

Homework 3

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2.6 F. Let a, b be positive real numbers. Set $x_0 = a$ and $x_{n+1} = (x_n^{-1} + b)^{-1}$ for $n \geq 0$.

(a) Prove that x_n is monotone decreasing.

proof

If x_n is monotone decreasing, then $x_n \geq x_{n+1}$ for all $n \geq 0$.

$$x_{n+1} = (x_n^{-1} + b)^{-1} = \frac{1}{\frac{1+bx_n}{x_n}} = \frac{x_n}{1+bx_n}$$

Note that if x_n and b are positive, then so is x_{n+1} . Now we are told that x_0 and b are positive, so we know that all x_n are positive. This means of course that $1 + bx_n > 1$ which in turn means that $x_n > \frac{x_n}{1+bx_n} = x_{n+1}$. Indeed it appears that not only is x_n monotone decreasing, it is strictly monotone decreasing. \square

(b) Prove that the limit exists and find it.

proof

As we noted in the previous proof, x_n is positive for all $n \geq 0$. This implies that $x_n > 0$ and is therefore bounded from below. Because x_n is monotone decreasing and bounded from below, it has a limit. \square

solution

$$L = \lim_{n \rightarrow \infty} x_{n+1} = \lim_{n \rightarrow \infty} (x_n^{-1} + b)^{-1} = \left(\left(\lim_{n \rightarrow \infty} x_n \right)^{-1} + b \right)^{-1} = (L^{-1} + b)^{-1}$$

$$L = \frac{1}{\frac{1}{L} + b}$$

$$1 = 1 + bL$$

$$0 = bL$$

So then $\lim_{n \rightarrow \infty} x_n = 0$.

G. Let $a_n = (\sum_{k=1}^n 1/k) - \log n$ for $n \geq 1$. **Euler's constant** is defined as $\gamma = \lim_{n \rightarrow \infty} a_n$. Show that $(a_n)_{n=1}^{\infty}$ is decreasing and bounded below by zero, and so this limit exists. HINT: Prove that $1/(n+1) \leq \log(n+1) - \log n \leq 1/n$

M. Suppose that $(a_n)_{n=1}^{\infty}$ has $a_n > 0$ for all n . Show that $\limsup a_n^{-1} = (\liminf a_n)^{-1}$.