1. Let

$$B = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \end{pmatrix} \right\} \text{ and } C = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

be bases of $\mathbb{R}^2$ and $\mathbb{R}^3$ (resp). Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 2y \\ -x \\ y \end{pmatrix}$$

and let

$$\mathbf{x} = \begin{pmatrix} -7 \\ 7 \end{pmatrix}$$

(a) Find the image $T(\mathbf{x})$ of the vector $\mathbf{x}$ under the action of T.

$T(\mathbf{x}) = \begin{pmatrix} 7 \\ 7 \\ 7 \end{pmatrix}$

(b) Find the change of basis matrices $P = M(1_{\mathbb{R}^2}, S_2, B)$ and $Q = M(1_{\mathbb{R}^3}, S_3, C)$.

$P = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}^{-1} \Rightarrow \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & -1 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & -7 & -2 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cc|cc} 1 & 0 & 1/7 & 3/7 \\ 0 & 1 & 2/7 & -1/7 \end{array}\right]$

$Q = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$

$P = \frac{1}{7} \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \qquad Q = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

(c) Use part (b) to compute $[\mathbf{x}]_B$ and $[T(\mathbf{x})]_C$.

$[\mathbf{x}]_B = P\mathbf{x} = \frac{1}{7} \begin{pmatrix} 14 \\ -21 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$

$[T(\mathbf{x})]_C = QT(\mathbf{x}) = \begin{pmatrix} 0 \\ 0 \\ 7 \end{pmatrix}$

(d) Find $M(T, B, C)$ using the method of Example 7.

$$\begin{aligned}
T \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix} &= c_{11} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_{21} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_{31} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\
&= 6 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + -3 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\
T \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix} &= c_{12} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_{22} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_{32} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\
&= 4 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + -2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + -1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}
\end{aligned}$$

$$M(T, B, C) = \begin{bmatrix} 6 & 4 \\ -3 & -2 \\ 2 & -1 \end{bmatrix}$$

(e) Find $M(T,B,C)$ using the method of example 10.

$$M(T,B,C)=QA_TP^{-1}=\begin{bmatrix}1&-1&0\\0&1&-1\\0&0&1\end{bmatrix}\begin{bmatrix}1&2\\-1&0\\0&1\end{bmatrix}\begin{bmatrix}1&3\\2&-1\end{bmatrix}=\begin{bmatrix}6&4\\-3&-2\\2&-1\end{bmatrix}$$

(f) Check our answer in part (c) by verifying that $M(T,B,C)[\mathbf{x}]_B=[T(\mathbf{x})]_C$.

$$\begin{bmatrix}6&4\\-3&-2\\2&-1\end{bmatrix}\begin{pmatrix}2\\-3\end{pmatrix}=\begin{pmatrix}0\\0\\7\end{pmatrix}$$

2. Let

$$B=\left\{\begin{pmatrix}-3\\2\end{pmatrix},\begin{pmatrix}4\\-2\end{pmatrix}\right\}\text{ and }C=\left\{\begin{pmatrix}-1\\2\end{pmatrix},\begin{pmatrix}2\\-2\end{pmatrix}\right\}$$

be bases of $\mathbb{R}^2$. If

$$M(T,B,B)=\begin{bmatrix}-2&7\\-3&7\end{bmatrix}$$

find $M(T,B,B)$.

Maybe we are meant to find $M(T,C,C)$?

$$\begin{aligned}M(T,B,B)&=\begin{bmatrix}-2&7\\-3&7\end{bmatrix}\\&=\begin{bmatrix}-3&4\\2&-2\end{bmatrix}^{-1}T\begin{bmatrix}-3&4\\2&-2\end{bmatrix}\\T&=\begin{bmatrix}-3&4\\2&-2\end{bmatrix}\begin{bmatrix}-2&7\\-3&7\end{bmatrix}\begin{bmatrix}-3&4\\2&-2\end{bmatrix}^{-1}\\T&=\begin{bmatrix}-3&4\\2&-2\end{bmatrix}\begin{bmatrix}-2&7\\-3&7\end{bmatrix}\begin{bmatrix}1&2\\1&\frac{3}{2}\end{bmatrix}\\M(T,C,C)&=\begin{bmatrix}-1&2\\2&-2\end{bmatrix}^{-1}\begin{bmatrix}1&-\frac{3}{2}\\2&4\end{bmatrix}\begin{bmatrix}-1&2\\2&-2\end{bmatrix}\\&=\begin{bmatrix}2&1\\-1&3\end{bmatrix}\end{aligned}$$

3. Prove Theorem 11. Let B_1,B_2 be bases for $\mathbb{R}^n$ and let C_1,C_2 be bases for $\mathbb{R}^m$. If $T:\mathbb{R}^n\to\mathbb{R}^m$ is a linear transformation, then $\operatorname{rank}(M(T,B_1,C_1))=\operatorname{rank}(M(T,B_2,C_2))$.

Now $M(T,B_1,C_1)[\mathbf{x}]_{B_1}=[T(\mathbf{x})]_{C_1}$ and $M(T,B_2,C_2)[\mathbf{x}]_{B_2}=[T(\mathbf{x})]_{C_2}$. But $[T(\mathbf{x})]_{C_1}$ and $[T(\mathbf{x})]_{C_2}$

Let $B_1=\{\mathbf{v}_{11},\dots,\mathbf{v}_{1n}\}$ and $B_2=\{\mathbf{v}_{21},\dots,\mathbf{v}_{2n}\}$ while $C_1=\{\mathbf{w}_{11},\dots,\mathbf{w}_{1m}\}$ and $C_2=\{\mathbf{w}_{21},\dots,\mathbf{w}_{2m}\}$

4. Let $T:\mathbb{R}^n\to\mathbb{R}^n$ be a linear transformation and suppose that $V\le\mathbb{R}^n$ is invariant under T (that is $T(V)\subseteq V$). Prove that there exists a basis B such that

$$M(T,B,B)=\begin{bmatrix}A&B\\0&C\end{bmatrix}$$

where A is a $\dim(V)\times\dim(V)$ matrix.

5. Let $T:\mathbb{R}^n\to\mathbb{R}^m$ be a linear transformation of $\operatorname{rank}(T)=r$. Prove there exist bases B,C of $\mathbb{R}^n,\mathbb{R}^m$ (resp) such that

$$M(T,B,C)=\begin{bmatrix}I_R&0\\0&0\end{bmatrix}$$

Do you see how this establishes the Rank Nullity Theorem?

6. Prove that $\sim$ is an equivalence relation on $\mathcal{M}_n$

   If we have some matrix A then $A = I_n A I_n = I_n A I_n^{-1}$ and so we have reflexivity. Now if $B = PAP^{-1}$ then $P^{-1}BP = A$ and so if $A \sim B$ then $B \sim A$. Now if $C = QBQ^{-1}$ and $B = PAP^{-1}$ then $C = (QP)A(P^{-1}Q^{-1})$. Because QP and $Q^{-1}P^{-1}$ are inverses, we know that $C \sim A$. Thus we have an equivalence relation.

7. Prove that $A \sim A'$ if and only if A and A' represent the same linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$. That is, $A \sim A'$ if and only if $A = M(T, B, B)$ and $A' = M(T, B', B')$ for some bases B, B' of $\mathbb{R}^n$.

   If $A = M(T, B, B)$ then $A = M(I_n, E_n, B)M(T, E_n, E_n)M(I_n, E_n, B)^{-1}$. And $A' = M(T, B', B') = M(I_n, E_n, B')M(T, E_n, E_n)M(I_n, E_n, B')^{-1}$. Thus $A \sim M(T, E_n, E_n) \sin A'$ and so by transitivity we have $A \sim A'$.

   If $A \sim A'$ then

8. Prove that if $A \sim B$, then $A^{-1} \sim B^{-1}$ and $A^T \sim B^T$.

   If $A \sim B$ then $A = PBP^{-1}$ for some P. Now $PBP^{-1}PB^{-1}P^{-1} = I$ and so $A^{-1} = PB^{-1}P^{-1}$. Thus $A^{-1} \sim B^{-1}$

   If $A \sim B$ then $A = PBP^{-1}$. Taking the transpose of both sides we have $A^T = (PBP^{-1})^T = (P^{-1})^T(PB)^T = (P^{-1})^T B^T P^T$ and so $A^T \sim B^T$