first order linear

$$\frac{\mathrm{d}y}{\mathrm{d}t} + p(t)y = q(t) \qquad \mu(t) = e^{\int p(t)\,\mathrm{d}t}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mu(t)y\right) = \mu(t)\frac{\mathrm{d}y}{\mathrm{d}t} + p(t)\mu(t)y \qquad \mu(t)y = \int \mu(t)q(t)\,\mathrm{d}t$$

exact

$$M(t,y)\,\mathrm{d}t + N(t,y)\,\mathrm{d}y = 0 \qquad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

$$\int M(t,y)\,\mathrm{d}t + \phi(y) = f(t,y) \qquad \phi'(y) = N(x,y) - \frac{\mathrm{d}}{\mathrm{d}y}\left(\int M(t,y)\,\mathrm{d}t\right)$$

$$\int M(t,y)\,\mathrm{d}t + \int \phi'(y)\,\mathrm{d}y = f(t,y)$$

Solution is $f(t,y) = C$

bernoulli

$$\frac{\mathrm{d}y}{\mathrm{d}t} + p(t)y = q(t)y^n \qquad \frac{1}{y^n}\frac{\mathrm{d}y}{\mathrm{d}t} + p(t)y^{1-n} = q(t)$$

$$w = y^{1-n} \qquad \frac{\mathrm{d}w}{\mathrm{d}t} = (1-n)\frac{1}{y^n}\frac{\mathrm{d}y}{\mathrm{d}t}$$

$$\frac{\mathrm{d}w}{\mathrm{d}t} + (1-n)p(t)w = (1-n)q(t)$$

Solve as first order linear, then back substitute

homogeneous

$$M(t,y)\,\mathrm{d}t + N(t,y)\,\mathrm{d}y = 0 \qquad M(xt,xy) + N(xt,xy) = x^n\left(M(t,y) + N(t,y)\right)$$

$$\mathrm{d}y = w\,\mathrm{d}t + t\,\mathrm{d}w \qquad \mathrm{d}t = w\,\mathrm{d}y + y\,\mathrm{d}w$$

Substitute with $y = wt$ if $N(t,y)$ is simpler and $t = wy$ if $M(t,y)$ is simpler. Solve as a separable equation

trigonometric identities

$$\sin x = \frac{1}{\csc x} \qquad \sin(-x) = -\sin x \qquad \cos(-x) = \cos x$$

$$\sin(u \pm v) = \sin u \cos v \pm \cos u \sin v \qquad \cos(u \pm v) = \cos u \cos v \mp \sin u \sin v \qquad \sin 2u = 2\sin u \cos u$$

$$\cos 2u = \cos^2 u - \sin^2 u \qquad \cos 2u = 2\cos^2 u - 1 \qquad \cos 2u = 1 - 2\sin^2 u$$

$$\sin^2 u = \frac{1 - \cos 2u}{2} \qquad \cos^2 u = \frac{1 + \cos 2u}{2}$$

$$\sin u \pm \sin(v) = 2\sin\left(\frac{u \pm v}{2}\right)\cos\left(\frac{u \mp v}{2}\right) \qquad \cos u + \cos v = 2\cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)$$

$$\cos u - \cos v = -2\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right) \qquad \sin u \sin v = \frac{1}{2}[\cos(u-v) - \cos(u+v)]$$

$$\cos u \cos v = \frac{1}{2}[\cos(u-v) + \cos(u+v)] \qquad \sin u \cos v = \frac{1}{2}[\sin(u+v) + \sin(u-v)]$$

Wronskian

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

y_1 and y_2 are linearly independent if $W \neq 0$

reduction of order

given $y'' + p(t)y' + q(t)y = 0$ and a known solution y_1 then full solution is given by

$$y_s = c_1 y_1 + c_2 y_2 = c_1 y_1 + c_2 v(t) y_1 \qquad v(t) = \int \frac{1}{{y_1}^2} e^{-\int p(t)\,\mathrm{d}t}\,\mathrm{d}t$$

second order linear homogeneous with constant coefficient

$$ay'' + by' + cy = 0 \to ar^2 + br + c = 0 \qquad r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$y_s = \begin{cases} c_1 e^{r_1 t} + c_2 e^{r_2 t} & r_1 \neq r_2 \\ (c_1 + c_2 t)e^{rt} & r_1 = r_2 \\ e^{\alpha t}\left[c_1 \cos(\beta t) + c_2 \sin(\beta t)\right] & r = \alpha \pm \beta i \end{cases}$$

method of undetermined coefficients

solution of $ay'' + by' + cy = f(t)$ is $y_s = y_h + y_p$ where y_h is solution to corresponding homogeneous equation

$$f(t) = t^m e^{\alpha t} \qquad \text{or} \qquad f(t) = t^m e^{\alpha t}\sin\beta t \quad \text{or} \quad f(t) = t^m e^{\alpha t}\cos\beta t$$

$$S = \{e^{\alpha t}, e^{\alpha t}t, e^{\alpha t}t^2, ..., e^{\alpha t}t^m\} \qquad S = \left\{\begin{matrix} e^{\alpha t}\sin\beta t, e^{\alpha t}\cos\beta t, te^{\alpha t}\sin\beta t, te^{\alpha t}\cos\beta t, \\ t^2 e^{\alpha t}\sin\beta t, t^2 e^{\alpha t}\cos\beta t, ..., t^m e^{\alpha t}\sin\beta t, t^m e^{\alpha t}\cos\beta t \end{matrix}\right\}$$

if $S_h \cap S_p \neq \emptyset$ then $S_p \to t^n S_p$. This will make y_h and y_p linearly independant. If $f(t)$ has more than one term then S_p is the union of the solution set for each term. Throw out constant coefficients in $f(t)$

$$y_p = a_1 S_p[1] + a_2 S_p[2] + ... + a_m S_p[m]$$

Solve for all a_n and we are done.

variation of parameters

$ay'' + by' + cy = f(t)$ for any $f(t)$. More general than undetermined coefficients. W refers to the Wronskian.

$$y_s = y_h + y_p \qquad y_h = c_1 y_1 + c_2 y_2 \qquad y_p = u_1 y_1 + u_2 y_2 \qquad {u_1}' = -\frac{y_2 f}{W} \qquad {u_2}' = \frac{y_1 f}{W}$$