first order linear

$$\frac{\mathrm{d}y}{\mathrm{d}t}+p(t)y=q(t) \qquad \mu(t)=e^{\int p(t)\,\mathrm{d}t} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\mu(t)y\right)=\mu(t)\frac{\mathrm{d}y}{\mathrm{d}t}+p(t)\mu(t)y \qquad \mu(t)y=\int \mu(t)q(t)\,\mathrm{d}t$$

exact $0=M(t,y)\,\mathrm{d}t+N(t,y)\,\mathrm{d}y$ $\qquad \dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial t} \qquad \displaystyle\int M(t,y)\,\mathrm{d}t+\phi(y)=f(t,y)$

$$\phi'(y)=N(x,y)-\frac{\mathrm{d}}{\mathrm{d}y}\left(\int M(t,y)\,\mathrm{d}t\right) \qquad f(t,y)=\int M(t,y)\,\mathrm{d}t+\int \phi'(y)\,\mathrm{d}y$$

Solution is $f(t,y)=C$

bernoulli $\qquad \dfrac{\mathrm{d}y}{\mathrm{d}t}+p(t)y=q(t)y^n \qquad \dfrac{1}{y^n}\dfrac{\mathrm{d}y}{\mathrm{d}t}+p(t)y^{1-n}=q(t)$

$$w=y^{1-n} \qquad \frac{\mathrm{d}w}{\mathrm{d}t}=(1-n)\frac{1}{y^n}\frac{\mathrm{d}y}{\mathrm{d}t} \qquad \frac{\mathrm{d}w}{\mathrm{d}t}+(1-n)p(t)w=(1-n)q(t)$$

Solve as first order linear, then back substitute
homogeneous

$$M(t,y)\,\mathrm{d}t+N(t,y)\,\mathrm{d}y=0 \qquad M(xt,xy)+N(xt,xy)=x^n\left(M(t,y)+N(t,y)\right)$$

$$\mathrm{d}y=w\,\mathrm{d}t+t\,\mathrm{d}w \qquad \mathrm{d}t=w\,\mathrm{d}y+y\,\mathrm{d}w$$

Substitute with $y=wt$ if $N(t,y)$ is simpler and $t=wy$ if $M(t,y)$ is simpler. Solve as a separable equation

population growth and decay $\qquad$ Logistical equation

$$y(t)=y_0e^{kt} \qquad y=\frac{ry_0}{ay_0+(r-ay_0)e^{-rt}}$$

Newton's law of cooling $\quad T(t)=\left(T_0-T_s\right)e^{kt}+T_s$

Newton's laws of motion $\quad v=v_0+at \qquad s=v_0t+\dfrac{1}{2}at^2 \qquad v^2={v_0}^2+2as$

Acceleration is $a=g=9.8\mathrm{m/sec}^2=32\mathrm{ft/sec}^2$ and position is s and velocity is v.
reduction of order given $y''+p(t)y'+q(t)y=0$ and a known solution y_1 then full solution is given by

$$y_s=c_1y_1+c_2y_2=c_1y_1+c_2v(t)y_1 \qquad v(t)=\int \frac{1}{{y_1}^2}e^{-\int p(t)\,\mathrm{d}t}\,\mathrm{d}t$$

second order linear homogeneous with constant coefficient

$$ay''+by'+cy=0\rightarrow ar^2+br+c=0 \qquad r=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$y_s=\begin{cases} c_1e^{r_1t}+c_2e^{r_2t} & r_1\neq r_2\\ (c_1+c_2t)e^{rt} & r_1=r_2\\ e^{\alpha t}\left[c_1\cos(\beta t)+c_2\sin(\beta t)\right] & r=\alpha\pm\beta i\end{cases}$$

method of undetermined coefficients solution of $ay''+by'+cy=f(t)$ is $y_s=y_h+y_p$ where y_h is solution to corresponding homogeneous equation

$$f(t)=t^me^{\alpha t} \quad \text{or} \quad f(t)=t^me^{\alpha t}\sin\beta t \quad \text{or} \quad f(t)=t^me^{\alpha t}\cos\beta t$$

$$S=\{e^{\alpha t},e^{\alpha t}t,e^{\alpha t}t^2,...,e^{\alpha t}t^m\} \qquad S=\left\{\begin{matrix} e^{\alpha t}\sin\beta t,e^{\alpha t}\cos\beta t,te^{\alpha t}\sin\beta t,te^{\alpha t}\cos\beta t,\\ t^2e^{\alpha t}\sin\beta t,t^2e^{\alpha t}\cos\beta t,...,t^me^{\alpha t}\sin\beta t,t^me^{\alpha t}\cos\beta t\end{matrix}\right\}$$

if $S_h \cap S_p \neq \emptyset$ then $S_p \to t^n S_p$. This will make y_h and y_p linearly independant. If $f(t)$ has more than one term then S_p is the union of the solution set for each term. Throw out constant coefficients in $f(t)$

$$y_p = a_1 S_p[1] + a_2 S_p[2] + ... + a_m S_p[m] \quad \text{Solve for all } a_n \text{ and we are done.}$$

variation of parameters $y'' + p(t)y' + q(t)y = f(t)$ for any $f(t)$. More general than undetermined coefficients. W refers to the Wronskian. Need to be able to find homogeneous solution for this to work.

$$y_s = y_h + y_p \qquad y_h = c_1 y_1 + c_2 y_2 \qquad y_p = u_1 y_1 + u_2 y_2 \qquad {u_1}' = -\frac{y_2 f}{W} \qquad {u_2}' = \frac{y_1 f}{W}$$

cauchy-euler $ax^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + bx\frac{\mathrm{d}y}{\mathrm{d}x} + cy = f(x) \qquad x = e^t \qquad t = \ln x \qquad x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} - \frac{\mathrm{d}y}{\mathrm{d}t} \qquad x\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}$

spring motion initial position is α initial velocity is β stretch is s $g = 32ft/s^2$ force(weight) is lb or N, mass is slugs or kg, length is ft or m, k is lb/ft or N/m and time is s. down is positive, up is negative

$$m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + kx = 0 \quad x(0) = \alpha \qquad x'(0) = \beta \qquad x(t) = \alpha \cos \omega t + \frac{\beta}{\omega} \sin \omega t \qquad \omega = \sqrt{\frac{k}{m}} \qquad F = ks \qquad F = mg = ma$$

Laplace

$$\mathcal{L}\{f(t)\} = \int_0^t e^{-st} f(t)\,\mathrm{d}t \qquad \mathcal{L}\{1\} = \frac{1}{s} \qquad \mathcal{L}\{e^{at}\} = \frac{1}{s-a} \qquad \mathcal{L}\{\sin kt\} = \frac{k}{s^2+k^2}$$

$$\mathcal{L}\{\cos kt\} = \frac{s}{s^2+k^2} \qquad \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \qquad \mathcal{L}\{e^{at} f(t)\} = F(s-a) \qquad \mathcal{L}\{t^n f(t)\} = (-1)^n \frac{\mathrm{d}^n F(s)}{\mathrm{d}s^n}$$

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0) \qquad \mathcal{L}\{f''(t)\} = s^2 \mathcal{L}\{f(t)\} - sf(0) - f'(0)$$

$$\mathcal{L}\{f'''(t)\} = s^3 \mathcal{L}\{f(t)\} - s^2 f(0) - sf'(0) - f''(0) \qquad f(t) * g(t) = \int_0^t f(t-v) g(v)\,\mathrm{d}v$$

$$\mathcal{L}\{f(t) * g(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$$

matrice

$$\mathbf{A}\mathbf{v} = \lambda \mathbf{v} \qquad (-1)^n \det(\mathbf{A} - \lambda \mathbf{I}) = 0 \qquad (\mathbf{A} - \lambda_n \mathbf{I})\mathbf{v}_n = 0 \qquad \lambda_{1,2} = \alpha \pm \beta i, \beta \neq 0 \quad \mathbf{v}_{1,2} = \mathbf{a} \pm \mathbf{b}i$$

$$\mathbf{X}' = \mathbf{A}\mathbf{X} \to \mathbf{X}(t) = c_1 \mathbf{v}_1 e^{\lambda_1 t} + c_2 \mathbf{v}_2 e^{\lambda_2 t} + \cdots c_n \mathbf{v}_n e^{\lambda_n t} \qquad \mathbf{X}_{1,2} = e^{\alpha t}(\mathbf{a} \cos \beta t \pm \mathbf{b} \sin \beta t)$$

trigonometric identities

$$\sin x = \frac{1}{\csc x} \qquad \sin(-x) = -\sin x \qquad \cos(-x) = \cos x$$

$$\sin(u \pm v) = \sin u \cos v \pm \cos u \sin v \qquad \cos(u \pm v) = \cos u \cos v \mp \sin u \sin v \qquad \sin 2u = 2 \sin u \cos u$$

$$\cos 2u = \cos^2 u - \sin^2 u \qquad \cos 2u = 2\cos^2 u - 1 \qquad \cos 2u = 1 - 2\sin^2 u$$

$$\sin^2 u = \frac{1 - \cos 2u}{2} \qquad \cos^2 u = \frac{1 + \cos 2u}{2}$$

$$\sin u \pm \sin(v) = 2 \sin\left(\frac{u \pm v}{2}\right) \cos\left(\frac{u \mp v}{2}\right) \qquad \cos u + \cos v = 2\cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)$$

$$\cos u - \cos v = -2 \sin\left(\frac{u+v}{2}\right) \sin\left(\frac{u-v}{2}\right) \qquad \sin u \sin v = \frac{1}{2}[\cos(u-v) - \cos(u+v)]$$

$$\cos u \cos v = \frac{1}{2}[\cos(u-v) + \cos(u+v)] \qquad \sin u \cos v = \frac{1}{2}[\sin(u+v) + \sin(u-v)]$$

separable

$$g(y)\,\mathrm{d}y = f(t)\,\mathrm{d}t \qquad g(y)y' = f(t) \qquad \int g(y)\,\mathrm{d}y = \int f(t)\,\mathrm{d}t$$

integration rules

$$\int e^{au}\sin(bu)\,\mathrm{d}u = e^{au}\frac{a\sin(bu) - b\cos(bu)}{b^2 + a^2} \qquad \int e^{au}\cos(bu)\,\mathrm{d}u = e^{au}\frac{b\sin(bu) + a\cos(bu)}{b^2 + a^2}$$

Wronskian

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

y_1 and y_2 are linearly independent if $W \neq 0$