Homework 3

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2.6 F. Let a, b be positive real numbers. Set $x_0 = a$ and $x_{n+1} = ({x_n}^{-1} + b)^{-1}$ for $n \geq 0$.

(a) Prove that x_n is monotone decreasing.

proof

If x_n is monotone decreasing, then $x_n \geq x_{n+1}$ for all $n \geq 0$.

$$x_{n+1} = ({x_n}^{-1} + b)^{-1} = \frac{1}{\frac{1+bx_n}{x_n}} = \frac{x_n}{1+bx_n}$$

Note that if x_n and b are positive, then so is x_{n+1}. Now we are told that x_0 and b are positive, so we know that all x_n are positive. This means of course that $1 + bx_n > 1$ which in turn means that $x_n > \frac{x_n}{1+bx_n} = x_n + 1$. Indeed it appears that not only is x_n monotone decreasing, it is strictly monotone decreasing. $\square$

(b) Prove that the limit exists and find it.

proof

As we noted in the previous proof, x_n is positive for all $n \geq 0$. This implies that $x_n > 0$ and is therefore bounded from below. Because x_n is monotone decreasing and bounded from below, it has a limit. $\square$

solution

$$L = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \left({x_n}^{-1} + b\right)^{-1} = \left(\left(\lim_{n\to\infty} x_n\right)^{-1} + b\right)^{-1} = \left(L^{-1} + b\right)^{-1}$$

$$\begin{aligned} L &= \frac{1}{\frac{1}{L} + b} \\ 1 &= 1 + bL \\ 0 &= bL \end{aligned}$$

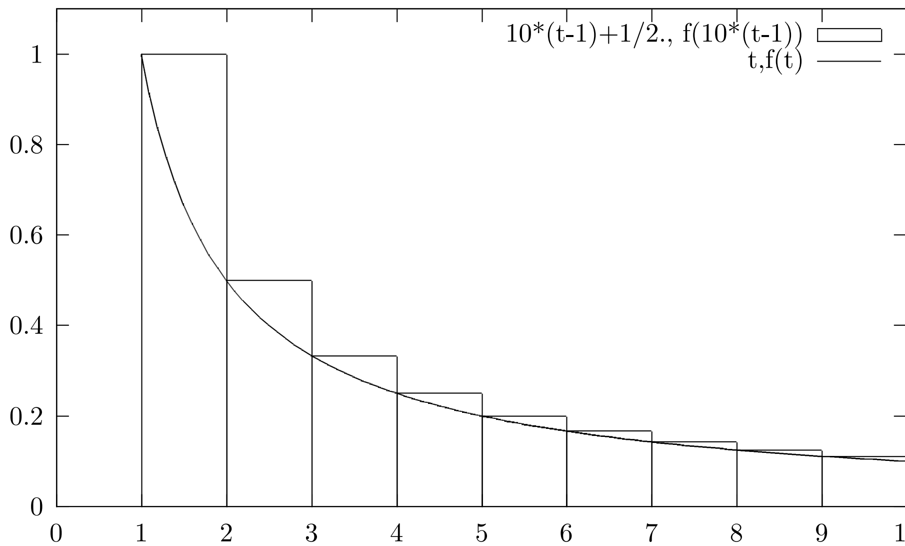
So then $\lim\limits_{n\to\infty} x_n = 0$.

G. Let $a_n = \left(\sum_{k=1}^{n} 1/k\right) - \log n$ for $n \geq 1$. **Euler's constant** is defined as $\gamma = \lim\limits_{n\to\infty} a_n$. Show that $(a_n)_{n=1}^{\infty}$ is decreasing and bounded below by zero, and so this limit exists. HINT: Prove that $1/(n+1) \leq \log(n+1) - \log n \leq 1/n$

argument

Now the trick is to realize that $\int_b^a \frac{1}{x}\mathrm{d}x = \log a - \log b$. In other words, the area under $\frac{1}{x}$ where x goes from b to a is $\log a - \log b$ So then $\int_1^n \frac{1}{x}\mathrm{d}x = \log n$ and $\int_n^{n+1} \frac{1}{x}\mathrm{d}x = \log(n+1) - \log n$.

Now lets imagine a series of boxes with a base of width one, and height $\frac{1}{k}$ positioned such that the left edge of the base is at k and the right edge is at $k+1$. So each of these boxes will have the area of their height, namely $\frac{1}{k}$. Further, the sum of the areas of these boxes will be $\sum_{k=1}^{n} \frac{1}{k}$. And our sequence then is the area of these boxes, with everything under the continuous line $1/x$ cut out.



Now we see that when $n = 1$ then we have a 1×1 box (area one). For $n = 2$ we add a box of size $\frac{1}{2}$ and cut out the curved section under $\frac{1}{x}$ from 1 to 2. Note that this line starts at the top left corner of the box of height 1 and ends at the point where the $\frac{1}{2}$ height box touches the height 1 box. Notice that we have cut out an area that is larger than $\frac{1}{2}$ but smaller than 1.
Lets generalize this observation a little:

$$
\begin{aligned}
a_{n+1} &= \left(\sum_{k=1}^{n+1} \frac{1}{k}\right) - \log(n+1) \\
&= \frac{1}{n+1} + \left(\sum_{k=1}^{n} \frac{1}{k}\right) - \log n - \log\left(1 + \frac{1}{n}\right) \\
&= \frac{1}{n+1} + a_n - (\log(n+1) - \log n) \\
&= a_n + \frac{1}{n+1} - \int_n^{n+1} \frac{1}{x}\mathrm{d}x
\end{aligned}
$$

Visually, every time we increase n we add a $\frac{1}{n+1}$ box, but carve out the the portion of the $\frac{1}{n}$ box that is under the line $1/x$. Note that the line goes from the top left of the $\frac{1}{n}$ box to where the $\frac{1}{n}$ box touches $\frac{1}{n+1}$. This means that while we are adding $\frac{1}{n+1}$, we are subtracting a number that is bigger than $\frac{1}{n+1}$ (and less than $\frac{1}{n}$). And so we see that our sequence decreases.

Futhermore, because $\log(n+1) - \log n$ is less than $\frac{1}{n}$ we always subtract less in the $n+1$ element of the sequence than we added in the n element of the sequence. Therefore, every element of the sequence will be greater than zero. Visually the boxes are always above the $\frac{1}{x}$ line so the area is always more than zero.

M. Suppose that $(a_n)_{n=1}^{\infty}$ has $a_n > 0$ for all n. Show that $\limsup {a_n}^{-1} = (\liminf a_n)^{-1}$.

proof

First we take a look at when a_n is unbounded. In this case we have defined $\liminf a_n = -\infty$. Naturally in this case $(\liminf a_n)^{-1}$ really has no meaning. We will then focus on the case where a_n is bounded.
Lets take some i, j such that $a_i \geq a_j$. Then if $a_i \geq a_j$ we know $\frac{1}{a_j} \geq \frac{1}{a_i}$. We define $b_n = \sup\{a_k : k \geq n\}$ for $n \geq 1$. Now we know that $b_n \geq a_k \forall k \geq n$. This implies that ${b_n}^{-1} \leq {a_k}^{-1} \forall k \geq n$. Another way of saying that is ${b_n}^{-1} = \inf\limits_{k \geq n} {a_k}^{-1}$. So then

$$\begin{aligned}
\lim_{n\to\infty} {b_n}^{-1} &= \lim_{n\to\infty} \left(\inf_{k \geq n} {a_k}^{-1} \right) \\
\left(\lim_{n\to\infty} b_n \right)^{-1} &= \liminf {a_n}^{-1} \\
\left(\lim_{n\to\infty} \sup_{k \geq n} a_k \right)^{-1} &= \liminf {a_n}^{-1} \\
(\limsup a_k)^{-1} &= \liminf {a_n}^{-1}
\end{aligned}$$

Boom. $\square$