Notes

February 24, 2014

lesson 12

definition

Given $f(x)$ on $\mathbb{R}$.

$$\mathcal{F}[f] = F(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i\xi x} f(x)\,\mathrm{d}x$$

Inverse transform recovers $f(x)$ from $F(\xi)$:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{+i\xi x} F(\xi)\,\mathrm{d}\xi \qquad \text{Deep Theorem}$$

problem

page 93

$$\begin{array}{rrrr} PDE & u_t = \alpha^2 u_{xx} & -\infty < x < \infty & 0 < t < \infty \\ IC & u(x,0) = \phi(x) & -\infty < x < \infty & \end{array}$$

apply $\mathcal{F}$ to pde. $U(\xi, t) = \mathcal{F}[u(x,t)]$. Use property 3 (derivative): $\mathcal{F}[u_{xx}] =$
How are $\mathcal{F}[f']$ and $\mathcal{F}[f]$ related? page 91.

$$\begin{aligned} \mathcal{F}[f'] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i\xi x} f'(x)\,\mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}} \left[e^{-i\xi x} f(x)\right]_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} (-i\xi) e^{-i\xi x} f(x)\,\mathrm{d}x \end{aligned}$$

note: implicit conditions on $f(x)$ to insure integrals exist $f(+\infty) = f(-\infty) = 0$

$$= 0 + i\xi \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i\xi x} f(x)\,\mathrm{d}x$$

property 3

$$\begin{aligned} \mathcal{F}[f'] &= i\xi \mathcal{F}[f] \\ \mathcal{F}[f''] &= \xi^2 \mathcal{F}[f] \end{aligned}$$

So $\mathcal{F}[u_{xx}] = -\xi^2 U(\xi, t)$. $\frac{\mathrm{d}U}{\mathrm{d}t} = -\alpha^2 \xi^2 U$ with $U(\xi, 0) = \phi(\xi)$

step 2

solve prolem $U(\xi, t) = \phi(\xi)e^{-\alpha^2\xi^2 t}$.

step 3

invert transform

property 4

convolution theorem

definition: given $f(x), gx)$ on $\mathbb{R}$

$$f * g(x) = 1\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x-s)g(s)\,\mathrm{d}s$$

theorem

$$\mathcal{F}[f * g(x)] = \mathcal{F}[f]\mathcal{F}[g] = F(\xi)G(\xi) \qquad \text{deep theorem}$$

sample calculation: find the convolution of two functions. Text example

$$\left.\begin{aligned} f(x) &= x \\ g(x) &= e^{-x^2} \end{aligned}\right\} f * g(x) = \frac{x}{\sqrt{2}}$$

$$\begin{aligned} f * g(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x-u)g(u)\,\mathrm{d}u \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} (x-u)e^{-u^2}\,\mathrm{d}u \\ &= \frac{1}{\sqrt{2\pi}} \left[x \int_{-\infty}^{+\infty} e^{-u^2}\,\mathrm{d}u - \int_{-\infty}^{+\infty} ue^{-u^2}\,\mathrm{d}u\right] && \text{odd integral} \\ &= \frac{x}{\sqrt{2\pi}}\sqrt{\pi}f(x) && = e^{-x^2} \\ g(x) &= x \\ f * g(x) &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-(x-u)^2} u\,\mathrm{d}u \end{aligned}$$

property:$f * g(x) = g * f(x)$ commutativity

$$\begin{aligned} f * g(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x-u)g(u)\,\mathrm{d}u \\ u &= -\infty \qquad \leftarrow x-u \\ x-u &= +\infty \end{aligned}$$

note that the definition in the book has a negative i and mathematica doesn't. and then that makes the signs flipped on the inverse transform as well. CAREFUL!!!

$$u(x,t) = \int_{-\infty}^{\infty} \phi(x-u)\frac{1}{\sqrt{\pi}}\frac{1}{2\alpha\sqrt{t}}e^{-\frac{u^2}{2/\alpha^2 t}}\,\mathrm{d}u$$

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}}\frac{1}{2\alpha\sqrt{t}}e^{-\frac{u^2}{2/\alpha^2 t}}\,\mathrm{d}u = \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}}\frac{1}{2\alpha\sqrt{t}}e^{-\frac{u^2}{2/\alpha^2 t}}\,\mathrm{d}u$$

$$u(x,t) = \int_{-\infty}^{\infty} \phi(x) f(x-u)\,\mathrm{d}u \text{ positive with unit area}$$