Subspaces of $\mathbb{R}^n$

Definition 1. Let V be a subset of $\mathbb{R}^n$. We call V a *subspace* of $\mathbb{R}^n$ if all three of the following conditions are met.
(1) $\mathbf{0} \in V$.
(2) $\mathbf{u}, \mathbf{v} \in V$ implies $\mathbf{u} + \mathbf{v} \in V$.
(3) $\mathbf{v} \in V$ and $\alpha \in \mathbb{R}$ implies $\alpha\mathbf{v} \in V$.
In this case, we my write $V \leq \mathbb{R}^n$.

Example 2. Let $\mathbf{x}_0, \mathbf{v} \in \mathbb{R}^n$ be nonzero and nonparallel vectors.
(1) $L(\mathbf{0}, \mathbf{v})$ is a subspace of $\mathbb{R}^n$. We must check three properties. (i) Since $\mathbf{0} = \mathbf{0} + 0\mathbf{v}$, we have that $\mathbf{0} \in L(\mathbf{0}, \mathbf{v})$. (ii) Suppose that $\mathbf{x}, \mathbf{y} \in L(\mathbf{0}, \mathbf{v})$. Then, $\mathbf{x} = \mathbf{0} + s\mathbf{v}$ and $\mathbf{y} = \mathbf{0} + t\mathbf{v}$ for some $s, t \in \mathbb{R}$. It follows that $\mathbf{x} + \mathbf{y} = \mathbf{0} + (s + t)\mathbf{v}$. Since $s + t \in \mathbb{R}$, we have that $\mathbf{x} + \mathbf{y} \in L(\mathbf{0}, \mathbf{v})$. (iii) Suppose that $\mathbf{x} \in L(\mathbf{0}, \mathbf{v})$ and $\alpha \in \mathbb{R}$. Then, $\mathbf{x} = \mathbf{0} + s\mathbf{v}$ for some $s \in \mathbb{R}$ and so $\alpha\mathbf{x} = \alpha(\mathbf{0} + s\mathbf{v}) = \alpha\mathbf{0} + \alpha(s\mathbf{v}) = \mathbf{0} + (\alpha s)\mathbf{v}$. It follows that $\alpha\mathbf{x} \in L(\mathbf{0}, \mathbf{v})$.
(2) $L(\mathbf{x}_0, \mathbf{v})$ is a not a subspace of $\mathbb{R}^n$. We show that $\mathbf{0} \notin L(\mathbf{x}_0, \mathbf{v})$. Suppose otherwise that $\mathbf{0} \in L(\mathbf{x}_0, \mathbf{v})$. Then there exists $t \in \mathbb{R}$ such that $\mathbf{0} = \mathbf{x}_0 + t\mathbf{v}$. If $t = 0$, then $\mathbf{0} = \mathbf{x}_0$ and we have a contradiction. It follows that $t \neq 0$ and so $\mathbf{v} = -\frac{1}{t}\mathbf{x}_0$ and again we have a contadiction since $\mathbf{x}_0, \mathbf{v}$ are nonparallel.

Theorem 3. If $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k \in \mathbb{R}^n$, then $\mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k) \leq \mathbb{R}^n$.

Proof. We must check three properties. You verified two of them ina previous homework (HW1)
(i) Since $\mathbf{0} = 0\mathbf{v}_1 + 0\mathbf{v}_2 + ... + 0\mathbf{v}_k$, we have $\mathbf{0} \in \mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k)$.
(ii) Choose any $\mathbf{v}, \mathbf{w} \in \mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k)$. There exist $c_i, d_i \in \mathbb{R}$ such that $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_k\mathbf{v}_k$ and $\mathbf{w} = d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + ... + d_k\mathbf{v}_k$. We have

$$\begin{aligned}
& \mathbf{v} + \mathbf{w} \\
= \quad & (c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_k\mathbf{v}_k) + (d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + ... + d_k\mathbf{v}_k) \\
= \quad & c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_k\mathbf{v}_k + d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + ... + d_k\mathbf{v}_k \quad \text{(A1)} \\
= \quad & c_1\mathbf{v}_1 + d_1\mathbf{v}_1 + c_2\mathbf{v}_2 + d_2\mathbf{v}_2 + ... + c_k\mathbf{v}_k + d_k\mathbf{v}_k \quad \text{(A2)} \\
= \quad & (c_1\mathbf{v}_1 + d_1\mathbf{v}_1) + (c_2\mathbf{v}_2 + d_2\mathbf{v}_2) + ... + (c_k\mathbf{v}_k + d_k\mathbf{v}_k) \quad \text{(A1)} \\
= \quad & (c_1 + d_1)\mathbf{v}_1 + (c_2 + d_2)\mathbf{v}_2 + ... + (c_k + d_k)\mathbf{v}_k \quad \text{(S3)} \\
\in \quad & \mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k).
\end{aligned}$$

(iii) Choose any $\mathbf{v} \in \mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k)$ and $c \in \mathbb{R}$. There exist $c_i \in \mathbb{R}$ such that $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_k\mathbf{v}_k$. We have

$$
\begin{aligned}
& c\mathbf{v} \\
= \; & c(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_k\mathbf{v}_k) \\
= \; & c(c_1\mathbf{v}_1) + c(c_2\mathbf{v}_2) + ... + c(c_k\mathbf{v}_k) \quad \text{(S2)} \\
= \; & (cc_1)\mathbf{v}_1 + (cc_2)\mathbf{v}_2 + ... + (cc_k)\mathbf{v}_k \quad \text{(S4)} \\
\in \; & \text{Span}(\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k).
\end{aligned}
$$

Definition 4. Let $A \in M_{m\times n}$ with columns $\mathbf{c}_1, \mathbf{c}_2, ..., \mathbf{c}_n$.
(1) We define the *nullspace* of A to be the set $N(A) = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$.
(2) We define the *column space* of A to be the set $C(A) = \text{Span}(\mathbf{c}_1, \mathbf{c}_2, ..., \mathbf{c}_n)$.

Theorem 5. Let $A \in M_{m\times n}$ with columns $\mathbf{c}_1, \mathbf{c}_2, ..., \mathbf{c}_n$.
(1) $N(A) \leq \mathbb{R}^n$.
(2) $C(A) \leq \mathbb{R}^m$.

Proof. It sufficed to check that (1). Observe that Theorem 3 and Definition 4(2) imply that $C(A) \leq \mathbb{R}^m$. (i) We have seen many times that $\mathbf{0} \in N(A)$. (ii) Choose any $\mathbf{v}, \mathbf{w} \in N(A)$. Then $A\mathbf{v} = \mathbf{0}$ and $A\mathbf{w} = \mathbf{0}$. It follows that $A\mathbf{v}+A\mathbf{w} = \mathbf{0}+\mathbf{0} = \mathbf{0}$. Using the usual matrix algebra, we find that $A(\mathbf{v}+\mathbf{w}) = \mathbf{0}$. It follows that $\mathbf{v}+\mathbf{w} \in N(A)$. (iii) Choose any $\mathbf{v} \in N(A)$ and $c \in \mathbb{R}$. Then $A\mathbf{v} = \mathbf{0}$ and so $c(A\mathbf{v}) = c\mathbf{0}$. Using the usual matrix algebra, we find that $A(c\mathbf{v}) = \mathbf{0}$. It follows that $c\mathbf{v} \in N(A)$.

Example 6. Let

$$
A = \begin{bmatrix} 3 & -1 & 2 & 7 \\ 2 & 1 & 3 & 3 \\ 2 & 2 & 4 & 2 \end{bmatrix}.
$$

(a) Find $N(A)$
(b) Find $C(A)$

Solution.
(a) Since

$$
rref(A) = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}
$$

we can see from the $rref$ that

$$
\begin{aligned}
x_1 + x_3 + 2x_4 &= 0 \\
x_2 + x_3 - x_4 &= 0.
\end{aligned}
$$

Then

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -x_3 - 2x_4 \\ -x_3 + x_4 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -x_3 \\ -x_3 \\ x_3 \\ 0 \end{pmatrix} + \begin{pmatrix} -2x_4 \\ x_4 \\ 0 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

In other words,

$$N(A) = \text{Span}\left(\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right)$$

(b) Bring the augmented matrix

$$\begin{bmatrix} 3 & -1 & 2 & 7 & b_1 \\ 2 & 1 & 3 & 3 & b_2 \\ 2 & 2 & 4 & 2 & b_3 \end{bmatrix}$$

to echelon form to get

$$\begin{bmatrix} 1 & 1 & 2 & 1 & \frac{1}{2}b_3 \\ 0 & -1 & -1 & 1 & b_2 - b_3 \\ 0 & 0 & 0 & 0 & b_1 - 4b_2 + \frac{5}{2}b_3 \end{bmatrix}.$$

Therefore, $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ must satisfy $b_1 - 4b_2 + \frac{5}{2}b_3 = 0$. That is,

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 4b_2 + \frac{5}{2}b_3 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 4b_2 \\ b_2 \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{5}{2}b_3 \\ 0 \\ b_3 \end{pmatrix} = b_2 \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} + b_3 \begin{pmatrix} \frac{5}{2} \\ 0 \\ 1 \end{pmatrix}$$

and so

$$C(A) = \text{Span}\left(\begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{5}{2} \\ 0 \\ 1 \end{pmatrix} \right)$$

Definition 7. Let $U, V \leq \mathbb{R}^n$.
(1) We define the *intersection* of U and V to be the set $U \cap V = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \in U$ and $\mathbf{x} \in V\}$.
(2) We define the *sum* of U and V to be the set $U + V = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \mathbf{u} + \mathbf{v}$ for some $\mathbf{u} \in U$ and $\mathbf{v} \in V\}$.
(3) We define the *orthogonal complement* of U to be the set $U^{\perp} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \cdot \mathbf{u} = 0$ for all $\mathbf{u} \in U\}$.

Theorem 8. Let $U, V \leq \mathbb{R}^n$. Then $U \cap V$, $U + V$, and $U^\perp$ are all subspaces of $\mathbb{R}^n$.

Proof. We verify that $U + V$ and $U^\perp$ are subspaces.
$(U + V)$ (i) Since U, V are subspaces, we have that $\mathbf{0} \in U$ and $\mathbf{0} \in V$. Since $\mathbf{0} = \mathbf{0} + \mathbf{0}$ we conclude that $\mathbf{0} \in U + V$. (ii) Suppose that $\mathbf{x}, \mathbf{y} \in U + V$. There exist $\mathbf{u}_1, \mathbf{u}_2 \in U$ and $\mathbf{v}_1, \mathbf{v}_2 \in V$ such that $\mathbf{x} = \mathbf{u}_1 + \mathbf{v}_1$ and $\mathbf{y} = \mathbf{u}_2 + \mathbf{v}_2$. Then

$$\begin{aligned} & \mathbf{x} + \mathbf{y} \\ = \; & (\mathbf{u}_1 + \mathbf{v}_1) + (\mathbf{u}_2 + \mathbf{v}_2) \\ = \; & (\mathbf{u}_1 + \mathbf{u}_2) + (\mathbf{v}_1 + \mathbf{v}_2) \text{ (vector algebra)} \\ \in \; & U + V. \end{aligned}$$

Indeed, since $\mathbf{u}_1, \mathbf{u}_2$ belong to the subspace U and $\mathbf{v}_1, \mathbf{v}_2$ belong to the subspace V, itmust be the case that $\mathbf{u}_1 + \mathbf{u}_2 \in U$ and $\mathbf{v}_1 + \mathbf{v}_2 \in V$ as needed. (iii) Suppose that $\mathbf{x} \in U + V$ and choose any $c \in \mathbb{R}$. There exist $\mathbf{u} \in U$ and $\mathbf{v} \in V$ such that $\mathbf{x} = \mathbf{u} + \mathbf{v}$. It follows that $c\mathbf{x} = c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v} \in U + V$. Again, $c\mathbf{u} \in U$ and $c\mathbf{v} \in V$ since U, V are subspaces.

Example 9. We compute $P(\mathbf{0}, \mathbf{u}, \mathbf{v})^\perp$ where $\mathbf{u} = (1, 2, 3)$ and $\mathbf{v} = (1, 0, 1)$. If $(x, y, z) \in P(\mathbf{0}, \mathbf{u}, \mathbf{v})^\perp$, then $(x, y, z) \cdot (1, 2, 3) = 0$ and $(x, y, z) \cdot (1, 0, 1) = 0$. We have a system of equations

$$\begin{aligned} x + 2y + 3z &= 0 \\ x + z &= 0. \end{aligned}$$

Then the corresponding matrix is

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \end{bmatrix} \Longrightarrow rref(A) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

The system is now

$$\begin{aligned} x + z &= 0 \\ y + z &= 0. \end{aligned}$$

and we find that
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -z \\ -z \\ z \end{pmatrix} = z \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}.$$

We have shown that if $\mathbf{x} = (-1, -1, 1)$, then $P(\mathbf{0}, \mathbf{u}, \mathbf{v})^\perp \subseteq L(\mathbf{0}, \mathbf{x})$. The reverse containment is the same and we have

$$P(\mathbf{0}, \mathbf{u}, \mathbf{v})^\perp \subseteq L(\mathbf{0}, \mathbf{x}) = \text{Span}(\mathbf{x})$$

Exercises 3.1: 6, 8, 9(b,c), 11, 15, 16. Please do 1 and 2 but you do not need to turn them in. They do make great test questions though.