Notes

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f is Riemann integrable if and only if $m*(D_f) = 0$ (we did this with contrapositive last time).

proof of converse

$D_r = \{x : \omega_f(x) \geq \frac{1}{n}\}$

$D_f = \bigcup_{n\in\mathbb{N}} J_{\frac{1}{n}}$ because $m*(D_f) = 0$ we know that $m*(J_{\frac{1}{n}}) = 0$

Let P be a partition with $\#(P) < S$ (mesh P is max Δ_i).

if $\epsilon > 0$ with $w_f(x) < \epsilon$ for all $x \in [a,b]$ there exists $S > 0$ such that $\Omega_f(T) < \epsilon$. If T is any closed interval with $m*(T) < S$.

?So if $m*(T) < S$ then $\Omega_f(T) < \frac{1}{n}$

break

$U(P,f) - L(P,f) = \sum\limits_{S_1}(M_i - m_i)\Delta_i + \sum\limits_{S_2}(M_i - m_i)\Delta_i$

$S_1 = \{[x_i, x_{i+1}] : J_{1/n} \cap (x_i, x_{i+1}) \neq \emptyset\}$ $S_2 = \{[x_i, x_{i+1}] : J_{1/n} \cap (x_i, x_{i+1}) = \emptyset\}$

oscillation on S_1 is small, maybe big on S_2

on S_2 we have $M_i - m_i < \frac{1}{n}$. and $\sum\limits_{S_2}(M_i - m_i)\Delta_i < \frac{1}{n}\sum\limits_{i=1}^{n}\Delta_i \leq \frac{1}{n}(b-a)$

function is bounded and so $M_i \leq M$ where M is upper bound for f and $m_i > m$ is lower bound

so on S_1 we havve $\sum\limits_{S_1}(M_i - m_i)\Delta_i \leq (M-m)\sum\limits_{S_1}\Delta_i \leq (M-m)\frac{1}{n}$

these intervals cover $J_{1/n}$. $m*(J_{1/n}) \leq \sum\limits_{S_1}(x_i - x_{i-1})$.

Any cover $U(a_i, b_i) \subseteq J_{1/n}$ with $|b_i - a_i| \subset S \to m*(J_{1/n}) \leq \sum b_i - a_i \leq m*(J_{1/n})) + \frac{1}{n}$

we choose a partition so that the subpartition reflects above. and then go through calculations and ge $U(Pf) - L(P,f) \leq \frac{(M-m)+(b-a)}{n}$

facts

1. if f is piecewise continuous on $[a,b]$ then f is Riemann integrable

2. $\chi_s(x) = \begin{cases} 1 & x \in S \\ 0 & x \notin S \end{cases}$

   where χ_s is discontinuous for any point on ∂S and continuous everywhere else

   $\partial S = \overline{S}/S^\circ$

   $\chi_C = \overline{C} = C \setminus \emptyset$

   $\partial(\chi_C) = C$

   and now $\chi_{\mathbb{Q}}$ and so boundary of rationals $\partial\mathbb{Q} = \mathbb{R}$

note that $\int \chi_s \mathrm{d}m = m*(S)$

f is simple if

1. range of $f = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ is finite
2. $E_k = \{x : \varphi(x) = \alpha_k\}$ is measurable.

notice that $\varphi(x) = \sum_{i=1}^{k} \alpha_k \chi_{E_i}(x)$
this is the canonical representation of φ and is unique.
$\sum_{i=1}^{3} \frac{1}{i}\chi_{E_i}$ with $E_1 = [0, \frac{1}{3}]$ E_2 and E_3 are other two thirds. no zeros in function, pairwise disjoint sets means canonical