Notes

November 5, 2014

5.4 compactness and extreme values

$y = x^2$ has min but no max and no supremum

$y = e^{-||\vec{x}||^2}$ has max at $x = 0$, no minimum, infimum at 0

when the domain $D \subseteq \mathbb{R}^n$ is unbounded a function need not attain it's max or min

$y = \frac{1}{x}$, $x \in (0, 1]$. domain is bounded, but not closed.

we like compact sets for domains, because this is when functions are guaranteed to obtain max and min (if they are continuous).

$$f(x) = \begin{cases} x & x \in [0, 1) \\ 0 & x = 1 \end{cases}$$

thm 5.4.4

let $C \subseteq \mathbb{R}^n$ be compact and let $f : C \to \mathbb{R}$ be continuous. then $\exists \vec{a}, \vec{b} \in C$ such that $\forall \vec{x} \in C$ $f(\vec{a}) \leq f(\vec{x} \leq f(\vec{b})$. ie f attains its min at $\vec{a}$ and it's max at $\vec{b}$.

we need something to prove this

5.4.3

Let C be a compact subset of $\mathbb{R}^n$ and let f be a continuous function from C into $\mathbb{R}^m$. then the image set $f(C)$ is compact.

a conintous function sends compact sets to compact sets.

pick a sequence $\{z_k\}$ such that $z_k \in f(C)$ for each k. we need to prove that this sequence has a convergent subsequence.

$z_k \in f(C) \Leftrightarrow z_k = f(c_k)$ for some $c_k \in C$ now $\{c_k\}$ has a convergent subsequence $\{c_{k_n}\}$ because C is compact. let $\lim c_{k_n} = x$, note that $x \in C$. since f is continuous, $\lim f(c_{k_n} = \lim z_{k_n} = f(x)$. so $\{z_{k_n}\}$ converges to $f(x) \in f(C)$ and so $f(C)$ is compact.

now to prove 5.4.4

proof 5.4.4

so $f(C) \subseteq \mathbb{R}$ is closed and bounded by 5.4.3. beccause it it bounded, then it has a supremum and an infimum. let $M = \sup f(C) \in \mathbb{R}, m = \inf f(C) \in \mathbb{R}$. given $\varepsilon = \frac{1}{n} \exists a_n \in f(C)$ such that $|a_n - M| < \varepsilon = \frac{1}{n}$. $\{a_n\}$ is a sequence of poins in $f(C)$ that converges to M. but $f(C)$ is closed so $M \in f(C)$ so $\exists b \in C$ such that $f(b) = M$. same proof for infimum.

5.5.9

let $f : C \to \mathbb{R}^m$ where $C \subseteq \mathbb{R}^n$ is compact. then f is uniformally continuous.

f is uniformly continuous if $\forall \varepsilon > 0 \exists r > 0$ such that $||f(x) - f(y)|| < \varepsilon$ whenever $||x - y|| < r$ for any $x, y \in$ domain of f.

proof

assume that f is not uniformly continuous then $\exists\varepsilon > 0$ such that $\forall r > 0$, we have some points x, y with $||x - y|| < r$ but $||f(x) - f(y)|| \geq \varepsilon$. in particular, let $r = \frac{1}{n}$, for each $n \in \mathbb{N}$. for each $r = \frac{1}{n} \exists x_n, y_n \in C$ such that $||x_n - y_n|| < \frac{1}{n}$ but $||f(x_n) - f(y_n)|| > \varepsilon$. $\{x_n\}$ is a sequence in C. there is a convergent subsequence x_{n_k} of $\{x_n\}$. let $a = \lim x_{n_k}$ then $||y_{n_k} - a|| \leq ||y_{n_k} - x_{n_k}|| + ||x_{n_k} - a||$. so $\{y_{n_k}\}$ also converges to a. so $f(x_n)$ and $f(y_n)$ converge to $f(a)$ and so $||f(x_n) - f(y_n)||$ converges to $||f(a) - f(a)|| = 0$ and so it is not possible for $x_{n_k} - y_{n_k}|| < \frac{1}{n_k}$ and $||f(x_{n_k} - f(y_{n_k}|| \geq \varepsilon$ for a fixed ε.

exercises

5.3.I

let f be a continuous real function defined on an open subset U of $\mathbb{R}^n$. show that $\{(\boldsymbol{x}, y) : \boldsymbol{x} \in U, y > f(\boldsymbol{x})\}$ is an open subset of $\mathbb{R}^{n+1}$.

two ways, prove that ball $B(x, r)$ exists for each $x \in A$ where $B(x, r) \subseteq A$

we could also prove that A^C is closed

let $(\vec{x}, y) \in A$. we know that $\vec{x} \in U$ and U is open. so $\exists r_1 > 0$ such that $B(\vec{x}, r_1) \subseteq U$. and $y \in (f(\vec{x}), \infty+) \in \mathbb{R}$. so $\exists r_2$ such that $(y - r_2, y + r_2) \in (f(\vec{x}), \infty+)$ and so $B(\vec{x}, r_1) \times (y - r_2, y + r_2) \supseteq U \times ($

know this for friday

f is continuous $\leftrightarrow \forall U$ open in $\mathbb{R}^m$ $f : S \to \mathbb{R}^m$, $f^{-1}(U)$ is open in S prove that this is equivalent to $\forall K$ closed in$\mathbb{R}^n$ $f^{-1}(K)$ is closed in S

5.4.G

fix $M \in \mathbb{N}$

$f(\overline{B(0, M)})$ then f is restricted to $\overline{B(0, M}$