Notes

April 16, 2014

finished lesson 23 (classification of PDEs-canonical form for hyperbolic PDEs). note: lesson 41 (canonical forms for parabolic ellipti pdes) moving outside text because classification $\neq$ solution. latst time: example of general I for hyperbolic pde $u_{xx} - u_{yy} = 0$ with data on $x + 2y = 0$

riemann's method for byperbolic pdes.

start

operation $< [u] := u_{\xi\eta} + a(\xi, \eta)u_\xi + b(\xi, \eta)u_\eta + c(\xi, \eta)u$

curve C:$\eta = \phi(\xi)$ for $-\infty < \xi < +\infty$ with $\phi'(\xi) < 0$ (alternate $\phi'(\xi) > 0$)

PDE $< [u] = F(\xi, \eta)$ on $\eta > \phi(\xi)$ on $\eta > \phi(\xi)$ (alternate $\eta < \phi(\xi)$)

ic $u(\xi, \phi(\xi)) = f(\xi)$ and $u_\xi(\xi, \phi(\xi)) = g(\xi)$

context start with hyperbolic pde with initial conditions and change to canonical coordinates ξ, η and get new form here

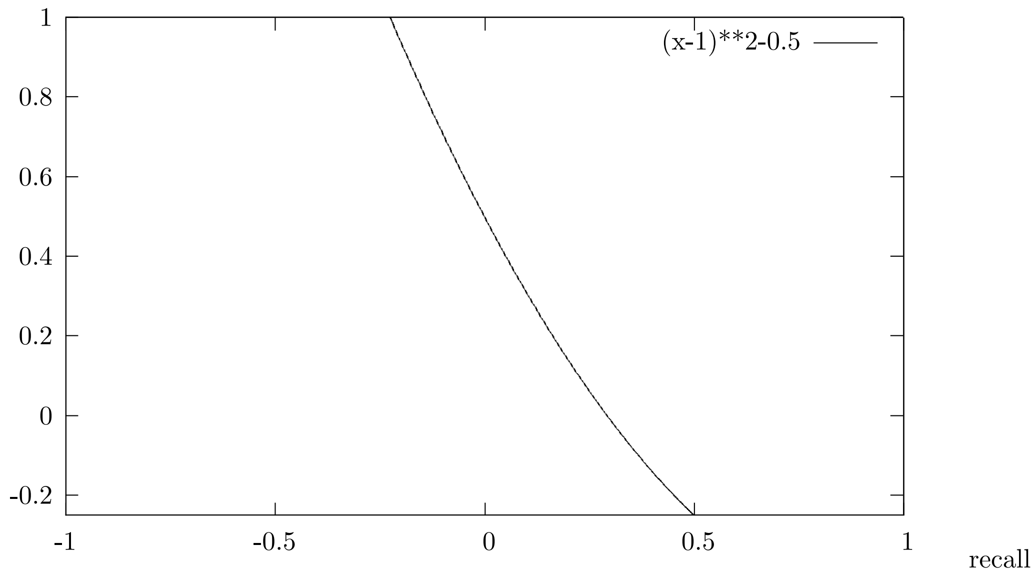
note: $u_\eta(\xi, \phi(\xi)$ is determined by initial condition.

$$\frac{\mathrm{d}}{\mathrm{d}\xi}u(\xi, \phi(\xi)) = u_\xi(\xi, \phi(\xi))\frac{\mathrm{d}\xi}{\mathrm{d}\xi} + u_\eta(\xi, \phi(\xi))\frac{\mathrm{d}\phi}{\mathrm{d}\xi} = \frac{\mathrm{d}f}{\mathrm{d}\xi}$$
$$g(\xi) + u_\eta(\xi, \phi(\xi))\phi'(\xi) = f'(\xi)$$

note: we will find the explicit riemann function for the constant coefficient case (a,b,c constant)

riemann's approach

Pick (ξ_0, η_0) and introducej auxiliary variables (x, y). We will describe the Riemann function $R(\xi_0, \eta_0; x, y)$ wich gives $u(\xi_0, \eta_0)$ that is $u(\xi, \eta)$ because (ξ_0, η_0) is arbitrary.



divergence thm

region R, boundary ∂R

$$\int\int_R \nabla\cdot\vec{F}\,\mathrm{d}x\mathrm{d}y = \int_{\partial R}\vec{F}\cdot\vec{n}\,\mathrm{d}s \qquad \vec{n} \text{ outer normal}$$

$$\int\int_R \left(\frac{\partial A}{\partial x}+\frac{\partial B}{\partial y}\right)\mathrm{d}x\mathrm{d}y = \int_{\partial R}(-B\,\mathrm{d}x + A\,\mathrm{d}y)$$

Identity for adjoint

$$vL[u]-uM[v] = \frac{\partial A}{\partial x}+\frac{\partial B}{\partial y} \text{ where } \begin{cases} A = \frac{1}{2}(vu_y - uv_y) + auv \\ B = \frac{1}{2}(vu_x - uv_x) + buv \end{cases}$$

$$L = \text{ as given } = u_{xy} + au_x + bu_+ cu$$

$$M[v] = v_{xy} - (av)_x - (bv)_y + cv$$

$$\begin{aligned}\int\int_{C_1C_2C_3}(vL[u]-uM[v])\mathrm{d}x\mathrm{d}y &= \int\int_{C_1C_2C_3}(\frac{\partial A}{\partial x}+\frac{\partial B}{\partial y})\mathrm{d}x\mathrm{d}y \\ &= \int\int_{C_1}(-B\mathrm{d}x + A\mathrm{d}y) \\ &\quad + \int\int_{C_2}(-B\mathrm{d}x + A\mathrm{d}y) \\ &\quad + \int\int_{C_3}(-B\mathrm{d}x + A\mathrm{d}y)\end{aligned}$$

of course $L[u] = F(x,y)$ and v is the Riemann function chosen to have special properties. $\int\int_{C_1}(-B\mathrm{d}x + A\mathrm{d}y)$ involves u, u_x, u_y on $y=\phi(x)$ and v, v_x, v_y on $y = \phi(x)$

$$\begin{aligned}\int_{C_2}(-B\mathrm{d}x + A\mathrm{d}y) &= \int_{C_2} A\mathrm{d}y \\ &= \int_{y=Q}^{y=P}\frac{1}{2}(vu_y - v_yu) + auv\,\mathrm{d}y\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2} vu\Big|_Q^P - \int_Q^P uv_y \,\mathrm{d}y + \int_Q^P auv \,\mathrm{d}y \\
&= \frac{1}{2}\left[v(P)u(P) - v(Q)u(Q)\right] + \int_Q^P u(v_y - av)\,\mathrm{d}y \\
\int_{C_3} (-B\mathrm{d}x + A\mathrm{d}y) &= \int_R^P \frac{1}{2}(vu_x - v_x u) + buv \,\mathrm{d}x \\
&= \frac{1}{2} vu\Big|_Q^P - \int_Q^P uv_y \,\mathrm{d}y + \int_Q^P auv \,\mathrm{d}y
\end{aligned}
$$