1. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x - 2y \\ 2x + y \end{pmatrix}$$

   (a) Find A_T

   $\begin{pmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \end{pmatrix}$

   (b) Is T 1-1? If not, does there exist a 1-1 map $P : \mathbb{R}^3 \to \mathbb{R}^2$? Justify.

   No, and no, $A_T \in \mathcal{M}_{2\times 3}$ which means that $\dim(\text{Null}(A_T)) \geq 1$. And so $\{0\}$ is a proper subset of $\text{Null}(A_T)$ and always will be regardless of our choice of A_T.

   (c) Is T onto? Justify.

   Yep, it's onto. The column space clearly has dimension 2, and so it is equivalent to $\mathbb{R}^2$

   (d) Find $\dim(\ker T)$ and $\dim(\text{Im}T)$

   The dimension of the column space is the same as the dimension of the image, and as we just pointed out, $\dim(\text{Col}A_T) = 2 = \dim(\text{Im}T)$. And the null space dimension is the number of columns of A_T minus the dimension of the column space, and so $\dim(\text{Null}T) = 1$

2. Let $S : \mathbb{R}^2 \to \mathbb{R}^4$ be given by

$$S \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x \\ y - 2x \\ 2x \\ x + 2y \end{pmatrix}$$

   Find a formula for $(S \circ T) : \mathbb{R}^3 \to \mathbb{R}^4$

$$\begin{pmatrix} 3 & 0 \\ -2 & 1 \\ 2 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & -6 & 0 \\ 0 & 5 & 0 \\ 2 & -3 & 0 \\ 5 & 0 & 0 \end{pmatrix}$$

3. Prove theorem 11: If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a bijective linear transformation, then $T^{-1} : R^n \to R^n$ is a linear transformation. Moreover, $A_{T^{-1}} = A_T^{-1}$

   We know that $T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T\mathbf{y}$. If we apply T^{-1} to both sides of the equation, then we have $T^{-1}(T(\mathbf{x}) + T(\mathbf{y})) = T^{-1}(T(\mathbf{x} + \mathbf{y})) = \mathbf{x} + \mathbf{y} = T^{-1}(T(\mathbf{x})) + T^{-1}(T(\mathbf{y}))$. Now we we observe $cT^{-1}(T(\mathbf{x})) = c\mathbf{x} = T^{-1}(T(c\mathbf{x}))$ and so T^{-1} is linear. Now we know that $T \circ T^{-1}$ maps any $\mathbf{x} \in \mathbb{R}^n$ to itself. And so if $A_{T^{-1}\circ T}\mathbf{x} = \mathbf{x}$ then $A_{T^{-1}\circ T} = I_n$. And so $A_{T^{-1}}A_T = I_n$. This means that $A_{T^{-1}=A_T^{-1}}$

4. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3x + 3y + z \\ 3x + 3y + z \\ 2x + 4y + z \end{pmatrix}$$

   Prove that T is an invertible map (1-1 and onto) and find a formula for $T^{-1} : \mathbb{R}^3 \to \mathbb{R}^3$.

$$\left(\begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 & 1 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|ccc} 0 & 3 & 1 & 3 & -2 & 0 \\ 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|ccc} 0 & 0 & 1 & 6 & -2 & -3 \\ 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 6 & -2 & -3 \end{array}\right)$$

So $\text{Null}(A_T) = \{\mathbf{0}\}$ and $\dim(\text{Col}(A_T)) = 3$ or $\dim(\text{Col}(A_T)) = \mathbb{R}^3$. Thus T is 1-1 and onto.

$$T^{-1}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -x+y \\ -x+z \\ 6x-2y-3z \end{pmatrix}$$

5. Construct a linear transformation $\rho : \mathbb{R}^2 \to \mathbb{R}^2$ that rotates every vector through an angle of $\theta = \frac{\pi}{2}$. Find the standard matrix A_ρ of the transformation and verify that ρ really does rotate the plane through $\theta = \frac{\pi}{2}$.

$$\rho\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix} \qquad A_\rho = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$r = \sqrt{x^2+y^2} \qquad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$$

$$\rho\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} \qquad = \begin{pmatrix} -r\sin\theta \\ r\cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} r\sin -\theta \\ r\cos -\theta \end{pmatrix} \qquad = \begin{pmatrix} r\sin\left(\theta + \frac{\pi}{2}\right) \\ r\cos\left(\theta + \frac{\pi}{2}\right) \end{pmatrix}$$

6. Let $B = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a *basis* of $\mathbb{R}^n$. Let $\{\mathbf{w}_1, \dots, \mathbf{w}_n\}$ be a *list* of n vectors in $\mathbb{R}^m$. Prove the following statements.

   (a) There exists a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ such that $T(\mathbf{v}_i) = \mathbf{w}_i$ for each index $i \le n$.
   We say B is the matrix $\begin{pmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_n \end{pmatrix}$ and C is the matrix $\begin{pmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_n \end{pmatrix}$. Now because $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ are linearly independent, we know that B has rank$n$ and is therefore nonsingular, or it has an inverse. Now $C = CI_n = (CB^{-1})B$. Thus we see that $CB^{-1}\mathbf{v}_i = \mathbf{w}_i$ and we have found our transformation.

   (b) If $S : \mathbb{R}^n \to \mathbb{R}^m$ is another linear transformation such that $S(\mathbf{v}_i) = \mathbf{w}_i$ for each index $i \le n$, then $S = T$.
   If $S \ne T$ then there exists some $x \in \mathbb{R}^n$ such that $T(\mathbf{x}) \ne S(\mathbf{x})$. Because the $\mathbf{v}_i$'s form a basis for $\mathbb{R}^n$, we can rewrite $\mathbf{x} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n$. And so we have

$$\begin{aligned} T(a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n) &\ne S(a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n) \\ a_1T(\mathbf{v}_1) + \cdots + a_nT(\mathbf{v}_n) &\ne a_1S(\mathbf{v}_1) + \cdots + a_nS(\mathbf{v}_n) \\ a_1\mathbf{w}_1 + \cdots + a_n\mathbf{w}_n &\ne a_1\mathbf{w}_1 + \cdots + a_n\mathbf{w}_n \end{aligned}$$

   That is absurd, so we must conclude that $T = S$

   (c) T is onto if and only if $\{\mathbf{w}_1, \dots, \mathbf{w}_n\}$ spans $\mathbb{R}^m$.
   We can express any element in the image of T as a linear combination of the images of the elements of B under T which means each element in the image of T can be expressed as a linear combination of $\{\mathbf{w}_1, \dots, \mathbf{w}_n\}$. Now if T is onto, then every element in $\mathbb{R}^m$ must then be in the span of our $\mathbf{w}$'s. We note that every linear combination of our $\mathbf{w}$'s can be expressed at a linear combination of our $T(\mathbf{v})$'s and so every linear combination of $\mathbf{w}$ is in the image of T. Now if T is not onto, then there is an element in $\mathbb{R}^m$ that is not in the image of T and therefore not in the span of our $\mathbf{w}$'s.

   (d) T is 1-1 iff $\{\mathbf{w}_1, \dots, \mathbf{w}_n\}$ is a linearly independent subset of $\mathbb{R}^m$.
   If T is not 1-1 then the nullspace of A_T is not trivial and there exists some $0 = A_T(a_1\mathbf{v}_1 + \cdots + a_n$ $mathbfv_n) = a_1A_T\mathbf{v}_1 + \cdots + a_nA_t\mathbf{v}_n = a_1\mathbf{w}_1 + \cdots + a_n\mathbf{w}_n$ where at least one $a_i \ne 0$. Thus our

w's are not linearly independent. Now if our **w**'s are not linearly independent, then there exists at least one $a_i \neq 0$ such that $0 = a_1\mathbf{w}_1 + \cdots + a_n\mathbf{w}_n = A_T(a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n)$ and we have found a nonzero element in the nullspace of A_T and so T is not 1-1.

(e) T is a bijection iff $\{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ is a basis of $\mathbb{R}^m$
From c and d we know that if T is 1-1 and onto then our **w**'s are a linearly independent and span $\mathbb{R}^m$. Also if our **w**'s are linearly independent and span $\mathbb{R}^m$ then T is 1-1 and onto. Thus by the definitions of basis and bijection we have our result.

7. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Prove the following statements.

We say A_T is the $m \times n$ matrix that is associate with T

(a) $\dim(\mathrm{Im}T) \leq n$.
ImT is the column space of A_T which is the rank of A_T which is less than n

(b) $n = \dim(\ker T) + \dim(\mathrm{Im}T)$
$\ker T$ is the nullspace of A_T and $\dim(\mathrm{Im}T)$ is the rank of A_T. The dimension of the rank plus the dimension of the nullspace is the width of a matrix.

(c) If T is 1-1 then $n \leq m$
If T is 1-1 then the nullspace of A_T is trivial, and so $\mathrm{rank}(A_T) = n$. Because the rank cannot be greater than either of a matrix's dimensions, then $n \leq m$

(d) If T is onto then $m \leq n$
If T is onto then $\mathrm{rank}(A_T) = \dim(\mathrm{Im}) = \dim(\mathbb{R}^m)$ and since the rank cannot be greater than either of the dimensions then $m \leq n$.

(e) If $n = m$, then T is onto if and only if T is a bijection if and only if T is 1-1.
We always assume that $n = m$. Now if T is onto then $\mathrm{Im}T = \mathbb{R}^m = \mathrm{Col}(A_T)$. Now because $\dim(\mathrm{Null}(A_T) + \dim(\mathrm{Col}T) = m = n$ then $\dim(\mathrm{Null}(A_T)) = 0$ and $\mathrm{Null}(A_T)) = \{0\}$. And so T is onto. Now if T is 1-1 then $\mathrm{Null}(A_T) = \{0\}$ and because $\dim(\mathrm{Null}(A_T) + \dim(\mathrm{Col}T) = m = n$ then $\mathrm{Col}A_T = \mathbb{R}^m$ and so T is onto. The bijective bit in the middle comes for free since T is 1-1 iff T is onto.