PDE B.

PDE.	$\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + \cos(\pi x)^2$	for	$0 < x < 1,$	$0 < t < \infty$
BC.	$u_x(0,t) = 0 = u_x(1,t)$	for		$0 < t < \infty$
IC.	$u(x,0) = 0$	for	$0 < x < 1$	

Solve PDE B completely. You may make use of your results from Problems 01 and 02.

$$\cos(\pi x)^2 = \sum_{n=0}^{\infty} f_n(t) X_n(x) = \sum_{n=0}^{\infty} f_n(t)\cos(n\pi x)$$

$$\int_0^1 \cos(m\pi x)\cos(\pi x)^2\,\mathrm{d}x = \int_0^1 \sum_{n=0}^{\infty} f_n(t)\cos(m\pi x)\cos(n\pi x)\,\mathrm{d}x$$

$$f_0(t) = \int_0^1 \cos(\pi x)^2\,\mathrm{d}x = \frac{1}{2} \quad \text{used computer here}$$

$$f_m(t) = 2\int_0^1 \cos(\pi x)^2\cos(m\pi x)\,\mathrm{d}x \quad m = 1,2,3,\ldots$$

and with a computer

$$f_m(t) = \frac{2(m^2-2)\sin(\pi m)}{\pi m^3 - 4\pi m}$$

simplifying because $m \in \mathbb{Z}$

$$f_m(t) = 0$$

well almost, check out the discontinuity at $m = 2$

$$f_2(t) = 2\int_0^1 \cos(\pi x)^2\cos(2\pi x)\,\mathrm{d}x$$

$$f_2(t) = \frac{1}{2}$$

substituting into original pde

$$\sum_{n=0}^{\infty} T_n'(t)\cos(n\pi x) = \frac{1}{2} + \frac{1}{2}\cos(2\pi x) - \sum_{n=0}^{\infty}(n\pi)^2 T_n(t)\cos(n\pi x)$$

$$-\sum_{n=0}^{\infty} n\pi T_n(t)\sin(n\pi 0) = 0 = -\sum_{n=0}^{\infty} n\pi T_n(t)\sin(n\pi 1)$$

$$-\sum_{n=0}^{\infty} n\pi T_n(t)0 = 0 = -\sum_{n=0}^{\infty} n\pi T_n(t)0$$

$$\sum_{n=0}^{\infty} T_n(0)\cos(n\pi x) = 0$$

$$\int_0^1 T_0(0)\cos(0)^2\,\mathrm{d}x = \int_0^1 0\cos(0)\,\mathrm{d}x$$

$$T_0(0) = 0$$

$$\int_0^1 T_m(0)\cos(m\pi x)^2\,\mathrm{d}x = 0 = \frac{1}{2}T_m(0) \quad \text{where } m = 1,2,3,\ldots$$

$$
\begin{aligned}
\frac{1}{2}+\frac{1}{2}\cos(2\pi x) &= \sum_{n=0}^{\infty}\left[T_n'(t)+(n\pi)^2T_n(t)\right]\cos(n\pi x)\\
\int_0^1\cos(0)\left[\frac{1}{2}+\frac{1}{2}\cos(2\pi x)\right]\,\mathrm{d}x &= T_0'(t)+(0\pi)^2T_0(t)\\
\frac{1}{2} &= T_0'(t)\\
c_1 &= \int T_0'(t)-\frac{1}{2}\,\mathrm{d}t = T_0(t)-\frac{1}{2}t\\
c_1 &= T_0(0)-\frac{1}{2}(0) = 0\\
T_0(t) &= \frac{1}{2}t\\
\frac{1}{2}+\frac{1}{2}\cos(2\pi x) &= \sum_{n=0}^{\infty}\left[T_n'(t)+(n\pi)^2T_n(t)\right]\cos(n\pi x)\\
\int_0^1\cos(m\pi x)\left[\frac{1}{2}+\frac{1}{2}\cos(2\pi x)\right]\,\mathrm{d}x &= T_m'(t)+(m\pi)^2T_m(t)\\
\frac{(m^2-2)\sin(\pi m)}{\pi m^3-4\pi m} &= 0 = T_m'(t)+(m\pi)^2T_m(t)\\
\mu(t) &= e^{\int(m\pi)^2\,\mathrm{d}t}\\
e^{m^2\pi^2t}T_m(t) &= \int e^{m^2\pi^2t}0\,\mathrm{d}t = c_1\\
T_m(0) &= c_1e^{-m^2\pi^2 0} = c_1 = 0\\
T_m(t) &= 0 \quad \text{for } m=1,3,4,5,\ldots\\
\int_0^1\cos(2\pi x)\left[\frac{1}{2}+\frac{1}{2}\cos(2\pi x)\right]\,\mathrm{d}x &= T_2'(t)+(2\pi)^2T_2(t)\\
\frac{1}{4} &= T_2'(t)+4\pi^2T_2(t)\\
e^{4\pi^2t}T_2(t) &= \frac{1}{4}\int e^{4\pi^2t}\,\mathrm{d}t = \frac{e^{4\pi^2t}}{16\pi^2}+c_1\\
T_2(t) &= c_1e^{-4\pi^2t}+\frac{1}{16\pi^2}\\
T_2(0) &= 0 = c_1e^{-4\pi^2 0}+\frac{1}{16\pi^2}\\
T_2(t) &= \frac{1}{16\pi^2}\left(1-e^{-4\pi t}\right)\\
u(x,t) &= \frac{1}{2}t\cos(0)+\frac{1}{16\pi^2}(1-e^{-4\pi^2t})\cos(2\pi x) = \frac{1}{2}t+\frac{1}{16\pi^2}(1-e^{-4\pi^2t})\cos(2\pi x)
\end{aligned}
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