Notes

February 7, 2014

homework

due next friday 14 feb

lesson 7 number 1

hw 10

find general series solution for PDE and BC's.

hw 11

solve the IC. Explain how orthogonality is used.

number 3

hw 12

find general series solution for PDE and BC's.

hw 13

solve the IC. note questions about steady-state behavior.

lesson 9 number 4

hw 14

find the general series solution for PDE and BC's.

hw 15

show the eigen functions are orthogonal and solve the IC.

lesson 9

$$
\begin{array}{lcr}
PDE & u_t = \alpha^2 u_{xx} + f(x,t), & 0 < x < 1, 0 < t < \infty \\
BC's & 0 = \alpha_1 u_x(0,t) + \beta_1 u(0,t), & 0 < t < \infty \\
 & 0 = \alpha_2 u_x(1,t) + \beta_2 u(1,t), & 0 < t < \infty \\
IC & u(x,0) = \phi(x), & 0 < x < 1 \\
PDE & u_t = \alpha^2 u_{xx} \rightarrow u = T(t)X(x) &
\end{array}
$$

$$\frac{T'(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)}$$

separation constant $= -\lambda$

assume const ≤ 0

$$\begin{aligned} u &= e^{-\alpha^2 \lambda t} X(x) \\ 0 &= X''(x) + \lambda^2 X(x) \\ &= \alpha_1 X'(0) + \beta_1 X(x) \\ &= \alpha_2 X'(1) + \beta_2 X(1) \end{aligned}$$

special case of sturm-louisville theorem (sp?)

results

there are nontrivial solutions $X_n(x)$ for a sequence of values $\lambda = \lambda_n$

$$\lambda_1 < \lambda_2 < \lambda_3 \text{ and } \lambda_n \to \infty$$

for

$$\lambda_m \neq \lambda_n, \int_0^1 X_m(x) X_n(x) \, \mathrm{d}x = 0 \text{ (orthagonality)}$$

we have $u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\alpha^2 \lambda_n t} X_n(x)$

How do we find c_n in $u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\alpha^2 \lambda_n t} X_n(x)$? because the original pde is linear, the sum of the solutions is a solution. similarly for the boundary conditions. Use orthogonality of $X_n(x)$'s and the initial condition $u(x,0) = \phi(x)$.

What about $u_t = \alpha^2 u_{xx} + f(x,t)$? Try solutions $u = \sum_{n=1}^{\infty} u_n(t) X_n(x)$. write $f(x,t) = \sum_{n=1}^{\infty} f(n(t) X_n(x)$. In the PDE: $\sum_{n=1}^{\infty} {u_n}'(t) X_n(x) = \alpha^2 \sum_{n=1}^{\infty} u_n(t) {X_n}''(x) + \sum_{n=1}^{\infty} f_n(t) X_n(x)$

Now ${x_n}''(x) = -\lambda_n X_n(x)$

For each n:

$${u_n}'(t) = -\alpha^2 \lambda_n u_n(t) + f_n(t)$$

note

$$f_n(t) \int_0^1 (X_n(x))^2 \, \mathrm{d}x = \int_0^1 f(x,t) X_n(x) \, \mathrm{d}x$$

since $f_n(t)$ is known we have

$$({u_n}' + \alpha^2 \lambda_n u_n = f_n(t)) e^{\alpha^2 \lambda_n t} \frac{\mathrm{d}}{\mathrm{d}t} \left(e^{\alpha^2 \lambda_n t} u_n(t) \right) = e^{\alpha^2 \lambda_n t} f_n(t)$$

$$\to e^{\alpha^2 \lambda_n t} u_n(t) - u_n(0) = \int_0^t e^{\alpha^2 \lambda_n u} f_n(u) \, \mathrm{d}u$$

$$\to u_n(t) = u_n(0) e^{-\alpha^2 \lambda_n t} + \int_0^t e^{-\alpha^2 \lambda_n (t-u)} f_n(u) \, \mathrm{d}u$$

note

initial condition is $\sum_{n=1}^{\infty} u_n(0) X_n(x) = \phi(x)$