Notes

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fundamental theorem of ring homomorphism

observation

let $\varphi : R \to S$ be a given ring homomorphism (R,S are commutative rings)

we want to define $\varepsilon : R[x] \to S$ ring homomorphism such that $\begin{cases} \varphi(r) = \varepsilon(r) & \forall r \in R \\ \varepsilon(x) = s & s \text{ fixed } \in S \end{cases}$

then $\varepsilon(a_0 + a_1 x + \cdots + a_n x^n) = \varphi(a_0) + \phi(a_1)s + \cdots + \varphi(a_n)s^n$

construction

we define $R_1 + R_2 + \cdots + R_n = \{(a_1, a_2, \ldots, a_n) : a_i \in R_i\}$ with component wise operations. this is a ring.

definition

given R a commutative ring, we define the characteristic of$R$ to be char R to be the smallest n such that $1_1 + 1_2 + ... + 1_n = 0$. If it doesn't exists we say char R is zero.

prop

the characteristic of an integral domain is either 0 or prime.

proof

if 0 then done, lets, assume that it's not prime.

then char $R = n = \alpha\beta$ and $0 = 1_1 + 1_2 + \cdots + 1_n = 1_1 + 1_2 + \cdots + 1_\alpha + 1_1 + \cdots + 1_\beta = \alpha\beta$. Because we are in an integral domain then $\alpha = 0$ or $\beta = 0$ and so we have a contradiction because $\alpha < n$ and the same for β and so we have a contradiction because n is the smallest to be 0 and so n is prime.

corollary

if K is a field then characteristic is 0 or prime