Notes

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14b

$m\mathbb{Z} \cdot n\mathbb{Z} = (mn)\mathbb{Z}$
$\subseteq$
let $x \in (n\mathbb{Z})(m\mathbb{Z})$ so $x = \sum_{i=1}^{k} a_i b_i$ and $n|a_i$ and $m|b_i$ and so $nm|a_i b_i$ and so $nm|x$ and so $n\mathbb{Z}m\mathbb{Z} \subseteq nm\mathbb{Z}$
$\supseteq$
let $y \in mn\mathbb{Z}$ and so $y = mnz_0 = (n \cdot 1)(m \cdot z_0) \in n\mathbb{Z}m\mathbb{Z}$

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gaussian integers are "complex" integers
$\mathbb{Z}[i]/\langle p \rangle = \{[a + bi] | a, b \in \mathbb{Z}\} = \{[a] + [b][i]\}$
$[a] + [b]i = [a'] + [b']i \to (a + bi) - (a' + bi) \in \langle p \rangle = p(c + di) \to a - a' = pc$ and $b - b' = pd \to a - a' \in \langle p \rangle \to [a] = [a']$ and similarly with b.

define

R comm ring and I ideal where $I \neq R$ then I prime ideal means that $ab \in I \to a \in I$or $b \in I$

example

$R = \mathbb{Z}$, $p \in \mathbb{Z}$ then $p\mathbb{Z}$ is a prime ideal because if $ab \in I$ then wlog $p|ab \to p|b \to b \in I$.

example

claim $n\mathbb{Z}$ prime ideal then n is prime
assume n not prime and $n \neq 0$. $n = \alpha\beta$ where $1 < \alpha < n, 1 < \beta < n, \alpha, \beta \in \mathbb{Z}$. then $\alpha\beta \in n\mathbb{Z}$ but $\alpha \notin n\mathbb{Z}$ and $\beta \notin n\mathbb{Z}$. notice that $n \neq 0$ is key here.

example

claim $\langle 0 \rangle$ is a prime ideal. $ab = 0 \to a = 0$ or $b = 0$ because R is an integral domain
observation: R commutative ring theen $R$is an integral domain iff $\langle 0 \rangle$is a prime ideal.

$\mathbb{Z}$

$n\mathbb{Z} \subseteq m\mathbb{Z} \Leftrightarrow m|n$
take$n = p$ prime nmber. $p\mathbb{Z} \subseteq m\mathbb{Z} \Leftrightarrow m|p \Leftrightarrow m \pm 1$ or $m = \pm p$ and so $m\mathbb{Z} = \mathbb{Z}$ or $m\mathbb{Z} = p\mathbb{Z}$

definition

if J is an ideal of R and $J \neq R$we say that J is a maximal ideal of R if for every ideal I of R $j \subseteq I \subseteq R \rightarrow I = J$ or $I = R$.

$J = p\mathbb{Z}$

note that $\langle 0 \rangle$ is not a maximal ideal.

claim

every maximal ideal is a prime ideal

proposition

let I be a proper ideal ofa commutative ring R (proper means different from ring itself). then I is maximum ideal iff R/I is a field. also I is a prime ideal iff R/I is an integral domain. finally I maximal implies I is a prime ideal.

proof 1

R/I field iff R/I has only the two trivial ideals (can you prove this?)
and this is true iff I is maximal.

proof 2

assume I is a prime ideal. then $[x][y] \in R/I$ then $[xy] = [0]$ in R/I and so $xy \in I$ means that $x \in I$ or $y \in I$ and so $[x] = 0$ or $[y] = 0$ so R/I is an integral domain.

proof 3

R/I integral domain
then $[xy] = [0]$ in R/I $[x][y] = [0]$ in R/I so $[x] = 0$ or$[y] = [0]$ then $x \in I$ or $y \in I$ hence I is a prime ideal.

thrm

if R is a principle ideal domain and P is a prime ideal different from zero. then P is maximal.

proof

$P \subseteq I \subseteq R$ write $P = aR$, $I = bR$. P is non-zero and so $a \neq 0$ and $P \in I \rightarrow aR \in bR$. then $a \in bR$ we write $a = br, r \in R$ then $a = br \in P$ and so $b \in P$ or $r \in P$ because P is prime ideal. if $b \in P$ then $a|b$ and so $I \subseteq P$ but $P \subseteq I$ and so $I = P$. if $r \in P$ then $r \in aR$ and $r = as, s \in R$ and then $a = br = b(as)$ and so $a(1 - bs) = 0$ but $a \neq 0$ and because we are in an integral domain then $1 - bs = 0$. and so $1 = bs \in I$ and $1 \in I$ and so $I = R$.