Homework

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3.2 19. Let G be a group, and let $a \in G$. The set $C(a) = \{x \in G | xa = ax\}$ of all elements of G that commute with a is called the **centralizer** of a.

(a) Show that $C(a)$ is a subgroup of G.
$C(a)$ contains the identity e because $ea = a = ae$. If we take any two $x_1, x_2 \in C(a)$ then $x_1x_2a = x_1ax_2 = ax_1x_2$ so $x_1x_2 \in C(a)$ and then $C(a)$ is closed. And finally, lets see if the inverse of some $x \in C(a)$ is in $C(a)$:

$$\begin{aligned} xa &= ax \\ xax^{-1} &= axx^{-1} = a \\ x^{-1}xax^{-1} &= x^{-1}a \\ ax^{-1} &= x^{-1}a \end{aligned}$$

So it looks like $C(a)$ is a subgroup of G.

(b) Show that $\langle a \rangle \subseteq C(a)$.
Note that $aa = aa$ so $a \in C(a)$ and we have just established that $C(a)$ is a group, so closure tells us that $a^n \in C(a) \forall n \in \mathbb{N}$. Also $a^0 = e \in C(a)$ and for all integers $n < 0$ then $a^n = (a^{-n})^{-1}$. Because a^{-n} is in the group, then it's inverse, a^n must also be. So $\langle a \rangle \subseteq C(a)$

(c) Compute $C(a)$ if $G = S_3$ and $a = (1, 2, 3)$.
We already know that $(1), (1, 2, 3) \in C(a)$ from parts (a) and (b).

$$\begin{aligned} (1,2)(1,2,3) &= (1,3) & (1,2,3)(1,2) &= (2,3) \\ (1,3)(1,2,3) &= (2,3) & (1,2,3)(1,3) &= (1,2) \\ (2,3)(1,2,3) &= (1,2) & (1,2,3)(2,3) &= (1,3) \\ (1,3,2)(1,2,3) &= (1) & (1,2,3)(1,3,2) &= (1) \end{aligned}$$

And so $C(a) = \{(1), (1, 2, 3), (1, 3, 2)\}$

(d) Compute $C(a)$ if $G = S_3$ and $a = (1, 2)$
We already know that $(1), (1, 2) \in C(a)$ and $(1, 2, 3) \notin C(a)$.

$$\begin{aligned} (1,3)(1,2) &= (1,3,2) & (1,2)(1,3) &= (1,2,3) \\ (2,3)(1,2) &= (1,2,3) & (1,2)(2,3) &= (1,3,2) \\ (1,3,2)(1,2) &= (1,3) & (1,2)(1,3,2) &= (2,3) \end{aligned}$$

So $C(a) = \{(1), (1, 2)\}$

21. Let G be a group. The set $Z(G) = \{x \in G | xg = gx \quad \forall g \in G\}$ of all elements that commute with every other element of G is called the **center** of G.

(a) Show that $Z(G)$ is a subgroup of G.

Showing that $Z(G)$ is a subgroup of G is nearly the same as showing that $C(a)$ is a subgroup of G. It is easy to see that for the identity e we have $eg = g = ge$ for all $g \in G$ and for any $x_1, x_2 \in Z(G)$ we can see that $x_1x_2g = x_1gx_2 = gx_1x_2$ and finally for some $x \in Z(G)$:

$$\begin{aligned} xg &= gx \\ xgx^{-1} &= gxx^{-1} \\ x^{-1}xgx^{-1} &= x^{-1}g \\ gx^{-1} &= x^{-1}g \end{aligned}$$

(b) Show that $Z(G) = \bigcap_{a \in G} C(a)$.
Because any $x \in Z(G)$ commutes with all $g \in G$ we know that $xa = ax$ for any $a \in G$ and so $x \in C(a)$ for all $a \in G$ and so $Z(G) \subseteq \bigcap_{a \in G} C(a)$. Now looking at it the other way, if we take some $x \in C(a_1)$, if there exists some a_2 such that $xa_2 \neq a_2x$ then $x \notin C(a_2)$ and therefore not in $\bigcap_{a \in G} C(a)$. Because this x has an element in G that it doesn't commute with it is also not in $Z(G)$. Now because if an element is not in $\bigcap_{a \in G}$ then it is not in $Z(G)$, we know that $\bigcap_{a \in G} C(a) \subseteq Z(G)$.

(c) Compute the center of S_3.
$C((1,2)) \cap C((1,2,3)) = \{(1)\}$ from number 19, and $(1) \in Z(S_3)$ from part (a), so $Z(S_3) = \{(1)\}$

3.3 11. Let G_1 and G_2 be groups, and let G be the direct product $G_1 \times G_2$.
Let $H = \{(x_1, x_2) \in G_1 \times G_2 | x_2 = e\}$ and let $K = \{(x_1, x_2) \in G_1 \times G_2 | x_1 = e\}$

(a) Show that H and K are subgroups of G.
We assume that $x_2 = e$ means that x_2 is the identity for the group G_2, say e_2. Similarly we assume that $x_1 = e$ means that x_1 is the identity for the group G_1, say e_1 Now of course $e_1 \in G_1$ and $e_2 \in G_2$. This means that $(e_1, e_2) \in H$ and $(e_1, e_2) \in K$. Further, because (e_1, e_2) is the identity for G it will be an identity for any subset of G. Which means that both H and K contain an identity because they are subsets of G. Now lets take some $x_1, y_1 \in G_1$ and $x_2, y_2 \in G_2$. Because $x_1y_1 \in G_1$ then $(x_1, e_1)(y_1, e_1) = (x_1y_1, e_2) \in H$ and therefore H is closed. Similarly $(e_1, x_2)(e_1, y_2) = (e_1, x_2y_2) \in K$ and K is closed. Now lets pick any $x_1 \in G_1$ and $x_2 \in G_2$. We know there exists some ${x_1}^{-1} \in G_1$ and some ${x_2}^{-1} \in G_2$. By extension then (x_1, e_2) and $({x_1}^{-1}, e_2)$ are both in H. Similarly: (e_1, x_2) and $(e_1, {x_2}^{-1})$ are both in K. Obviously, $(x_1, e_2)({x_1}^{-1}, e_2) = (e_1, e_2) \in H$. And also similarly, $(e_1, x_2)(e_1, {x_2}^{-1}) = (e_1, e_2) \in K$.

(b) Show that $HK = KH = G$.
Lets take any $(x_1, e_2) \in H$ and any $(e_1, x_2) \in K$. Then for any $(x_1, e_2)(e_1, x_2) \in HK$ we see that $(x_1, e_2)(e_1, x_2) = (x_1e_1, e_2x_2) = (e_1x_1, x_2e_2) = (e_1, x_2)(x_1, e_2) \in KH$. And for any $(e_1, x_2)(x_1, e_2) \in KH$ we have $(e_1, x_2)(x_1, e_2) = (e_1x_1, x_2e_2) = (x_1, x_2) \in G$. And so $HK \subseteq KH \subseteq G$.
Now we take any $(x_1, x_2) \in G$. Then $(x_1, x_2) = (e_1x_1, x_2e_2) = (e_1, x_2)(x_1, e_2) \in KH$. And obviously for any $(e_1, x_2)(x_1, e_2) \in KH$ we have $(e_1, x_2)(x_1, e_2) = (e_1x_1, x_2e_2) = (x_1e_1, e_2x_2) = (x_1, e_2)(e_1, x_2) \in HK$. And so $G \subseteq KH \subseteq HK$.

(c) Show that $H \cap K = \{(e, e)\}$.
If $(x_1, x_2) \in H$ then $x_2 = e_2$ and if $(x_1, x_2) \in K$ then $x_1 = e_1$ and so if $(x_1, x_2) \in H$ and $(x_1, x_2) \in K$ then $(x_1, x_2) = (e_1, e_2)$ and so $H \cap K = \{(e_1, e_2)\}$

3.4 27. Using the definition of a group homomorphism given in Exercise 26, let $\phi : G_1 \to G_2$ be a group homomorphism. We define the **kernel** of ϕ to be

$$\ker(\phi) = \{x \in G_1 | \phi(x) = e\}$$

Prove that $\ker(\phi)$ is a subgroup of G_1.

First we establish that the identity is an element of $\ker(\phi)$.

$$\begin{aligned}
\phi(e) &= \phi(ee) \\
\phi(e)e &= \phi(ee) \\
\phi(e)e &= \phi(e)\phi(e) \\
e &= \phi(e)
\end{aligned}$$

So $e \in \ker(\phi)$. Now lets take some $x \in \ker(\phi)$. Then $e = \phi(e) = \phi(xx^{-1}) = \phi(x)\phi(x^{-1}) = e\phi(x^{-1}) = \phi(x^{-1})$ and so $x^{-1} \in \ker(\phi)$ for all $x \in \ker(\phi)$. And finally if we have some $x, y \in \ker(\phi)$ then $\phi(xy) = \phi(x)\phi(y) = ee = e$ and so $xy \in \ker(\phi)$ and we have closure which is the last requirement for a group.