8.4

D. Does $\sum_{n=1}^{\infty} \frac{1}{x^2+n^2}$ converge uniformly on the whole real line?

We know $0 \leq \sum_{n=1}^{\infty} \frac{1}{x^2+n^2} < \sum_{n=1}^{\infty} \frac{1}{n^2}$. Because $\frac{1}{n^2}$ is convergent, then $\frac{1}{x^2+n^2}$ must also be convergent.

This also gives us uniform convergence, because for every $\varepsilon > 0$ there exists an N such that $0 \leq || \sum_{i=k+1}^{l} \frac{1}{x^2+i^2} \leq || \sum_{i=k+1}^{l} \frac{1}{i^2} \leq \varepsilon$ for every $l > k \geq N$ regardless of our choice of x.

E. Show that if $\sum_{n=1}^{\infty} |a_n| < \infty$, then $\sum_{n=1}^{\infty} a_n \cos nx$ converges uniformly on $\mathbb{R}$.

Because $0 \leq |\cos nx| \leq 1$ then $|a_n \cos nx| \leq |a_n|$. Now we know that $|a_n|$ converges and so then for any $\varepsilon > 0$ there exists an N such that $\sum_{i=k+1}^{l} |a_n| < \varepsilon$ for any $l > k \geq N$. But $\sum_{i=k+1}^{l} |a_n \cos nx| \leq \sum_{i=k+1}^{l} |a_n| < \varepsilon$ regardless of our choice of x. And since $|a_n \cos nx|$ converges uniformly, then we get $a_n \cos nx$ converging uniformly for free.

F. (a) Let $f_n(x) = \frac{x^2}{(1+x^2)^n}$ for $x \in \mathbb{R}$. Evaluate the sum $S(x) = \sum_{n=0}^{\infty} f_n(x)$.

At $x = 0$ the sum is 0. At all other values we have a geometric series which converges to $\frac{x^2}{1-(\frac{1}{1+x^2})} = \frac{x^2}{\frac{x^2}{1+x^2}} = 1 + x^2$

(b) Is this convergence uniform? For which values $a < b$ does this series converge uniformly on $[a, b]$?

The convergence is not uniform. Our series converges to a discontinuous function ($0 < 1 + x^2$), and so it is not uniformly continuous, by theorem 8.4.4.

We take the derivative

$$
\begin{aligned}
\frac{\partial}{\partial x} \frac{x^2}{(1+x^2)^n} &= 2x(1+x^2)^{-n} - nx^2(1+x^2)^{-n-1}2x \\
&= \frac{2x(1+x^2)}{(1+x^2)(1+x^2)^n} - \frac{2nx^3}{(1+x^2)(1+x^2)^n} \\
&= \frac{2x(1+x^2-nx^2)}{(1+x^2)(1+x^2)^n} \\
&= \frac{2x(1+x^2(1-n))}{(1+x^2)(1+x^2)^n}
\end{aligned}
$$

So the denominator of our derivative has no zeros, and our numerator has zeros at $x = 0$ at $x = \pm\frac{1}{\sqrt{n-1}}$. Zero is obviously a minimum because the function has no negative terms. And $\frac{1}{\sqrt{n-1}}$ is less than 1 for all $n > 2$. So if comparing $x = \frac{1}{\sqrt{n-1}}$ and $x = 1$ when $n = 3$ we see that

$$\frac{\frac{1}{n-1}}{(1+\frac{1}{n-1})^3} ? \frac{1}{(1+1)^3}$$
$$\frac{\frac{1}{n-1}}{(\frac{n}{n-1})^3} ? \frac{1}{2^3}$$
$$\frac{1}{n-1}\left(\frac{n-1}{n}\right)^3 ? \frac{1}{8}$$
$$\frac{(n-1)^2}{n^3} ? \frac{1}{8}$$
$$\frac{2^2}{3^3} ? \frac{1}{8}$$
$$0.\overline{148} > .125$$

And so $\frac{1}{\sqrt{n-1}}$ is a maximum. Observe that

$$\frac{\frac{1}{n-1}}{\left(1+\frac{1}{n-1}\right)^n} = \frac{1}{n-1}\left(\frac{n-1}{n}\right)^n = \frac{(n-1)^{n-1}}{n^n}$$

We have a higher degree on the bottom, so this will converge to zero. And so we have uniform convergence on $[a, \infty)$ for all $a > 0$. And of course $(-\infty, -a]$ or any subinterval of these.

H. Suppose that $a_k(x)$ are continuous functions on $[0, 1]$, and define $s_n(x) = \sum_{k=1}^{n} a_k(x)$. Show that if (s_n) converges uniformly on $[0, 1]$, then (a_n) converges uniformly to 0.

If we assume that (a_n) does not converge uniformly to 0. Then either it does not converge at all or it converges to some non-zero value for some x. Let us assume that $\lim_{n\to\infty} a_n(b) = c$ for some $c \neq 0$. Then $\lim_{n\to\infty} s_n(b) = \infty$

J. Let (f_n) be a sequence of functions defined on $\mathbb{N}$ such that $\lim_{k\to\infty} f_n(k) = L_n$ exists for each $n \geq 0$. Suppose that $||f_n||_\infty \leq M_n$, where $\sum_{n=0}^{\infty} M_n < \infty$. Define a function $F(k) = \sum_{n=0}^{\infty} f_n(k)$. Prove that $\lim_{k\to\infty} F(k) = \sum_{n=0}^{\infty} L_n$. Hint: Think of f_n as a function g_n on $\{\frac{1}{k} : k \geq 1\} \cup 0$. How will you define $g_n(0)$?

8.5

A.

B.