PDE A.

PDE.	$\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$	for	$0 < x < 1,$	$0 < t < \infty$
BC.	$u_x(0,t) = 0 = u_x(1,t)$	for		$0 < t < \infty$
IC.	$u(x,0) = f(x)$	for	$0 < x < 1$	

For PDE A, apply separation of variables and, for separated solutions $u = T(t)X(x)$, analyze the associateed eigenvalue problem $X''(x) = \lambda X(x)$ and determine the eigenfunctions (or their nonexistence) for the cases:

$$
\begin{aligned}
u &= T(t)X(x) \\
\frac{\partial u}{\partial t} &= T'(t)X(x) \\
\frac{\partial u}{\partial x} &= X'(x)T(t) \\
\frac{\partial^2 u}{\partial x^2} &= X''(x)T(t) \\
T'(t)X(x) &= X''(x)T(t)
\end{aligned}
$$

t, and x are independant of eachother, therefore:

$$
\begin{aligned}
\frac{T'(t)}{T(t)} &= \frac{X''(x)}{X(x)} = \lambda \\
T'(t) - \lambda T(t) &= 0 \\
\omega(t) &= e^{\int -\lambda \, \mathrm{d}t} \\
\omega(t)T(t) &= \int 0 \, \mathrm{d}t = c_3 \\
T(t) &= c_3 e^{\lambda t} \\
X''(x) - \lambda X(x) &= 0 \\
X'' - \lambda X &= 0 \\
r^2 + 0r - \lambda &= 0 \\
r &= \frac{-0 \pm \sqrt{0^2 - 4(-\lambda)}}{2} \\
&= \pm\sqrt{\lambda}
\end{aligned}
$$

(a) $\lambda = +\mu^2 > 0$

$$
\begin{aligned}
r &= \pm\mu \\
X(x) &= c_1 e^{\mu x} + c_2 e^{-\mu x} \\
u_x &= X'(x)T(t) = \left(c_1 \mu e^{\mu x} - c_2 \mu e^{-\mu x}\right) T(t) \\
u_x(0,t) &= 0 = u_x(1,t) \\
(c_1\mu - c_2\mu)\, T(t) &= 0 = \left(c_1 \mu e^{\mu} - c_2 \mu e^{-\mu}\right) T(t)
\end{aligned}
$$

note that if $T(t) = 0$ then we are dealing with the trivial case $u(x,t) = 0$ which is not what we are looking for, so we say that $T(t) \neq 0$

$$
\begin{aligned}
c_1\mu - c_2\mu &= 0 = c_1\mu e^{\mu} - c_2\mu e^{-\mu} && \mu \neq 0 \\
c_1 - c_2 &= 0 && c_1 = c_2
\end{aligned}
$$

$$\begin{aligned}
c_1e^{\mu} - c_1e^{-\mu} &= 0 \\
e^{\mu} &= e^{-\mu} \\
e^{2\mu} &= 1 \\
\ln(e^{2\mu}) &= \ln(1) = 2\mu = 0 \\
\mu &= 0
\end{aligned}$$

But we have defined $\mu^2 > 0$ so we have no solutions.

(b) $\lambda = 0$

$$\begin{aligned}
r &= \pm\sqrt{0} = 0 \\
X(x) &= (c_1 + c_2x)e^{0x} = c_1 + c_2x \\
u_x(0,t) &= 0 = u_x(1,t) \\
c_2T(t) &= 0 = c_2T(t)
\end{aligned}$$

Again we take $T(t) \neq 0$

$$\begin{aligned}
c_2 &= 0 \\
X(x) &= c_1 & T(t) &= c_3e^{0t} = c_3 \\
u(x,t) &= c_1 \cdot c_3 = c_4
\end{aligned}$$

So we have one eigenfunction, $u(x,t) = c_0$

(c) $\lambda = -\mu^2 < 0$

$$\begin{aligned}
r &= \pm\sqrt{-\mu^2} = \pm\mu i \\
X(x) &= c_1\cos(\mu x) + c_2\sin(\mu x) \\
X'(x) &= -c_1\mu\sin(\mu x) + c_2\mu\cos(\mu x) \\
u_x(0,t) &= 0 = u_x(1,t) \\
[c_2\mu\cos(0) - c_1\mu\sin(0)]T(t) &= 0 = [c_2\mu\cos(\mu) - c_1\mu\sin(\mu)]T(t)
\end{aligned}$$

Taking $T(t) \neq 0$

$$\begin{aligned}
c_2\mu &= 0 = c_2\mu\cos(\mu) - c_1\mu\sin(\mu) & \mu > 0 \to c_2 = 0 \\
-c_1\mu\sin(\mu) &= 0
\end{aligned}$$

Avoiding the trivial solution requires $\sin(\mu) = 0$

$$\begin{aligned}
\mu &= n\pi & n = 1, 2, 3, \ldots \\
T(t) &= c_3e^{-\mu^2t} = c_3e^{-n^2\pi^2t} \\
u_n(x,t) &= c_ne^{-n^2\pi^2t}\cos(n\pi x)
\end{aligned}$$

PDE A.

PDE.	$\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$	for	$0 < x < 1,$	$0 < t < \infty$
BC.	$u_x(0,t) = 0 = u_x(1,t)$	for		$0 < t < \infty$
IC.	$u(x,0) = f(x)$	for	$0 < x < 1$	

For PDE A, determine the full solution for the general initial condition $f(x)$. State how orthogonality is used and how coefficients in the series expansion are determined using $f(x)$

$$\begin{aligned}
u_0(x,t) &= c_0 \\
u_n(x,t) &= c_n e^{-n^2\pi^2 t}\cos(n\pi x) && n = 1,2,3,\ldots \\
u(x,t) &= c_0 + \sum_{n=1}^{\infty} c_n e^{-n\pi^2 t}\cos(n\pi x) \\
u(x,t) &= \sum_{n=0}^{\infty} c_n e^{-n\pi^2 t}\cos(n\pi x) \\
f(x) = u(x,0) &= \sum_{n=0}^{\infty} c_n \cos(n\pi x) && \text{let } m \in \mathbb{Z} \\
\int_0^1 f(x)\cos(m\pi x)\,\mathrm{d}x &= \int_0^1 \cos(m\pi x)\sum_{n=0}^{\infty} c_n \cos(n\pi x)\,\mathrm{d}x \\
\text{let } n &= m = 0 \\
\int_0^1 c_0\cos(0)^2\,\mathrm{d}x &= c_0 \\
\text{let } n &= m \neq 0 \\
\int_0^1 c_m\cos(m\pi x)^2\,\mathrm{d}x &= \frac{1}{2}c_m\int_0^1 2\cos(m\pi x)^2\,\mathrm{d}x \\
&= \frac{1}{2}c_m\int_0^1 1 + \cos(2m\pi x)\,\mathrm{d}x \\
&= \frac{1}{2}c_m \left| x + \frac{1}{2m\pi}\sin(2m\pi x)\right|_0^1 \\
&= \frac{1}{2}c_m + \frac{1}{2}c_m\frac{\sin(2m\pi)}{2m\pi} = \frac{1}{2}c_m && m \in \mathbb{Z} \to \sin(2m\pi) = 0
\end{aligned}$$

and to establish orthogonality let $n \neq m$

$$\begin{aligned}
\int_0^1 c_m\cos(n\pi x)\cos(m\pi x)\,\mathrm{d}x &= \frac{1}{2}c_m\int_0^1 2\cos(n\pi x)\cos(m\pi x)\,\mathrm{d}x \\
&= \frac{1}{2}c_m\int_0^1 \cos(n\pi x - m\pi x) + \cos(n\pi x + m\pi x)\,\mathrm{d}x \\
&= \frac{1}{2}c_m\left[\frac{1}{(n-m)\pi x}\sin((n-m)\pi x)\right. \\
&\qquad \left. + \frac{1}{(n+m)\pi x}\sin((n+m)\pi x)\right]_0^1 \\
&= \frac{1}{2}c_m\left[\frac{1}{(n-m)\pi}\sin((n-m)\pi)\right. \\
&\qquad \left. + \frac{1}{(n+m)\pi}\sin((n+m)\pi)\right] && n-m, n+m \in \mathbb{Z}
\end{aligned}$$

$$= \frac{1}{2} c_m \cdot 0 = 0 \qquad \sin((m \pm n)\pi) = 0$$

And now we have everything we need to determine the full solution. Since we can find the coefficient to the mth term by multiplying $\cos(m\pi x)$ and then integrating, letting orthogonality kill off all the extra terms.

$$u(x,t) = \sum_{n=0}^{\infty} c_n e^{-n\pi^2 t} \cos(n\pi x)$$

$$c_0 = \int_0^1 f(x)\,\mathrm{d}x$$

$$c_m = 2\int_0^1 f(x)\cos(m\pi x)\,\mathrm{d}x \qquad \text{where } m = 1, 2, 3, \ldots$$

PDE B.

PDE.	$\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + \cos(\pi x)^2$	for	$0 < x < 1,$	$0 < t < \infty$
BC.	$u_x(0,t) = 0 = u_x(1,t)$	for		$0 < t < \infty$
IC.	$u(x,0) = 0$	for	$0 < x < 1$	

Solve PDE B completely. You may make use of your results from Problems 01 and 02.

$$\cos(\pi x)^2 = \sum_{n=0}^{\infty} f_n(t) X_n(x) = \sum_{n=0}^{\infty} f_n(t)\cos(n\pi x)$$
$$\int_0^1 \cos(m\pi x)\cos(\pi x)^2 \,\mathrm{d}x = \int_0^1 \sum_{n=0}^{\infty} f_n(t)\cos(m\pi x)\cos(n\pi x)\,\mathrm{d}x$$
$$f_0(t) = \int_0^1 \cos(\pi x)^2 \,\mathrm{d}x = \frac{1}{2} \quad \text{used computer here}$$
$$f_m(t) = 2\int_0^1 \cos(\pi x)^2 \cos(m\pi x)\,\mathrm{d}x \quad m = 1,2,3,\ldots$$

and with a computer

$$f_m(t) = \frac{2(m^2-2)\sin(\pi m)}{\pi m^3 - 4\pi m}$$

simplifying because $m \in \mathbb{Z}$

$$f_m(t) = 0$$

well almost, check out the discontinuity at $m = 2$

$$f_2(t) = 2\int_0^1 \cos(\pi x)^2 \cos(2\pi x)\,\mathrm{d}x$$
$$f_2(t) = \frac{1}{2}$$

substituting into original pde

$$\sum_{n=0}^{\infty} T_n'(t)\cos(n\pi x) = \frac{1}{2} + \frac{1}{2}\cos(2\pi x) - \sum_{n=0}^{\infty} (n\pi)^2 T_n(t)\cos(n\pi x)$$
$$-\sum_{n=0}^{\infty} n\pi T_n(t)\sin(n\pi 0) = 0 = -\sum_{n=0}^{\infty} n\pi T_n(t)\sin(n\pi 1)$$
$$-\sum_{n=0}^{\infty} n\pi T_n(t)0 = 0 = -\sum_{n=0}^{\infty} n\pi T_n(t)0$$
$$\sum_{n=0}^{\infty} T_n(0)\cos(n\pi x) = 0$$
$$\int_0^1 T_0(0)\cos(0)^2\,\mathrm{d}x = \int_0^1 0\cos(0)\,\mathrm{d}x$$
$$T_0(0) = 0$$
$$\int_0^1 T_m(0)\cos(m\pi x)^2\,\mathrm{d}x = 0 = \frac{1}{2}T_m(0) \quad \text{where } m = 1,2,3,\ldots$$

$$\begin{aligned}
\frac{1}{2}+\frac{1}{2}\cos(2\pi x) &= \sum_{n=0}^{\infty}\left[T_n'(t)+(n\pi)^2T_n(t)\right]\cos(n\pi x)\\
\int_0^1 \cos(0)\left[\frac{1}{2}+\frac{1}{2}\cos(2\pi x)\right]\,\mathrm{d}x &= T_0'(t)+(0\pi)^2T_0(t)\\
\frac{1}{2} &= T_0'(t)\\
c_1 &= \int T_0'(t)-\frac{1}{2}\,\mathrm{d}t = T_0(t)-\frac{1}{2}t\\
c_1 &= T_0(0)-\frac{1}{2}(0)=0\\
T_0(t) &= \frac{1}{2}t\\
\frac{1}{2}+\frac{1}{2}\cos(2\pi x) &= \sum_{n=0}^{\infty}\left[T_n'(t)+(n\pi)^2T_n(t)\right]\cos(n\pi x)\\
\int_0^1 \cos(m\pi x)\left[\frac{1}{2}+\frac{1}{2}\cos(2\pi x)\right]\,\mathrm{d}x &= T_m'(t)+(m\pi)^2T_m(t)\\
\frac{(m^2-2)\sin(\pi m)}{\pi m^3-4\pi m} &= 0 = T_m'(t)+(m\pi)^2T_m(t)\\
\mu(t) &= e^{\int (m\pi)^2\,\mathrm{d}t}\\
e^{m^2\pi^2t}T_m(t) &= \int e^{m^2\pi^2t}0\,\mathrm{d}t = c_1\\
T_m(0) &= c_1e^{-m^2\pi^2 0}=c_1=0\\
T_m(t) &= 0 \quad \text{for } m=1,3,4,5,\ldots\\
\int_0^1 \cos(2\pi x)\left[\frac{1}{2}+\frac{1}{2}\cos(2\pi x)\right]\,\mathrm{d}x &= T_2'(t)+(2\pi)^2T_2(t)\\
\frac{1}{4} &= T_2'(t)+4\pi^2T_2(t)\\
e^{4\pi^2t}T_2(t) &= \frac{1}{4}\int e^{4\pi^2t}\,\mathrm{d}t = \frac{e^{4\pi^2t}}{16\pi^2}+c_1\\
T_2(t) &= c_1e^{-4\pi^2t}+\frac{1}{16\pi^2}\\
T_2(0) &= 0 = c_1e^{-4\pi^2 0}+\frac{1}{16\pi^2}\\
T_2(t) &= \frac{1}{16\pi^2}\left(1-e^{-4\pi t}\right)\\
u(x,t) &= \frac{1}{2}t\cos(0)+\frac{1}{16\pi^2}(1-e^{-4\pi^2t})\cos(2\pi x) = \frac{1}{2}t+\frac{1}{16\pi^2}(1-e^{-4\pi^2t})\cos(2\pi x)
\end{aligned}$$

PDE C.

PDE.	$\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$	for	$0 < x < \infty,$	$0 < t < \infty$
BC.	$\dfrac{\partial u}{\partial x}(0,t) = u(0,t) - \dfrac{1}{\sqrt{\pi t}}$	for		$0 < t < \infty$
IC.	$u(x,0) = 0$	for	$0 < x < \infty$	

Solve PDE C completely by a Laplace transform with respect to t. Use the BC as stated – do not transform to homogeneous BC. (The necessary inverse Laplace transform is not in the textbook table but is on the handout list of transforms.)

$$
\begin{aligned}
sU(x) - 0 &= \frac{\mathrm{d}^2 U}{\mathrm{d}x^2}(x) \\
\frac{\mathrm{d}U}{\mathrm{d}x}(0) &= U(0) - \mathcal{L}\left\{\frac{1}{\sqrt{\pi t}}\right\} \\
&= U(0) - \frac{1}{\sqrt{s}} \qquad \text{used computer} \\
0 &= \frac{\mathrm{d}^2 U}{\mathrm{d}x^2}(x) - sU(x) \\
0 &= r^2 + 0r - s \\
r &= \frac{\pm\sqrt{4s}}{2} = \pm\sqrt{s} \\
U(x) &= c_1 e^{x\sqrt{s}} + c_2 e^{-x\sqrt{s}} \\
U'(x) &= c_1\sqrt{s}e^{x\sqrt{s}} - c_2\sqrt{s}e^{-x\sqrt{s}} \\
U'(0) &= c_1\sqrt{s} - c_2\sqrt{s} = c_1 + c_2 - \frac{1}{\sqrt{s}} \\
c_1\sqrt{s} - c_1 &= c_2 + c_2\sqrt{s} - \frac{1}{\sqrt{s}}
\end{aligned}
$$

used computer to help find convenient values

$$
\begin{aligned}
c_1(\sqrt{s} - 1) &= \frac{1}{s + \sqrt{s}} + \frac{\sqrt{s}}{s + \sqrt{s}} - \frac{1}{\sqrt{s}} \\
c_1(\sqrt{s} - 1) &= \frac{\sqrt{s} + s}{\sqrt{s}(s + \sqrt{s})} - \frac{s + \sqrt{s}}{\sqrt{s}(s + \sqrt{s})} \\
c_1 &= 0 \\
c_2 &= \frac{1}{s + \sqrt{s}} \\
U(x) &= \frac{1}{s + \sqrt{s}} e^{-x\sqrt{s}}
\end{aligned}
$$

from handout

$$
u(x,t) = e^{x+t}\mathrm{erfc}\left(\sqrt{t} + \frac{x}{2\sqrt{t}}\right)
$$

PDE D.1

PDE.	$\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2}$	for	$0 < x < \infty,$	$0 < t < \infty$
BC.	$w(0,t) = 1$	for		$0 < t < \infty$
IC.	$w(x,0) = 0$	for	$0 < x < \infty$	

Solve PDE D.1 by Laplace transforming with respect to t. In particular, show that the Laplace transform of the solution $w(x,t)$ is $W(x,s) = \frac{1}{s}e^{-x\sqrt{s}}$ and then obtain the solution $w(x,t)$ (use tables).

$$\begin{aligned}
sW(x) - 0 &= \frac{\mathrm{d}^2 W}{\mathrm{d}x^2} \\
W(0) &= \mathcal{L}\{1\} = \frac{1}{s} \\
0 &= \frac{\mathrm{d}^2 W}{\mathrm{d}x^2} - sW(x) \\
0 &= r^2 + 0r - s \\
r &= \frac{\pm\sqrt{4s}}{2} = \pm\sqrt{s} \\
W(x) &= c_1 e^{x\sqrt{s}} + c_2 e^{-x\sqrt{s}} \\
W(0) &= c_1 + c_2 = \frac{1}{s} \\
c_1 = 0 \quad c_2 &= \frac{1}{s} \\
W(x) &= \frac{1}{s} e^{-x\sqrt{s}}
\end{aligned}$$

from handout

$$w(x,t) = \operatorname{erfc}\left(\frac{x}{2\sqrt{t}}\right)$$

PDE D.2

PDE.	$\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$	for	$0 < x < \infty,$	$0 < t < \infty$
BC.	$u(0,t) = f(t)$	for		$0 < t < \infty$
IC.	$u(x,0) = 0$	for	$0 < x < \infty$	

Apply the Laplace transform with respect to t to PDE D.2 to obtain the relation $U(x,s) = sF(s)W(x,s)$, where $U(x,s)$ is the Laplace transform of the solution $u(x,t), F(s)$ is the transform of $f(t)$ and $W(x,s)$ is the transform of problem 5. Show that the result leads to the formula

$$u(x,t) = f(0)\text{erfc}\left(\frac{x}{2\sqrt{t}}\right) + t\int_0^1 \text{erfc}\left(\frac{x}{2\sqrt{t}}(1-u)^{-1/2}\right) f'(tu)\text{d}u$$

$$\begin{aligned}
sU(x) - 0 &= \frac{\text{d}^2 U}{\text{d}^2 x} \\
U(0) &= F(s) \\
0 &= \frac{\text{d}^2 U}{\text{d}^2 x} - sU(x) \\
U(x) &= c_1 e^{x\sqrt{s}} + c_2 e^{-x\sqrt{s}} \\
U(0) &= F(s) = c_1 + c_2 \\
c_1 = 0 \quad c_2 &= F(s) \\
U(x) &= F(s)e^{-x\sqrt{s}} = \frac{s}{s}F(s)e^{-x\sqrt{s}} \\
W(s) &= \frac{1}{s}e^{-x\sqrt{s}} \\
U(x) &= sF(s)W(x)
\end{aligned}$$

And now we do the reverse transform

$$\begin{aligned}
U(x) &= (sF(s) - f(0) + f(0))W(x) \\
&= (sF(s) - f(0))W(x) + f(0)W(x) \\
u(x,t) &= f(0)\text{erfc}\left(\frac{x}{2\sqrt{t}}\right) + \int_0^t f'(u)\,\text{erfc}\left(\frac{x}{2\sqrt{(t-u)}}\right)\,\text{d}u \\
&= f(0)\text{erfc}\left(\frac{x}{2\sqrt{t}}\right) + t\int_0^1 f'(tu)\,\text{erfc}\left(\frac{x}{2\sqrt{(t-tu)}}\right)\,\text{d}u \\
&= f(0)\text{erfc}\left(\frac{x}{2\sqrt{t}}\right) + t\int_0^1 f'(tu)\,\text{erfc}\left(\frac{x}{2\sqrt{t}\sqrt{1-u}}\right)\,\text{d}u
\end{aligned}$$

This seems too easy, so maybe I'm not understanding a term properly or something but I believe that the eigenvalue for the eigenfunction $E_{n,m}(r,\theta) = \sin(n\theta)J_n(k_{n,m}r)$ is just $k_{n,m}$

$$\int_0^{\pi}\int_0^1 E_{n,m}(r,\theta)E_{q,p}(r,\theta)r\mathrm{d}r\mathrm{d}\theta$$

let $n \neq q$

$$\begin{aligned}
\int_0^{\pi}\int_0^1 E_{n,m}(r,\theta)E_{q,p}(r,\theta)r\mathrm{d}r\mathrm{d}\theta &= \int_0^{\pi}\sin(n\theta)\sin(q\theta)\left[\int_0^1 J_n(k_{n,m}r)J_q(k_{q,pr)r\,\mathrm{d}r}\right]\mathrm{d}\theta \\
&= \left[\int_0^1 J_n(k_{n,m}r)J_q(k_{q,pr)r\,\mathrm{d}r}\right]\int_0^{\pi}\sin(n\theta)\sin(q\theta)\,\mathrm{d}\theta \\
&= \left[\int_0^1 J_n(k_{n,m}r)J_q(k_{q,pr)r\,\mathrm{d}r}\right]\left(-\frac{(q-n)\sin(q\pi+n\pi)+(-q-n)\sin(q\pi-n\pi)}{2q^2-2n^2}\right) \\
&= \left[\int_0^1 J_n(k_{n,m}r)J_q(k_{q,pr)r\,\mathrm{d}r}\right]\cdot 0 \\
&= 0
\end{aligned}$$

now let $(n,m)=(q,p)$

$$\begin{aligned}
\int_0^{\pi}\int_0^1 E_{n,m}(r,\theta)E_{q,p}(r,\theta)r\mathrm{d}r\mathrm{d}\theta &= \left[\int_0^1 J_n(k_{n,m}r)J_q(k_{q,pr)r\,\mathrm{d}r}\right]\int_0^{\pi}\sin(n\theta)\sin(q\theta)\,\mathrm{d}\theta \\
&= \left[\int_0^1 J_n(k_{n,m}r)^2 r\,\mathrm{d}r\right]\int_0^{\pi}\sin(n\theta)^2\,\mathrm{d}\theta \\
&= -\frac{\sin(2\pi n)-2\pi n}{4n}\left[\int_0^1 J_n(k_{n,m}r)^2 r\,\mathrm{d}r\right] \\
&= \frac{\pi}{2}\left[\int_0^1 J_n(k_{n,m}r)^2 r\,\mathrm{d}r\right] \\
&= \frac{\pi}{2}\left[\frac{1}{2}(J_n'(k_{n,m}))^2\right] = \frac{\pi}{4}J_n'(k_{n,m})^2 \quad \text{formula from class}
\end{aligned}$$

let $n=q$ and $m \neq p$

$$\begin{aligned}
\int_0^{\pi}\int_0^1 E_{n,m}(r,\theta)E_{q,p}(r,\theta)r\mathrm{d}r\mathrm{d}\theta &= \left[\int_0^1 J_n(k_{n,m}r)J_q(k_{q,pr)r\,\mathrm{d}r}\right]\int_0^{\pi}\sin(n\theta)^2\,\mathrm{d}\theta \\
&= \frac{\pi}{2}\left[\int_0^1 J_n(k_{n,m}r)J_q(k_{q,pr)r\,\mathrm{d}r}\right]
\end{aligned}$$

we take as given that $k_{n,m} \neq k_{q,p}$ and use the formula from my notes

$$\begin{aligned}
a &= k_{n,m} \\
b &= k_{q,p} \\
\int_0^{\pi}\int_0^1 E_{n,m}(r,\theta)E_{q,p}(r,\theta)r\mathrm{d}r\mathrm{d}\theta &= \frac{\pi}{2}\left[\int_0^1 J_n(ar)J_n(br)r\,\mathrm{d}r\right] \\
&= \frac{\pi}{2}\left[\frac{r}{b+a}\frac{1}{b-a}(aJ_n'(ar)J_n(br)-bJ_n(ar)J_n'(br))\right]_0^1 \\
&= \frac{\pi}{2}\left[\frac{1}{b+a}\frac{1}{b-a}(aJ_n'(a)J_n(b)-bJ_n(a)J_n'(b))\right. \\
&\qquad \left.-\frac{0}{b+a}\frac{1}{b-a}(aJ_n'(ar)J_n(br)-bJ_n(ar)J_n'(br))\right] \\
J_n(a) &= J_n(b) = 0 \\
\int_0^{\pi}\int_0^1 E_{n,m}(r,\theta)E_{q,p}(r,\theta)r\mathrm{d}r\mathrm{d}\theta &= 0
\end{aligned}$$