Homework 8

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November 7, 2014

5.1. G, H, M
5.2. G*, H*

5.1 G. Suppost the $f : \mathbb{R}^n \to \mathbb{R}$ is continuous. If there are $\boldsymbol{x} \in \mathbb{R}^n$ and $C \in \mathbb{R}$ such that $f(\boldsymbol{x}) < C$, then prove that there is $r > 0$ such that for all $\boldsymbol{y} \in \boldsymbol{B}_r(\boldsymbol{x}), f(\boldsymbol{y}) < C$

We know that for every $\varepsilon > 0$ there exists an $r > 0$ such that for all $\boldsymbol{x}, \boldsymbol{y}$ with $||\boldsymbol{x}-\boldsymbol{y}|| < r$ we have $|f(\boldsymbol{x}) - f(\boldsymbol{y})| < \varepsilon$. Now if we say $f(\boldsymbol{x}) + \varepsilon = C$. That is $f(\boldsymbol{x}) = C - \varepsilon$. Then $|C - \varepsilon - f(\boldsymbol{y})| < \varepsilon$ or $C - \varepsilon - \varepsilon < f(\boldsymbol{y}) < C - \varepsilon + \varepsilon$. And so for this ε we have $f(\boldsymbol{y}) < C$. Because the function is continuous we can find our r as required.

H. Suppose that functions f, g, h mapping $S \subset \mathbb{R}^n$ into $\mathbb{R}$ satisfy $f(\boldsymbol{x}) \le g(\boldsymbol{x}) \le h(\boldsymbol{x})$ for $\boldsymbol{x} \in S$. Suppose that c is a limit point of S and $\lim_{\boldsymbol{x}\to\boldsymbol{c}} f(\boldsymbol{x}) = \lim_{\boldsymbol{x}\to\boldsymbol{c}} h(\boldsymbol{x}) = L$. Show that $\lim_{\boldsymbol{x}\to\boldsymbol{c}} g(\boldsymbol{x}) = L$.

For any ε we can find r_1, r_2 such that

$$\begin{aligned} &|h(\boldsymbol{x}) - L| < \varepsilon && \text{whenever} && 0 < ||\boldsymbol{x} - \boldsymbol{c}|| < r_1 \\ &|f(\boldsymbol{x}) - L| < \varepsilon && \text{whenever} && 0 < ||\boldsymbol{x} - \boldsymbol{c}|| < r_2 \end{aligned}$$

$$\begin{aligned} -\varepsilon < h(\boldsymbol{x}) - L < \varepsilon \\ -\varepsilon < f(\boldsymbol{x}) - L < \varepsilon \\ f(\boldsymbol{x}) \le g(\boldsymbol{x}) \le h(\boldsymbol{x}) \\ f(\boldsymbol{x}) - L \le g(\boldsymbol{x}) - L \le h(\boldsymbol{x}) - L \end{aligned}$$

and so when

$$0 < ||\boldsymbol{x} - \boldsymbol{c}|| < \min\{r_1, r_2\}$$

then

$$\begin{aligned} &-\varepsilon < f(\boldsymbol{x} - L \le g(\boldsymbol{x}) - L \le h(\boldsymbol{x}) - L < \varepsilon \\ &-\varepsilon < g(\boldsymbol{x}) - L < \varepsilon \end{aligned}$$

M. Consider the linear transformation A on $\mathbb{R}^4$ given by the matrix $A = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$

(a) Compute the Lipschits constant obtained in corollary 5.1.7.

$$
\begin{aligned}
C &= \left(\sum_{i=1}^{4}\sum_{j=1}^{4}|a_{ij}|^2\right)^{1/2} \\
&= \left(\sum_{i=1}^{4}\sum_{j=1}^{4}\frac{1}{4}\right)^{1/2} \\
&= \left(\sum_{i=1}^{4} 1\right)^{1/2} \\
C &= 2
\end{aligned}
$$

(b) Show that $||A\boldsymbol{x}|| = ||x||$ for all $\boldsymbol{x} \in \mathbb{R}^4$. Deduce that the optimal Lipschitz constant is 1. Hint: The columns of A form and orthonormal basis for $\mathbb{R}^4$

$$
\begin{aligned}
(x_1+x_2+x_3+x_4)^2 =& x_1^2 + x_1x_2 + x_1x_3 + x_1x_4 + x_2x_1 + x_2^2 + x_2x_3 + x_2x_4 \\
&+ x_3x_1 + x_3x_2 + x_3^2 + x_3x_4 + x_4x_1 + x_4x_2 + x_4x_3 + x_4^2 \\
&= {x_1}^2 + {x_2}^2 + {x_3}^2 + {x_4}^2 + x_1{x_2}^2 + x_1{x_3}^2 + x_1{x_4}^2 + x_2{x_3}^2 + x_2{x_4}^2 + x_3{x_4}^2 \\
(x_1-x_2+x_3-x_4)^2 =& {x_1}^2 + {x_2}^2 + {x_3}^2 + {x_4}^2 - x_1{x_2}^2 + x_1{x_3}^2 - x_1{x_4}^2 - x_2{x_3}^2 + x_2{x_4}^2 - x_3{x_4}^2 \\
(x_1+x_2-x_3-x_4)^2 =& {x_1}^2 + {x_2}^2 + {x_3}^2 + {x_4}^2 + x_1{x_2}^2 - x_1{x_3}^2 - x_1{x_4}^2 - x_2{x_3}^2 - x_2{x_4}^2 + x_3{x_4}^2 \\
(x_1-x_2-x_3+x_4)^2 =& {x_1}^2 + {x_2}^2 + {x_3}^2 + {x_4}^2 - x_1{x_2}^2 - x_1{x_3}^2 + x_1{x_4}^2 + x_2{x_3}^2 - x_2{x_4}^2 - x_3{x_4}^2
\end{aligned}
$$

$$
\begin{aligned}
&(x_1+x_2+x_3+x_4)^2 + \\
&(x_1-x_2+x_3-x_4)^2 + \\
&(x_1+x_2-x_3-x_4)^2 + \\
&\quad (x_1-x_2-x_3+x_4)^2 = 2\cdot({x_1}^2+{x_2}^2+{x_3}^2+{x_4}^2)
\end{aligned}
$$

$$
||A\boldsymbol{x}|| = \sqrt{\frac{1}{2}\cdot 2\cdot({x_1}^2+{x_2}^2+{x_3}^2+{x_4}^2)} = ||\boldsymbol{x}||
$$

Of course then $||A\boldsymbol{x} - A\boldsymbol{y}|| = ||A(\boldsymbol{x}-\boldsymbol{y})|| = ||\boldsymbol{x}-\boldsymbol{y}|| \leq 1\cdot||\boldsymbol{x}-\boldsymbol{y}||$ and so the optimal Lipschitz constant is 1.

5.2 G. (A monotone convergence test for functions.) Suppose that f is an increasing function on (a,b) that is bounded above. Prove that the one-sided limit $\lim\limits_{x\to b-} f(x)$ exists.

Lets say the least upper bound of $f(x)$ on (a,b) is L. We seek to show that $\lim\limits_{x\to b^-} x = L$. This means that for every $\varepsilon > 0$ there exists an $r > 0$ such that $|f(x) - L| < \varepsilon$ for every $b - r < x < b$. Why don't we assume instead that for any $r \in (a,b)$ such that $x \leq b - r$ or $x \geq b$ we can find some $\varepsilon > 0$ such that $|f(x) - L| \geq \varepsilon$. Naturally we can throw out $x \geq b$ condition. Now observe that for any $c \in (a,b)$ if $x \leq c$ then $f(x) \leq f(c)$ because $f(x)$ is increasing. This means that $f(x) \leq f(b-r)$ and so $|f(x) - L| = L - f(x) \geq L - f(b-r) = \varepsilon$. And so we have found our ε as required.

H. Define f on $\mathbb{R}$ by $f(x) = x\chi_{\mathbb{Q}}(x)$. Show that f is continuous at 0 and that this is the *only* point where f is continuous.

First note that $f(0) = 0$. And so we seek to show that for every $\varepsilon > 0$ there is a $r > 0$ such that for all $x \in \mathbb{R}$ with $|x| < r$ we have $|f(x)| < \varepsilon$. Lets just make $r = \varepsilon$. Now if $x \in \mathbb{Q}$ then $f(x) = x$

and if $|x| < \varepsilon$ then $|f(x)| < \varepsilon$. But if $x \in \mathbb{R} \setminus \mathbb{Q}$ then $f(x) = 0$ and $0 \leq |x| < \varepsilon$ and so we are still good.

Now lets assume $a \neq 0$. Let's fix $\varepsilon = |\frac{a}{2}|$ and assume we can find some r such that for all $x \in \mathbb{R}$ with $|x - a| < r$ we will have $|f(x) - f(a)| < \varepsilon$.

Lets assume that $a > \varepsilon$ is rational. Then for any r we can find some $x = a + \frac{r}{\sqrt{2}}$ where $a + \frac{r}{\sqrt{2}} - a = \frac{r}{\sqrt{2}} < r$ but $f(x) = 0$ and $f(a) = a$ and so $|f(x) - f(a)| = f(a) = a > \varepsilon$. Similarly if $a < -\varepsilon$ is rational then $x = a - \frac{r}{\sqrt{2}}$ will give us $|f(x) - f(a)| > \varepsilon$.

Now we assume that $a > \varepsilon$ is irrational. Now we can find a rational number between any two real numbers. Lets say $a < x < a + r$ where $x \in \mathbb{Q}$. Then $|x - a| < r$ but

$$|f(x) - f(a)| = f(x) = x > a > \frac{a}{2} = \varepsilon$$

Similarly if $a < -1$ and irrational then for $a - r < x < a$ and $x \in \mathbb{Q}$ we have $|x - a| < r$ and $|f(x) - f(a)| > \frac{a}{2}$. And so for all $a \neq 0$ we know that f is not continuous