7.3

F. Let $\mathbb{R}^n$ have the max morm $||\mathbf{x}||_\infty = \max\{|x_i| : 1 \leq i \leq n\}$. Let K be the unit ball of V and let $v = (2, 0, \ldots, 0)$. Find all closest points to v in K.

We need $\min\{||\mathbf{v} - \mathbf{x}|| : \mathbf{x} \in K\}$. We know that $|v_i - x_i| = x_i$ if $i \neq 1$ so let us look at v_1. We just need $\min |v_1 - x_1|$. Since $-1 \leq x_1 \leq 1$ then we can't do better than $x_1 = 1$. So the closes points to v in K are $\{(1, x_2, \ldots, x_n) : |x_i| \leq 1\}$

7.4

B. Show that every iiner product space stisfies the parallelogram law:

$$||x + y||^2 + ||x - y||^2 = 2||x||^2 + 2||y||^2 \text{ for all } x, y \in V$$

$$\begin{aligned}
||x + y||^2 &= \langle x + y, x + y\rangle = \langle x, x\rangle + 2\langle x, y\rangle + \langle y, y\rangle \\
||x - y||^2 &= \langle x - y, x - y\rangle = \langle x, x\rangle - 2\langle x, y\rangle + \langle y, y\rangle \\
||x + y||^2 + ||x - y||^2 &= 2\langle x, x\rangle + 2\langle y, y\rangle = 2(||x||^2 + ||y||^2) \\
||x + y||^2 + ||x - y||^2 &= 2||x||^2 + 2||y||^2
\end{aligned}$$

C. Minimize the quantity $||x||^2 - 2t\langle x, y\rangle + t^2||y||^2$ over $t \in \mathbb{R}$. You will see why we chose t as we did in th proof of the Cauchy-Schwarz inequality.

First we need to find any critical points.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(||x||^2 - 2t\langle x, y\rangle + t^2||y||^2\right) = -2\langle x, y\rangle + 2t||y||^2$$

Notice that the derivative has no asymptotes or discontinuities. It's only zero is at $t = \frac{\langle x,y\rangle}{||y||^2}$. The second derivative is $2||y||^2 \geq 0$ and so the expression is convex with a minimum at $\frac{\langle x,y\rangle}{||y||^2}$

G. A normed vector space is **strictly convex** if $||u|| = ||v|| = ||u+v)/2|| = 1$ for vectors $u, v \in V$ implies that $u = v$

(a) Show that an inner product space is always strictly convex.

We assume that $||u|| = ||v|| = ||(u + v)||/2|| = 1$. Then

$$\begin{aligned}
1 &= ||(u + v)/2|| \\
1^2 &= (\frac{1}{2}||(u + v)||)^2 \\
1 &= \frac{1}{4}||(u + v)||)^2
\end{aligned}$$

$$4 = 2 + 2 = ||u + v||^2 = 2||u||^2 + 2||v||^2 - ||u - v||^2$$
$$0 = ||y - v||^2 = \langle u - v, u - v \rangle = \langle 0, 0 \rangle$$

Thus $u = v$

(b) Show that $\mathbb{R}^2$ with the norm $||(x, y)||_\infty = \max\{|x|, |y|\}$ is not strictly convex.

We take $(1, 1)$ and $(1, 0)$. Then $||(1, 1)|| = ||(1, 0)|| = ||(2, 1)/2|| = 1$ but $(1, 1) \neq (1, 0)$.