1.1 5. Let ℓ be the line given parametrically by $\mathbf{x} = (1,3)+t(-2,1), t \in \mathbb{R}$. Which of the following points lie on ℓ? Give your reasoning.

No magic, just algebra, if we can work out a true equation it's on the line. If we work out a false equation, it's not.

(a) $\mathbf{x} = (-1,4)$ leads to $(-1,4) = (1,3)+t(-2,1)$ and $(-1-1, 4-3) = (-2,1) = t(-2,1)$. If we let $t = 1$ then the equation holds, thus the point lies on the line

(b) $\mathbf{x} = (7,0)$ leads to $(7-1, 0-3) = (6,-3) = t(-2,1)$. So we let $t = -3$ to make the equation hold and find that this point also lies on the line.

(c) $\mathbf{x} = (6,2)$ leads to $(6-1, 2-3) = (5,-1) \neq t(-2,1)$ and so the point is not on the line.

6. Find a parametric equation of each of the following lines:

(a) $3x_1 + 4x_2 = 6$

$$\begin{aligned} x_2 &= -\frac{3}{4}x_1 + \frac{6}{4} \\ (x_1, x_2) &= (0, \frac{6}{4}) + t(-3,4) \\ \mathbf{x} &= (2,0) + t(-3,4) \end{aligned}$$

(c) the line with the slope 2/5 that passes through $A = (3,1)$

$$\mathbf{x} = (3,1) + t(5,2)$$

(d) the line through $A = (-2,1)$ parallel to $\mathbf{x} = (1,4) + t(3,5)$

$$\mathbf{x} = (-2,1) + t(3,5)$$

(h) the line through $(1,1,0,-1)$ parallel to $\mathbf{x} = (2+t, 1-2t, 3t, 4-t)$

$$\begin{aligned} \mathbf{x} &= (2+t, 1-2t, 3t, 4-t) \\ &= (2,1,0,4) + t(1,-2,3,-1) \\ \mathbf{x}' &= (1,1,0,-1) + t(1,-2,3,-1) \end{aligned}$$

7. Suppose $\mathbf{x} = \mathbf{x}_0 + t\mathbf{v}$ and $\mathbf{y} = \mathbf{y}_0 + s\mathbf{w}$ are two parametric representations of the same line ℓ in $\mathbb{R}$.

(a) Show that there is a scalar t_0 so that $\mathbf{y}_0 = \mathbf{x}_0 + t_0\mathbf{v}$
By definition 2.2 the line goes through $\mathbf{y}_0$ and $\mathbf{x}_0$. Because $\mathbf{y}_0 \in \ell = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \mathbf{x}_0 + t\mathbf{v} \text{ for some } t \in \mathbb{R}\}$ then there is some $t_0 \in \mathbb{R}$ such that $\mathbf{y}_0 = \mathbf{x} = \mathbf{x}_0 + t_0\mathbf{v}$

(b) Show that $\mathbf{v}$ and $\mathbf{w}$ are parallel.
We choose the point $\mathbf{y}_0 + \mathbf{w} \in \ell$. Note that because this point is on our line, then there exists some $t_1 \in \mathbb{R}$ such that $\mathbf{y}_0 + \mathbf{w} = \mathbf{x}_0 + t_1\mathbf{v}$. And we have already established that we have some $t_0 \in \mathbb{R}$ such that $\mathbf{y}_0 = \mathbf{x}_0 + t_0\mathbf{v}$. Making the substitution we have $(\mathbf{x}_0 + t_0\mathbf{v}) + \mathbf{w} = \mathbf{x}_0 + t_1\mathbf{v}$. Now using the algebraic properties of vectors and scalars from theorem 1.9:

$$\begin{aligned} (\mathbf{x}_0 + t_0\mathbf{v}) + \mathbf{w} &= \mathbf{x}_0 + t_1\mathbf{v} \\ \mathbf{w} &= \mathbf{x}_0 - \mathbf{x}_0 + t_1\mathbf{v} - t_0\mathbf{v} \\ \mathbf{w} &= (t_1 - t_0)\mathbf{v} \end{aligned}$$

We know from the definition of a line that $\mathbf{v}$ and $\mathbf{w}$ are not $\mathbf{0}$. So the above equation means that $\mathbf{v}$ and $\mathbf{w}$ fit the definition of parallel.

10. Find a parametric equation of each of the following planes:

(a) the plane containing the point $(-1,0,1)$ and the line $\mathbf{x} = (1,1,1) + t(1,7,-1)$
We know that the plane contains the points $(-1,0,1)$ and $(1,1,1)$. Importantly $(-1,0,1)$ is not on our line. So we just pick $(1,1,1)$ to be $\mathbf{x}_0$ and then choose the direction vector from our line to be one of the spanning vectors for the plane. Plugging it all into the definition of a plane we get $(-1,0,1) = (1,1,1) + s\mathbf{u} + t(1,7,-1)$. Because $(-1,0,1)$ is not on our line, so if we use $t = 0$ and $s = 1$ then applying theorem 1.9 and 11 should give us a second spanning vector.

$$\begin{aligned}
\mathbf{x} &= \mathbf{x}_0 + s\mathbf{u} + t\mathbf{v} \\
(-1,0,1) &= (1,1,1) + \mathbf{u} + 0(1,7,-1) \\
(-1,0,1) - (1,1,1) &= \mathbf{u} \\
(-2,-1,-2) &= \mathbf{u} \\
\mathcal{P}(\mathbf{x}_0, \mathbf{u}, \mathbf{v}) &= \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = (1,1,1) + s(-2,1,-2) + t(1,7,-1) \quad \forall s,t \in \mathbb{R}\}
\end{aligned}$$

Note that we cannot multiply $(1,7,-1)$ by any real number and get $(-2,1,-2)$. Thus $(-2,1,-2)$ and $(1,7,-1)$ are not parallel and the above equation fits all the criteria for our plane.

(d) the plane in $\mathbb{R}^4$ containing the points $(1,1,-1,4),(2,3,0,1)$ and $(1,2,2,3)$
So we should be able to just pick a point from the three for an $\mathbf{x}_0$ or $\mathbf{y}_0$ and then find the equation for lines through this point and the other two points. That will give us our spanning vectors. So lets pick $(1,1,-1,4) = \mathbf{x}_0 = \mathbf{y}_0$ and do the algebra.

$$\begin{aligned}
\mathbf{x} &= \mathbf{x}_0 + t\mathbf{v} & \mathbf{y} &= \mathbf{y}_0 + s\mathbf{u} \\
(2,3,0,1) &= (1,1,-1,4) + t\mathbf{v} & (1,2,2,3) &= (1,1,-1,4) + s\mathbf{u} \\
t\mathbf{v} &= (2,3,0,1) - (1,1,-1,4) & s\mathbf{u} &= (1,2,2,3) - (1,1,-1,4) \\
\text{choose } t &= 1 & \text{choose } s &= 1 \\
\mathbf{v} &= (1,2,1,-3) & \mathbf{u} &= (0,1,3,-1)
\end{aligned}$$

I claim that the zero in $\mathbf{u}$ means that it is obviously not parallel to $\mathbf{v}$ which has no zeroes. Thus we have our plane in $\mathcal{P}\left((1,1,-1,4),(1,2,1,-3),(0,1,3,-1)\right)$

20. Assume that $\mathbf{u}$ and $\mathbf{v}$ are parallel vectors in $\mathbb{R}^n$. Prove that $\text{Span}(\mathbf{u},\mathbf{v})$ is a line.
First we note that $\text{Span}(\mathbf{u},\mathbf{v}) = \{\mathbf{x} \in \mathbb{R}^n : a\mathbf{u}+b\mathbf{v} \quad \forall a,b \in \mathbb{R}\}$. Of course we know that because $\mathbf{u}$ and $\mathbf{v}$ are parallel, then $\exists c \in \mathbb{R}$ such that $\mathbf{u} = c\mathbf{v}$. And so we have $a\mathbf{u}+b\mathbf{v} = a(c\mathbf{v})+b\mathbf{v} = (ac+b)\mathbf{v}$. Let $ac+b = t$ and $\mathbf{x}_0 = (0,0,0)$. Then $(ac+b)\mathbf{v} = \mathbf{x}_0+t\mathbf{v}$. Thus $\text{Span}(\mathbf{u},\mathbf{v}) = \{x \in \mathbb{R}^n : \mathbf{x}_0+t\mathbf{v}\}$. From the definition of parallel we know that $\mathbf{v} \neq \mathbf{0}$ and so $\text{Span}(\mathbf{u},\mathbf{v})$ fits the definition of a line.
□

21. Suppose $\mathbf{v},\mathbf{w} \in \mathbb{R}^n$ and c is a scalar. Prove that $\text{Span}(\mathbf{v}+c\mathbf{w},\mathbf{w}) = \text{Span}(\mathbf{v},\mathbf{w})$. (See the blue box on p. 12.)
Sometimes it's easier to let the math speak for itself, and so from the definition of span and from our established algebraic properties:

$$\begin{aligned}
\text{Span}(\mathbf{v}+c\mathbf{w}) &= \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = a(\mathbf{v}+c\mathbf{w}) + b(\mathbf{w}) \quad \forall a,b \in \mathbb{R}\} \\
&= \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = a\mathbf{v} + (ac+b)\mathbf{w} \quad \forall a,b \in \mathbb{R}\}
\end{aligned}$$

Now of course if we fix any $a,c \in \mathbb{R}$ then we know that $\mathbb{R} = \{ac+b : \forall b \in \mathbb{R}\}$. Lets just say $d = ac+b$. And so:

$$\begin{aligned}
\{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = a\mathbf{v} + (ac+b)\mathbf{w} \quad \forall a,b \in \mathbb{R}\} &= \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = a\mathbf{v} + d\mathbf{w} \quad \forall a,d \in \mathbb{R}\} \\
&= \text{Span}(\mathbf{v},\mathbf{w})
\end{aligned}$$

□

22. Suppose the vectors $\mathbf{v}$ and $\mathbf{w}$ are both linear combinations of $\mathbf{v}_1, \dots, \mathbf{v}_k$.

    (a) Prove that for any scalar $c, c\mathbf{v}$ is a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_k$.
    Say $\mathbf{v} = a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k$ for some $a_1, \dots, a_k \in \mathbb{R}$. Then $c\mathbf{v}$ must be $c(a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k) = ca_1\mathbf{v}_1 + \cdots + ca_k\mathbf{v}_k$ and so $c\mathbf{v}$ is a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_k$.

    (b) Prove that $\mathbf{v} + \mathbf{w}$ is a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_k$
    Say $\mathbf{w} = b_1\mathbf{v}_1 + \cdots + b_k\mathbf{v}_k$. Then $\mathbf{v} + \mathbf{w} = a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k + b_1\mathbf{v}_1 + \cdots + b_k\mathbf{v}_k = (a_1 + b_1)\mathbf{v}_1 + \cdots + (a_k + b_k)\mathbf{v}_k$. Naturally this is a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_k$.

23. Consider the line $\ell : \mathbf{x} = \mathbf{x}_0 + r\mathbf{v}(r \in \mathbb{R})$ and the plane $\mathcal{P} : \mathbf{x} = s\mathbf{u} + t\mathbf{v}(s, t \in \mathbb{R})$. Show that if ℓ and $\mathcal{P}$ intersect, then $\mathbf{x}_0 \in \mathcal{P}$
    If $\ell \cap \mathcal{P} \neq \emptyset$ then there exists some $r_0, s_0, t_0 \in \mathbb{R}$ such that $\mathbf{x}_0 + r_0\mathbf{v} = s_0\mathbf{u} + t_0\mathbf{v}$. Of course then $\mathbf{x}_0 = s_0\mathbf{u} + (t_0 - r_0)\mathbf{v}$. And because $t_0 - r_0 \in \mathbb{R}$ then $s_0\mathbf{u} + (t_0 - r_0)\mathbf{v} \in \mathcal{P}$. So $\mathbf{x}_0$ is clearly on our plane.

24. Consider the lines $\ell : \mathbf{x} = \mathbf{x}_0 + t\mathbf{v}$ and $m : \mathbf{x} = \mathbf{x}_1 + s\mathbf{u}$. Show that ℓ and m intersect if and only if $\mathbf{x}_0 - \mathbf{x}_1$ lies in $\text{Span}(\mathbf{u}, \mathbf{v})$.
    First let us assume that $\ell \cap m \neq \emptyset$. Then $\exists t_0, s_0 \in \mathbb{R}$ such that $\mathbf{x}_0 + t_0\mathbf{v} = \mathbf{x}_1 + s_0\mathbf{u}$. Simple manipulation leads to $\mathbf{x}_0 - \mathbf{x}_1 = s_0\mathbf{u} + (-t_0)\mathbf{v}$. And since $\mathbf{x}_0 - \mathbf{x}_1$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$ then $\mathbf{x}_0 - \mathbf{x}_1 \in \text{Span}(\mathbf{u}, \mathbf{v})$
    And if we assume that $\mathbf{x}_0 - \mathbf{x}_1 \in \text{Span}(\mathbf{u}, \mathbf{v})$? Well then $\mathbf{x}_0 - \mathbf{x}_1 = c_1\mathbf{u} + c_2\mathbf{v}$. For some $c_1, c_2 \in \mathbb{R}$. Again, simple manipulation leads to $\mathbf{x}_0 + (-c_2)\mathbf{v} = \mathbf{x}_1 + c_1\mathbf{u}$. Thus we have found two ways to represent the point where the two lines intersect. $\square$

25. Suppose $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ are nonparallel vectors. (Recall definition on p.3.)

    (a) Prove that if $s\mathbf{x} + t\mathbf{y} = \mathbf{0}$ then $s = t = 0$. (*Hint:* Show that neither $s \neq 0$ nor $t \neq 0$ is possible.)
    We first note that we have no definition of nonparallel, only parallel. Parallel vectors can not be zero, which kind of implies that nonparallel vectors might be. One can easily find a counterexample to the assertion if either vector is zero. I do not think that is the point of the exercise though, so we will assume that neither vector is the zero vector.
    Now let us assume that $s \neq 0$. Then with some simple algebra we go from $s\mathbf{x} + t\mathbf{y} = 0$ to $\mathbf{x} = -\frac{t}{s}\mathbf{y}$. Thus $\mathbf{x}$ and $\mathbf{y}$ fit the definition of parallel. But they are not parallel, and so we know that $s = 0$. Similarly, assuming $t \neq 0$ leads to $\mathbf{y} = -\frac{s}{t}\mathbf{x}$ and so we know that $t = 0$. $\square$

    (b) Prove that if $a\mathbf{x} + b\mathbf{y} = c\mathbf{x} + d\mathbf{y}$, then $a = c$ and $b = d$
    If we rearrange our equation a little, we arrive at $(a - c)\mathbf{x} = (d - b)\mathbf{y}$. Similarly to above we must now add to our premise that $\mathbf{x}$ and $\mathbf{y}$ are not zero. Now if we assume that $a \neq c$ then we find that $a - c \neq 0$. This allows us to arrange our equation as $\mathbf{x} = \frac{d-b}{a-c}\mathbf{y}$. Similarly, assuming $b \neq d$ leads us to $\mathbf{y} = \frac{a-c}{d-b}\mathbf{x}$. Either way we must conclude that $\mathbf{x}$ and $\mathbf{y}$ are parallel. But this is not true, and so we know that $a = c$ and $b = d$.

28. Verify algebraically that the following properties of vector arithmetic hold. (Do so for $n = 2$ if the general case is too intimidating.) Give the geometric interpretation of each property.

    (d) For each $\mathbf{x} \in \mathbb{R}^n$, there is a vector $-\mathbf{x}$ so that $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$
    Let $\mathbf{x} = (x_1, \dots, x_n)$. We know that for each $x_i \in \mathbb{R}$ there exists some $-x_i \in \mathbb{R}$ such that $x_i + (-x_i) = 0$. Let us define $-\mathbf{x} = (-x_1, \dots, -x_n)$. Then

    $$\begin{aligned}\mathbf{x} + (-\mathbf{x}) &= (x_1, \dots, x_n) + (-x_1, \dots, -x_n)\\ &= (x_1 + (-x_1), \dots, x_n + (-x_n))\\ &= (0, \dots, 0) = \mathbf{0}\end{aligned}$$

    Geometrically, for every vector, there exists another parallel vector with equal magnitude, and opposite direction.