Notes

2 fevrier, 2015

quiz

for a simple function, the range needs to be finite and the inverse image needs to be measurable for all ranges

χ_E continuous except on $\partial E = E^C \setminus E^\circ$

E is nonmeasurable if χ_E is not simple.

les notes

if φ is simple then r a n $\varphi = \{\alpha_1, \dots, \alpha_n\}$, $E_i = \varphi^{-1}(\{\alpha_i\})$ then $\varphi = \sum_{i=1}^{n} \alpha_i \chi_{E_i}$

example $\sum_{n=1}^{5} \chi_{[n,n+1]}$

one on $[1, 6]$ everywhere but $\{2, 3, 4, 5\}$ where it is 2.

this is not in canonical form. rewritten in cannonical form is

$E_1 = (-\infty, 1) \cup (6, \infty)$, $E_2 = [1, 2) \cup (2, 3) \cup (3, 4) \cup (4, 5) \cup (5, 6]$, $E_3 = \{2, 3, 4, 5\}$

$\varphi = 0\chi_{E_1} + 1\chi_{E_2} + 2\chi_{E_3}$

now $\sum_{i=1}^{n} \beta_i \chi_{E_i}$ and $B_1 = E_1 \cap \cdots \cap E_n$ to $(\sum_{i=1}^{n} \beta_1)\chi B_1 + \dots$ and on with every single set getting thrown away and on with every possible combination of two sets getting thrown away and 3 and so on

E_1, E_2, E_3 look at $B_1 = E_1 \cap E_2 \cap E_3$, $B_2 = E_2 \cap E_3) \setminus (E_1 \cap E_2 \cap E_3)$, $B_3 = E_1 \cap E_3) \setminus (E_1 \cap E_2 \cap E_3)$, $B_4 = E_1 \cap E_2) \setminus (E_1 \cap E_2 \cap E_3)$, and so on

note that $B_i \cap B_j = \emptyset$

$\varphi = \sum_{i=1}^{n} \alpha_i \chi_{E_i}$

definition of integral

only applies to simple functions $\sum \varphi \mathrm{d}m \sum_{i=1}^{n} \alpha_i m * (E_i)$

now $\int \chi_{[0,\infty]} \mathrm{d}m = 0m * (-\infty, 0) + m * [0, \infty]$

tweak: any function f is 0 outside some bounded interval E. $\int_E f \mathrm{d}m$

propositions

if φ, ψ are simple then $\int_E (\alpha\varphi + \beta\psi)\mathrm{d}m = \alpha \int_E \varphi \mathrm{d}m + \beta \int_E \psi \mathrm{d}m$

proof

$$\int \underbrace{\alpha \sum \alpha_i \chi_{E_i}}_{\varphi} + \underbrace{\beta \sum \beta_i \chi_{F_i}}_{\psi}$$

move alphas and betas into sum, merge sums by changing χ_{E_i} and χ_{F_i} to $\chi_{E_i \cap F_j}$

proposition

if φ, ψ are simple with $\varphi \leq \psi$ then $\int_E \varphi \leq \int_E \psi$

proof, $(\psi - \varphi) \geq 0$. find canonical. $\sum_{i=1}^{n} \alpha_i \chi_{E_i}$ notice $\alpha_i \geq 0$ and $\int_E \psi - \varphi \mathrm{d}m = \sum_{i=1}^{n} \alpha_i m * (E_i) \geq 0$ and so $\int_E \psi - \int_E \varphi = \int_E (\psi - \varphi) \mathrm{d}m \geq 0$ and so $\int_E \psi \geq \int_E \phi$

break

if φ, ψ is simple and $m * \{x : \varphi(x) \neq \psi(x)\} = 0$ then $\int_E \varphi = \int_E \psi$

$(\varphi - \psi) = 0\chi_{[\{x:\varphi(x)=\psi(x)\}]} + (mess)\chi_{\{x:\varphi(x)\neq\psi(x)\}}$

almost everywhere

$f = g$ almost everywhere if $m * (\{x : f(x) \neq g(x)\}) = 0$

definition

is measurable if for any $a \in \mathbb{R}$ we have $\{x : f(x) \geq a\}$ is measurable