Jon Allen
HW 19

Let X and T be physical variables for distance and time. Consider the following general diffusion problem for $u(X,T)$:

$$\begin{aligned}
&\text{PDE} && \frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\alpha x^2} + F(X,T) && 0 < X < L, && 0 < T < +\infty \\
&\text{BC} && G_1(T) = \alpha_1 L \frac{\partial u}{\partial X}(0,T) + \beta_1 u(0,T) && && 0 < T < +\infty \\
& && G_2(T) = \alpha_2 L \frac{\partial u}{\partial X}(L,T) + \beta_2 u(L,T) && && \\
&\text{IC} && u(X,0) = \Phi(X) && 0 < X < L &&
\end{aligned}$$

Note:

$$\alpha_1{}^2 + \beta_1{}^2 \neq 0 \qquad \alpha_2{}^2 + \beta_2{}^2 \neq 0$$

(a) If the units for X and T are [cm] and [sec] respectively (and u is taken as temerature with units [deg]), what are the units for L, α^2, F, ϕ, and for $\alpha_1, \beta_2, \alpha_2, \beta_2$?

$$\begin{aligned}
\frac{\text{deg}}{\text{sec}} &= \alpha^2 \frac{\text{deg}}{\text{cm}^2} + F & \alpha \cdot \text{cm}\frac{\text{deg}}{\text{cm}} + \beta \cdot \text{deg} &= \text{deg} \\
F &= \frac{\text{deg}}{\text{sec}} & \alpha \cdot \text{deg} = \beta \cdot \text{deg} &= \text{deg} \\
\alpha^2 &= \frac{\text{cm}^2}{\text{sec}} & \alpha_{1,2} = \beta_{1,2} = 1 &= \text{dimensionless} \\
L &= \text{cm} & \phi(X) &= \text{deg}
\end{aligned}$$

Define dimensionless variables x, t by $x = X/L$ and $t = \frac{\alpha^2}{L^2}T$. Define $w(x,t) = u(X,T)$

(b) Find $\frac{\partial u}{\partial T}, \frac{\partial u}{\partial X}, \frac{\partial^2 u}{\partial X^2}$ in terms of $\frac{\partial w}{\partial t}, \frac{\partial w}{\partial x}, \frac{\partial^2 w}{\partial x^2}$.

$$\begin{aligned}
T &= \frac{L^2}{\alpha^2} t & X &= xL \\
\frac{\partial u}{\partial T} &= \frac{\partial}{\partial T}\left(w(x,t)\right) & \frac{\partial u}{\partial X} &= \frac{\partial}{\partial X}(w(x,t)) \\
&= \frac{\partial w}{\partial\left(\frac{L^2}{\alpha^2}t\right)} & &= \frac{\partial}{\partial(xL)}(w(x,t)) \\
\frac{\partial u}{\partial T} &= \frac{\alpha^2}{L^2}\frac{\partial w}{\partial t} & \frac{\partial u}{\partial X} &= \frac{1}{L}\frac{\partial w}{\partial x} \\
\frac{\partial^2 u}{\partial X^2} &= \frac{\partial}{\partial X}\left(\frac{1}{L}\frac{\partial w}{\partial x}\right) & \frac{\partial^2 u}{\partial X^2} &= \frac{1}{L}\frac{\partial}{\partial\,(xL)}\left(\frac{\partial w}{\partial x}\right) \\
&= \frac{1}{L^2}\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right) & &= \frac{1}{L^2}\frac{\partial^2 w}{\partial x^2}
\end{aligned}$$

(c) Show that the PDE can be written as

$$\text{PDE} \qquad \frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + f(x,t) \qquad 0 < x < 1, \qquad 0 < t < +\infty$$

PDE

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= \alpha^2 \frac{\partial^2 u}{\alpha x^2} + F(X,T) && 0 < X < L, && 0 < T < +\infty \\
\frac{\alpha^2}{L^2}\frac{\partial w}{\partial t} &= \alpha^2 \frac{1}{L^2}\frac{\partial^2 w}{\partial x^2} + F(X,T) && 0 < xL < L, && 0 < \frac{L^2}{\alpha^2} t < +\infty \\
\frac{\partial w}{\partial t} &= \frac{\partial^2 w}{\partial x^2} + \frac{L^2}{\alpha^2} F(X,T) && 0 < x < 1, && 0 < t < +\infty \\
\frac{\partial w}{\partial t} &= \frac{\partial^2 w}{\partial x^2} + \frac{L^2}{\alpha^2} F\left(Lx, \frac{L^2}{\alpha^2} t\right) && 0 < x < 1, && 0 < t < +\infty \\
\frac{\partial w}{\partial t} &= \frac{\partial^2 w}{\partial x^2} + f(x,t) && 0 < x < 1, && 0 < t < +\infty
\end{aligned}
$$

What is $f(x,t)$ in terms of $F(X,T)$?

$$
f(x,t) = \frac{L^2}{\alpha^2} F\left(Lx, \frac{L^2}{\alpha^2} t\right)
$$

(d) Show hat the BC can be written as

BC

$$
\begin{aligned}
\alpha_1 \frac{\partial w}{\partial x}(0,t) + \beta_1 w(0,t) &= g_1(t) && 0 < t < +\infty \\
\alpha_2 \frac{\partial w}{\partial x}(1,t) + \beta_2 w(1,t) &= g_2(t) && 0 < t < +\infty
\end{aligned}
$$

BC

$$
\begin{aligned}
\alpha_1 L \frac{\partial u}{\partial X}(0,T) + \beta_1 u(0,t) &= G_1(T) && 0 < T < +\infty \\
\alpha_1 L \frac{1}{L}\frac{\partial w}{\partial x}(0,t) + \beta_1 w(0,t) &= G_1\left(\frac{L^2}{\alpha^2} t\right) && 0 < \frac{L^2}{\alpha^2} t < +\infty \\
\alpha_1 \frac{\partial w}{\partial x}(0,t) + \beta_1 w(0,t) &= g_1(t) && 0 < t < +\infty \\
\alpha_2 L \frac{\partial u}{\partial X}(L,T) + \beta_2 u(L,t) &= G_2(T) && \\
\alpha_2 L \frac{1}{L}\frac{\partial w}{\partial x}\left(\frac{L}{L}, t\right) + \beta_2 w\left(\frac{L}{L}, t\right) &= G_2\left(\frac{L^2}{\alpha^2} t\right) && 0 < \frac{L^2}{\alpha^2} t < +\infty \\
\alpha_2 \frac{\partial w}{\partial x}(1,t) + \beta_2 w(1,t) &= g_2(t) && 0 < t < +\infty
\end{aligned}
$$

What are $g_1(t)$ and $g_2(t)$ in terms of $G_1(T)$ and $G_2(T)$?

$$
g_1(t) = G_1\left(\frac{L^2}{\alpha^2} t\right) \qquad\qquad g_2(t) = G_2\left(\frac{L^2}{\alpha^2} t\right)
$$

(e) Show that the IC can be written as

IC

$$
\begin{aligned}
w(x,0) &= \phi(x) && 0 < x < 1
\end{aligned}
$$

IC

$$
\begin{aligned}
u(X,0) &= \Phi(X) && 0 < X < L \\
u(X,0) = w(x,0) &= \Phi(Lx) && 0 < Lx < L
\end{aligned}
$$

$$w(x,0) = \phi(x) \qquad\qquad 0 < x < 1$$

What is $\phi(x)$ in terms of $\Phi(X)$?

$$\phi(x) = \Phi(Lx) = \Phi(X)$$