8.1

G. Find all intervals on which the sequence $f_n(x) = \frac{x^{2n}}{n+x^{2n}}, n \geq 1$, converges uniformly.

This function is asymptotic with $f(x) = 1$ and $f_n(0) = 0$. Further, we notice that if $|x| \leq 1$ then we can make the denominator infinitely large, but the numerator will never be larger than 1, therefore for any εwe can find some N such that if $n \geq N$ then $f_n(x) \leq \varepsilon$ and so on the interval $[-1, 1]$ we see that f_n converges to $f = 0$. Now if $|x| > 1$ then we notice that the second derivatives of the top and bottom are both the same, so L'Hôpital tells us that we must converge to $f = 1$. Now our problem spot is $-1, 1$. Our maximum f_n in the case of $|x| \leq 1$ is at the ones. Obviously we can make $\frac{1}{n+1}$ as small as we want, so any subinterval of $[-1, 1]$ will converge uniformly. Now lets pick $\varepsilon = \frac{1}{2}$. If we can find some $f_n(x) \leq \frac{1}{2}$ for some $x \in (1, \infty)$ and for any n then we have a problem.

$$\begin{aligned}
&\frac{1}{2} = \frac{x^{2n}}{n + x^{2n}} \\
&n = x^{2n} \\
&x = \pm \sqrt[2n]{n}
\end{aligned}$$

That's a strange number but it is bigger than one, so the function does not converge uniformly the interval $(1, \infty)$. We should be fine though if we choose any subinterval of $[a, \infty)$ where $a > 1$. And the same thing for negatives.

H. Suppose that $f_n : [0, 1] \to \mathbb{R}$ is a sequence of C^1 functions (i.e., functions with continuous derivatives) that converges pointwise to a function f. If there is a constant M such that $||f_n'||_\infty \leq M$ for all n, then prove that (f_n) converges to f uniformly.

If we assume that M

I. Prove **Dini's Theorem**: if f and f_n are continuous functions on $[a, b]$ such that $f_n \leq f_{n+1}$ for all $n \geq 1$ and (f_n) converges to f pointwise, then (f_n) converges to f uniformly.

Hint: Work with $g_n = f - f_n$ which decrease to 0. Show that for any point x_0 and $\varepsilon > 0$, there are an integer N and a positive $r > 0$ such that $g_N(x) \leq \varepsilon$ on $(x_0 - r, x_0 + r)$. If convergence is not uniform, say $\lim ||g_n||_\infty = d > 0$, find x_n such that $\lim g_n(x_n) = d$. Obtain a contradiction.

If we can show that $g_n = f - f_n$ converges uniformly to $g = 0$ then we will have an equivalent result. Naturally if f and f_n are continuous functions, then g_n must also be continuous. Thus we know that for any $\varepsilon > 0$ and $x_0 \in [a, b]$ we can find some $r > 0$ and N for all x such that $|x - x_0| < r$ will satisfy $|g_N(x) - g_N(x_0)| < \varepsilon$. That is to say we can find some range

$(x_0 - r, x_0 + r)$ where $g_N(x) \leq \varepsilon$. And because g_n is monotonic, then $g_k(x) \leq \varepsilon$ for all $k \geq N$ and $x \in (x_0 - r, x_0 + r)$

J. Find an example which shows that Dini's Theorem is false if $[a, b]$ is replaced with a non compact subset of $\mathbb{R}$.

If we take $f_n(x) = -\left(\frac{x}{n}\right)^2$ then we have a function that converges pointwise to $f(x) = 0$ and $-\left(\frac{x}{n}\right)^2 \leq -\left(\frac{x}{n+1}\right)^2$ for all $n \in \mathbb{N}$ and $x \in [0, \infty)$. If we have a compact subset we can get uniform convergence with this thing, but if we look at say $[0, \infty)$ then we see that no matter how small ε is, if we go far enough out, we can always find some $x \in [0, \infty)$ such that $f_n(x) > \varepsilon$ for any n no matter how big.

K. (a) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous. Let $f_n(x) = f(x + 1/n)$. Prove that f_n converges uniformly to f on $\mathbb{R}$.

If we choose any $\varepsilon > 0$ then we know we can find some $\delta > 0$ so that if $||x - y|| < \delta$ then $||f(x) - f(y)|| < \varepsilon$. That is uniform continuity. So then we just need to pick N large enough that $1/N < \delta$ and we have $||f(x) - f(x + 1/n)|| < \varepsilon$ for all $n \geq N$.

(b) Does this remain true if f is just continuous? Prove it or provide a counterexample.

It does not remain true. Take $f(x) = x^2$ for example. If we choose $\varepsilon = 1$ then we should be able to find some N so that $|x^2 - (x+1/n)^2| < 1$ for all $x \in \mathbb{R}$ and $n \geq N$. But this implies that $|2x/n + 1/n^2| < 1$ which is clearly false for all $x \geq \frac{1}{2n}$ no matter how big we make n.