integer set, and the point of considering the extrapolation error evaluated under the L^p norm. We show that there exists a minimax extrapolator for the class of spectral densities ϵ-contaminated by unknown spectral densities under our conditions. When the spectral distribution function is not absolutely continuous to the Lebesgue measure, we show that there exists an approximate extrapolator such that its maximal extrapolation error is arbitrarily close to the minimax error in the L^p interpolation problem.

1. Introduction. We consider minimax problem for the linear extrapolation of second-order stationary processes among the class that their spectral structures are vaguely specified. In [5], a specialized method for one step ahead prediction problem is introduced. The solution is applied to the interpolation problem in [13]. Also, [4] considered the minimax prediction when the spectral densities are contained in a convex set. [14] provided a game-theoretic approach to the minimax problem when the spectral densities in the Hilbert space. The minimax problem for linear extrapolation, however, is still an unexplored field although the linear extrapolation problem, initially developed in the Kolmogorov-Wiener theory of prediction, has been considered for a long time when the spectral structure of the stationary process is specified.

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Let $H(\omega)$ be the spectral distribution of the process $\{X_t\}$. If the spectral distribution function $H(\omega)$ is absolutely continuous with respect to the Lebesgue measure on $(-\pi, \pi]$, then $h(\omega)$ denotes its spectral density. One of results for the linear extrapolation problem, as an evolution from the observation set $\{X_t; t = -1, -2, \dots\}$ of the process to the different observation sets S, is considered in [11]. The linear extrapolation problem can be regarded as a minimization problem of the distance between 1 and $h(\omega)$, which is in the manifold generated by the observation set S.

Another direction of the linear extrapolation problem is the minimization of L^p error. [12] introduced new methods for two dual extremal problems in the Hardy spaces H^p. Based on the duality between the functional space in [10], the main approach to derive the extrapolator in L^p has been considered in [9, 1] and reference therein. Besides, minimizing L^p error has many potential applications. In statistics, quantile regression, corresponding to minimization of L^1 error developed in [8], has been more and more popular alternative to ordinary least squares regression. An L^1 approach to spectral analysis for stationary process has been considered in [2]. Our results in this paper are prospective to contribute to mathematics, probability and statistical inference in a variety of phases. (See [7] and references therein.)

We develop the theory of minimax extrapolation in this paper. The organization of the paper is given as follows. In Section 2, we give the sufficient conditions for the existence of minimax extrapolator in L^p. In Section 3, we show the minimax extrapolators in two examples of prediction and interpolation problems with the line of our sufficient conditions. In Section 4, we show that even if the spectral distribution $H(\omega)$ is not absolutely continuous with respect to the Lebesgue measure, there is still an approximate extrapolator for the process such that its maximal extrapolation error is arbitrarily close to the minimax error in the L^p interpolation problem.

2. Minimax extrapolation problem. In this section, we consider the minimax extrapolation problem in L^p for stationary processes. Suppose the spectral distribution function $H(\omega)$ of the stationary process $\{X_t; t \in \mathbb{Z}\}$ is absolutely continuous with respect to the Lebesgue measure on $(-\pi, \pi]$ and has the density function $h(\omega)$. Under certain conditions in the following, we provide the representation of minimax extrapolator for the minimax extrapolation problem in $L^p(h)$, where $h(\omega)$ is defined in a class of ϵ-contaminated spectral densities. Suppose $\mathscr{D}$ is the class of spectral density functions of which the integrals are 1.

An extrapolation problem in L^p is formulated by an extremal problem

$$\sigma_p^S(h) \equiv \inf_{\phi \in \mathscr{L}(S)} \int_{-\pi}^{\pi} |1-\phi(\omega)|^p h(\omega)\, \mathrm{d}\omega. \tag{2.1}$$

Here, $\mathscr{L}(S)$ is the completion of the linear hull of the set $\{\mathrm{e}^{\mathrm{i}j\omega};\, j \in S\}$, a subspace of $L^p(h)$, with respect to $L^p(h)$ norm. S is a subset of the set of all integers $\mathbb{Z}$ excluding 0, $\mathbb{Z} \setminus \{0\}$. Several different choices of the set S correspond to different extrapolation problems. Examples are given below.

Example 1. We can choose the set S as follows.

- $S_1 = \{\cdots, -3, -2, -1\}$: the one-step ahead prediction problem;
- $S_2 = \{\cdots, -3, -2, -1\} \cup \{1, 2, 3, \cdots\}$: the interpolation problem;
- $S_3 = \{\ldots, -3, -2, -1\} \setminus \{-k\}$: the incomplete past prediction problem;
- $S_4 = \{\ldots, -k-2, -k-1, -k\}$: the k-step ahead prediction problem.

For any linear extrapolator $\phi \in \mathscr{L}(S)$, define the extrapolation error $e_p^S(\phi, h)$ for the spectral density h by

$$e_p^S(\phi, h) = \int_{-\pi}^{\pi} |1-\phi(\omega)|^p h(\omega)\, \mathrm{d}\omega. \tag{2.2}$$

The optimal extrapolator $\phi^{h,S}(\omega)$, then, satisfies

$$\sigma_p^S(h) = \inf_{\phi \in \mathscr{L}(S)} e_p^S(\phi, h) = \int_{-\pi}^{\pi} |1-\phi^{h,S}(\omega)|^p h(\omega)\, \mathrm{d}\omega. \tag{2.3}$$

In general, it is difficult to determine the optimal extrapolator $\phi^{h,S}(\omega)$, since it not only depends on the density h but also depends on the set $S \subset \mathbb{Z}$.

Next, we formulate the problem of minimax extrapolation as follows. In most of natural situations, the spectral density h is not known *a priori*. Let $\mathscr{D}$ denote a certain class of density distributions supported by the interval $\Lambda \equiv (-\pi, \pi]$. Suppose $h(\omega)$ belongs to a class $\mathscr{F}$ of ϵ-contaminated spectral densities, that is,

$$\mathscr{F} = \{h \in \mathscr{D};\, h(\omega) = (1-\epsilon) f(\omega) + \epsilon g(\omega),\, g(\omega) \in \mathscr{D}\},$$

where $f(\omega)$ is a known spectral density distribution and $g(\omega)$ denotes a variable function which ranges over the class $\mathscr{D}$.

In such situation, we adopt the minimax principle as the criterion of the optimality. We define the minimax linear extrapolator by ϕ^*, which minimizes the maximal extrapolation error with respect to h, i.e.,

$$\max_{h \in \mathscr{F}} e_p^S(\phi^*, h) = \min_{\phi \in \mathscr{L}(S)} \max_{h \in \mathscr{F}} e_p^S(\phi, h). \tag{2.4}$$

We give the following two conditions to solve the problem (2.4) of minimax extrapolation. It is shown that when the conditions are both satisfied, the minimax linear extrapolator corresponds to the optimal extrapolator of one element in $\mathscr{F}$. Let $\operatorname{ess.sup} f$ denote the essential supremum of the function f, and μ the Lebesgue measure on Λ.

Condition 1. Define the set $A(h)$ by

$$A(h) = \{\omega : |1 - \phi^{h,S}(\omega)|^p = \operatorname{ess.} \sup_{\omega \in (-\pi,\pi]} |1 - \phi^{h,S}(\omega)|^p\}.$$

For brevity, denote $A(h)$ by A hereafter. As for the set A, it holds that

$$\mathcal{G} = \{g \in \mathscr{D}; \mu(A) > 0 \text{ for } h = (1-\varepsilon)f + \varepsilon g\} \neq \emptyset.$$

Condition 2. There exists $g^* \in \mathcal{G}$ such that g^* is distributed on $A(h^*)$, where $h^* = (1-\varepsilon)f + \varepsilon g^*$.

Under Conditions 1 and 2, we have the following theorem.

THEOREM 2.1. *If Condition 1 and Condition 2 hold, we have*

$$\min_{\phi \in \mathscr{L}(S)} \max_{h \in \mathscr{F}} e_p^S(\phi, h) = e_p^S(\phi^{h^*,S}, h^*), \tag{2.5}$$

and thus the minimax linear extrapolator ϕ^* is $\phi^* = \phi^{h^*,S}$.

PROOF. Consider the maximization problem of $\max_{h \in \mathscr{F}} e_p^S(\phi, h)$. It is equivalent to find the maximizer of

$$(1-\varepsilon)e_p^S(\phi, f) + \varepsilon \max_{g \in \mathscr{D}} e_p^S(\phi, g). \tag{2.6}$$

From Hölder's inequality, it holds that for any $\phi \in \mathscr{L}(S)$,

$$e_p^S(\phi, g) \leq \||1-\phi|^p\|_{L^\infty} \|g\|_{L^1}. \tag{2.7}$$

Under Condition 1, $e_p^S(\phi^{h,S}, h)$ can attain its upper bound if $g \in \mathcal{G}$; otherwise, there is no such $h \in \mathscr{F}$. Note that the equation in (2.7) holds if g is distributed on the set $A = \{\omega : |1 - \phi(\omega)|^p = \operatorname{ess.sup}_\omega |1-\phi|^p\}$. Now, from condition 2, we have

$$\max_{g \in \mathscr{D}} e_p^S(\phi^{h^*,S}, g) = e_p^S(\phi^{h^*,S}, g^*).$$

Thus, from (2.6),

$$\max_{h\in\mathscr{F}} e_p^S(\phi^{h^*,S},h) = (1-\varepsilon)e_p^S(\phi^{h^*,S},f)+\varepsilon\max_{g\in\mathscr{D}} e_p^S(\phi^{h^*,S},g) = e_p^S(\phi^{h^*,S},h^*). \tag{2.8}$$

Besides, from (2.2) and (2.3),

$$e_p^S(\phi^{h^*,S},h^*) = \min_{\phi\in\mathscr{L}(S)} e_p^S(\phi,h^*). \tag{2.9}$$

Combining (2.8) and (2.9), we have

$$\max_{h\in\mathscr{F}} e_p^S(\phi^{h^*,S},h) = \min_{\phi\in\mathscr{L}(S)} e_p^S(\phi,h^*).$$

Furthermore, for arbitrary $\phi\in\mathscr{L}(S)$,

$$\max_{h\in\mathscr{F}} e_p^S(\phi,h) \ge e_p^S(\phi,h^*) \ge \min_{\phi\in\mathscr{L}(S)} e_p^S(\phi,h^*) = \max_{h\in\mathscr{F}} e_p^S(\phi^{h^*,S},h).$$

Consequently, (2.5) holds since $\phi^{h^*,S}\in\mathscr{L}(S)$, which concludes Theorem 2.1. □

3. Two examples. In Section 3, we discuss two examples by using our new conditions. To consider the set A in both conditions, we have to derive the optimal extrapolator. After the derivation, we discuss the minimax problem (2.4) in each case.

Let $\mathcal{D}$ be the unit disc with central 0 and $H^p(\mathcal{D})$ be the Hardy space which consists of holomorphic functions f on the open unit disk with

$$\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(r\mathrm{e}^{\mathrm{i}\theta})|^p\,\mathrm{d}\theta\right\}^{1/p}$$

remaining bounded as $r\to 1^-$. Define $\|f\|_p=\lim_{r\to1^-}\{\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(r\mathrm{e}^{\mathrm{i}\theta})|^p\,\mathrm{d}\theta\}^{1/p}$ as the norm of f in $H^p(\mathcal{D})$. From now on, we suppose $1/p+1/q=1$.

3.1. *Case $S=S_1$.* Let us consider the prediction problem for $S=S_4$. Take $\bar{S}_4=\{k,k+1,k+2,\dots\}$.

THEOREM 3.1. *Suppose that h is nonnegative and integrable, $\log h\in L^1$, $p\ge 1$. Let $\mathcal{E}(z)$ be*

$$\mathcal{E}(z)=\exp\Big(\frac{1}{2\pi p}\int_{-\pi}^{\pi}\frac{\mathrm{e}^{\mathrm{i}\lambda}+z}{\mathrm{e}^{\mathrm{i}\lambda}-z}\log h(\mathrm{e}^{\mathrm{i}\lambda})\,\mathrm{d}\lambda\Big)$$

such that $\mathcal{E}^p$ *is the outer function of* h *and thus* $\mathcal{E}^p(\mathrm{e}^{\mathrm{i}\omega}) = h(\omega)$ *a.e. For the following minimal problem*

$$\sigma_p^{S_4}(h) = \min_{\phi \in \mathscr{L}(S_4)} \int_{-\pi}^{\pi} |1 - \phi(\omega)|^p h(\omega)\, \mathrm{d}\omega,$$

the optimal predictor satisfies

$$\bar{\phi}(\omega)\, \mathcal{E}(\mathrm{e}^{\mathrm{i}\omega}) = \mathcal{E}(\mathrm{e}^{\mathrm{i}\omega}) - (p_k(\mathcal{E}^{p/2}))^{2/p}(\mathrm{e}^{\mathrm{i}\omega})$$

and

$$\sigma_p^{S_4}(h) = \sum_{j=0}^{k-1} |c_j|^2,$$

where $p_k(\mathcal{E}^{p/2})(z) \equiv \sum_{j=0}^{k-1} c_j z^j$ *has no zeros in* $\mathcal{D}$*. Especially, it always holds when* $p = 2$*, no matter which* $p_k(\phi^{p/2})(z)$ *has no zeros in* $\mathcal{D}$ *or not.*

Proof. Since $\mathcal{E}$ and $\mathcal{E}^{-1}$ are analytic in the unit disc, $\mathcal{E}\mathscr{L}(\bar{S}_4) = \mathscr{L}(\bar{S}_4)$. Besides, $\mathrm{e}^{-\mathrm{i}k\lambda}\mathscr{L}(\bar{S}_4) = H^p(\mathcal{D})$. Then we have that

$$\begin{aligned}
\sigma_p^{S_4}(h)^{1/p} &= \min_{\phi \in \mathscr{L}(S_4)} \{\int_{-\pi}^{\pi} |1 - \phi(\omega)|^p h(\omega)\, \mathrm{d}\omega\}^{1/p} \\
&= \min_{\phi \in \mathscr{L}(\bar{S}_4)} \{\int_{-\pi}^{\pi} |\mathcal{E}(\mathrm{e}^{\mathrm{i}\omega}) - \mathcal{E}(\mathrm{e}^{\mathrm{i}\omega})\phi(\omega)|^p\, \mathrm{d}\omega\}^{1/p} \\
&= \min_{l \in \mathscr{L}(\bar{S}_4)} \{\int_{-\pi}^{\pi} |\mathcal{E}(\mathrm{e}^{\mathrm{i}\omega}) - l(\omega)|^p\, \mathrm{d}\omega\}^{1/p} \\
&= \min_{g \in H^p(\mathcal{D})} \{\int_{-\pi}^{\pi} |\mathrm{e}^{-\mathrm{i}k\omega}\mathcal{E}(\mathrm{e}^{\mathrm{i}\omega}) - g(\omega)|^p\, \mathrm{d}\omega\}^{1/p}
\end{aligned}$$

where $l = \mathcal{E}\phi$ and $g = \mathrm{e}^{-\mathrm{i}k\omega} l$.

Let $p_k(\mathcal{E}^{p/2}) = f$. Decompose $\mathcal{E}$ by $\mathcal{E} = \mathcal{E} - f^{2/p} + f^{2/p}$. Note that $z^{-k}(\mathcal{E} - f^{2/p}) \in H^p(\mathcal{D})$ since f has no zeros in the unit disc. Thus the extremal problem is equivalent to

$$\inf_{g \in H^p(\mathcal{D})} \|\mathrm{e}^{-\mathrm{i}k\omega} f^{2/p} - g\|_p. \tag{3.1}$$

Redefine $g = (\mathcal{E}\phi - \mathcal{E} + f^{2/p})\mathrm{e}^{-\mathrm{i}k\lambda}$. From Theorem 8.1([3]),

$$\begin{aligned}
\inf_{g \in H^p(\mathcal{D})} \|\mathrm{e}^{-\mathrm{i}k\omega} f^{2/p} - g\|_p &= \sup_{\substack{K(z) \in H^q(\mathcal{D}), \\ \|K(z)\|_q = 1}} \frac{1}{2\pi} \Big| \int_{|z|=1} z^{-k} f^{2/p}(z) K(z)\, \mathrm{d}z \Big| \\
&= \sup_{K(z) \in H^q(\mathcal{D})} \frac{|\int_{|z|=1} z^{-k} f^{2/p}(z) K(z)\, \mathrm{d}z|}{2\pi \|K(z)\|_q}.
\end{aligned}$$

Let $K^*(\mathrm{e}^{\mathrm{i}\omega})$ be

$$K^*(\mathrm{e}^{\mathrm{i}\omega}) = \frac{f(\mathrm{e}^{\mathrm{i}\omega}) \cdot \overline{f(\mathrm{e}^{\mathrm{i}\omega})}\mathrm{e}^{\mathrm{i}(k-1)\omega}}{f^{2/p}(\mathrm{e}^{\mathrm{i}\omega})}.$$

Notice that $K^*(z) \in H^q(\mathcal{D})$, and $\|K^*\|_q$ is $(1/2\pi)(\int f \cdot \bar{f}\,\mathrm{d}\theta)^{1/q}$ when $p > 1$, and $1/2\pi$ when $p = 1$. Also, when $g = 0$,

$$\|\mathrm{e}^{-\mathrm{i}k\omega} f^{2/p} - 0\|_p = \frac{|\int_{|z|=1} z^{-k} f^{2/p} K^*(z)\,\mathrm{d}z|}{2\pi\|K^*(z)\|_q}.$$

Meanwhile,

$$\begin{aligned}
\|\mathrm{e}^{-\mathrm{i}k\omega} f^{2/p} - 0\|_p &\geq \inf_{g \in H^p(\mathcal{D})} \|\mathrm{e}^{-\mathrm{i}k\omega} f^{2/p} - g\|_p \\
&= \sup_{K(z) \in H^q(\mathcal{D})} \frac{|\int_{|z|=1} z^{-k} f^{2/p}(z) K(z)\,\mathrm{d}z|}{2\pi\|K\|_q} \\
&\geq \frac{|\int_{|z|=1} z^{-k} f^{2/p}(z) K^*(z)\,\mathrm{d}z|}{2\pi\|K^*(z)\|_q}.
\end{aligned}$$

Therefore the function $g = 0$ is the minimizer of the extremal problem (3.1). From the definition of g, we obtain that ϕ satisfies

$$\bar{\phi}(\omega)\,\mathcal{E}(\mathrm{e}^{\mathrm{i}\omega}) = \mathcal{E}(\mathrm{e}^{\mathrm{i}\omega}) - (p_k(\mathcal{E}^{p/2}))^{2/p}(\mathrm{e}^{\mathrm{i}\omega}),$$

as the conclusion in Theorem 3.1. □

Let us consider the minimax linear prediction problem (2.4) for the observations set S_1 in the following.

Example 2. Suppose $S = S_1$. Note that $S_1 = S_4$ if $k = 1$ for S_4. For arbitrary $h \in \mathscr{F}$, from Theorem 3.1,

$$|1 - \phi^{h,S_1}(\omega)|^p = \frac{1}{h(\omega)} \exp \frac{1}{2\pi} \int_{-\pi}^{\pi} \log h(\omega)\,\mathrm{d}\omega \quad a.e. \tag{3.2}$$

Let the set A be $\{\omega : h(\omega) = \operatorname{ess.inf}_{\omega \in (-\pi,\pi]} h(\omega)\}$ for $h \in \mathscr{F}$. Assume that there exists $h^* \in \mathscr{F}$ such that $h^* = (1-\varepsilon)f + \varepsilon g^*$ satisfies Condition 1 and Condition 2. Set

$$m = \operatorname{ess.}\inf_{\omega} h^*(\omega), \quad E_m = A, \quad \text{and} \quad F_m = (-\pi, \pi] - A.$$

Accordingly, $\mu(E_m) > 0$ and g^* is distributed on the set E_m. Thus,

$$g^*(\omega) = \begin{cases} 0 & \text{for } \omega \in F_m, \\ \frac{m-(1-\varepsilon)f(\omega)}{\varepsilon} & \text{for } \omega \in E_m. \end{cases}$$

From $g^* \in \mathscr{D}$, that is, the integral of g^* is 1, m should satisfy the following equation

$$\int_{E_m} \frac{m - (1-\varepsilon)f(\omega)}{\varepsilon} \, \mathrm{d}\omega = 1. \tag{3.3}$$

Also, from the fact that g^* is nonnegative, $\mu(E_m) > 0$ and (3.2), the interval $(-\pi, \pi]$ should be divided by E_m and F_m as

$$\begin{aligned} E_m &= \{\omega \in (-\pi, \pi];\, m \geq (1-\epsilon)f(\omega)\}; \\ F_m &= \{\omega \in (-\pi, \pi];\, m < (1-\epsilon)f(\omega)\}. \end{aligned} \tag{3.4}$$

Noticing that $(1/\varepsilon)\int_{E_m} m - (1-\varepsilon)f(\omega)\,\mathrm{d}\omega$ is an increasing and continuous function with respect to m in the domain $[0, \infty)$, the solution of (3.3) and (3.4) exists uniquely. Thus, if we take

$$h^*(\omega) = \begin{cases} (1-\varepsilon)f(\omega) & \text{for } \omega \in F_m, \\ m & \text{for } \omega \in E_m, \end{cases} \tag{3.5}$$

then the predictor ϕ^{h^*, S_1} is the minimax linear predictor. The solution in [5] is a special case of our result for $p = 2$.

3.2. *Case $S = S_2$.* Consider the case $S = S_2$. The case corresponds to the interpolation problem in L^p. The optimal interpolator is given as follows.

THEOREM 3.2. *For $S = S_2$ and $1/p + 1/q = 1$, the following result holds.*

(i) *When $p > 1$, $h^{-q/p} \in L^s$ for some $s > 1$,*

$$\sigma_p^{S_2}(h) = (2\pi)^p \Big(\int_{-\pi}^{\pi} h(\omega)^{-q/p} \, \mathrm{d}\omega\Big)^{-p/q}.$$

The optimal interpolator is

$$\phi^{h, S_2}(\omega) = 1 - \frac{h(\omega)^{-q/p}}{(1/2\pi)\int_{-\pi}^{\pi} h(\omega)^{-q/p}\,\mathrm{d}\omega} \quad a.e.$$

(ii) *When $p = 1$,*

$$\sigma_p^{S_2}(h) = \frac{2\pi}{\|h^{-1}\|_{L^\infty}}.$$

PROOF. From the definition of $\sigma_p^{S_2}(h)$,

$$\begin{aligned} \sigma_p^{S_2}(h) &= \inf_{\phi \in \mathscr{L}(S_2)} \int_{-\pi}^{\pi} |1 - \phi(\omega)|^p h(\omega)\,\mathrm{d}\omega \\ &= \inf_{\phi \in \mathscr{L}(S_2)} \frac{\int_{-\pi}^{\pi} |1 - \phi(\omega)|^p h(\omega)\,\mathrm{d}\omega}{(2\pi)^{-p} |\int_{-\pi}^{\pi} (1 - \phi(\omega))\,\mathrm{d}\omega|^p}. \end{aligned}$$

Replace each $1-\phi$ with a member of $\mathscr{L}(S_2\cup\{0\})$, divided by its constant term. Then we obtain

$$\begin{aligned}
\sigma_p^{S_2}(h) &= \inf_{\phi\in\mathscr{L}(S_2\cup\{0\}),\int_{-\pi}^{\pi}\phi\,\mathrm{d}\omega\neq 0} \frac{\int_{-\pi}^{\pi}|\phi(\omega)|^p h(\omega)\,\mathrm{d}\omega}{(2\pi)^{-p}|\int_{-\pi}^{\pi}\phi(\omega)\,\mathrm{d}\omega|^p} \\
&= \left(\sup_{\substack{\phi\in\mathscr{L}(S_2\cup\{0\}),\\ \int_{-\pi}^{\pi}\phi(\omega)\,\mathrm{d}\omega\neq 0}} \frac{(2\pi)^{-p}|\int_{-\pi}^{\pi}\phi(\omega)\,\mathrm{d}\omega|^p}{\int_{-\pi}^{\pi}|\phi(\omega)|^p h(\omega)\,\mathrm{d}\omega}\right)^{-1} \\
&= \left(\sup_{\substack{\phi(\omega)\in\mathscr{L}(S_2\cup\{0\}),\\ \int_{-\pi}^{\pi}\phi(\omega)\,\mathrm{d}\omega\neq 0}} \frac{(2\pi)^{-p}|\int_{-\pi}^{\pi}\phi(\omega)h(\omega)^{1/p}h(\omega)^{-1/p}\,\mathrm{d}\omega|^p}{\int_{-\pi}^{\pi}|\phi(\omega)h(\omega)^{1/p}|^p\,\mathrm{d}\omega}\right)^{-1}.
\end{aligned}$$

Thus,

$$\sigma_p^{S_2}(h)^{1/p} = 2\pi\left(\sup_{\substack{\phi(\omega)\in\mathscr{L}(S_2\cup\{0\}),\\ \int_{-\pi}^{\pi}\phi(\omega)\,\mathrm{d}\omega\neq 0}} \frac{\int_{-\pi}^{\pi}\phi(\omega)h(\omega)^{1/p}h(\omega)^{-1/p}\,\mathrm{d}\omega}{\|\phi h^{1/p}\|_{L^p}}\right)^{-1}.$$

When $h(\omega)^{-1/p}\in L^q$, we can regard $h^{-1/p}$ as a function in L^p. Thus from Theorem 7.1([3]), we have

$$\sigma_p^{S_2}(h)^{1/p} = 2\pi\left(\inf_{\substack{\phi\in\mathscr{L}(S_2\cup\{0\}),\\ \int\phi\,\mathrm{d}\omega\neq 0,\int\phi h^{1/p}g\,\mathrm{d}\omega\equiv 0}} \|h^{-1/p}-g\|_{L^q}\right)^{-1}.$$

Since for arbitrary $\phi\in\mathscr{L}(S_2\cup\{0\}), \int\phi\,\mathrm{d}\omega\neq 0$ and $\int\phi h^{1/p}g\,\mathrm{d}\omega\equiv 0$, g must be 0 a.e. Thus we have, when $p>1$,

$$\sigma_p^{S_2}(h) = (2\pi)^p(\int_{-\pi}^{\pi}h^{-q/p}(\omega)\,\mathrm{d}\omega)^{-p/q}.$$

Furthermore, from Carleson's theorem([6]), if $h^{-q/p}\in L^s$ for some $s>1$, it holds that

$$\lim_{N\to\infty}\sum_{n=-N}^{N}\widehat{h^{-q/p}}(n)\cdot\mathrm{e}^{\mathrm{i}n\omega} = h^{-q/p}(\omega)\quad a.e.$$

Thus it is not difficult to see that the optimal interpolator is

$$\phi^{h,S_2}(\omega) = 1-\frac{h^{-q/p}(\omega)}{(1/2\pi)\int_{-\pi}^{\pi}h^{-q/p}(\omega)\,\mathrm{d}\omega}\quad a.e.$$

Besides, we have, when $p = 1$,

$$\sigma_p^{S_2}(h) = \frac{2\pi}{\|h^{-1}\|_{L^\infty}}.$$

□

Let us consider the minimax interpolation problem in the following.

EXAMPLE 3. Suppose $S = S_2$. From Theorem 3.2, it holds that

$$|1 - \phi^{h,S_2}(\omega)|^p = \frac{(2\pi)^p h^{-q}(\omega)}{(\int_{-\pi}^{\pi} h^{-q/p}(\omega)\,\mathrm{d}\omega)^p} \quad a.e.$$

Let the set A be equal to $\{\omega : h^q(\omega) = \operatorname{ess.inf}_{\omega\in(-\pi,\pi]} h^q(\omega)\}$, i.e.

$$\{\omega : h(\omega) = \operatorname{ess.} \inf_{\omega\in(-\pi,\pi]} h(\omega)\}.$$

We find that there exists $h^* \in \mathscr{F}$ such that h^* satisfies Condition 1 and Condition 2 if h^* is defined by (3.5). The minimax linear interpolator is ϕ^{h^*,S_2}. We omit the detail since the proof is similar to the case $S = S_1$. Especially, the result in [13] is also the special case of ours for $p = 2$.

4. Approximate minimax extrapolator. In this section, we consider the minimax extrapolation problem for two cases which are not included in Theorem 2.1. When spectral distribution function H is absolutely continuous with respect to the Lebesgue measure, the corresponding density function h exists, and then, Theorem 2.1 makes sense. On the other hand, when H has jumps, we have to modify the details of Theorem 2.1.

Let $\mathcal{J}_0$ be the class of all the distribution functions on $(-\pi, \pi]$. Let $\mathcal{J}$ be the class of all the absolutely continuous distribution functions on $(-\pi, \pi]$. Two cases considered in Section 4 are

(i) $\mathcal{H} = \{H : H = (1-\varepsilon)F + \varepsilon G, F \in \mathcal{J}_0, G \in \mathcal{J}\}$
(ii) $\mathcal{H}_0 = \{H : H = (1-\varepsilon)F + \varepsilon G, F \in \mathcal{J}, G \in \mathcal{J}_0\}$

in each following subsection.

4.1. *The case (i).* We only care about the case that the spectral distribution function $F(\omega)$ has jumps. Let $F(\omega) = \int_{-\pi}^{\omega} f(\tau)\,\mathrm{d}\tau + \sum_{\lambda_i\le\omega} \Delta F(\lambda_i)$, where $\Delta F(\lambda_i)$ stands for the saltus at the jump point λ_i. It is well known that the jump points are countable. Without loss of generality, we assume that there is only one jump at point λ_1.

PROPOSITION 4.1. *Suppose $\delta(\omega)$ is defined by*

$$\delta(\omega-\lambda_1)=\begin{cases}1 & \text{for } \omega=\lambda_1,\\ 0 & \text{otherwise.}\end{cases}$$

If $\delta(\omega)\in\mathscr{L}(S)$, then the minimax extrapolator ϕ^* is

$$\phi^*(\omega)=\begin{cases}1 & \text{for } \lambda=\lambda_1,\\ \phi^{h^*,S}(\omega) & \text{otherwise.}\end{cases}\tag{4.1}$$

PROOF. As in Theorem 2.1, $\phi^{h^*,S}$ satisfies conditions 1 and 2. The set $A(h^*)$ is not affected by the definition (4.1) of $\phi^*(\omega)$ if $\delta(\omega)\in\mathscr{L}(S)$. $\square$

EXAMPLE 4. Let us consider the minimax interpolation problem for the class $\mathcal{H}$. Here, we show that $\delta(\omega-\lambda_1)\in\mathscr{L}(S_2)$. Let $\delta_m(\omega-\lambda_1)=\sum_{t=-m}^{-1}\mathrm{e}^{\mathrm{i}t(\omega-\lambda_1)}/(2m)+\sum_{t=1}^{m}\mathrm{e}^{\mathrm{i}t(\omega-\lambda_1)}/(2m)$. Then, we find that

$$\begin{aligned}&\int_{-\pi}^{\pi}|\delta(\omega-\lambda_1)-\delta_m(\omega-\lambda_1)|^p\,\mathrm{d}H(\omega)\\ &\quad=\int_{(-\pi,\pi]\setminus\{\lambda_1\}}\left|\frac{\sin\{(m+1/2)(\omega-\lambda_1)\}-\sin\{(1/2)(\omega-\lambda_1)\}}{2m\sin\{(\omega-\lambda_1)/2\}}\right|^p\mathrm{d}H(\omega)\\ &\qquad\qquad+\int_{\{\lambda_1\}}0\,\mathrm{d}H(\omega).\end{aligned}$$

Since the integrand in the right hand side tends to 0 pointwise as $m\to\infty$ from Lebesgue's convergence theorem, the integral converges to 0. Thus, $\int_{-\pi}^{\pi}|\delta(\omega-\lambda_1)-\delta_m(\omega-\lambda_1)|^p\,\mathrm{d}H(\omega)\to 0$ implies that $\|\delta(\omega-\lambda_1)-\delta_m(\omega-\lambda_1)\|_{L^p(\mathrm{d}H)}\to 0$, which in turn means that $|\|\delta(\omega-\lambda_1)\|_{L^p(\mathrm{d}H)}-\|\delta_m(\omega-\lambda_1)\|_{L^p(\mathrm{d}H)}|\to 0$. Thus $\delta(\omega-\lambda_1)$ belongs to $\mathscr{L}(S_2)$.

4.2. *The case (ii).* In this case, without loss of generality, we may assume that F is absolutely continuous and has a density f. Let $f(-\pi)=\lim_{x\to-\pi^+}f(x)$, then f is defined on a closed interval. If $G\in\mathscr{J}_0$, then G may have jumps at any uncertain points. Thus, in general, it holds that

$$\max_{H\in\mathcal{H}_0}e_p^S(\phi^*,\mathrm{d}H)>e_p^S(\phi^{h^*,S},h^*),$$

where ϕ^* is the minimax linear extrapolator for the class $\mathcal{H}$.

For some S, however, we can construct a predictor $\tilde{\phi}$ such that the maximum prediction error in $\mathcal{H}_0$ can be sufficiently close to the minimax prediction error in $\mathcal{H}$. We will leave it as an open problem that under what conditions for the set S, for any $\delta > 0$, there exists a extrapolator $\tilde{\phi}$ such that

$$\max_{H \in \mathcal{H}_0} e_p^S(\tilde{\phi}, \mathrm{d}H) < e_p^S(\phi^{h^*,S}, h^*) + \delta. \tag{4.2}$$

As a subproblem, (4.2) holds if $S = S_1$ or $S = S_2$. For $S = S_1$, (4.2) can be shown in line with the argument in [5]. Therefore, we provide the proof for the case $S = S_2$ in the following.

Theorem 4.2. *For any $\delta > 0$, there exists a extrapolator $\tilde{\phi}$ such that*

$$\max_{H \in \mathcal{H}_0} e_p^{S_2}(\tilde{\phi}, \mathrm{d}H) < e_p^{S_2}(\phi^{h^*,S_2}, h^*) + \delta.$$

Notice that the optimal interpolator $\phi^{h^*,S_2}(\omega)$ for $h^* \in \mathcal{H}$ is

$$\phi^{h^*,S_2}(\omega) = 1 - \frac{h^*(\omega)^{-q/p}}{(1/2\pi)\int_{-\pi}^{\pi} h^*(\omega)^{-q/p}\,\mathrm{d}\omega} \quad a.e. \tag{4.3}$$

Here, the equation (4.3) only holds almost everywhere. We approximate f in h^* by a differentiable function $\tilde{f}$. Let m and $\tilde{m}$ be the functionals $m(f)$ and $m(\tilde{f})$ determined by the solution of (3.3) with $f(\omega)$ and $\tilde{f}(\omega)$. We provide Lemmas 4.3 and 4.4 before the proof of Theorem 4.2.

Lemma 4.3. *For any $\delta > 0$, there exists $\sigma > 0$ such that if*

$$\int_{-\pi}^{\pi} |f(\omega)^q - \tilde{f}(\omega)^q|\,\mathrm{d}\omega < \sigma,$$

then it holds that

$$|m - \tilde{m}| = |m(f) - m(\tilde{f})| < \delta.$$

Proof. As (3.4), we can define the set E_m and $\tilde{E}_{\tilde{m}}$ by

$$\begin{aligned} E_m &= \{\omega \in (-\pi, \pi];\, m \geq (1-\epsilon) f(\omega)\}; \\ \tilde{E}_{\tilde{m}} &= \{\omega \in (-\pi, \pi];\, \tilde{m} \geq (1-\epsilon) \tilde{f}(\omega)\}. \end{aligned}$$

Also, suppose $F_m = \Lambda \setminus E_m$ and $\tilde{F}_{\tilde{m}} = \Lambda \setminus \tilde{E}_{\tilde{m}}$. Without loss of generality, suppose $\tilde{m} \geq m$, and then

$$|m - \tilde{m}| = \frac{\int_{E_m} |m - \tilde{m}|\,\mathrm{d}\omega}{\int_{E_m} \mathrm{d}\omega} = \frac{\int_{E_m} (\tilde{m} - m)\,\mathrm{d}\omega}{\mu(E_m)}; \tag{4.4}$$

otherwise, consider the integration on $\tilde{E}_{\tilde{m}}$. Note that $\mu(E_m) > 0$ and $\mu(\tilde{E}_{\tilde{m}}) > 0$. Let us consider the numerator at the right hand side in (4.4). From (3.3), it holds that

$$\int_{E_m} m - (1-\epsilon) f(\omega)\,\mathrm{d}\omega = \int_{\tilde{E}_{\tilde{m}}} \tilde{m} - (1-\epsilon)\tilde{f}(\omega)\,\mathrm{d}\omega.$$

Thus,

$$\begin{aligned}
\int_{E_m} (\tilde{m} - m)\,\mathrm{d}\omega &= \int_{E_m} \{\tilde{m} - (1-\epsilon) f(\omega)\}\,\mathrm{d}\omega + \int_{E_m} \{(1-\epsilon) f(\omega) - m\}\,\mathrm{d}\omega \\
&= \int_{E_m} \{\tilde{m} - (1-\epsilon) f(\omega)\}\,\mathrm{d}\omega + \int_{\tilde{E}_{\tilde{m}}} \{(1-\epsilon)\tilde{f}(\omega) - \tilde{m}\}\,\mathrm{d}\omega \\
&= \int_{E_m \cap \tilde{E}_{\tilde{m}}} (1-\epsilon)\{\tilde{f}(\omega) - f(\omega)\} d\omega \\
&\qquad + \int_{E_m \cap \tilde{F}_{\tilde{m}}} \{\tilde{m} - (1-\epsilon) f(\omega)\}\,\mathrm{d}\omega \\
&\qquad\quad + \int_{\tilde{E}_{\tilde{m}} \cap F_m} \{(1-\epsilon)\tilde{f}(\omega) - \tilde{m}\}\,\mathrm{d}\omega \\
&\le \int_{E_m} (1-\epsilon)|\tilde{f}(\omega) - f(\omega)|\,\mathrm{d}\omega.
\end{aligned}$$

Note that, on $\tilde{E}_{\tilde{m}}$, $(1-\epsilon)\tilde{f}(\omega) - \tilde{m} \le 0$. Therefore,

$$\int_{E_m} |\tilde{m} - m|\,\mathrm{d}\omega \le \int_{-\pi}^{\pi} (1-\epsilon)|\tilde{f}(\omega) - f(\omega)|\,\mathrm{d}\omega.$$

Since it holds that

$$\int_{-\pi}^{\pi} |f(\omega)^q - \tilde{f}(\omega)^q|\,\mathrm{d}\omega \ge \int_{-\pi}^{\pi} |f(\omega) - \tilde{f}(\omega)|^q\,\mathrm{d}\omega \ge \left(\int_{-\pi}^{\pi} |f(\omega) - \tilde{f}(\omega)|\,\mathrm{d}\omega\right)^q,$$

it is easy to see that $|m - \tilde{m}| < \delta$, if we take $\sigma = \{2\pi(1-\epsilon)\}^{-q}\delta^q$. □

Lemma 4.4. *For any $\delta > 0$, if*

$$\int_{-\pi}^{\pi} |f(\omega)^q - \tilde{f}(\omega)^q|\,\mathrm{d}\omega < \sigma, \tag{4.5}$$

then it holds that

$$\max_{H \in \mathcal{H}_0} \|1 - \phi^{\tilde{h}^*, S_2}(\omega)\|^p_{L^p(\mathrm{d}H)} < \max_{H \in \mathcal{H}} \|1 - \phi^{h^*, S_2}(\omega)\|^p_{L^p(\mathrm{d}H)} + \delta.$$

Proof. For (4.5), if we can show that for any $\sigma^* > 0$,

$$\left| m^{-q} - \tilde{m}^{-q} \right| < \sigma^*, \tag{4.6}$$

$$\Big| \Big(\int_{-\pi}^{\pi} h^*(\omega)^{-q/p} \, \mathrm{d}\omega \Big)^{-p} - \Big(\int_{-\pi}^{\pi} \tilde{h}^*(\omega)^{-q/p} \, \mathrm{d}\omega \Big)^{-p} \Big| < \sigma^*, \tag{4.7}$$

and

$$\Big| \int \frac{(1-\varepsilon) f(\omega)}{h^*(\omega)^q} \, \mathrm{d}\omega - \int \frac{(1-\varepsilon) f(\omega)}{\tilde{h}^*(\omega)^q} \, \mathrm{d}\omega \Big| < \sigma^*, \tag{4.8}$$

then Lemma 4.4 holds since

$$\begin{aligned} &\max_{H \in \mathcal{H}} \| 1 - \phi^{h^*, S_2}(\omega) \|^p_{L^p(\mathrm{d}H)} \\ &\qquad = \frac{(2\pi)^p}{(\int_{-\pi}^{\pi} h^*(\omega)^{-q/p} \, \mathrm{d}\omega)^p} \Big\{ \int_{-\pi}^{\pi} (1-\varepsilon) f(\omega) \cdot h^*(\omega)^{-q} \, \mathrm{d}\omega + \frac{\varepsilon}{m^q} \Big\}, \end{aligned}$$

and

$$\begin{aligned} &\max_{H \in \mathcal{H}_0} \| 1 - \phi^{\tilde{h}^*, S_2}(\omega) \|^p_{L^p(\mathrm{d}H)} \\ &\qquad = \frac{(2\pi)^p}{(\int_{-\pi}^{\pi} \tilde{h}^*(\omega)^{-q/p} \, \mathrm{d}\omega)^p} \Big\{ \int_{-\pi}^{\pi} (1-\varepsilon) f(\omega) \cdot \tilde{h}^*(\omega)^{-q} \, \mathrm{d}\omega + \frac{\varepsilon}{\tilde{m}^q} \Big\}. \end{aligned}$$

From Lemma 4.3, (4.6) holds obviously. Note that

$$\begin{aligned} &\int_{-\pi}^{\pi} | h^*(\omega)^q - \tilde{h}^*(\omega)^q | \, \mathrm{d}\omega \\ &\qquad < (1-\varepsilon) \int_{-\pi}^{\pi} | f(\omega)^q - \tilde{f}(\omega)^q | \, \mathrm{d}\omega + \int_{-\pi}^{\pi} | m^q - \tilde{m}^q | \, \mathrm{d}\omega. \end{aligned}$$

If we take the function $\tilde{f}$ satisfying $\int_{-\pi}^{\pi} | f^q(\omega) - \tilde{f}^q(\omega) | \, \mathrm{d}\omega < \eta$ and $|m^q(\omega) - m^q(\nu)| < \eta$ simultaneously, then it holds that

$$\int_{-\pi}^{\pi} | h^*(\omega)^q - \tilde{h}^*(\omega)^q | \, \mathrm{d}\omega < \sigma^*.$$

Also, we have

$$\begin{aligned} \Big(\int_{-\pi}^{\pi} | h^*(\omega)^{q/p} - \tilde{h}^*(\omega)^{q/p} | \, \mathrm{d}\omega \Big)^p &\le \int_{-\pi}^{\pi} | h^*(\omega)^{q/p} - \tilde{h}^*(\omega)^{q/p} |^p \, \mathrm{d}\omega \\ &\le \int_{-\pi}^{\pi} | h^*(\omega)^q - \tilde{h}^*(\omega)^q | \, \mathrm{d}\omega < \sigma^*. \end{aligned}$$

Thus, (4.7) holds since

$$\int_{-\pi}^{\pi} |h^*(\omega)^{-q/p} - \tilde{h}^*(\omega)^{-q/p}| \,\mathrm{d}\omega \leq \frac{\int_{-\pi}^{\pi} |h^*(\omega)^{q/p} - \tilde{h}^*(\omega)^{q/p}| \,\mathrm{d}\omega}{m^{q/p} \cdot \tilde{m}^{q/p}}.$$

As for (4.8), it also holds since

$$\begin{aligned}
&\Big| \int_{-\pi}^{\pi} \frac{(1-\varepsilon) f(\omega)}{h^*(\omega)^q} \,\mathrm{d}\omega - \int_{-\pi}^{\pi} \frac{(1-\varepsilon) f(\omega)}{\tilde{h}^*(\omega)^q} \,\mathrm{d}\omega \Big| \\
\leq \quad & \int_{E_m} \Big| \frac{h^*(\omega)^q - \tilde{h}^*(\omega)^q}{m^{q-1} \cdot \tilde{h}^*(\omega)^q} \Big| \,\mathrm{d}\omega + \int_{F_m} \Big| \frac{h^*(\omega)^q - \tilde{h}^*(\omega)^q}{h^*(\omega)^{q-1} \tilde{h}^*(\omega)^q} \Big| \,\mathrm{d}\omega \\
\leq \quad & \int_{-\pi}^{\pi} \Big| \frac{h^*(\omega)^q - h^*(\omega)^q}{h^*(\omega)^{q-1} \cdot \tilde{m}} \Big| \,\mathrm{d}\omega.
\end{aligned}$$

Therefore, Lemma 4.4 holds if we take a common σ such that (4.6), (4.7) and (4.8) hold simultaneously. $\square$

Proof. For any $\delta > 0$ and integrable function $f(\omega)^q$ on the closed interval, there exists a differentiable function $\tilde{f}$ such that

$$|f(\omega)^q - \tilde{f}(\omega)^q| < \delta.$$

Thus, we can find a differentiable function $\tilde{f}$ so that Lemma 4.3 holds. Then it holds that $\tilde{h}^*$ has left and right derivatives at every points. The Fourier series converge to the average of the left and right limits every where since $(\tilde{h}^*)^{-q/p}$ also has left and right derivatives. The extrapolator $\phi^{\tilde{h}^*, S_2}$ convergences to $1 - \frac{\tilde{h}^*(\omega)^{-q/p}}{(1/2\pi)\int_{-\pi}^{\pi} \tilde{h}^*(\omega)^{-q/p} \,\mathrm{d}\omega}$ everywhere but π and $-\pi$. Let $\tilde{\phi}$ be

$$\tilde{\phi}(\omega) = \begin{cases} 1 - \frac{\tilde{h}^*(\omega)^{-q/p}}{(1/2\pi)\int_{-\pi}^{\pi} \tilde{h}^*(\omega)^{-q/p} \,\mathrm{d}\omega}, & \text{when } |\omega| < -\pi, \\ 1 & \text{when } \omega = \pi \text{ or } -\pi, \end{cases}$$

then we find that for arbitrary δ, there exists $\tilde{\phi}$ such that

$$\max_{H \in \mathcal{H}_0} e_p^{S_2}(\tilde{\phi}, \mathrm{d}H) < e_p^{S_2}(\phi^{h^*, S_2}, h^*) + \delta.$$

$\square$

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