9.1

B. Show that every subset of a discrete metric space is both open and closed.

We have a discrete metric d on a set X. Now we take $U \subset X$. For any $x \in U$ we have $B(x, r) \subset U$ if $r \leq 1$ because the ball will contain only the point x. Note that this is trivially true even if $U = \emptyset$ because there is no $x \in U$ that does not have a ball around it. Now because our choice of U was arbitrary we know that all subsets of X are open. And the complements of any subsets of X are themselves subsets of X, and so they are open. But they are the complement of an open set, and so they must be closed. Thus every subset of a discrete metric space is both open and closed.

D. Prove Theorem 9.1.7

Let f map a metric space (X, ρ) into a metric space (Y, ρ). The following are equivalent:

(1) f is continuous on X;

(2) for every sequence (x_n) with $\lim_{n\to\infty} x_n = a \in X$, we have $\lim_{n\to\infty} f(x_n) = f(x)$; and

(3) $f^{-1}(U) = \{x \in X : f(x) \in U\}$ is open in X for every open set U in Y.

We start by assuming that f is continuous on X.

H. Two metrics ρ and σ on a set X are **equivalent** if there are constants $0 < c < C$ such that $c\rho(x, y) \leq \sigma(x, y) \leq C\sigma(x, y)$ for all $x, y \in X$

(a) Prove that equivalent metrics are topologically equivalent

(b) Prove that equivalent metrics have the saame Cauchy sequences

(c) Give examples of topologically equivalent metrics that are not equivalent

K. Recal the 2-adic metric of examples 9.1.2 (4) and 9.1.5 (4). Extend it to $\mathbb{Q}$ by setting $\rho_2(a/b, a/b) = 0$ and, if $a/b \neq c/d$, then $\rho_2(a/b, c/d) = 2^{-e}$, where e is the unique integer such that $a/b - c/d = 2^e(f/g)$ and both f and g are odd integers

(a) Prove that ρ_2 is a metric on $\mathbb{Q}$

(b) Show that the sequence of integers $a_n = (1 - (-2)^n)/3$ converges in $(\mathbb{Q}, \rho_2)$

(c) Find the limit of $\dfrac{n!}{n!+1}$ in this metric.