Notes

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irrationals $\subseteq \bigcup_{i=1}^{\infty}(-i,i) = \mathbb{R}$
$\{0\} \subseteq$

outer measure of a sum

$E \subseteq \mathbb{R}$ then $m*(E) = \inf\{\sum_{i=1}^{\infty} b_i = a_i : E \subseteq \bigcup_{i=1}^{\infty}(a_i, b_i)\}$

example

$m*(\{x\}) = 0$
$\{x\} \subseteq (x-\epsilon, x+\varepsilon$ for any $\varepsilon > 0$, $m*(t) \leq 2\varepsilon$ for any $\epsilon > 0 \to 0$

1. note that $m*(E) \subseteq [0,\infty]$, no negative
2. the empty set has measure zero

   $m*(\emptyset) = 0$

   $\emptyset \subseteq \bigcup_{i=1}^{\infty}(a_i, b_i)$
3. $A \subseteq B \Rightarrow m*(A) \leq m*(B)$

   proof

   if $B \subseteq \bigcup_{i=1}^{\infty}(a_i, b_i)$ then $A \subseteq \bigcup_{i=1}^{\infty}(a_i, b_i)$
   $\inf\{\sum b_i - a_i | A \subseteq \bigcup(a_i, b_i\} \leq \inf\{\sum c_i - d_i | B \subseteq \bigcup(c_i, d_i\}$
   cant do this with contradiction, that's the "actual way"

   example

   $m*([a,b]) = b - a$?
   $\inf\{\sum(b_i - a_i) : [a,b] \subseteq \bigcup(a_i, b_i\}$
   proof

$$\begin{aligned} [a,b] &\subseteq (a-\epsilon, b+\epsilon) \text{ for all } \epsilon > 0 \\ m*([a,b]) &\subseteq (b+\epsilon) - (a-\epsilon) \text{ for all } \epsilon > 0 \\ &\subseteq (b-a) + 2\epsilon \end{aligned}$$

so $m*([a,b]) \subseteq b-a$

heine-borel theorem (HBT) compact set is closed and bounded

compact:every open cover has finite subcovers? check this

if $[a,b] \subseteq \bigcup (a_i, b_i)$

wlog $[a,b] \subseteq \bigcup^n (a_i, b_i)$

$$\begin{aligned}
a_1 &< a \\
a_2 &< b_1 \\
a_3 &< b_2 \\
a_4 &< b_3 \\
\vdots a_n & \qquad\qquad\qquad\qquad < b_{n-1} \\
b &< b_n
\end{aligned}$$

these are overlapping covers, they make a sequence, it is important that they overlap

sum to infinity is bigger than sum to n. $\sum(b_1 - a_i) + (b_2 - a_2) + \cdots + (b_n - a_n) \geq (a_2 - a_1) + (a_3 - a_2) \ldots (a_n - a_n - 1) + (b_n - a_n) = b_n - a_1 \geq b - a$

and so $m*([a,b]) \geq b-a$ and we already have less than so it's equal

example 2

$m*((a,b)) = b-a$ second part of limhof? thm, uniqueness

notice that the different sets are the same size

example 3

$m*([a,\infty)) = \infty$ notice that the measure of any subset is greater than or equal and so $[a, a+k] \subseteq [a,\infty)$ for all k and so $k = m*[a, a+k] \leq m*[a,\infty) \forall k$

example 4,5

$m*\mathbb{Q}, m*\mathbb{C}-\mathbb{R}$

4. $m*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum\limits_{i=1}^{\infty} m*(A_i)$

   proof

   let $\epsilon > 0$ then for all i there is $\{(a_i^j, b_i^j\}_{i=1}^{\infty}$ such that $m*A_i \leq \sum b_i^j - a_i^j \leq m*A_j + \frac{\epsilon}{2^j}$

   notice that $\{\{a_i^j, b_i^j\}_{i=1}^{\infty}\}_{j=1}^{\infty}$ is countable collection of intervals with $\bigcup A_j \subseteq \bigcup_{j=1}(\bigcup_{i=1}(a_i^j, b_i^j))$ $m*A_j \leq \sum_{j=1}(\sum_{i=1}(a_i^j, b_i^j))$

   this is called countable subadditivity