Excercise Set 1

1. Prove the matrix
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

   is invertible if and only if $ad - bc \neq 0$. Find A^{-1} in this case by solving a system of two equations with two unknowns.

   We need some matrix A^{-1} such that $AA^{-1} = I_2$. Assume A^{-1} exists, then let $A^{-1} = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$ and we have

$$AA^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$$
$$= \begin{bmatrix} aa' + bc' & ab' + bd' \\ ca' + dc' & cb' + dd' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

In particular

$$aa' + bc' = 1 \qquad\qquad\qquad ca' + dc' = 0$$

Now if $c = 0$ then $d = 0$ or $c' = 0$. We know that $d \neq 0$ because $cb' + dd' = 1$. And so if $c = 0$ then $c' = 0$. But then $aa' + bc' = 1$ so $a \neq 0$ in this case. We will proceed in two cases then for $c \neq 0$ and $a \neq 0$.

If we assume that $c \neq 0$ then we have

$$ca' + dc' = 0 \qquad\qquad a' + \frac{d}{c}c' = 0 \qquad\qquad -aa' - \frac{ad}{c}c' = 0$$

Now if we add $aa' + bc' = 1$ to the above result we get $\left(b - \frac{ad}{c}\right)c' = 1$. Solving for c' gives us $c' = -\frac{c}{ad-bc}$. Using we can solve $ca' + dc' = 0$ to find $a' = \frac{d}{ad-bc}$. Moving on we have

$$cb' + dd' = 1 \qquad\qquad b' + \frac{d}{c}d' = \frac{1}{c} \qquad\qquad -ab' - \frac{ad}{c}d' = -\frac{a}{c}$$

Adding in $ab' + bd' = 0$ leads us to $\frac{bc-ad}{c}d' = \frac{-a}{c}$, or $d' = \frac{a}{ad-bc}$. And finally substituting d' in for $cb' + dd' = 1$ leads to $cb' + d\frac{a}{ad-bc} = cb' + 1 + \frac{bc}{ad-bc} = 1$ or $b' = -\frac{b}{ad-bc}$.

The case when $a \neq 0$ is similar.

$$ab' + bd' = 0 \qquad\qquad\qquad cb' + dd' = 1$$
$$-cb' - \frac{cb}{a}d' = 0 \qquad\qquad\qquad cb' + dd' = 1$$
$$\left(d - \frac{cb}{a}\right)d' = 1 \qquad\qquad\qquad d' = \frac{a}{ad - bc}$$

If we continue as we did when $c \neq 0$ then we will arrive at $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ just as we did when $c \neq 0$.

In this way we see that when we assume that A^{-1} exists and that $ad - bc = 0$ we arrive at a contradiction. Namely that $\frac{1}{ad-bc}$ is undefined and therefore A^{-1} can't exist. However, if we assume that $ad - bc \neq 0$ we find our inverse, and so it must exist. Thus we have proven both directions of the hypothesis. □

2. Let

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & -3 \end{bmatrix}$$

(a) Show that $A^3 + 3A^2 - 2A - I_3 = \mathbf{0}$

$$A^3 + 3A^2 - 2A - I_3 = 0$$
$$(A^2 + 3A - 2I_3)A = I_3$$

$$\left(\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & -3 \end{bmatrix}^2 + 3\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & -3 \end{bmatrix} + \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \right) A = I_3$$

$$\left(\begin{bmatrix} 0 & 1 & -3 \\ 0 & 2 & -5 \\ 1 & -3 & 11 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 3 \\ 3 & 0 & 6 \\ 0 & 3 & -9 \end{bmatrix} + \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \right) A = I_3$$

$$\begin{bmatrix} -2 & 1 & 0 \\ 3 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & -3 \end{bmatrix} = I_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

(b) Use part (a) to see that A is invertible and compute A^{-1}

Obviously $A^{-1} = A^2 + 3A - 2I_3 = \begin{bmatrix} -2 & 1 & 0 \\ 3 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

3. Let $A \in \mathcal{M}_n$ be a diagonal matrix. Prove that A is invertible if and only if $\text{ent}_{ii}(A) \neq 0$ for all $i \leq n$. Find A^{-1} in this case.

Let us assume that there exists some $\text{ent}_{ii}(A) = 0$ and that A^{-1} exists with $\text{ent}_{jk}(A^{-1}) = a_{jk}^{-1}$ and $\text{ent}_{jk}(A) = a_{jk}$. Then $\text{ent}_{ii}(AA^{-1}) = \sum_{k=1}^{n} a_{ik}a_{ki}^{-1} = \sum_{k=1}^{n} 0a_{ki}^{-1} = 0$. But $AA^{-1} = I_n$ and $\text{ent}_{ii}(I_n) = 1$ and so we have a contradiction.

Now lets assume that $\text{ent}_{ii}(A) = c_i$ with $i \leq n$ and $c_i \neq 0$. Lets choose another diagonal matrix and name it A^{-1}. We will assign $\text{ent}_{ii}(A^{-1}) = c_i^{-1}$. Now we see that if we choose any $i \leq n, j \leq n$ such that $j \neq i$ then we see that $\text{ent}_{ij}(AA^{-1}) = \sum_{k=1}^{n} a_{ik}a_{kj}^{-1} = 0 + \cdots + a_{ii}a_{ij}^{-1} + a_{ij}a_{jj}^{-1} + \ldots 0 = 0 + \cdots + a_{ii}0 + 0a_{jj}^{-1} + \ldots 0 = 0$. But $\text{ent}_{ii}(AA^{-1}) = \sum_{k=1}^{n} a_{ik}a_{ki}^{-1} = 0 + \cdots + a_{ii}a_{ii}^{-1} + \cdots + 0 = cc^{-1} = 1$. And so we see that AA^{-1} is a diagonal matrix with all ones for its entries. But that is I_n and so we have found the inverse of A^{-1}. $\square$

4. If $A, B \in \mathcal{M}_\setminus$ are invertible such that $A + B \neq 0$, does it follow that $(A + B)^{-1}$ exists? Prove or find a counterexample.

$$\begin{bmatrix} 3 & 2 \\ 8 & 4 \end{bmatrix} + \begin{bmatrix} 3 & 2 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 12 & 16 \end{bmatrix}$$

Notice that $3 \cdot 4 - 8 \cdot 2 = -4$ and $3 \cdot 4 - 2 \cdot 4 = 4$ but $6 \cdot 8 - 4 \cdot 12 = 6 \cdot 4 \cdot 2 - 4 \cdot 2 \cdot 6 = 0$ and so by excercise 1 we have found a counterexample.

Excercise Set 2

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \end{bmatrix}$$

1. For the matrix A in the example above, determine $E_{3 \to c3}$ where c is any nonzero real number.

$$E_{3 \to c3} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2. For the matrix A in the example above, determine $E_{4 \to 4+c2}$ where c is any nonzero real number.

$$E_{4 \to 4+c2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & c & 0 & 1 \end{bmatrix}$$

3. Prove that each of $E_{i \leftrightarrow k}, E_{i \to ci}, E_{i \to i+ck}$ is invertible by finding an inverse. Prove that the inverse of an elementary matrix is an elementary matrix.

   We assume that $i \neq k$. We posit that $E_{i \leftrightarrow k}$ is the identity matrix, save that $\text{ent}_{ii}E_{i \leftrightarrow k} = \text{ent}_{kk}E_{i \leftrightarrow k} = 0$ and $\text{ent}_{ki}E_{i \leftrightarrow k} = \text{ent}_{ik}E_{i \leftrightarrow k} = 0$

   We shall now verify this claim. First we choose some $j \notin \{i, k\}$ and any l. Now we see that for any $A \in \mathcal{M}_n$ we have $\text{ent}_{jl}(E_{i \leftrightarrow k}A) = \sum\limits_{m=1}^{n} \text{ent}_{jm}(E_{i \leftrightarrow k})a_{ml} = a_{jl}$ because all entries in the elementary matrix row are zero except the j element which is one. And so as we wish $E_{i \leftrightarrow k}$ leaves everything off of rows i and k undisturbed. Now lets see what happens for $\text{ent}_{il}(E_{i \leftrightarrow k}A)$ and $\text{ent}_{kl}(E_{i \leftrightarrow k}A)$.

   $$\text{ent}_{il}(E_{i \leftrightarrow k}A) = \sum_{m=1}^{n} \text{ent}_{im}(E_{i \leftrightarrow k})a_{ml} = a_{kl}$$

   $$\text{ent}_{kl}(E_{i \leftrightarrow k}A) = \sum_{m=1}^{n} \text{ent}_{km}(E_{i \leftrightarrow k})a_{ml} = a_{il}$$

   Thus we see that the i and k rows of A and its product with the elementary matrices have been switched. This occurs because $\text{ent}_{ki}(E_{i \leftrightarrow k}) = \text{ent}_{ik}(E_{i \leftrightarrow k}) = 1$ but all other entries on these two rows is zero.

   Now that we know what $E_{i \leftrightarrow k}$ is, we see that for $i \neq l$ we have $\text{ent}_{il}(E_{i \leftrightarrow k}E_{i \leftrightarrow k}) = \text{ent}_{kl}(E_{i \leftrightarrow k}) = 0$. And we have $\text{ent}_{ii}(E_{i \leftrightarrow k}E_{i \leftrightarrow k}) = \text{ent}_{ki}(E_{i \leftrightarrow k}) = 1$. Similarly we have $\text{ent}_{kl}(E_{i \leftrightarrow k}E_{i \leftrightarrow k}) = \text{ent}_{il}(E_{i \leftrightarrow k}) = 0$ when $k \neq l$ and $\text{ent}_{kk}(E_{i \leftrightarrow k}E_{i \leftrightarrow k}) = \text{ent}_{ik}(E_{i \leftrightarrow k}) = 1$ when $k = l$. Thus $E_{i \leftrightarrow k}^2 = I_n$ and so this matrix is it's own inverse.

   For the matrix $E_{i \to ci}$ we must have $c \neq 0$. Now if we take the identity matrix, but change ent_{ii} to be c then I claim we have our elementary matrix. As above, when $j \neq i$ we have $\text{ent}_{jk}(E_{i \to ci}A) = a_{jk}$ for any $A \in \mathcal{M}_n$ and $k \leq n$. And so our matrix leaves all rows but i untouched. Now if we examine row i we see that $\text{ent}_{ik}(E_{i \to ci}A) = \sum\limits_{l=1}^{n} \text{ent}_{il}(E_{i \to ci})a_{li} = ca_{ik}$ as desired. So we have found our elementary matrix.

   Now let us examine $E_{i \leftrightarrow (1/c)i}E_{i \leftrightarrow ci}$. We have already seen that $\text{ent}_{jk}(E_{i \leftrightarrow (1/c)i}E_{i \to ci}) = \text{ent}_{jk}(E_{i \to ci}) = \text{ent}_{jk}(I_n)$. And $\text{ent}_{ik}(E_{i \leftrightarrow (1/c)i}E_{i \to ci}) = \frac{1}{c}\text{ent}_{ik}(E_{i \to ci})$. Of course when $k \neq i$ then this is zero and

when $k = i$ then this entry will be $\frac{1}{c}c = 1$. Thus we have exactly the identity matrix. And so we have found our inverse.

Finally we look at $E_{i \to i+ck}$. As before we start with the identity matrix. This time the only entry on I_n that we change is $\text{ent}_{ik}(E_{i \to i+ck}) = c$.

Obviously if we have $j \neq i, l \leq n$ and any $A \in \mathcal{M}_n$ then $\text{ent}_{jl}(E_{i \to i+ck}A) = \sum\limits_{m=1}^{n} \text{ent}_{jm}(E_{i \to i+ck})a_{ml} = a_{jl}$. But $\text{ent}_{il}(E_{i \to i+ck}A) = \sum\limits_{m=1}^{n} \text{ent}_{im}(E_{i \to i+ck})a_{ml} = a_{il} + ca_{kl}$. Fantastic, we have found something that takes the entry from row k and multiplies it by c and adds it to the entry in row i, then places the result in row i. This then is the elementary matrix we require.

Now if we take $E_{i \to i-ck}$ for $j \neq i, l \leq n$ we have

$$\text{ent}_{jl}(E_{i \to i-ck}E_{i \to i+ck}) = \sum_{m=1}^{n} \text{ent}_{jm}(E_{i \to i-ck})\text{ent}_{jm}(E_{i \to i+ck}) = \text{ent}_{jl}(I_n)$$

And $\text{ent}_{il}(E_{i \to i-ck}E_{i \to i+ck}) = \sum\limits_{m=1}^{n} \text{ent}_{im}(E_{i \to i-ck})\text{ent}_{im}(E_{i \to i+ck}) = \text{ent}_{il}(E_{i \to i+ck}) - c \cdot \text{ent}_{kl}(E_{i \to i+ck})$. Of course if $i = l$ then this will give us $1 - 0 = 1$ and if $k = l$ then we have $c - c = 0$. This will leave us with our identity. And so we have our inverse.

For each of the three elementary matrices we have found an inverse, and thus one must exist. Further, these inverses are simply elementary matrices themselves. $\square$

4. Define a relation $\sim$ on $\mathcal{M}_{m \times n}$ given by $A \sim B$ if and only if there exists a $P \in \mathcal{M}_{m \times n}$ such that $A = PB$ where $P \in \mathcal{M}_{m \times n}$ is a product of elementary matrices.

We know that $A = I_n A$ and I_n is an elementary matrix and so this relation is reflexive.

We also know that every elementary matrix has an inverse and that inverse is an elementary matrix. And so if $P = E_1 E_2 \ldots E_n$ then P^{-1} exists. In fact $P^{-1} = E_n^{-1} \ldots E_1^{-1}$ which is the product of elementary matrices also. Therefore if $A = PB$ then $P^{-1}A = B$ and so commutativity is preserved with this relation.

Now of we have P_1, P_2 both products of elementary matrices, then it follows that $P_1 P_2$ is also the product of elementary matrices. And so if $A = P_1 B$ and $B = P_2 C$ then $A = P_1(P_2 C) = (P_1 P_2)C$ and so the relation is transitive.

Those are the three properties we need for the relation to fit the definition of an equivalence relation. $\square$