

8.5

A. Determine the interval of convergence of the following power series:

(f) $\sum_{n=0}^{\infty} x^{n!}$.

We first compare x^n to $x^{n!}$. If $|x| < 1$ then $|x^{n!}| < |x^n|$ and if $|x| > 1$ then $|x^{n!}| > |x^n|$. Of course if $|x| = 1$ then $|x^n| = 1 = |x^{n!}|$.

Now examining $\sum_{n=0}^{\infty} x^n$ we see that $\lim_{n \rightarrow \infty} |1|^{1/n} = 1$ and so our radius of convergence is 1.

Now $\sum_{n=0}^{\infty} x^n$ is a geometric series, and so it converges only if $|x| < 1$. And so by comparison $\sum_{n=0}^{\infty} x^{n!}$ has an interval of convergence of $(-1, 1)$

B. Find a power series $\sum_{n=0}^{\infty} a_n x^n$ that has a different *interval* of convergence than $\sum_{n=0}^{\infty} n a_n x^{n-1}$.

We choose $a_n = \frac{1}{n+1}$ and $\lim_{n \rightarrow \infty} \frac{n+1}{n+2} = 1$. Our radius of convergence then is 1. $\sum_{n=0}^{\infty} \frac{x^n}{n+1}$ converges at -1 by the alternating series test. Now $\sum_{n=0}^{\infty} \frac{1}{2n} < \sum_{n=0}^{\infty} \frac{1}{n+1}$. But $\frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{n}$ diverges and so $\sum_{n=0}^{\infty} \frac{1}{n+1}$ diverges. And so our interval of convergence is $[-1, 1)$. Now $\sum_{n=0}^{\infty} \frac{n}{n+1} x^{n-1}$ has the same radius of convergence. Now $\sum_{n=0}^{\infty} \frac{n}{n+1} = \sum_{n=0}^{\infty} 1 - \frac{1}{n+1}$. But $\lim_{n \rightarrow \infty} 1 - \frac{1}{n+1} = 1$ and so this series diverges at 1. And similarly $(-1)^{n-1} - \frac{(-1)^{n-1}}{n+1}$ alternately approaches 1 and -1 as n goes to infinity. And so because $(-1)^{n-1} \frac{n}{n+1}$ has no limit, the series can not converge. Thus our interval of convergence is $(-1, 1)$

10.1

C. Let f satisfy the hypotheses of Taylor's Theorem at $x = a$.

(a) Show that $\lim_{x \rightarrow a} \frac{f(x) - P_n(x)}{(x-a)^n} = 0$.

$$\begin{aligned} \lim_{x \rightarrow a} \left| \frac{f(x) - P_n(x)}{(x-a)^n} \right| &= \lim_{x \rightarrow a} \left| \frac{R_n(x)}{(x-a)^n} \right| \\ &\leq \lim_{x \rightarrow a} \left| \frac{M(x-a)^{n+1}}{(n+1)!(x-a)^n} \right| \end{aligned}$$

$$\begin{aligned}
&= \frac{M}{(n+1)!} \lim_{x \rightarrow a} |(x-a)| \\
&= \frac{M}{(n+1)!} 0 = 0
\end{aligned}$$

(b) If $Q(x) \in \mathbb{P}_n$ and $\lim_{x \rightarrow a} \frac{f(x) - Q(x)}{(x-a)^n} = 0$, prove that $Q = P_n$.

Because $\lim_{x \rightarrow a} \frac{f(x) - Q(x)}{(x-a)^n} = 0$ and $\lim_{x \rightarrow a} \frac{f(x) - P_n(x)}{(x-a)^n} = 0$ it follows that

$$\begin{aligned}
\lim_{x \rightarrow a} \frac{f(x) - Q(x)}{(x-a)^n} - \lim_{x \rightarrow a} \frac{f(x) - P_n(x)}{(x-a)^n} &= 0 \\
\lim_{x \rightarrow a} \frac{f(x) - Q(x) - (f(x) - P_n(x))}{(x-a)^n} &= 0 \\
\lim_{x \rightarrow a} \frac{P_n(x) - Q(x)}{(x-a)^n} &= 0
\end{aligned}$$

Recalling that $P_n(X), Q(x) \in \mathbb{P}_n$

$$\begin{aligned}
\lim_{x \rightarrow a} \frac{P_n(x) - Q(x)}{(x-a)^n} &= \lim_{x \rightarrow a} \sum_{i=0}^n \frac{a_i x^i}{(x-a)^n} \\
\lim_{x \rightarrow a} \sum_{i=0}^n \frac{a_i x^i}{(x-a)^n} &= \sum_{i=0}^n \lim_{x \rightarrow a} \frac{a_i x^i}{(x-a)^n}
\end{aligned}$$

Now if we assume $P_n(x) \neq Q(x)$ then there exists some $a_i \neq 0$. $\frac{a_i x^i}{(x-a)^n}$ does not converge as $x \rightarrow a$, and so neither does $\frac{P_n(x) - Q(x)}{(x-a)^n}$, which is contrary to our assumption.

F. Let $f(x) = \log x$.

(a) Find the Taylor series of f about $x = 1$.

$$\begin{aligned}
f'(x) &= \frac{1}{x} & f''(x) &= -\frac{1}{x^2} \\
f^{(3)}(x) &= \frac{2}{x^3} & f^{(k)}(x) &= \frac{(-1)^{k+1}(k-1)!}{x^k} \\
P_n(x) &= \sum_{k=1}^n \frac{(-1)^{k+1}(k-1)!}{1^k k!} (x-1)^k & P_n(x) &= \sum_{k=1}^n \frac{(-1)^{k+1}(x-1)^k}{k}
\end{aligned}$$

(b) What is the radius of convergence of this series?

$$\lim_{k \rightarrow \infty} \left| \frac{(-1)^{k+2}k}{(-1)^{k+1}(k+1)} \right| = \lim_{k \rightarrow \infty} \frac{k}{(k+1)} = 1 = R$$

- (c) What happens at the two endpoints of the interval of convergence? Hence find a series converging to $\log 2$.

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}(-1)^k}{k} = \sum_{k=1}^{\infty} \frac{-1}{k} = \infty$$

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}(2-1)^k}{k} = -\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$$

So the series does not converge at 0, but it does at 2, and the series is above.

- (d) By observing that $\log 2 = \log 4/3 - \log 2/3$, find another series converging to $\log 2$. Why is this series more useful?

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}(\frac{4}{3}-1)^k}{k} - \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(\frac{2}{3}-1)^k}{k}$$

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{3^k k} + \sum_{k=1}^{\infty} \frac{1}{3^k k}$$

We know that our error is less than $\frac{M|x-1|^{n+1}}{(n+1)!}$ where $M \geq |f^{(n+1)}(x)|$

- I. Let $f(x) = (1+x)^{-1/2}$

- (a) Find a formula for $f^{(k)}(x)$. Hence show that

$$f^{(k)}(0) = \binom{-\frac{1}{2}}{k} := \frac{-\frac{1}{2}(-\frac{1}{2}-1)\cdots(-\frac{1}{2}+1-k)}{k!} = \frac{(-1)^k(2k)!}{2^{2k}(k!)^2} = \left(\frac{-1}{4}\right)^k \binom{2k}{k}.$$

- (b) Show that the Taylor series for f about $x = 0$ is $\sum_{k=0}^{\infty} \binom{2k}{k} \left(\frac{-x}{4}\right)^k$, and compute the radius of convergence.
- (c) Show that $\sqrt{2} = 1.4f(-0.02)$. Hence compute $\sqrt{2}$ to 8 decimal places.
- (d) Express $\sqrt{2} = 1.415f(\varepsilon)$, where ε is expressed as a fraction in lowest terms. Use this to obtain an alternating series for $\sqrt{2}$. How many terms are needed to estimate $\sqrt{2}$ to 100 decimal places?

10.2

- D. Suppose that f is a continuous function on $[0, 1]$ such that $\int_0^1 f(x)x^n dx = 0$ for all $n \geq 0$. Prove that $f = 0$. HINT: Use the Weierstrass Theorem to show that $\int_0^1 |f(x)|^2 dx = 0$