Math 329

Exam 1

Directions: Attempt all of the problems. Show all work for full credit. Each exercise is worth 20 points. Good luck and just do the best you can.

1. Consider the matrix

$$A = \begin{bmatrix} 3 & -1 & 2 & 7 \\ 2 & 1 & 3 & 3 \\ 2 & 2 & 4 & 2 \end{bmatrix}.$$

(a) Determine rank(A).
(b) Find the general solution in standard form of the equation $A\mathbf{x} = \mathbf{0}$.
(c) Find all vectors $\mathbf{b} \in \mathbb{R}^3$ such that $A\mathbf{x} = \mathbf{b}$ is consistent.

Solution.
(a) Since

$$rref(A) = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

it is easy to see that rank$(A) = 2$.
(b) We can see from the $rref$ that

$$\begin{aligned} x_1 + x_3 + 2x_4 &= 0 \\ x_2 + x_3 - x_4 &= 0. \end{aligned}$$

The free variables are x_3 and x_4. We have

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -x_3 - 2x_4 \\ -x_3 + x_4 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -x_3 \\ -x_3 \\ x_3 \\ 0 \end{pmatrix} + \begin{pmatrix} -2x_4 \\ x_4 \\ 0 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

In other words,

$$\text{Sol}(\mathcal{H}) = \text{Span}\left(\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right)$$

(c) Bring the augmented matrix

$$\begin{bmatrix} 3 & -1 & 2 & 7 & b_1 \\ 2 & 1 & 3 & 3 & b_2 \\ 2 & 2 & 4 & 2 & b_3 \end{bmatrix}$$

to echelon form to get

$$\begin{bmatrix} 1 & 1 & 2 & 1 & \frac{1}{2}b_3 \\ 0 & -1 & -1 & 1 & b_2 - b_3 \\ 0 & 0 & 0 & 0 & b_1 - 4b_2 + \frac{5}{2}b_3 \end{bmatrix}.$$

Therefore, $\mathbf{b} = (b_1, b_2, b_3)$ must satisfy

$$b_1 - 4b_2 + \frac{5}{2}b_3 = 0$$

2. Let A be an 4×5 matrix and let $\mathcal{H}$ be the homogeneous system $A\mathbf{x} = \mathbf{0}$. Prove (carefully) that if $\text{rank}(A) = 3$, then $\text{Sol}(\mathcal{H})$ is a plane in $\mathbb{R}^5$.

Solution. Since $\text{rank}(A) = 3$, the $rref(A)$ has 3 pivot variables and two free variables. Without loss of generality (there are other options), we assume that

$$rref(A) = \begin{bmatrix} 1 & 0 & 0 & b_{14} & b_{15} \\ 0 & 1 & 0 & b_{24} & b_{25} \\ 0 & 0 & 1 & b_{34} & b_{35} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We now have the equivalent system

$$\begin{aligned} x_1 + b_{14}x_4 + b_{15}x_5 &= 0 \\ x_2 + b_{24}x_4 + b_{25}x_5 &= 0 \\ x_3 + b_{34}x_4 + b_{35}x_5 &= 0. \end{aligned}$$

Solving for the pivot variables, we have

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -b_{14}x_4 - b_{15}x_5 \\ -b_{24}x_4 - b_{25}x_5 \\ -b_{34}x_4 - b_{35}x_5 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -b_{14}x_4 \\ -b_{24}x_4 \\ -b_{34}x_4 \\ x_4 \\ 0 \end{pmatrix} + \begin{pmatrix} -b_{15}x_5 \\ -b_{25}x_5 \\ -b_{35}x_5 \\ 0 \\ x_5 \end{pmatrix} = x_4 \begin{pmatrix} -b_{14} \\ -b_{24} \\ -b_{34} \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -b_{15} \\ -b_{25} \\ -b_{35} \\ 0 \\ 1 \end{pmatrix}.$$

Therefore, if

$$\mathbf{u} = \begin{pmatrix} -b_{14} \\ -b_{24} \\ -b_{34} \\ 1 \\ 0 \end{pmatrix} \text{ and } \mathbf{v} = \begin{pmatrix} -b_{15} \\ -b_{25} \\ -b_{35} \\ 0 \\ 1 \end{pmatrix}$$

then $\text{Sol}(\mathcal{H}) = P(\mathbf{0}, \mathbf{u}, \mathbf{v})$

3. Prove that if $A, B \in \mathcal{M}_{m\times n}$ and $rref(A) = rref(B)$ then there exists an invertible matrix E such that $A = EB$. Is the converse true?

Proof. Let $R = rref(A) = rref(B)$. Then $R = PA$ where P is a product of elementary matrices. Similarly, $R = QB$ where Q is a product of elementary matrices. It follows that $PA = QB$. Since elementary matrices are invertible, the product P must be invertible (by a theorem from the notes). It follows that $A = P^{-1}(QB) = (P^{-1}Q)B$. Since P^{-1} and Q are both invertible, their product $E = P^{-1}Q$ is invertible. The result follows.

4. A square matrix $A \in \mathcal{M}_n$ is called *upper-triangular* if $\text{ent}_{ij}(A) = 0$ whenever $i > j$. Prove that the product of two upper-triangular matrices is upper-triangular.

Proof. Suppose that $A, B \in \mathcal{M}_n$ are upper-triangular. Then

$$\begin{aligned}
& \mathbf{r}_i(AB) \\
= \ & a_{i1}\mathbf{r}_1(B) + a_{i2}\mathbf{r}_2(B) + ... + a_{i,i-1}\mathbf{r}_{i-1}(B) + a_{ii}\mathbf{r}_i(B) + ... + a_{in}\mathbf{r}_n(B) \\
= \ & 0\mathbf{r}_1(B) + 0\mathbf{r}_2(B) + ... + 0\mathbf{r}_{i-1}(B) + a_{ii}\mathbf{r}_i(B) + ... + a_{in}\mathbf{r}_n(B) \quad \text{(since } A \text{ is upper-triangular)} \\
= \ & a_{ii}(0, ..., 0, b_{ii}, ..., b_{in}) + ... + a_{in}(0, ..., 0, ..., 0, b_{nn}) \quad \text{(since } B \text{ is upper-triangular)} \\
= \ & (0, ..., 0, a_{ii}b_{ii}, ..., a_{ii}b_{in}) + ... + (0, ..., 0, ..., 0, a_{in}b_{nn}) \\
= \ & (0, ..., 0, a_{ii}b_{ii}, ..., a_{ii}b_{in} + a_{in}b_{in}).
\end{aligned}$$

The first $i - 1$ entries of the ith row of AB are all 0. It follows that AB is upper-triangular.

5. Let A be an $m \times n$ matrix and let $\mathcal{H}$ be the homogeneous system $A\mathbf{x} = \mathbf{0}$. Prove (carefully) that if $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k \in \mathrm{Sol}(\mathcal{H})$, then $\mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k) \subseteq \mathrm{Sol}(\mathcal{H})$.

Proof. Choose any $\mathbf{u} \in \mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k)$ and write $\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_k\mathbf{v}_k$. Using the usual matrix algebra, we have

$$
\begin{aligned}
& A\mathbf{u} \\
= \; & A(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_k\mathbf{v}_k) \\
= \; & A(c_1\mathbf{v}_1) + A(c_2\mathbf{v}_2) + ... + A(c_k\mathbf{v}_k) \\
= \; & c_1(A\mathbf{v}_1) + c_2(A\mathbf{v}_2) + ... + c_k(A\mathbf{v}_k) \\
= \; & c_1\mathbf{0} + c_2\mathbf{0} + ... + c_k\mathbf{0} \\
= \; & \mathbf{0}.
\end{aligned}
$$

Therefore, $\mathbf{u} \in \mathrm{Sol}(\mathcal{H})$.

6. Suppose that $A, B \in \mathcal{M}_n$ are square matrices such that $BA = I_n$. Prove:
(a) $A\mathbf{x} = \mathbf{0}$ has a unique solution.
(b) A is invertible and $A^{-1} = B$.

Proof.
(a) Using the usual matrix algebra, we have

$$\begin{aligned} A\mathbf{u} &= \mathbf{0} \\ &\Rightarrow B(A\mathbf{u}) = B\mathbf{0} \ \text{ (since matrix mult is well-defined)} \\ &\Rightarrow (BA)\mathbf{u} = \mathbf{0} \ \text{ (since matrix mult is associative)} \\ &\Rightarrow I_n\mathbf{u} = \mathbf{0} \ \text{ (since } BA = I_n\text{)} \\ &\Rightarrow \mathbf{u} = \mathbf{0} \ \text{ (Identity).} \end{aligned}$$

(a) Using a main theorem, we know that $A\mathbf{x} = \mathbf{0}$ has a unique solution is equivalent to the fact that A is invertible. We have

$$\begin{aligned} BA &= I_n \\ &\Rightarrow (BA)A^{-1} = I_nA^{-1} \\ &\Rightarrow B(AA^{-1}) = I_nA^{-1} \\ &\Rightarrow BI_n = I_nA^{-1} \\ &\Rightarrow BI_n = I_nA^{-1} \\ &\Rightarrow B = A^{-1} \end{aligned}$$

7. Prove that if A is a square matrix with a column of all zeros, then A is not invertible.

Proof. Suppose that $\mathbf{r}_i(A) = (0, 0, ..., 0)$ and that A is invertible. Then there exists a square matrix B such that $AB = I_n$. But

$$\begin{aligned}
& \mathbf{r}_i(AB) \\
= \; & a_{i1}\mathbf{r}_1(B) + a_{i2}\mathbf{r}_2(B) + ... + a_{in}\mathbf{r}_n(B) \\
= \; & 0\mathbf{r}_1(B) + 0\mathbf{r}_2(B) + ... + \mathbf{r}_n(B) \\
= \; & (0, 0, ..., 0) \\
\neq \; & \mathbf{e}_i \\
= \; & \mathbf{r}_i(I_n)
\end{aligned}$$

which is absurd.

8. Let $A \in \mathcal{M}_n$ be a symmetric matrix and let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Prove that if $A\mathbf{x} = \lambda_1 \mathbf{x}$ and $A\mathbf{y} = \lambda_2 \mathbf{y}$ for some $\lambda_1 \neq \lambda_2 \in \mathbb{R}$, then $\mathbf{x}$ and $\mathbf{y}$ are orthogonal.

Proof. Since A is symmetric $A = A^{\mathrm{T}}$. By a theorem in the notes

$$A\mathbf{x} \cdot \mathbf{y} = \mathbf{x} \cdot A^{\mathrm{T}}\mathbf{y} = \mathbf{x} \cdot A\mathbf{y}.$$

It follows that

$$(\lambda_1 \mathbf{x}) \cdot \mathbf{y} = \mathbf{x} \cdot (\lambda_2 \mathbf{y})$$

and so

$$\lambda_1(\mathbf{x} \cdot \mathbf{y}) = \lambda_2(\mathbf{x} \cdot \mathbf{y}).$$

It follows that

$$\lambda_1(\mathbf{x} \cdot \mathbf{y}) - \lambda_2(\mathbf{x} \cdot \mathbf{y}) = 0$$

from which it follows that

$$(\lambda_1 - \lambda_2)(\mathbf{x} \cdot \mathbf{y}) = 0.$$

Since $\lambda_1 - \lambda_2 \neq 0$, it follows that

$$\mathbf{x} \cdot \mathbf{y} = 0.$$

9. A square matrix $A \in \mathcal{M}_n$ is called orthogonal if $A^{\mathrm{T}}A = I_n$. Prove that the columns of A are unit vectors that are orthogonal to one another. That is, prove that

$$\mathbf{c}_i \cdot \mathbf{c}_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Proof. Since $A^{\mathrm{T}}A = I_n$, we have that

$$\mathrm{ent}_{ij}(A^{\mathrm{T}}A) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}.$$

But

$$\mathrm{ent}_{ij}(A^{\mathrm{T}}A) = \mathbf{r}_i(A^{\mathrm{T}}) \cdot \mathbf{c}_j(A) = \mathbf{c}_i(A) \cdot \mathbf{c}_j(A).$$

It follows that

$$\mathbf{c}_i(A) \cdot \mathbf{c}_j(A) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}.$$

This makes the orthogonal part clear. Since $\mathbf{c}_i(A) \cdot \mathbf{c}_i(A) = 1$,it must be the case that $\|\mathbf{c}_i(A)\|^2 = 1$. Therefore, $\|\mathbf{c}_i(A)\| = 1$ as needed.

10. Let $\mathbf{x}$ and $\mathbf{y}$ be arbitrary nonzero vectors. Prove that if $r = \|\mathbf{x}\|$ and $s = \|\mathbf{y}\|$, then $s\mathbf{x} + r\mathbf{y}$ bisects the angle between $\mathbf{x}$ and $\mathbf{y}$.

Proof. Let θ be the angle between $\mathbf{x}, \mathbf{y}$. It is enough to check that the angle θ_x between $\mathbf{x}$ and $s\mathbf{x} + r\mathbf{y}$ is equal to the angle θ_y between $\mathbf{y}$ and $s\mathbf{x} + r\mathbf{y}$. We have

$$\begin{aligned}
& \theta_x \\
= \ & \cos^{-1}\left(\frac{\mathbf{x} \cdot (s\mathbf{x} + r\mathbf{y})}{\|\mathbf{x}\| \, \|s\mathbf{x} + r\mathbf{y}\|}\right) \\
= \ & \cos^{-1}\left(\frac{s(\mathbf{x} \cdot \mathbf{x}) + r(\mathbf{x} \cdot \mathbf{y})}{\|\mathbf{x}\| \, \|s\mathbf{x} + r\mathbf{y}\|}\right) \\
= \ & \cos^{-1}\left(\frac{s \|\mathbf{x}\|^2 + r \|\mathbf{x}\| \, \|\mathbf{y}\| \cos\theta}{\|\mathbf{x}\| \, \|s\mathbf{x} + r\mathbf{y}\|}\right) \\
= \ & \cos^{-1}\left(\frac{s \|\mathbf{x}\| + r \|\mathbf{y}\| \cos\theta}{\|s\mathbf{x} + r\mathbf{y}\|}\right) \\
= \ & \cos^{-1}\left(\frac{\|\mathbf{x}\| \, \|\mathbf{y}\| + \|\mathbf{x}\| \, \|\mathbf{y}\| \cos\theta}{\|s\mathbf{x} + r\mathbf{y}\|}\right)
\end{aligned}$$

and

$$\begin{aligned}
& \theta_y \\
= \ & \cos^{-1}\left(\frac{\mathbf{y} \cdot (s\mathbf{x} + r\mathbf{y})}{\|\mathbf{y}\| \, \|s\mathbf{x} + r\mathbf{y}\|}\right) \\
= \ & \cos^{-1}\left(\frac{s(\mathbf{x} \cdot \mathbf{y}) + r(\mathbf{y} \cdot \mathbf{y})}{\|\mathbf{y}\| \, \|s\mathbf{x} + r\mathbf{y}\|}\right) \\
= \ & \cos^{-1}\left(\frac{s \|\mathbf{x}\| \, \|\mathbf{y}\| \cos\theta + r \|\mathbf{y}\|^2}{\|\mathbf{y}\| \, \|s\mathbf{x} + r\mathbf{y}\|}\right) \\
= \ & \cos^{-1}\left(\frac{s \|\mathbf{x}\| \cos\theta + r \|\mathbf{y}\|}{\|s\mathbf{x} + r\mathbf{y}\|}\right). \\
= \ & \cos^{-1}\left(\frac{\|\mathbf{x}\| \, \|\mathbf{y}\| \cos\theta + \|\mathbf{x}\| \, \|\mathbf{y}\|}{\|s\mathbf{x} + r\mathbf{y}\|}\right)
\end{aligned}$$