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2.4

26

Solve $\left(2t\sin y - 2ty\sin(t^2)\right)\mathrm{d}t + \left(t^2\cos y + \cos(t^2)\right)\mathrm{d}y = 0$

$$M(t,y) = 2t\sin y - 2ty\sin(t^2) \qquad N(t,y) = t^2\cos y + \cos(t^2)$$
$$\frac{\partial M}{\partial y} = 2t\cos y - 2t\sin(t^2) \qquad \frac{\partial N}{\partial t} = 2t\cos y - 2t\sin(t^2)$$

The equation is exact, we proceed by integrating M

$$\int \left(2t\sin y - 2ty\sin(t^2)\right)\,\mathrm{d}t = t^2\sin y + y\cos(t^2) + \phi(y)$$
$$N(x,y) = \frac{\mathrm{d}}{\mathrm{d}y}\left(t^2\sin y + y\cos(t^2) + \phi(y)\right)$$
$$t^2\cos y + \cos(t^2) = t^2\cos y + \cos(t^2) + \phi'(y)$$
$$\phi'(y) = 0,\ \phi(y) = k$$
$$t^2\sin y + y\cos(t^2) + k = C_1,\ C = C_1 - k$$
$$t^2\sin y + y\cos(t^2) = C$$

38

Solve the IVP $\left(\cos^2 t - \sin^2 t + y\right)\mathrm{d}t + (\sec y\tan y + t)\,\mathrm{d}y = 0, y(0) = 0$

$$M(t,y) = \cos^2 t - \sin^2 t + y \qquad N(t,y) = \sec y\tan y + t$$
$$\frac{\partial M}{\partial y} = 1 \qquad \frac{\partial N}{\partial t} = 1$$

$$\int \left(\cos^2 t - \sin^2 t + y\right)\,\mathrm{d}t = \int \cos^2 t\,\mathrm{d}t - \int \sin^2 t\,\mathrm{d}t + ty + \phi(y)$$
$$u = \cos t,\ \mathrm{d}v = \cos t\,\mathrm{d}t,\ \mathrm{d}u = -\sin t\,\mathrm{d}t,\ v = \sin t$$
$$= \cos t\sin t - \int \left(-\sin^2 t\right)\,\mathrm{d}t - \int \sin^2 t\,\mathrm{d}t + ty + \phi(y)$$
$$N(x,y) = \frac{\mathrm{d}}{\mathrm{d}y}\left(\cos t\sin t + ty + \phi(y)\right)$$
$$\sec y\tan y + t = t + \phi'(y)$$
$$\phi(y) = \int \frac{\sin y}{\cos^2 y}\,\mathrm{d}y = \int -u^{-2}\,\mathrm{d}u$$
$$= \frac{1}{\cos y}$$
$$\cos t\sin t + ty + \sec y = C = \cos(0)\sin(0) + 0\cdot 0 + \frac{1}{\cos(0)} = 1$$
$$\cos t\sin t + ty + \sec y = 1$$

2.5

5

Solve the Bernoulli equation $y' - 2y = y^{-\frac{1}{2}} \cos t$

$$\begin{aligned}
v = y^{3/2},\ y = v^{2/3},\ y' &= \frac{2}{3} v^{-1/3} v' \\
\frac{2}{3} v^{-1/3} v' - 2v^{2/3} &= v^{-1/3} \cos t \\
\frac{2}{3} v' - 2v &= \cos t \\
v' - 3v &= \frac{3}{2} \cos t \qquad\qquad e^{\int -3\,\mathrm{d}t} = e^{-3t} \\
e^{-3t} v &= \int \left(\frac{3}{2} e^{-3t} \cos t \right) \mathrm{d}t = \frac{3}{2} \int \left(e^{-3t} \cos t \right) \mathrm{d}t \\
&= \frac{3}{2} \cdot \frac{e^{-3t}}{9+1} (-3 \cos t + \sin t) + C \\
v &= \frac{3}{20} (-3 \cos t + \sin t) + C \cdot e^{3t} \\
y^{3/2} &= \frac{3}{20} (-3 \cos t + \sin t) + C \cdot e^{3t}
\end{aligned}$$

$$y^{3/2} + \frac{9}{20} \cos t - \frac{3}{20} \sin t - C e^{3t} = 0$$

19

Solve $\left(ty - y^2\right) \mathrm{d}t + t\,(t - 3y)\,\mathrm{d}y = 0$

$$M(t, y) = ty - y^2 \qquad\qquad N(t, y) = t^2 - 3ty$$

$$M(xt, xy) = xtxy - (xy)^2 = x^2(ty - y^2) \qquad\qquad N(xt, xy) = (xt)^2 - 3xtxy = x^2(t^2 - 3ty)$$

Substitute $t = vy$ and $\mathrm{d}t = v\,\mathrm{d}y + y\,\mathrm{d}v$

$$(vy^2 - y^2)(v\,\mathrm{d}y + y\,\mathrm{d}v) + (v^2y^2 - 3vy^2)\,\mathrm{d}y = 0 = v^2y^2\,\mathrm{d}y + vy^3\,\mathrm{d}v - vy^2\,\mathrm{d}y - y^3\,\mathrm{d}v + v^2y^2\,\mathrm{d}y - 3vy^2\,\mathrm{d}y$$

$$\begin{aligned}
(v^2y^2 - vy^2 + v^2y^2 - 3vy^2)\,\mathrm{d}y = (-vy^3 + y^3)\,\mathrm{d}v &= (2v^2y^2 - 4vy^2)\,\mathrm{d}y \\
\frac{1}{y}(2v^2 - 4v)\,\mathrm{d}y &= (1 - v)\,\mathrm{d}v \\
\frac{1}{y}\,\mathrm{d}y &= -\frac{1}{2} \cdot \frac{v - 1}{v^2 - 2v}\,\mathrm{d}v \\
\int \frac{1}{y}\,\mathrm{d}y &= -\frac{1}{4} \int \frac{2v - 2}{v^2 - 2v}\,\mathrm{d}v \quad u = v^2 - 2v,\ \mathrm{d}u = 2v - 2\,\mathrm{d}v \\
\ln|y| &= -\frac{1}{4} \int \frac{1}{u}\,\mathrm{d}u = -\frac{1}{4} \ln\left|v^2 - 2v\right| + C_0 \\
e^{\ln|y|} &= e^{-\frac{1}{4}\ln\left|v^2 - 2v\right| + C_0} \qquad\qquad C = e^{C_0} \\
y &= C \cdot (v^2 - 2v)^{-\frac{1}{4}} \\
y &= C \cdot \left(\frac{t^2}{y^2} - 2\frac{t}{y} \right)^{-\frac{1}{4}}
\end{aligned}$$