separable

$$g(y)\,\mathrm{d}y = f(t)\,\mathrm{d}t \qquad g(y)y' = f(t) \qquad \int g(y)\,\mathrm{d}y = \int f(t)\,\mathrm{d}t$$

first order linear

$$\frac{\mathrm{d}y}{\mathrm{d}t} + p(t)y = q(t) \qquad \mu(t) = e^{\int p(t)\,\mathrm{d}t}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mu(t)y\right) = \mu(t)\frac{\mathrm{d}y}{\mathrm{d}t} + p(t)\mu(t)y \qquad \mu(t)y = \int \mu(t)q(t)\,\mathrm{d}t$$

exact

$$M(t,y)\,\mathrm{d}t + N(t,y)\,\mathrm{d}y = 0 \qquad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

$$\int M(t,y)\,\mathrm{d}t + \phi(y) = f(t,y) \qquad \phi'(y) = N(x,y) - \frac{\mathrm{d}}{\mathrm{d}y}\left(\int M(t,y)\,\mathrm{d}t\right)$$

$$\int M(t,y)\,\mathrm{d}t + \int \phi'(y)\,\mathrm{d}y = f(t,y)$$

Solution is $f(t,y) = C$

bernoulli

$$\frac{\mathrm{d}y}{\mathrm{d}t} + p(t)y = q(t)y^n \qquad \frac{1}{y^n}\frac{\mathrm{d}y}{\mathrm{d}t} + p(t)y^{1-n} = q(t)$$

$$w = y^{1-n} \qquad \frac{\mathrm{d}w}{\mathrm{d}t} = (1-n)\frac{1}{y^n}\frac{\mathrm{d}y}{\mathrm{d}t}$$

$$\frac{\mathrm{d}w}{\mathrm{d}t} + (1-n)p(t)w = (1-n)q(t)$$

Solve as first order linear, then back substitute

homogeneous

$$M(t,y)\,\mathrm{d}t + N(t,y)\,\mathrm{d}y = 0 \qquad M(xt,xy) + N(xt,xy) = x^n\left(M(t,y) + N(t,y)\right)$$

$$\mathrm{d}y = w\,\mathrm{d}t + t\,\mathrm{d}w \qquad \mathrm{d}t = w\,\mathrm{d}y + y\,\mathrm{d}w$$

Substitute with $y = wt$ if $N(t,y)$ is simpler and $t = wy$ if $M(t,y)$ is simpler. Solve as a separable equation

population

growth and decay

$$y(t) = y_0e^{kt}$$

Logistical equation

$$y = \frac{ry_0}{ay_0 + (r - ay_0)e^{-rt}}$$

Newton's law of cooling

$$T(t) = (T_0 - T_s)\, e^{kt} + T_s$$

Newton's laws of motion

Acceleration is $a = g = 9.8\text{m}/\text{sec}^2 = 32\text{ft}/\text{sec}^2$ and position is s and velocity is v.

$$\begin{aligned} v &= v_0 + at \\ s &= v_0 t + \frac{1}{2}at^2 \\ v^2 &= {v_0}^2 + 2as \end{aligned}$$

trigonometric identities

$$\sin x = \frac{1}{\csc x} \qquad \sin(-x) = -\sin x \qquad \cos(-x) = \cos x$$

$$\sin(u \pm v) = \sin u \cos v \pm \cos u \sin v \qquad \cos(u \pm v) = \cos u \cos v \mp \sin u \sin v \qquad \sin 2u = 2 \sin u \cos u$$

$$\cos 2u = \cos^2 u - \sin^2 u \qquad \cos 2u = 2\cos^2 u - 1 \qquad \cos 2u = 1 - 2\sin^2 u$$

$$\sin^2 u = \frac{1 - \cos 2u}{2} \qquad \cos^2 u = \frac{1 + \cos 2u}{2}$$

$$\sin u \pm \sin(v) = 2 \sin\left(\frac{u \pm v}{2}\right) \cos\left(\frac{u \mp v}{2}\right) \qquad \cos u + \cos v = 2\cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)$$

$$\cos u - \cos v = -2\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right) \qquad \sin u \sin v = \frac{1}{2}[\cos(u-v) - \cos(u+v)]$$

$$\cos u \cos v = \frac{1}{2}[\cos(u-v) + \cos(u+v)] \qquad \sin u \cos v = \frac{1}{2}[\sin(u+v) + \sin(u-v)]$$

integration rules

$$\int e^{au} \sin(bu)\,\mathrm{d}u = e^{au}\frac{a\sin(bu) - b\cos(bu)}{b^2 + a^2} \qquad \int e^{au}\cos(bu)\,\mathrm{d}u = e^{au}\frac{b\sin(bu) + a\cos(bu)}{b^2 + a^2}$$

Wronskian

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

y_1 and y_2 are linearly independent if $W \neq 0$

reduction of order

given $y'' + p(t)y' + q(t)y = 0$ and a known solution y_1 then full solution is given by

$$y_s = c_1 y_1 + c_2 y_2 = c_1 y_1 + c_2 v(t) y_1 \qquad v(t) = \int \frac{1}{{y_1}^2} e^{-\int p(t)\,\mathrm{d}t}\,\mathrm{d}t$$

second order linear homogeneous with constant coefficient

$$ay'' + by' + cy = 0 \to ar^2 + br + c = 0 \qquad r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$y_s = \begin{cases} c_1 e^{r_1 t} + c_2 e^{r_2 t} & r_1 \neq r_2 \\ (c_1 + c_2 t)e^{rt} & r_1 = r_2 \\ e^{\alpha t}\left[c_1 \cos(\beta t) + c_2 \sin(\beta t)\right] & r = \alpha \pm \beta i \end{cases}$$

method of undetermined coefficients

solution of $ay'' + by' + cy = f(t)$ is $y_s = y_h + y_p$ where y_h is solution to corresponding homogeneous equation

$$f(t) = t^m e^{\alpha t} \qquad \text{or} \qquad f(t) = t^m e^{\alpha t} \sin \beta t \quad \text{or} \quad f(t) = t^m e^{\alpha t} \cos \beta t$$

$$S = \{e^{\alpha t}, e^{\alpha t}t, e^{\alpha t}t^2, ..., e^{\alpha t}t^m\} \qquad S = \left\{ \begin{matrix} e^{\alpha t} \sin \beta t, e^{\alpha t} \cos \beta t, te^{\alpha t} \sin \beta t, te^{\alpha t} \cos \beta t, \\ t^2 e^{\alpha t} \sin \beta t, t^2 e^{\alpha t} \cos \beta t, ..., t^m e^{\alpha t} \sin \beta t, t^m e^{\alpha t} \cos \beta t \end{matrix} \right\}$$

if $S_h \cap S_p \neq \emptyset$ then $S_p \to t^n S_p$. This will make y_h and y_p linearly independant. If $f(t)$ has more than one term then S_p is the union of the solution set for each term. Throw out constant coefficients in $f(t)$

$$y_p = a_1 S_p[1] + a_2 S_p[2] + ... + a_m S_p[m]$$

Solve for all a_n and we are done.

variation of parameters

$y'' + p(t)y' + q(t)y = f(t)$ for any $f(t)$. More general than undetermined coefficients. W refers to the Wronskian. Need to be able to find homogeneous solution for this to work.

$$y_s = y_h + y_p \qquad y_h = c_1 y_1 + c_2 y_2 \qquad y_p = u_1 y_1 + u_2 y_2 \qquad {u_1}' = -\frac{y_2 f}{W} \qquad {u_2}' = \frac{y_1 f}{W}$$

cauchy-euler

$$ax^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + bx \frac{\mathrm{d}y}{\mathrm{d}x} + cy = f(x) \qquad \text{or} \qquad ax^2 y'' + bxy' + cy = f(x)$$

$$x = e^t \qquad t = \ln x \qquad x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} - \frac{\mathrm{d}y}{\mathrm{d}t} \qquad x \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}$$

spring motion

$$m \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + kx = 0 \qquad x(0) = \alpha \qquad x'(0) = \beta \qquad x(t) = \alpha \cos \omega t + \frac{\beta}{\omega} \sin \omega t$$

$$\omega = \sqrt{\frac{k}{m}} \qquad F = ks \qquad F = mg = ma$$

initial position is α initial velocity is β stretch is s $g = 32ft/s^2$ force(weight) is lb or N, mass is slugs or kg, length is ft or m, k is lb/ft or N/m and time is s. down is positive, up is negative

Laplace

$$\mathcal{L}\{f(t)\} = \int_0^t e^{-st} f(t)\,\mathrm{d}t \qquad \mathcal{L}\{1\} = \frac{1}{s} \qquad \mathcal{L}\{e^{at}\} = \frac{1}{s-a} \qquad \mathcal{L}\{\sin kt\} = \frac{k}{s^2+k^2}$$

$$\mathcal{L}\{\cos kt\} = \frac{s}{s^2+k^2} \qquad \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \qquad \mathcal{L}\{e^{at} f(t)\} = F(s-a) \qquad \mathcal{L}\{t^n f(t)\} = (-1)^n \frac{\mathrm{d}^n F(s)}{\mathrm{d}s^n}$$

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0) \qquad \mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$$

$$\mathcal{L}\{f'''(t)\} = s^3\mathcal{L}\{f(t)\} - s^2 f(0) - sf'(0) - f''(0) \qquad f(t) * g(t) = \int_0^t f(t-v) g(v)\,\mathrm{d}v$$

$$\mathcal{L}\{f(t) * g(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$$

matrice

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v} \quad \det(\mathbf{A} - \lambda\mathbf{I}) = 0 \quad (\mathbf{A} - \lambda_n \mathbf{I})\mathbf{v}_n = 0 \quad \mathbf{X}' = \mathbf{A}\mathbf{X} \to \mathbf{X}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} + \cdots c_n\mathbf{v}_n e^{\lambda_n t}$$