Quantile tests in frequency domain for sinusoid models

Yan Liu

Abstract For second order stationary processes, the spectral distribution function is uniquely determined by the autocovariance functions of the processes. We define the quantiles of the spectral distribution function and propose two estimators for the quantiles. Asymptotic properties of both estimators are elucidated and the difference from the quantile estimators in time domain is also indicated. We construct a testing procedure of quantile tests from the asymptotic distribution of the estimators and strong statistical power is shown in our numerical studies.

Keywords Frequency domain · Stationary process · Quantile test · Sinusoid models · Periodogram · Asymptotic distribution

1 Introduction

Nowadays, the quantile based estimation becomes a notable method in statistics. Not only statistical inference for the quantile of cumulative distribution function is considered, the quantile regression, a method taking place of the ordinary regression, is also broadly used for statistical inference. (See Koenker (2005).) In the area of time series analysis, however, the quantile based inference is still undeveloped yet. A fascinating approach in frequency domain, called "quantile periodogram" is proposed and studied in Li (2008, 2012). The method associated with copulas, quantiles and ranks are developed in Dette et al. (2015).

As there exists a well-behaved spectral distribution function for second order stationary process, we introduce the quantile of the spectral distribution and develop a statistical inference theory for it. We also propose a quantile test in frequency domain to test the dependence structure of second order stationary process, since the spectral distribution function is uniquely determined by the autocovariance functions of the process.

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In the context of time series analysis, Whittle (1952) mentioned that "the search for periodicities" constituted the whole of time series theory. He proposed an estimation method based on a nonlinear model driven by a simple harmonic component. After the work, to estimate the frequency has been a remarkable statistical analysis. A sequential literature by Whittle (1952), Walker (1971), Hannan (1973), Rice and Rosenblatt (1988) and Quinn and Thomson (1991) investigated the method proposed by Whittle (1952) and pointed out the misunderstandings in Whittle (1952), respectively. The noise structure is also generalized from independent and identically distributed white noise to the second order stationary process. The main result in those works revealed the properties of the periodogram and showed that the convergence factor of the estimator for the frequencies is $n^{3/2}$, which is different from well known order $n^{1/2}$, although the asymptotic distribution of the method is Gaussian.

Quinn and Hannan (2001) reviewed all the results above and proposed an alternative approach based on an iterative ARMA method. In reality, they found that the nonlinear model for $\{y_t\}$ with a peculiar frequency structure plus stationary process $\{x_t\}$, called "sinusoid models", such that $y_t = A\cos(\lambda t+\phi)+x_t$ can be rewritten, by the trigonometric relation, as $y_t - \beta y_{t-1} + y_{t-2} = x_t - \alpha x_{t-1} + x_{t-2}$, where $\alpha = \beta$ depend on the peculiar frequency. The method can be summarized by estimating β for given α and substituting β for α until both α and β converge.

Different from all the methods above, we employ the check function to estimate quantiles, the frequencies of spectral distribution function, for second order stationary process. In view of correspondence between the spectral density function and the periodogram for the stationary process, we first directly apply the objective function to the bare periodogram. It is expected the asymptotic normality of the approach from the result by Hosoya (1989) on the bracketing condition in frequency case. The approach for estimating in frequency domain certainly has the consistency for the true value. However, asymptotic normality of the quantile estimator based on the bare periodogram does not hold, which is obviously different from the quantile estimation theory in time domain. We give the results on the asymptotic properties of the estimator and modified the estimator. The modified estimator, by the method of smoothing, is asymptotically normal distributed. We extended our result to the sinusoid models and applied the asymptotic distribution to the quantile tests in the frequency domain.

The notations and symbols used in this paper are listed in the following: for a vector or a matrix A, A_j and A_{ij}, respectively, denote the jth and the (i,j)th element of corresponding vector and matrix; A' denotes the transpose of the matrix A; $\mathrm{cum}(X_1,\ldots,X_n)$ denotes the joint cumulant of the random variables $\{X_1,\ldots,X_n\}$; for stationary process $\{X_t\}$, the joint cumulant $\mathrm{cum}_X(u_1,\ldots,u_{n-1})$ simply denotes $\mathrm{cum}(X_t,X_{t+u_1},\ldots,X_{t+u_{n-1}})$; L^p denotes the space of complex-valued functions on $[-\pi,\pi]$, equipped with L^p norm $\|g\|_p$, i.e., $\{\int_{-\pi}^{\pi}|g(\omega)|^p d\omega\}^{1/p}$; $\mathbb{1}(\cdot)$ denotes the indicator function; e denotes the Napier's constant; I_d denotes the d-dimensional identity matrix; $\xrightarrow{\mathcal{P}}$ and $\xrightarrow{\mathcal{L}}$ denote the convergence in probability and the convergence in law, respectively.

2 Preliminaries

In this section, we review the spectral distribution functions of second order stationary processes and introduce the quantiles of the spectral distribution functions. Suppose $\{X_t\,;\,t \in \mathbb{Z}\}$ is a zero mean second order stationary process with finite autocovariance function $R_X(h) = \text{Cov}(X_{t+h}, X_t)$, for $h \in \mathbb{Z}$. From Herglotz's theorem, there exists a right continuous, non-decreasing, bounded distribution function $F_X(\omega)$ on $[-\pi, \pi]$ for the autocovariance function $R_X(h)$ of the process such that

$$R_X(h) = \int_{-\pi}^{\pi} e^{-ih\omega} F_X(d\omega), \quad (h \in \mathbb{Z}).$$

Explicitly, the spectral distribution function $F_X(\omega)$ is represented by

$$F_X(\omega) = \lim_{n \to \infty} \frac{1}{2\pi} \sum_{h=-n}^{n} R_X(h) \frac{\exp(-i\omega h) - 1}{-ih}. \tag{1}$$

The structure of the second order stationary process can be discriminated by their own spectral distribution function $F_X(\omega)$. Below, we give 4 figures of spectral distribution functions of second order Gaussian stationary processes, including White noise, MA(1) process with coefficient 0.9, AR(1) process with coefficient 0.9 and -0.9.
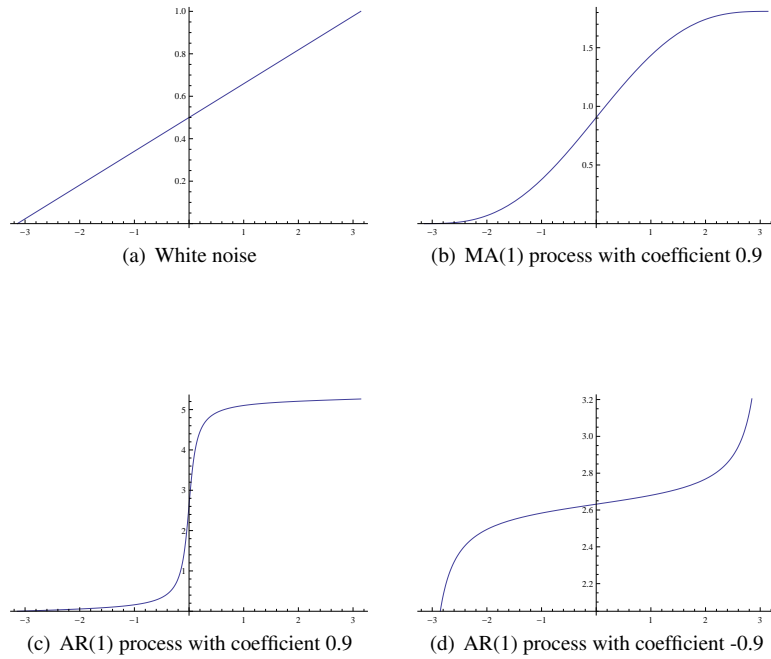

(a) White noise

(b) MA(1) process with coefficient 0.9

(c) AR(1) process with coefficient 0.9

(d) AR(1) process with coefficient -0.9

Fig. 1 Figures of spectral distributions of second order stationary processes

To be specific, if the spectral distribution function $F_X(\omega)$ is absolutely continuous with respect to Lebesgue measure, then $\{X_t\}$ has the spectral density $f_X(\omega)$, which is corresponding to the hth autocovariance function $R_X(h)$ by

$$R_X(h) = \int_{-\pi}^{\pi} e^{-ih\omega} f_X(\omega) d\omega.$$

Next, we introduce the pth quantile λ_p of the spectral distribution function $F_X(\omega)$. For simplicity, write $R_X(0) = \Sigma_X$. Note that the spectral distribution function $F_X(\omega)$ takes value on $[0, \Sigma_X]$. The generalized inverse distribution function $F_X^{-1}(\psi)$ for $0 \leq \psi \leq \Sigma_X$ is defined by

$$F_X^{-1}(\psi) = \inf\{\omega \,;\, F_X(\omega) \geq \psi\}.$$

For $0 \leq p = \Sigma_X^{-1}\psi \leq 1$, we define the pth quantile λ_p as

$$\lambda_p := F_X^{-1}(\Sigma_X^{-1}\psi) = \inf\{\omega \,;\, F_X(\omega)\Sigma_X^{-1} \geq p\}. \tag{2}$$

Define $\Lambda = [-\pi, \pi]$. In the following, we show that the pth quantile λ_p can be defined by the minimizer of the following objective function $S(\theta)$, i.e.,

$$S(\theta) = \int_{-\pi}^{\pi} \rho_p(\omega - \theta) F_X(d\omega), \tag{3}$$

where $\rho_\tau(u)$, called "the check function" (e.g. Koenker (2005)), is defined as

$$\rho_\tau(u) = u(\tau - \mathbb{1}(u < 0)).$$

Theorem 1 *Suppose $\{X_t;\, t \in \mathbb{Z}\}$ is a zero mean second order stationary process with spectral distribution function $F_X(\omega)$. Define $S(\theta)$ by (3). Then the pth quantile λ_p of the spectral distribution $F_X(\omega)$ is a minimizer of $S(\theta)$. Furthermore, λ_p is unique and satisfies*

$$\lambda_p = \inf\{\omega \in \Lambda;\, S(\omega) = \min_{\theta \in \Lambda} S(\theta)\}. \tag{4}$$

The representation (4) of the pth quantile λ_p of the spectral distribution function $F_X(\omega)$ is useful when we consider the estimation theory of λ_p. From the definition of the spectral distribution function $F_X(\omega)$, $F_X(\omega)$ is uniquely determined by the autocovariance function $R_X(h)$ ($h \in \mathbb{Z}$). Accordingly, the dependence structure of the second order stationary process $\{X_t;\, t \in \mathbb{Z}\}$ can be discriminated by the pth quantile λ_p since $\lambda_p \neq \lambda_p'$ if $p \neq 0, 1/2, 1$ and $F_X(\omega) \neq cF_X'(\omega)$, $c \in \mathbb{R}$.

Let us consider the estimation procedure for λ_p. Suppose the observation stretch of the process is defined by $\{X_t;\, 1 \leq t \leq n\}$. The parameter space for the pth quantile λ_p is defined by Λ. λ_p is in the interior of Λ. The objective function $S_n(\theta)$ for estimation can be defined by

$$S_n(\theta) = \int_{-\pi}^{\pi} \rho_p(\omega - \theta) I_{n,X}(\omega) d\omega, \tag{5}$$

where $I_{n,X}(\omega)$ is the periodogram based on the observation stretch, and defined by

$$I_{n,X}(\omega) = \frac{1}{2\pi n}\left|\sum_{j=1}^{n} X_j e^{ij\omega}\right|^2. \tag{6}$$

Hence, the estimator $\hat{\lambda}_p$ for λ_p can be defined by

$$\hat{\lambda}_p \equiv \hat{\lambda}_{p,n} = \arg\min_{\theta \in \Lambda} S_n(\theta). \tag{7}$$

3 Asymptotic distribution of $\hat{\lambda}_p$ for stationary processes

In this section, we consider the asymptotic properties of the estimator $\hat{\lambda}_p$ defined by (7) for stationary process $\{X_t;\ t \in \mathbb{Z}\}$ under the following assumptions.

Assumption 1   (i) $\{X_t\}$ *is a zero mean, strictly stationary real valued process, all of whose moments exist with*

$$\sum_{u_1,\ldots,u_{k-1}=-\infty}^{\infty} |\mathrm{cum}_X(u_1,\ldots,u_{k-1})| < \infty, \quad \text{for } k = 2,3,\ldots.$$

(ii) $f_X(\omega) \in Lip(\alpha)$ *for* $\alpha > 1/2$.

Under Assumption 1, the fourth order spectral density is defined by

$$Q_X(\omega_1,\omega_2,\omega_3) = \frac{1}{(2\pi)^3}\sum_{t_1,t_2,t_3=-\infty}^{\infty} \exp\{-i(\omega_1 t_1 + \omega_2 t_2 + \omega_3 t_3)\}\mathrm{cum}_X(t_1,t_2,t_3).$$

First, we show the consistency of the estimator $\hat{\lambda}_p$ under Assumption 1.

Theorem 2 *Suppose* $\{X_t;\ t \in \mathbb{Z}\}$ *satisfies Assumption 1 and the pth quantile* λ_p *of the spectral distribution of* $\{X_t\}$ *is defined by (2). If* $\hat{\lambda}_p$ *is defined by (7), then we have*

$$\hat{\lambda}_p \xrightarrow{\mathcal{P}} \lambda_p.$$

The consistency of the estimator (7) is not difficult to expect. The result, however, requires the continuity of the spectral distribution function $F_X(\omega)$, a strong assumption, if we stand on the estimator (7). We will modify the estimator (7) by a new estimator later to loose Assumption 1.

Next, we investigate the asymptotic distribution of the estimator $\hat{\lambda}_p$. We impose the following assumption on $\{X_t\}$ instead of Assumption 1, which is stronger than Assumption 1.

Assumption 2 $\{X_t\}$ *is a zero mean, strictly stationary real valued process, all of whose moments exist with*

$$\sum_{u_1,\ldots,u_{k-1}=-\infty}^{\infty}\left(1 + \sum_{j=1}^{k-1}|u_j|\right)|\mathrm{cum}_X(u_1,\ldots,u_{k-1})| < \infty, \quad \text{for } k = 2,3,\ldots.$$

The asymptotic distribution of $\hat{\lambda}_p$ is given as follows.

Theorem 3 *Suppose $\{X_t; t \in \mathbb{Z}\}$ satisfies Assumption 2 and the pth quantile λ_p of the spectral distribution of $\{X_t\}$ is defined by (2). If $\hat{\lambda}_p$ is defined by (7), then we have*

$$\sqrt{n}\ (\hat{\lambda}_p - \lambda_p) \xrightarrow{\mathcal{L}} \mathscr{E}^{-2}\mathcal{N}(0, \sigma^2),$$

where $\mathscr{E}$ is a random variable distributed as exponential distribution with mean $f_X(\lambda_p)$ and

$$\sigma^2 = \pi p^2 \int_{-\pi}^{\pi} f_X(\omega)^2 d\omega + 2\pi(1 - 4p) \int_{-\pi}^{\lambda_p} f_X(\omega)^2 d\omega$$

$$+ 2\pi \Big\{ \int_{-\pi}^{\lambda_p} \int_{-\pi}^{\lambda_p} Q_X(\omega_1, \omega_2, -\omega_2) d\omega_1 d\omega_2$$

$$+ \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} p^2 Q_X(\omega_1, \omega_2, -\omega_2) d\omega_1 d\omega_2$$

$$- 2p \int_{-\pi}^{\lambda_p} \int_{-\pi}^{\pi} Q_X(\omega_1, \omega_2, -\omega_2) d\omega_1 d\omega_2 \Big\}.$$

The random variables $\mathscr{E}$ and $\mathcal{N}$ are correlated according to a quantity concerning with the third order cumulants of the process $\{X_t\}$. If the process $\{X_t\}$ is Gaussian or symmetric around 0, then $\mathscr{E}$ and $\mathcal{N}$ are independent.

Although the estimator $\hat{\lambda}_p$, defined by (7), is consistent, the asymptotic distribution of $\hat{\lambda}_p$ is very hard to use in practice. A modified estimator $\hat{\lambda}_p^*$ will given in the next section for quantile tests.

4 Hypotheses testing for sinusoid models

In this section, we consider the following testing problem $(\star)$,

$$H : Y_t = X_t$$

versus

$$A : Y_t = \sum_{j=1}^{J} R_j \cos(\lambda_j t + \phi_j) + X_t, \tag{8}$$

where $\{X_t\}$ is a zero mean second order stationary process with finite autocovariance function $R_X(h)$ as before. $\{\phi_j\}$ is uniformly distributed on $(-\pi, \pi)$, independent of $\{X_t\}$. $\{R_j\}$ and $\{\lambda_j\}$ are real constants. In addition, suppose there exists at least one R_j such that $R_j \neq 0$. In the alternative, the autocovariance function $R_Y(h)$ of $\{Y_t\}$ is

$$R_Y(h) = \frac{1}{2} \sum_{j=1}^{J} R_j^2 \cos(\lambda_j h) + R_X(h).$$

From (1), the spectral distribution function $F_Y(\omega)$ is represented by

$$F_Y(\omega) = \frac{1}{2} \sum_{j=1}^{J} R_j^2 \mathcal{H}(\omega - \lambda_j) + F_X(\omega),$$

where $\mathcal{H}(\omega)$ is so called Heaviside step function such that

$$\mathcal{H}(\omega) = \begin{cases} 1, & \text{if } \omega \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

As for the alternative hypothesis, $F_Y(\omega) \neq F_X(\omega)$ if $\omega \neq -\pi$, 0 or π.

As what we have seen in Section 3, the asymptotic distribution of the estimator $\hat{\lambda}_p$ is peculiar with stronger assumptions while it acts like a sandwich form. We will modify $\hat{\lambda}_p$ by the method of smoothing. We introduce the modified quantile estimator $\hat{\lambda}_p^*$ for the spectral distribution function of the sinusoid models $\{Y_t\}$ and test the null hypothesis H by quantile test below.

Let us first introduce an extension of periodogram (6) by

$$I_{n,Y}^*(\omega) = \sum_{|h|<n} C_n^Y(h) \exp(-ih\omega),$$

where $C_n^Y(h)$ is the sample autocovariance of $\{Y_t\}$. The smoothed periodogram is defined based on a window function $A(\omega)$ such that

$$\hat{f}_Y(\omega) = \frac{1}{2\pi} \sum_{|h|\leq m} \phi\left(\frac{h}{m}\right) \int_{-\pi}^{\pi} I_{n,Y}^*(\lambda) \exp(-ih(\omega - \lambda))d\lambda. \tag{9}$$

Assumptions on the window function $\phi(\omega)$ are given as follows.

Assumption 3 *Let $\phi(\omega)$ satisfy*

  (i) $m \to \infty$ *and* $m/n \to 0$ *as* $n \to \infty$.
  (ii) $\phi(0) = 1$.
 (iii) $\phi(-\omega) = \phi(\omega)$ *and* $|\phi(\omega)| \leq 1$ *for all* $\omega \in \Lambda$.
  (iv) $\phi(\omega) = 0$ *for* $|\omega| > 1$.
  (v) *The pair* (ϕ, f_Y) *satisfies* $\phi(\cdot)f_Y(\cdot) \in \mathcal{L}^u$ *for some* u, $1 < u \leq 2$, *and suppose that there exists* $c > 0$ *such that*

$$\sup_{|\lambda|<\epsilon} \|\phi(\cdot)\{f_Y(\cdot) - f_Y(\cdot - \lambda)\}\|_u = O(\epsilon^c)$$

  as $\epsilon \to 0$.

Let us introduce the modified quantile estimator $\hat{\lambda}_p^*$. Following (6), define the objective function $S_n^*(\theta)$ by

$$S_n^*(\theta) = \int_{-\pi}^{\pi} \rho_p(\omega - \theta)\hat{f}_Y(\omega)d\omega.$$

The modified estimator $\hat{\lambda}_p^*$, then, is

$$\hat{\lambda}_p^* = \arg\min_{\theta \in \Lambda} S_n^*(\theta). \tag{10}$$

Theorem 4 *Suppose $\{Y_t; t \in \mathbb{Z}\}$ is defined by (8). The pth quantile λ_p of the spectral distribution of $\{Y_t\}$ is defined by (2). If $\hat{\lambda}_p^*$ is defined by (10), then we have*

$$\hat{\lambda}_p^* \xrightarrow{\mathcal{P}} \lambda_p.$$

The consistency of the modified estimator (10) do not require the continuity of the spectral distribution function $F_Y(\omega)$, which can be considered as a stronger result than Theorem 2. We can use the modified estimator $\hat{\lambda}_p^*$ in practice as a method to test the hypothesis of sinusoid models since $F_Y(\omega)$ is uniquely determined by its autocovariance function $R_Y(h)$.

Let us introduce quantile tests in frequency domain for sinusoid models. The hypothesis testing problem $(\star)$ can be changed into a general testing problem

$$H : \hat{\lambda}_p^* = \lambda_p$$

versus

$$A : \hat{\lambda}_p^* \neq \lambda_p.$$

Here, we consider the asymptotic distribution of the estimator $\hat{\lambda}_p^*$.

Assumption 4 *The spectral distribution function $F_Y(\omega)$ has a density $f_Y(\omega)$ in a neighborhood of λ_p and $f_Y(\omega)$ is continuous at λ_p with $0 < f_Y(\lambda_p) < \infty$.*

This assumption is not so strong since the jump points in the distribution are countable at most. It is possible to choose a proper quantile or multiple quantiles as our interest to implement the hypothesis testing.

The asymptotic distribution of the modified estimator $\hat{\lambda}_p^*$ is given below.

Theorem 5 *Suppose $\{Y_t; t \in \mathbb{Z}\}$ is defined by (8). The pth quantile λ_p of the spectral distribution of $\{Y_t\}$ is defined by (2). If $\hat{\lambda}_p^*$ is defined by (10), then we have*

$$\sqrt{n}(\hat{\lambda}_p^* - \lambda_p) \rightarrow_d \mathcal{N}(0, \sigma^2), \tag{11}$$

where

$$
\begin{aligned}
\sigma^2 = f(\lambda_p)^{-2} \Big[&\pi p^2 \int_{-\pi}^{\pi} \phi(\omega)^2 f_Y(\omega) f_X(\omega) d\omega \\
&+ 2\pi(1 - 4p) \int_{-\pi}^{\lambda} \phi(\omega)^2 f_Y(\omega) f_X(\omega) d\omega \\
+ 2\pi \Big\{ &\int_{-\pi}^{\lambda_p} \int_{-\pi}^{\lambda_p} \phi(\omega)^2 Q_X(\omega_1, \omega_2, -\omega_2) d\omega_1 d\omega_2 \\
+ &\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} p^2 \phi(\omega)^2 Q_X(\omega_1, \omega_2, -\omega_2) d\omega_1 d\omega_2 \\
&- 2p \int_{-\pi}^{\lambda_p} \int_{-\pi}^{\pi} \phi(\omega)^2 Q_X(\omega_1, \omega_2, -\omega_2) d\omega_1 d\omega_2 \Big\} \Big].
\end{aligned}
$$

Theorem 5 holds for sinusoid models so it also can be applied to the null hypothesis.

Let us introduce the testing procedure for the quantile problem above. From Theorem 5, we have the following result. Let μ_p be the pth quantile of the spectral distribution of $\{Y_t\}$ in the alternative hypothesis.

Corollary 1 *Suppose $\{Y_t; t \in \mathbb{Z}\}$ is defined by (8). The pth quantile λ_p of the spectral distribution of $\{Y_t\}$ is defined by (2) and $\hat{\lambda}_p^*$ is defined by (10). From (11),*

(i) *Under the null hypothesis H, $\sqrt{n}(\hat{\lambda}_p^* - \lambda_p)/\sigma \xrightarrow{\mathcal{L}} \mathcal{N}(0,1)$;*

(ii) *Under the alternative hypothesis A, $\sqrt{n}(\hat{\lambda}_p^* - \lambda_p)/\sigma \xrightarrow{\mathcal{L}} \mathcal{N}(\mu_p - \lambda_p, 1)$.*

The hypothesis is rejected if $\sqrt{n}|\hat{\lambda}_p^* - \lambda_p|/\sigma > \Phi_{1-\alpha/2}$, where $\Phi_{1-\alpha/2}$ is the $1-\alpha/2$ percentage point of a standard normal distribution.

5 Numerical Studies

In this section, we implement the numerical studies to confirm the theoretical results in Sections 3 and 4.

5.1 Numerical results for estimator $\hat{\lambda}_p$

First, we focus on the consistency of the estimator $\hat{\lambda}_p$ defined by (7). Second order stationary processes considered here are Gaussian white noise model, Gaussian MA(1) process with coefficient 0.9, Gaussian AR(1) model with coefficient 0.9 and Gaussian AR(1) model with coefficient -0.9. The spectral distribution functions for these four models are given in Figure 1. The dependence structures of them are obviously different.

We estimated the quantile λ_p of the spectral distribution function by 30 samples, generated from each Gaussian stationary process. The numerical results of the estimator $\hat{\lambda}_p$ only for $0.5 \leq p \leq 1$ are listed in Table 1, since the spectral distribution functions of real-valued stationary processes are symmetric.

Table 1 the estimated quantiles $\hat{\lambda}_p$ of the spectral distribution with 30 samples

p	White noise	MA(1)	AR(1) with 0.9	AR(1) with -0.9
0.5	0.000	0.000	0.000	0.000
0.6	0.305	0.211	0.026	2.940
0.7	1.187	0.576	0.055	3.030
0.8	1.564	0.891	0.092	3.074
0.9	2.093	1.235	0.190	3.109
1.0	3.142	3.142	3.142	3.142

We can see that the results in Table 1 correspond to Figure 1 in Section 2. That is to say, the quantile of the spectral distribution function reflects the traits of stationary processes. Furthermore, we can make use of $\hat{\lambda}_p$ to seize the traits.

In general asymptotic theory, if the estimator is asymptotically normal, then the estimates will be improved when the sample sizes get large. However, as what we have shown in Section 3, the estimator $\hat{\lambda}_p$ based on the bare periodogram is not asymptotically normal. We next give the results in the white noise case with different sample size to see the phenomenon. The sample sizes are set to be 30, 50, 100 and 200.

Table 2 the estimated quantiles $\hat{\lambda}_p$ in white noise case with different numbers of samples

$p \backslash n$	30	50	100	200
0.5	0.000	0.000	0.000	0.000
0.6	0.305	0.663	0.366	0.745
0.7	1.187	1.226	0.966	1.260
0.8	1.564	1.990	1.602	1.881
0.9	2.093	2.440	2.251	2.334
1.0	3.142	3.142	3.142	3.142

From Table 2, we can see the accuracy is not quite improved when the sample size gets large. This numerical result supports the theoretical results given in Theorem 3 in Section 3, since, not only a normal distribution inside the asymptotic distribution of $\hat{\lambda}_p$, the asymptotic distribution is also influenced by exponential distributed random variable.

At last, we would like to look at the behavior of the estimator $\hat{\lambda}_p$ for sinusoid models. In addition to the same settings of X_t given above, we add a harmonic component m_t in the model with $\omega_0 = \pi/2$, i.e.

$$Y_t = m_t + X_t, \tag{12}$$

where m_t is defined in the following way: with uniformly distributed ϕ on $[-\pi, \pi]$

$$m_t = 1/2 \cos(\omega_0 t + \phi).$$

As already known, the spectral distribution function of $\{Y_t\}$ has a large change at the certain frequency $\omega_0 = \pi/2$. Still, we estimated the quantile λ_p by 30 samples, generated from the sinusoid models (12). Compared with the results in Table 1, we can see that the estimated quintiles are pulled around to the frequency ω_0 from Table 3. Accordingly, even in the sinusoid models, the quantile λ_p shows the phase of the spectral distribution function. We can grasp them from the quantile estimator $\hat{\lambda}_p$.

Table 3 the estimated quantiles $\hat{\lambda}_p$ of the spectral distribution with 30 samples

p	White noise	MA(1)	AR(1) with 0.9	AR(1) with -0.9
0.5	0.000	0.000	0.000	0.000
0.6	1.399	0.412	0.030	2.610
0.7	1.513	0.789	0.065	3.014
0.8	1.577	1.254	0.116	3.066
0.9	1.679	1.582	1.152	3.106
1.0	3.142	3.142	3.142	3.142

5.2 Statistical power of quantile tests in frequency domain

Next, we implement quantile tests in frequency domain to see the performance of our testing procedure. The Bartlett window is used for our purpose to smooth the periodogram (6). To know the quantile λ_p for each model is very difficult, so we fixed $p = 0.7$ and $p = 0.8$ and numerically calculated λ_p in advance.

Also, the theoretical result of the asymptotic variance σ^2 is also difficult to calculate. We used the unbiased variance $\hat{\sigma}$ of the estimator in 100 simulations. The significant level α is set to be 0.1.

We set λ_p as the true quantile for the null hypothesis. Under the alternative models (Gaussian white noise model, Gaussian MA(1) model, Gaussian AR(1) models as before), 50 samples are generated to estimate the quantile by the estimator $\hat{\lambda}_p^*$.

Table 4 Statistical power of quantile tests for $\lambda_{0.7}$ with 50 samples

H \ A	White noise	MA(1)	AR(1) with 0.9	AR(1) with -0.9
White noise	–	0.99	1.00	1.00
MA(1)	1.00	–	0.99	1.00
AR(1) with 0.9	1.00	1.00	–	1.00
AR(1) with -0.9	1.00	1.00	1.00	–

Table 5 Statistical power of quantile tests for $\lambda_{0.8}$ with 50 samples

H \ A	White noise	MA(1)	AR(1) with 0.9	AR(1) with -0.9
White noise	–	1.00	1.00	1.00
MA(1)	1.00	–	0.99	1.00
AR(1) with 0.9	1.00	1.00	–	1.00
AR(1) with -0.9	1.00	1.00	1.00	–

As what we can see from both Tables 4 and 5, the statistical power is much high. One reason to explain this result is that the dependence structures of these four models are quite different. When p is closer to 0.5 or 1, or the dependence structures of models are more similar, then the statistical power will be lower.

6 Proofs of Theorems

In this section, we provide proofs of theorems in the previous sections.

Proof (Theorem 1) First, we confirm the existence of the minimizer of $S(\theta)$. The right derivative of $S(\theta)$ is

$$S'_+(\theta) \equiv \lim_{\epsilon \to +0} \frac{S(\theta + \epsilon) - S(\theta)}{\epsilon} = F_Y(\theta) - p\Sigma_Y.$$

From (2), we have

$$S'_+(\theta) \begin{cases} < 0, & \text{for } \theta < \lambda_p, \\ \geq 0, & \text{for } \theta \geq \lambda_p. \end{cases}$$

Thus, the minimizer of $S(\theta)$ exists and $S(\lambda_p) = \min_{\theta \in \Lambda} S(\theta)$. The uniqueness of λ_p and the representation (4) follow (2). □

Proof (Theorem 2) Let m be the minimum of $S(\theta)$. The convexity of $S_n(\theta)$ is shown by the positiveness of the second derivative of $S_n(\theta)$, i.e.,

$$\frac{\partial^2}{\partial \theta^2} S_n(\theta) = I_{n,X}(\theta) > 0 \quad \text{a.s.}$$

Now, let us consider the pointwise limit of $S_n(\theta)$. Actually, for each $\theta \in \Lambda$,

$$|S_n(\theta) - S(\theta)| \leq \left| \int_{-\pi}^{\pi} \rho_p(\omega - \theta)(I_{n,X}(\omega) - EI_{n,X}(\omega))d\omega \right|$$
$$+ \left| \int_{-\pi}^{\pi} \rho_p(\omega - \theta)(EI_{n,X}(\omega) - f_X(\omega))d\omega \right|.$$

The first term in right hand side converges to 0 in probability, which can be shown by the summability of the fourth order cumulants under Assumption 1 (i). The second term in right hand side converges to 0 under Assumption 1 (ii). (See Hannan (1970); Hosoya and Taniguchi (1982)). By the Convexity Lemma in Pollard (1991),

$$\sup_{\theta \in K} |S_n(\theta) - S(\theta)| \xrightarrow{\mathcal{P}} 0, \tag{13}$$

for any compact subset $K \subset \Lambda$.

Let $B(\lambda_p)$ be any open neighborhood of λ_p. From the uniqueness of zero of $S(\theta)$, there exists an $\epsilon > 0$ such that $\inf_{\mu \in \Lambda/B(\lambda)} |S(\mu)| > m + \epsilon$. Thus, with probability tending to 1,

$$\inf_{\theta \in \Lambda/B(\lambda_p)} S_n(\theta) \geq \inf_{\theta \in \Lambda/B(\lambda_p)} S(\theta) - \sup_{\theta \in \Lambda/B(\lambda_p)} |S(\theta) - S_n(\theta)| > m,$$

where it is implied by (13) that the second term can be chosen arbitrarily small. The conclusion follows that with probability tending to 1, $S_n(\hat{\lambda}_p) \leq m - \epsilon^*$ by the pointwise convergence of $S_n(\theta)$ in probability. □

To prove Theorem 3, we first consider asymptotic variance of

$$T_n(\lambda) \equiv n^\beta \int_{\lambda}^{\lambda + n^{-\beta}} I_{n,X}(\omega)d\omega. \tag{14}$$

The asymptotic variance can be classified as the following lemma.

Lemma 1 *Suppose $\{X(t)\}$ satisfies Assumption 2. Let $T_n(\lambda)$ be defined as (14). Then the asymptotic variance of $T_n(\lambda)$ is given by*

$$\lim_{n \to \infty} \mathrm{Var}(T_n(\lambda)) = \begin{cases} 0, & \text{if } \beta < 1, \\ f_X(\lambda)^2, & \text{if } \beta = 1, \\ \infty, & \text{if } \beta > 1. \end{cases}$$

Proof Let $a_n = n^\beta$. Divide $T_n(\lambda)$ by

$$a_n \int_{-\pi}^{\lambda + a_n^{-1}} I_{n,X}(\omega) d\omega - a_n \int_{-\pi}^{\lambda} I_{n,X}(\omega) d\omega.$$

The variances of both two parts and their covariance are given by

$$\mathrm{Var}\left(a_n \int_{-\pi}^{\lambda + a_n^{-1}} I_{n,X}(\omega) d\omega\right) =$$

$$\frac{a_n^2}{n} 2\pi \left(\int_{-\pi}^{\lambda + a_n^{-1}} f_X(\omega)^2 d\omega + \int_{-\pi}^{\lambda + a_n^{-1}} \int_{-\pi}^{\lambda + a_n^{-1}} Q_X(\omega_1, \omega_2, -\omega_2) d\omega_1 d\omega_2\right),$$

$$\mathrm{Var}\left(a_n \int_{-\pi}^{\lambda} I_{n,X}(\omega) d\omega\right) =$$

$$\frac{a_n^2}{n} 2\pi \left(\int_{-\pi}^{\lambda} f_X(\omega)^2 d\omega + \int_{-\pi}^{\lambda} \int_{-\pi}^{\lambda} Q_X(\omega_1, \omega_2, -\omega_2) d\omega_1 d\omega_2\right),$$

and

$$\mathrm{Cov}\left(a_n \int_{-\pi}^{\lambda + a_n^{-1}} I_{n,X}(\omega) d\omega, a_n \int_{-\pi}^{\lambda} I_{n,X}(\omega) d\omega\right)$$

$$= \frac{a_n^2}{n} 2\pi \left(\int_{-\pi}^{\lambda} f_X(\omega)^2 d\omega + \int_{-\pi}^{\lambda} \int_{-\pi}^{\lambda + a_n^{-1}} Q_X(\omega_1, \omega_2, -\omega_2) d\omega_1 d\omega_2\right).$$

As a result, the variance of $T_n(\lambda)$ is

$$\mathrm{Var}(T_n(\lambda)) = \frac{a_n^2}{n} 2\pi \left(\int_{\lambda}^{\lambda + a_n^{-1}} f_X(\omega)^2 d\omega \right.$$

$$+ \int_{\lambda}^{\lambda + a_n^{-1}} \int_{-\pi}^{\lambda + a_n^{-1}} Q_X(\omega_1, \omega_2, -\omega_2) d\omega_1 d\omega_2$$

$$\left. - \int_{-\pi}^{\lambda} \int_{\lambda}^{\lambda + a_n^{-1}} Q_X(\omega_1, \omega_2, -\omega_2) d\omega_1 d\omega_2\right). \quad (15)$$

We can see the result from (15) by cases:

(i) if $a_n = n^\beta$ where $0 < \beta < 1$, then the limiting variance of $T_n(\lambda)$ is

$$\mathrm{Var}(T_n(\lambda)) \to 0,$$

(ii) if $a_n = n^\beta$ where $\beta > 1$, then the limiting variance of $T_n(\lambda)$ is

$$\mathrm{Var}(T_n(\lambda)) \to \infty,$$

(iii) if $a_n = n^\beta$ where $\beta = 1$, then the limiting variance of $T_n(\lambda)$ is

$$\mathrm{Var}(T_n(\lambda)) \to f_X(\lambda)^2.$$

Thus, the conclusion holds.   □

Remark 1 The result in Lemma 1 seems surprising at first glance, since it may be expected that (14) do not depend on the order of factor n^β. However, the phenomenon can be explained in a heuristic way. Returning back to the definition of $T_n(\lambda)$, the quantity

$$\int_\lambda^{\lambda+n^{-\beta}} I_{n,X}(\omega)d\omega$$

is approximated by the following discrete statistic

$$\frac{2\pi}{n} \sum_{\lambda \le 2\pi s/n \le \lambda+n^{-\beta}} I_{n,X}\left(\frac{2\pi s}{n}\right). \tag{16}$$

Looking at the number of periodograms $I_{n,X}(\lambda_s)$ with different frequencies, we can find that (16) depends on the order of $n^{-\beta}$. If $0 < \beta < 1$, then more and more periodograms will be involved in the summation as n increases. Conversely, if $\beta > 1$, then the interval for the frequency will be much smaller as n increases. Only the case $\beta = 1$ keeps the same order between the number of periodograms and the length of the interval, and therefore only one periodogram $I_{n,X}(2\pi s/n)$ is involved in the summation.

Next, we have to consider the domain of periodogram on the lattice as in Brillinger (2001). That is to say, for any $\omega \in [-\pi, \pi]$, define periodogram $I_{n,X}(\omega)$ discretely by $I_{n,X}(\omega_k)$, where ω_k is defined as the closest frequency of the multiple of $2\pi/n$. It is easy to see that

$$|I_{n,X}(\omega) - I_{n,X}(\omega_k)| = o_p(1).$$

Lemma 2 *If $\omega_k \neq -\pi, 0, \pi$, then the random vector*

$$\sqrt{n}\left(\frac{1}{n}\sum_{t=1}^n X_t \cos(\omega_k t), \frac{1}{n}\sum_{t=1}^n X_t \sin(\omega_k t)\right)'$$

has a joint asymptotic normal distribution with the covariance matrix $1/2\, \Sigma_X I_2$.

Proof Obvious.   □

Then, let $C_n(m)$ be the sample autocovariance, i.e.

$$C_n(m) = \frac{1}{n-m}\sum_{s=1}^{n-m} X_s X_{s+m}.$$

The joint distribution of the random vector $\sqrt{n}\,(C_n(1)-R_X(1), \cdots, C_n(l)-R_X(l),$ $(1/n)\sum_{t=1}^n X_t \cos(\omega_k t), (1/n)\sum_{t=1}^n X_t \sin(\omega_k t))'$ will be considered in the next lemma. The result is applied to show the asymptotic distribution of $\sqrt{n}(\hat{\lambda}_p - \lambda_p)$.

Lemma 3 *Under Assumptions 2, the asymptotic joint distribution of the sample autocovariances and the trigonometric transforms ($\omega_k \neq -\pi, 0, \pi$) of samples is given by*

$$\sqrt{n} \begin{pmatrix} C_n(1) - R_X(1) \\ \vdots \\ C_n(l) - R_X(l) \\ \frac{1}{n} \sum_{t=1}^{n} X_t \cos(\omega_k t) \\ \frac{1}{n} \sum_{t=1}^{n} X_t \sin(\omega_k t) \end{pmatrix} \xrightarrow{\mathcal{L}} \mathcal{N}(\mathbf{0}, \begin{pmatrix} V & \Delta_3 & \Delta_3 \\ \Delta_3' & \frac{1}{2}\Sigma_X & 0 \\ \Delta_3' & 0 & \frac{1}{2}\Sigma_X \end{pmatrix}), \qquad (17)$$

where the matrix V is given by

$$V_{m_1 m_2} = 2\pi \int_{-\pi}^{\pi} f_X^2(\omega)\{\exp(-i(m_2 - m_1)\omega) + \exp(i(m_2 + m_1)\omega)\}d\omega$$

$$+ (2\pi)^{-2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \exp(im_1\omega_1 + im_2\omega_2)Q_X(\omega_1, -\omega_2, \omega_2)d\omega_1 d\omega_2.$$

The l-vector Δ_3 is a quantity defined in the proof, which is related to the third order cumulants of the stochastic process $\{X_t\}$.

Proof The statement will be shown by Cramér-Wold device. Suppose $\mathbf{q} = (q_1, \ldots, q_{l+2})$ and $(X_{n+1}, \ldots, X_{n+l})$ is generated from the stationary process $\{X_t; t \in \mathbb{Z}\}$. Then, we can define a random vector $\tilde{S}_t$ as

$$\tilde{S}_t = (X_t X_{t+1} - R_X(1), \cdots, X_t X_{t+l} - R_X(l), X_t \cos(\omega_k t), X_t \sin(\omega_k t))'.$$

Denote the left hand side of (17) by S_n. It is not difficult to see that

$$\left| \frac{1}{n} \sum_{t=1}^{n} \tilde{S}_t - S_n \right| \xrightarrow{\mathcal{P}} 0,$$

since $(X_{n+1}, \ldots, X_{n+l})$ is bounded. Let us consider the random variable $\mathbf{q}'\tilde{S}_t$. It holds that $E(\mathbf{q}'\tilde{S}_t) = 0$. Denote the variance of $\mathbf{q}'\tilde{S}_t$ by $s_n = \mathrm{Var}(\mathbf{q}'\tilde{S}_t)$. Under Assumption 2, we can find that, from Hosoya and Taniguchi (1982),

$$\mathrm{Cov}(C_n(i), C_n(j)) = O\left(\frac{1}{n}\right),$$

for $i, j = 1, 2, \ldots, l$, from Lemma 2,

$$\mathrm{Var}\left(\frac{1}{n} \sum_{t=1}^{n} X_t \cos(\omega_k t)\right) = O\left(\frac{1}{n}\right), \quad \mathrm{Var}\left(\frac{1}{n} \sum_{t=1}^{n} X_t \sin(\omega_k t)\right) = O\left(\frac{1}{n}\right),$$

and for any $1 \leq m \leq l$,

$$\mathrm{Cov}\left(C_n(m), n^{-1} \sum_{t=1}^{n} X_t \cos(\omega_k t)\right)$$

$$= n^{-1}(n - m)^{-1} \sum_{s=1}^{n-m} \sum_{t=1}^{n} \cos(\omega_k t)\mathrm{cum}(X_s, X_{s+m}, X_t). \quad (18)$$

Under Assumption 2, the right hand side of (18) can be bounded by

$$n^{-1}(n-m)^{-1}\sum_{s=1}^{n-m}\sum_{t=1}^{n}\cos(\omega_k t)\mathrm{cum}(X_s, X_{s+m}, X_t)$$

$$\leq \frac{1}{n}\sum_{k=1-n}^{n-1}\left(1-\frac{|k|}{n}\right)\mathrm{cum}_X(m,k) = O\Big(\frac{1}{n}\Big).$$

Thus, for any $\epsilon > 0$,

$$n^{-1}\sum_{t=1}^{n}E((\boldsymbol{j}'\tilde{S}_t)^2\mathbb{1}(|\boldsymbol{j}'\tilde{S}_t| > n^{1/2}\epsilon)) \to 0,$$

as $n \to \infty$. Now if we define

$$\Delta_3(m) \equiv \lim_{n\to\infty}\frac{1}{n}\sum_{s=1}^{n}\sum_{t=1}^{n}\cos(\omega_k t)\mathrm{cum}(X_s, X_{s+m}, X_t),$$

then $\Delta_3 = (\Delta_3(1), \ldots, \Delta_3(l))'$. By Lindeberg's central limit theorem, $n^{-1/2}\sum_{t=1}^{n}\boldsymbol{q}'\tilde{S}_t$ is asymptotically Gaussian distributed. The conclusion follows Cramér-Wold device. $\square$

Following Lemma 1, Lemma 2 and Lemma 3, we give the proof of Theorem 3.

Proof (Theorem 3) Consider the following process

$$M_n(\delta) = n\left\{S_n(\lambda_p - \frac{\delta}{\sqrt{n}}) - S_n(\lambda_p)\right\}.$$

By Knight's identity (see Knight (1998)), we have

$$M_n(\delta) = -\delta\sqrt{n}\left\{\int_{-\pi}^{\pi}(p - \mathbb{1}(\omega < \lambda_p))(I_{n,X}(\omega) - f_X(\omega))d\omega\right\}$$

$$+ \int_{-\pi}^{\pi}\int_{0}^{\delta/\sqrt{n}}n\left(\mathbb{1}(\omega \leq \lambda_p + s) - \mathbb{1}(\omega \leq \lambda_p)I_{n,X}(\omega)dsd\omega\right.$$

$$= M_{n1}(\delta) + M_{n2}(\delta), \quad (\text{say}).$$

Under Assumption 2, we have, by Theorem 7.6.3 in Brillinger (2001),

$$M_{n1}(\delta) \xrightarrow{\mathcal{L}} -\delta\mathcal{N}(0, \sigma^2),$$

where

$$\sigma^2 = \pi p^2\int_{-\pi}^{\pi}f_X(\omega)^2 d\omega + 2\pi(1-4p)\int_{-\pi}^{\lambda_p}f_X(\omega)^2 d\omega$$

$$+ 2\pi\Big\{\int_{-\pi}^{\lambda_p}\int_{-\pi}^{\lambda_p}Q_X(\omega_1, \omega_2, -\omega_2)d\omega_1 d\omega_2$$

$$+ \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}p^2 Q_X(\omega_1, \omega_2, -\omega_2)d\omega_1 d\omega_2$$

$$- 2p\int_{-\pi}^{\lambda_p}\int_{-\pi}^{\pi}Q_X(\omega_1, \omega_2, -\omega_2)d\omega_1 d\omega_2\Big\}.$$

From Lemma 1, in view of

$$n \int_{\lambda_p}^{\lambda_p + n^{-1}} I_{n,X}(\omega) d\omega \to I_{n,X}(\lambda_p) \quad \text{a.s.,} \tag{19}$$

we will use (19) to evaluate $M_{n2}(\delta)$. The second term $M_{n2}(\delta)$ can be evaluated by

$$
\begin{aligned}
M_{n2}(\delta) &= \int_0^{\delta/\sqrt{n}} \int_{\lambda_p}^{\lambda_p + s} n\, I_{n,X}(\omega) d\omega ds \\
&= \int_0^{\delta\sqrt{n}} \left(n \int_{\lambda_p}^{\lambda_p + t/n} I_{n,X}(\omega) d\omega \right) dt \\
&= \int_0^{\delta\sqrt{n}} t\, I_{n,X}(\lambda_p)\, dt \\
&= \frac{1}{2}\, I_{n,X}(\lambda_p)\, \delta^2 \quad \text{a.s.}
\end{aligned}
$$

This term, actually, does not converge in probability, but has an asymptotic exponential distribution $\mathscr{E}$, which has mean $f(\lambda)$. Applying continuous mapping theorem to the result in Lemma 3, the following joint distribution converges in distribution, i.e.,

$$\begin{pmatrix} M_{n1}(\delta) \\ M_{n2}(\delta) \end{pmatrix} \xrightarrow{\mathcal{L}} \begin{pmatrix} \mathcal{N} \\ \mathscr{E} \end{pmatrix}.$$

Then by continuous mapping theorem again, we obtain

$$M_n(\delta) \xrightarrow{\mathcal{L}} M(\delta) = -\delta\mathcal{N} + \frac{1}{2}\delta\mathscr{E}^2,$$

which is minimized by $\delta = \mathscr{E}^{-1}\mathcal{N}$. In conclusion,

$$\sqrt{n}(\hat{\lambda}_p - \lambda) \xrightarrow{\mathcal{L}} \mathscr{E}^{-2}\mathcal{N}(0, \sigma^2),$$

From Lemma 3, it can be seen that the dependence relationship between random variables $\mathscr{E}$ and $\mathcal{N}$ depends on Δ_3, i.e., the third cumulants of the process $\{X_t\}$. If $\{X_t\}$ is Gaussian or symmetric around 0, then $\Delta_3 = 0$, which implies that $\mathscr{E}$ and $\mathcal{N}$ are independent. $\qquad \square$

Below, we provide the proof of Theorem 4. First, an extension of Lemma A2.2 in Hosoya and Taniguchi (1982) is given in the following.

Lemma 4 *Assume $\sum_{j_1,j_2,j_3=-\infty}^{\infty} |Q_X(j_1, j_2, j_3)| < \infty$. For any square-integrable function $\phi(\omega)$,*

$$\int_{-\pi}^{\pi} (I_{n,Y}(\omega) - EI_{n,Y}(\omega))\phi(\omega) d\omega \xrightarrow{\mathcal{P}} 0. \tag{20}$$

Proof Let

$$\tilde{\phi}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi(\omega) \exp(in\omega) d\omega.$$

From Hosoya and Taniguchi (1982) and Li et al. (1994), it holds that

$$\mathrm{Var}\left\{ \int_{-\pi}^{\pi} (I_{n,Y}(\omega) - EI_{n,Y}(\omega))\phi(\omega)d\omega \right\} =$$

$$\frac{1}{n^2} \sum_{t_1,t_2,t_3,t_4=1}^{n} \tilde{\phi}(t_1 - t_2)\tilde{\phi}(t_3 - t_4)\{ R_Y(t_3 - t_1)R_Y(t_4 - t_2)$$

$$+ R_Y(t_4 - t_1)R_Y(t_3 - t_2) + Q_Y(t_2 - t_1, t_3 - t_1, t_4 - t_1) \}$$

$$= \frac{2\pi}{n} \int_{-\pi}^{\pi} (\phi(\omega)\overline{\phi(\omega)} + \phi(\omega)\overline{\phi(-\omega)}) f_Y(\omega) f_X(\omega) d\omega$$

$$+ \frac{2\pi}{n} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \phi(\omega_1)\phi(-\omega_2) Q_X(\omega_1, \omega_2, -\omega_2) d\omega_1 d\omega_2.$$

Here, $f_Y(\omega) = f_d(\omega) + f_X(\omega)$ if we suppose $f_d(\omega) = \frac{1}{4\pi} R_j^2 \delta(\omega - \lambda_j)$, where $\delta(\omega)$ is the Dirac delta function. From Chebyshev's inequality, (20) holds.

Proof (Theorem 4) We only have to show the pointwise limit of $S_n^*(\theta)$ is given by $S(\theta)$. The rest of argument for the proof follows the proof of Theorem 2. Note that $\hat{f}_Y(\omega)$ has a representation such that

$$\hat{f}_Y(\omega) = \int_{-\pi}^{\pi} \phi(\omega - \lambda) I_{n,Y}^*(\lambda) d\lambda.$$

Similarly, we have

$$|S_n^*(\theta) - S(\theta)| \le \left| \int_{-\pi}^{\pi} \rho_p(\omega - \theta)(\hat{f}_Y(\omega) - E\hat{f}_Y(\omega))d\omega \right|$$

$$+ \left| \int_{-\pi}^{\pi} \rho_p(\omega - \theta) E(\hat{f}_Y(\omega))d\omega - \int_{-\pi}^{\pi} \left(\int_{-\pi}^{\pi} \rho_p(\omega - \theta)\phi(\omega - \lambda)d\omega \right) F_Y(d\lambda) \right|.$$

The first term in right hand side converges to 0 in probability, which can be seen from Lemma 4. Under Assumption 3 (v), we see that the second term in right hand side converges to 0 from Theorem 1.1 in Hosoya (1997). $\qquad\square$

Last, we give the proof of Theorem 5.

Proof (Theorem 5) Consider the following process

$$M_n^*(\delta) = n \left\{ S_n^*(\lambda_p - \frac{\delta}{\sqrt{n}}) - S_n^*(\lambda_p) \right\}.$$

By Knight's identity, we have

$$
\begin{aligned}
M_n^*(\delta) = -\delta\sqrt{n}\Bigg\{ & \int_{-\pi}^{\pi}(p-\mathbb{1}(\omega<\lambda_p))\hat{f}_Y(\omega)d\omega \Bigg\} \\
& + \int_{-\pi}^{\pi}\int_0^{\delta/\sqrt{n}} n\left(\mathbb{1}(\omega\leq\lambda_p+s)-\mathbb{1}(\omega\leq\lambda_p)\hat{f}_Y(\omega)dsd\omega\right. \\
= & \ M_{n1}^*(\delta)+M_{n2}^*(\delta), \quad \text{(say)}.
\end{aligned}
$$

From Lemma 4, we can see that

$$
M_{n1}^*(\delta) \xrightarrow{\mathcal{L}} -\delta\mathcal{N}(0,\tilde{\sigma}^2),
$$

where σ^2 is

$$
\begin{aligned}
\tilde{\sigma}^2 = & \ \pi p^2\int_{-\pi}^{\pi}\phi(\omega)^2 f_Y(\omega)f_X(\omega)d\omega + 2\pi(1-4p)\int_{-\pi}^{\lambda}\phi(\omega)^2 f_Y(\omega)f_X(\omega)d\omega \\
& + 2\pi\Big\{\int_{-\pi}^{\lambda_p}\int_{-\pi}^{\lambda_p} Q_X(\omega_1,\omega_2,-\omega_2)d\omega_1 d\omega_2 \\
& + \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} p^2 Q_X(\omega_1,\omega_2,-\omega_2)d\omega_1 d\omega_2 \\
& \hspace{4em} - 2p\int_{-\pi}^{\lambda_p}\int_{-\pi}^{\pi} Q_X(\omega_1,\omega_2,-\omega_2)d\omega_1 d\omega_2\Big\}.
\end{aligned}
$$

As for the second term $M_{n2}(\delta)$, we have, under Assumptions 3 and 4,

$$
\begin{aligned}
M_{n2}^*(\delta) & = \int_0^{\delta/\sqrt{n}}\int_{\lambda_p}^{\lambda_p+s} n\,\hat{f}_Y(\omega)d\omega ds \\
& = \int_0^{\delta\sqrt{n}}\Big(n\int_{\lambda_p}^{\lambda_p+t/n}\hat{f}_Y(\omega)d\omega\Big)dt \\
& \xrightarrow{\mathcal{P}} \frac{1}{2}\,f_Y(\lambda_p)\,\delta^2.
\end{aligned}
$$

Applying continuous mapping theorem to M_n, we obtain

$$
M_n^*(\delta) \xrightarrow{\mathcal{L}} M^*(\delta) = -\delta\mathcal{N} + \frac{1}{2}\,f_Y(\lambda_p)\,\delta^2,
$$

which is minimized by $\delta = f_Y(\lambda_p)^{-1}\mathcal{N}$. Therefore,

$$
\sqrt{n}(\hat{\lambda}_p-\lambda) \xrightarrow{\mathcal{L}} \mathcal{N}(0,f_Y(\lambda_p)^{-2}\tilde{\sigma}^2),
$$

and the asymptotic variance σ^2 in Theorem 5 is $\sigma^2 = f_Y(\lambda_p)^{-2}\tilde{\sigma}^2$.   □

References

Brillinger DR (2001) Time Series: Data Analysis and Theory. Vol. 36; Siam

Dette H, Hallin M, Kley T, Volgushev S et al. (2015) Of copulas, quantiles, ranks and spectra: An $l_-\{1\}$-approach to spectral analysis. Bernoulli 21(2):781–831

Hannan E (1970) Multiple Time Series. ; John Wiley & Sons

Hannan EJ (1973) The estimation of frequency. Journal of Applied probability 10:510–519

Hosoya Y (1989) The bracketing condition for limit theorems on stationary linear processes. The Annals of Statistics 17(1):401–418

Hosoya Y (1997) A limit theory for long-range dependence and statistical inference on related models. The Annals of Statistics 25(1):105–137

Hosoya Y, Taniguchi M (1982) A central limit theorem for stationary processes and the parameter estimation of linear processes. The Annals of Statistics 10:132–153

Knight K (1998) Limiting distributions for l_1 regression estimators under general conditions. The Annals of Statistics 26(2):755–770

Koenker R (2005) Quantile Regression. ; number 38 Cambridge university press

Li TH (2008) Laplace periodogram for time series analysis. Journal of the American Statistical Association 103(482):757–768

Li TH (2012) Quantile periodograms. Journal of the American Statistical Association 107(498):765–776

Li TH, Kedem B, Yakowitz S (1994) Asymptotic normality of sample autocovariances with an application in frequency estimation. Stochastic Processes and their Applications 52(2):329–349

Pollard D (1991) Asymptotics for least absolute deviation regression estimators. Econometric Theory 7(02):186–199

Quinn BG, Hannan EJ (2001) The estimation and tracking of frequency. Vol. 9; Cambridge University Press

Quinn BG, Thomson PJ (1991) Estimating the frequency of a periodic function. Biometrika 78(1):65–74

Rice JA, Rosenblatt M (1988) On frequency estimation. Biometrika 75(3):477–484

Walker AM (1971) On the estimation of a harmonic component in a time series with stationary independent residuals. Biometrika 58(1):21–36

Whittle P (1952) Tests of fit in time series. Biometrika 39(3):309–318