NUMERICAL SEMIGROUPS, MARKOV BASES, AND SIMILAR STRUCTURES IN EXTENSIONS OF $\mathbb{N}$

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Abstract. The purpose of this paper is to explore numerical semigroups. We will cover bijections between numerical semigroups, Markov bases, and lattice ideals. We will go on to investigate analogous semigroups in natural extensions of $\mathbb{N}$, and we will see that most standard numerical semigroup properties break down quickly.

1. Introduction

There is a special kind of sub-semigroup of $\mathbb{N}$ called a numerical semigroup. These semigroups have proven to be interesting as they are closely related to more complicated objects. In order to understand these objects we will need to use tools from many branches of mathematics. These include statistics, abstract algebra, linear algebra, and combinatorics.

Because all of these disciplines will be represented we will endeavor to keep our notation consistent. For example, we will use the letter S to represent a numerical semigroup. The letter A will be used to represent the generating set of S and the letters m and M will represent $\min(A)$ and $\max(A)$ respectively.

When we first encounter Markov bases, we will be using statistics and linear algebra. That means tables and matrices. In this section we will use A to represent a matrix. However this matrix will be used to generate a numerical semigroup and so the use of the letter remains consistent.

Once we have dealt with Markov bases we will have created a map from numerical semigroups through Markov bases to lattice ideals and back again. This achieves our goal of using these humble semigroups to consider more complicated objects.

We are then free to imagine what else we can achieve with these objects. We will began exploring natural extensions of $\mathbb{N}$ and there we will see some surprising results. First though, we need to see the numerical semigroups on their own.

2. Numerical Semigroups

We begin with a set $A = \{n_1, \dots, n_k : n_i \in \mathbb{N}\}$. We can form an additive semigroup $S \subseteq \mathbb{N}$ with elements of the form $a_1 n_1 + \cdots + a_k n_k \in S$ where $a_i \in \mathbb{N}$. It is sometimes useful to represent A as a vector $\mathbf{n} \in \mathbb{N}^k$ and S as a span over vectors in $\mathbb{N}^k$. We say that $S = \langle A \rangle = \{\mathbf{n} \cdot \mathbf{a} : \forall \mathbf{a} \in \mathbb{N}^k\}$. Suppose that $n_i = 0$ for some $n_i \in A$. Then n_i does not contribute to S and $\langle A \rangle \cong \langle A \setminus \{n_i\} \rangle$ as a semigroup. Going forward we assume that $n_i \neq 0$ for all i.

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As before, we let $A = \{n_1, \dots, n_k\}$. We will say $c = \gcd(n_1, \dots, n_k)$ and $A' = \{\frac{n_1}{c}, \dots, \frac{n_k}{c}\}$. Consider the following function:

$$\begin{aligned} \varphi :&\langle A\rangle \to \langle A'\rangle \\ n_i &\mapsto \frac{n_i}{c} \end{aligned}$$

Now because φ maps the generators of $\langle A\rangle$ to the generators of $\langle A'\rangle$ and is invertible, we know that $\langle A\rangle \cong \langle A'\rangle$. Further, we know that $\gcd(A') = 1$.

Example 1. *The semigroup $\langle 2\rangle$ is $\{0, 2, 4, \dots\}$. This semigroup is isomorphic to $\langle 2/2\rangle = \langle 1\rangle = \{0, 1, 2, \dots\} = \mathbb{N}$*

Henceforth, we will restrict ourselves to semigroups whose generators have a greatest common denominator of one. This restriction has some interesting consequences.

Lemma 1. *Given $A = \{n_1, \dots, n_k\}$ with $\gcd(A) = 1$, the set of linear combinations of A is the set of all integers.*

Proof. We say $E = \{a_1n_1 + \cdots + a_kn_k > 0 : a_i \in \mathbb{Z} \text{ and } n_i \in A\}$. Obviously $n_1 \in E$. Because E has at least one element and a lower bound, there must be a smallest element in E. We say the smallest element of E is $s = a_1n_1 + \cdots + a_kn_k$. We choose some $n_i \in A$. We note that $s \le n_i$ and divide n_i by s to obtain $n_i = sq + r$ where $q, r \in \mathbb{N}$ and $0 \le r < s$. This means that

$$\begin{aligned} r &= n_i - sq \\ &= n_i - (a_1n_1 + \cdots + a_kn_k)q \\ &= n_i - qa_1n_1 - \cdots - qa_kn_k \\ &= -qa_1n_1 - \cdots - qa_in_i + n_i - \cdots - a_kn_k \\ &= -qa_1n_1 - \cdots - (qa_i - 1)n_i - \cdots - a_kn_k \end{aligned}$$

Thus $r \in E \cup \{0\}$. But $r < s$ and s is the smallest element of E so $r = 0$. This means that $n_i = sq$ for all n_i, but the only number that divides all n_i is 1, and so $s = 1$. And since every multiple of s is a linear combination of the elements of A, we have our result. □

Of course studying a more complicated version of $\mathbb{Z}$ is not very interesting. This is why we have been limiting the generators of our semigroups to $\mathbb{N}$. There is another much less obvious property that we can obtain from this result however. It allows us to create an upper bound for the complement in $\mathbb{N}$ of these semigroups. The immediate consequence of this is that the complement in $\mathbb{N}$ of these semigroups is finite.

Theorem 1. *A semigroup $S = \langle A\rangle$ where $A = \{n_1, \dots, n_k\}$ with $\gcd(A) = 1$ has a finite complement in $\mathbb{N}$.*

Proof. We know from our previous lemma that we can find some $a_1, \dots, a_k \in \mathbb{Z}$ such that $1 = a_1n_1 + \cdots + a_kn_k$. Let $\mathbf{a}^+ = \{a_i : a_i > 0\}$ and $\mathbf{a}^- = \{a_j : a_j < 0\}$. Then $1 = \sum\limits_{a_i \in \mathbf{a}^+} a_in_i + \sum\limits_{a_i \in \mathbf{a}^-} a_jn_j$ or $1 - \sum\limits_{a_j \in \mathbf{a}^-} a_jn_j = \sum\limits_{a_i \in \mathbf{a}^+} a_in_i$. Now because $a_j < 0$ for all $a_j \in \mathbf{a}^- < 0$ and $a_j < 0$ for all $a_j \in \mathbf{a}^+ > 0$ then $\sum\limits_{a_i \in \mathbf{a}^+} a_in_i \in S$ and $-\sum\limits_{a_i \in \mathbf{a}^-} a_in_i \in S$. If we say $c = -\sum\limits_{a_i \in \mathbf{a}^-} a_in_i$ then we have found c and $c + 1$ which are both elements of S.

We claim that for any $n \geq (c-1)(c+1)$ then $n \in S$. To verify this claim, we divide n by c. This leads us to $n = cq + r$ where $q, r \in \mathbb{N}$ and $0 \leq r < c$. We know that $cq+r \geq (c-1)(c+1) = (c-1)c+(c-1)$. Further $r \leq (c-1)$. This means that $cq \geq (c-1)c$ or $q \geq c-1 \geq r$. But $n = cq + r = qc + r + rc - rc = (q-r)c + r(c+1)$. We know that $q \geq r \geq 0$ and so $q - r \geq 0$ and $r \geq 0$. We also know that c and $c+1$ are both in S. This means that $n \in S$. Thus $\mathbb{N} \setminus S \subseteq \{n \in \mathbb{N} : n < (c+1)(c-1)\}$, which is finite.

$\square$

Example 2. *The semigroup generated by* $\{2, 3\}$ *is* $S = \{0, 2, 3, 4, \dots\}$. *Obviously* $\mathbb{N} \setminus S = \{1\}$ *which is finite.*

Now that we have constructed this object and explored the logic behind it, we give it a name and a formal definition.

Definition 1. [7] A numerical semigroup (NSG) is a nonempty subset of $\mathbb{N}$ that is closed under addition, contains the zero element, and whose complement in $\mathbb{N}$ is finite.

One thing to note is that $\mathbb{N}$ itself is a numerical semigroup.

Example 3. *The semigroup generated by* $\{1\}$ *is* $\{0, 1, 2, \dots\} = \mathbb{N}$. *Obviously* $\mathbb{N} \setminus \mathbb{N} = \emptyset$ *which is finite, thus* $\mathbb{N}$ *is a NSG.*

But in example 1 we saw that the even numbers, which are a subset of $\mathbb{N}$ are actually not a NSG. This leads to the following observation: while the elements of the generating set for a NSG are required to have a greatest common denominator of 1, any subset of the generating set need not meet that restriction. So a sub-semigroup of an NSG may not actually be an NSG.

Example 4. *The semigroup generated by* $\{6, 10, 15\}$ *is*

$$S = \{0, 6, 10, 12, 15, 16, 18, 20, 21, 22, 24, 25, 26, 27, 28, 30, \dots\}$$

. We see that

$$\mathbb{N} \setminus S = \{1, 2, 3, 4, 7, 8, 9, 11, 13, 14, 17, 19, 23, 29\}$$

is finite, and S *is a NSG.*

The sub-semigroup of $< 6, 10, 15 >$ *generated by* $\{6, 10\}$ *is* $\{0, 6, 10, 12, 16, 18, 20, 22, \dots\}$. *This semigroup, has an infinite complement in* $\mathbb{N}$ *and is therefore not a NSG, even though it is a sub-semigroup of a NSG.*

2.1. Frobenius Numbers

The fact that NSGs have a finite complement in $\mathbb{N}$ has a consequence. Namely that for any numerical semigroup S there exists some minimal natural number $F(S)$ such that if $n > F(S)$ then $n \in S$. In other words, this number $F(S)$ is the largest natural number that is not in our semigroup. This is called the Frobenius number. The number $F(S) + 1$ is referred to as the conductor[7].

As we saw in the proof for Theorem 1, if we have two consecutive numbers $c, c+1 \in S$ then $c^2 - 1$ provides an upper bound for $F(S)$. How much better can we do? There are formulas for some specific cases, but there are no known formulas for every case. Here we present an algorithm for finding the Frobenius number of any NSG.

The structure of an NSG has some trivial but very useful properties. The first thing of note, is that if we find the conductor than we have found the Frobenius number and vice versa. If we choose some $s_1 \in S$, then there exists at least one s_2 such that $s_1 < s_2 \leq s_2 + M$. In fact, for any $n_i \in A, s_1 \in S$ we know that $s_1 < s_1 + n_i \leq s_1 + M$. We use this fact to provide a working interval for our algorithm of $[s_i, s_i + M]$.

Let us suppose we have found m sequential elements in S starting with $s_1 \in S$. We know that $s_1 + i \in S$ for all $0 \leq i \leq m - 1$. Choose any $s_2 > s_1$. Then $s_2 = s_1 + j$ for some $j \in \mathbb{N}$. If we divide j by m then we obtain $j = q \cdot m + r$ where $0 \leq r < m$. From the definition of a semigroup, we know that if $s_1 \in S$ then $s_1 + q \cdot m \in S$. It also follows that $s_1 + r \in S$. Putting thse two facts together, we have $(s_1 + r) + q \cdot m \in S$. This means that $s_2 \in S$.

Thus the smallest number in A which begins a consecutive sequence of m elements in A is the conductor. Our algorithm begins with a known lower bound for our conductor. We increase this bound until we find a sequence of m consecutive elements in S. This reveals our conductor.

We know that every NSG contains zero. Zero is also the smallest element of any NSG and so it is our first candidate for the conductor. We say $c_1 = 0$ and begin our algorithm by constructing a working set $F_1 = \{c_1 + n_1, \ldots, c_1 + n_k\}$.

We say $c_i = c_1$ and $F_i = F_1$ and we begin iterating our algorithm. As we iterate we will seek to ensure that $|F_i| \leq M$.

We check to see if the smallest m elements of F_i are sequential. If they are, then we have met our termination criteria, and $F(S) = c_i - 1$. If we have not met the termination criteria, then we choose our next conductor candidate. We know that we have accounted for all elements in our semigroup at this point up to $c_i + m$. It is then safe to discard any elements of our semigroup below $\min(F_i)$. We choose our next lower bound for our conductor to be $c_{i+1} = \min(F_i)$ and define $F'_i = F_i \setminus \{\min(F_i)\}$.

We have assumed that $c_i + m$ elements are accounted for, and so we need to ensure this for the next iteration. Thus we say $F_{i+1} = F'_i \cup \{c_{i+1} + n_j : \forall n_j \in A\}$. Now we are ready to iterate again, and so return to the termination criteria step for F_{i+1}

The maximum time this algorithm takes to find the Frobenius number is a linear multiple of the number of elements in the semigroup and the size of the Frobenius number. There may be faster ways of finding this number, but for our purposes, it is simple to implement and lends itself to a great deal of optimization when run on a binary computer system.[2]

The following is a pseudocode implementation of the algorithm. Given a numerical semigroup $S = \langle A \rangle$ where $A = \{n_1, \ldots, n_k\}$, the algorithm takes A as input and produces the conductor c of the NSG as output.

`INPUT:` $A = \{n_1, \ldots, n_k\}$ `where` $\gcd(n_1, \ldots, n_k) = 1$

  $c := 0$
  $F := A$
  `WHILE` $[c+1, c+\min(A)-1] \not\subset F$
    $c := \min(F)$
    $F := F \setminus \{\min(F)\}$
    $F := F \cup \{c + n_i : \forall n_i \in A\}$
  `ENDWHILE`

`OUTPUT: The conductor of` $\langle A \rangle$ `is` c

Example 5. *We let $\langle A \rangle = \langle 5, 7, 9 \rangle$. Then $c_1 = 0$ and $F_1 = \{5, 7, 9\}$ while $m = 5$. We iterate our algorithm in the table below:*

Conductor	*Working NSG Window*	*Termination*
$c_1 = 0$	$F_1 = \{5, 7, 9\}$	$\{1, 2, 3, 4\} \not\subset F_1$
$c_2 = 5$	$F_2 = \{7, 9, 10, 12, 14\}$	$\{6, 7, 8, 9\} \not\subset F_2$
$c_3 = 7$	$F_3 = \{9, 10, 12, 14, 15\}$	$\{8, 9, 10, 11\} \not\subset F_3$
$c_4 = 9$	$F_4 = \{10, 12, 14, 15, 16, 18\}$	$\{10, 11, 12, 13\} \not\subset F_4$
$c_5 = 10$	$F_5 = \{12, 14, 15, 16, 17, 18, 19\}$	$\{11, 12, 13, 14\} \not\subset F_5$
$c_6 = 12$	$F_6 = \{14, 15, 16, 17, 18, 19, 21\}$	$\{12, 13, 14, 15\} \not\subset F_6$
$c_7 = 14$	$F_7 = \{15, 16, 17, 18, 19, 21, 23\}$	$\{15, 16, 17, 18\} \subset F_7$

And so we see that 13 *is the Frobenius number for* $\langle 5, 7, 9 \rangle$

3. Markov Bases

The semigroups that we have been examining are relatively simple objects. In the upcoming section on bijections we will see how we can relate them to less straightforward algebraic and combinatorial objects. However in order to have that discussion we will need to understand a tool called the Markov basis.

Markov bases play a central role in the recent field of algebraic statistics. A seminal paper[4] on this field introduced the idea of a Markov basis for log linear statistical models and related them to commutative algebra. This work has been applied in many fields and has been particularly active in computational biology[5]. However, we will be glossing over the statistical role of these bases and will instead focus on their algebraic properties. First we need to get a few definitions and some notation out of the way.

Coming from statistics, Markov bases were developed to be used with tables. Tables are naturally expressed as matrices, and the Markov basis revolves around this. We noted earlier that a NSG can be represented as the span of vectors in $\mathbb{N}^k$. We can similarly represent the generators of a numerical semigroup $S = \langle n_1, \ldots, n_k \rangle$ as a matrix $A = \begin{bmatrix} n_1 & \cdots & n_k \end{bmatrix}$. Then for every element $s \in S$ we can say $s = A\mathbf{u}$ for some $\mathbf{u} \in \mathbb{N}^k$.

Definition 2. [5] The set of tables

$$\mathcal{F}(\mathbf{u}) = \left\{ \mathbf{v} \in \mathbb{N}^k : A\mathbf{v} = A\mathbf{u} \right\}$$

is called the *fiber* of a contingency table $\mathbf{u} \in \mathcal{T}(n)$ with respect to the model $\mathcal{M}_A$

Contingency tables and matrix models are specific to statistics. The thing we should take from this definition, is that a fiber $\mathcal{F}(\mathbf{u})$ of an element $A\mathbf{u}$ of our semigroup A is the set of vectors $\left\{ \mathbf{v} \in \mathbb{N}^k : A\mathbf{v} = A\mathbf{u} \right\}$

Example 6. *For the numerical semigroup $A = \begin{bmatrix} 3 & 4 & 5 \end{bmatrix}$ we will find the fiber corresponding to the element $A\mathbf{u} = 8$.*

We need to find all solutions to the equation $\begin{bmatrix} 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} x & y & z \end{bmatrix} = 8$ or $3x+4y+5z = 8$ where $x, y, z \in \mathbb{N}$. We find that $3 + 5$ and $4 \cdot 2$ are the only two possible solutions, and so for $A\mathbf{u} = 8$ we see that $\mathcal{F}(\mathbf{u}) = \{(1, 0, 1), (0, 2, 0)\}$.

Before we give the definiton of a Markov basis, we note that the literature often refers to the elements of a Markov basis as *moves*[5, p.16]

Definition 3. [5] Let $\mathcal{M}_A$ be the log-linear model associated with a matrix A whose integer kernel we denote by $\ker_{\mathbb{Z}}(A)$. A finite subset $\mathcal{B} \subset \ker_{\mathbb{Z}}(A)$ is a *Markov basis* for $\mathcal{M}_A$ if for all $\mathbf{u} \in \mathcal{T}(n)$ and all pairs $\mathbf{v}, \mathbf{v}' \in \mathcal{F}(\mathbf{u})$ there exists a sequence $\mathbf{u}_1, \ldots, \mathbf{u}_L \in \mathcal{B}$ such that

$$\mathbf{v}' = \mathbf{v} + \sum_{k=1}^{L} \mathbf{u}_k \text{ and } \mathbf{v} + \sum_{k=1}^{l} \mathbf{u}_k \geq 0 \text{ for all } l = 1, \ldots, L.$$

Example 7. *We will find the Markov basis which corresponds to the numerical semigroup $A = \langle 3, 4, 5 \rangle$. First we generate the fibers which correspond to the elements of our NSG.*

	3	4	5
3	1	0	0
4	0	1	0
5	0	0	1
6	2	0	0
7	1	1	0
8	1	0	1
8	0	2	0
9	3	0	0
9	0	1	1
10	2	1	0
10	0	0	2

We are particularly interested in fibers with more than one element. These are the fibers associated with the elements $8, 9$ and 10. From the definition of the Markov basis, we know that the product of A and an element of the Markov basis is 0. Furthermore, we know that we can add a sequence of elements of the Markov basis to any of the elements of a fiber to obtain any other element of that fiber. Taking the difference of two elements from the same fiber will meet both of these criteria.

Thus if $A\mathbf{u} = 8$ then $\mathcal{F}(\mathbf{u}) = \{(1,0,1),(0,2,0)\}$ and so $(-1,2,-1) \in \mathcal{B}$. If $A\mathbf{v} = 9$ then $\mathcal{F}(\mathbf{v}) = \{(3,0,0),(0,1,1)\}$ and so $(3,-1,-1) \in \mathcal{B}$. And if $A\mathbf{w} = 10$ then $\mathcal{F}(\mathbf{w}) = \{(2,1,0),(0,0,2)\}$ and so $(-2,-1,2) \in \mathcal{B}$. And so we have built our Markov basis.

$$\mathcal{B} = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 2 & -1 \\ -2 & -1 & 2 \end{bmatrix}$$

We acknowledge that we have given no explanation as to why we stopped searching for elements of the Markov basis. In general there is no easy termination criteria when searching for these. However, we will come back to this example later, and show that in this case we have found all the elements of our basis.

We also have the fundamental theorem of Markov bases which provides a direct relation to a lattice ideal.

Theorem 2. [3, p. 54] *A finite set of moves $\mathcal{B}$ is a Markov basis for A if and only if the set of binomials $\{p^{\mathbf{z}^+} - p^{\mathbf{z}^-} | \mathbf{z} \in \mathcal{B}\}$ generates the toric ideal I_A.*

4. Bijections

As we mentioned in the last section, NSGs are simple to study. It would be useful if we could consider more complicated objects as NSGs. One such object is the lattice ideal.

4.1. Ideals

We will begin by looking at common mappings for monomials and binomials. We will continue on to connect these mappings to our NSGs. This will allow us to consider lattice ideals as NSGs.

Consider $\varphi : \mathbb{N}^n \to k[x_1, \ldots, x_k]$ given by $\alpha \mapsto \mathbf{x}^\alpha$.

Lemma 2. *Let S be a NSG. Then $\varphi(S)$ is the set of monic monomials of a monomial ideal of $k[x]$, where φ is as above.*

Proof. Let $S = \langle n_1, \ldots, n_k \rangle$ and let $I = \langle x^{n_1}, \ldots, x^{n_k} \rangle$. We choose $X \in I$ and $s \in S$ where $X = \prod_{i=1}^{k} (x_i^{n_i})^{a_i}$ while $s = \sum_{i=1}^{k} a_i n_i$. We observe that $\varphi(s) = \varphi(\sum_{i=1}^{k} a_i n_i) = \prod_{i=1}^{k} x_i^{a_i n_i} = \prod_{i=1}^{k} (x_i^{n_i})^{a_i} = X$. □

If we look at $\mathbb{Z}$ instead of $\mathbb{N}$ then we have an analogous map with binomials. In the binomial case we start with some $\mathbf{z} \in \mathbb{Z}^n$. We define $\mathbf{z} = (z_1, \ldots, z_n)$. We say that $\mathbf{z}^+ = \mathbf{z} \vee \mathbf{0}$ gives us a vector where $z_i^+ = \max(z_i, 0)$. Similarly $\mathbf{z}^- = \mathbf{z} \wedge \mathbf{0}$ is a vector where $z_i^- = \min(z_i, 0)$. We can map an element of $\mathbb{Z}^n$ to a binomials over field k by $\varphi : \mathbb{Z}^n \to k[x_1 \ldots x_n]$ where $\mathbf{z} \mapsto \mathbf{x}^{\mathbf{z}^+} - \mathbf{x}^{\mathbf{z}^-}$.

If we wish for the map to be bijective, then we need some addition restrictions. Let us assume that k has characteristic 2. This means that $x - y = x + y$ for any $x, y \in k$. Then $\varphi(1, -1) = x - y = x + y = \varphi(1, 1)$ and $\varphi(1, 1) = x + y = y - x = \varphi(-1, 1)$. This map is obviously not one to one, and so we must restrict ourselves to fields who have characteristic other than 2.

Now consider the binomial $x + x^2$. This binomial is not in the image of φ. Bijection requires surjection, and so we restrict our codomain to be the set of pure binomials in $k[x_1, \ldots, x_n]$

4.2. Markov Bases

We wish to move from $\mathbb{N}$ to $k[x_1, \ldots, x_n]$. The tool we need for this is the Markov basis. These bases are troublesome to compute, but theorem 2 tells us that they provide a map from $\mathbb{N}$ to $\mathbb{Z}^n$. This combined with our above discussion on binomials will give us all that we need to complete this map.

Each element s of a numerical semigroup $S = \langle n_1, \ldots, n_k \rangle$ takes the form $s = a_1 n_1 + \cdots + a_k n_k$. Now we examine the vector $(a_1, \ldots, a_k)$. Note that any given $s \in S$ may have more than one vector associated with it, and the set of these vectors make up the fiber over s.

Example 8. $S = \langle 3, 4, 5 \rangle$. *Notice that* $8 = 2 \cdot 4 = (3, 4, 5) \cdot (0, 2, 0)$ *and* $8 = 3 + 5 = (3, 4, 5) \cdot (1, 0, 1)$. *Thus the vectors* $(0, 2, 0)$ *and* $(1, 0, 1)$ *make up the fiber over* $8 \in S$.

Two vectors $\mathbf{a} = (a_1, \ldots, a_k)$ and $\mathbf{b} = (b_1, \ldots, b_k)$ from the same fiber are connected if there exists some i such that $a_i > 0$ and $b_i > 0$. Furthermore, if $\mathbf{a}$ is connected to $\mathbf{b}$ and $\mathbf{b}$ is connected to $\mathbf{c}$ then $\mathbf{a}$ is connected to $\mathbf{c}$.

We are looking for elements of our NSG which have multiple associated but disconnected vectors. Once we have found two disconnected vectors associated with an element of our

NSG, then we subtract them to find an element of our Markov basis. We continue until we have found the elements of our Markov basis guaranteed by Theorem 2.

Example 9. *We will find the Markov basis which corresponds to the numerical semigroup* $\langle 3, 4, 5\rangle$. *First we generate a list of vectors which correspond to the elements of our NSG.*

	3	4	5		3	4	5
3	1	0	0	11	2	0	1
4	0	1	0	11	1	2	0
5	0	0	1	12	4	0	0
6	2	0	0	12	0	3	0
7	1	1	0	12	1	1	1
8	1	0	1	13	3	1	0
8	0	2	0	13	1	0	2
9	3	0	0	13	0	2	1
9	0	1	1	14	2	2	0
10	2	1	0	14	0	1	2
10	0	0	2	14	3	0	1

Now we see that $8, 9, 10$ *all have two associated but disconnected vectors. The fibers of* $11, 12, 13$ *and* 14 *are all connected. We further observe that the Frobenius number of this semigroup is 2. Now the fiber of 12 contains the special vector* $(1, 1, 1)$. *If we choose any element* $n > 14$ *of our semigroup, then we can say* $n = 12 + n'$. *We know that* n' *is also in our semigroup because* 2 *is our Frobenius number. Thus the fiber of every element* $n > 14$ *contains a vector that is the sum of* $(1, 1, 1)$ *and some other vector. Thus the fibers of all elements greater than* 10 *are in fact connected. And so we can subtract the vectors in the fibers of* $8, 9, 10$ *to obtain the following.*

$$\begin{bmatrix} 3 & -1 & -1 \\ -1 & 2 & -1 \\ -2 & -1 & 2 \end{bmatrix}$$

This is all the elements of our Markov basis.

It is convenient to write our Markov basis as a matrix, whose rows consist of the elements of the Markov basis.

4.3. Smith Normal Form

Now we have found a map from NSGs to pure binomial ideals of $k[x_1, \dots, x_n]$ where k is not characteristic 2. If we wish to consider the NSG and the binomial ideals the same objects, then given a Markov basis, we should be able to find an associated NSG. Let $\mathcal{B}$ be the matrix with rows corresponding to the vectors of a Markov basis. Every row $\mathbf{v}$ of $\mathcal{B}$ consists of two vectors $\mathbf{a}, \mathbf{b}$ where $\mathbf{v} = \mathbf{a} - \mathbf{b}$. Let $S = \langle n_1, \dots, n_k\rangle$ and define the vector $\mathbf{n} = (n_1, \dots, n_k)$. Now if S is the semigroup associated with $\mathcal{B}$, then by construction $\mathbf{n} \cdot \mathbf{a} = \mathbf{n} \cdot \mathbf{b}$. And so $\mathbf{n} \cdot \mathbf{v} = 0$. This means that if we have $\mathcal{B}$, we can find our numerical semigroup S by finding a nontrivial solution to $\mathcal{B}\mathbf{n} = 0$.

The usual tools of linear algebra are not useful here. The major problem is that we are dealing with matrices in $\mathbb{Z}$ instead of $\mathbb{R}$ and so we do not have division available in general. The tool we need is the Smith normal form.

Definition 4. If we are given some matrix A whose entries are in a principal ideal domain, then we can find some matrices U, V, B such that

$$UAV = B = \begin{bmatrix} b_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & b_r \end{bmatrix} \text{ with } b_i | b_{i+1}$$

We call B the *Smith normal form* of A.[1]

If there exists some $\mathbf{n} \neq \mathbf{0}, \mathcal{B} \in \mathcal{M}_{q\times r}$ such that $\mathcal{B}\mathbf{n} = \mathbf{0}$ then $\operatorname{rank}(\mathcal{B}) < r$. And so if we have $U\mathcal{B}V = C$ where C is the Smith normal form of $\mathcal{B}$ then $C = \begin{bmatrix} c_1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & c_{r-1} & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix}$. Because the last column of C is $\mathbf{0}$ we know that if $\mathbf{v}$ is the last column of V then $\mathcal{B}\mathbf{v} = \mathbf{0}$. The semigroup generated by the coordinates of $\mathbf{v}$ must be isomorphic to a unique numerical semigroup generated by the coordinates of some vector $\mathbf{n} = \alpha\mathbf{v}$ where $\alpha \in \mathbb{Q}$ and the greatest common divisor of the coordinates of $\mathbf{n}$ is 1.

Example 10. *Let us find the Smith normal form of* $\mathcal{B} = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 2 & -1 \\ -2 & -1 & 2 \end{bmatrix}$. *We start with* $U'\mathcal{B}V' = I_{\mathcal{B}}\mathcal{B}I_{\mathcal{B}}$. *We then use the standard row and column operations on* $\mathcal{B}$ *while recording the row operations on* U' *and the column operations on* V' *to find our Smith normal form, along with the* U *and* V. *Remember we are working over* $\mathbb{Z}$ *and so we will only be adding, subtracting, and multiplying, not dividing.*

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & -1 & -1 \\ -1 & 2 & -1 \\ -2 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & -1 & -1 \\ -1 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & -3 \\ -1 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 1 & 3 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & -3 \\ 0 & 5 & -4 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 1 & 3 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 1 & 3 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \\ 0 & 1 & 5 \end{bmatrix}$$

And so $\begin{bmatrix} 1 & 2 & 0 \\ 1 & 3 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 & -1 \\ -1 & 2 & -1 \\ -2 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \\ 0 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Observe that the Markov basis was the basis obtained in a previous example from the NSG $\langle 3, 4, 5\rangle$*. We see that the last column of* V *is* $(3, 4, 5)$ *which corresponds exactly to the NSG we began with.*

Calculating these matrices is tedious, time consuming and error prone. Fortunately computers well suited to these kinds of calculations. One software package that can easily compute these matrices is Xcas[6].

4.4. Numerical Semigroups in Lattice Theory

We have already discussed NSGs, Markov basis, and binomials. Now we examine lattice ideals, which can be expressed as binomial ideals.

Definition 5. [8] A *lattice* is a partially ordered set in which every two elements have a unique least upper bound and a unique greatest lower bound.

Example 11. *The space* $\mathbb{N}^2$ *is a lattice with supremum and infinum for any two elements which belong to it. Notice that* $(1, 2)$ *and* $(2, 1)$ *have a lower bound of* $(1, 1)$ *and an upper bound of* $(2, 2)$*.*

Example 12. *We can form a lattice if we order* $\mathbb{N}$ *by division. The least common multiple forms a least upper bound and an greatest lower bound is formed by the greatest common denominator.*

The fundamental theorem of Markov bases (theorem 2) states that there is a bijection between a lattice ideal and a Markov basis. As we have already seen, Markov bases are in bijection with NSGs.

Thus we see that every NSG corresponds to a lattice ideal.

Example 13. *We begin with a lattice ideal in* $k[x, y, x]$ *with* $char(k) \neq 2$

$$I_\Lambda = \langle x^3 - yz, y^2 - xz, z^2 - xy\rangle$$

We can map this generating set to $\mathbb{Z}^3$ *in the usual way.*

$$\begin{cases} x^3 - yz \\ y^2 - xz \\ z^2 - x^2y \end{cases} \Rightarrow \begin{bmatrix} 3 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

Theorem 2 is key here. It guarantees that these vectors are actually a Markov basis. As we saw in the previous two sections, this basis corresponds to the numerical semigroup $\langle 3, 4, 5\rangle$*. So we see*

$$\begin{cases} x^3 - yz \\ y^2 - xz \\ z^2 - xy \end{cases} \mapsto \langle 3, 4, 5\rangle$$

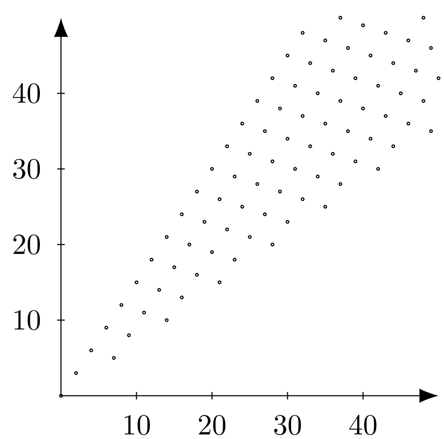
The reverse map is similarly straightforward.

5. Semigroups in Extensions of $\mathbb{N}$

The complex numbers are a natural extension of the real numbers as the Gaussian integers are a natural extension of the real integers. In this section, we consider the corresponding extension of the natural numbers. We will explore the NSG construction in terms of the Gaussian integers and discuss the similarities and differences.

We take two NSGs $\langle A\rangle, \langle B\rangle$ where $A = \{x_1, \ldots, x_k\}$ and $B = \{y_1, \ldots, y_k\}$. These semigroups can be used to form a "Gaussian numerical semigroup" $\langle G\rangle$ where $G = \{x_1 + y_1 i, \ldots, x_k + y_k i\}$. For the real and imaginary parts individually, we observe that we can generate all numbers greater than the respective Frobenius numbers. However, unlike the component NSGs, the GNSG never achieves any kind of finite complement. This is because for any $n > F(\langle A\rangle)$ there are a only a finite number of ways in which we can combine the elements of A to add up to n. Combining the elements of B in the same ways will then only produce a finite number of outputs. This means that for any $n \in \langle A\rangle$ we can only generate a finite number of $n + mi \in \langle G\rangle$. Surprisingly, we have the opposite case from the NSG. Rather than having a finite complement in any infinite subset of the positive Gaussian integers, we have an infinite complement. The complement of the GNSG remains infinite over the entire $\mathbb{N}^2$ lattice.

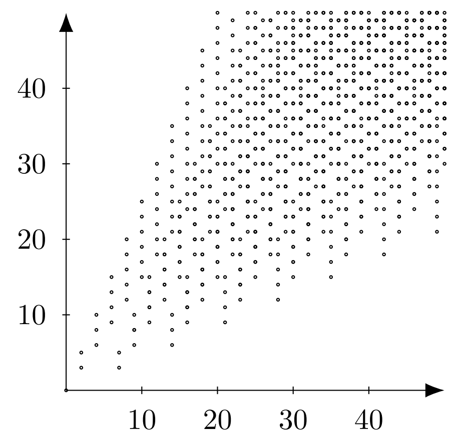
Example 14. *We consider $\langle 2, 7\rangle$ and $\langle 3, 5\rangle$. These NSGs have Frobenius numbers 5 and 7 respectively*[2]. *The following is a plot in $\mathbb{N}^2$ of the numbers generated by $\langle 2 + 3i, 7 + 5i\rangle$.*



This notion of a GNSG does impose a sort of ordering on our elements that is not present in the original numerical semigroup. This ordering is caused by linking elements of our two NSGs in the generation of the GNSG. In addition, combining two NSGs with differently sized generating sets could be problematic.

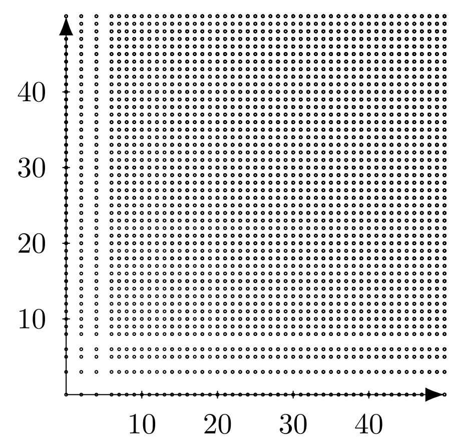
These problems can be overcome with the creation of a semigroup in the form of $\langle A \oplus B\rangle$. Combining NSGs whose generating sets are different sizes is explicitly handled in this case, and the two semigroups are allowed to generate their elements independently. Unfortunately we still have the same problem of only being able to generate any number in a finite number of ways. This leaves us with the same problem of the semigroup retaining its infinite complement over all of $\mathbb{N}^2$

Example 15. *We consider $\langle 2, 7\rangle$ and $\langle 3, 5\rangle$. These NSGs have Frobenius numbers 5 and 7 respectively*[2]. *The following is a plot in $\mathbb{N}^2$ of the numbers generated by $\langle 2 + 3i, 7 + 5i, 2 + 5i, 7 + 3i\rangle$.*



If we wish to create a semigroup in $\mathbb{N}[i]$ with some form of a finite complement, then we must allow the semigroup generators to include zero for either the real or complex components. Thus our semigroup would be $\langle x_1, \ldots, x_k, y_1 i, \ldots, y_l i\rangle$. But this is just the direct sum of our original NSGs. We can find any point $x + yi \in \langle A\rangle \oplus \langle B\rangle$ so long as $x > F(A)$ and $y > F(B)$.

Example 16. *We consider $\langle 2, 7\rangle$ and $\langle 3, 5\rangle$. These NSGs have Frobenius numbers 5 and 7 respectively*[2]*. The following is a plot in $\mathbb{N}^2$ of the numbers generated by $\langle 3i, 5i, 2, 7\rangle$.*



6. Conclusion

Extending the concept of a NSG into the Gaussian integers breaks everything we have seen so far. There may be ways to fix this problem and we have discussed two alternative approaches to this problem. We do not know if any of the relationships between NSGs and lattice ideals and Markov bases can be applied in this new space. Going forward it remains to examine these Gaussian semigroups and see if we can find analogues to lattices and Markov bases in the Gaussian integers.

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