Notes

January 28, 2015

last quiz answers: ?

on board

f bounded on $[a, b]$
$D_f = \{x : f \text{ is not continuous at } x\}$

theorem

f is Riemann integrable iff $m * (D_f) = 0$

definition

if $T \subseteq [a, b]$ oscillation of f on T

$$\Omega_f(T) = \sup\{f(x) - f(y) : x, y \in T\}$$

for example

$f(x) = x^2, T = [0, 5]$ then oscillation is $\Omega_f = 25$
$f(x) = x^2, T = \mathbb{Q}^C \cap [0, 5]$ then oscillation is 25 but this time $5 \notin T$ and $0 \notin T$ and so we need the supremum

we are interested in how much the oscillation happens as we approach x

def

$\omega_f(x) = \lim_{h \to 0^+} \Omega_f(B(x, h) \cap [a, b])$

facts

1. $\omega_f(x) = 0$ iff f is continuous at x

   ### theorem

   let $\varepsilon > 0$ be given. if $\omega_f(x) < \varepsilon$ for all $x \in [a, b]$ then $\exists \delta > 0$ so that when $\Omega_f(T) < \varepsilon$ for any closed interval $T \subseteq [a, b]$ with $m * (T) < \delta$

for every $x \in [a, b]$ there is $B(x, \delta_x)$ such that $\Omega_f(B(x, \delta_x) \cap [a, b]) < \varepsilon$. $\{B(x, \delta_x)\}_{x \in [a,b]}$ is an open cover for $[a, b]$ there is $x_1, \ldots, x_n \in [a, b]$ with $[a, b] \subseteq \bigcup_{i=1}^n B(x_i, \frac{\delta_{x_i}}{2})$. $\delta = \min\{\frac{\delta_{x_i}}{2}\}$

if $m * (T) < \delta$ then any two points are are within δ of x_i. $T \cap B(x_i, \frac{\delta_{x_i}}{2}) \neq \emptyset$ so $T \subseteq B(x_i, \delta_{x_i})$ and so $\Omega_f(T) < \varepsilon$

lemma

let $J_\varepsilon = \{x \in [a,b] : \omega_f(x) \geq \varepsilon\}$ then J_ε is closed and $D_f = \bigcap\limits_{n=1}^{\infty} J_{\frac{1}{n}}$

second part is just lemma? continuous=measure 0

first part $y \in J_\varepsilon^C$ then $\omega_f(y) < \varepsilon$ there is δ such that $\Omega_f(B(y,\delta) \cap [a,b] < \varepsilon$, $B(y,\delta) \subseteq J_\varepsilon^C$

if $z \in B(y,\delta)$ consider $bB(z,\delta)$ where δ' is chosen so that $B(z,\delta') \subseteq B(y,\delta)$. notice that $\omega_f(z) \geq \Omega_f(B(z,\delta') \cap [a,b]) \leq \Omega_f(B(y,\delta) \cap [a,b] < \varepsilon$ so $z \notin J_\varepsilon$.

now for proof of thrm

we want to show that f is reimann integrable iff $m*(D_f) = 0$.

assume $m*(D_f) > 0$. $D = \bigcup_{r=1}^{\infty} J_{\frac{1}{r}}$ with $J_{\frac{1}{r}} = \{x : \omega_f(x) \geq \frac{1}{r}\}$

these are closed sets, closed sets are measurable, and this is countable union so the whole thing (D) is measurable. and since $m*(D_f) > 0$ there is $N > 0$ such that $m*(J_{\frac{1}{n}}) > 0$

if $J_N \subseteq \bigcup_{i=1}^{\infty}(a_i,b_i)$ then $\sum b_i - a_i > \varepsilon$ for some $\varepsilon > 0$.

now let P be a partition.

$U(P,f) - L(P,f)$ (upper-lower) $= \sum\limits_{k=1}^{n} M_k - m_k\delta_k$ check page 114ish

$= \left[\sum\limits_{s_1}(M_k - m_k)delta_k\right] + \left[\sum\limits_{s_2} M_k - m_k delta_k\right]$

$s_1 = J_{1/N} \cap (x_{k-1}, x_k) \neq \emptyset$

$s_2 = J_{1/N} \cap (x_{k-1}$

1. $D \subseteq \bigcup_{k \in S_1}(x_{k-1}, x_k)$

   $M_k(f) - m_k(f) \geq \frac{1}{N}$ for all $k \in S$

   $\sum\limits_{k \in S_1} \delta_k > \varepsilon$

   $U(f_i P) - L(f,P) \geq \sum\limits_{s_1} M_k - m_k)\delta_k \geq \frac{1}{N}\sum\limits_{s_1}\delta_k = \frac{1}{N}\varepsilon$ so f not integrable