Notes

9 février, 2015

quiz

completeness theorem in 8.2 is super important in functional analysis, but less important here.

gist of proof is that if you have a sequence f_n that is cauchy in $||\cdot||_\infty$ then ...

reminder

f_n, f are continuous. $\{f_n\} : [a,b] \to \mathbb{R}$ where $f_n \leq f_{n+1}$ and $f_n \to f$ pointwise then $f_n \to f$ uniformly

proof

$g_n = f - f_n$ and $g_n \to 0$ pointwise then $g_{n+1} \leq g_n$.

for every x_0 there is N_0 and r_{x_0} such that $g_{N_0}(x) \leq \varepsilon$ on $(x_0 - r_{x_0}, x_0 + r_{x_0})$

and so $g_{N_0} < \varepsilon$ so we find r_0 such that $|g_{N_0}(x) - g_{N_0}(y)| < \varepsilon$ for all y with $|x-y| < r_0$.

we get this from continuity.

notice $\subseteq \bigcup_{x\in[a,b[}(x - r_x, x + r_x)$ by compactness there $x_1, \ldots, x_n$ with $[a,b] \subseteq \bigcup_{i=1}^{n}(x_i - r_{x_i}, x_i + r_{x_i})$

if $n = \max\{N_{x_i} : 1 \leq i \leq n\}$

$g_N(x) \leq g_{N_{x_i}}(x) \leq \varepsilon$ on $(x_i - r_{x_i}, x_i + r_{x_i})$

$g_N() \leq \varepsilon$ for all $x \in [a,b]$ so $||g_n - 0||_{\infty<\varepsilon}$ for all $n > N$. thus $g_n \to 0$ uniformally.

important arguments are compactness which allowed us to switch from infinite to finite and something else i missed

theorem

$f_n : S \to \mathbb{R}^m$ with $f_n \to f$ uniformally on S if f_n continuous for all n then f continuous

proof

fix x, given $\epsilon > 0$ find δ such that $||f(x) - f(y)|| < \epsilon$ wwhen $||x-y|| < \delta$

using definition of uniform continuity:

give $\epsilon > 0$ there is N such that $||f_j - f||_\infty < \varepsilon/3$ when $j \geq N$.

using definition of continuity of f_N:

given $\epsilon > 0$ there is $\delta_1 > 0$ such that $||f_N(x) - f_N(y)|| < \epsilon/3$ when $||x-y|| < \delta_1$

if $||x-y|| < \delta_1$

$$\begin{aligned}
||f(x) - f(y)|| &= ||f(x) - f_N(x) + f_N(x) - f_N(y) + f_N(y) - f(y)|| \\
&\leq ||f(x) - f_N(x)|| + ||f(y) - f_N(y)|| + ||f_N(y) - f_N(x)|| \\
&\leq ||f - f_n||_\infty + ||f_N(x) - f_N(y)|| + |f - f_N||_\infty \\
&\leq \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon
\end{aligned}$$

corrolary

if f_n converges to f uniformally on S and f_n is uniformally continuous, then so is f.

the above is the standard proof for uniform convergence.

general idea: want property P for f and we know that f_n had P. then we say $||f - f_N||_\infty$ (close). $f = f - f_N + f_N$ so as long as property is preserved by smallness, we are solid.

proposition

if $f_n \to f$ uniformally on S and f_n bounded for all n then so is f.

proof

let $\epsilon = 1$. find N such that $||f - f_n||_\infty < 1$ for all $n \geq N$. now $||f(x)|| = ||f(x) - f_N(x) + f_N(x)|| \leq ||f(x) - f_N(x)|| + ||f_N(x)|| \leq ||f - f_n||\infty + ||f_N(x)|| < 1 + M_N$ and so bounded

example

if $f_n \to f$ uniformly on S $g_n \to g$ uniformly on S then $f_n + g_n \to f + g$ uniformly on S.

proof

given $\epsilon > 0$ find N such that $||(f_n + g_n) - (f + g)|| < \epsilon$ when $n \geq N$.

homework subtlety: $||f_n + g_n - (f + g)|| \leq ||f_n(x) - f(x)|| + ||g_n(x) - g(x)|| < ||f_n - f||_\infty + ||g_n - g||_\infty$ independent of x and N.

there is an N such that $||f - f_N||_\infty < \epsilon/2$ and $||g - g_N||_\infty < \epsilon/2$

for any x then $||(f_n(x) + g_n(x)) - (f(x) + g(x))||_\infty < ||f_n - f||_\infty + ||g_n - g||_\infty$