10.5

G. Recall that a norm is strictly convex if $||x|| = ||y|| = ||(x+y)/2||$ implies that $x = y$.

  (a) Suppose that V is a vector space with a strictly convex norm and M is a finite-dimensional subspace of V. Prove that each $v \in V$ has a unique closest point in M.

  We choose two points $u, w \in M$ such that $||u-v|| = ||w-v|| \le ||z-v||$ for all $z \in M$. In particular $||\frac{u+w}{2} - v|| \ge ||u - v||$. Some algebraic manipulation gives us $||\frac{u+w}{2} - v|| = ||\frac{u-v+w-v}{2}|| = \frac{1}{2}||(u-v) + (w - v)|| \le \frac{1}{2}||u - v|| + \frac{1}{2}||w - v|| = ||u - v||$. And so $||\frac{u-v+w-v}{2}|| = ||u - v|| = ||w - v||$ and because V is strictly convex then we know that $u - v = w - v$ or $u = w$. And so we know there is only one closest point to v in M

  (b) Prove that an inner product norm is strictly convex.

  First, observe the following identity:

  $$||x + y||^2 = \langle x + y, x + y\rangle = \langle x, x\rangle + 2\langle x, y\rangle + \langle y, y\rangle$$
  $$||x - y||^2 = \langle x - y, x - y\rangle = \langle x, x\rangle - 2\langle x, y\rangle + \langle y, y\rangle$$
  $$||x + y||^2 + ||x - y||^2 = 2\langle x, x\rangle + 2\langle y, y\rangle = 2(||x||^2 + ||y||^2)$$
  $$||x + y||^2 = 2(||x||^2 + ||y||^2) - ||x - y||^2$$

  Now if we assume that $||x|| = ||y|| = ||(x + y)/2|| = c$ then we can use the above identity to obtain:

  $$c^2 = \frac{1}{4}||x + y||^2$$
  $$4c^2 = ||x + y||^2$$
  $$2c^2 + 2c^2 = 2||x||^2 + 2||y||^2 - ||x - y||^2 0 \qquad = ||x - y||^2$$
  $$0 = \langle x - y, x - y\rangle$$

  But $\langle x - y, x - y\rangle = 0$ if and only if $x - y = 0$ so $x = y$

  (c) Show by example that $C[0, 1]$ is not strictly convex.

  $\cos x, 1 \in C[0, 1]$ and $||\cos x||_\infty = ||1||_\infty = ||(1 + \cos x)/2||_\infty = 1$ but $\cos x \ne 1$