Invertible Matrices

Definition 1. Let $A \in \mathcal{M}_n$ be a square matrix. Then A is called an invertible matrix if there exists a matrix $B \in \mathcal{M}_n$ such that $AB = I_n = BA$.

Example 2. Let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

We compute an inverse of A. We need a matrix

$$B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$$

such that

$$AB = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x+z & w+y \\ z & w \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

We obtain the equations

$$\begin{aligned} x+z &= 1 \\ w+y &= 0 \\ z &= 0 \\ w &= 1. \end{aligned}$$

Solving, we find that $x = 1$, $y = -1$, $z = 0$, $w = 1$ and so

$$B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

Theorem 3. If $A \in \mathcal{M}_n$ has an inverse, then this inverse is unique.

Proof. Suppose that B and C are both inverses of A. Then

$$\begin{aligned} & B \\ = \; & I_n B \quad \text{(Identity)} \\ = \; & (CA)B \quad \text{(Since } C \text{ is an inverse of } A\text{)} \\ = \; & C(AB) \quad \text{(Associative)} \\ = \; & CI_n \quad \text{(Since } B \text{ is an inverse of } A\text{)} \\ = \; & C \quad \text{(Identity).} \end{aligned}$$

Remark. If $A \in \mathcal{M}_n$ is invertible, then we write A^{-1} for this unique inverse of A.

Theorem. Let $A, B \in \mathcal{M}_n$ be invertible matrices. Then,
(1) A^{-1} is invertible and $(A^{-1})^{-1} = A$.
(2) AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

Proof.
(1) The $AA^{-1} = I_n = A^{-1}A$ says that A is an inverse of A^{-1}. But $(A^{-1})^{-1}$ is certainly an inverse of A^{-1}. It follows from uniqueness that $(A^{-1})^{-1} = A$.
(2) The equations

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}I_nB = B^{-1}B = I_n$$

and

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AI_nA^{-1} = AA^{-1} = I_n$$

say two things. First, they say that AB is invertible since we have found $C \in \mathcal{M}_n$ such that $(AB)C = I_n = C(AB)$. The equations also say that $B^{-1}A^{-1}$ is an inverse of AB. But $(AB)^{-1}$ is the unique inverse of AB so that $(AB)^{-1} = B^{-1}A^{-1}$.

Exercises

1. Prove that the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if and only if $ad - bc \neq 0$. Find A^{-1} in this case by solving a system of two equations with two unknowns.

2. Let

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & -3 \end{bmatrix}.$$

(a) Show that $A^3 + 3A^2 - 2A - I_3 = \mathbf{0}$.
(b) Use part (a) to see that A is invertible and compute A^{-1}.

3. Let $A \in \mathcal{M}_n$ be a diagonal matrix. Prove that A is invertible if and only if $\mathrm{ent}_{ii}(A) \neq 0$ for all $i \leq n$. Find A^{-1} in this case.

4. If $A, B \in \mathcal{M}_n$ are invertible such that $A + B \neq 0$, does it follow that $(A + B)^{-1}$ exists. Prove or find a counterexample.

Elementary Matrices

Definition. There are three *elementary row operations* on a matrix $A \in \mathcal{M}_{m\times n}$.
(1) $E_{i\leftrightarrow k} \in \mathcal{M}_m$ switches $\mathbf{r}_i(A)$ and $\mathbf{r}_k(A)$.
(2) $E_{i\to ci} \in \mathcal{M}_m$ replaces $\mathbf{r}_i(A)$ with $c\mathbf{r}_i(A)$.
(3) $E_{i\to i+ck} \in \mathcal{M}_m$ replaces $\mathbf{r}_i(A)$ with $\mathbf{r}_i(A) + c\mathbf{r}_k(A)$.

Recall. If $A \in \mathcal{M}_{m\times p}$ and $B \in \mathcal{M}_{p\times n}$, then $\mathbf{r}_i(AB) = \mathbf{r}_i(A)B$. In particular, we have that $\mathbf{r}_i(AB) = a_{i1}\mathbf{r}_1(B) + a_{i2}\mathbf{r}_2(B) + ... + a_{ip}\mathbf{r}_p(B)$. so that $\mathbf{r}_i(AB) \in \text{Span}(\mathbf{r}_1(B), \mathbf{r}_2(B), ..., \mathbf{r}_p(B))$.

Example. What is $E_{2\leftrightarrow 4}$ if

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \end{bmatrix}?$$

Well, first note that we need $E_{2\leftrightarrow 4}A$ to be the same as A but only with the second and fourth rows swapped. So we know that $E_{2\leftrightarrow 4}$ is a 4×4 matrix

$$E_{2\leftrightarrow 4} = \begin{bmatrix} e_{11} & e_{12} & e_{13} & e_{14} \\ e_{21} & e_{22} & e_{23} & e_{24} \\ e_{31} & e_{32} & e_{33} & e_{34} \\ e_{41} & e_{42} & e_{43} & e_{44} \end{bmatrix}.$$

According to our formula for multiplication.

$$\mathbf{r}_i(E_{2\leftrightarrow 4}A) = e_{i1}\mathbf{r}_1(A) + e_{i2}\mathbf{r}_2(A) + ... + e_{im}\mathbf{r}_m(A).$$

Now $\mathbf{r}_1(E_{2\leftrightarrow 4}A)$ is no different than $\mathbf{r}_1(A)$. That is,

$$\mathbf{r}_1(E_{2\leftrightarrow 4}A) = e_{11}\mathbf{r}_1(A) + e_{12}\mathbf{r}_2(A) + e_{13}\mathbf{r}_3(A) + e_{14}\mathbf{r}_4(A) = \mathbf{r}_1(A).$$

So take

$$e_{11} = 1, e_{12} = 0, e_{13} = 0, e_{14} = 0.$$

Now on to $\mathbf{r}_2(E_{2\leftrightarrow 4}A)$:

$$\mathbf{r}_2(E_{2\leftrightarrow 4}A) = e_{21}\mathbf{r}_1(A) + e_{22}\mathbf{r}_2(A) + e_{23}\mathbf{r}_3(A) + e_{24}\mathbf{r}_4(A) = \mathbf{r}_4(A)$$

so take

$$e_{21} = 0, e_{22} = 0, e_{23} = 0, e_{24} = 1.$$

Similar computations for $\mathbf{r}_3(E_{2\leftrightarrow 4}A)$ and $\mathbf{r}_4(E_{2\leftrightarrow 4}A)$ give

$$E_{2\leftrightarrow 4} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Exercises

1. For the matrix A in the example above, determine $E_{3\to c3}$ where c is any nonzero real number.

2. For the matrix A in the example above, determine $E_{4\to 4+c2}$ where c is any nonzero real number.

3. Prove that each of $E_{i\leftrightarrow k}, E_{i\to ci}, E_{i\to i+ck}$ is invertible by finding an inverse. Prove that the inverse ofan elementary matrix is an elementary matrix.

4. Define a relation $\sim$ on $\mathcal{M}_{m\times n}$ given by $A \sim B$ if and only if there exists a $P \in \mathcal{M}_{m\times m}$ such that $A = PB$ where $P \in \mathcal{M}_{m\times n}$ is a product of elementary matrices. Prove that $\sim$ is an equivalence relation.