## 9.1

B. Show that every subset of a discrete metric space is both open and closed.

We have a discrete metric $d$ on a set $X$. Now we take $U \subset X$. For any $x \in U$ we have $B(x, r) \subset U$ if $r \leq 1$ because the ball will contain only the point $x$. Note that this is trivially true even if $U = \emptyset$ because there is no $x \in U$ that does not have a ball around it. Now because our choice of $U$ was arbitrary we know that all subsets of $X$ are open. And the complements of any subsets of $X$ are themselves subsets of $X$, and so they are open. But they are the complement of an open set, and so they must be closed. Thus every subset of a discrete metric space is both open and closed.

D. Prove Theorem 9.1.7

Let $f$ map a metric space $(X, \rho)$ into a metric space $(Y, \rho)$. The following are equivalent:

(1) $f$ is continuous on $X$;

(2) for every sequence $(x_n)$ with $\lim\limits_{n\to\infty} x_n = a \in X$, we have $\lim\limits_{n\to\infty} f(x_n) = f(x)$; and

(3) $f^{-1}(U) = \{x \in X : f(x) \in U\}$ is open in $X$ for every open set $U$ in $Y$.

We start by assuming that $f$ is continuous on $X$. Now we know that for every $\delta > 0$ there exists some $N$ such that $\rho(x_n, a) < \delta$ for all $n \geq N$. Thus for every

H. Two metrics $\rho$ and $\sigma$ on a set $X$ are **equivalent** if there are constants $0 < c < C$ such that $c\rho(x, y) \leq \sigma(x, y) \leq C\rho(x, y)$ for all $x, y \in X$

(a) Prove that equivalent metrics are topologically equivalent

(b) Prove that equivalent metrics have the saame Cauchy sequences

We begin with some Cauchy sequence $(x_n) \in \rho$. Then for every $\varepsilon/C > 0$ there exists some $N$ such that $\rho(x_i, x_j) < \varepsilon/C$. But $\sigma(x_i, x_j) \leq C\rho(x_i, x_j) < \varepsilon$ and so the sequence is Cauchy in $\sigma$. Now let us assume that our sequence is Cauchy in $\sigma$. Then for every $c\varepsilon > 0$ there exists some $N$ such that $c\rho(x_i, x_j) \leq \sigma(x_i, x_j) < c\varepsilon$ and so certainly $\rho(x_i, x_j) < \varepsilon$.

(c) Give examples of topologically equivalent metrics that are not equivalent

K. Recall the 2-adic metric of examples 9.1.2 (4) and 9.1.5 (4). Extend it to $\mathbb{Q}$ by setting $\rho_2(a/b, a/b) = 0$ and, if $a/b \neq c/d$, then $\rho_2(a/b, c/d) = 2^{-e}$, where $e$ is the unique integer such that $a/b - c/d = 2^e(f/g)$ and both $f$ and $g$ are odd integers

(a) Prove that $\rho_2$ is a metric on $\mathbb{Q}$
if $a/b \neq c/d$ then $a/b - c/d = \frac{ad-cb}{db}$. Now $ad - cb = 2^i f$ for some odd $f$ and $db = 2^j g$ for some odd $g$. Then $a/b - c/d = 2^{i-j}(f/g)$. Of course $2^{i-j}$ is non-zero and so $\rho_2(a/b, c/d) \neq 0$.
Now we assume that $a/b - c/b = 2^e \frac{f}{g}$. Then $c/d - a/b = 2^e(-f/g)$ and so $\rho_2(x, y) = \rho)2(y, x)$.
And finally, if $\rho_2(a/b, c/d) = 2^{-i+l}$, $\rho_2(a/b, e/f) = 2^{-k+l}$ and $\rho_2(c/d, e/f) = 2^{-j+l}$ then $a/b - c/d = (adf - bcf)/bdf$ and $c/d - e/f = (bcf - bde)/bdf$ while $a/b - e/f = (adf - bde)/bdf = (adf - bcf)/bdf + (bcf - bde)/bdf$. Now we see that $\rho_2(a/b, e/f) = 2^{-i-j+l} \leq 2^{-i-j+2l} = 2^{-i+l} + 2^{-j+l}$

(b) Show that the sequence of integers $a_n = (1 - (-2)^n)/3$ converges in $(\mathbb{Q}, \rho_2)$

(c) Find the limit of $\dfrac{n!}{n! + 1}$ in this metric.