Notes

November 11, 2014

$f : \mathbb{R}^n \to \mathbb{R}^m$ is continous iff $\forall C$ is closed in R^m then $f^{-1}(C)$ is closed in $\mathbb{R}^n$.o

assume f is continuous. let $c \in \mathbb{R}^n$ be closed. we want to show $f^{-1}(C)$ is closed.

first we pick a sequence $x_k \in f^{-1}(C)$. or $f(x_k) \in C$. we are picking x_k so that it converges to something (say y). now because $f(x)$ is continous, then $f(x_k) \to f(y)$. and because $f(y) \in C$ then C is closed.

true false

$f : \mathbb{R}^n \to \mathbb{R}^m$ is continous

1. fi $A \subseteq \mathbb{R}^n$ is open then $f(A)$ is open
2. fi $A \subseteq \mathbb{R}^n$ is closed then $f(A)$ is closed
3. fi $A \subseteq \mathbb{R}^n$ is compact then $f(A)$ is compact
4. fi $A \subseteq \mathbb{R}^m$ is open then $f^{-1}(A)$ is open
5. fi $A \subseteq \mathbb{R}^m$ is compact then $f(A)$ is compact

unions, intersections of $f$or f^{-1} over two sets. $f(A^c) = (f(A))^c$ and same with f^{-1}.

5.5.A

note that $g'(x)$ is unbounded on $(0, \infty)$. so we cannot sed the "bounded slope " argueent for uniform continuity. need to show $\exists r = r(\varepsilon)$ such that if $|x - y| < r$ then $|\sqrt{x} - \sqrt{y}| < \varepsilon$

if we prove this then $\sqrt{x} - \sqrt{y} \le \sqrt{x - y}\ \sqrt{r} = \varepsilon$

$$x - y = (\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y}) \ge (\sqrt{x} - \sqrt{y})^2$$
$$\sqrt{x - y} \ge \sqrt{x} - \sqrt{y}$$

5.5.D

show that $f(x) = x^p$ is not uniformally continuous on $\mathbb{R}$ is $p > 1$

$$\frac{x^p - y^p}{x - y} = |xx^{p-1} - y^p| = xx^{p-1} - x^{p-1}y + x^{p-1}y - y^p|le|x^{p-1}||x - y| + |y||x^{p-1} - y^{p-1}|$$

T is continouous, A is closed, it T(A) closed?

no, hint, look for $T : \mathbb{R}^2 \to \mathbb{R}$ or $\mathbb{R}^2$ where T is a linear transform (matrix), example $f(x) = \frac{1}{x^2+1}$ on $\mathbb{R}$, $f(\mathbb{R}) = (0.1]$

5.4.J

let A be compact in $\mathbb{R}^n$. show that $\forall x \in \mathbb{R}^n \exists a \in A$ such that a is the closest point from A so x. that is $||x - a|| \leq ||x - y|| \forall y \in A$.

proof

x is a fixed point. $f(y) = ||x - y||$. A is compact, if we showthat f is continous then f has min on A.

if $||z - y|| < r$ then $|f(z) - f(y)| < \varepsilon$. $|||x - z|| - ||x - y||| \leq ||z - y||$. need $r = \epsilon$

5.5.HI

let f be continuous on $(0, 1]$ show f is uniformallly continuous iff $\lim_{x \to 0^+} f(x)$ exists.

asssuming f is uniformally continuous, $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < r$. take x_k any decreasing sequence converging to zero. we use the cauchy condition. $\forall r \exists N$ such that $|x_k - x_l| < r$ if $k, l \geq N$. so $|f(x_k) - f(x_l)| < \varepsilon$ then $f(x_k)$ is cauchy. and so $\lim_{k \to \infty} f(x_k) = L = \lim_{x \to 0^+} f(x)$

other way, we need to shwo that r does not depend on ε