# Homework 6

Jon Allen

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4.2 B. If a sequence $(\boldsymbol{x}_n)_{n=1}^{\infty}$ in $\mathbb{R}^n$ satisfies $\sum_{n\geq 1} ||\boldsymbol{x}_n - \boldsymbol{x}_{n+1}|| < \infty$, show that it is a Cauchy sequence.

Let's say $\sum_{n\geq 1} ||\boldsymbol{x}_n - \boldsymbol{x}_{n+1}|| = L$. Then for every $\varepsilon > 0$ there exists some $N$ such that $\sum\limits_{n=1}^{N-1} ||\boldsymbol{x}_n - \boldsymbol{x}_{n+1}|| > L - \varepsilon$ and by extension $\varepsilon > \sum\limits_{n=N}^{\infty} ||\boldsymbol{x}_n - \boldsymbol{x}_{n+1}|| > \sum\limits_{k=n}^{m-1} ||\boldsymbol{x}_k - \boldsymbol{x}_{k+1}||$ for all $m > n \geq N$. And with the triangle inequality and the observation that our series is telescoping we quickly see that

$$\varepsilon > \sum_{k=n}^{m-1} ||\boldsymbol{x}_k - \boldsymbol{x}_{k+1}|| \geq \left|\left|\sum_{k=n}^{m-1} \boldsymbol{x}_k - \boldsymbol{x}_{k+1}\right|\right| = ||\boldsymbol{x}_n - \boldsymbol{x}_m||$$

Which is the very definition of a Cauchy sequence. Well almost, I guess to be complete I should point out that $||\boldsymbol{x}_n - \boldsymbol{x}_n|| = 0 < \varepsilon$ and $||\boldsymbol{x}_n - \boldsymbol{x}_m|| = ||\boldsymbol{x}_m - \boldsymbol{x}_n|| < \varepsilon$. So our inequality holds for all $m, n \geq N$, not just $m > n \geq N$

C. (a) Give an example of a Cauchy sequence for which the condition of Exercise B fails.

$$a_n = \frac{(-1)^n}{n}$$

$$\begin{aligned}
\sum_{n\geq 1} \left| \frac{(-1)^n}{n} - \frac{(-1)^{n+1}}{n+1} \right| &= \sum_{n\geq 1} \left| (-1)^n \left( \frac{1}{n} - \frac{-1}{n+1} \right) \right| \\
&= \sum_{n\geq 1} \left| \frac{1}{n} + \frac{1}{n+1} \right| \\
&> \sum_{n\geq 1} \frac{1}{n} = \infty
\end{aligned}$$

(b) However, show that every Cauchy sequence $(\boldsymbol{x}_n)_{n=1}^{\infty}$ has a subsequence $(\boldsymbol{x}_{n_i})_{i=1}^{\infty}$ such that $\sum_{i\geq 1} \left|\left|\boldsymbol{x}_{n_i} - \boldsymbol{x}_{n_{i+1}}\right|\right| < \infty$

First we choose $\boldsymbol{x}_{N_1}$ such that $||\boldsymbol{x}_m - \boldsymbol{x}_n|| < \frac{1}{2}$ for all $m, n \geq N_1$. We then proceed, choosing $\boldsymbol{x}_{N_i}$ such that $||\boldsymbol{x}_m - \boldsymbol{x}_n|| < \frac{1}{2^i}$ for all $m, n \geq N_i$. Now then $\sum_{i\geq 1} ||\boldsymbol{x}_{N_i} - \boldsymbol{x}_{N_i}|| < \sum_{i\geq 1} \frac{1}{2^i} = -1 + \sum_{i\geq 0} \frac{1}{2^i} = -1 + \frac{1}{1-\frac{1}{2}} = 1 < \infty$ as required.

4.3 B. Let $(\boldsymbol{a}_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{R}^k$ with $\lim\limits_{n\to\infty} \boldsymbol{a}_n = \boldsymbol{a}$. Show that $\{\boldsymbol{a}_n : n \geq 1\} \cup \{\boldsymbol{a}\}$ is closed.

Let's say $A = \{\boldsymbol{a}_n : n \geq 1\} \cup \{\boldsymbol{a}\}$. If $A$ is not closed, then we can form a sequence from the elements of $A$ that converge on some point $\boldsymbol{b} \in \mathbb{R}^k$ where $\boldsymbol{b} \notin A$.

Now lets assume that there is no element in $A$ that is closest to $\boldsymbol{b}$. Then for every $\varepsilon > 0$ then we could find some $L$ such that $||\boldsymbol{a}_{n_l} - \boldsymbol{b}|| < \varepsilon$ where $n_l \geq L$ and $\boldsymbol{a}_{n_l}$ is a subsequence of $\boldsymbol{a}_n$. Of course this is the definition of a limit. Unfortunately we know that all subsequences of $\boldsymbol{a}_n$

must converge to $\boldsymbol{a}$. Of course $\boldsymbol{b} \notin A$ so $\boldsymbol{b} \neq \boldsymbol{a}$. This contradiction means that we can find some $\boldsymbol{a}_m \in A$ that is closest to $\boldsymbol{b}$.

Great, now lets say the sequence that converges on $\boldsymbol{b}$ is $\boldsymbol{a}_j$. Now we know that the distance from any element in $A$ to $\boldsymbol{b}$ is at least $||\boldsymbol{a}_m - \boldsymbol{b}||$. Lets pick $\varepsilon = \frac{||\boldsymbol{a}_m - \boldsymbol{b}||}{2}$. Then for all $\boldsymbol{a}_j$ we have $||\boldsymbol{a}_j - \boldsymbol{b}|| > \varepsilon$ and so $\boldsymbol{b}$ can not be a limit. And so we have closure by contradiction.

D. If $A$ is a bounded subset of $\mathbb{R}$, show that $\sup A$ and $\inf A$ belong to $\overline{A}$.

Well $\sup A \geq \inf A$ and so $\inf A \leq a_n = \sup A - \frac{1}{n}(\sup A - \inf A) \leq \sup A$ for all $n \in \mathbb{N}\backslash\{0\}$. Notice that $a_n \in A$ for all $n$ and $\lim\limits_{n\to\infty} a_n = \sup A$. Similarly $\inf A \leq b_n = \inf A + \frac{1}{n}(\sup A - \inf A) \leq \sup A$. We see that $b_n \in A$ for every $n$ and $\lim\limits_{n\to\infty} b_n = \inf A$. And so because $\overline{A}$ contains all the limit points of $A$ then the supremum and infimum are in the closure.

J. Show that if $U$ is open and $A$ is closed, the $U \setminus A = \{\boldsymbol{x} \in U : \boldsymbol{x} \notin A\}$ is open. What can be said about $A \setminus U$?

If $U$ is open, then $U'$ is closed. And since $U'$ is closed and $A$ is closed, then $U' \cup A$ is closed. And the complement of $U' \cup A$ is open. But notice that the complement of $U' \cup A$ is $U \setminus A$. And so $U \setminus A$ is open as required.

$A \setminus U$ is equal to the complement of $A' \cup U$ which is the union of two open sets. But we don't know anything about the closure of the union of open sets in general. If $A \cap U = \emptyset$ then $A \setminus U = A$ which is closed. But if $A = [0, 2]$ and $U = [1, 2)$ then $A \setminus U = [0, 1) \cup \{2\}$ which is open.

K. Suppose that $A$ and $B$ are closed subsets of $\mathbb{R}$

(a) Show that the product set $A \times B = \{(x, y) \in \mathbb{R}^2 : x \in A \text{ and } y \in B\}$ is closed.
Lets suppose that $A \times B$ is open. Then there exists some sequence $(\boldsymbol{x}_n)_{n=1}^{\infty}$ such that every $\boldsymbol{x}_n \in A \times B$ and $\lim\limits_{n\to\infty} \boldsymbol{x}_n = \boldsymbol{x}$ where $\boldsymbol{x} \notin A \times B$. We know that $\boldsymbol{x}_n$ only converges to a point if each of it's coefficients converge. So if $\boldsymbol{x} = (x_1, x_2)$ then $\lim\limits_{n\to\infty} x_{k,1} = x_1$. Because $A$ is closed we know that $x_1 \in A$. Similarly $\lim\limits_{n\to\infty} x_{k,2} = x_2$. And again, because $B$ is closed we know that $x_2 \in B$. Well, if $x_1 \in A$ and $x_2 \in B$ then $\boldsymbol{x} = (x_1, x_2) \in A \times B$. Whoops, that contradicts our assumption. I guess $A \times B$ is closed after all.

(b) Likewise show that if both $A$ and $B$ are open, then $A \times B$ is open.
If $A$ is open, then there exists some sequence $a_n$ where $a_n \in A$ for all $n$ but $\lim\limits_{n\to\infty} a_n = a \notin A$. Similarly, if $B$ is open, then there exists some sequence $b_n$ where $b_n \in B$ for all $n$ but $\lim\limits_{n\to\infty} b_n = b \notin B$. Now we define a sequence $\boldsymbol{x}_n = (a_n, b_n)$ in $A \times B$. We know that $\lim\limits_{n\to\infty} \boldsymbol{x}_n = \boldsymbol{x} = (a, b) \notin A \times B$. And so we have found a sequence in $A \times B$ with a limit outside of $A \times B$ and then by definition $A \times B$ is open.