Eigenvectors and Diagonalization

Definition 1. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. A subspace $V \leq \mathbb{R}^n$ is called *invariant* under T if $T(V) \subseteq V$.

Examples 2. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation.
(1) If $V = \ker T$, then $T(V) = \{\mathbf{0}\} \subseteq V$.
(2) If $V = \operatorname{Im} T$, then $T(\mathbf{b}) = T(T(\mathbf{x})) \in V$ for all $\mathbf{b} \in V$.
(3) It is clear that $\mathbb{R}^n$ and $\{\mathbf{0}\}$ are invariant subspaces.

Definition 3. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. A real number scalar $\lambda \in \mathbb{R}$ is called an *eigenvalue* of T, if there eists a nonzero vector $\mathbf{v} \in \mathbb{R}^n$ such that $T(\mathbf{v}) = \lambda \mathbf{v}$.

Theorem 4. The following statements are equivalent for a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$.
(1) T has an eigenvalue $\lambda \in \mathbb{R}$.
(2) The linear transformation $T - \lambda 1_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n$ given by $(T - \lambda 1_{\mathbb{R}^n})(\mathbf{x}) = T(\mathbf{x}) - \lambda \mathbf{x}$ is not invertible.
(3) There exists a subspace $V \leq \mathbb{R}^n$ such that $\dim(V) = 1$ and V is invariant under T.

Proof. We prove (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1).
(1) $\Rightarrow$ (2) If T has an eigenvalue λ, then there eists a nonzero vector $\mathbf{v} \in \mathbb{R}^n$ such that $T(\mathbf{v}) = \lambda \mathbf{v}$. It follows that $T(\mathbf{v}) - \lambda \mathbf{v} = \mathbf{0}$ and so $(T - \lambda 1_{\mathbb{R}^n})(\mathbf{v}) = \mathbf{0}$. But then, $\mathbf{v} \in \ker(T - \lambda 1_{\mathbb{R}^n})$. That is, $\ker(T - \lambda 1_{\mathbb{R}^n})$ is non-trivial and therefore, $T - \lambda 1_{\mathbb{R}^n}$ is not invertible.
(2) $\Rightarrow$ (3) If $T - \lambda 1_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n$ is not invertible, then it is not 1-1 since any transformation $\mathbb{R}^n \to \mathbb{R}^n$ is 1-1 if and only if it is onto. It follows that $\ker(T - \lambda 1_{\mathbb{R}^n})$ is non-trivial and so we can find an non-zero $\mathbf{v} \in \mathbb{R}^n$ such that $(T - \lambda 1_{\mathbb{R}^n})(\mathbf{v}) = \mathbf{0}$. This is the same thing as saying $T(\mathbf{v}) = \lambda \mathbf{v}$. Take $V = \mathbb{R}\mathbf{v}$ and observe that $T(V) \subseteq V$ since $T(c\mathbf{v}) = cT(\mathbf{v}) = c(\lambda \mathbf{v}) = (c\lambda)\mathbf{v} \in \mathbb{R}\mathbf{v}$.

Corollary 5. The linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$.has eigenvalue $\lambda = 0$ if and only if T is not invertible.

Definition 6. Let $\lambda \in \mathbb{R}$ be an eigenvalue of the transformation $T : \mathbb{R}^n \to \mathbb{R}^n$. The subspace $\ker(T - \lambda 1_{\mathbb{R}^n}) = \{\mathbf{v} \in \mathbb{R}^n : T(\mathbf{v}) = \lambda \mathbf{v}\}$ is called the *eigenspace* of T and its elements are called *eigenvectors*.

Theorem 7. Let V be a finite dimensional vector space and let $T : \mathbb{R}^n \to \mathbb{R}^n$. If $\{\lambda_1, ..., \lambda_m\}$ is the set of distinct eigenvalues of T, and if $S = \{\mathbf{v}_1, ..., \mathbf{v}_m\}$ is a set of corresponding eigenvectors ($T(\mathbf{v}_j) = \lambda_j \mathbf{v}_j$), then S is linearly independent.

Proof. Suppose that S is dependent and let $N = \{k \in \mathbb{Z}^+ : \mathbf{v}_k \in \text{Span}(\mathbf{v}_1, ..., \mathbf{v}_{k-1})\}$. Since S is dependent, N is non-empty. Indeed, if $\mathbf{v}_k \notin \text{Span}(\mathbf{v}_1, ..., \mathbf{v}_{k-1})$ for all $k \leq m$, then S is independent as you should verify. By Well-ordering, we can select a least positive integer l in N so that

$$\mathbf{v}_l = \gamma_1 \mathbf{v}_1 + ... + \gamma_{l-1} \mathbf{v}_{l-1}. \tag{1}$$

Take equation (1) and apply T to both sides to get

$$T(\mathbf{v}_l) = \gamma_1 T(\mathbf{v}_1) + ... + \gamma_{l-1} T(\mathbf{v}_{l-1})$$

or

$$\lambda_l \mathbf{v}_l = \gamma_1 \lambda_1 \mathbf{v}_l + ... + \gamma_{l-1} \lambda_{l-1} \mathbf{v}_{l-1}. \tag{2}$$

Back to (1) again, multiply both sides by λ_l to get

$$\lambda_l \mathbf{v}_l = \gamma_1 \lambda_l \mathbf{v}_1 + ... + \gamma_{l-1} \lambda_l \mathbf{v}_{l-1}. \tag{3}$$

Subtracting equations (3) - (2), we get

$$0 = \gamma_1 (\lambda_l - \lambda_1) \mathbf{v}_1 + ... + \gamma_1 (\lambda_l - \lambda_{l-1}) \mathbf{v}_{l-1}.$$

By minimality of l, the set $\{\mathbf{v}_1, ..., \mathbf{v}_{l-1}\}$ is linearly independent so that

$$\gamma_i (\lambda_l - \lambda_i) = 0 \quad \text{for each } i \leq l - 1.$$

Since the eigenvalues $\lambda_1, ..., \lambda_l$ are all distinct, we have that $\lambda_l - \lambda_i \neq 0$ for each $i \leq l - 1$. It follows that $\gamma_i = 0$ for each $i \leq l - 1$. Equation (1) then implies that $\mathbf{v}_l = \mathbf{0}$ contradicting the fact that $\mathbf{v}_l$ is *nonzero.*

Corollary 8. If $\{\lambda_1, ..., \lambda_m\}$ is the set of distinct eigenvalues of $T : \mathbb{R}^n \to \mathbb{R}^n$, then $m \leq n$.

Proof. Let $S = \{\mathbf{v}_1, ..., \mathbf{v}_m\}$ be a set of corresponding eigenvectors ($T(\mathbf{v}_j) = \lambda_j \mathbf{v}_j$). Since S is independent, $m \leq \dim(\mathbb{R}^n) \leq n$.

Theorem 9. Suppose that $T : \mathbb{R}^n \to \mathbb{R}^n$ and $\{\lambda_1, ..., \lambda_m\}$ is the set of distinct eigenvalues of T. Then V has a basis consisting entirely of eigenvectors of T if and only if there exists a basis B of V such that

$$M(T, B, B) = \begin{bmatrix} \mu_1 & 0 & \cdots & 0 \\ 0 & \mu_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mu_n \end{bmatrix} \quad \text{where each } \mu_j \in \{\lambda_1, ..., \lambda_m\}.$$

Proof. ($\Rightarrow$) Suppose that $B = \{\mathbf{v}_1, ..., \mathbf{v}_n\}$ is a basis consisting of eigenvectors, say $T(\mathbf{v}_j) = \mu_j \mathbf{v}_j$ where $\mu_j \in \{\lambda_1, ..., \lambda_m\}$. Then

$$[\mathbf{v}_j]_B = \mathbf{e}_j \quad \text{and} \quad [T(\mathbf{v}_j)]_B = \mu_j \mathbf{e}_j$$

and so

$$M(T, B, B) = \begin{bmatrix} \mu_1 & 0 & \cdots & 0 \\ 0 & \mu_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mu_n \end{bmatrix}.$$

($\Leftarrow$) Suppose that B is a basis of $\mathbb{R}^n$ such that $M(T, B, B)$ has the given form. Then

$$T(\mathbf{v}_j) = M(T, B, B)\,[\mathbf{v}_j]_B = \mu_j \mathbf{v}_j$$