Math 436 — Cataland cartography

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Definition. The *Catalan numbers* are defined as $C_n = \dfrac{1}{n+1}\dbinom{2n}{n}$ for $n \geq 0$. The first few Catalan numbers are $1, 1, 2, 5, 14, 42, 132, 429, \ldots$.

Among the 207 known interpretations (see `http://www-math.mit.edu/~rstan/ec/catalan.pdf` and `http://www-math.mit.edu/~rstan/ec/catadd.pdf`), the Catalan numbers count:

A. $2 \times n$ standard Young tableaux (fillings of a $2 \times n$ diagram with $\{1, 2, \ldots, 2n\}$ using each number once and such that the rows and columns are increasing).

B. Noncrossing matchings on $2n$ vertices (pairings of the numbers $\{1, 2, \ldots, 2n\}$ arranged around a circle such that no two lines cross).

C. Dyck paths from $(0, 0)$ to $(2n, 0)$ (taking only diagonal steps $(1, 1)$ and $(1, -1)$ and never going below the x-axis).

D. Ballot sequences, that is, the number of sequences $a_1, a_2, \ldots a_{2n}$ of n 1's and n -1's whose partial sums are always nonnegative, $a_1 + a_2 + \cdots + a_k \geq 0$.

E. Binary parenthesizations (arrangements of n left parentheses '(' and n right parentheses ')' so that all the parentheses match).

F. Multiplication schemes for noncommutative variables, that is, the number of ways to insert parentheses into the sequence $a_1, a_2, \ldots, a_{n+1}$ to get a multiplication equivalent to $(a_1(a_2(a_3(\cdots(a_{n-1}(a_n a_{n+1}))\cdots))))$ by the associative law.

G. Triangulations of a convex $(n+2)$-gon (into n triangles by $n-1$ diagonals that do not intersect in their interior).

Exercises:

1. Find (or recall from the first day worksheet) bijections between A, B, C and E.
2. Find bijections from D and F to any of A, B, C, or E.
3. Pick one of these bijections to write down carefully. Prove it is a bijection.

4. Find a bijection from G to F via the following steps.

   (a) First work through the following example. Draw a triangulation of an octagon. Pick one side to be the base. Label the other sides circularly $a_1, a_2, \ldots, a_7$. Progressively label the diagonals of your triangulation by multiplying the labels of the other two sides of the triangle, keeping the variables in order and keeping track of the order of multiplication using the parentheses. After labelling all the diagonals, label the base in the same way.

   (b) Convince yourself that this map yields a bijection for general n. What is the inverse map? (Work through an example in the opposite direction.)

We now have bijections between all of these sets! Last week, you proved that G satisfies the Catalan recurrence, $C_n = C_0C_{n-1}+C_1C_{n-2}+\cdots C_{n-1}C_0$, so by these bijections we know that all of these sets satisfy this recurrence. On the last worksheet, we used this recurrence to derive the Catalan counting formula, but it is easier to derive the counting formula directly.

5. Prove that D is counted by the Catalan numbers via the following steps:

   (a) What is the total number of sequences of n 1's and n -1's (without the restriction that the partial sums are nonnegative)?

   (b) If a sequence of n 1's and n -1's satisfies the property that the partial sums are nonnegative, call it acceptable, and unacceptable otherwise. Let A_n count the number of acceptable sequences and U_n the number of unacceptable sequences. So

   $$A_n + U_n =$$

   (c) Given an unacceptable sequence, let k be the *first spot* in the sequence where the partial sum is negative, so $a_1 + a_2 + \cdots + a_k < 0$. So what does a_k equal? And what does $a_1 + a_2 + \cdots a_{k-1}$ equal?

   (d) What is the parity of k? Why?

   (e) Reverse the sign of each of $a_1, \ldots a_k$. How many 1's and -1's does the sequence have now? Is this process reversible?

   (f) So what does U_n equal?

   (g) Find A_n by subtracting U_n from your answer to (a) and simplifying.