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HW 13
Lesson 7 problem 3. Solve the IC. Note the questions about steady state behavior.

$$
\begin{array}{llll}
PDE & u_t = u_{xx} & 0 < x < 1 & 0 < t < \infty \\
BCs & \begin{cases} u_x(0,t) = 0 \\ u_x(1,t) = 0 \end{cases} & & 0 < t < \infty \\
IC & u(x,0) = x & 0 \le x \le 1 &
\end{array}
$$

$$\lambda_n = n\pi \qquad u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n{}^2 t} \cos(\lambda_n x)$$

$$u(x,0) = x = \sum_{n=1}^{\infty} a_n \cos(\lambda_n x) \qquad \int_0^1 \xi \cos(\lambda_m \xi)\,\mathrm{d}\xi = \sum_{n=1}^{\infty} a_n \int_0^1 \cos(\lambda_n \xi)\cos(\lambda_m \xi)\,\mathrm{d}\xi$$

Because $\{\cos(\lambda_i x)\}_{0 \le i \le n}$ are orthogonal functions we can convert the above equation into the following.

$$
\begin{aligned}
\int_0^1 \xi \cos(\lambda_m \xi)\,\mathrm{d}\xi &= a_m \int_0^1 \cos(\lambda_m \xi)^2\,\mathrm{d}\xi \\
&= a_m \cdot \frac{\sin(2\lambda_m) + 2\lambda_m}{4\lambda_m} \\
&= a_m \frac{\lambda_m + \sin(\lambda_m)\cos(\lambda_m)}{2\lambda_m}
\end{aligned}
$$

Recall that we discovered in HW 12 that $\sin(\lambda) = 0$

$$
\begin{aligned}
&= \frac{a_m}{2} \\
a_n &= 2 \int_0^1 \xi \cos(\lambda_n \xi)\,\mathrm{d}\xi \\
&= 2 \left[\frac{\lambda_n x \sin(\lambda_n x) + \cos(\lambda_n x)}{\lambda_n{}^2} \right]_0^1 \\
&= 2 \left[\frac{\lambda_n 1 \sin(\lambda_n 1) + \cos(\lambda_n 1)}{\lambda_n{}^2} - \frac{\lambda_n 0 \sin(\lambda_n 0) + \cos(\lambda_n 0)}{\lambda_n{}^2} \right]_0^1 \\
&= 2 \left[\frac{\cos(\lambda_n)}{\lambda_n{}^2} - \frac{1}{\lambda_n{}^2} \right] = 2 \left[\frac{\cos(\lambda_n) - 1}{\lambda_n{}^2} \right] \\
&= 2 \left[\frac{\cos(n\pi) - 1}{(n\pi)^2} \right] \\
&= \frac{2((-1)^n - 1)}{(n\pi)^2} \\
u(x,t) &= \sum_{n=1}^{\infty} \frac{2((-1)^n - 1)}{(n\pi)^2} e^{-(n\pi)^2 t} \cos(n\pi x)
\end{aligned}
$$