Notes

April 16, 2014

lesson 23

example continued

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$$
\begin{aligned}
0 &= y^2 u_{xx} - x^2 u_{yy} \\
\xi &\equiv \text{constant} \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{x}{y} \to \xi = x^2 + y^2 & \frac{\mathrm{d}y}{\mathrm{d}x}\frac{1}{2A}\left(B - \sqrt{B^2 - 4AC}\right) \\
\eta &\equiv \text{constant} \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{x}{y} \to \eta = x^2 - y^2 & \frac{\mathrm{d}y}{\mathrm{d}x}\frac{1}{2A}\left(B + \sqrt{B^2 - 4AC}\right)
\end{aligned}
$$

new PDE

$$
\begin{aligned}
\hat{G} &= \hat{A} u_{\xi\xi} + \hat{B} u_{\xi\eta} + \hat{C} u_{\eta\eta} + \hat{D} u_\xi + \hat{E} u_\eta + \hat{F} u \\
\hat{A} &= \hat{C} = 0 & \text{by construction of } \xi, \eta
\end{aligned}
$$

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$$
\begin{aligned}
\hat{B} &= 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y + \eta_x) + 2C\xi_y\eta_y \\
&= 16x^2y^2 \\
\hat{D} &= A\xi_{xx} + B\xi_{xy} + C\xi_{yy} - D\xi_x + E\eta_x \\
&= (y^2)(2) + 0 + (-x^2)(+2) + 0 = 2(y^2 - x^2) \\
\hat{E} &= A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y \\
&= (y^2)(2) + 0 + (-x^2)(-2) + 0 = 2(x^2 + y^2)
\end{aligned}
$$

new PDE

$$
\begin{aligned}
0 &= 16x^2y^2 u_{\xi\eta} + 2(y^2 - x^2)u_\xi + 2(x^2 + y^2)u_\eta \\
0 &= 16 \cdot \frac{1}{4}(\xi^2 - \eta^2)u_{\xi\eta} - 2\eta u_\xi + 2\xi u_\eta \\
u_{\xi\eta} &= \frac{\eta u_\xi - \xi u_\eta}{2(\xi^2 - \eta^2)}
\end{aligned}
$$

riemann's method

method of solution is riemann's method. this is not in text.

to prepare ourself for this we need to think of hyperbolic equations in a more general setting

$$\begin{array}{llll} \text{PDE} & 0 = c^2 u_{xx} - u_{yy} & -\infty < x < \infty & 0 < y < \infty \\ \text{IC} & u(x,0) = f(x) & -\infty < x < \infty & \\ & u_y(x,0) = g(x) & & \end{array}$$

$$\text{PDE} \qquad 0 = u_{xx} - u_{yy}$$

IC on line $y = -\frac{1}{2}x$

$$\begin{array}{ll} \text{IC} & u(x,y) = f(x) \\ & u_y(x,y) = g(x) \end{array}$$

change variables

$$\begin{aligned} \xi &= x - y \\ \eta &= x + y \\ u_{\xi\eta} &= 0 \\ u &= F(\xi) + G(\eta) \\ u &= F(x-y) + G(x+y) \end{aligned}$$

now match IC

$$\begin{aligned} u(x, -\frac{1}{2}x) = f(x) &= F\left(\frac{3}{2}x\right) + G\left(\frac{1}{2}x\right) \\ u_y &= -F'(x-y) + G'(x+y) \\ u_y(x, -\frac{1}{2}x) = g(x) &= -F'\left(\frac{3}{2}x\right) + G'\left(\frac{1}{2}x\right) \\ \int_{x_0}^{x} g(s)\,\mathrm{d}s &= -\frac{2}{3}F\left(\frac{3}{2}x\right) + 2G(\frac{1}{2}x) \\ \frac{2}{3}f(x) + \int_{x_0}^{x} g(s)\,\mathrm{d}s &= \frac{8}{3}G\left(\frac{1}{2}x\right) - 2f(x) + \int_{x_0}^{x} g(s)\,\mathrm{d}s = -\frac{8}{3}F(\frac{3}{2}x) \\ G\left(\frac{1}{2}x\right) &= \frac{1}{4}f(x) + \frac{3}{8}\int_{x_0}^{x} g(s)\,\mathrm{d}s \\ G\left(x\right) &= \frac{1}{4}f(2x) + \frac{3}{8}\int_{x_0}^{2x} g(s)\,\mathrm{d}s \\ F\left(\frac{3}{2}x\right) &= \frac{3}{4}f(x) - \frac{3}{8}\int_{x_0}^{x} g(s)\,\mathrm{d}s \\ F\left(x\right) &= \frac{3}{4}f(\frac{2}{3}x) - \frac{3}{8}\int_{x_0}^{\frac{2}{3}x} g(s)\,\mathrm{d}s \\ u(x,y) &= F(x-y) + G(x+y) \end{aligned}$$

$$
\begin{aligned}
&= \frac{3}{4} f\left(\frac{2}{3}(x-y)\right) - \frac{3}{8}\int_{x_0}^{\frac{2}{3}(x-y)} g(s)\,\mathrm{d}s \\
&\quad + \frac{1}{4} f\left(2(x+y)\right) + \frac{3}{8}\int_{x_0}^{2(x+y)} g(s)\,\mathrm{d}s
\end{aligned}
$$

check

$$
\begin{aligned}
u_{xx} - u_{yy} &= 0 && \text{because has form } F(x-y) + G(x+y) \\
u(x, -\frac{1}{2}x) &= \frac{3}{4} f\left(\frac{2}{3}(x + \frac{1}{2}x)\right) - \frac{3}{8}\int_{x_0}^{\frac{2}{3}(x-y)} g(s)\,\mathrm{d}s \\
&\quad + \frac{1}{4} f\left(2(x - \frac{1}{2}x)\right) + \frac{3}{8}\int_{x_0}^{2(x+y)} g(s)\,\mathrm{d}s \\
&= f(x) \\
u_y &= \frac{3}{4} \cdot -\frac{2}{3} f'(\frac{2}{3}(x-y)) - \frac{3}{8} \cdot \frac{2}{3} g(\frac{2}{3}(x-y)) \\
&\quad + \frac{1}{4}(2) f'(2(x+y)) + \frac{3}{8}(2) g(2(x+y)) \\
u_y &= -\frac{1}{2} f'(\frac{2}{3}(x-y)) + \frac{1}{4} g(\frac{2}{3}(x-y)) \\
&\quad + \frac{1}{2} f'(2(x+y)) + \frac{3}{4} g(2(x+y)) \\
u_y(x, -\frac{1}{2}x) &= g(x)
\end{aligned}
$$