Homework 8

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5.1. G, H, M
5.2. G*, H*

5.1 G. Suppost the $f : \mathbb{R}^n \to \mathbb{R}$ is continuous. If there are $\boldsymbol{x} \in \mathbb{R}^n$ and $C \in \mathbb{R}$ such that $f(\boldsymbol{x}) < C$, then prove that there is $r > 0$ such that for all $\boldsymbol{y} \in \boldsymbol{B}_r(\boldsymbol{x})$, $f(\boldsymbol{y}) < C$

H. Suppose that functions f, g, h mapping $S \subset \mathbb{R}^n$ into $\mathbb{R}$ satisfy $f(\boldsymbol{x}) \leq g(\boldsymbol{x}) \leq h(\boldsymbol{x})$ for $\boldsymbol{x} \in S$. Suppose that c is a limit point of S and $\lim_{\boldsymbol{x}\to\boldsymbol{c}} h(\boldsymbol{x}) = L$. Show that $\lim_{\boldsymbol{x}\to\boldsymbol{c}} g(\boldsymbol{x}) = L$.

M. Consider the linear transformation A on $\mathbb{R}^4$ given by the matrix $A = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$

(a) Compute the Lipschits constant obtained in corollary 5.1.7.

(b) Show that $||A\boldsymbol{x}|| = ||x||$ for all $\boldsymbol{x} \in \mathbb{R}^4$. Deduce that the optimal Lipschitz constant is 1. Hint: The columns of A form and orthonormal basis for $\mathbb{R}^4$

5.2 G. (A monotone convergence test for functions.) Suppose that f is an increasing function on (a, b) that is bounded above. Prove that the one-sided limit $\lim\limits_{x\to b-} f(x)$ exists.

H. Define f on $\mathbb{R}$ by $f(x) = x\chi_{\mathbb{Q}}(x)$. Show that f is continuous at 0 and that this is the *only* point where f is continuous.