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Generate the 6-tuples of 0s and 1s by using the base 2 arithmetic generating scheme and identify them with subsets of the set $\{x_5, x_3, x_2, x_1, x_0\}$.

000000→ $\emptyset$ 000001→ $\{x_0\}$ 000010→ $\{x_1\}$ 000011→ $\{x_1, x_0\}$ 000100→ $\{x_2\}$ 000101→ $\{x_2, x_0\}$ 000110→ $\{x_2, x_1\}$ 000111→ $\{x_2, x_1, x_0\}$ 001000→ $\{x_3\}$ 001001→ $\{x_3, x_0\}$ 001010→ $\{x_3, x_1\}$ 001011→ $\{x_3, x_1, x_0\}$ 001100→ $\{x_3, x_2\}$ 001101→ $\{x_3, x_2, x_0\}$ 001110→ $\{x_3, x_2, x_1\}$ 001111→ $\{x_3, x_2, x_1, x_0\}$ 010000→ $\{x_4\}$ 010001→ $\{x_4, x_0\}$ 010010→ $\{x_4, x_1\}$ 010011→ $\{x_4, x_1, x_0\}$ 010100→ $\{x_4, x_2\}$ 010101→ $\{x_4, x_2, x_0\}$ 010110→ $\{x_4, x_2, x_1\}$ 010111→ $\{x_4, x_2, x_1, x_0\}$ 011000→ $\{x_4, x_3\}$ 011001→ $\{x_4, x_3, x_0\}$ 011010→ $\{x_4, x_3, x_1\}$ 011011→ $\{x_4, x_3, x_1, x_0\}$ 011100→ $\{x_4, x_3, x_2\}$ 011101→ $\{x_4, x_3, x_2, x_0\}$ 011110→ $\{x_4, x_3, x_2, x_1\}$ 011111→ $\{x_4, x_3, x_2, x_1, x_0\}$ 100000→ $\{x_5\}$ 100001→ $\{x_5, x_0\}$ 100010→ $\{x_5, x_1\}$ 100011→ $\{x_5, x_1, x_0\}$ 100100→ $\{x_5, x_2\}$ 100101→ $\{x_5, x_2, x_0\}$ 100110→ $\{x_5, x_2, x_1\}$ 100111→ $\{x_5, x_2, x_1, x_0\}$ 101000→ $\{x_5, x_3\}$ 101001→ $\{x_5, x_3, x_0\}$ 101010→ $\{x_5, x_3, x_1\}$ 101011→ $\{x_5, x_3, x_1, x_0\}$ 101100→ $\{x_5, x_3, x_2\}$ 101101→ $\{x_5, x_3, x_2, x_0\}$ 101110→ $\{x_5, x_3, x_2, x_1\}$ 101111→ $\{x_5, x_3, x_2, x_1, x_0\}$ 110000→ $\{x_5, x_4\}$ 110001→ $\{x_5, x_4, x_0\}$ 110010→ $\{x_5, x_4, x_1\}$ 110011→ $\{x_5, x_4, x_1, x_0\}$ 110100→ $\{x_5, x_4, x_2\}$ 110101→ $\{x_5, x_4, x_2, x_0\}$ 110110→ $\{x_5, x_4, x_2 x_1\}$ 110111→ $\{x_5, x_4, x_2 x_1, x_0\}$ 111000→ $\{x_5, x_4, x_3\}$ 111001→ $\{x_5, x_4, x_3, x_0\}$ 111010→ $\{x_5, x_4, x_3, x_1\}$ 111011→ $\{x_5, x_4, x_3, x_1, x_0\}$ 111100→ $\{x_5, x_4, x_3, x_2\}$ 111101→ $\{x_5, x_4, x_3, x_2, x_0\}$ 111110→ $\{x_5, x_4, x_3, x_2, x_1\}$ 111111→ $\{x_5, x_4, x_3, x_2, x_1, x_0\}$

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For each of the subsets (a), (b), (c), and (d) in the preceding exercise, determine the subset that immediately *precedes* it in the base 2 arithmetic generating scheme.

(a)

$\{x_4, x_1, x_0\} = 00010011 \leftarrow 00010010$, or $\{x_4, x_1\}$.

(b)

$\{x_7, x_5, x_3\} = 10101000 \leftarrow 10100111$, or $\{x_7, x_5, x_2, x_1, x_0\}$

(c)

$\{x_7, x_5, x_4, x_3, x_2, x_1, x_0\} = 10111111 \leftarrow 10111110$, or $\{x_7, x_5, x_4, x_3, x_2, x_1\}$

(d)

$\{x_0\} = 00000001 \leftarrow 00000000$, or $\emptyset$

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Which subset of $\{x_7, x_6, \ldots, x_1, x_0\}$ is 150th on the list of subsets of S when the base 2 arithmetic generating scheme is used? 200th? 250th? (As in Section 4.3, the places on the list are numbered beginning with 0.)

$2^0 = 1 \quad 2^1 = 2 \quad 2^2 = 4 \quad 2^3 = 8 \quad 2^4 = 16 \quad 2^5 = 32 \quad 2^6 = 64 \quad 2^7 = 128$

$150 = 128 + 16 + 4 + 2 \to \{x_7, x_4, x_2, x_1\}$ $\quad 200 = 128 + 64 + 8 \to \{x_7, x_6, x_3\}$

$250 = 128 + 64 + 32 + 16 + 8 + 2 \to \{x_7, x_6, x_5, x_4, x_3, x_1\}$

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Determine the reflected Gray code of order 6.

000000 000001 000011 000010 000110 000111 000101 000100 001100 001101 001111 001110 001010 001011 001001 001000 011000 011001 011011 011010 011110 011111 011101 011100 010100 010101 010111 010110 010010 010011 010001 010000 110000 110001 110011 110010 110110 110111 110101 110100 111100 111101 111111 111110 111010 111011 111001 111000 101000 101001 101011 101010 101110 101111 101101 101100 100100 100101 100111 100110 100010 100011 100001 100000

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Determine the predecessors of each of the 9-tuples in Exercise 23 in the reflected Gray code of order 9.

(a)

010100110 $\leftarrow$ 010100010

(b)

110001100 $\leftarrow$ 110000100

(c)

111111111 $\leftarrow$ 111111110

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Generate the 2-subsets of $\{1, 2, 3, 4, 5, 6\}$ in lexicographic order by using the algorithm described in Section 4.4.

12	23	35
13	24	36
14	25	45
15	26	46
16	34	56

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Determine the 7-subset of $\{1, 2, \ldots, 15\}$ that immediately follows 1,2,4,6,8,14,15 in the lexicographic order. Then determine the 7-subset that immediately precedes 1,2,4,6,8,14,15.

Since 14 and 15 are as high as we can go we increment the 8 and start counting from there.

$$1, 2, 4, 6, 8, 14, 15 \text{ is followed by } 1, 2, 4, 6, 9, 10, 11$$

Since we can't decrement the 15 to 14 because we already have a 14 we decrement the 14 to 13 and leave the 15 since it is the max and we want it to roll over on the next count up.

$$1, 2, 4, 6, 8, 14, 15\text{is preceded by}1, 2, 4, 6, 8, 13, 15$$

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Generate the 3-permutations of $\{1, 2, 3, 4, 5\}$

123	124	125	134	135	145	234	235	245	345
132	142	152	143	153	154	243	253	254	354
312	412	512	413	513	514	423	523	524	534
321	421	521	431	531	541	432	532	542	543
231	241	251	341	351	451	342	352	452	453
213	214	215	314	315	415	324	325	425	435

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In which position does the subset 2489 occur in the lexicographic order of the 4-subsets of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$?
Using theorem 4.4.2 we have:

$$\binom{9}{4} - \binom{7}{4} - \binom{5}{3} - \binom{1}{2} - \binom{0}{1} = \frac{9!}{4!(9-4)!} - \frac{7!}{4!(7-4)!} - \frac{5!}{4!(5-3)!} - 0 - 0 = 81$$

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Consider the r-subsets of $\{1, 2, \dots, n\}$ in lexicographic order.

(a)

What are the first $(n - r + 1)$ r-subsets?

$$\begin{gathered}
\{1, 2, \dots, (r-1), r\} \\
\{1, 2, \dots, (r-1), (r+1)\} \\
\vdots \\
\{1, 2, \dots, (r-1), n\}
\end{gathered}$$

The first $(n - r + 1)$ subsets all contain $1, 2, \dots, (r - 1)$ and then one last element that is all the numbers $r, \dots, n$

(b)

What are the last $(r + 1)$ r-subsets?

$$\begin{aligned}
\{(n-r+1)(n-r+2)\dots(n-1)(n)\} \\
\{(n-r)(n-r+2)\dots(n-1)(n)\} \\
\{(n-r)(n-r+1)(n-r+3)\dots(n-1)(n)\} \\
\{(n-r)(n-r+1)\dots(n-1)\}
\end{aligned}$$

Basically the last subset is the last r elements. Each of the preceeding r subsets decreases by one the element that corresponds with how far from the last subset we are.

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The *complement* $\bar{A}$ of an r-subset A of $\{1,2,\ldots,n\}$ is the $(n-r)$-subset of $\{1,2,\ldots,n\}$, consisting of all those elements that do not belong to A. Let $M = \binom{n}{r}$, the number of r-subsets and, at the same time, the number of $(n-r)$-subsets of $\{1,2,\ldots,n\}$. Prove that, if

$$A_1, A_2, A_3, \ldots, A_M$$

are the r-subsets in lexicographic order, then

$$\overline{A_M}, \ldots, \overline{A_3},\ \overline{A_2},\ \overline{A_1}$$

are the $(n-r)$-subsets in lexicographic order.

lemma

Reversing theorem 4.4.1 from the text we get the following lemma.

Let $a_1a_2\ldots a_r$ be an r-subset of $\{1,2,\ldots,n\}$. The last r-subset in the lexicographic ordering is $(n-r+1)(n-r+2)\ldots n$. Assume that $a_1a_2\ldots a_r \neq 12\ldots r$. Let k be the largest integer such that $a_k > 1$ and $a_k - 1$ is different from each of $a_1,\ldots,a_{k-1}$ then the r-subset that is the immediate predecessor of $a_1a_2\ldots a_r$ in the lexicographic ordering is $a_1\ldots a_{k-1}(a_k-1)(n-r+k+1)\ldots n$

Proof

First let's take care of the trivial cases.

Let $n < r$. Then $A_1 = A_M = \emptyset$ and $\overline{A_1} = \overline{A_M} = \{1,2,\ldots,n\}$. Since there is only one subset we are done.

Let $n = r$. Then $A_1 = A_M = \{1,2,\ldots,n\}$ and $\overline{A_1} = \overline{A_M} = \emptyset$. Since there is only one subset we are done.

Let $n > r$

Let's look at A_1. It is $\{1,2,\ldots,r\}$. We also know that A_M is $\{(n-r+1),(n-r+2),\ldots,n\}$ so $\overline{A_M}$ is $\{1,2,\ldots,(n-r)\}$ which is by definition the first $(n-r)$-subset in lexicographical order.

Now we find that A_{M-1} is $\{n-r, n-r+2,\ldots,n\}$. The complement of this is $\{1,2,\ldots,n-r-1,n-r+1\}$. And the successor to $\overline{A_M}$ is $\{1,2,\ldots,n-r-1,n-r+1\}$ So we see that the hypothesis holds for A_M and A_{M-1}

Now lets take some random subset $A_i = a_1a_2\ldots a_{n-r}$. Determine k as in the lemma above. Then

$$a_1a_2\ldots a_r = a_1\ldots a_{k-1}a_k(n-r+k+1)\ldots n$$

where

$$a_k - 1 > a_{k-1}$$

Thus the immediate predecessor of $a_1a_2\ldots a_r$ is $A_{i-1} = a_1\ldots a_{k-1}(a_k-1)(n-r+k+1)\ldots n$

Let us take $\overline{A_i}$. We know that $a_k(a_k+1)\ldots(a_k+r-k-1)(a_k+r-k) \notin \overline{A_i}$ and $a_1\ldots a_{k-1} \notin \overline{A_i}$ but that $a_k - 1 \in \overline{A_i}$. Further we know that $(a_k+r-k+1)\ldots n \in \overline{A_i}$. Notice that if we "shift" this $r-k$ places we have $a_k+1\ldots n-r+k$ and $|a_k+1\ldots n-r+k| = |(a_k+r-k+1)\ldots n|$. We don't really care anything about the part of $\overline{A_i}$ before a_k so we'll just call it $b_1\ldots b_p$. So the successor to $\overline{A_i}$ is $b_1\ldots b_p a_k(a_k+1)(a_k+2)\ldots(n-r+k)$. And if we examine $\overline{A_{i-1}}$ we will have the same $b_1\ldots b_p$ that $\overline{A_i}$ has since we didn't change anything before a_k. And we have $(a_k-1)(n-r+k+1)\ldots n \notin \overline{A_{i-1}}$. Anything leftover must be in $\overline{A_{i-1}}$ so we have $\overline{A_{i-1}} = b_1\ldots b_p a_k(a_k+1)\ldots(n-r+k)$. Which is precisely the successor of $\overline{A_i}$ and so we have proven our result by induction.