1.2

1. For each of the following pairs of vectors $\mathbf{x}$ and $\mathbf{y}$, calculate $\mathbf{x}\cdot\mathbf{y}$ and the angle θ between the vectors.

   d. $\mathbf{x} = (1, 4, -3), \mathbf{y} = (5, 1, 3)$

$$||\mathbf{x}||||\mathbf{y}||\cos\theta = \mathbf{x}\cdot\mathbf{y} \qquad \cos\theta = \frac{\mathbf{x}\cdot\mathbf{y}}{||\mathbf{x}||||\mathbf{y}||} \qquad \theta = \cos^{-1}\frac{\mathbf{x}\cdot\mathbf{y}}{||\mathbf{x}||||\mathbf{y}||}$$

$$\theta = \cos^{-1}\frac{5+4-9}{||\mathbf{x}||||\mathbf{y}||} \qquad \theta = \cos^{-1} 0 \qquad \theta = \frac{\pi}{2}$$

2. For each pair in exercise 1, calculate $\text{proj}_{\mathbf{y}}\mathbf{x}$ and $\text{proj}_{\mathbf{x}}\mathbf{y}$

   d. The vectors are orthogonal. The projection of either onto the other is the zero vector.

7. Suppose $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $||\mathbf{x}|| = \sqrt{2}$, $||\mathbf{y}|| = 1$, and the angle between x and y is $3\pi/4$. Show that the vectors $2\mathbf{x} + 3\mathbf{y}$ and $\mathbf{x} - \mathbf{y}$ are orthogonal.

   Using proposition 2.1 we see that $(2\mathbf{x}+3\mathbf{y})\cdot(\mathbf{x}-\mathbf{y}) = 2\mathbf{x}\cdot\mathbf{x} - 2\mathbf{x}\cdot\mathbf{y} + 3\mathbf{y}\cdot\mathbf{x} - 3\mathbf{y}\cdot\mathbf{y} = 2||\mathbf{x}||^2 + \mathbf{x}\cdot\mathbf{y} - 3||\mathbf{y}||^2$. From the definition of the angle between two vectors we can further simplify this expression to $2\cdot 2 + ||\mathbf{x}||||\mathbf{y}||\cos\frac{3\pi}{4} + 3 = 4 + \sqrt{2}\cdot(-\frac{\sqrt{2}}{2}) - 3 = 0$ and so the vectors are orthogonal.

10. Let $\mathbf{x} = (1, 1, 1, \ldots, 1) \in \mathbb{R}^n$ and $\mathbf{y} = (1, 2, 3, \ldots, n) \in \mathbb{R}^n$. Let θ_n be the angle between $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$. Find $\lim_{n\to\infty}\theta_n$. (The formulas $1 + 2 + \cdots + n = n(n+1)/2$ and $1^2 + 2^2 + \cdots + n^2 = n(n+1)(2n+1)/6$ may be useful.)

    We know that by definition $\theta_n = \cos^{-1}\frac{\mathbf{x}\cdot\mathbf{y}}{||\mathbf{x}||||\mathbf{y}||}$ and so

$$\begin{aligned}
\lim_{n\to\infty}\theta_n &= \lim_{n\to\infty}\cos^{-1}\frac{\mathbf{x}\cdot\mathbf{y}}{||\mathbf{x}||||\mathbf{y}||} \\
&= \lim_{n\to\infty}\cos^{-1}\frac{1\cdot 1 + 1\cdot 2 + \cdots + 1\cdot n}{\sqrt{n}\sqrt{1^2+2^2+\cdots+n^2}} \\
&= \lim_{n\to\infty}\cos^{-1}\frac{\sqrt{6}n(n+1)}{2\sqrt{n}\sqrt{n(n+1)(2n+1)}} \\
&= \lim_{n\to\infty}\cos^{-1}\sqrt{\frac{3(n+1)}{2(2n+1)}} = \lim_{n\to\infty}\cos^{-1}\sqrt{\frac{3(n+1/2+1/2)}{4(n+1/2)}} \\
&= \lim_{n\to\infty}\cos^{-1}\sqrt{\frac{3}{4}\left(1+\frac{1/2}{n+1/2}\right)} = \cos^{-1}\sqrt{\frac{3}{4}\left(1+\lim_{n\to\infty}\frac{1/2}{n+1/2}\right)} \\
&= \cos^{-1}\sqrt{\frac{3}{4}(1+0)} = \frac{\pi}{6}
\end{aligned}$$

11. Suppose $\mathbf{x}, \mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^n$ and $\mathbf{x}$ is orthogonal to each of the vectors $\mathbf{v}_1, \ldots \mathbf{v}_k$. Show that $\mathbf{x}$ is orthogonal to any linear combination $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$.

    We know that $\mathbf{x}$ is orthogonal if and only if the dot product is zero. So lets just find it.

$$\begin{aligned}
\mathbf{x}\cdot(c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k) &= \mathbf{x}\cdot(c_1\mathbf{v}_1) + \cdots + \mathbf{x}\cdot(c_k\mathbf{v}_k) \\
&= c_1(\mathbf{x}\cdot\mathbf{v}_1) + \cdots + c_k(\mathbf{x}\cdot\mathbf{v}_k)
\end{aligned}$$

    But then $\mathbf{x}$ is orthogonal to $\mathbf{v}_i$ for all $0 < i \le k$. Which leads us to $\mathbf{x}\cdot\mathbf{v}_i = 0$ and $\mathbf{x}\cdot(c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k) = c_1\cdot 0 + \cdots + c_k\cdot 0 = 0$. And we have our result.

13. Use the algebraic properties of the dot product to show that

$$||\mathbf{x}+\mathbf{y}||^2 + ||\mathbf{x}-\mathbf{y}||^2 = 2(||\mathbf{x}||^2 + ||\mathbf{y}||^2).$$

Interpret the result geometrically.

$$\begin{aligned}||\mathbf{x}+\mathbf{y}||^2+||\mathbf{x}-\mathbf{y}||^2 &= ||\mathbf{x}+\mathbf{y}||^2+||\mathbf{x}+(-\mathbf{y})||^2\\ &= (||\mathbf{x}||^2+2\mathbf{x}\cdot\mathbf{y}+||\mathbf{y}||^2)+(||\mathbf{x}||^2+2\mathbf{x}\cdot(-\mathbf{y})+||-\mathbf{y}||^2)\\ &= ||\mathbf{x}||^2+2\mathbf{x}\cdot\mathbf{y}+||\mathbf{y}||^2+||\mathbf{x}||^2-2\mathbf{x}\cdot\mathbf{y}+||\mathbf{y}||^2\\ &= ||\mathbf{x}||^2+||\mathbf{y}||^2+||\mathbf{x}||^2+||\mathbf{y}||^2\\ &= 2(||\mathbf{x}||^2+||\mathbf{y}||^2)\end{aligned}$$

14. Use the dot product to prove the law of cosines: As shown in Figure 2.8.

$$c^2=a^2+b^2-2ab\cos\theta$$

Let $\overline{CB}=\mathbf{a},\overline{CA}=\mathbf{b}$, and $\overline{BA}=\mathbf{c}$. Notice that $\mathbf{c}=\mathbf{b}-\mathbf{a}$. And so $c=||\mathbf{b}-\mathbf{a}||$, $a=||\mathbf{a}||$, and $b=||\mathbf{b}||$. Using corollary 2.3 from the notes and definition 2.9 from the notes we have

$$\begin{aligned}c^2&=||\mathbf{b}-\mathbf{a}||^2\\ &=||\mathbf{b}||^2-2\mathbf{b}\cdot\mathbf{a}+||\mathbf{a}||^2\\ &=||\mathbf{a}||^2+||\mathbf{b}||^2-2\mathbf{b}\cdot\mathbf{a}\frac{||\mathbf{a}||||\mathbf{b}||}{||\mathbf{a}||||\mathbf{b}||}\\ &=a^2+b^2-2ab\frac{\mathbf{a}\cdot\mathbf{b}}{||\mathbf{a}||||\mathbf{b}||}\\ &=a^2+b^2-2ab\cos\theta\end{aligned}$$

Boom.□

17. If $\mathbf{x}=(x_1,x_2)\in\mathbb{R}^2$, set $\rho(\mathbf{x})=(-x_2,x_1)$.

   a. Check that $\rho(\mathbf{x})$ is orthogonal to $\mathbf{x}$. (Indeed, $\rho(\mathbf{x})$ is obtained by rotating $\mathbf{x}$ an angle $\pi/2$ counterclockwise.)

$$\mathbf{x}\cdot\rho(\mathbf{x})=(x_1,x_2)\cdot(-x_2,x_1)=-x_1x_2+x_1x_2=0$$

   b. Given $\mathbf{x},\mathbf{y}\in\mathbb{R}^2$,show that $\mathbf{x}\cdot\rho(\mathbf{y})=-\rho(\mathbf{x})\cdot\mathbf{y}$. Interpret this statement geometrically.
   Let $\mathbf{y}=(y_1,y_2)$. Then

$$\mathbf{x}\cdot\rho(\mathbf{y})=(x_1,x_2)\cdot(-y_2,y_1)=-x_1y_2+x_2y_1=-(-x_2y_1+x_1y_2)=-\rho(\mathbf{x})\cdot\mathbf{y}$$

18. Prove the *triangle inequality*: For any vectors $\mathbf{x},\mathbf{y}\in\mathbb{R}^n$, $||\mathbf{x}+\mathbf{y}||\le||\mathbf{x}||+||\mathbf{y}||$. (*Hint:* Use the dot product to calculate $||\mathbf{x}+\mathbf{y}||^2$.)
We know from Cauchy-Schwartz that $|\mathbf{x}\cdot\mathbf{y}|\le||\mathbf{x}||||\mathbf{y}|||$ and that $\mathbf{x}\cdot\mathbf{y}\le|\mathbf{x}\cdot\mathbf{y}|$. Double both sides and we have $2\mathbf{x}\cdot\mathbf{y}\le2||\mathbf{x}||||\mathbf{y}||$. Of course $\mathbf{x}\cdot\mathbf{x}=||\mathbf{x}||^2\ge0$ and $\mathbf{y}\cdot\mathbf{y}=||\mathbf{y}||^2\ge0$ which leads us to $\mathbf{x}\cdot\mathbf{x}+2\mathbf{x}\cdot\mathbf{y}+\mathbf{y}\cdot\mathbf{y}\le||\mathbf{x}||^2+2||\mathbf{x}||||\mathbf{y}||+||\mathbf{y}||^2$. Factoring we get $(\mathbf{x}+\mathbf{y})\cdot(\mathbf{x}+\mathbf{y})=||\mathbf{x}+\mathbf{y}||^2\le(||\mathbf{x}||+||\mathbf{y}||)^2$. And finally, because $||\mathbf{x}+\mathbf{y}||\ge0$ and $||\mathbf{x}||+||\mathbf{y}||\ge0$ it is safe to say that $||\mathbf{x}+\mathbf{y}||\le||\mathbf{x}||+||\mathbf{y}||$

1.3 4. Find a normal vector to the given hyperplane and use it to find the distance from the origin to the hyperplane.

   a. $\mathbf{x}=(-1,2)+t(3,2)$

$$\begin{aligned}(-2,3)\cdot\mathbf{x}&=(-2,3)\cdot(-1,2)+t(3,2)\cdot(-2,3)\\ (-2,3)\cdot\mathbf{x}&=7\end{aligned}$$

$$||\text{proj}_{\mathbf{a}}\mathbf{x}_0|| = \frac{|c|}{||\mathbf{a}||} = \frac{7}{\sqrt{4+9}} = \frac{7\sqrt{13}}{13}$$

And so we have a normal vector of $(-2, 3)$ and a distance of $\frac{7\sqrt{13}}{13}$

b. The plane in $\mathbb{R}^3$ given by the equation $2x_1 + x_2 - x_3 = 5$

$$(2, 1, -1) \cdot (x_1, x_2, x_3) = 5$$
$$||\text{proj}_{\mathbf{a}}\mathbf{x}_0|| = \frac{5}{\sqrt{4+1+1}}$$

And so we have a normal of $(2, 1, -1)$ and a distance of $\frac{5\sqrt{6}}{6}$

c. The plane passing through $(1, 2, 2)$ and orthogonal to the line $\mathbf{x} = (3, 1, -1) + t(-1, 1, -1)$

$$(-1, 1, -1) \cdot \mathbf{x} = (-1, 1, -1) \cdot (1, 2, 2) = -1$$
$$||\text{proj}_{\mathbf{a}}\mathbf{x}_0|| = \frac{1}{\sqrt{1+1+1}}$$

Looks like a normal of $(-1, 1, -1)$ and a distance of $\frac{\sqrt{3}}{3}$

d. The plane passing through $(2, -1, 1)$ and orthogonal to the line $\mathbf{x} = (3, 1, 1) + t(-1, 2, 1)$
The normal is $(-1, 2, 1)$ and has a distance of $\frac{|-2-2+1|}{\sqrt{4+1+1}} = \frac{3\sqrt{6}}{6}$

e. The plane spanned by $(1, 1, 4)$ and $(2, 1, 0)$ and passing through $(1, 1, 2)$

$$\begin{aligned} a_1 + a_2 + 4a_3 &= 0 & 2a_1 + a_2 &= 0 \\ a_1 - 2a_1 + 4a_3 &= 0 & 4a_3 &= a_1 \\ a_3 &= 1 & a_1 &= 4 \\ a_2 &= -8 & \mathbf{a} &= (4, -8, 1) \end{aligned}$$

So our normal vector is $(4, -8, 1)$ and our distance is $\frac{|4-8+2|}{\sqrt{16+64+1}} = \frac{2}{9}$

f. The plane spanned by $(1, 1, 1)$ and $(2, 1, 0)$ and passing through $(3, 0, 2)$

$$\begin{aligned} a_1 + a_2 + a_3 &= 0 & 2a_1 + a_2 &= 0 \\ a_1 &= 1 & a_2 &= -2 \\ a_3 &= 1 \end{aligned}$$

Normal is $(1, -2, 1)$ and distance is $\frac{3+2}{\sqrt{1+4+1}} = \frac{5\sqrt{6}}{6}$.

g. The hyperplane in $\mathbb{R}^4$ spanned by $(1, -1, 1, -1), (1, 1, -1, -1)$ and $(1, -1, -1, 1)$ and passing through $(2, 1, 0, 1)$

$$\begin{aligned} a_1 - a_2 - a_3 + a_4 &= 0 & a_1 - a_2 + a_3 - a_4 &= 0 & a_1 + a_2 - a_3 - a_4 &= 0 \\ a_1 - a_2 + a_3 &= a_4 & a_1 + a_2 - a_3 &= a_4 & 2a_3 &= 2a_2 \\ a_1 - a_2 - a_2 + (a_1 - a_2 + a_2) &= 0 & 2a_1 &= 2a_2 & a4 &= a_1 + a_1 - a_1 \end{aligned}$$

So we let $(1, 1, 1, 1)$ be the normal vector and $\frac{|2+1+0+1|}{\sqrt{4}} = 2$

6. a. Give the general solution of the equation $x_1 + 5x_2 - 2x_3 = 0$ in $\mathbb{R}^3$ (as a linear combination of two vectors, as in the text).
   Note that $\mathbf{0}$ is on the plane and so we just need to find two vectors that aren't parallel and are on the plane, and we have our solution. Lets take $(-3, 1, 1)$ and $(2, 0, 1)$. Then our equation is $\mathbf{x} = x_2(-3, 1, 1) + x_3(2, 0, 1)$
   b. Find a specific solution of the equation $x_1 + 5x_2 - 2x_3 = 3$ in $\mathbb{R}^3$; give the general solution.
   c. Give the general solution of the equation $x_1 + 5x_2 - 2x_3 + x_4 = 0$ in $\mathbb{R}^4$. Now give the general solution of the equation $x_1 + 5x_2 - 2x_3 + x_4 = 3$
7. The equation $2x_1 - 3x_2 = 5$ defines a line in $\mathbb{R}^2$.
   a. Give a normal vector $\mathbf{a}$ to the line.
   b. Find the distance from the origin to the line by using projection.
   c. Find the point on the line closest to the origin by using the parametric equation of the line through $\mathbf{0}$ with direction vector $\mathbf{a}$. Double-check your anser to part *b*.
   d. Find the distance from the point $\mathbf{w} = (3, 1)$ to the line by using projection.
   e. Find the point on the line closest to $\mathbf{w}$ by using the parametric equation of the line through $\mathbf{w}$ with direction vector $\mathbf{a}$. Double-check your answer to part *d*
9. The equation $2x_1 + 2x_2 - 3x_3 + 8x_4 = 6$ defines a hyperplane in $\mathbb{R}^4$.
   a. Give a normal vector $\mathbf{a}$ to the hyperplane.
   b. Find the distance from the origin to the hyperplane using projection.
   c. Find the point on the plane closest to the origin by using the parametric equation of the line through $\mathbf{0}$ with direction vector $\mathbf{a}$. Double-check your answer to part *b*.
   d. Find the distance from the point $\mathbf{w} = (1, 1, 1, 1)$ to the hyperplane by using projection.
   e. Find the pointon the plane closest to $\mathbf{w}$ by using the parametric equation of the line through $\mathbf{w}$ with direction vector $\mathbf{a}$. Double-check your answer to part *d*

10.

11.

12.

13.