Chapter 3

4.

Show that if $n+1$ integers are chosen from the set $\{1, 2, \ldots, 2n\}$, then there are always two which differ by 1.

proof

Since we want all the integers to differ by more than one, we can only pickevery other integer from $\{1, 2, \ldots, 2n\}$. This gives us a maximum of n integers. Since we are choosing $n+1$ integers, we know that at least two of them must differ by only one.$\square$

5.

Show that if $n+1$ distinct integers are chosen from the set $\{1, 2, \ldots, 3n\}$, then there are always two which differ by at most 2.

proof

Since we want all the integers to differ by more than two, we can only pickevery third integer from $\{1, 2, \ldots, 3n\}$. This gives us a maximum of n integers. Since we are choosing $n+1$ integers, we know that at least two of them must differ by two or less.$\square$

6.

Generalize Exercises 4 and 5.

hypothesis

If $n+1$ distinct integers are chosen from the set $\{1, 2, \ldots, mn\}$ where m is a positive integer then there are always two which differ by at most $m-1$.

proof

We can select at most n integers which have a difference of m or more. Since we are selecting $n+1$ integers then we must have at least two which differ by $m-1$ or less.$\square$

8.

Use the pigeonhole principle to prove that the decimal expansion of a rational number m/n eventually is repeating. For example,

$$\frac{34,478}{99,900} = 0.345125125125\cdots.$$

proof

We assume the $n > 0$ because if $n = 0$ we don't really have a rational number and if $n < 0$ we can simply multiply by $\frac{-1}{-1}$ to make n positive.

Now we will start building our decimal representation of our number by dividing.

$$m = q_0 n + r_0, \qquad 0 \leq r_0 \leq n - 1$$

Now we have an integer part q_0 and our fractional decimal part $\frac{r_0}{n}$. We expand our fractional decimal digits by multiplying successive remainders by 10 and dividing by n repeatedly. So for our $\frac{1}{10^i}$ place we have:

$$r_{i-1} \cdot 10 = q_i n + r_i$$

This will give us q_i which is the digit in the $\frac{1}{10^i}$th spot. Notice that because we are dividing by n our remainders will always satisfy $0 \leq r_i \leq n - 1$. Now lets take some sequence of n remainders from our fractional expansion. Say $r_i, \ldots, r_{i+n-1}$. Because we have n remainders which can have $n - 1$ possible values, we know from the pigeon hole principle that at least two of these remainders are the same. Lets pick r_j from the sequence $r_i, \ldots, r_{i+n-1}$ such that $r_j = r_i$. Applying the division algorithm to obtain:

$$r_i \cdot 10 = q_{i+1} n + r_{i+1} = r_j \cdot 10$$
$$r_j \cdot 10 = q_{j+1} n + r_{j+1} = r_i \cdot 10$$

We know that because the remainders and quotients of division are unique $q_{i+1} = q_{j+1}$ and $r_{i+1} = r_{j+1}$. Given this we can say from induction that for any integer $1 \leq k$

$$q_{i+k} = q_{j+k}$$
$$r_{i+k} = r_{j+k}$$

And in fact $q_{i+k} = q_{j+k} = q_{i+2(j-i)+k} = q_{i+3(j-i)+k} = q_{i+4(j-i)+k} = \ldots$ and so we see that fractional part of the decimal representation of the ratio will at some point begin to repeat.$\square$

12.

Show by example that the conclusion of the Chinese rmainder theorem (Application 6) need not hold when m and n are not relatively prime.

Take 3 and 9 for m and n. Take 2 and 4 for a and b. Then we should be able to find an x such that:

$$\begin{aligned} x &= 3p + 2 \\ x &= 9q + 4 \\ 3p + 2 &= 9q + 4 \\ 3p &= 9q + 2 \\ 3 &\mid 3p \\ 3 &\nmid 9q + 2 \\ 3p &\neq 9q + 2 \end{aligned}$$

So we see that x does not exist