Notes

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pde

$$u_{tt} = c^2 \nabla^2 u \text{ on } 0 < r < 1, 0 < \theta < 2\pi, 0 < t < \infty$$

bc

$$u = 0 \text{ on edge}$$

ic

$$\begin{aligned} u(r,\theta,0) &= f(r,\theta) \\ u_t(r,\theta,0) &= g(r,\theta) \end{aligned}$$

sep variables

$$\begin{aligned} u &= T(t)U(r,\theta) \\ \frac{T''}{c^2 T} &= \frac{\nabla^2 U}{U} = -\lambda^2 \le 0 \end{aligned}$$

note: $\frac{\nabla^2 U}{U} = \lambda^2 > 0$ has no sulutions with use of modified bessel functions.

note: $-\lambda^2 = 0$ was eliminated using euler's diffeq

get

$$U_{n,m}(r,\theta) = J_n(k_{n,m} r)\underbrace{(a\sin(n\theta) + b\cos(n\theta))}_{=A\cos(n(\theta-\theta_0))}$$

with

$$\begin{aligned} \lambda_{n,m} &= k_{n,m} \\ n &= 0, 1, 2, \ldots \\ m &= 1, 2, 3, \ldots \\ k_{n,m} &= m^{\text{th}} \text{ positive root of } J_n(x) \end{aligned}$$

$$T_{n,m}(t) = \cos(k_{n,m}ct), \sin(k_{n,m}ct)$$

general solution

$$u = \sum_{\substack{n\geq 0 \\ m\geq 1}} J_n(k_{n,m}r)(\cos(k_{nm}ct)(a_{nm}\sin(n\theta) + b_{nm}\cos(n\theta)) + \sin(k_{nm}ct)(c_{nm}\sin(n\theta) + d_{nm}\cos(n\theta))$$

main question: how to find codfficients a,b,c,d?

lab observations: we have frequencies $\frac{k_{nm}c}{2\pi}$ associated with spatial functions $U_{n,m}(r,\theta) = J_n(k_{nm}r)\cos(n(\theta-\theta_0))$

refer to page 237 for pictures.

m refers to which zero of the bessel function

$$\begin{aligned} n &= 0 & U_{0,m} &= J_0(k_{0,m}r)\cdot 1 \\ m &= 1 & U_{0,1} &= J_0(k_{0,1}r) \end{aligned}$$

so for this, the drumhead going up and down in center, no nodal lines, sand just falls off.

$$\begin{aligned} n &= 0 & U_{0,m} &= J_0(k_{0,m}r)\cdot 1 \\ m &= 2 & U_{0,2} &= J_0(k_{0,2}r) \end{aligned}$$

going in and out on center in opposite time to edge,nodal line at $r = \frac{k_{01}}{k_{02}}$

$$\begin{aligned} n &= 0 & U_{0,m} &= J_0(k_{0,m}r)\cdot 1 \\ m &= 3 & U_{0,3} &= J_0(k_{0,3}r) \end{aligned}$$

going in and out on center in opposite time to edge,nodal lines at $r = \frac{k_{01}}{k_{03}}$ and $r = \frac{k_{02}}{k_{03}}$

etc.

$$\begin{aligned} n &= 3 & U_{3,m} &= J_3(k_{3,m}r)\cos(3(\theta-\theta_0)) \\ m &= 1 & U_{3,1} &= J_3(k_{3,1}r)\cos(3(\theta-\theta_0)) \end{aligned}$$

three radial nodal lines separated by $\frac{2\pi}{3}$ from the cosine term. First bessel zero at outer edge from the bessel term.

$$\begin{aligned} n &= 3 & U_{3,m} &= J_3(k_{3,m}r)\cos(3(\theta-\theta_0)) \\ m &= 2 & U_{3,2} &= J_3(k_{3,2}r)\cos(3(\theta-\theta_0)) \end{aligned}$$

still three radial nodal lines, and now one circular nodal line.

back to the general solution. we want to find coefficients. orthoganality relation on p 239.

$$\int_0^1 rJ_0(k_{0i}r)J_0(k_{0j}r)\,\mathrm{d}r = \begin{cases} 0 & i\neq j \\ \frac{1}{2}{J_1}^2(k_{oi}) & i = j \end{cases}$$

we are deriving orthogonality for helmholtz equation

$$\nabla^2 U + \lambda^2 U = 0 \text{ on } R$$
$$U = 0 \text{ on } \partial R$$

have 2 solutions fo λ (U_λ) and μ (U_μ).

claim

if $\lambda \neq \mu$, then $\int\int_R U_\lambda U_\mu \, \mathrm{d}a = 0$

proof

start with $\lambda^2 \int\int U_\lambda U_{gm} \, \mathrm{d}a = -\int\int_R \nabla^2 U_\lambda U_\mu \, \mathrm{d}a$. Note that $\nabla \cdot (\nabla U_\lambda \cdot U_\mu) = (\nabla^2 U_\lambda) U_\mu + \nabla U_\lambda \cdot \nabla U_\mu$ where nablaUlambda is vector and Umu is function

$$
\begin{aligned}
&= \int\int_R \left[-\nabla \cdot \left(\left(\nabla U_\lambda\right) U_\mu + \nabla U_\lambda \cdot \nabla U_\mu\right] \mathrm{d}a \\
&= -\int_{\partial R} U_\mu \nabla U_\lambda \, \mathrm{d}a + \int\int_R \nabla U_\lambda \cdot \nabla U_\mu \, \mathrm{d}a
\end{aligned}
$$