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HW 12
Lesson 7 problem 3. Find general series solution for PDE and BC's.

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\begin{array}{llll}
PDE & u_t = u_{xx} & 0 < x < 1 & 0 < t < \infty \\
BCs & \begin{cases} u_x(0,t) = 0 \\ u_x(1,t) = 0 \end{cases} & & 0 < t < \infty \\
IC & u(x,0) = x & 0 \le x \le 1 &
\end{array}
$$

$$
\begin{aligned}
u(x,t) &= X(x)T(t) & u_t = u_{xx} &= XT' = X''T \\
\frac{XT'}{XT} &= \frac{X''T}{XT} & \frac{T'}{T} &= \frac{X''}{X} = \mu \\
T' - \mu T &= 0 & X'' - \mu X &= 0
\end{aligned}
$$

We will just assume $\mu \le 0$.

$$
\begin{aligned}
\mu &= -\lambda^2 \\
T' + \lambda^2 T &= 0 & X'' + \lambda^2 X &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\int \lambda^2 \, \mathrm{d}t} T\right) &= e^{\int \lambda^2 \, \mathrm{d}t} T' + \lambda^2 e^{\int \lambda^2 \, \mathrm{d}t} T & r^2 + 0r + \lambda^2 &= 0 \\
e^{\int \lambda^2 \, \mathrm{d}t} T &= \int e^{\int \lambda^2 \, \mathrm{d}t} \cdot 0 \, \mathrm{d}t = A & \frac{-0 \pm \sqrt{0^2 - 4\lambda^2}}{2} &= r \\
T &= Ae^{-\lambda^2 t} & 0 \pm \lambda i &= r \\
& & B\cos(\lambda x) + C\sin(\lambda x) &= X
\end{aligned}
$$

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\begin{aligned}
u(x,t) = XT &= e^{-\lambda^2 t}\left[A\sin(\lambda x) + B\cos(\lambda x)\right] & \lambda e^{-\lambda^2 t}\left[A\cos(\lambda x) - B\sin(\lambda x)\right] &= u_x \\
u_x(0,t) &= \lambda e^{-\lambda^2 t}\left[A\cos(\lambda 0) - B\sin(\lambda 0)\right] = 0 & \lambda e^{-\lambda^2 t}\left[A\cos(\lambda) - B\sin(\lambda)\right] &= u_x(1,t) = 0 \\
A\lambda e^{-\lambda^2 t} &= 0 \\
A &= 0 & -B\lambda e^{-\lambda^2 t}\sin(\lambda) &= 0 \\
& & \sin(\lambda) &= 0 \\
\lambda_n &= n\pi & u(x,t) &= \sum_{n=1}^{\infty} a_n e^{-\lambda_n{}^2 t}\cos(\lambda_n x)
\end{aligned}
$$