Sample Capstone Paper

by Math Student

1. Introduction.

In general it is rather easy to write a capstone paper, provided that you've done a good and rigorous research on a nice topic under the supervision of a faculty.

In this article we will give a sample on paper writing in LaTeX and hopefully it will be helpful.

2. Basic Construction.

In this section we will demonstrate the construction of paragraphs, writing some mathematical scripts, writing formulas in display form, etc.

Real numbers are denoted by $\mathbb{R}$, positive rationals are denoted by $\mathbb{Q}^+$, (Cartesian) product of positive integers and positive rationals is denoted by $\mathbb{Z}^+ \times \mathbb{Q}^+$

An operation (or a function) is denoted, for example, by $\oplus : \mathbb{Q}^+ \times \mathbb{N} \to \mathbb{C}$.

A formula in display mode is given by

$$\tfrac{p}{q} + \tfrac{p'}{q'} = \tfrac{p+p'}{q+q'},$$

where $\frac{p}{q}$, $\frac{p'}{q'} \in \mathbb{Q}^+$.

If one wants to insert a picture or a graph, it's done like the following example:

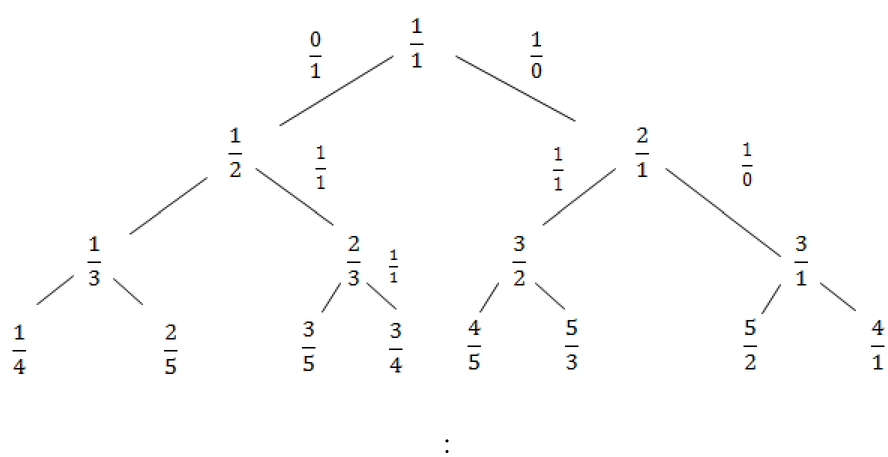

Figure.1 Stern-Brocot tree

One can write a statement, its proof and end sign of the proof as follows:

Fact 1. *Let $I = \mathbb{R}^+ \cup \{\infty, 0\}$. If the map $\eta : I \to [0,1]$, is defined by*

$$\eta(x) = \frac{x}{x+1}, \;\; x \in \mathbb{R}^+, \quad \eta(\infty) = 1.$$

Proof. It suffices to prove that $\eta(x) + \eta(x') = \eta(x + x')$. To show this, let $x = \frac{p}{q}$ and $x' = \frac{r}{s}$, then,

$$\eta(x)+\eta(x')=\frac{\frac{p}{q}}{\frac{p}{q}+1}+\frac{\frac{r}{s}}{\frac{r}{s}+1}=\frac{\frac{p}{q}}{\frac{p+q}{q}}+\frac{\frac{r}{s}}{\frac{r+s}{s}}=\frac{p}{p+q}\oplus\frac{r}{r+s}=\frac{p+r}{p+q+r+s}.$$

$$\eta(x+x')=\eta(\tfrac{p}{q}+\tfrac{r}{s})=\eta(\tfrac{p+r}{q+s})=\frac{\frac{p+q}{r+s}}{\frac{p+q}{r+s}+1}=\frac{\frac{p+q}{r+s}}{\frac{p+q+r+s}{r+s}}=\frac{p+q}{p+q+r+s}.$$

Therefore, $\eta(x)+\eta(x')=\eta(x+x')$. ■

Remark. As it is seen in Fact.1 one should not forget to end a proof without the sign that indicates that it's the case. For, you cal also use the command: ■

One can also use a different command to write a formula in display mode as:

Example 1. *Notice that* $\frac{3}{2},\frac{11}{9}\in\mathcal{T}\setminus\mathcal{F}$. *Then*

$$\eta(x)=\frac{\frac{3}{2}}{\frac{3}{2}+1}=\frac{\frac{3}{2}}{\frac{5}{2}}=\frac{6}{10}=\frac{3}{5}\in\mathcal{F},\ \textit{and}$$
$$\eta(x)=\frac{\frac{11}{9}}{\frac{11}{9}+1}=\frac{\frac{11}{9}}{\frac{20}{9}}=\frac{99}{180}=\frac{11}{20}\in\mathcal{F}.$$

You can write definitions as:

Definition. The *mediant* of $\frac{p}{q}$ and $\frac{p'}{q'}$ is defined as $\frac{p+p'}{q+q'}$ and need not be in the lowest terms.

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You can give citations within the article by using the command [3] [1] [2] [4] [5] [6].

3. Continuing with Additional Sections.

Typically a paper begins with an Abstract, Introduction, a section where basic tools and techniques are introduced, and subsequent sections where the subject is developed.

Definition. A code that corrects all error patterns of weight at most t and does not correct any error pattern of weight $t+1$ is called a t *error-correcting code.*

Example 2. *Given* $C=\{000,010,101,111\}$. *C has distance 3 so by theorem 1.2, C detects all error patterns of weight 1 or 2, but does not detect error patterns of weight 3. Hence, C is a 2 error-correcting code.*

Definition. Any expression of the form

$$x=a_0+\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{\ddots+\cfrac{1}{a_n+\cfrac{1}{\ddots}}}}}$$

is called an **infinite continued fraction** will be denoted by $[a_0;a_1,a_2,\dots]$.

Example 3. *Let x be the Golden Ratio $\frac{1+\sqrt{5}}{2} \approx \frac{3.236}{2}$, then we observe that*

$$\begin{gathered} 3.236 = \mathbf{1}\cdot 2 + 1.236 \\ 2 = \mathbf{1}\cdot 1.236 + 0.764 \\ 1.236 = \mathbf{1}\cdot 0.764 + 0.472 \\ 0.764 = \mathbf{1}\cdot 0.472 + 0.292\ . \\ \vdots \end{gathered}$$

Therefore, $\frac{1+\sqrt{5}}{2}$=[1;1,1,1,$\cdots$] which can be proved by letting

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\ddots}}}},$$

which is the continued fraction generated by [1;1,1,1,$\cdots$]. The continued fraction above can also be written as $x = 1 + \frac{1}{x}$, since the 1's are infinite, and this implies that $x^2 - x - 1 = 0$. Therefore the continued fraction would have to equal the positive solution to this polynomial, which is $\frac{1+\sqrt{5}}{2}$ by the quadratic formula. Recall that this is the classical definition of the Golden Ratio.

Now, having continued fraction defined, we will explore some of its basic properties.

Proposition 1. *For any finite continued fraction we have*

$$[a_0; a_1, \ldots, a_n] = \begin{cases} [a_0; a_1, \ldots, a_{n-1} + 1] \; if \; a_n = 1 \\ [a_0; a_1, \ldots, a_n - 1, 1] \; if \; a_n \neq 1 \end{cases}$$

Proof. If $a_n = 1$, then

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots \cfrac{1}{a_{n-1} + \cfrac{1}{1}}}}} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots \cfrac{1}{a_{n-1} + 1}}}}, \text{ since } \frac{1}{1} = 1.$$

Therefore, $[a_0; a_1, \ldots, a_{n-1}, 1] = [a_0; a_1, \ldots, a_{n-1} + 1]$. Similarly, if $a_n \neq 1$, then

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots \cfrac{1}{a_n}}}} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots \cfrac{1}{(a_n - 1) + 1}}}}$$

$$= a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots \cfrac{1}{a_n - 1 + \cfrac{1}{1}}}}}.$$

Therefore, $[a_0; a_1, \ldots, a_n] = [a_0; a_1, \ldots, a_n - 1, 1]$ if $a_n \neq 1$. ■

Theorem 1. *For any $x \in \mathbb{R}$, there exists a unique continued fraction $[a_0, a_1, a_2, \ldots]$ such that*

$$x = [a_0, a_1, a_2, \ldots].$$

Furthermore, this continued fraction is finite if $x \in \mathbb{Q}$.

Proof. Since the assertion is trivial if x is an integer, we will assume that x is not an integer. Let a_0 be the greatest integer less than or equal to x. Then, $x = a_0 + \frac{1}{r_1}$. Since $\frac{1}{r_1} = x - a_0 < 1$, r_1 is not an integer and $r_1 > 1$. In general, if r_n is not an integer, let a_n be the greatest integer less than or equal to r_n, and define $r_n = a_n - \frac{1}{r_{n+1}}$. Then $r_{n+1} > 1$. This process continues as long as r_n's are not integers. Now, from $x = a_0 + \frac{1}{r_1}$, we have $x = [a_0; r_1]$, and in general, $r_n = a_n - \frac{1}{r_{n+1}}$ implies that

$$x = [a_0; a_1, a_2, \ldots, a_{n-1}, r_n], \ n \geq 1,$$

provided that $r_1, r_2, \ldots, r_n$ are not integers.

Now, if x is a rational number, then each r_n is also a rational number. If $r_n = \frac{a}{b}$ form some $a, b \in \mathbb{Z}$, then $r_n - a_n = \frac{a - ba_n}{b} = \frac{c}{b}$, where $c < b$ since $r_n - a_n < 1$. If $c = 0$, i.e., a_n is an integer, then r_n is an integer and we are done. If $c \neq 0$, then $r_n = a_n - \frac{1}{r_{n+1}}$ implies that $r_{n+1} = \frac{b}{c}$, and hence, r_{n+1} has smaller denominator than that of r_n. Hence, continuing in this manner, eventually we will reach $a_m = r_m$ for some m. This fact, together with the representation $x = [a_0; a_1, a_2, \ldots, a_{n-1}, r_n]$, imply that x has a finite continued fraction representation.

If x is irrational, then all r_n's are irrational, and hence, the process above continues indefinitely. Letting $[a_0; a_1, \ldots, a_n] = \frac{p_n}{q_n}$, we have $x = [a_0; a_1 \ldots, a_{n-1}, r_n]$. By Fact 3,

$$x = \frac{p_{n-1} r_n + + p_{n-2}}{q_{n-1} r_n + q_{n-2}}, \quad n \geq 2.$$

Since $\frac{p_n}{q_n} = \frac{p_{n-1} a_n + + p_{n-2}}{q_{n-1} a_n + q_{n-2}}$, we have

$$\begin{aligned} \left| x - \frac{p_n}{q_n} \right| &= \left| \frac{(p_{n-1} q_{n-2} - q_{n-1} p_{n-2})(r_n - a_n)}{(q_{n-1} r_n + q_{n-2})(q_{n-1} a_n + q_{n-2})} \right| \\ &< \frac{1}{|(q_{n-1} r_n + q_{n-2})(q_{n-1} a_n + q_{n-2})|} < \frac{1}{q_n^2}, \end{aligned}$$

since $|r_n - a_n| < 1$ and $|p_{n-1} q_{n-2} - q_{n-1} p_{n-2}| < 1$ be Fact 4. Hence, it follows from Fact 7 that $\lim_n \frac{p_n}{q_n} = x$.

It remains to prove the uniqueness of the representation. Assume that

$$[a_0; a_1, a_2, \ldots] = x = [a'_0; a'_1, a'_2, \ldots].$$

Obviously, $a_0 = a'_0$ =greatest integer less than or equal to x. If $a_i = a'_i$ for $1 \le i \le k$, then $p_i = p'_i$ and $q_i = q'_i$ for $1 \le i \le k$. Thus,

$$x = \frac{p_n r_{n+1} + p_{n-1}}{q_n r_{n+1} + q_{n-1}} = \frac{p'_n r'_{n+1} + p'_{n-1}}{q'_n r'_{n+1} + q'_{n-1}} = \frac{p_n r'_{n+1} + p_{n-1}}{q_n r'_{n+1} + q_{n-1}},$$

which implies that $r_{n+1} = r'_{n+1}$. Since a_{n+1} =greatest integer less than or equal to r_{n+1} and a'_{n+1} =greatest integer less than or equal to r'_{n+1}, we obtain that $a_{n+1} = a'_{n+1}$ by induction. ■

Remark. From Theorem above, we deduce that a rational number x has a a finite continued fraction while an irrational number has an infinite sequence continued fraction representation.

Definition: The sum in LaTeX is written as $\sum\limits_{n=0}^{\infty} a_n$.

4. Matrix Representation and Coding.

There is a 1-1 correspondence between the elements in the $\mathbb{Q}^+$ and elements of the set of 2×2 matrices with integer entries $SL(2, \mathbb{Z})$. This correspondence is very instrumental in studying the dynamics on continued fractions.

We can describe each rational number as a matrix of the parent fractions [2],

$$x = \frac{p}{q} + \frac{p'}{q'} \equiv \begin{pmatrix} p & p' \\ q & q' \end{pmatrix}$$

This association defines a function $\Phi : \mathcal{F} \to SL(2, \mathbb{Z})$ by

$$\Phi(\frac{p}{q} + \frac{p'}{q'}) = \begin{pmatrix} p & p' \\ q & q' \end{pmatrix}.$$

Example 4. $\Phi(\frac{9}{11}) = \Phi(\frac{5}{6} + \frac{4}{5}) = \begin{pmatrix} 5 & 4 \\ 6 & 5 \end{pmatrix}$.

This matrix representation has the following property. Let

$$\frac{1}{2} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} =: L \quad \text{and} \quad \frac{2}{1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} =: R.$$

Then it follows that

Fact 2. $\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix} =: L^k$, *and* $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} =: R^k$.

Proof. Trivial. ■

Theorem 2. [5] [6] [1] *Let H be a parity-check matrix for a linear code C. Then C has distance d iff any set of $d-1$ rows of H are linearly independent and at least one set of d rows of H is linearly dependent.*

Proof. If G is a generator matrix for C, place G in RREF. Rearrange the columns of the RREF so that the leading columns come first and form an identity matrix. The result is a matrix G' in standard form which is a generator matrix for a code C' equivalent to C. ■

Having the infinite string coding of any real number in $\{L, R\}^{\mathbb{N}}$ defined, we can exploit some structural properties of $\{L, R\}^{\mathbb{N}}$ to study its deeper features.

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