Homework 4

Jon Allen

September 26, 2014

Section 2.7: G, H, J*
2.8: B, C, E*, H (for C, you can use what you know about series from Calculus 2).

2.7 G. Let $(x_n)_{n=1}^{\infty}$ be a sequence of real numbers. Suppose that there is a real number L such that $L = \lim_{n\to\infty} x_{3n-1} = \lim_{n\to\infty} x_{3n+1} = \lim_{n\to\infty} x_{3n}$. Show that $\lim_{n\to\infty} x_n$ exists and equals L.

Let us take any $\varepsilon > 0$. Now we find N_1 such that $|x_{3n} - L| < \varepsilon$ for all $n \geq N_1$. And find N_2 such that $|x_{3n+1} - L| < \varepsilon$ for all $n \geq N_2$. Finally find N_3 such that $|x_{3n-1} - L| < \varepsilon$ for all $n \geq N_3$. My brain hurts and I don't feel like working out the logic for an optimal N so let's take $N = 3 \cdot \max\{N_1, N_2, N_3\} + 1$. Then $|x_n - L| < \varepsilon$ for all $n \geq N$ and we have our result.

H. Let $(x_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$. Suppose that there is a number L such that every subsequence $(x_{n_k})_{k=1}^{\infty}$ has a subsubsequence $\left(x_{n_{k(l)}}\right)_{l=1}^{\infty}$ with $\lim_{n\to\infty} x_{n_{k(l)}} = L$. Show that the whole sequence converges to L. Hint: If not, you could find a subsequence bounded away from L.

If (x_n) does not converge to L then there would be a subsequence that doesn't converge on L and which does not contain a subsequence that converges to L. Because we can always find such a subsubsequence we know the whole thing must converge.

J. Suppose $(x_n)_{n=1}^{\infty}$ is a sequence in $\mathbb{R}$, and that L_k are real numbers with $\lim_{k\to\infty} L_k = L$. If for each $k \geq 1$, there is a subsequence of $(x_n)_{n=1}^{\infty}$ converging to L_k, show that some subsequence converges to L. Hint: Find an increasing sequence n_k such that $|x_{n_k} - L| < \frac{1}{k}$.

We choose (x_{n_k}) from (x_n) such that as k increases, x_{n_k} comes from the subsequence of (x_n) that approaches L_k and is bigger than $x_{n_{k-1}}$. So then as k increases, L_k approaches L and so will x_{n_k}.

2.8 B. Give a sequence (a_n) such that $\lim_{n\to\infty} |a_n - a_{n+1}| = 0$, but the sequence does not converge.

$a_n = \log n$

C. Let (a_n) be a sequence such that $\lim_{N\to\infty} \sum_{n=1}^{N} |a_n - a_{n+1}| < \infty$. Show that (a_n) is Cauchy.

First we note that $\lim_{N\to\infty} \sum_{n=1}^{N} |a_n - a_{n+1}| \geq 0$ because we are adding only nonnegative terms. So our limit is in the reals (particularly it's not $-\infty$). Lets call it L.

$$\lim_{N\to\infty} \sum_{n=1}^{N} |a_n - a_{n+1}| = L$$

So then for all $\varepsilon \in \mathbb{R}$ where $\varepsilon > 0$ there exists some $N_1 \in \mathbb{N}$ such that for all $N \geq N_1$ we have

$$\left| \sum_{n=1}^{N} |a_n - a_{n+1}| - L \right| < \varepsilon$$

And so then playing a bit when $N > N_1$ we come up with

$$\left| \sum_{n=1}^{N_1} |a_n - a_{n+1}| + \sum_{n=N_1+1}^{N} |a_n - a_{n+1}| - L \right| < \varepsilon$$

$$\left|\sum_{n=N_1+1}^{N}|a_n - a_{n+1}| - \left(L - \sum_{n=1}^{N_1}|a_n - a_{n+1}|\right)\right| < \varepsilon$$

Note that because our series is weakly increasing, L must be greater than our series for all N. This leads us to the following observations.

$$\begin{aligned}
L - \sum_{n=1}^{N_1}|a_n - a_{n+1}| &< \varepsilon \\
\sum_{n=1}^{N_1}|a_n - a_{n+1}| + \sum_{n=N_1+1}^{N}|a_n - a_{n+1}| &< L \\
\sum_{n=N_1+1}^{N}|a_n - a_{n+1}| &< L - \sum_{n=1}^{N_1}|a_n - a_{n+1}| \\
\sum_{n=N_1+1}^{N}|a_n - a_{n+1}| &< L - \sum_{n=1}^{N_1}|a_n - a_{n+1}| < \varepsilon \\
\sum_{n=N_1+1}^{N}|a_n - a_{n+1}| &< \varepsilon \\
\left|\sum_{n=N_1+1}^{N} a_n - a_{n+1}\right| < \sum_{n=N_1+1}^{N}|a_n - a_{n+1}| &< \varepsilon \\
|(a_{N_1+1} - a_{N_1+2}) + (a_{N_1+2} - a_{N_1+3}) + \cdots + (a_{N-1} - a_N)\ + (a_N - a_{N+1})| &< \varepsilon \\
|a_{N_1+1} - a_{N+1}| &< \varepsilon
\end{aligned}$$

And so we see that for every $\varepsilon > 0$ there is an integer N such that $|a_n - a_m| < \epsilon$ for all $m, n \geq N$. $\square$

E. Suppose that (a_n) is a sequence such that $a_{2n} \leq a_{2n+2} \leq a_{2n+3} \leq a_{2n+1}$ for all $n \geq 0$. Show that this sequence is Cauchy if and only if $\lim\limits_{n\to\infty} |a_n - a_{n+1}| = 0$

   Lets partition our sequence into even and odd elements. We notice that $a_{2n} \leq a_{2N+1}$ for any $N \in \mathbb{N}$ and for all $n \geq N$. Further we notice that $a_{2n+1} \geq a_{2N}$ for any $N \in \mathbb{N}$ and for all $n \geq N$. We also observe that the evens are monotone increasing ($a_{2n} \leq a_{2n+2}$) while the odds are monotone decreasing ($a_{2n+1} \geq a_{2n+3}$).

   Now because the evens are bounded above by a_1 and the odds are bounded below by a_0 we know that each of our sequences has a limit. Lets say $\lim\limits_{n\to\infty} a_{2n} = L_2$ and $\lim\limits_{n\to\infty} a_{2n+1} = L_1$. By definition then for all $\frac{\varepsilon}{2} > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$

$$\begin{aligned}
|a_{2n} - L_2| &< \frac{\varepsilon}{2} \\
|a_{2n+1} - L_1| &< \frac{\varepsilon}{2} \\
|a_{2n} - L_2| + |a_{2n+1} - L_1| &< \varepsilon \\
|a_{2n} - L_2| + |L_1 - a_{2n+1}| &< \varepsilon \\
|a_{2n} - L_2 + L_1 - a_{2n+1}| \leq |a_{2n} - L_2| + |L_1 - a_{2n+1}| &< \varepsilon \\
|(a_{2n} - a_{2n+1}) - (L_2 - L_1)| &< \varepsilon \\
||a_{2n} - a_{2n+1}| - |L_2 - L_1|| \leq |(a_{2n} - a_{2n+1}) - (L_2 - L_1)| &< \varepsilon
\end{aligned}$$

$$||a_{2n} - a_{2n+1}| - |L_2 - L_1|| < \varepsilon$$

So then $\lim_{n\to\infty} |a_{2n} - a_{2n+1}| = |L_2 - L_1|$. Similarly we can show $\lim_{n\to\infty} |a_{2n+1} - a_{2n+2}| = |L_2 - L_1|$. And so we have that $\lim_{n\to\infty} |a_n - a_{n+1}| = |L_2 - L_1|$.

Let us assume that $\lim_{n\to\infty} |a_n - a_{n+1}| = 0$. Then $L_1 = L_2 = L$. Now because the odds provide an upper bound for the evens, and the evens provide a lower bound for the odds, we know from the squeeze theorem that $\lim_{n\to\infty} a_n = L$. So our sequence is convergent and is therefore Cauchy.

Now let's assume that our sequence is Cauchy. Then for every $\varepsilon > 0$ there is an integer N such that $|a_n - a_m| < \varepsilon$ for all $m, n \geq N$. In particular

$$|a_n - a_{n+1}| < \varepsilon$$
$$||a_n - a_{n+1}| - 0| \leq |a_n - a_{n+1}| < \varepsilon$$

And so $\lim |a_n - a_{n+1}| = 0$. □

H. Let $a_0 = 0$ and set $a_{n+1} = \cos(a_n)$ for $n \geq 0$. Try this on your calculator (use radian mode!).

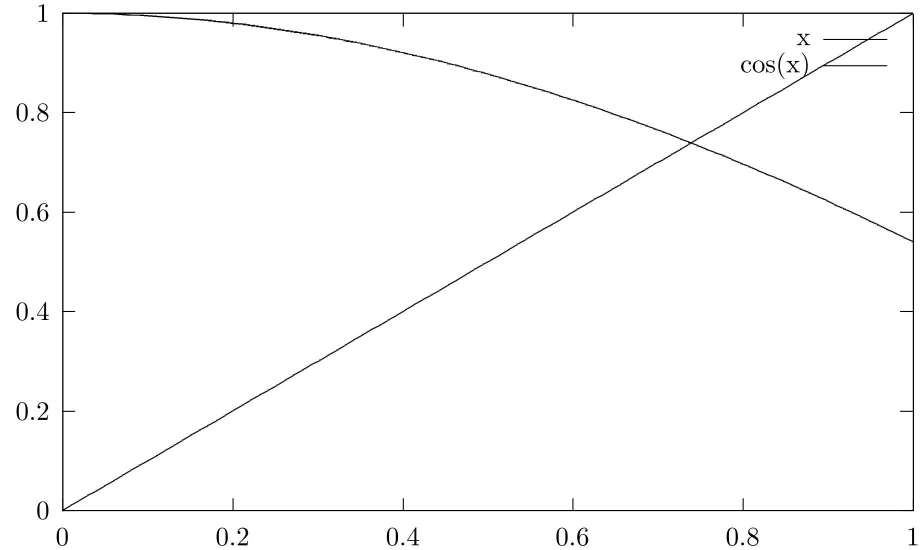
(a) Show that $a_{2n} \leq a_{2n+2} \leq a_{2n+3} \leq a_{2n+1}$ for all $n \geq 0$.
We know that $a_{n+1} = \cos a_n$. First we note that a_n is bounded by $[0, 1]$. We know this because $\cos 0$ gives us a 1 back, which is the maximum value $\cos a_n$ can give us. Furthermore, $0 \leq \cos x \leq 1$ for all $0 \leq x \leq \pi/2$. So a_n can never escape the $[0, 1]$ bounds. Lets assume that $a_n < L$ where $0 = L - \cos L$. Then $\cos a_n > L$. Similarly, if $a_n > L$ then $\cos a_n < L$. So if $a_n < L$ then $a_{n+1} = \cos a_n > L$ and $a_{n+2} = \cos\cos a_n < L$. Because $a_0 = 0$ we know that $a_0 < L$ and by extension $a_{2n} < L$ while $a_{2n+1} > L$ for all n.
Now we assume that $a_2n < a_{2n+2}$. So $a_{2n} < \cos\cos a_{2n}$ and $\cos a_{2n} > \cos\cos\cos a_{2n}$ because in our domain, as the angle gets smaller, the cosine gets larger. Thus $a_{2n+1} > \cos\cos a_{2n+1} = a_{2n+3}$ and $a_{2n+2} < a_{2n+4}$. Notice that $\cos 0 = 1$ and $\cos 1 > \cos \pi/2 = 0$. So then $a_0 < a_2$. Good enough for a basis. And induction then gives us our result for all elements in the sequence.

(b) Use the Mean Value Theorem to find an explicit number $r < 1$ such that $|a_{n+2} - a_{n+1}| \leq r\,|a_n - a_{n+1}|$ for all $n \geq 0$. Hence show that this sequence is Cauchy.

$$\frac{|a_{n+2} - a_{n+1}|}{|a_n - a_{n+1}|} = \left|\frac{\cos\cos a_n - \cos a_n}{\cos a_n - a_n}\right|$$
$$1 \geq \left|\frac{(\sin x + 1)\cos(\cos(x)) + \sin(x)(\cos x - x)\sin(\cos x) - \cos x - x\sin x}{(x - \cos x)^2}\right|$$

And solve for x
I'd like to note that the result in part (a) combined with the squeeze theorem actually already shows that it has a limit and is Cauchy. It even tells us what the limit is, as shown in the next part.

(c) Describe the limit geometricly as the intersection point of two curves.
It is the intersection of $\cos x$ and x


x
cos(x)
0
0.2
0.4
0.6
0.8
1