Jon Allen
HW 10
Lesson 7 problem 1. Find general series solution for PDE and BC's.

$$
\begin{array}{llll}
PDE & u_t = u_{xx} & 0 < x < 1 & 0 < t < \infty \\
BCs & \begin{cases} u(0,t) = 0 \\ u_x(1,t) = 0 \end{cases} & & 0 < t < \infty \\
IC & u(x,0) = x & 0 \le x \le 1 &
\end{array}
$$

$$
\begin{aligned}
u(x,t) &= X(x)T(t) & u_t = u_{xx} &= XT' = X''T \\
\frac{XT'}{XT} &= \frac{X''T}{XT} & \frac{T'}{T} &= \frac{X''}{X} = \mu \\
T' - \mu T &= 0 & X'' - \mu X &= 0
\end{aligned}
$$

First μ is not positive as that would cause $T(t)$ to grow to infinity and therefore u to grow to infinity which doesn't make physical sense. Let's see if $\mu = 0$

$$
\begin{aligned}
X' &= 0 & X(x) &= A + Bx \\
u(0,t) &= X(0)T(t) & u_x(1,t) &= X'(1)T(t) \\
&= AT(t) = 0 & &= BT(t) = 0 \\
A &= 0 & B &= 0
\end{aligned}
$$

Since this gives only the trivial solution ($u(x,t) = 0$) which is not interesting, we will just assume $\mu < 0$.

$$
\begin{aligned}
\mu &= -\lambda^2 & & \\
T' + \lambda^2 T &= 0 & X'' + \lambda^2 X &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\int \lambda^2 \,\mathrm{d}t} T\right) &= e^{\int \lambda^2 \,\mathrm{d}t} T' + \lambda^2 e^{\int \lambda^2 \,\mathrm{d}t} T & r^2 + 0r + \lambda^2 &= 0 \\
e^{\int \lambda^2 \,\mathrm{d}t} T &= \int e^{\int \lambda^2 \,\mathrm{d}t} \cdot 0 \,\mathrm{d}t = A & \frac{-0 \pm \sqrt{0^2 - 4\lambda^2}}{2} &= r \\
T &= Ae^{-\lambda^2 t} & 0 \pm \lambda i &= r \\
& & B\cos(\lambda x) + C\sin(\lambda x) &= X
\end{aligned}
$$

$$
\begin{aligned}
u(x,t) = XT &= e^{-\lambda^2 t}\left[A\sin(\lambda x) + B\cos(\lambda x)\right] & \lambda e^{-\lambda^2 t}\left[A\cos(\lambda x) - B\sin(\lambda x)\right] &= u_x \\
u(0,t) &= e^{-\lambda^2 t} B = 0 & \lambda e^{-\lambda^2 t}\left[A\cos(\lambda) - B\sin(\lambda)\right] &= u_x(1,t) = 0 \\
B &= 0 & A\lambda e^{-\lambda^2 t}\cos(\lambda) &= 0 \\
& & \cos(\lambda) &= 0 \\
\lambda_n &= \frac{2n-1}{2}\pi & u(x,t) &= \sum_{n=1}^{\infty} a_n e^{-{\lambda_n}^2 t} \sin(\lambda_n x)
\end{aligned}
$$

Jon Allen
HW 11
Lesson 7 problem 1. Solve the IC. Explain how orthogonality is used.

$$\begin{array}{llll} PDE & u_t = u_{xx} & 0 < x < 1 & 0 < t < \infty \\ BCs & \begin{cases} u(0,t) = 0 \\ u_x(1,t) = 0 \end{cases} & & 0 < t < \infty \\ IC & u(x,0) = x & 0 \le x \le 1 & \end{array}$$

$$\lambda_n = \frac{2n-1}{2}\pi \qquad u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n{}^2 t} \sin(\lambda_n x)$$

$$u(x,0) = x = \sum_{n=1}^{\infty} a_n \sin(\lambda_n x) \qquad \int_0^1 \xi \sin(\lambda_m \xi)\,\mathrm{d}\xi = \sum_{n=1}^{\infty} a_n \int_0^1 \sin(\lambda_n \xi)\sin(\lambda_m \xi)\,\mathrm{d}\xi$$

Because $\{\sin(\lambda_i x)\}_{0 \le i \le n}$ are orthogonal functions we can convert the above equation into the following.

$$\begin{aligned} \int_0^1 \xi \sin(\lambda_m \xi)\,\mathrm{d}\xi &= a_m \int_0^1 \sin(\lambda_m \xi)^2\,\mathrm{d}\xi \\ &= a_m \cdot -\frac{\sin(2\lambda_m) - 2\lambda_m}{4\lambda_m} \\ &= a_m \frac{\lambda_m - \sin(\lambda_m)\cos(\lambda_m)}{2\lambda_m} \end{aligned}$$

Recall that we discovered in HW 10 that $\cos(\lambda) = 0$

$$\begin{aligned} &= \frac{a_m}{2} \\ a_n &= 2\int_0^1 \xi \sin(\lambda_n \xi)\,\mathrm{d}\xi \\ &= 2\left[\frac{\sin(\lambda_n x) - \lambda_n x\cos(\lambda_n x)}{\lambda_n{}^2}\right]_0^1 \\ &= 2\left[\frac{\sin(\lambda_n 1) - \lambda_n 1\cos(\lambda_n 1)}{\lambda_n{}^2} - \frac{\sin(\lambda_n 0) - \lambda_n 0\cos(\lambda_n 0)}{\lambda_n{}^2}\right] \\ &= 2\left[\frac{\sin(\lambda_n)}{\lambda_n{}^2} - \frac{0}{\lambda_n{}^2}\right] = 2\left[\frac{\sin(\lambda_n)}{\lambda_n{}^2}\right] \\ &= 2\left[\frac{\sin\left(\frac{2n-1}{2}\pi\right)}{\left(\frac{2n-1}{2}\pi\right)^2}\right] \\ &= \frac{8\sin\left(\frac{2n-1}{2}\pi\right)}{(2n-1)^2\pi^2} \\ &= -1^{(n+1)}\frac{8}{(2n-1)^2\pi^2} \\ u(x,t) &= \sum_{n=1}^{\infty} -1^{(n+1)}\frac{8}{(2n-1)^2\pi^2} e^{-\lambda_n{}^2 t}\sin(\lambda_n x) \\ &= \sum_{n=1}^{\infty} -1^{(n+1)}\frac{8}{(2n-1)^2\pi^2} e^{-\left(\frac{2n-1}{2}\pi\right)^2 t}\sin\left[\left(\frac{2n-1}{2}\pi\right)x\right] \end{aligned}$$

Jon Allen
HW 12
Lesson 7 problem 3. Find general series solution for PDE and BC's.

$$
\begin{array}{llll}
PDE & u_t = u_{xx} & 0 < x < 1 & 0 < t < \infty \\
BCs & \begin{cases} u_x(0,t) = 0 \\ u_x(1,t) = 0 \end{cases} & & 0 < t < \infty \\
IC & u(x,0) = x & 0 \le x \le 1 &
\end{array}
$$

$$
\begin{aligned}
u(x,t) &= X(x)T(t) & u_t = u_{xx} &= XT' = X''T \\
\frac{XT'}{XT} &= \frac{X''T}{XT} & \frac{T'}{T} &= \frac{X''}{X} = \mu \\
T' - \mu T &= 0 & X'' - \mu X &= 0
\end{aligned}
$$

We will just assume $\mu \le 0$.

$$
\begin{aligned}
\mu &= -\lambda^2 \\
T' + \lambda^2 T &= 0 & X'' + \lambda^2 X &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\int \lambda^2\,\mathrm{d}t} T\right) &= e^{\int \lambda^2\,\mathrm{d}t} T' + \lambda^2 e^{\int \lambda^2\,\mathrm{d}t} T & r^2 + 0r + \lambda^2 &= 0 \\
e^{\int \lambda^2\,\mathrm{d}t} T &= \int e^{\int \lambda^2\,\mathrm{d}t} \cdot 0\,\mathrm{d}t = A & \frac{-0 \pm \sqrt{0^2 - 4\lambda^2}}{2} &= r \\
T &= Ae^{-\lambda^2 t} & 0 \pm \lambda i &= r \\
& & B\cos(\lambda x) + C\sin(\lambda x) &= X
\end{aligned}
$$

$$
\begin{aligned}
u(x,t) = XT &= e^{-\lambda^2 t}\left[A\sin(\lambda x) + B\cos(\lambda x)\right] & \lambda e^{-\lambda^2 t}\left[A\cos(\lambda x) - B\sin(\lambda x)\right] &= u_x \\
u_x(0,t) &= \lambda e^{-\lambda^2 t}\left[A\cos(\lambda 0) - B\sin(\lambda 0)\right] = 0 & \lambda e^{-\lambda^2 t}\left[A\cos(\lambda) - B\sin(\lambda)\right] &= u_x(1,t) = 0 \\
A\lambda e^{-\lambda^2 t} &= 0 \\
A &= 0 & -B\lambda e^{-\lambda^2 t}\sin(\lambda) &= 0 \\
& & \sin(\lambda) &= 0 \\
\lambda_n &= n\pi & u(x,t) &= \sum_{n=1}^{\infty} a_n e^{-\lambda_n{}^2 t}\cos(\lambda_n x)
\end{aligned}
$$

Jon Allen
HW 13
Lesson 7 problem 3. Solve the IC. Note the questions about steady state behavior.

$$
\begin{array}{llll}
PDE & u_t = u_{xx} & 0 < x < 1 & 0 < t < \infty \\
BCs & \begin{cases} u_x(0,t) = 0 \\ u_x(1,t) = 0 \end{cases} & & 0 < t < \infty \\
IC & u(x,0) = x & 0 \le x \le 1 &
\end{array}
$$

$$\lambda_n = n\pi \qquad u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n{}^2 t} \cos(\lambda_n x)$$

$$u(x,0) = x = \sum_{n=1}^{\infty} a_n \cos(\lambda_n x) \qquad \int_0^1 \xi \cos(\lambda_m \xi)\,\mathrm{d}\xi = \sum_{n=1}^{\infty} a_n \int_0^1 \cos(\lambda_n \xi)\cos(\lambda_m \xi)\,\mathrm{d}\xi$$

Because $\{\cos(\lambda_i x)\}_{0 \le i \le n}$ are orthogonal functions we can convert the above equation into the following.

$$
\begin{aligned}
\int_0^1 \xi \cos(\lambda_m \xi)\,\mathrm{d}\xi &= a_m \int_0^1 \cos(\lambda_m \xi)^2\,\mathrm{d}\xi \\
&= a_m \cdot \frac{\sin(2\lambda_m) + 2\lambda_m}{4\lambda_m} \\
&= a_m \frac{\lambda_m + \sin(\lambda_m)\cos(\lambda_m)}{2\lambda_m}
\end{aligned}
$$

Recall that we discovered in HW 12 that $\sin(\lambda) = 0$

$$
\begin{aligned}
&= \frac{a_m}{2} \\
a_n &= 2\int_0^1 \xi \cos(\lambda_n \xi)\,\mathrm{d}\xi \\
&= 2\left[\frac{\lambda_n x \sin(\lambda_n x) + \cos(\lambda_n x)}{\lambda_n{}^2}\right]_0^1 \\
&= 2\left[\frac{\lambda_n 1 \sin(\lambda_n 1) + \cos(\lambda_n 1)}{\lambda_n{}^2} - \frac{\lambda_n 0 \sin(\lambda_n 0) + \cos(\lambda_n 0)}{\lambda_n{}^2}\right]_0^1 \\
&= 2\left[\frac{\cos(\lambda_n)}{\lambda_n{}^2} - \frac{1}{\lambda_n{}^2}\right] = 2\left[\frac{\cos(\lambda_n) - 1}{\lambda_n{}^2}\right] \\
&= 2\left[\frac{\cos(n\pi) - 1}{(n\pi)^2}\right] \\
&= \frac{2((-1)^n - 1)}{(n\pi)^2} \\
u(x,t) &= \sum_{n=1}^{\infty} \frac{2((-1)^n - 1)}{(n\pi)^2} e^{-(n\pi)^2 t} \cos(n\pi x)
\end{aligned}
$$

Jon Allen
HW 14
Lesson 9 problem 4. Find general series solution for PDE and BC's.

$$
\begin{array}{rlll}
PDE & u_t = u_{xx} + \sin(\pi x) & 0 < x < 1 & 0 < t < \infty \\
BCs & \begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases} & & 0 < t < \infty \\
IC & u(x,0) = 0 & 0 \le x \le 1 &
\end{array}
$$

We need to find the coefficients $T_n(t)$ in

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t)\sin(n\pi x)$$

Substituting into the original problem we have

$$\sum_{n=1}^{\infty} T_n(t)\sin(n\pi x) = -\sum_{n=1}^{\infty}(n\pi)^2 T_n(t)\sin(n\pi x) + \sum_{n=1}^{\infty} f_n(t)\sin(n\pi x)$$
$$\sum_{n=1}^{\infty} T_n(t)\sin 0 = 0$$
$$\sum_{n=1}^{\infty} T_n(t)\sin(n\pi) = 0$$
$$\sum_{n=1}^{\infty} T_n(0)\sin(n\pi) = 0$$

We can rewrite the pde to get

$$\sum_{n=1}^{\infty} T_n(t)\sin(n\pi x) + (n\pi)^2 T_n(t)\sin(n\pi x) - f_n(t)\sin(n\pi x) = 0$$

Jon Allen
HW 15
We substitute this expansion into the problem to get, notice we establish orthagonality with different values of n

$$T_n' + (n\pi)^2 T_n = f_n(t) = 2\int_0^1 \sin(\pi x)\sin(n\pi x)\,\mathrm{d}x = \begin{cases} 1 & n = 1 \\ 0 & n \neq 1 \end{cases}$$
$$T_n(0) = 2\int_0^1 0\,\mathrm{d}\xi = 0$$

We really only have two cases to worry about

$$(n=1) \qquad \left.\begin{aligned} T_1' + \pi^2 T_1 &= 1 \\ T_1(0) &= 0 \end{aligned}\right\} \Rightarrow \qquad \begin{aligned} e^{\int \pi^2\,\mathrm{d}t} T_1 &= \int 1\,\mathrm{d}t \\ T_1 &= xe^{-\pi^2 t} + c_1 \\ &= xe^{-\pi^2 t} \end{aligned}$$

$$(n \geq 2) \qquad \left.\begin{aligned} T_2' + \pi^2 T_2 &= 0 \\ T_2(0) &= 0 \end{aligned}\right\} \Rightarrow \qquad \begin{aligned} e^{\int \pi^2\,\mathrm{d}t} T_2 &= \int 0\,\mathrm{d}t \\ T_2 &= c_1 e^{-\pi^2 t} \\ &= 0 \end{aligned}$$

So our solution looks like

$$u(x,t) = xe^{-t\pi^2}\sin(\pi x)$$