1. Let $f_n(x) = \frac{x^2}{(1+x^2)^n}$ for all $x \in \mathbb{R}$. For what intervals $[a,b]$ does the series $\sum\limits_{n=0}^{\infty} f_n(x)$ converge uniformly?

   First we note that $1 \le (1+x^2)^n$ for all x and $n \ge 0$ so $\frac{1}{1+x^2} < 1$ for all x. And so

$$
\begin{aligned}
\sum_{n=0}^{k} \frac{x^2}{(1+x^2)^n} &= x^2 \left(\frac{1-(\frac{1}{1+x^2})^n}{1-\frac{1}{1+x^2}} \right) \\
&= x^2 \left(\frac{\frac{(1+x^2)^n-1}{(1+x^2)^n}}{\frac{1+x^2-1}{1+x^2}} \right) \\
&= \frac{((1+x^2)^n-1)(1+x^2)}{(1+x^2)^n} \\
&= 1+x^2-\frac{1}{(1+x^2)^{n-1}} \\
\sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n} &= x^2 \left(\frac{1}{1-\frac{1}{1+x^2}} \right) \\
&= x^2 \left(\frac{1}{\frac{1+x^2-1}{1+x^2}} \right) \\
&= 1+x^2
\end{aligned}
$$

   Now to determine uniform convergence, we look at $\lim\limits_{k\to\infty} ||f_k - f||_\infty$

$$
\begin{aligned}
\lim_{k\to\infty} ||f_k - f||_\infty &= \lim_{k\to\infty} \left|\left| 1+x^2-\frac{1}{(1+x^2)^{k-1}} - (1+x^2) \right|\right|_\infty \\
&= \lim_{k\to\infty} \left|\left| \frac{1}{(1+x^2)^k} \right|\right|_\infty \\
1 &> \frac{1}{(1+x^2)^n} \\
&\therefore \lim_{k\to\infty} ||f_k - f||_\infty = 0
\end{aligned}
$$

   Notice that we obtained this result without restricting the domain of x at all. And so f_n converges uniformly on any $[a,b] \subseteq (-\infty,\infty)$

2. For $x \neq -1$ evaluate the sum $\sum\limits_{n=0}^{\infty} \left(\frac{x-7}{x+1}\right)^n$

   We first note that as x gets close to -1 then $x+1$ gets close to zero and $x-7$ gets close to -8 and so as we approach from the right, $x+1$ is positive and our term gets very largely negative. Similarly as we approach from the left our term gets very largely positive. Furthermore, as x gets very large or very largely negative, then $\frac{x-7}{x+1}$ gets close to one. So we have identified two asymptotes, one vertical at $x=-1$ and one horizontal at $\frac{x-7}{x+1}=1$. We have a geometric series if $\left|\frac{x-7}{x+1}\right|<1$. Okay, now let us assume that $x+1<0$. That is $x<-1$. Well actually we just figured out what that graph of this term looks like, and if $x<-1$ then our term is always above it's asymptote at 1 and so there are no solutions in this case. Assuming $x+1>0$ we have

$$\begin{aligned}-1 &< \frac{x-7}{x+1} < 1\\ -x-1 &< x-7 < x+1\end{aligned}$$

we drop the last term because derpaderp

$$\begin{aligned}-x-1 &< x-7\\ -2x &< -6\\ x &> 3\end{aligned}$$

And so our term converges in the interval $(3,\infty)$.

Now what about $(-1,3]$? Well referring to what we figured out about how this graph looks, we know that $\frac{x-7}{x+1}\le -1$ on this interval. And so we have an alternating series with $\sum(-1)^n\left(\frac{x-7}{x-1}\right)^n$. Of course $\frac{x-7}{x-1}\ge 1$ and so $\left(\frac{x-7}{x-1}\right)^{n+1}\ge\left(\frac{x-7}{x-1}\right)^n$ which means our alternating series diverges.

Now going back to geometric convergence, we find

$$\begin{aligned}\sum_{n=0}^{\infty}\left(\frac{x-7}{x+1}\right)^n &= \frac{1}{1-\frac{x-7}{x+1}}\\ &= \frac{1}{\frac{8}{x+1}}\\ &= \frac{x+1}{8}\end{aligned}$$

And so we have $\sum\limits_{n=0}^{\infty}\left(\frac{x-7}{x+1}\right)^n=\frac{x+1}{8}$ when $x\in(3,\infty)$ and it is divergent when $x\in(-\infty,-1)\cup(-1,3)$

References

1. The convergence tests may be in the book, and in my notes from last semester, and I should know them, but I googled them, because that was easiest.

   https://www.math.hmc.edu/calculus/tutorials/convergence/