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HW 15

We substitute this expansion into the problem to get, notice we establish orthagonality with different values of n

$$T_n' + (n\pi)^2 T_n = f_n(t) = 2\int_0^1 \sin(\pi x)\sin(n\pi x)\,\mathrm{d}x = \begin{cases} 1 & n = 1 \\ 0 & n \neq 1 \end{cases}$$
$$T_n(0) = 2\int_0^1 0\,\mathrm{d}\xi = 0$$

We really only have two cases to worry about

$$(n = 1) \qquad \left.\begin{aligned} T_1' + \pi^2 T_1 &= 1 \\ T_1(0) &= 0 \end{aligned}\right\} \Rightarrow \qquad \begin{aligned} e^{\int \pi^2\,\mathrm{d}t} T_1 &= \int 1\,\mathrm{d}t \\ T_1 &= xe^{-\pi^2 t} + c_1 \\ &= xe^{-\pi^2 t} \end{aligned}$$

$$(n \geq 2) \qquad \left.\begin{aligned} T_2' + \pi^2 T_2 &= 0 \\ T_2(0) &= 0 \end{aligned}\right\} \Rightarrow \qquad \begin{aligned} e^{\int \pi^2\,\mathrm{d}t} T_2 &= \int 0\,\mathrm{d}t \\ T_2 &= c_1 e^{-\pi^2 t} \\ &= 0 \end{aligned}$$

So our solution looks like

$$u(x,t) = xe^{-t\pi^2}\sin(\pi x)$$