Graph Theory Homework

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Definitions

path A graph of order n and size $n-1$ whose vertices can be labeled by $v_1, v_2, \ldots, v_n$ and whose edges are $v_1 v_{i+1}$ for $i = 1, 2, \ldots, n-1$.

cycle A graph of order n and size n whose vertices can be labeled by $v_1, v_2, \ldots, v_n$ and whose edges are $v_1 v_n$ and $v_1 v_{i+1}$ for $i = 1, 2, \ldots, n-1$.

isomorphism If G and H are graphs and $\phi : V(G) \to V(H)$ is a bijective function such that two vertices u and v are adjacent in G if and only if $\phi(u)$ and $\phi(v)$ are adjacent in H. The function ϕ is an isomorphism.

subgraph Let G and H be graphs. Then if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ then H is a subgraph of G. That is to say, H is a subgraph of G if G contains all the vertices and edges of H.

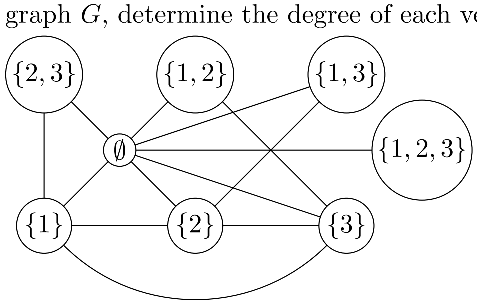
regular graph A graph whose vertices all have the same degree.

bipartate graph A graph whose vertices can be partitioned into two sets in such a way that every edge of the graph joins vertices from both sets.

complement A complement of a graph G is the graph $\overline{G}$ which has the same vertex set as G and where any two vertices are adjacent if and only if these vertices are not adjacent in G.

Exercises

1.1 2. A graph $G = (V, E)$ of order 8 has the power set of the set $S = \{1, 2, 3\}$ as its vertex set, that is V is the set of subsets of S. Two vertices A and B of V are adjacent if $A \cap B = \emptyset$. Draw the graph G, determine the degree of each vertex of G and determine the size of G.



$$\deg \emptyset = 7 \qquad \deg\{1\} = \deg\{2\} = \deg\{3\} = 3$$
$$\deg\{1,2,3\} = 1 \qquad \deg\{1,2\} = \deg\{2,3\} = \deg\{1,3\} = 2$$

The size $|E(G)|$ of G is $7 + 1 + 9 + 6 = 23$

3. A graph G of order 26 and size 58 has 5 vertices of degree 4, 6 vertices of degree 5 and 7 vertices of degree 6. The remaining vertices of G all have the same degree. What is this degree?

$$\begin{aligned} 26 - 5 - 6 - 7 &= 8 \\ 116 - 5 \cdot 4 - 6 \cdot 5 - 7 \cdot 6 &= 24 \\ 24 \div 8 &= 3 \end{aligned}$$

   The remaining 8 vertices have degree 3.

4. A graph of G has order $n = 3k + 3$ for some positive integer k. Every vertex of G has degree $k + 1, k + 2$ or $k + 3$. Prove that G has at least $k + 3$ vertices of degree $k + 1$ or at least $k + 1$ vertices of degree $k + 2$ or at least $k + 2$ vertices of degree $k + 3$.

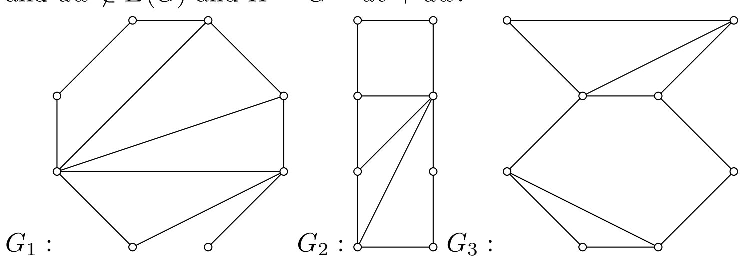
11. Prove for every graph G and every integer $r \geq \Delta(G)$ that there exists an r-regular graph containing G as an induced subgraph.

13. Determine all bipartite graphs G such that $\overline{G}$ is bipartite

18. Let G be a self-complementary graph of order n, where $n \equiv 1 \mod 4$. Prove that G contains an odd number of vertices of degree $(n - 1)/2$.

1.2

6. Let G and H be two graphs that are neither empty nor complete. The graph H is said to be obtained from G by an **edge rotation** if G contains three vertices u, v, and w where $uv \in E(G)$ and $uw \notin E(G)$ and $H \cong G - uv + uw$.



   (a) Show that the graph $G_2$of figure 1.33 is obtained from G_1 by an edge rotation.

   (b) Show that G_3 of figure 1.33 cannot be obtained from G_1 by an edge rotation.

7. Determine whether the following sequences are graphical. If so, construct a graph with the appropriate degree sequence.

   (a) 4,4,3,2,1

   (b) 3,3,2,2,2,2,1,1

   (c) 7,7,6,5,4,4,3,2

   (d) 7,6,6,5,4,3,2,1

   (e) 7,4,3,3,2,2,2,1,1,1

10. For which integers $x (0 \leq x \leq 7)$, if any, is the sequence $7, 6, 4, 3, 2, 1, x$ graphical?

15. Two finite sequences s_1 and s_2 of nonnegative integers are called **bigraphical** if there exists a bipartite graph G with partite sets V_1 and V_2 such that s_i lists the degrees of the vertices of G in V_i for $i = 1, 2$. Prove that the sequences $s_1 : a_1, a_2, \ldots, a_r$ and $s_2 : b_1, b_2, \ldots, b_t$ of nonnegative integers with $r \geq 2, a_1 \geq a_2 \geq \cdots \geq a_r, b_1 \geq b_2 \geq \cdots \geq b_t, 0 < a_1 \leq t$ and $0 < b_1 \leq r$ are bigraphical if and only if the sequences $s'_1 : a_2, a_3, \ldots, a_r$ and $s'_2 : b_1 - 1, b_2 - 1, \ldots, b_{a_1} - 1, b_{a_1+1}, \ldots, b_t$ are bigraphical