Excercise Set 1

1. Prove the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if and only if $ad - bc \neq 0$. Find A^{-1} in this case by solving a system of two equations with two unknowns.

We need some matrix A^{-1} such that $AA^{-1} = I_2$. Assume A^{-1} exists, then let $A^{-1} = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$ and we have

$$\begin{aligned} AA^{-1} &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \\ &= \begin{bmatrix} aa' + bc' & ab' + bd' \\ ca' + dc' & cb' + dd' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

In particular

$$aa' + bc' = 1 \qquad\qquad ca' + dc' = 0$$

Now if $c = 0$ then $d = 0$ or $c' = 0$. We know that $d \neq 0$ because $cb' + dd' = 1$. And so if $c = 0$ then $c' = 0$. But then $aa' + bc' = 1$ so $a \neq 0$ in this case. We will proceed in two cases then for $c \neq 0$ and $a \neq 0$.

If we assume that $c \neq 0$ then we have

$$ca' + dc' = 0 \qquad\qquad a' + \frac{d}{c}c' = 0 \qquad\qquad -aa' - \frac{ad}{c}c' = 0$$

Now if we add $aa' + bc' = 1$ to the above result we get $\left(b - \frac{ad}{c}\right) c' = 1$. Solving for c' gives us $c' = -\frac{c}{ad-bc}$

The case when $a \neq 0$ is similar.

$$\begin{aligned} ab' + bd' &= 0 & cb' + dd' &= 1 \\ -cb' - \frac{cb}{a}d' &= 0 & cb' + dd' &= 1 \\ \left(d - \frac{cb}{a}\right) d' &= 1 & d' &= \frac{a}{ad - bc} \end{aligned}$$

2. Let

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & -3 \end{bmatrix}$$

(a) Show that $A^3 + 3A^2 - 2A - I_3 = \mathbf{0}$

$$\begin{aligned} A^3 + 3A^2 - 2A - I_3 &= 0 \\ (A^2 + 3A - 2I_3)A &= I_3 \end{aligned}$$

$$\left(\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & -3 \end{bmatrix}^2 + 3\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & -3 \end{bmatrix} + \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}\right) A = I_3$$

$$\left(\begin{bmatrix} 0 & 1 & -3 \\ 0 & 2 & -5 \\ 1 & -3 & 11 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 3 \\ 3 & 0 & 6 \\ 0 & 3 & -9 \end{bmatrix} + \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}\right) A = I_3$$

$$\begin{bmatrix} -2 & 1 & 0 \\ 3 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & -3 \end{bmatrix} = I_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

(b) Use part (a) to see that A is invertible and compute A^{-1}

Obviously $A^{-1} = A^2 + 3A - 2I_3 = \begin{bmatrix} -2 & 1 & 0 \\ 3 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

3. Let $A \in \mathcal{M}_n$ be a diagonal matrix. Prove that A is invertible if and only if $\text{ent}_{ii}(A) \neq 0$ for all $i \leq n$. Find A^{-1} in this case.

4. If $A, B \in \mathcal{M}_\backslash$ are invertible such that $A + B \neq 0$, does it follow that $(A+B)^{-1}$ exists? Prove or find a counterexample.

Excercise Set 2

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \end{bmatrix}$$

1. For the matrix A in the example above, determine $E_{3\to c3}$ where c is any nonzero real number.

2. For the matrix A in the example above, determine $E_{4\to 4+c2}$ where c is any nonzero real number.

3. Prove that each of $E_{i\leftrightarrow k}, E_{i\to ci}, E_{i\to i+ck}$ is invertible by finding an inverse. Prove that the inverse of an elementary matrix is an elementary matrix.

4. Define a relation $\sim$ on $\mathcal{M}_{m\times n}$ given by $A \sim B$ if and only if there exists a $P \in \mathcal{M}_{m\times n}$ such that $A = PB$ where $P \in \mathcal{M}_{m\times n}$ is a product of elementary matrices.