15(a) Let $A \in M_{m\times n}$ with column vectors $\mathbf{c}_1, \mathbf{c}_2, ..., \mathbf{c}_n \in \mathbb{R}^m$. Prove that if $\mathbf{c}_1 + \mathbf{c}_2 + ... + \mathbf{c}_n = \mathbf{0}$, then $\text{rank}(A) < n$.

Proof. Recall that

$$A \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + ... + x_n\mathbf{c}_n.$$

Since

$$\mathbf{c}_1 + \mathbf{c}_2 + ... + \mathbf{c}_n = \mathbf{0}$$

we have that

$$1\mathbf{c}_1 + 1\mathbf{c}_2 + ... + 1\mathbf{c}_n = \mathbf{0}$$

and so

$$A \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \mathbf{0}.$$

But then the equation $A\mathbf{x} = \mathbf{0}$ has more than one solution. By Theorem 17 of the notes, we know that $\text{rank}(A) = \text{rank}[A \mid \mathbf{0}] = n$ implies $A\mathbf{x} = \mathbf{0}$ has exactly one solution. Therefore, it must be the case that $\text{rank}(A) < n$.