Homework

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Section 2.3: 12. Section 3.1: 2, 9. Section 3.2: 6.

2.3 12. Prove that (a, b) cannot bewritten as a product of two cycles of length three.
(a, b) is a product of an odd number of transpositions. A cycle of length three is a product of an even numbe of transpositions. We know from theorem 2.3.11 that if a permutation is written with a certain number of transpositions, then writing it in another way will have the same parity as our first. This means that we can never write a transposition as a product of any number of cycles of length three as this will always wind up being the product of an even number of transpositions, not the odd number required.

3.1 2. For each binary operation $*$ defined on a set below, determine whether or not $*$ gives a group structure on the set. If it is not a group, say which axioms fail to hold.

(a) Define $*$ on $\mathbb{Z}$ by $a * b = ab$.
There is only one identity element, which we know is 1. There are no solutions to $0 \cdot x = 1$ so the element 0 has no inverse, thus multiplication and the integers do not make a group.

(b) Define $*$ on $\mathbb{Z}$ by $a * b = \max\{a, b\}$.
Let e be the identity element. Then $\max(e, e-1)$ gives us e not $e-1$. So by contradicttion, there is no identity element, and so the max operation doesn't make a group with the integers.

(c) Define $*$ on $\mathbb{Z}$ by $a * b = a - b$.
Associativity fails, for example $(1-2)-3 = -4$ but $1-(2-3) = 0$. So subtraction does not form a group with the integers.

(d) Define $*$ on $\mathbb{Z}$ by $a * b = |ab|$.
There is no indentity element, because there is no identity element for negative numbers. For example, there is no solution to the equation $|-2 \cdot e| = -2$. So taking the absolute value of the product of two numbers does not form a group with the integers.

(e) Define $*$ on $\mathbb{R}^+$ by $a * b = ab$.
Any positive real number multiplied by any positive real number will be positive, and so multiplication is a binary operation on $\mathbb{R}^+$. Multiplication is also associative under $\mathbb{R}^+$. The identity element–one–is in $\mathbb{R}^+$. And finally, if $a \in \mathbb{R}^+$ then $\frac{1}{a} \in \mathbb{R}^+$ and $a \cdot \frac{1}{a} = 1$. So $\mathbb{R}+$ forms a group with multiplication.

(f) Define $*$ on $\mathbb{Q}$ by $a * b = ab$.
There is no multiplicative inverse for 0 in the rationals. This fails in the exact same way as part (a) fails.

9. Let $G = \{x \in \mathbb{R} | x > 0 \text{ and } x \neq 1\}$. Define the operation $*$ on G by $a * b = a^{\ln b}$, for all $a, b \in G$. Prove that G is an abelian group under the operation $*$.

proof

If we take any $b \in \mathbb{R}$ such that $b > 0$ then $\ln b \in \mathbb{R}$. Furthermore, if we take any $a, b \in \mathbb{R}$ such that $a > 0$ then $a^b \in \mathbb{R}$ and $a^b > 0$. Note that $1 \notin G$ and $a^0 = 1 \forall a \in G$. But $0 = \ln 1$ and $1 \notin G$. So then G is closed under our operation.

Proving associativity is pretty straighforward, using the usual exponent and logarithm rules.

$$a * (b * c) = a * b^{\ln c} = a^{\ln b^{\ln c}} = a^{(\ln b)\cdot(\ln c)} = (a^{\ln b})^{\ln c} = (a * b)^{\ln c} = (a * b) * c$$

The identity is actually Euler's number. We often use e to represent a generic identity. Here we are using the letter e to represent our particular identity–Euler's number.

$$e * a = e^{\ln a} = a = a^1 = a^{\ln e} = a * e$$

Now we are just left to ensure we can always find an inverse $a^{-1} \in G$ for any $a \in G$.

$$a * a^{-1} = e = a^{\ln a^{-1}} \qquad \ln a^{\ln a^{-1}} = \ln e \qquad \ln a \cdot \ln a^{-1} = 1$$

$$\ln a^{-1} = \frac{1}{\ln a} \qquad e^{\ln a^{-1}} = e^{\frac{1}{\ln a}} \qquad a^{-1} = e^{\frac{1}{\ln a}}$$

We can take e to any power and we will get a positive real back. We can not get $e^0 = 1$ because there is not number divided by zero which will give us 0. We also do not have toworry about $e^{\frac{1}{0}}$ because $0 = \ln a$, has only 1 as a solution and $1 \notin G$.

$$a * a^{-1} = a^{\ln e^{\frac{1}{\ln a}}} = a^{\frac{\ln e}{\ln a}} = a^{\log_a e} = e$$
$$a^{-1} * a = (e^{\frac{1}{\ln a}})^{\ln a} = e^{\frac{\ln a}{\ln a}} = e^1 = e$$

Well we definitely have a group. Is it commutative?

$$a * b = a^{\ln b} = e * a^{\ln b} = e^{\ln a^{\ln b}} = e^{(\ln b)(\ln a)} = e^{\ln b^{\ln a}} = e * b^{\ln a} = b^{\ln a} = b * a$$

Yep, it is commutative and therefore abelian.

3.2 6. Let $G = GL_2(\mathbb{R})$.

(a) Show that $T = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \middle| ad \neq 0 \right\}$ is a subgroup of G.

First we choose an arbitrary element of T, call it T_1 and find it's inverse.

$$\left[\begin{array}{cc|cc} a_1 & b_1 & 1 & 0 \\ 0 & d_1 & 0 & 1 \end{array}\right] \quad \left[\begin{array}{cc|cc} a_1 & b_1 & 1 & 0 \\ 0 & 1 & 0 & \frac{1}{d_1} \end{array}\right] \quad \left[\begin{array}{cc|cc} a_1 & 0 & 1 & -\frac{b_1}{d_1} \\ 0 & 1 & 0 & \frac{1}{d_1} \end{array}\right] \quad \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{a_1} & -\frac{b_1}{a_1 d_1} \\ 0 & 1 & 0 & \frac{1}{d_1} \end{array}\right]$$

We note that two implications of $ad \neq 0$ are that $a_1 \neq 0$ and $d_1 \neq 0$. This is all we need to say that ${T_1}^{-1}$ is in T (although we needn't demonstrate that for this proof).

$$\begin{aligned} {T_1}^{-1}T_1 &= \begin{bmatrix} \frac{1}{a_1} & -\frac{b_1}{a_1 d_1} \\ 0 & \frac{1}{d_1} \end{bmatrix} \begin{bmatrix} a_1 & b_1 \\ 0 & d_1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{a_1}{a_1} & \frac{b_1}{a_1} - \frac{d_1 b_1}{a_1 d_1} \\ 0 & \frac{d_1}{d_1} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned} \qquad \begin{aligned} T_1{T_1}^{-1} &= \begin{bmatrix} a_1 & b_1 \\ 0 & d_1 \end{bmatrix} \begin{bmatrix} \frac{1}{a_1} & -\frac{b_1}{a_1 d_1} \\ 0 & \frac{1}{d_1} \end{bmatrix} \\ &= \begin{bmatrix} \frac{a_1}{a_1} & -\frac{a_1 b_1}{a_1 d_1} + \frac{b_1}{d_1} \\ 0 & \frac{d_1}{d_1} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Now we take another arbitrary element of T, say T_2 and check to make sure $T_2{T_1}^{-1}$ is in T.

$$T_2 = \begin{bmatrix} a_2 & b_2 \\ 0 & d_2 \end{bmatrix}$$

$$\begin{aligned} T_2{T_1}^{-1} &= \begin{bmatrix} a_2 & b_2 \\ 0 & d_2 \end{bmatrix} \begin{bmatrix} \frac{1}{a_1} & -\frac{b_1}{a_1 d_1} \\ 0 & \frac{1}{d_1} \end{bmatrix} \\ &= \begin{bmatrix} \frac{a_2}{a_1} & -\frac{a_2 b_1}{a_1 d_1} + \frac{b_2}{d_1} \\ 0 & \frac{d_2}{d_1} \end{bmatrix} \end{aligned}$$

Similarly to our previous observation, we note that $a_1 \neq 0$, $d_1 \neq 0$, $a_2 \neq 0$, and $d_2 \neq 0$. And this is enough to show that $T_2{T_1}^{-1} \in T$. And so by 3.2.3 in the textbook, T is a subgroup of $GL_2(\mathbb{R})$

(b) Show that $D = \left\{ \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \middle| ad \neq 0 \right\}$ is a subgroup of G.

Let us take T_1 from the part (a) and set $b_1 = 0$. Lets call this D_1. So then ${D_1}^{-1} = \begin{bmatrix} \frac{1}{a_1} & 0 \\ 0 & \frac{1}{d_1} \end{bmatrix}$. We note that both D_1 and ${D_1}^{-1}$ are in D. And taking another arbitrary element from D, say D_2:

$$\begin{aligned} D_2 &= \begin{bmatrix} a_2 & 0 \\ 0 & d_2 \end{bmatrix} \\ D_2{D_1}^{-1} &= \begin{bmatrix} a_2 & 0 \\ 0 & d_2 \end{bmatrix} \begin{bmatrix} \frac{1}{a_1} & 0 \\ 0 & \frac{1}{d_1} \end{bmatrix} \\ &= \begin{bmatrix} \frac{a_2}{a_1} & 0 \\ 0 & \frac{d_2}{d_1} \end{bmatrix} \end{aligned}$$

And similarly to part (a) we notice that $a_1 \neq 0$, $d_1 \neq 0$, $a_2 \neq 0$, and $d_2 \neq 0$ and so $D_2{D_1}^{-1} \in D$ and D is a subgroup of $GL_2\left(\mathbb{R}\right)$