Notes

February 7, 2014

homework

due next friday 14 feb

lesson 7 number 1

hw 10

find general series solution for PDE and BC's.

hw 11

solve the IC. Explain how orthogonality is used.

number 3

hw 12

find general series solution for PDE and BC's.

hw 13

solve the IC. note questions about steady-state behavior.

lesson 9 number 4

hw 14

find the general series solution for PDE and BC's.

hw 15

show the eigen functions are orthogonal and solve the IC.

lesson 9

$$
\begin{array}{lll}
PDE & u_t = \alpha^2 u_{xx} + f(x,t), & 0 < x < 1, 0 < t < \infty \\
BC's & 0 = \alpha_1 u_x(0,t) + \beta_1 u(0,t), & 0 < t < \infty \\
 & 0 = \alpha_2 u_x(1,t) + \beta_2 u(1,t), & 0 < t < \infty \\
IC & u(x,0) = \phi(x), & 0 < x < 1 \\
PDE & u_t = \alpha^2 u_{xx} \rightarrow u = T(t)X(x) &
\end{array}
$$

$$\frac{T'(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)}$$

separation constant $= -\lambda$

assume const ≤ 0

$$\begin{aligned} u &= e^{-\alpha^2 \lambda t} X(x) \\ 0 &= X''(x) + \lambda^2 X(x) \\ &= \alpha_1 X'(0) + \beta_1 X(x) \\ &= \alpha_2 X'(1) + \beta_2 X(1) \end{aligned}$$

special case of sturm-louisville theorem (sp?)

results

there are nontrivial solutions $X_n(x)$ for a sequence of values $\lambda = \lambda_n$

$$\lambda_1 < \lambda_2 < \lambda_3 \text{ and } \lambda_n \to \infty$$

for

$$\lambda_m \neq \lambda_n, \int_0^1 X_m(x) X_n(x)\,\mathrm{d}x = 0 \text{ (orthagonality)}$$

we have $u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\alpha^2 \lambda_n t} X_n(x)$

How do we find c_n in $u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\alpha^2 \lambda_n t} X_n(x)$? because the original pde is linear, the sum of the solutions is a solution. similarly for the boundary conditions. Use orthogonality of $X_n(x)$'s and the initial condition $u(x,0) = \phi(x)$.

What about $u_t = \alpha^2 u_{xx} + f(x,t)$? Try solutions $u = \sum_{n=1}^{\infty} u_n(t) X_n(x)$. write $f(x,t) = \sum_{n=1}^{\infty} f(n(t) X_n(x)$. In the PDE: $\sum_{n=1}^{\infty} u_n{}'(t) X_n(x) = \alpha^2 \sum_{n=1}^{\infty} u_n(t) X_n{}''(x) + \sum_{n=1}^{\infty} f_n(t) X_n(x)$

Now $x_n{}''(x) = -\lambda_n X_n(x)$

For each n:

$$u_n{}'(t) = -\alpha^2 \lambda_n u_n(t) + f_n(t)$$

note

$$f_n(t) \int_0^1 (X_n(x))^2\,\mathrm{d}x = \int_0^1 f(x,t) X_n(x)\,\mathrm{d}x$$

since $f_n(t)$ is known we have

$$(u_n{}' + \alpha^2 \lambda_n u_n = f_n(t)) e^{\alpha^2 \lambda_n t} \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\alpha^2 \lambda_n t} u_n(t)\right) = e^{\alpha^2 \lambda_n t} f_n(t)$$

$$\to e^{\alpha^2 \lambda_n t} u_n(t) - u_n(0) = \int_0^t e^{\alpha^2 \lambda_n u} f_n(u)\,\mathrm{d}u$$

$$\to u_n(t) = u_n(0) e^{-\alpha^2 \lambda_n t} + \int_0^t e^{-\alpha^2 \lambda_n (t-u)} f_n(u)\,\mathrm{d}u$$

note

initial condition is $\sum_{n=1}^{\infty} u_n(0) X_n(x) = \phi(x)$

Notes

February 10, 2014

Lesson 10

use of integral transforms

sine or fourier transform?

$$\mathcal{F}_s[f] = \frac{2}{\pi}\int_0^\infty f(t)\sin(\omega t)\,\mathrm{d}t = F(\omega)$$
$${\mathcal{F}_s}^{-1}[F] = \int_0^\infty F(\omega)\sin(\omega t)\,\mathrm{d}\omega = f(t)$$

Laplace

$$\mathcal{L}[f] = F(s) = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t$$

Start with a problem in terms of (say) time t. Original solutioon $f(t)$. Transform it to an equation (in s). Solve for $F(s)$. Inversion $F(s) \to f(t)$. Simplest way of inverstion: Use a table of transforms.

Laplace	$f(t)$	$F(s)$
	t^p	$\frac{T(p+1)}{S^{p+}}$
	$\cos(at)$	$\frac{s}{s^2+a^2}$
	$\vdots$	$\vdots$

note

when you first see laplace transforms you do not see an inversion formula written down. This is because it turns out that the inversion formula requires complex analysis. Even to write down.

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)e^t\,\mathrm{d}s$$

This is covered in chapter 13. *extra credit for using this inversion formula. Talk to him about it*

some properties of sine transform

$$
\begin{aligned}
\mathcal{F}_s[f'] &= \frac{2}{\pi}\int_0^\infty f'(t)\sin(\omega t)\,\mathrm{d}t \\
&= \frac{2}{\pi}\left(f(t)\sin(\omega t)|_0^\infty - \int_0^\infty f(t)\cdot\omega\cos(\omega t)\,\mathrm{d}t\right) \\
&= \frac{2}{\pi}\left(0 - 0 - \omega\int_0^\infty f(t)\cdot\cos(\omega t)\,\mathrm{d}t\right) \\
&= -\omega\mathcal{F}_c[f] \\
\mathcal{F}_s[f''] &= -\omega\mathcal{F}_c[f'] = -\omega^2\mathcal{F}_s[f] - \frac{2\omega}{\pi}f(0) \\
\mathcal{F}_c[f'] &= +\omega\mathcal{F}_s[f] - \frac{2}{\pi}f(0)
\end{aligned}
$$

example p. 77

PDE	$u_t = \alpha^2 u_{xx}$	$0 < x < \infty$	$0 < t < \infty$
BC	$u(0,t) = A$		$0 < t < \infty$
IC	$u(x,0) = 0$	$0 < x < \infty$	

indefinite length rod. starts at zero temp.

use integral transforms to solve this. specifically the sine transform.

$$
\begin{aligned}
\mathcal{F}_s[f] &= \frac{2}{\pi}\int_0^\infty f(t)\sin(\omega t)\,\mathrm{d}t = F(\omega) \\
\mathcal{F}_s[u_t] &= \alpha^2\mathcal{F}_s[u_{xx}] \\
&= \alpha^2(-\omega^2 U(\omega,t) + \frac{2}{\pi}\omega A u(0.t))
\end{aligned}
$$

note:

$$
\begin{aligned}
\frac{2}{\pi}\int_0^\infty \sin(\omega x)\frac{\partial u}{\partial t}(x,t)\,\mathrm{d}x &= \frac{\partial}{\partial t}\left[\frac{2}{\pi}\int_0^\infty]sin(\omega x)u(x,t)\,\mathrm{d}x\right] \\
U(\omega,t) &= C(\omega)e^{-\alpha^2\omega^2 t} + \frac{2}{\pi}\frac{A}{\omega}
\end{aligned}
$$

as $t \to 0^+$

$$
\begin{aligned}
U(\omega,t) &\to \frac{2}{\pi}\int_0^\infty \sin(\omega t)\cdot 0\,\mathrm{d}t \text{ initial condition} \\
0 &= C(\omega) + \frac{2}{\pi}\frac{A}{\omega} \\
U(\omega,t) &= \frac{2}{\pi}\frac{A}{\omega}\left(1 - e^{-\alpha^2\omega^2 t}\right) \text{ sine transform of the solution } u(x,t)
\end{aligned}
$$

Notes

February 12, 2014

lesson 10

outside of class

sine transform

$$\begin{cases} \mathcal{F}_s[f] = F(\omega) = \dfrac{2}{\pi}\displaystyle\int_0^\infty \sin(\omega x) f(x)\,\mathrm{d}x \\ {\mathcal{F}_s}^{-1}[F] = f(t) = \displaystyle\int_0^\infty \sin \omega x) F(\omega)\,\mathrm{d}\omega \end{cases}$$

$$\begin{aligned} U(\omega, t) &= \frac{2A}{\pi\omega}\left(1 - e^{-\alpha^2\omega^2 t}\right) && \leftarrow \text{sine transform of u(x,t)} \\ u(x,t) &= A\ erfc\left(\frac{x}{2\alpha\sqrt{t}}\right) && \leftarrow \text{look up in a table} \end{aligned}$$

complementary error function

$$\begin{array}{cccc} PDE & u_t = \alpha^2 u_{xx} & 0 < x < \infty, & 0 < t < \infty \\ BC & u(0,t) = A & & 0 < t < \infty \\ IC & u(x,0) = 0 & 0 < x < \infty & \end{array}$$

converted PDE in x, t to ODE in t (with ω as a parameter)

now the outside of class bit

we want to find

$$\begin{aligned} u(x,t) &= \int_0^\infty \sin(\omega x) U(\omega, t)\,\mathrm{d}\omega \\ &= \int_0^\infty \sin(\omega x)\frac{2A}{\pi\omega}\left(1 - e^{-\alpha^2 t\omega^2}\right)\,\mathrm{d}\omega \end{aligned}$$

two pieces

$$I_1(x) - \int_0^\infty \frac{\sin(\omega x)}{\omega}\,\mathrm{d}w \qquad\qquad I_2 = \int_0^\infty \frac{\sin(\omega x)}{\omega} e^{-\alpha^2 t\omega^2}\,\mathrm{d}\omega$$

note:

$$\frac{\sin(\omega x}{\omega} e^{-\alpha^2 t\omega^2} \to x \text{ as } \omega \to 0$$

$$\int_0^\infty \frac{\sin(\omega x}{\omega}\,\mathrm{d}x = -\frac{1}{x}\cos(\omega x)\frac{1}{\omega}\mid_1^\infty + \int_1^\infty \frac{1}{x}\cos(\omega x)\frac{-\mathrm{d}\omega}{\omega^2}$$
$$\left|\frac{\sin(\omega)}{\omega}\right| \leq \frac{1}{\omega} \qquad \left|\frac{\cos(\omega x}{\omega^2}\right| \leq \frac{1}{\omega^2}$$

very slow convergence, and oscillating, difficult to do numerically. approaches from calc 2:

1) elementary antiderivatives - NO
2) series expansion - get answers that are infinite series
3) convert to a differential equation and solve that
4) contour integration (in complex analysis)

$$I_1(x) = \int\limits_{\omega=0}^{\omega=\infty} \frac{sin(\omega x}{\omega}\,\mathrm{d}\omega = \int_0^\infty \frac{\sin(\omega)}{\omega}\,\mathrm{d}\omega \text{ constant}$$

define

$$\begin{aligned}
f(s) &= \int_0^\infty e^{s\omega}\frac{\sin(\omega}{\omega}\,\mathrm{d}\omega \\
\lim_{s\to 0} f(0) &= I_1(const) \\
f'(s) &= \frac{\mathrm{d}}{\mathrm{d}s}\int_0^i nftye^{-s\omega}\frac{\sin(\omega)}{\omega}\,\mathrm{d}\omega \\
&= \int_0^\infty \frac{\partial}{\partial s}\overbrace{\underbrace{\left(e^{-s\omega}\frac{\sin(\omega)}{\omega}\right)}}\,\mathrm{d}\omega \\
&= -\int_0^\infty e^{-s\omega}\sin(\omega)\,\mathrm{d}\omega \text{ can be done by integrating by parts} \\
&= -\frac{1}{1+s^2} \\
f(s) &= -\arctan(s) + C \\
\lim_{s\to\infty} f(s) &= 0 \\
0 &= c - \lim_{s\to\infty}\arctan(s) \\
c &= \frac{\pi}{2}
\end{aligned}$$

back to the second piece

$$\begin{aligned}
\omega &= \frac{s}{\alpha\sqrt{t}} \\
I_2 &= \int\limits_{s=0}^{\infty} \frac{\sin\left(\frac{sx}{\alpha\sqrt{t}}\right)}{s}e^{-s^2}\,\mathrm{d}s \\
\beta &= \frac{x}{\alpha\sqrt{t}} > 0 \\
I_2(\beta) &= \int_0^\infty \frac{\sin(\beta s}{s}e^{-s^2}\,\mathrm{d}s
\end{aligned}$$

Notes

February 14, 2014

homework

hw 16

lesson 10 exercise 3. Find the cosine transform $U(\omega, t)$ of the solution $u(x, t)$.

hw 17

find the inverse transform $u(x, t)$. Hint use mathematica

hw 18

plot the solutioon $u(x, t)$ for $t = 0.01, 0.1, 1.0$ (take $\alpha^2 = 1$)

homework help

lesson 9. non-homogeneous pde. $u_t = \alpha u_{xx} + f(x, t)$ find fundamental solutions of the form $T(t)X(x)$ where $X(x)$ and λ are eigendata for the homogeneous BC.

He wrote $-\lambda^2$ originally instead of λ.

The separation constant occurs

$$\frac{T'(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)} = \mu \text{ the separation constant}$$

To find the separation constant study the hom. BC. It's okay to restrict to ≤ 0

question?

$$I = \int_0^\infty \frac{\sin(\omega x}{\omega} e^{-\alpha^2\omega^2 t} \,\mathrm{d}\omega = \int_0^\infty \frac{\sin(\frac{x}{\alpha\sqrt{t}} s)}{s} e^{-s^2} \,\mathrm{d}s$$
$$I(\beta) = \int_0^\infty \frac{\sin(\beta s}{s} e^{-s^2} \,\mathrm{d}s \qquad \beta = \frac{x}{\alpha\sqrt{t}}$$

Do we know any special values for $I(\beta)$?

$$\beta = 0 \qquad\qquad I = 0$$
$$= \int_0^\infty \frac{\sin(\beta s)}{s} \,\mathrm{d}s = \int_0^\infty \frac{\sin(s)}{s} \,\mathrm{d}s = \frac{\pi}{2}$$
$$s \leftarrow \frac{s}{\beta}$$

Notes

February 19, 2014

leftover

sine integral transform (evaluating integrals explicitly)
find:

$$I = \int_0^\infty \frac{\sin(x\omega}{\omega} e^{-\alpha^2 t\omega^2}\,\mathrm{d}\omega, \qquad x > 0$$

convert to

$$I = I(\beta) = \int_0^\infty \frac{\sin(\beta s}{s} e^{-s^2}\,\mathrm{d}s \qquad \text{write } \beta = \frac{x}{\alpha\sqrt{t}} > 0$$

note

$$I(\beta) = \int_0^\infty \frac{\sin(s)}{s} e^{-s^2/\beta^2}\,\mathrm{d}s \quad \to \quad 0 \text{ as } \beta \to 0$$

$$\to \quad \frac{\pi}{2} \text{ as } \beta \to +\infty$$

end note

$$I'(\beta) = \frac{\mathrm{d}}{\mathrm{d}\beta}\int_0^\infty \cos(\beta s)e^{-s^2}\,\mathrm{d}s \qquad u = e^{-s^2} \qquad \mathrm{d}v = \cos(\beta s)\mathrm{d}s$$

$$\mathrm{d}u = -2se^{-s^2}\mathrm{d}s \qquad v = \frac{1}{\beta}\sin(\beta s)$$

$$\begin{aligned}
I'(\beta) &= e^{-s^2}\frac{1}{\beta}\sin(\beta s)\Big|_0^\infty - \int_0^\infty \frac{\sin(\beta s)}{\beta}(-2se^{-s^2})\,\mathrm{d}s \\
&= +\frac{2}{\beta}\int_0^\infty \sin(\beta s)se^{-s^2}\,\mathrm{d}s \\
&= \frac{2}{\beta}\left(-I''(\beta)\right) \\
I''(\beta) &= \int_0^\infty \sin(\beta s)se^{-s^2}\,\mathrm{d}s = -\frac{\beta}{2}I'(\beta) \\
I'(\beta) &= c_1 e^{-\beta^2/4} \\
I(\beta) &= c_2 - c_1\int_\beta^\infty e^{t^2/4}\,\mathrm{d}t \text{ note the integration starting at } \beta \\
&= c_2 - 0 \\
&= \frac{\pi}{2} - c_1\int_\beta^\infty e^{-t^2/4}\,\mathrm{d}t
\end{aligned}$$

$$I(0) = \frac{\pi}{2} - c_1 \int_0^\infty e^{-t^2/4}\,\mathrm{d}t$$

fact $\int_0^\infty e^{-x^2}\,\mathrm{d}x = \frac{\sqrt{\pi}}{2}$

$$\frac{1}{2}\int_0^\infty e^{-t^4}\,\mathrm{d}t = \frac{\sqrt{\pi}}{2}$$

$$I(\beta) = \frac{\pi}{2} - \frac{\sqrt{\pi}}{2}\int_x^\infty e^{-x^2/4}\,\mathrm{d}x$$

note error function (erf)

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-t^2}\,\mathrm{d}t \to 1 \text{ as } x \to \infty$$

$$\mathrm{erfc}(x) = 1 - \mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\,\mathrm{d}t$$

graph on page 79

end note

$$x = 2t \qquad \mathrm{d}x = 2\mathrm{d}t$$

$$0 = \frac{\pi}{2} - c_2\sqrt{\pi} \qquad \frac{\sqrt{\pi}}{2} = c_1$$

$$\begin{aligned}
I(\beta) &= \frac{\pi}{2} - \frac{\sqrt{\pi}}{2}\int_{t/2\cdot 2t?}^\infty 2e^{-t^2}\,\mathrm{d}t \\
&= \frac{\pi}{2} - \frac{\sqrt{\pi}}{2}\cdot 2\cdot\frac{\sqrt{\pi}}{2}\left(\frac{2}{\sqrt{\pi}}\int_{2t}^\infty e^{-u^2}\,\mathrm{d}u\right) \\
&= \frac{\pi}{2} - \frac{\pi}{2}\mathrm{erfc}\left(\frac{\beta}{2}\right)
\end{aligned}$$

solution on p79

$$u(x,t) = A\mathrm{erfc}\left(\frac{x}{2\alpha\sqrt{t}}\right)$$

last homework problem (hw09)

we can do this without paying attention to formula's at all because the idea is so simple.

$$u_x(0,t) = 0 = f(t)$$
$$u_x(1,t) + hu(1,t) = 1 = g(t)$$

introduce $u(x,t) = \omega(x,t) +$ adjustment. This adjustment is chosen to obtain hetorgeneous boundary conditions ($f(t) = g(t) = 0$). Take adjustment to be $+a(t) + bt)x$ because original boundary values (0 and 1) lie on a line.

$$u = \omega + a(t) + b(t)x$$
$$\omega_x(0,t) + b(t) + 0 = f(t)$$
$$(\omega_x(1,t) + b(t)) + h(\omega(1,t) + a(t) + b(t)\cdot 1) = g(t)$$

Notes

February 21, 2014

Notes

February 24, 2014

lesson 12

definition

Given $f(x)$ on $\mathbb{R}$.

$$\mathcal{F}[f] = F(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i\xi x} f(x) \,\mathrm{d}x$$

Inverse transform recovers $f(x)$ from $F(\xi)$:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{+i\xi x} F(\xi) \,\mathrm{d}\xi \qquad \text{Deep Theorem}$$

problem

page 93

$$\begin{array}{rccc} PDE & u_t = \alpha^2 u_{xx} & -\infty < x < \infty & 0 < t < \infty \\ IC & u(x,0) = \phi(x) & -\infty < x < \infty & \end{array}$$

apply $\mathcal{F}$ to pde. $U(\xi, t) = \mathcal{F}[u(x,t)]$. Use property 3 (derivative): $\mathcal{F}[u_{xx}] =$
How are $\mathcal{F}[f']$ and $\mathcal{F}[f]$ related? page 91.

$$\begin{aligned} \mathcal{F}[f'] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i\xi x} f'(x) \,\mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}} \left[e^{-i\xi x} f(x) \right]_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} (-i\xi) e^{-i\xi x} f(x) \,\mathrm{d}x \end{aligned}$$

note: implicit conditions on $f(x)$ to insure integrals exist $f(+\infty) = f(-\infty) = 0$

$$= 0 + i\xi \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i\xi x} f(x) \,\mathrm{d}x$$

property 3

$$\begin{aligned} \mathcal{F}[f'] &= i\xi \mathcal{F}[f] \\ \mathcal{F}[f''] &= \xi^2 \mathcal{F}[f] \end{aligned}$$

So $\mathcal{F}[u_{xx}] = -\xi^2 U(\xi, t)$. $\frac{\mathrm{d}U}{\mathrm{d}t} = -\alpha^2 \xi^2 U$ with $U(\xi, 0) = \phi(\xi)$

step 2

solve prolem $U(\xi,t)=\phi(\xi)e^{-\alpha^2\xi^2 t}$.

step 3

invert transform

property 4

convolution theorem

definition: given $f(x), gx)$ on $\mathbb{R}$

$$f*g(x)=1\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x-s)g(s)\,\mathrm{d}s$$

theorem

$$\mathcal{F}[f*g(x)]=\mathcal{F}[f]\mathcal{F}[g]=F(\xi)G(\xi) \qquad \text{deep theorem}$$

sample calculation: find the convolution of two functions. Text example

$$\left.\begin{aligned} f(x)&=x\\ g(x)&=e^{-x^2}\end{aligned}\right\} f*g(x)=\frac{x}{\sqrt{2}}$$

$$\begin{aligned}
f*g(x)&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x-u)g(u)\,\mathrm{d}u\\
&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}(x-u)e^{-u^2}\,\mathrm{d}u\\
&=\frac{1}{\sqrt{2\pi}}\left[x\int_{-\infty}^{+\infty}e^{-u^2}\,\mathrm{d}u-\int_{-\infty}^{+\infty}ue^{-u^2}\,\mathrm{d}u\right] && \text{odd integral}\\
&=\frac{x}{\sqrt{2\pi}}\sqrt{\pi}f(x) && =e^{-x^2}\\
g(x)&=x\\
f*g(x)&=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}e^{-(x-u)^2}u\,\mathrm{d}u
\end{aligned}$$

property:$f*g(x)=g*f(x)$ commutativity

$$\begin{aligned}
f*g(x)&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x-u)g(u)\,\mathrm{d}u\\
u&=-\infty \qquad \leftarrow x-u\\
x-u&=+\infty
\end{aligned}$$

note that the definition in the book has a negative i and mathematica doesn't. and then that makes the signs flipped on the inverse transform as well. CAREFUL!!!

$$u(x,t)=\int_{-\infty}^{\infty}\phi(x-u)\frac{1}{\sqrt{\pi}}\frac{1}{2\alpha\sqrt{t}}e^{-\frac{u^2}{2/\alpha^2 t}}\,\mathrm{d}u$$

$$\int_{-\infty}^{\infty}\frac{1}{\sqrt{\pi}}\frac{1}{2\alpha\sqrt{t}}e^{-\frac{u^2}{2/\alpha^2 t}}\,\mathrm{d}u=\int_{-\infty}^{\infty}\frac{1}{\sqrt{\pi}}\frac{1}{2\alpha\sqrt{t}}e^{-\frac{u^2}{2/\alpha^2 t}}\,\mathrm{d}u$$

$$u(x,t) = \int_{-\infty}^{\infty} \phi(x) f(x-u)\,\mathrm{d}u \text{ positive with unit area}$$

Notes

February 26, 2014

lesson 13

laplace transform for $ft)$ on $0 \leq t < \infty$

$$\mathcal{L}\{f\} = F(x) = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t$$

inverse transform

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)e^{st}\,\mathrm{d}s$$

note $F(s)$ will typically be defined on $s \geq s_0$ (as described in introductory courses). $F(x)$ is analytic on $\mathrm{Re}(s_s) \geq s_0$ on half-planes in $\mathbb{C}$.

page 101

$$\begin{aligned}
&PDE \quad & u_t = u_{xx} \quad & 0 \leq x < \infty, \quad & 0 < t < \infty \\
&BC \quad & u_x(0,t) - u(0,t) = 0 \quad & & 0 < t < \infty \\
&IC \quad & u(x,0) = u_0 \quad & &
\end{aligned}$$

u_x is temperature gradient. $u_x = u$. $-cu_x$ is heat flow (in positive direction). when $u > 0$ heat flows out of the rod and if $u < 0$ then heat is flowing into rod. if the BC had a + instead of a $-$ we would have an unstable condition where more heat means the heat increases at a greater rate and boom. extra credit for this neh?

$$U(x,s) = \mathcal{L}\{u(x,t)\}$$
$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$

$$\begin{aligned}
&PDE \quad & sU(x,s) - u(x,0) = U_{xx}(x,s) \\
& & \frac{\mathrm{d}^2U}{\mathrm{d}x^2} - sU = -u_0 \\
&BC \quad & U_x(0,s) - U(0,s) = 0
\end{aligned}$$

solution of DE. Can assume s is positive (and large)

$$U(x,s) = c_1 \text{ etc from pg 102}$$

Notes

March 3, 2014

lesson 13 laplace transform

sample problem

Get a transform $U(s,t)$ for a solution (last page of lesson 13) and text gives $u(x,t)$ and says “follows from tables”. Turns out it’s not in the texts, tables. Not available in Mathematica either.

We will take a couple of days on approaches to finding inverse laplace transforms. Started talking about this on the 28th.

elementary inversion based

Find laplace transforms for t^p and $t^n e^{at} \begin{cases} \cos(bt) \\ \sin(bt) \end{cases}$ that occur in circuit analysis

the general answer is

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} F(x)\, \mathrm{d}s$$

this is the general inversion formula for laplace transforms. Requires essentialuse of complex analysis. the typical result is an infinite series of an integral representation for $f(t)$

approach 1

expand $F(s)$ in reciprocal powers and invert termwise. $\frac{F(p+1)}{s^{p+1}} \to t^p$

example

$$\begin{aligned} F(s) &= (s^2+1)^{-1/2} \text{ in text table} \\ \text{Find } f(t) \\ \text{Found } f(t) &= \sum_{n=0}^{\infty} \frac{(-t^2/4)^n}{(n!)^2} \end{aligned}$$

approach 1 give series for $f(t)$.

note this happens to be $J_0(t)$

approach 2

using $F(s)$, try to find an equation (typically differential equatin) for $f(t)$.

example

$$
\begin{aligned}
F(s) &= (s^2+1)^{-1/2} && = \mathcal{L}\{f(t)\} \\
F'(s) &= -\frac{1}{2}(s^2+1)^{-3/2}(2s) && = \mathcal{L}\{-t\cdot f(t)\} \\
&= -\frac{1}{s^2+1}(s^2+1)^{-1/2} \\
&= -\frac{s}{s^2+1}F(s) \\
(s^2+1)F'(s)+sF(s) &= 0 \\
sF(s)-f(0) & && = \mathcal{L}\{f'(t)\} \\
F'(s) & && = \mathcal{L}\{-t\cdot f(t)\} \\
s^2F'(s)-s(-t\cdot f(t))_{t=0}-\left(\frac{\mathrm{d}}{\mathrm{d}t}(-t\cdot f(t))\right) & && = \mathcal{L}\left\{\frac{\mathrm{d}^2}{\mathrm{d}t^2}(-t\cdot f(t)\right\} \\
s^2G(s)-sg(0)-g'(0) & && = \mathcal{L}\{g''(t)\} \\
\mathcal{L}\{f'(t)-tf(t)-\frac{\mathrm{d}^2}{\mathrm{d}t^2}(-tf(t))\} & \\
+f(0)\underbrace{-s(tf(t))_{t=0}}_{=0}-\frac{\mathrm{d}}{\mathrm{d}t}(tf(t))_{t=0}-(f(t)+tf'(t))_{t=0} &= 0 \\
-tf'(t) \text{ as } t\to 0 &
\end{aligned}
$$

we assume $f(0)$ exists, and that $\lim_{t\to 0^+} tf'(t)=0$. *AFTER* solving for $f(t)$ wee can check that these conditions hold.

$$
\begin{aligned}
\mathcal{L}\{f'(t)-tf(t)-\frac{\mathrm{d}^2}{\mathrm{d}t^2}(tf(t))-(tf''(t)+2f'(t)\} &= 0 \\
tf''(t)+f'(t)+tf(t) &= 0
\end{aligned}
$$

bessel de

$$
z^2\frac{\mathrm{d}^2w}{\mathrm{d}z^2}+z\frac{\mathrm{d}w}{\mathrm{d}z}+(z^2+\mu^2)w=0
$$

$$
\begin{aligned}
\mu &= \text{order} \\
J_\mu(z) &\text{ Bessel function of first kind (order } \mu) \\
Y_\mu(z) &\text{ Bessel function of second kind (order } \mu)
\end{aligned}
$$

note dlmf.nist.gov is reference for standard functions.

Notes

March 3, 2014

$$\begin{aligned} L &= \text{length} \\ \alpha^2 &= \text{diffusion} \\ &= \left[\frac{\text{cm}^2}{\text{sec}}\right] \end{aligned}$$

homework, in #20 we are deriving when change of variables will and won't work. #21 is deriving a little more complicated version for the remaining cases.

oh snap the equation i couldn't read last time was the gamma function. $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}$. We understan $n!$ what is $x!$

$$\begin{aligned} n! &= \int_0^\infty e^{-t}t^{x-1}\,\mathrm{d}t \\ x! &= \int_0^\infty e^{-t}t^{x}\,\mathrm{d}t,\ x > -1 \\ \Gamma(n+1) &= n! \\ \Gamma\left(\frac{1}{2}\right) &= \pi^{1/2} \\ \Gamma(x+1) &= x\Gamma(x) \\ t^p &\to \frac{\Gamma(p+1)}{s^{p+1}} \end{aligned}$$

okay on to dlmf.nist.gov handbook of mathematical functions? written/editted by by abramowitz and stegan. part of a government project to get a standardized reference of mathematical functions. update to it is NIST Handbook of Mathematical Functions. Which is precisely this web site. *THE* reference for special functions. reference we want is 5.2 and 5.3, famma function. 5.12 beta function is relevant to homework exercises.

lesson 13

page 102

obtained $U(x,s) = u_0\left(\frac{1}{s} - \frac{\sqrt{s}}{s(\sqrt{s}+1)}e^{-x\sqrt{s}}\right)$. Text claims that $u(x,t) = u_0(1 - (\text{erfc}(\frac{x}{2\sqrt{2}}) - \text{erfc}(\sqrt{t} + \frac{x}{2\sqrt{t}})e^{x+t})$. The text tables are not adequate. Mathematica does not handle it.

note:

$$\frac{\sqrt{s}}{s(\sqrt{s}+1)}e^{-a\sqrt{s}} \qquad (a > 0)$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{s}(\sqrt{s}+1)} e^{-a\sqrt{s}} \\
&= (\frac{1}{\sqrt{s}} - \frac{1}{\sqrt{s}+1}) e^{-a\sqrt{s}}
\end{aligned}
$$

case 1

$$F(s) = \frac{1}{\sqrt{s}} e^{-\sqrt{s}} \to f(t)? \text{ in table}$$

approach

$$G(s) = e^{-\sqrt{s}} \to g(t)$$

note

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}(e^{-\sqrt{s}}) &= e^{-\sqrt{s}} \cdot -\frac{1}{2}\frac{1}{\sqrt{s}} = -\frac{1}{2}F(s) \\
\frac{\mathrm{d}}{\mathrm{d}s}(e^{-\sqrt{s}}) &= \frac{\mathrm{d}}{\mathrm{d}s}(G(s)) \to -tg(t) = -\frac{1}{2}f(t) \\
G'(s) &= -\frac{1}{2}s^{-1/2}e^{-\sqrt{s}} \to -tg(t)
\end{aligned}
$$

$$\vdots$$

$$
\begin{aligned}
4sG''(s) + 2G'(s) - G(s) &= 0 \\
4sG''(s) - t^2 g(t) \mid_{t=0} + 2G'(s) - G(s) &= 0 \\
4\frac{\mathrm{d}}{\mathrm{d}t}\left[t^2 g(t)\right] + 2(-tg(t) - g(t) &= 0 \\
4t^2 \frac{\mathrm{d}g}{\mathrm{d}t} + 6tg(t) - g(t) &= 0 \\
\frac{\mathrm{d}g}{\mathrm{d}t} + \left(\frac{3}{2t} - \frac{1}{4t^2}\right) g &= 0
\end{aligned}
$$

integrating factor

$$\mu = t^{3/2} \cdot e^{1/4t}$$

Notes

March 7, 2014

homework 14-15
sturm-lionville expansions. The last stage of separation of variales (on finite intevals)

$$\text{PDE} \qquad u_t = u_{xx} + \underbrace{f(x,t)}_{\sin(\lambda,x)} \qquad \text{on } 0 < x < 1$$

$$\text{BC} \qquad u(0,t) = 0$$
$$u_x(1,t) + u(1,t) = 0$$

standard approach (incorrect). theeigenfunctions are $\sin(n\pi x) = X_n(x)$ do not satisfy the BC. The problem is driven by the boundary condition.

correct answer looks like this?

$$u(x,t) = \sum c_n T_n(t) X_n(x)$$

where $X_n(x)$ satisfy the BC. and $X_n(x)$ satisfies the separated equation $\frac{X_n{}''}{X_n(s)} = -\lambda_n$.

In lesson 9, there is a detailed example that has

$$\begin{aligned} \text{BC} \qquad & u(0,t) = 0 \\ & u(1,t) = 0 \\ \text{PDE} \qquad & u_t = u_{xx} + f(x,t) \end{aligned}$$

solution is written out in detail and $X_n(x) = \sin(n\pi x)$ appear (because of the BC).

In class, BC looked more like (see notes on 2/7)

$$\begin{aligned} \text{PDE} \qquad & u_t = \alpha^2 u_{xx} + f(x,t) \\ \text{BC} \qquad & 0 = \alpha_1 u_x(0,t) + \beta_1 u(0,t) \\ & 0 = \alpha_2 u_x(1,t) + \beta_2 u(1,t) \\ \text{IC} \qquad & u(x,0) = \phi(x) \end{aligned}$$

see page 65-66 in text (81-82). Step 1 on page 66. set $u(x,t) = \sum c_n T_n(t) X_n(x)$ where $X_n(x)$ are eigenfunctions for the homogeneous PDE (and homogeneous BC)

Try $T(t)X(x)$. $u_t = \alpha^2 u_{xx}$ gives $\frac{T'(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda^2 \le 0$. So $X''(x) + \lambda^2 X(x) = 0$.

$$\begin{aligned} \text{BC} \qquad & X(0) = 0 \\ & X'(1) + X(1) = 0 \end{aligned}$$

$$\begin{aligned} X(x) &= a\cos(\lambda x) + b\sin(\lambda x) \\ X(0) &= 0 = a\cos(0) + b\sin(0) \to a = 0 \\ X'(x) &= +b\lambda\cos(\lambda x) \end{aligned}$$

$$
\begin{aligned}
X'(1) + X(1) = 0 \\
X_n(x) &= \sin(\lambda_n x) \text{ where } \tan(\lambda) = -\lambda \\
u(x,t) &= \sum_{n=1}^{\infty} T_n(t)\sin(\lambda_n x) \\
f(x,t) &= \sum f_n(t)\sin(\lambda_n x) \\
{T_n}'(t) &= \alpha_2(-{\lambda_n}^2)T_n(t) + f_n(t)
\end{aligned}
$$

coefficient of $\sin(\lambda_n x)$

we need $\lambda\cos(\lambda) + \sin(\lambda) = 0, \lambda > 0$. $\cos(\lambda) = 0$? no, else $\sin(\lambda) = 0$. So divide out by cosine to get $\lambda + \tan(\lambda) = 0$. Showed that eigenfunctions $\sin(\lambda x)$ are orthogonal.

Homework due date extended to 3/14

Notes

March 10, 2014

lesson 13

$U(x,s) = \frac{u_0}{s} - \frac{u_0}{s(\sqrt{s}+1}e^{-x\sqrt{s}} \to u(x,t) = u_0\left[1 - (\mathrm{erfc}(\frac{x}{2\sqrt{t}}) - \mathrm{erfc}(\sqrt{t} + \frac{x}{2\sqrt{t}})e^{x+t}\right]$

mixup, he wrote $\frac{u_0}{s} - \frac{u_0\sqrt{s}}{s(\sqrt{s}+1)}e^{-\sqrt{s}x}$ but want $F(s) = \frac{1}{s(\sqrt{s}+1)}e^{-a\sqrt{s}} \to f(t) =?$ and then $F(\frac{s}{a^2}) = \frac{a^3}{s(\sqrt{s}+a)}e^{-\sqrt{s}} \to a^2f(a^2t)$ concentrate on $G(s) = \frac{e^{-s}}{s(\sqrt{s}+a)} = \frac{(\sqrt{s}-a)e^{-\sqrt{s}}}{s(s-a^2)}$ and then $(s-a^2)G(s) = \frac{1}{\sqrt{s}}e^{-\sqrt{s}} - \frac{a}{s}e^{-\sqrt{s}}$ step 3 solve $a'(t) - a^2g(t) = \dots$

number 19 from handout

$$f(t) = \mathrm{erfc}\left(\frac{a}{2\sqrt{t}}\right)$$

in process of inverting (notes from 2 lectures back) $F(s) = e^{-\sqrt{s}}$ we obtained a DE for $f(t)$.

$$\begin{aligned}
f(t) &= c_1t^{-3/2}e^{-\frac{1}{4t}} \\
\lim_{s\to\infty} se^{-\sqrt{s}} &= f(0) \\
\lim_{s\to 0^+} e^{-\sqrt{s}} &= 1 = \int_0^\infty f(t)\,\mathrm{d}t \\
\int_0^\infty f(t)\,\mathrm{d}t &= c\int_0^\infty t^{-3/2}e^{-\frac{1}{4t}}\,\mathrm{d}t = 1 \\
\Gamma(x) &= \int_0^\infty e^{-t}t^{x-1}, \mathrm{d}t \\
u &= \frac{1}{4t} \\
t &= \frac{1}{4u} \\
\mathrm{d}t &= -\frac{1}{4u^2}\mathrm{d}u \\
\int_0^\infty f(t)\,\mathrm{d}t &= c\int_0^\infty 4u^{3/2}e^{-u}\frac{-1}{4u^2}\,\mathrm{d}t \\
1 &= c\int_0^\infty e^{-u}u^{-1/2}4^{1/2}\,\mathrm{d}u \\
&= 2c\int_0^\infty e^{-u}u^{-1/2}\,\mathrm{d}u = 2c\Gamma\left(\frac{1}{2}\right) \\
c &= \frac{1}{2\Gamma(1/2)} = \frac{1}{2\sqrt{\pi}}
\end{aligned}$$

$$
\begin{aligned}
F(s) &= e^{-\sqrt{s}} \to f(t) \\
F'(s) &= -\frac{1}{2\sqrt{s}}e^{-\sqrt{s}} \to -tf(t) \\
\frac{1}{\sqrt{s}} &\to 2tf(t) = 2t\frac{1}{2\sqrt{\pi}}t^{-3/2}e^{-\frac{1}{rt}} \\
e^{-\sqrt{s}} &\to \frac{1}{2\sqrt{\pi}}t^{-3/2}e^{-1/(4t)} \\
\frac{1}{\sqrt{s}} &\to \frac{1}{\sqrt{\pi}}t^{-1/2}e^{-1/(4t)} \\
F(s) &\to f(t) \\
\frac{1}{s}F(s) &\to \int_0^t f(u)\,\mathrm{d}u \\
\frac{1}{s}e^{-\sqrt{s}} &\to \frac{1}{2\sqrt{gp}}\int_{n=0}^{u=t} e^{-1/(4u)}u^{-3/2}\,\mathrm{d}u \\
&\vdots \\
g'(t) - a^2g(t) &= \frac{1}{\sqrt{\pi t}}e^{1/(4t)} - a\cdot\mathrm{erfc}\left(\frac{1}{2\sqrt{t}}\right)
\end{aligned}
$$

multiply by e^{-a^2t} which is integrating factor

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-a^t}\right) =
$$

Notes

March 12, 2014

$$\begin{aligned} G(s) &= \frac{1}{s}\frac{1}{\sqrt{s}+a}e^{-\sqrt{s}} \\ (s-a^2)G(s) &= \frac{1}{\sqrt{s}}e^{-\sqrt{s}} - \frac{a}{s}e^{-\sqrt{s}} \end{aligned}$$

assuming $g(0) = 0$

$$\frac{\mathrm{d}g}{\mathrm{d}t} - a^2 g(t) = \frac{1}{\sqrt{\pi t}}e^{-1/4t} - a\mathrm{erfc}(\frac{1}{2\sqrt{t}})$$

with $g(0) = 0$ integrating factor $\omega = e^{-a^2 t}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-a^2 t}g(t)\right) = \frac{e^{-a^2 t}}{\sqrt{\pi t}e^{-1/4t}} - ae^{-a^2 t}\mathrm{erfc}(\frac{1}{2\sqrt{t}}) \qquad \text{decays exponentially}$$

instead of integrating from $0 \to t$ rewrite as $t \to \infty$ with arbitrary constant

$$\begin{aligned} \int_0^t &= -\int_t^\infty + C \\ \int_0^t \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-a^2 t}g(t)\right) &= -\int_t^\infty \frac{e^{-a^2 t}}{\sqrt{\pi t}}e^{-1/4t} - ae^{-a^2 t}\mathrm{erfc}(\frac{1}{2\sqrt{t}}) \\ e^{-a^2 t}g(t) - 0 &= -\int_t^\infty \frac{1}{\sqrt{\pi t}}e^{-a^2 u - 1/4u}\,\mathrm{d}u + \int_t^\infty ae^{-a^2 u}\mathrm{erfc}(\frac{1}{2\sqrt{u}})\,\mathrm{d}u + C \\ \int_t^\infty e^{a^2 u}\mathrm{erfc}\left(\frac{1}{2\sqrt{u}}\right)\,\mathrm{d}u &= \int_t^\infty \mathrm{erfc}\left(\frac{1}{2\sqrt{t}}\mathrm{d}(\frac{-1}{a^2}e^{-a^2 u})\right) \\ &= \mathrm{erfc}(\frac{1}{2\sqrt{u}})\frac{-1}{a^2}e^{-a^2 u}\Big|_t^\infty + \frac{1}{a^2}\int_t^\infty e^{-a^2 u}\mathrm{d}(\mathrm{erfc}(\frac{1}{2\sqrt{u}})) \\ &= \frac{1}{a^2}e^{-a^2 t}\mathrm{erfc}(\frac{1}{2\sqrt{t}}) + \frac{1}{a^2}\int_t^\infty e^{-a^2 u}\left(\frac{-2}{\sqrt{\pi}}e^{-1/4u}\underbrace{\frac{1}{2}(-\frac{1}{2})u^{-3/2}\mathrm{d}u}_{\mathrm{d}(1/2\sqrt{u})}\right) \\ e^{-a^2 t}g(t) - 0 &= -\int_t^\infty \frac{1}{\sqrt{\pi u}}e^{-a^2 u - \frac{1}{4u}} + \frac{1}{a}e^{-a^2 t}\mathrm{erfc}(\frac{1}{2\sqrt{t}} + \frac{1\ 2}{4a\sqrt{\pi}}\int_t^\infty e^{-a^2 u - 1/4u}\frac{\mathrm{d}u}{u^{3/2}} \qquad \vdots \\ g(t) &= \frac{1}{a}\mathrm{erfc}\left(\frac{1}{2\sqrt{t}}\right) - \frac{1}{a}e^{a^2 t + a}\cdot\mathrm{erfc}\left(a\sqrt{t} + \frac{1}{2\sqrt{t}}\right) + C \end{aligned}$$

$$
\begin{aligned}
g(0) = 0 = 0 - \frac{1}{a}e^{a} \cdot 0 + C \\
C = 0
\end{aligned}
$$

homework 23

$$
\begin{aligned}
g(t) &= \int_0^t e^t \cdot h(u)\mathrm{d}u \\
f(t) &= g'(t) \\
g(t) &= e^t \cdot \int_0^t h(u)\mathrm{d}u
\end{aligned}
$$

Notes

March 14, 2014

very interesting use of laplace transforms. straight from the text

lesson 14 page 122

duhamel's principle.

easy problem

$$
\begin{aligned}
\text{PDE} \qquad & w_t = w_{xx} \\
\text{BC} \qquad & w(0,t) = 0 \\
& w(1,t) = 1 \\
\text{IC} \qquad & w(x,0) = 0
\end{aligned}
$$

subsubsection*hard problem

$$
\begin{array}{cccc}
 & & & z = u + v \\
\text{PDE} & u_t = u_{xx} & v_t = v_{xx} & z_t = z_{xx} \\
\text{BC} & u(0,t) = 0 & v(0,t) = 0 & z(0,t) = 0 \\
 & u(1,t) = g(t) & v(1,t) = 0 & z(1,t) = g(t) \\
\text{IC} & u(x,0) = 0 & v(x,0) = \phi(x) & z(x,0) = \phi(x)
\end{array}
$$

$\mathcal{L}$ with respect to time

$$
\begin{aligned}
sW(x,s) - \underbrace{w(x,0)}_{\to 0} &= W_{xx}(x,s) \\
W_{xx} - SW &= 0 \text{ on } 0 < x < 1 & U_{xx} - sU &= 0 \\
W(0,s) = 0 \text{ and } W(1,s) &= \frac{1}{s} & U(0,s) = 0 \text{ and } U(1,s) &= \\
W &= c_1 \sinh(\sqrt{s}x) + c_2 \cosh(\sqrt{s}x) \\
\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - sy &= 0 & y &= e^{rx} \\
& & r^2 - s &= 0 \\
y &= a_1 e^{-\sqrt{s}x} + a_2 e^{\sqrt{s}x} \\
\sinh(z) &= \frac{1}{2}(e^{z} - e^{-z})
\end{aligned}
$$

$$
\begin{aligned}
\cosh(z) &= \frac{1}{2}(e^z + e^{-z}) \\
W &= c_1 \sinh(\sqrt{s}x) \\
\frac{1}{s} &= c_1 \sinh(\sqrt{s}) \\
W(x,s) &= \frac{1}{s}\frac{\sinh(\sqrt{s}x)}{\cosh(\sqrt{s})} \\
U &= c_1 \sinh(\sqrt{s}x) \\
G(s) &= c_1 \sinh(\sqrt{s}) \\
U(x,s) &= G(s)\frac{\sinh(\sqrt{s}x}{\sinh(\sqrt{s})} = G(s)sW(x,s)
\end{aligned}
$$

note: $sW(x,s) - \underbrace{w(x,0)}_{\to 0} = \mathcal{L}\{w_t\}$

$$
\begin{aligned}
u(x,t) &= \int_0^t g(t-u)w_t(x,u)\,\mathrm{d}u \\
&= g(t-u)w(x,u)\mid_0^t - \int_0^t (-g'(t-u)w(x,u)\,\mathrm{d}u
\end{aligned}
$$

page 107 (124)

$$
w(x,t) = x + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-(n\pi)^2 t}\sin(n\pi x) \text{ from eigenfunction expansion}
$$

homework #27 lesson 14, exercise 4 $g(t) = \sin(t)$. take $\alpha^2 = 1$. due friday, 28 march.

Notes

March 24, 2014

lesson 16 wave equation

obtain $u_{tt} = \alpha^2 u_{xx}$ and other forms.

1740s "vibrating string"

equilibrium position

violin string. tension T is grams cm/sec^2. density ρ is grams/cm. we are looking at segemnt of string $x_0 - \frac{\Delta x}{2} < x_0 < x_0 + \frac{\Delta x}{2}$. mass is $\rho \Delta x$ and $u(x,t)$ is vertical displacement from equilibrium. Assumptions, difference in length from Δx causes no change in mass. horizontal shifts are negligible.

this is page 125(141) in text.

Newton says $F = ma$. So $\rho \Delta x u_{tt}$ =vertical force on segment arising from string forces+etc.

Tension acts tangentially to $x_0 - \frac{\Delta x}{2}$ and $x_0 + \frac{\Delta x}{2}$. θ is angle from tangential line to horizontal with θ_1 being on the left. Tension= $T \sin\theta_2 - T \sin\theta_2$. notice that $\tan\theta = u_x$ so

$$\sin\theta = \frac{u_x}{\sqrt{1 + {u_x}^2}}$$

.

So we have

$$\rho \Delta x u_{tt} = T \left[\left(\frac{u_x}{\sqrt{1 + {u_x}^2}} \right)_{x_0 + \frac{\Delta x}{2}} - \left(\frac{u_x}{\sqrt{1 + {u_x}^2}} \right)_{x_0 - \frac{\Delta x}{2}} \right]$$

$$\rho u_{tt} = \frac{T}{\Delta x} \Delta \left(\frac{u_x}{\sqrt{1 + {u_x}^2}} \right) + \frac{\text{etc}}{\Delta x}$$

$$\rho u_{tt} = T \frac{\partial}{\partial x} \left(\frac{u_x}{\sqrt{1 + {u_x}^2}} \right) + \lim_{\Delta x \to 0} \frac{\text{etc}}{\Delta x}$$

note

$$\frac{\mathrm{d}}{\mathrm{d}x} \left[u_x (1 + {u_x}^2)^{-1/2} \right] = u_{xx}(1 + u_x^2) + u_x(-1/2)...$$

$$\rho u_{tt} = T \frac{u_{xx}}{(1 + {u_x}^2)^{3/2}} + \lim_{\Delta x \to 0} (\text{etc}/\Delta x)$$

more assumptions:tension remains essentially constant and $|u_x|$ is small

Since $|u_x|$ is small we have

$$\rho u_{tt} = T u_{xx} + \lim_{\Delta x \to 0} \frac{\text{etc}}{\Delta x}$$

other forces on segment might be

1. $F(x,t)$ is vertical force per unit length (gravity)

2. $-\gamma u$ is elastic restoring force per unit length
3. $-\beta u_t$ is frictional force (of medium) per unit length
4. etc is additional forces on segment of length Δx

$= F(x_0, t)\Delta x - \gamma u \Delta x - \beta u_t \Delta x$
α="velocity"

PDE	$u_{tt} = \alpha^2 u_{xx}$	$0 < t < \infty$ various x-internal
BC		
IC	$u(x,0) = \phi(x)$ $u_t(x,0) = \psi(x)$	

question: are there travelling wave solutions? this means a function that depends on x and t such that $u(x,t) = f(x - ct)$. So wave is being translated through time with speed c.

$$\begin{aligned}
u(x,t) &= f(x - ct) \\
u_x &= f'(x - ct) \\
u_{xx} &= f''(x - ct) \\
u_t &= f'(x - ct)(-c) \\
u_{tt} &= f''(x - ct)(-c)^2
\end{aligned}$$

back substitute into pde

$$\begin{aligned}
f''(x - ct)c^2 &= \alpha^2 f''(x - ct) \\
c &= \pm\alpha
\end{aligned}$$

Notes

March 26, 2014

lesson 17

this stuff is like the prototype solution setting up method of characteristic lines, we'll see more of this later.

Today we look at Alembert's solution

$$
\begin{array}{lllll}
\text{PDE} & u_{tt} = c^2 u_{xx} & c > 0 & \infty < x < +\infty & 0 < t < \infty \\
\text{IC} & \left.\begin{array}{r} u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{array}\right\} & & \infty < x < +\infty &
\end{array}
$$

change independent variables

$$
\begin{aligned}
\xi &= x + ct \\
\eta &= x - ct
\end{aligned}
$$

point will get $u_{\xi\eta}$

See page 130(146) for more explanation

Notes

March 28, 2014

result from last time (D'Alemberts' solution)

$$\begin{array}{lll} \text{PDE} & u_{tt} = c^2 u_{xx} & -\infty < x < +\infty \qquad 0 < t < \infty \\ \text{IC} & u(x,0) = f(x) & \\ & u_t(x,0) = g(x) & \\ \hline & u(x,t) = \frac{1}{2}(f(x-ct)+f(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\mathrm{d}s & \end{array}$$

18 properties of this solution

Case 1.

$$\text{IC} \qquad \left.\begin{aligned} u(x,0) &= f(x) \\ u_t(x,0) &= 0 \end{aligned}\right\} \text{ Solution } u(x,t) = \frac{1}{2}\left[\underbrace{f(x-ct)}_{\text{wave moving right}} + \underbrace{f(x+ct)}_{\text{wave moving left}}\right]$$

Case 2.

$$\text{IC} \qquad \left.\begin{aligned} u(x,0) &= 0 \\ u_t(x,0) &= g(x) \end{aligned}\right\} \text{ Solution } u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,\mathrm{d}s$$

value of g over a widening interval

see graphs on pages 139-141 (155-157)

$$\begin{array}{llll} \text{PDE} & u_{tt} = c^2 u_{xx} & 0 < x < \infty & 0 < t < \infty \\ \text{BC} & u(0,t) = 0 & & 0 < t < \infty \\ \text{IC} & u(x,0) = f(x) & 0 < x < \infty & \\ & u_t(x,0) = g(x) & & \end{array}$$

as last time $u(x,t) = \phi(x-ct) + \psi(x+ct)$ general solution − IC's and BC are not used.

Match IC:

$$\begin{aligned} \phi(x) + \psi(x) &= f(x) & 0 < x < \infty \\ -c\phi'(x) + c\psi'(x) &= g(x) & \rightarrow -\phi(x) + +\psi(x) \\ & & = \frac{1}{c}\int_0^x g(s)\,\mathrm{d}s + K \end{aligned}$$

$$\phi(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(s)\,\mathrm{d}s + \frac{k}{2}$$
$$\psi(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(s)\,\mathrm{d}s + \frac{k}{2}$$
$$u(x,t) = \phi(x-ct) + \psi(x+ct)$$

k cancel out. $x + ct > 0$ so $\psi(x + ct)$ is no problem. $x - ct$ changes sign.

What is $\phi(x - ct)$ when $x - ct < 0$?

now lets look at the boundary condition

BC	$u(0,t) = 0 = \phi(-ct) + \psi(ct)$ for	$0 < t < \infty$
for	$-\infty < x < 0,$	$\phi(x) = -\psi(-x)$
for	$x - ct > 0,$	$u(x,t) = \phi(x-ct) + \psi(x+ct)$
		$= \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,\mathrm{d}s$
for	$x - ct < 0,$	$u(x,t) = -\psi(x-ct) + \psi(x+ct)$
		$= \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,\mathrm{d}s$

more on page 143(159)

homework #28 & #29 due next friday (first friday of april) lession 17 exercise 3 and exercise 4

Notes

April 2, 2014

lesson 19 (bc)

lesson 20 solution by separation of variables

PDE	$u_{tt} = c^2 u_{xx}$	$0 < x < L$	$0 < t < \infty$
BC		$x = 0, x = L$	
IC	$u(x,0) = f(x)$		
	$u_t(x,0) = g(x)$		

19

names book gives for boundaries

1. controlled endpoints

$$u(0,t) = g_1(t)$$
$$u(L,t) = g_2(t)$$

   1st kind, dirichlet

2. force at boundaries

$$u_x(0,t) = g_1(t)$$
$$u_x(L,t) = g_2(t)$$

   snd kind newmann

3. elastic attachement

$$u_x(0,t) + cu(0,t) = g_1(t)$$
$$u_x(L,t) + cu(L,t) = g_2(t)$$

   3rd kind, mixed or robin

using terminology like transverse, longitudinal and torsional waves. Transverse waves vibrate perpindicular to reference axis or direction of motion. Like a jump rope. Longitudinal waves vibrate parallel to reference axis/direction of motion. Like a slinky. Torsional waves represent rotational vibrations about reference axis or direction of motion.

2nd kind

recall that vertical force= $T \cdot \sin(\theta) = T \cdot \frac{\tan(\theta)}{\sec(\theta)} = T \cdot \frac{x_x}{\sqrt{1+{u_x}^2}}$. Zero vertical force at enpoint means $T \cdot \frac{x_x}{\sqrt{1+{u_x}^2}} = 0$ or $u_x = 0$. So frictionless endpoints. When there is an applied force on the endpoints, then the derivative picks up something.

Note that in longitudinal displacement (slinky example) maximum displacement is at the top, because that is where the most force is. as you go down, less force, less displacement.

3rd kind

page 150. Still frictionless, but elastic attachment.

vertical force of spring is $-hl\sin(\theta) = -hl_0\frac{\sin(\theta)}{\cos(\theta)} = -hu = T\frac{u_x}{\sqrt{1+{u_x}^2}}$. Assum $\sqrt{1+{u_x}^2} \approx 1$ then $Tu_x = -hu$. Note that l is the length of the little spring contraption and l_0 is the initial (short) length.

20

separation of variables

$$\begin{aligned} u &= X(x)T(t) \\ X(x)T''(t) &= c^2X''(x)T(t) \\ \frac{T''(t)}{c^2T(t)} &= \frac{X''(x)}{X(x)} = \lambda \end{aligned}$$

λ is a constant, separation constant

$$\begin{aligned} -\infty < \lambda < \infty& \\ T'' - c^2\lambda T = 0, \quad X'' - \lambda X &= 0 \\ 0 < t < \infty \quad 0 < x < L& \\ \frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \mu^2 = 0 \quad e^{\mu x}, e^{-\mu x}& \\ \frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 0 = 0 \quad 1, x& \\ \frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \mu^2 = 0 \quad \cos(\mu x), \sin(\mu x)& \end{aligned}$$

cases are

1. $\lambda = \mu^2$ positive
2. $\lambda = 0$
3. $\lambda = -\mu^2$ negative

earlier, $\lambda = \mu^2$ was eliminated on the ground that it gave exponntially increasing solutions in time t. Time solutions now $e^{+\mu ct}, e^{-\mu ct}$.

The real reason $\mu^2 > 0$ can be eliminated is that the BC for $X(x)$ cannot be satisfied. $\lambda = \mu^2 > 0$ $X'' - \mu^2X = 0$

$$\begin{aligned} X &= c_1\cosh(\mu x) + c_2\sinh(\mu x) \\ X(0) &= 0 = c_1 \cdot 1 + c_2 \cdot 0 \quad c_1 = 0 \end{aligned}$$

$$X = c_2 \sinh(\mu x)$$
$$X(L) = 0 = c_2 \sinh(\mu L)$$

want nontrivial solution $c_2 \neq 0$. $\sinh(\mu L) = 0$, only solution is $\mu = 0$ and since $\mu^2 > 0$ we have no solutions.

Notes

April 4, 2014

lesson 20 (continuing from last time

$$
\begin{array}{llll}
\text{PDE} & u_{tt} = c^2 u_{xx} & 0 < x < L & 0 < t < \infty \\
\text{BC} & \left.\begin{array}{r} u(0,t) = 0 \\ u(L,t) = 0 \end{array}\right\} & & 0 < t < \infty \\
\text{IC} & \left.\begin{array}{r} u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{array}\right\} & 0 < x < L &
\end{array}
$$

note $c^2 = T/\rho$ tension over mass density.

Separated solutions $u(x,t) = T(t)X(x)$

$$
\text{PDE} \qquad \frac{T''}{c^2 T} = \frac{X''}{X} = \lambda
$$

$$
\text{cases} \begin{cases}
\lambda = \mu^2 > 0 & \text{only trivial solutions for } X(x) \text{ from BC} \\
\lambda = 0 & X'' = 0 \; X = c_1 + c_2 x \text{ BC } X(0) = 0 = c_1 \; X(L) = 0 = 0 + c_2 L \to c_2 = 0 \\
\lambda = \mu^2 < 0 & X'' + \mu^2 X = 0 \; X = c_1 \cos(\mu x) + c_2 \sin(\mu x) \text{ BC } X(0) = 0 = c_1 \cdot 1 + c_2 \cdot 0 \to c_1 = 0 \\
 & X(L) = 0 = c_2 \sin(\mu L) \to \sin(\mu L) = 0
\end{cases}
$$

So $\mu L = n\pi$ give $\mu_n = n\pi/L$ for $n = 1, 2, 3, \ldots$

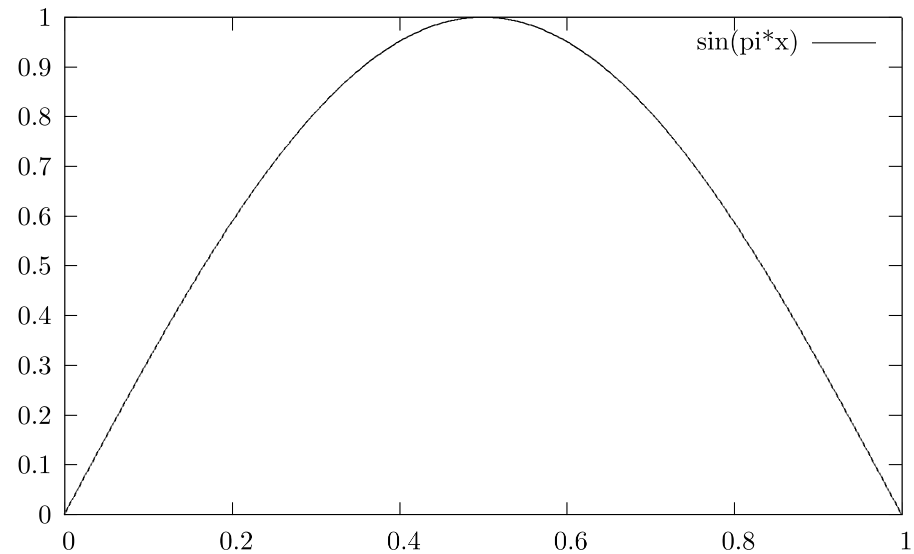
Have $X_n(x) = \sin(\mu_n x) = \sin(n\pi \frac{x}{L})$ nontrivial solutions (eigenfunctions)

Use $\frac{T''(t)}{c^2 T(t)} = -\mu_n{}^2 \; T''(t) + c^2 \mu_n{}^2 T(t) = 0$

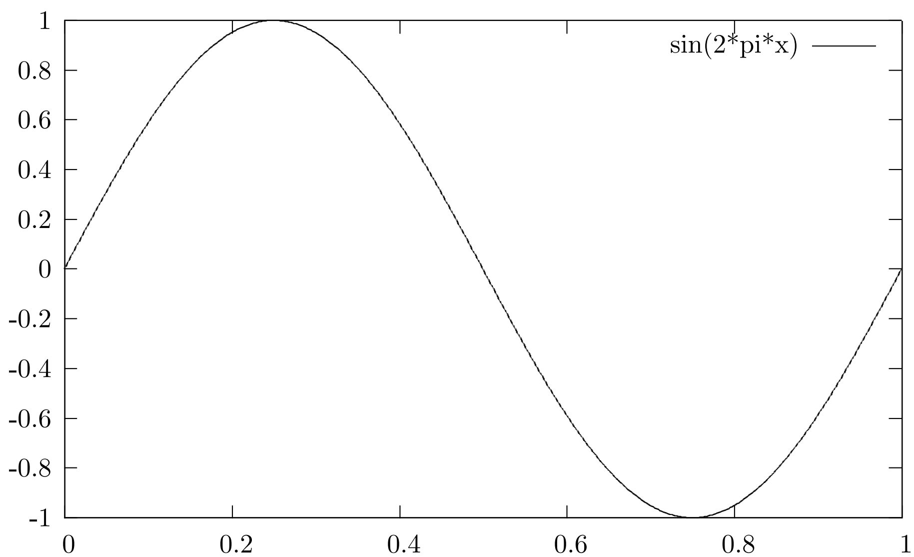
$T(t) = a_n \cos(c\mu_n t) + b_n \sin(c\mu_n t)$

Separated solutions $u_n(x,t) = \left[a_n \cos(n\pi \frac{ct}{L}) + b_n \sin(n\pi \frac{x}{L})\right] \sin(n\pi \frac{x}{L}$.

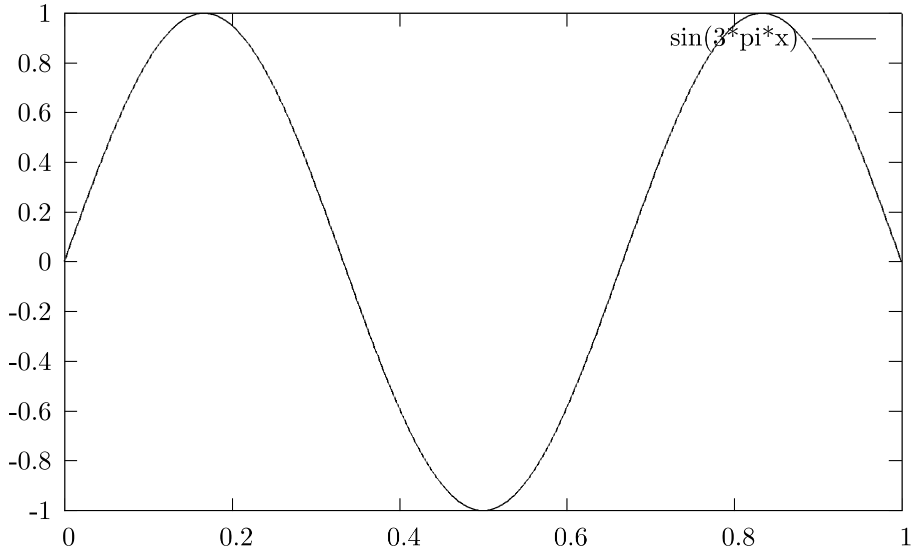
$\sin(2\pi\omega t)$ where ω is frequency (oscillations/second) Hz. Frequency $\omega_n = n\frac{c}{2L} = n\sqrt{\frac{T}{\rho}} \cdot \frac{1}{2L}$ for $n = 1, 2, 3, \ldots$


sin(pi*x)
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.2
0.4
0.6
0.8
1

n=1


sin(2*pi*x)
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
0.2
0.4
0.6
0.8
1

n=2



n=3

All solutions have periods ω_1, this is why same pitch happens when plucked in different places.

page 154

Mersenne Laws for Strings

mid-1600s, First person to determine frequency of a pitch. middle c is $256/2^8$ Hz.

1. Frequency is proportional to root of tension $\alpha\ \sqrt{T}$
2. Frequency is inversely proportional to length. $\alpha\ 1/L$
3. Freqency is inversely proportion to root of density. $\alpha\ \frac{1}{\sqrt{\rho}}$

Fix L, ρ, and for low T freq= $k_0\sqrt{\frac{T}{\rho}}\frac{1}{L}$

$\sin(\omega t)$

don't forget to add -shell-escape to the plugin

Notes

April 7, 2014

lesson 20

vibrating strings on the finite interval.

PDE	$u_{tt} = c^2 u_{xx}$	$0 < x < L$	$0 < t < \infty$
BC	$u(0,t) = 0 = u(L,t)$		$0 < t < \infty$
IC	$\begin{cases} u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \end{cases}$	$0 < x < L$	

looked for separated solutions $u = T(t)X(x)$

found $u_n(x,t) = [a_n \sin(n\pi ct/L) + b_n \cos(n\pi ct/L)] \sin(n\pi x/L)$

text p 157

General solution $u(x,t) = \sum_{n=1}^{\infty} [a)n \sin(n\pi ct/L) + b_n \cos(n\pi ct/L)] \sin(n\pi x/L)$ satisfies PDE (linear/hom) and BC(linear hom)

set $t = 0$ so $u(x,t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x/L) = f(x)$.

Recal $\int_0^L \sin(n\pi x/L)\sin(m\pi x/L)\mathrm{d}x = 0, m \neq n$ and $\int_0^L \sin(n\pi x/L) = L/2$.

$$\int_0^L \sum_{n=1}^{\infty} b_n \sin(n\pi x/L)\mathrm{d}x = \int_0^L f(x)\mathrm{d}x$$
$$\int_0^L \sin(m\pi x/L) \sum_{n=1}^{\infty} b_n \sin(n\pi x/L)\mathrm{d}x = \int_0^L \sin(m\pi x/L) f(x)\mathrm{d}x$$

blah blah, page 157

$$u(x,t) = \sum_{n=1}^{\infty} (a_n \sin(n\pi ct/L) + b_n \cos(n\pi ct/L)) \sin(n\pi x/L$$
$$u_t(x,t) = \sum_{n=1}^{\infty} (n\pi \frac{c}{L} a_n \cos(n\pi ct/L) - n\pi \frac{c}{L} b_n \sin(n\pi ct/L)) \sin(n\pi x/L$$
$$u_t(x,t) = \sum_{n=1}^{\infty} n\pi \frac{c}{L} a_n \sin(n\pi x/L) = g(x)$$

even in something like $\sin(\mu_n \frac{x}{L}$ where $\mu + tan(\mu) = 0$ orthoganality will still be present. Sturm-Lionville theory gives this.

lesson 21

the vibrating beam (4th order PDE)

$$\begin{array}{rlll} \text{PDE} & u_{tt} = \alpha^2 u_{xxxx} & 0 < x < L & 0 < t < \infty \\ \text{BC} & u(0,t) = 0 = u(L,t) & & 0 < t < \infty \\ & u_{xx}(0,t) = 0 = u_{xx}(L,t) & & \\ \text{IC} & \begin{cases} u(x,0) & = f(x) \\ u_t(x,0) & = g(x) \end{cases} & 0 < x < L & \end{array}$$

HW will involve exer 1 on p 166-167 $u(0,t) = 0, u_{xx}(1,t) = 0, u_x(1,t) = 0, u_{xxx}(1,t) = 0$ free end p 166.
set $u = T(t)X(x)$

$$\frac{T''}{-\alpha^2 T} = \frac{X''''}{X} = \lambda \qquad \text{separation constant}$$

$$X'''' - \lambda X = 0 \qquad \text{consider bc}$$

assume $\lambda > 0$ since $\lambda \leq 0$ cannot satisfy bc

$$\begin{aligned} \lambda &= \omega^2 > 0 \\ X^{(4)} - \omega^2 X &= 0 \\ X &= e^{rx} \\ X^{(4)} &= r^4 e^{rx} - \omega^2 e^{rx} \\ r^4 - \omega^2 &= 0 \\ r^2 &= \pm\omega \\ r &= \pm\sqrt{\omega}, \pm i\sqrt{\omega} \\ e^{\pm\sqrt{\omega}x}&, e^{\pm i\sqrt{\omega}x} \\ X(x) &= C\cos(\sqrt{\omega}x) + D\sin(\sqrt{\omega}x) + E\cosh(\sqrt{\omega}x) + F\sinh(\sqrt{\omega}x \end{aligned}$$

apply bc

$$\begin{aligned} X(0) &= 0 = C + E \\ X'' &= -C\omega\cos(\sqrt{\omega}x) - D\omega\sin(\sqrt{\omega}x) + E\omega\cosh(\sqrt{\omega}x) + F\omega\sinh(\sqrt{\omega}x) \\ X''(0) = 0 &= -C\omega + E\omega \\ C &= E = 0 \end{aligned}$$

now at $x = L$

Notes

April 9, 2014

lesson 21

PDE	$u_{tt} = -u_{xxxx}$	$0 < x < 1$	$0 < t < \infty$
BC	$u(0,t) = 0 = u(1,t)$		$0 < t < \infty$
	$u_{xx}(0,t) = 0 = u_{xx}(1,t)$		
IC	$u(x,0) = f(x)$	$0 < x < 1$	
	$u_t(x,0) = g(x)$		

$u = T(t)X(x)$ give $\frac{T''}{-T} = \frac{X''''}{X} = \lambda$
assume $\lambda > 0$ ($\lambda \leq 0$ can be eliminated by using bc)
Have $X'''' - \omega^2 X = 0$ with characteristic roots $\pm\sqrt{\omega}, \pm i\sqrt{\omega}$ and solutions:

$$= C\cos(\sqrt{\omega}x) + D\sin(\sqrt{\omega}x) + E\cosh(\sqrt{\omega}x + F\sinh(\sqrt{\omega}x)$$
$$X'' = \omega(-C\cos(\sqrt{\omega}x) - D\sin(\sqrt{\omega}x) + E\cosh(\sqrt{\omega}x + F\sinh(\sqrt{\omega}x))$$
$$x = 0 \to C = E = 0$$

$$x = 1 \to \left.\begin{aligned} D\sin(\sqrt{\omega}) + F\sinh(\sqrt{\omega}) &= 0 \\ \omega(-D\sin(\sqrt{\omega}) + F\sinh(\sqrt{\omega})) &= 0 \end{aligned}\right\} \begin{aligned} &2F\sinh(\sqrt{\omega}) = 0 \implies F = 0 \\ &2D\sin(\sqrt{\omega}) = 0 \implies \begin{array}{c}\text{nontrivial when } \sin(\sqrt{\omega}) = 0 \\ \text{and } \sqrt{\omega} = n\pi \text{ for } n = 1,2,3,\dots\end{array} \end{aligned}$$

Have $X_n(x) = \sin(n\pi x)$ for $n = 1,2,3,\dots$

$$\frac{T_n{}''}{-T_n} = \omega_n^2 = (n\pi)^4$$
$$T_n'' + (n\pi)^4 T_n = 0$$
$$cos((n\pi)^2 t), \sin((n\pi)^2 t)$$

Have $T_n X_n = a_n \sin((n\pi)^2 t) + b_n \cos((n\pi)^2 t)$ frequencies $\frac{n^2}{2}\pi, n = 1,2,3,\dots$
for pde and bc at $t = 0$

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) \to \text{use orthogonality}$$
$$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} a_n (n\pi)^2 \sin(n\pi x)$$

he will give us a homework problem that will look like this

PDE

BC

IC

hopefully the previous problem will help us work out the homework problem

lesson 22

dimensional analysis

object moving through a fluid (air). question: frictional force actin on the object. expect the force to be related to velocity V. want drag force F_D in terms of V velocity of object. A is "characteristic" area associated with object (analysis should hold for similar objects). ρ is fluid density

$$F_D \text{ units } \left[\frac{\text{mass} \cdot \text{length}}{\text{time}^2}\right]$$
$$V \text{ units } \left[\frac{\text{length}}{\text{time}}\right]$$
$$A \text{ units } \left[\text{length}^2\right]$$
$$\rho \text{ units } \left[\frac{\text{mass}}{\text{length}^3}\right]$$

we are looking for a dimensionless combination

$$\frac{F_D}{\rho} \text{ units } \left[\frac{\text{length}^4}{\text{time}^2}\right]$$
$$\frac{F_D}{\rho V^2} \text{ units } \left[\text{length}^2\right]$$
$$\frac{F_D}{\rho A V^2} = \text{dimensionless}$$
$$F_D = C_D \cdot \rho A V^2 \text{ where } C_D \text{ is dimensionless constant to be measured by experiment}$$

V^2-law for drag

Notes

April 11, 2014

lesson 21

vibrating beam - done (hw in prep-later)

lesson 22

dimensionless form - practice in past hw - discussion of dimensional analysis.

lesson 23

classification of pde's

hw 33 lesson 23 exercise 3. hw 34 lesson 23 exercise 4. hw 35 lesson 23 exercise 5. due fri april 18.

general second order linear PDE (in x,y).

$$Au_{xx} + Bu_{xy} + C_{yy} + Du_x + Eu_y + F_u = G$$

$A, \ldots, G$ are functions of x, y. Basic question:can we simplify by changing variable:$\xi = \xi(x, y)$ and $\eta = \eta(x, y)$?

new equations in new variables look like:

$$\hat{A}u_{\xi\xi} + \hat{B}u_{\xi\eta} + \hat{C}u_{\eta\eta} + \hat{D}u_{\xi} + \hat{E}u_{\eta} + \hat{F}u = \hat{G}$$

$$\begin{aligned} u_x &= u_\xi \xi_x + u_\eta \eta_x \\ u_y &= u_\xi \xi_y + u_\eta \eta_y \end{aligned}$$

the following are mentioned on page 177 exercise 2

$$\begin{aligned} u_{xx} &= u_{\xi\xi}{\xi_x}^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}{\eta_x}^2 + u_\xi\xi_{xx} + u_\eta\eta_{xx} \\ u_{xy} &= \frac{\partial}{\partial y}(u_x) \\ &= \frac{\partial}{\partial y}(u_\xi\xi_x + u_\eta\eta_x) \\ &= \frac{\partial}{\partial y}(u_\xi)\xi_x + u_\xi\xi_{xy} + \frac{\partial}{\partial y}(u_\eta)\eta_x + u_\eta \cdot \eta_{xy} \\ &= (u_{\xi\xi}\xi_y + u_{\xi\eta\eta_y})\xi_x + u_\xi\xi_{xy} + (u_{\eta\xi}\xi_y + u_{\eta\eta}\eta_y)\eta_x + u_\eta ? \\ &= u_{\xi\xi}\xi_x\xi_y + u_{\xi\eta}(\xi_x\eta_y + \eta_x\xi_y) + u_{\eta\eta} \cdot \eta_x\eta_y + u_\xi\xi_{xy} + u_\eta\eta_{xy} \\ u_{yy} &= u_{\xi\xi}{\xi_y}^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}{\eta_y}^2 + u_\xi\xi_{yy} + u_\eta\eta_{yy} \end{aligned}$$

the hat equations is obtained by expanding out the non-hat equations with the above. So $\hat{A} = u_{\xi\xi}(A{\xi_x}^2 + B\xi_x\xi_y + C{\xi_y}^2)$ and so on.

try setting $\hat{A} = 0 = \hat{C}$ 2 conditions to determine $\xi(,y), \eta(x,y)$

$$\hat{A} = 0 = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2$$
$$\hat{C} = 0 = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2$$

notice that a curve $\xi(x,y)$ =constant.

$$\frac{\mathrm{d}}{\mathrm{d}x}(\xi(x,y)) = \xi_x + \xi_y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\xi_x}{\xi_y}$$

these classifications come from the general quadratic in two variables:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

which gives you ellipse($B^2 - 4AC < 0$), hyperboles($B^2 - 4AC > 0$), parabolas($B^2 - 4AC = 0$)

Notes

April 14, 2014

homework due on 25 april now

lesson 23

classification of pde's

PDE	$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$	$A = A(x,y), \dots, G = G(x,y)$
Idea:	New variables $\xi = \xi(x,y), \eta = \eta(x,y)$	
PDE	$\hat{A}u_{\xi\xi} + \hat{B}u_{\xi\eta} + \hat{C}u_{\eta\eta} + \hat{D}u_\xi + \hat{E}u_\eta + \hat{F}u = \hat{G}$	$\hat{A}, \dots, \hat{G}$ on page 177
Idea:	$\hat{A} = 0 = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2$	
	$\hat{C} = 0 = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2$	
curves:	$\xi = \text{constant have } \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\xi_x}{\xi_y} \to A(\frac{\mathrm{d}y}{\mathrm{d}x})^2 - B\frac{\mathrm{d}y}{\mathrm{d}x} + C = 0$	
curves:	$\eta = \text{constant have } \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\eta_x}{\eta_y} \to A(\frac{\mathrm{d}y}{\mathrm{d}x})^2 - B\frac{\mathrm{d}y}{\mathrm{d}x} + C = 0$	

in agreement with the text we take

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2A}\left(B - \sqrt{B^2 - 4AC}\right) \qquad \xi \equiv \text{const}$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2A}\left(B + \sqrt{B^2 - 4AC}\right) \qquad \eta \equiv \text{const}$$

define hyperbolic pde as $B^2 - 4AC > 0$ and parabolis is $B^2 - 4AC = 0$ and elliptic is $B^2 - 4AC > 0$

example

$$u_{tt} = c^2 u_{xx} \qquad t = y$$
$$c^2 u_{xx} - u_{yy} = 0$$
$$A = c^2 \quad b = 0 \quad c = -1 \quad D = E = F = G = 0$$
$$\xi \equiv \text{const} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2C^2}\left[0 - \sqrt{0 + 4C^2}\right]$$
$$= -\frac{1}{c}$$
$$y = -\frac{x}{c} + \text{const determines } \xi \equiv$$
$$\eta \equiv \text{const} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2C^2}\left[0 + \sqrt{0 + 4C^2}\right]$$

$$
\begin{aligned}
y &= \frac{x}{c} + \text{const determines } \eta \equiv \\
&= \frac{1}{c} \\
\xi &= x + cy \\
\eta &= x - cy
\end{aligned}
$$

what is new pde?

$$
\begin{aligned}
\hat{A}u_{\xi\xi} + \hat{B}u_{\xi\eta} + \hat{C}u_{\eta\eta} + \ldots &= \hat{G} \\
0 + \hat{B} + 0 + 0 &= 0 \quad \text{from 23.6 on page 177}
\end{aligned}
$$

new pde

$$
\begin{aligned}
4c^2 u_{\xi\eta} &= 0 \quad \text{or} \\
u_{\xi\eta} &= 0 \text{ this has general solutioon}
\end{aligned}
$$

example

$$
\begin{aligned}
y^2 u_{xx} - x^2 u_{yy} &= 0 \\
A = y^2 \quad b &= 0 \quad c = -x^2 \quad D = E = F = G = 0 \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{2y^2}\left[0 - \sqrt{0 + 4y^2x^2}\right] = -\frac{x}{y} \\
x^2 + y^2 &= a_1 \\
\xi &= x^2 + y^2 \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{2y^2}\left[0 + \sqrt{0 + 4y^2x^2}\right] = \frac{x}{y} \\
x^2 - y^2 &= a_2 \\
\eta &= x^2 - y^2
\end{aligned}
$$

note typo in text around page 179, formulas on bottom of page not possible, sign errors

$$
\begin{aligned}
\hat{A} &= \hat{C} = 0 \\
\hat{B} &= 2y^2 2x 2y + 0 + (-2x^2)(2y)(-2y) \\
&\vdots \\
16x^2y^2 u_{\xi\eta} + \hat{D}u_\xi + \hat{E}u_\eta &= 0
\end{aligned}
$$

change coefficients from xy to $\xi\eta$

Notes

April 16, 2014

lesson 23

example continued

page 179

$$
\begin{aligned}
0 &= y^2 u_{xx} - x^2 u_{yy} \\
\xi &\equiv \text{constant} \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{x}{y} \to \xi = x^2 + y^2 && \frac{\mathrm{d}y}{\mathrm{d}x}\frac{1}{2A}\left(B - \sqrt{B^2 - 4AC}\right) \\
\eta &\equiv \text{constant} \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{x}{y} \to \eta = x^2 - y^2 && \frac{\mathrm{d}y}{\mathrm{d}x}\frac{1}{2A}\left(B + \sqrt{B^2 - 4AC}\right)
\end{aligned}
$$

new PDE

$$
\begin{aligned}
\hat{G} &= \hat{A}u_{\xi\xi} + \hat{B}u_{\xi\eta} + \hat{C}u_{\eta\eta} + \hat{D}u_{\xi} + \hat{E}u_{\eta} + \hat{F}u \\
\hat{A} &= \hat{C} = 0 && \text{by construction of } \xi, \eta
\end{aligned}
$$

formulas page 177

$$
\begin{aligned}
\hat{B} &= 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y + \eta_x) + 2C\xi_y\eta_y \\
&= 16x^2y^2 \\
\hat{D} &= A\xi_{xx} + B\xi_{xy} + C\xi_{yy} - D\xi_x + E\eta_x \\
&= (y^2)(2) + 0 + (-x^2)(+2) + 0 = 2(y^2 - x^2) \\
\hat{E} &= A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y \\
&= (y^2)(2) + 0 + (-x^2)(-2) + 0 = 2(x^2 + y^2)
\end{aligned}
$$

new PDE

$$
\begin{aligned}
0 &= 16x^2y^2u_{\xi\eta} + 2(y^2 - x^2)u_\xi + 2(x^2 + y^2)u_\eta \\
0 &= 16 \cdot \frac{1}{4}(\xi^2 - \eta^2)u_{\xi\eta} - 2\eta u_\xi + 2\xi u_\eta \\
u_{\xi\eta} &= \frac{\eta u_\xi - \xi u_\eta}{2(\xi^2 - \eta^2)}
\end{aligned}
$$

riemann's method

method of solution is riemann's method. this is not in text.

to prepare ourself for this we need to think of hyperbolic equations in a more general setting

$$\begin{array}{rlll} \text{PDE} & 0 = c^2 u_{xx} - u_{yy} & -\infty < x < \infty & 0 < y < \infty \\ \text{IC} & u(x,0) = f(x) & -\infty < x < \infty & \\ & u_y(x,0) = g(x) & & \end{array}$$

$$\text{PDE} \qquad 0 = u_{xx} - u_{yy}$$

IC on line $y = -\frac{1}{2}x$

$$\begin{array}{rl} \text{IC} & u(x,y) = f(x) \\ & u_y(x,y) = g(x) \end{array}$$

change variables

$$\begin{aligned} \xi &= x - y \\ \eta &= x + y \\ u_{\xi\eta} &= 0 \\ u &= F(\xi) + G(\eta) \\ u &= F(x-y) + G(x+y) \end{aligned}$$

now match IC

$$\begin{aligned} u(x, -\frac{1}{2}x) = f(x) &= F\left(\frac{3}{2}x\right) + G\left(\frac{1}{2}x\right) \\ u_y &= -F'(x-y) + G'(x+y) \\ u_y(x, -\frac{1}{2}x) = g(x) &= -F'\left(\frac{3}{2}x\right) + G'\left(\frac{1}{2}x\right) \\ \int_{x_0}^{x} g(s)\,\mathrm{d}s &= -\frac{2}{3}F\left(\frac{3}{2}x\right) + 2G(\frac{1}{2}x) \\ \frac{2}{3}f(x) + \int_{x_0}^{x} g(s)\,\mathrm{d}s = \frac{8}{3}G\left(\frac{1}{2}x\right) &- 2f(x) + \int_{x_0}^{x} g(s)\,\mathrm{d}s = -\frac{8}{3}F(\frac{3}{2}x) \\ G\left(\frac{1}{2}x\right) &= \frac{1}{4}f(x) + \frac{3}{8}\int_{x_0}^{x} g(s)\,\mathrm{d}s \\ G\left(x\right) &= \frac{1}{4}f(2x) + \frac{3}{8}\int_{x_0}^{2x} g(s)\,\mathrm{d}s \\ F\left(\frac{3}{2}x\right) &= \frac{3}{4}f(x) - \frac{3}{8}\int_{x_0}^{x} g(s)\,\mathrm{d}s \\ F\left(x\right) &= \frac{3}{4}f(\frac{2}{3}x) - \frac{3}{8}\int_{x_0}^{\frac{2}{3}x} g(s)\,\mathrm{d}s \\ u(x,y) &= F(x-y) + G(x+y) \end{aligned}$$

$$
\begin{aligned}
&= \frac{3}{4} f\left(\frac{2}{3}(x-y)\right) - \frac{3}{8}\int_{x_0}^{\frac{2}{3}(x-y)} g(s)\,\mathrm{d}s \\
&\quad + \frac{1}{4} f\left(2(x+y)\right) + \frac{3}{8}\int_{x_0}^{2(x+y)} g(s)\,\mathrm{d}s
\end{aligned}
$$

check

$$
\begin{aligned}
u_{xx} - u_{yy} &= 0 && \text{because has form } F(x-y) + G(x+y) \\
u(x, -\frac{1}{2}x) &= \frac{3}{4} f\left(\frac{2}{3}(x + \frac{1}{2}x)\right) - \frac{3}{8}\int_{x_0}^{\frac{2}{3}(x-y)} g(s)\,\mathrm{d}s \\
&\quad + \frac{1}{4} f\left(2(x - \frac{1}{2}x)\right) + \frac{3}{8}\int_{x_0}^{2(x+y)} g(s)\,\mathrm{d}s \\
&= f(x) \\
u_y &= \frac{3}{4} \cdot -\frac{2}{3} f'(\frac{2}{3}(x-y)) - \frac{3}{8} \cdot \frac{2}{3} g(\frac{2}{3}(x-y)) \\
&\quad + \frac{1}{4}(2) f'(2(x+y)) + \frac{3}{8}(2) g(2(x+y)) \\
u_y &= -\frac{1}{2} f'(\frac{2}{3}(x-y)) + \frac{1}{4} g(\frac{2}{3}(x-y)) \\
&\quad + \frac{1}{2} f'(2(x+y)) + \frac{3}{4} g(2(x+y)) \\
u_y(x, -\frac{1}{2}x) &= g(x)
\end{aligned}
$$

Notes

April 16, 2014

finished lesson 23 (classification of PDEs-canonical form for hyperbolic PDEs). note: lesson 41 (canonical forms for parabolic ellipti pdes) moving outside text because classification $\neq$ solution. latst time: example of general I for hyperbolic pde $u_{xx} - u_{yy} = 0$ with data on $x + 2y = 0$

riemann's method for byperbolic pdes.

start

operation $< [u] := u_{\xi\eta} + a(\xi,\eta)u_\xi + b(\xi,\eta)u_\eta + c(\xi,\eta)u$

curve C:$\eta = \phi(\xi)$ for $-\infty < \xi < +\infty$ with $\phi'(\xi) < 0$ (alternate $\phi'(\xi) > 0$)

PDE $< [u] = F(\xi,\eta)$ on $\eta > \phi(\xi)$ on $\eta > \phi(\xi)$ (alternate $\eta < \phi(\xi)$)

ic $u(\xi,\phi(\xi)) = f(\xi)$ and $u_\xi(\xi,\phi(\xi)) = g(\xi)$

context start with hyperbolic pde with initial conditions and change to canonical coordinates ξ,η and get new form here

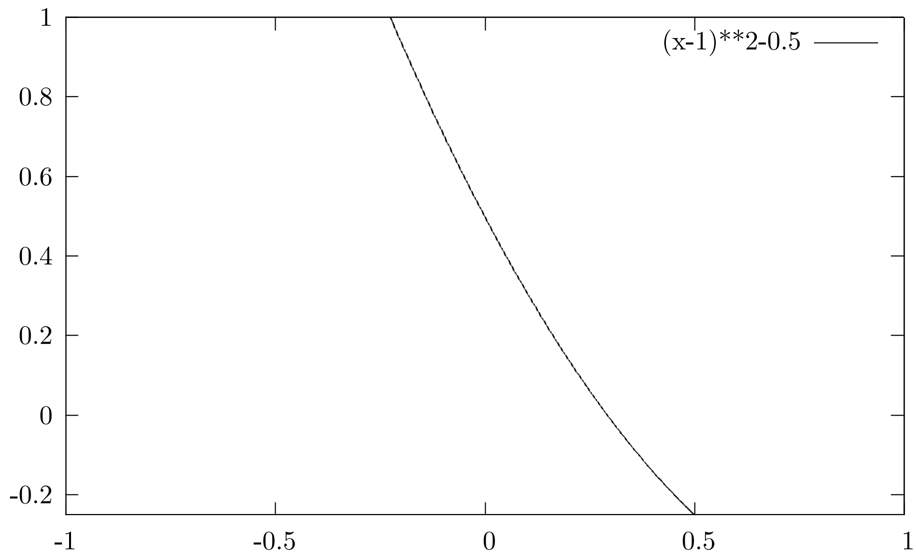
note: $u_\eta(\xi,\phi(\xi)$ is determined by initial condition.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\xi}u(\xi,\phi(\xi)) &= u_\xi(\xi,\phi(\xi))\frac{\mathrm{d}\xi}{\mathrm{d}\xi} + u_\eta(\xi,\phi(\xi))\frac{\mathrm{d}\phi}{\mathrm{d}\xi} = \frac{\mathrm{d}f}{\mathrm{d}\xi} \\ g(\xi) + u_\eta(\xi,\phi(\xi))\phi'(\xi) = f'(\xi)\end{aligned}$$

note: we will find the explicit riemann function for the constant coefficient case (a,b,c constant)

riemann's approach

Pick (ξ_0,η_0) and introducej auxiliary variables (x,y). We will describe the Riemann function $R(\xi_0,\eta_0;x,y)$ wich gives $u(\xi_0,\eta_0)$ that is $u(\xi,\eta)$ because (ξ_0,η_0) is arbitrary.



recall

divergence thm

region R, boundary ∂R

$$\int\int_R \nabla\cdot\vec{F}\,\mathrm{d}x\mathrm{d}y = \int_{\partial R}\vec{F}\cdot\vec{n}\,\mathrm{d}s \qquad \vec{n} \text{ outer normal}$$

$$\int\int_R\left(\frac{\partial A}{\partial x}+\frac{\partial B}{\partial y}\right)\mathrm{d}x\mathrm{d}y = \int_{\partial R}(-B\,\mathrm{d}x + A\,\mathrm{d}y)$$

Identity for adjoint

$$vL[u]-uM[v] = \frac{\partial A}{\partial x}+\frac{\partial B}{\partial y} \text{ where } \begin{cases} A = \frac{1}{2}(vu_y - uv_y) + auv \\ B = \frac{1}{2}(vu_x - uv_x) + buv \end{cases}$$

$$L = \text{ as given} = u_{xy} + au_x + bu_+cu$$

$$M[v] = v_{xy} - (av)_x - (bv)_y + cv$$

$$\begin{aligned}\int\int_{C_1C_2C_3}(vL[u]-uM[v])\mathrm{d}x\mathrm{d}y &= \int\int_{C_1C_2C_3}(\frac{\partial A}{\partial x}+\frac{\partial B}{\partial y})\mathrm{d}x\mathrm{d}y \\ &= \int\int_{C_1}(-B\mathrm{d}x + A\mathrm{d}y) \\ &\quad + \int\int_{C_2}(-B\mathrm{d}x + A\mathrm{d}y) \\ &\quad + \int\int_{C_3}(-B\mathrm{d}x + A\mathrm{d}y)\end{aligned}$$

of course $L[u] = F(x,y)$ and v is the Riemann function chosen to have special properties. $\int\int_{C_1}(-B\mathrm{d}x + A\mathrm{d}y)$ involves u, u_x, u_y on $y = \phi(x)$ and v, v_x, v_y on $y = \phi(x)$

$$\begin{aligned}\int_{C_2}(-B\mathrm{d}x + A\mathrm{d}y) &= \int_{C_2}A\mathrm{d}y \\ &= \int_{y=Q}^{y=P}\frac{1}{2}(vu_y - v_yu) + auv\,\mathrm{d}y\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2} vu\Big|_Q^P - \int_Q^P uv_y \, \mathrm{d}y + \int_Q^P auv \, \mathrm{d}y \\
&= \frac{1}{2}\left[v(P)u(P) - v(Q)u(Q)\right] + \int_Q^P u(v_y - av) \, \mathrm{d}y
\end{aligned}
$$

$$
\begin{aligned}
\int_{C_3} (-B\mathrm{d}x + A\mathrm{d}y) &= \int_R^P \frac{1}{2}(vu_x - v_x u) + buv \, \mathrm{d}x \\
&= \frac{1}{2} vu\Big|_Q^P - \int_Q^P uv_y \, \mathrm{d}y + \int_Q^P auv \, \mathrm{d}y
\end{aligned}
$$

Notes

April 25, 2014

riemanns method

pde $L[u] = u_{\xi\eta} + a(\xi,\eta)u_\xi + b(\xi,\eta)u_\eta + c(\xi,\eta)u = F(\xi,\eta)$. Boundary conditions $u(\xi, phi(\xi)) = f(\xi), u_\xi(\xi,\phi(\xi)) = g(\xi)$ for curve $C : \eta = \phi(\xi)$ with $\phi'(\xi) < 0$.

auxiliary problem, we obtained conditions defining $v(x,y)$. (ξ_0,η_0) lies above curve $C = C_1$, C_2 is vertical line from C_1 to (ξ_0,η_0) and C_3 is horizontal line.

pde $M[v] = 0$ in (at least in region $C_1C_2C_3$) ($M[v] = v_{xy} - (av)_x - (bv)_y + cv$. Boundary conditions $v_y - av = 0$ on C_2 and $v_x - bv = 0$ on C_3 and $v(\xi_0,\eta_0) = 1$.

last time $\int\int_{C_1C_2C_3} v(x,y)F(x,y)\,\mathrm{d}x\mathrm{d}y = \int_{C_1} [(v_x - bv)\,\mathrm{d}x - u(v_y - av)\,\mathrm{d}y] - \frac{1}{2}(u(Q)v(Q) + u(R)v(R)) + u(\xi_0,\eta_0$. Representation for $u(\xi_0,\eta_0)$ in terms of boundary data and $v(x,y)$

$v(x,y)$ from this problem is the riemann function $R(\xi_0,\eta_0;x,y)$.

assume a,b,c are constants, the riemann function for $L[u] = u_{\xi\eta} + au_\xi + bu_\eta + cu$ can be found explicitly. $M[v] = v_{xy} - av_x - bv_y + cv = 0$. $v = e^{bx+ay}\cdot w$, $v_x = (w_x + bw)e^{bx+ay}$, $v_y = (wy + aw)e^{bx+ay}$ and $v_{xy} = (w_{xy} + aw_x + bw_y + baw)e^{bx+ay}$.

$$\begin{aligned}
(w_{xy} + aw_x + bw_y + abw) - a(w_x + bw) - b(w_y + aw) + cw &= 0\\
w_{xy} - abw + cw = 0 &= w_{xy} + (c - ab)w\\
(w_x + bw) - bw &= 0\\
w_x(x,\eta_0) &= 0 \text{ for } x \le \xi_0\\
w(\xi_0,\eta_0) &= e^{-(b\xi_0 + a\eta_0)}\\
w(x,\eta_0) &= e^{-(b\xi_0 + a\eta_0)} \text{ for } x \le \xi_0\\
v_y - av &= 0 \text{ for } x = \xi_0\\
(w_y + aw) - aw &= 0 \text{ for } y \le \eta_0\\
w(\xi_0,\eta_0) &= e^{-(b\xi_0 + a\eta_0)}\\
w(\xi_0,y) &= e^{-(b\xi_0 + a\eta_0)}
\end{aligned}$$

w is a constant along C_2 and C_3 so divide off the constant to get $w = 1$. wait! $v = e^{bx+ay}\frac{w}{e^{b\xi_0 + a\eta_0}}$ so this change gives $w = 1$ on the boundary.

Idea: maybe there is a solution of one symmetric variable. $z = (\xi_0 - x)(\eta_0 - y) \ge 0$. Try $w = h(z)$

$$\begin{aligned}
w_x &= h'(z)z_x & z_x &= -(\eta_0 - y)\\
w_y &= h'(z)z_y & z_y &= -(\xi_0 - x)\\
w_{xy} &= h'(z)z_{xy} + h''(x)z_xz_y & x_xz_y &= z, z_{xy=1}
\end{aligned}$$

Notes

April 28, 2014

solution $w(\xi_0, \eta_o; x, y)$ is related to riemann function $v(\xi_0, \eta_0; x, y)$ for original problem by $v = e^{b(x-\xi_0)+a(y-\eta_0)} w(\xi_0, \eta_0; x, y)$.

Introduced $x = (\xi_0 - x)(\eta_0 - y)$

$$w(x, y) = h(z)$$
$$zh''(z) + h'(z) + (c + ab)h(z) = 0$$

with $h(0) = 1$ want $h(z)$ on $z \geq 0$ try $h(z) = \sum_{n=0}^{\infty} h_n z^n$

$$\begin{aligned}
0 &= z \sum_{n=0}^{\infty} h_n n(n-1) z^{n-2} + \sum_{n=0}^{\infty} h_n n z^{n-1} + (c+ab) \sum_{n=0}^{\infty} h_n z^n \\
&= \sum_{n=0}^{\infty} h_n n^2 z^{n-1} + \sum_{n=0}^{\infty} (c+ab) h_n z^n \\
&= \sum_{n+1=1}^{\infty} h_{n+1} (n+1)^2 z^n + \sum_{n=0}^{\infty} (c+ab) h_n z^n
\end{aligned}$$

for $n \geq 0$ $(n+1)^2 h_{n+1} = (ab - c) h_n, h_0 = 1$

$n \geq 1$

$$h_n = \frac{ab-c}{n^2} h_{n-1} = \frac{ab-c}{n^2} \cdot \frac{ab-c}{(n-1)^2} \dots$$

so $h(z) = \sum_{n=0}^{\infty} \frac{(ab-c)^n}{n!n!} z^n$ that is $w(x, y) = \sum_{n=0}^{\infty} \frac{(ab-c)^n}{n!n!} (\xi_0 - x)^n (\eta_0 - y)^n$

note series converges for $|z| < \infty$

note $J_0(x) = \sum_{n=0}^{\infty} \frac{(-x^2/4)^n}{n!n!}$ and $I_0(x) = \sum_{n=0}^{\infty} \frac{(x^2/4)^n}{n!n!}$ so $h(z)$ can be written as J_0 or I_0 depending on sign of $ab - c$

lesson 30

vibrating drumhead

PDE	$u_{tt} = c^2(u_{xx} + u_{yy})$	$0 \leq r \leq 1$	$0 < \theta < 2\pi$	
BC	$u(1, \theta, t) = 0$		$0 < \theta < 2\pi$	$t > 0$
IC	$u(r, \theta, 0) = f(r, \theta)$			
	$u_t(r, \theta, 0) = g(r, \theta)$			

$\nabla^2 u = u_{xx} + u_{yy}$ (cartesian) is laplacian operator $= u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}$ (polar) remember $x = r\cos(\theta)$ and y is multiple of r also.

we will use separation of variables

$$u = U(r,\theta)T(t)$$
$$U = R(r)\Theta(\theta)$$

eigenfunction $U = R(r)$. note that this is a circle. nodel line. add in Θ and get radial nodel lines ($U = R(r)\Theta(\theta)$)

chladni came up with sprinkling sand on surface of these things.

$$u_r = u_x \cos(\theta)$$

$$\text{PDE} \qquad \frac{T''(t)}{T(t)} = \left[U_{rr} + \frac{1}{r}U_r + \frac{1}{r^2}U_{\theta\theta}\right] = \text{separation constant} = -\lambda^2 \text{ will assume less than 0}$$

$$T'' + c^2\lambda^2 T = 0 \leftarrow \text{trig solution}$$
$$U_{rr} + \frac{1}{r}U_r + \frac{1}{r^2}U_{\theta\theta} + \lambda^2 U = 0$$
$$U = R(r)\Theta(\theta)$$
$$R''(r) + \frac{1}{r}R'(r) + \frac{1}{r^2}R(r)\frac{\Theta''}{\Theta} + \lambda^2 R = 0$$

notice

$$\frac{\Theta''(\theta)}{Theta(\theta)} = \text{function of } r$$
$$= \text{function of } \theta$$
$$= \text{constant}$$
$$\Theta'' + \mu^2\Theta = 0$$

Θ must be 2π periodic

$$\cos(\mu\theta), \sin(\mu\theta)$$

by periocity

$$\mu = 1, 2,$$
$$\Theta(\theta) = a_n\cos(n\theta) + b_n\sin(n\theta)$$

Notes

April 30, 2014

lesson 30

$$
\begin{aligned}
\text{pde} \qquad & u_{tt} = c^2\nabla^2 u = c^2(u_{rr} + \frac{1}{r}u_r + 1\frac{1}{r^2}u_{\theta\theta}) \\
\text{bc} \qquad & u(1,\theta,t) = 0 \\
\text{ic} \qquad & u(r,\theta,0) = f(r,\theta) \\
& u_t(r,\theta,0) = g(r,\theta)
\end{aligned}
$$

$$
\begin{aligned}
u = T(t)U(r,\theta) \to \frac{T''(t)}{c^2T(t)} = \frac{\nabla^2 U}{U} &= -\lambda^2 \leq 0 \\
\nabla^2 U + \lambda^2 U = 0 & \text{ Helmholtz equation} \\
U &= R(r)\Theta(\theta) \\
0 &= R''(r) + \frac{1}{r}R'(r) + \left(\frac{1}{r^2}\frac{\Theta''(\theta)}{\Theta(\theta)} + \lambda^2\right)R(r)
\end{aligned}
$$

solutionsmust be 2π periodicin θ

$$
\begin{aligned}
\Theta''(\theta) + n^2\Theta(\theta) &= 0 \\
n &= 0,1,2,\ldots \\
& \cos(n\theta), \sin(n\theta)
\end{aligned}
$$

$$
0 = R''(r) + \frac{1}{r}R'(r) + \left(\lambda^2 - \frac{n^2}{r^2}\right)R(r)
$$

---

eigenvalues λ will besuch that $R(1) = 0$. Condition $R(r)$ should be bounded at $r = 0$

$$
\begin{aligned}
\lambda &= 0 \text{ not an eigenvalue} \\
0 &= R'' + \frac{1}{r}R' - \frac{n^2}{r^2}R \\
0 &= r^2R'' + rR' - n^2R \text{ Euler de} \\
R &= r^p \\
p(p-1)r^p + pr^p - n^2r^p &= 0 \\
p^2 - n^2 &= 0 \quad p = \pm n
\end{aligned}
$$

if $n = 1, 2, \ldots$

$$R$$

if $n = 0$

bessel function website is at http://dlmf.nist.gov/10.2
$J_v(z)$ bessel functions of first kind, bounded at origin
$Y_v(z)$ bessel functions of second kind, unbounded.
also look at http://dlmf.nist.gov/10.8
blah....

λ must satisfy $J_n(\lambda) = 0$ mathematica function name for bessel: BesselJZero[n,k] is k^{th} zero of the Bessel function $J_n(x)$. add //N for numeric output

$$U(r,\theta) = J)n(k_{n,m}r)(a_n \cos(n\theta) + b_n \sin(n\theta))$$

for

$$n = 0, 1, 2, \ldots$$
$$m = 1, 2, 3, \ldots$$

here $\lambda = k_{n,m}$

$$T''(t) + c^2 {k_{n,m}}^2 T(t) = 0$$

has solutions $\cos(k_{n,m}ct), \sin(k_{n,m}ct)$

frequency $\frac{k_{n,m}c}{2\pi}$ figure 30.3 $n =$ the number of zeros of the trig part of $U(r,\theta) = J_n(k_{n,m}r)(a_n \cos(n\theta) + b_n \sin(n\theta))$

Notes

May 2, 2014

pde

$$u_{tt} = c^2\nabla^2 u \text{ on } 0 < r < 1, 0 < \theta < 2\pi, 0 < t < \infty$$

bc

$$u = 0 \text{ on edge}$$

ic

$$\begin{aligned} u(r,\theta,0) &= f(r,\theta) \\ u_t(r,\theta,0) &= g(r,\theta) \end{aligned}$$

sep variables

$$\begin{aligned} u &= T(t)U(r,\theta) \\ \frac{T''}{c^2T} &= \frac{\nabla^2 U}{U} = -\lambda^2 \leq 0 \end{aligned}$$

note: $\frac{\nabla^2 U}{U} = \lambda^2 > 0$ has no sulutions with use of modified bessel functions.

note: $-\lambda^2 = 0$ was eliminated using euler's diffeq

get

$$U_{n,m}(r,\theta) = J_n(k_{n,m}r)(\underbrace{a\sin(n\theta) + b\cos(n\theta)}_{=A\cos(n(\theta-\theta_0))})$$

with

$$\begin{aligned} \lambda_{n,m} &= k_{n,m} \\ n &= 0, 1, 2, \ldots \\ m &= 1, 2, 3, \ldots \\ k_{n,m} &= m^{\text{th}} \text{ positive root of } J_n(x) \end{aligned}$$

$$T_{n,m}(t) = \cos(k_{n,m}ct), \sin(k_{n,m}ct)$$

general solution

$$u = \sum_{\substack{n\geq 0 \\ m\geq 1}} J_n(k_{n,m}r)(\cos(k_{nm}ct)(a_{nm}\sin(n\theta) + b_{nm}\cos(n\theta)) + \sin(k_{nm}ct)(c_{nm}\sin(n\theta) + d_{nm}\cos(n\theta))$$

main question: how to find codfficients a,b,c,d?

lab observations: we have frequencies $\frac{k_{nm}c}{2\pi}$ associated with spatial functions $U_{n,m}(r,\theta) = J_n(k_{nm}r)\cos(n(\theta - \theta_0))$

refer to page 237 for pictures.

m refers to which zero of the bessel function

$$\begin{aligned} n &= 0 & U_{0,m} &= J_0(k_{0,m}r)\cdot 1 \\ m &= 1 & U_{0,1} &= J_0(k_{0,1}r) \end{aligned}$$

so for this, the drumhead going up and down in center, no nodal lines, sand just falls off.

$$\begin{aligned} n &= 0 & U_{0,m} &= J_0(k_{0,m}r)\cdot 1 \\ m &= 2 & U_{0,2} &= J_0(k_{0,2}r) \end{aligned}$$

going in and out on center in opposite time to edge,nodal line at $r = \frac{k_{01}}{k_{02}}$

$$\begin{aligned} n &= 0 & U_{0,m} &= J_0(k_{0,m}r)\cdot 1 \\ m &= 3 & U_{0,3} &= J_0(k_{0,3}r) \end{aligned}$$

going in and out on center in opposite time to edge,nodal lines at $r = \frac{k_{01}}{k_{03}}$ and $r = \frac{k_{02}}{k_{03}}$

etc.

$$\begin{aligned} n &= 3 & U_{3,m} &= J_3(k_{3,m}r)\cos(3(\theta - \theta_0)) \\ m &= 1 & U_{3,1} &= J_3(k_{3,1}r)\cos(3(\theta - \theta_0)) \end{aligned}$$

three radial nodal lines separated by $\frac{2\pi}{3}$ from the cosine term. First bessel zero at outer edge from the bessel term.

$$\begin{aligned} n &= 3 & U_{3,m} &= J_3(k_{3,m}r)\cos(3(\theta - \theta_0)) \\ m &= 2 & U_{3,2} &= J_3(k_{3,2}r)\cos(3(\theta - \theta_0)) \end{aligned}$$

still three radial nodal lines, and now one circular nodal line.

back to the general solution. we want to find coefficients. orthoganality relation on p 239.

$$\int_0^1 rJ_0(k_{0i}r)J_0(k_{0j}r)\,\mathrm{d}r = \begin{cases} 0 & i \neq j \\ \frac{1}{2}{J_1}^2(k_{oi}) & i = j \end{cases}$$

we are deriving orthogonality for helmholtz equation

$$\begin{aligned} \nabla^2 U + \lambda^2 U &= 0 \text{ on } R \\ U &= 0 \text{ on } \partial R \end{aligned}$$

have 2 solutions fo λ (U_λ) and μ (U_μ).

claim

if $\lambda \neq \mu$, then $\int\int_R U_\lambda U_\mu \, \mathrm{d}a = 0$

proof

start with $\lambda^2 \int\int U_\lambda U_{gm} \, \mathrm{d}a = -\int\int_R \nabla^2 U_\lambda U_\mu \, \mathrm{d}a$. Note that $\nabla \cdot (\nabla U_\lambda \cdot U_\mu) = (\nabla^2 U_\lambda) U_\mu + \nabla U_\lambda \cdot \nabla U_\mu$ where nablaUlambda is vector and Umu is function

$$
\begin{aligned}
&= \int\int_R \left[-\nabla \cdot \left(\left(\nabla U_\lambda\right) U_\mu + \nabla U_\lambda \cdot \nabla U_\mu\right]\right. \mathrm{d}a \\
&= -\int_{\partial R} U_\mu \nabla U_\lambda \, \mathrm{d}a + \int\int_R \nabla U_\lambda \cdot \nabla U_\mu \, \mathrm{d}a
\end{aligned}
$$

Notes

May 5, 2014

pde

$$u_{tt} = c^2\nabla^2 u \text{ on } 0 \leq x \leq, 0 < \theta < 2\pi, 0 < t < \infty$$

bc

$$u(1, \theta, t) = 0$$

ic

$$u(r, \theta, 0) = f(r, \theta)$$
$$u_t(r, \theta, 0) = g(r, \theta)$$

sep of var

$$u = T(t)U(t, \theta)$$

helmholtz

$\nabla^2 U + \lambda^2 U = 0$ has solutions

$$\nabla^2 U_{n,m} + {\lambda_{n,m}}^2 U_{n,m} = 0$$
where $\lambda_{n,m} = k_{n,m} = m^{\text{th}}$ positive zero of $J_n(x)$
$$U_{n,m}(t, \theta) = J_n(k_{nm}r)\cos(n\theta), J_n(k_{nm}r)\sin(n\theta$$

$$u_{tt} = c^2\nabla^2 u \text{ wave eqnfor 2d and 3d}$$

$$\begin{cases} 2d & \nabla^2 u = u_{xx} + u_{yy} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} \\ 3d & \nabla^2 u = u_{xx} + u_{yy} + u_{zz} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz} \end{cases}$$

bessel functions called zylinderfunktionen in german because of usefullness in cylinders (and he's german), measured distance to a star, funktionen introduced by dirichlet in 1829.

fact

for $(n_1, m_1) \neq (n_2, m_2)$, $k_{n_1,m_1} \neq k_{n_2,m_2}$ (a bit deep)

fact

for $(n_1, m_1) \neq (n_2, m_2)$, $\int\int_{x^2+y^2+z^2} U_{n_1,m_1}(r, \theta)U_{n_2,m_2}(r, \theta)r \,\mathrm{d}r\mathrm{d}\theta = 0$
eigenfunction for $k_{n_1,m_1} \neq k_{n_2,m_2}$are orthogonal
consequence of divergence theorem

general solution

$$u(r,\theta,t) = \sum_{n\geq 0, m\geq 1} J_n(k_{nm}r)\left[\cos(k_{nm}ct)(a_{nm}\cos(n\theta) + b_{nm}\sin(n\theta)) + \sin(k_{nm}ct)(c_{nm}\cos(n\theta) + d_{nm}\sin(n\theta))\right]$$

satisfies pde and bc. how to satisfy ic?

at $t = 0$ $u(t,\theta,0) = f(r,\theta) = \sum_{n\geq 0, m\geq 1} J_n(k_{nm}r)(a_{nm}\cos(n\theta) + b_{nm}\sin(n\theta))$ and $u_r(t,\theta,0) = g(r,\theta)$ is similar for d_{nm} and c_{nm}

to find a_{NM}

$$\begin{aligned}\int_{\theta=0}^{2\pi}\int_{r=0}^{1} f(r,\theta)\cdot J_N(k_{NM}r)\cos(N\theta)r\,\mathrm{d}r\mathrm{d}\theta &= a_{NM}\int_{\theta=0}^{2\pi}\int_{r=0}^{1} J_N(k_{NM})^2\cos(N\theta)^2 r\,\mathrm{d}r\mathrm{d}\theta \\ &= \int_0^{2\pi}\cos(N\theta)^2\,\mathrm{d}\theta\int_0^1 J_N(k_{NM}r)^2 r\,\mathrm{d}r\end{aligned}$$

to find b_{NM}

$$\begin{aligned}\int_{\theta=0}^{2\pi}\int_{r=0}^{1} f(r,\theta)\cdot J_N(k_{NM}r)\sin(N\theta)r\,\mathrm{d}r\mathrm{d}\theta &= a_{NM}\int_{\theta=0}^{2\pi}\int_{r=0}^{1} J_N(k_{NM})^2\sin(N\theta)^2 r\,\mathrm{d}r\mathrm{d}\theta \\ &= \int_0^{2\pi}\sin(N\theta)^2\,\mathrm{d}\theta\int_0^1 J_N(k_{NM}r)^2 r\,\mathrm{d}r\end{aligned}$$

note

$$\begin{aligned}\int_0^{2\pi}\cos(N\theta)^2\,\mathrm{d}\theta &= \int_0^{2\pi}\cos(N\theta)^2\,\mathrm{d}\theta = \pi \text{ where } n = 1,2,3,\ldots \\ \int_0^{2\pi}\cos(N\theta)^2\,\mathrm{d}\theta &= \int_0^{2\pi}\cos(N\theta)^2\,\mathrm{d}\theta = 0 \text{ where } n = 0\end{aligned}$$

page 239

$$\int_0^1 J_0(k_{0i}r)J_0(k_{0j}r)r\,\mathrm{d}r = \begin{cases} 0 & \text{for } i \neq j \\ \frac{1}{2}J_{1j}(k_{0i})^2 & \text{for } i = j\end{cases}$$

will obtain ageneralformula for

$$\int_0^1 J_N(k_{NM}r)^2 r\,\mathrm{d}r \text{ text gives } N = 0$$

$J_v(z)$ satisfies $z^2\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + z\frac{\mathrm{d}w}{\mathrm{d}z} + (z^2 - v^2)w = 0$. antiderivative for $\int J_v(ar)J_v(br)r\,\mathrm{d}r$ for $a \neq b$ $(a,b > 0)$

write $y_a(r) = J_v(ar)$ then $y_z(r)$ satisfies $r^2\frac{\mathrm{d}^2 y_a}{\mathrm{d}r^2} + r\frac{\mathrm{d}y_a}{\mathrm{d}r} + (a^2r^2 - v^2)y_a = 0$. note $z = ar$. want

$$\begin{aligned} I &= \int y_a(r)y_b(r)r\,\mathrm{d}r \\ b^2 I &= \int y_a(r)(b^2 r y_b(r))\,\mathrm{d}r \\ &= \int y_a(r)\left[\frac{v^2}{r}y_b(r) - (ry_b')'\right] i\,\mathrm{d}r\end{aligned}$$

note

$$
\begin{aligned}
\int y_a (r y_b')' \mathrm{d}r &= y_a r y_b' - \int (y_a' r) y_b' \mathrm{d}r \\
&= y_a r y_b' - y_a' r y_b + \int y_b (r y_a')' \mathrm{d}r \\
b^2 I &= \int \left[y_a \frac{v^2}{r} y_b - y_b (r y_a')' \right] \mathrm{d}r + r(y_a' y_b - y_a y_b') \\
y_b(\frac{v^2}{r} y_a - (r y_a')') &= a^2 r y_a
\end{aligned}
$$

note

$$
\begin{aligned}
r(r y_b')' - v^2 y_b + b^2 r^2 y_b &= 0 \\
b^2 \int y_a y_b r \, \mathrm{d}r &= a^2 \int y_a y_b r \, \mathrm{d}r + r(y_a' y_b - y_a y_b') \leftarrow \frac{\mathrm{d}y_a}{\mathrm{d}r}
\end{aligned}
$$

Notes

May 7, 2014

final

turn in to mel, math dept secretary by noon on thursday.

last time

finding the n^{th} derivative of $J_v(x)$. $J_v(x)$ is bessel function 1st kind, order v. Satisfies $z^2w''(z) + zw'(z) + (z^2 - v^2)w(z) = 0$. showed

1. $(b^2 - a^2)\int J_v(ar)J_v(br)r\,\mathrm{d}r = r\left(aJ_v{}'(ar)J_v(br)b - J_v(ar)J_v{}'(br)\right)$
2. $\int (J_v(ar))^2 r\,\mathrm{d}r$

$$\begin{aligned}
\int J_v(ar)J_v(br)r\,\mathrm{d}r &= \frac{r}{b+a}\frac{1}{b-a}\left(aJ_v{}'(ar)J_v(br) - bJ_v(ar)J_v{}'(br)\right)\\
\int J_v(ar)^2 r\,\mathrm{d}r &= \frac{r}{2a}\frac{\mathrm{d}}{\mathrm{d}b}(\text{ above })\\
&= \frac{r}{2a}\left[aJ_v{}'(ar)rJ_v{}'(ar) - J_v(ar)J_\nu{}'(ar) - aJ_\nu(ar)J_\nu{}''(ar)r\right]\\
&= \frac{r}{2a}\left[ar(J_\nu{}'(ar))^2 - J_\nu(ar)\left(arJ_\nu{}''(ar) + J_\nu{}'(ar)\right)\right]\\
&= \frac{r}{2a}\left[ar(J_\nu{}'(ar))^2 - (J_\nu(ar)^2(\frac{\nu^2-(ar)^2}{ar}\right]
\end{aligned}$$

book page 239

$$\begin{aligned}
\int_0^1 J_0(k_{oi}r)^2 r\,\mathrm{d}r &= \frac{1}{2}J_1{}^2(k_{oi})\\
\int_0^1 J_n(k_{nm}r)r\,\mathrm{d}r &= \frac{1}{2}\left[(J_n{}'(k_{nm})^2 + (J_n(k_{nm}))^2\right] - \frac{n^2}{2k_{nm}{}^2}(J_n(k_{nm}))^2 = \frac{1}{2}(J_n'(k_{nm}))^2\\
\int_0^1 J_0(k_{0m}r)^2 r\,\mathrm{d}r &= \frac{1}{2}J_0'(k_{om})^2
\end{aligned}$$