Notes

March 7, 2014

homework 14-15
sturm-lionville expansions. The last stage of separation of variales (on finite intevals)

$$\text{PDE} \qquad u_t = u_{xx} + \underbrace{f(x,t)}_{\sin(\lambda,x)} \qquad \text{on } 0 < x < 1$$

$$\begin{aligned} \text{BC} \qquad u(0,t) &= 0 \\ u_x(1,t) + u(1,t) &= 0 \end{aligned}$$

standard approach (incorrect). theeigenfunctions are $\sin(n\pi x) = X_n(x)$ do not satisfy the BC. The problem is driven by the boundary condition.

correct answer looks like this?

$$u(x,t) = \sum c_n T_n(t) X_n(x)$$

where $X_n(x)$ satisfy the BC. and $X_n(x)$ satisfies the separated equation $\frac{X_n{}''}{X_n(s)} = -\lambda_n$.

In lesson 9, there is a detailed example that has

$$\begin{aligned} \text{BC} \qquad & u(0,t) = 0 \\ & u(1,t) = 0 \\ \text{PDE} \qquad & u_t = u_{xx} + f(x,t) \end{aligned}$$

solution is written out in detail and $X_n(x) = \sin(n\pi x)$ appear (because of the BC).

In class, BC looked more like (see notes on 2/7)

$$\begin{aligned} \text{PDE} \qquad & u_t = \alpha^2 u_{xx} + f(x,t) \\ \text{BC} \qquad & 0 = \alpha_1 u_x(0,t) + \beta_1 u(0,t) \\ & 0 = \alpha_2 u_x(1,t) + \beta_2 u(1,t) \\ \text{IC} \qquad & u(x,0) = \phi(x) \end{aligned}$$

see page 65-66 in text (81-82). Step 1 on page 66. set $u(x,t) = \sum c_n T_n(t) X_n(x)$ where $X_n(x)$ are eigenfunctions for the homogeneous PDE (and homogeneous BC)

Try $T(t)X(x)$. $u_t = \alpha^2 u_{xx}$ gives $\frac{T'(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda^2 \leq 0$. So $X''(x) + \lambda^2 X(x) = 0$.

$$\begin{aligned} \text{BC} \qquad & X(0) = 0 \\ & X'(1) + X(1) = 0 \end{aligned}$$

$$\begin{aligned} X(x) &= a\cos(\lambda x) + b\sin(\lambda x) \\ X(0) &= 0 = a\cos(0) + b\sin(0) \to a = 0 \\ X'(x) &= +b\lambda\cos(\lambda x) \end{aligned}$$

$$
\begin{aligned}
X'(1) + X(1) &= 0 \\
X_n(x) &= \sin(\lambda_n x) \text{ where } \tan(\lambda) = -\lambda \\
u(x,t) &= \sum_{n=1}^{\infty} T_n(t)\sin(\lambda_n x) \\
f(x,t) &= \sum f_n(t)\sin(\lambda_n x) \\
{T_n}'(t) &= \alpha_2(-{\lambda_n}^2)T_n(t) + f_n(t)
\end{aligned}
$$

coefficient of $\sin(\lambda_n x)$

we need $\lambda\cos(\lambda) + \sin(\lambda) = 0, \lambda > 0$. $\cos(\lambda) = 0$? no, else $\sin(\lambda) = 0$. So divide out by cosine to get $\lambda + \tan(\lambda) = 0$. Showed that eigenfunctions $\sin(\lambda x)$ are orthogonal.

Homework due date extended to 3/14