Math 329

Exam 1

Directions: Attempt all of the problems. Show all work for full credit. Each exercise is worth 20 points. Good luck and just do the best you can.

1. Consider the matrix

$$A = \begin{bmatrix} 3 & -1 & 2 & 7 \\ 2 & 1 & 3 & 3 \\ 2 & 2 & 4 & 2 \end{bmatrix}.$$

(a) Determine rank(A).
(b) Find the general solution in standard form of the equation $A\mathbf{x} = \mathbf{0}$.
(b) Find all vectors $\mathbf{b} \in \mathbb{R}^4$ such that $A\mathbf{x} = \mathbf{0}$ is consistent.

2. Let A be an 4×5 matrix and let $\mathcal{H}$ be the homogeneous system $A\mathbf{x} = \mathbf{0}$. Prove (carefully) that if $\mathrm{rank}(A) = 3$, then $\mathrm{Sol}(\mathcal{H})$ is a plane in $\mathbb{R}^5$.

3. Prove that if $A, B \in \mathcal{M}_{m\times n}$ and $rref(A) = rref(B)$ then there exists an invertible matrix E such that $A = EB$. Is the converse true?

4. A square matrix $A \in \mathcal{M}_n$ is called *upper-triangular* if $\mathrm{ent}_{ij}(A) = 0$ whenever $i > j$. Prove that the product of two upper-triangular matrices is upper-triangular.

5. Let A be an $m \times n$ matrix and let $\mathcal{H}$ be the homogeneous system $A\mathbf{x} = \mathbf{0}$. Prove (carefully) that if $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k \in \mathrm{Sol}(\mathcal{H})$, then $\mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k) \subseteq \mathrm{Sol}(\mathcal{H})$.

6. Suppose that $A, B \in \mathcal{M}_n$ are square matrices such that $BA = I_n$. Prove:
(a) $A\mathbf{x} = \mathbf{0}$ has a unique solution.
(b) A is invertible and $A^{-1} = B$.

7. Prove that if A is a square matrix with a column of all zeros, then A is not invertible.

8. Let $A \in \mathcal{M}_n$ be a symmetric matrix and let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Prove that if $A\mathbf{x} = \lambda_1\mathbf{x}$ and $A\mathbf{y} = \lambda_1\mathbf{y}$ for some $\lambda_1 \neq \lambda_2 \in \mathbb{R}$, then $\mathbf{x}$ and $\mathbf{y}$ are orthogonal.

9. A square matrix $A \in \mathcal{M}_n$ is called orthogonal if $A^{\mathrm{T}} A = I_n$. Prove that the columns of A are unit vectors that are orthogonal to one another. That is, prove that

$$\mathbf{c}_i \cdot \mathbf{c}_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

10. Let $\mathbf{x}$ and $\mathbf{y}$ be arbitrary nonzero vectors. Prove that if $r = \|\mathbf{x}\|$ and $s = \|\mathbf{y}\|$, then $s\mathbf{x} + r\mathbf{y}$ bisects the angle between $\mathbf{x}$ and $\mathbf{y}$.