Notes

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6.3 how non-planar are you?

if we draw a nonplanar graph in a plane, what must occur? we have crossing edges.

the number of such crossings indicates how nonplanar you are

when drawing a graph, we will use 5 conventions.

1. no edge will cross itself
2. at any crossing, only two edges are involved
3. no edge goes through a vertex
4. adjacent edges never cross
5. edges cross at most once

definition: using these conventions to draw a graph G, the minumal number of crossings is called **the crossing number** and is denoted $\mathrm{cr}(G)$

minimizing the crossing number is challenging to find, but it exists.

thm

if G is a graph with $|V(G)| = n \geq 3$ and $|E(G)| = m$, then $\mathrm{cr}(G) \geq m - 3n + 6$.

proof

Draw G in the plane with minimal crossings.

if $\mathrm{cr}(G) > 0$ then we have crossings. draw a vertex at each crossing. Call this graph G' and note that it is planar.

$|V(G')| = |V(G)| + \mathrm{cr}(G)$ and $|E(G')| = |E(G)| + 2\mathrm{cr}(G)$. Now we know that $|E(G')| \leq 3|V(G')| - 6$ (thm 6.3) and $\mathrm{cr}(G) \geq m - 3n + 6$

for complete graphs, we have other bounds

thm

$\mathrm{cr}(K_n) \geq \frac{1}{5}\binom{n}{4}$ and $n \geq 5$

thm

$\mathrm{cr}(K_n) \leq \frac{1}{4}\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor\lfloor\frac{n-2}{2}\rfloor\lfloor\frac{n-3}{2}\rfloor$

thm

previous theorem is equal (sharp) for $1 \leq n \leq 12$

sixth convention

all edges are straight (rectilineary embeddings)

fact

most bounds are now shot with these rectilinear embeddings

thm

for planar graphs, the crossing number of the rectilinear embedding is also 0.

notation

rectilinear crossing number is $\overline{\mathrm{cr}}(G)$