Homework

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Section 3.8: 7, 12, 19, 22

3.8

7. Let H be a subggroup of G, and let $a \in G$. Show that aHa^{-1} is a subgroup of G that is isomorphic to H.
   $e \in H \to aea^{-1} = e \in aHa^{-1}$. So H is non-empty. Let $h, g \in H$.

$$(aga^{-1})^{-1} = ((ag)(a^{-1}))^{-1} = (a^{-1})^{-1}(ag)^{-1} = ag^{-1}a^{-1}$$
$$(aha^{-1})(ag^{-1}a^{-1}) = aheg^{-1}a^{-1} = ahg^{-1}a^{-1}$$

   Because $g \in H$ we know that $g^{-1} \in H$ and then $hg^{-1} \in H$ so $ahg^{-1}a^{-1} \in aHa^{-1}$.

$$aha^{-1}aga^{-1} = ahga^{-1}$$

   So it's homeomorphic. Now if we assume that $aha^{-1} = ah'a^{-1}$ cancelling left and right easily shows that $h = h'$ and so it is injective. And obviously any $aha^{-1} \in H$ has a preimage of $h \in H$ and so we have surjectivity falling out of the definition of the group. And that is the final condition for it to be an isomorphism.

12. Let H and K be normal subgroups of G such that $H \cap K = \langle e \rangle$. Show that $hk = kh$ for all $h \in H$ and $k \in K$.
    Take any $k \in K$ and $h \in H$. Because for any $g \in G$ we know that $ghg^{-1} \in H$. We also know that $gkg^{-1} \in K$. So because $k \in G$ we have $h = khk^{-1}$ and then $hk = khk^{-1}k = kh$

19. Show that $(\mathbb{Z} \times \mathbb{Z})/\langle(0,1)\rangle$ is an infinite cyclic group.
    Notice that $\langle(0,1)\rangle = \{0\} \times \mathbb{Z}$ because $\langle 1 \rangle = \mathbb{Z}$ and $\langle 0 \rangle = \{0\}$. If we take $\{0\} \times \mathbb{Z} + (a,0)$ for some $a \in \mathbb{Z}$ then we have the set $\{a\} \times \mathbb{Z}$. Also note that $\bigcup_{a\in\mathbb{Z}} \{a\} \times \mathbb{Z} = \mathbb{Z} \times \mathbb{Z}$ and $\{a\} \times \mathbb{Z} \cap \{b\} \times \mathbb{Z} = \emptyset$ for all $a \neq b$. This means that $\{a\} \times \mathbb{Z}$ partitions $\mathbb{Z} \times \mathbb{Z}$ and cosets of $\langle(0,1)\rangle$ have the form $(a,0) + \langle(0,1)\rangle$. And $(a,0) \in \langle(1,0)\rangle$. And so $(\mathbb{Z} \times \mathbb{Z})/\langle(0,1)\rangle$ is generated by $(1,0) + \langle(0,1)\rangle$. This makes the group cyclic. And obviously the number of elements in the group is the number of ways we can pick our $(a,0)$ which is the number of ways we can pick an integer. And so we have an infinite cyclic group.

22. Show that $\mathbb{R}^\times/\langle -1 \rangle$ is isomorphic to the group of positive real numbers under multiplication.
    Note that $\langle -1 \rangle = \{-1, 1\}$. And so the factor group has the form $\{-a, a\}$ where $a \in \mathbb{R}$. Notice also that if $b = -a$ then $\{-a, a\} = \{b, -b\}$ and so $\mathbb{R}^\times/\langle -1 \rangle = \{\{-a, a\} : a \in \mathbb{R}^+\}$. Let us define $\phi : \mathbb{R}^\times/\langle -1 \rangle \to \mathbb{R}$ as $\phi : \{-a, a\} \to a$.

$$\phi(\{-a,a\} \cdot \{-b,b\}) = \phi(\{-a \cdot -b, -a \cdot b, a \cdot -b, a \cdot b\}) = \phi(\{-ab, ab\}) = ab = \phi(\{-a,a\}) \cdot \phi(\{-b,b\})$$

    Clearly we have a homeomorphism. I think surjectivity is clear, any $a \in \mathbb{R}^+$ has a preimage $\{-a, a\}$ in our factor set. Now if $\phi(\{-a,a\}) = \phi(\{-b,b\})$ then $a = b$. If $a = b$ then $\{-a,a\} = \{-b,b\}$ and so we have injectivity. Thus our factor set is isomorphic to the positive integers.