Homework

Jon Allen

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Section 4.3: 9, 12 Section 4.4: 4 (d), 17

4.3

9. If we multiply two elements $[a+bx], [c+dx]\mathbb{R}[x]/\langle x^2+x+1\rangle$ then we get $ac+(ad+bc)x+bdx^2$. Note that $x^2+x+1 \equiv 0 \mod x^2+x+1$ and so $x^2 \equiv -x-1 \mod x^2+x+1$ which means $ac+(ad+bc)x+bdx^2 = ac-bd+(ad+bc-bd)x$

   We first need to construct $\phi : \mathbb{C} \to \mathbb{R}[x]/\langle x^2+x+1\rangle$. We know that $\phi(1)=1$ so we have half done. Because $i^2=-1$ we need to find $\phi(i)^2=-1$. Now $(a+bx)^2 = a^2+2abx+b^2x^2 = a^2-b^2+(2ab-b^2)x = -1$ and so

$$\begin{aligned}
2ab - b^2 &= 0 \\
a^2 - b^2 &= -1 \\
2a &= b \\
a^2 - 4a^2 &= a^2(1-4) = -1 \\
a^2 &= \frac{1}{3}
\end{aligned}$$

   We can choose $a = \frac{1}{\sqrt{3}}$ or $a = -\frac{1}{\sqrt{3}}$. We choose $a = \frac{1}{\sqrt{3}}$ and say $\phi(a+bi) = a+b(\frac{1}{\sqrt{3}}+\frac{2}{\sqrt{3}}x) = a+bz$.

$$\begin{aligned}
\phi((a+bi)+(c+di)) &= \phi(a+c+(b+d)i) \\
&= a+c+(b+d)z \\
&= a+bz+c+dz \\
&= \phi(a+bi)+\phi(c+di) \\
\phi((a+bi)(c+di)) &= \phi(ac-bd^2+(ad+bc)i) \\
&= ac-bd+(ad+bc)z \\
\phi(a+bi)\phi(c+di) &= (a+bz)(c+dz) \\
&= ac+(ad+bc)z+bdz^2 \\
&= ac+(ad+bc)z-bd \\
&= ac-bd+(ad+bc)z
\end{aligned}$$

   I claim that surjectivity is obvious because both fields are reals plus something times a real.

12. We build need to construct $\phi : \mathbb{Q}(\sqrt{2}) \to \mathbb{Q}[x]/\langle x^2-2\rangle$. We already have $\phi(1)=1$ and so we are half done. First lets find $\phi(\sqrt{2})^2 = \phi(2)$ Note that $x^2 \equiv 2 \mod x^2-2$

$$\begin{aligned}
(a+bx)^2 &= a^2+2abx+b^2x^2 \\
&= a^2+2b^2+2abx \\
2 &= a^2+2b^2 \\
0 &= a
\end{aligned}$$

$$1 = b^2$$

We choose $b = 1$ and so $\phi(a + b\sqrt{2}) = a + bx$. Which is obvious to me now, but I already did the work, so I'm gonna leave it.

$$\begin{aligned}
\phi(a + b\sqrt{2}) + \phi(c + d\sqrt{2}) &= a + c + (b + d)x = \phi(a + c + (b + d)\sqrt{2} \\
\phi(a + b\sqrt{2}) \cdot \phi(c + d\sqrt{2}) &= ac + (ad + bc)x + bdx^2 = ac + 2bd + (ad + bc)x \\
\phi((a + b\sqrt{2}) \cdot (c + d\sqrt{2})) &= \phi(ac + 2bd + (ad + bc)\sqrt{2}) = ac + 2bd + (ad + bc)x
\end{aligned}$$

4.4 4. (d) $x^2 + 2x - 5$
$(x - 1)^2 + 2x - 2 - 5 = x^2 - 2x + 1 + 2x - 2 - 5 = x^2 - 6$
$(x + 1)^2 + 2x + 2 - 5 = x^2 + 2x + 1 + 2x + 2 - 5 = x^2 + 4x - 2$
So in either shift we have 2 dividing all the coefficients, but not 1 and 4 does not divide 6 or two. So it is irreducible

17. (a)

$$\begin{aligned}
f'(x) &= 6x^5 + 3x^2 \\
f''(x) &= 30x^4 + 6x \\
f^3(x) &= 120x^3 + 6 \\
f^4(x) &= 360x^2 \\
f^5(x) &= 720x \\
f^6(x) &= 720 \\
f(x + c) &= x^6 + 6cx^5 + 15c^2x^4 + (20c^3 + 1)x^3 + (15c^4 + 3c)x^2 + (6c^5 + 3c^2)x + c^6 + c^3 + 1
\end{aligned}$$

We see immediately that $x - 1$ doesn't work because $(-1)^6 + (-1)^3 + 1 = 1$ which doesn't give us anything to divide. $x + 1$ gives us a divisor of 3 that works. $x - 2$ gives us $a_0 = 3 \cdot 19$ which works for 3 again. $x + 2$ gives us $a_0 = 73$ which is prime, so that doesn't work.