Jon Allen
HW 19
Let X and T be physical variables for distance and time. Consider the following general diffusion problem for $u(X,T)$:

$$\begin{aligned}
&\text{PDE} && \frac{\partial u}{\partial t} = \alpha^2\frac{\partial^2 u}{\alpha x^2} + F(X,T) && 0 < X < L, && 0 < T < +\infty \\
&\text{BC} && \alpha_1 L\frac{\partial u}{\partial X}(0,T) + \beta_1 u(0,t) = G_1(T) && && 0 < T < +\infty \\
& && \alpha_2 L\frac{\partial u}{\partial X}(L,T) + \beta_2 u(L,t) = G_1(T) && && \\
&\text{IC} && u(X,0) = \phi(X) && 0 < X < L &&
\end{aligned}$$

Note:

$$\alpha_1{}^2 + \beta_1{}^2 \neq 0 \qquad\qquad \alpha_2{}^2 + \beta_2{}^2 \neq 0$$

(a) If the units for X and T are [cm] and [sec] respectively (and u is taken as temerature with units [deg]), what are the units for L, α^2, F, ϕ, and for $\alpha_1, \beta_2, \alpha_2, \beta_2$?

$$\begin{aligned}
\frac{\text{deg}}{\text{sec}} &= \alpha^2\frac{\text{deg}}{\text{cm}^2} + F & \alpha\cdot\text{cm}\frac{\text{deg}}{\text{cm}} + \beta\cdot\text{deg} &= \text{deg} \\
F &= \frac{\text{deg}}{\text{sec}} & \alpha\cdot\text{deg} = \beta\cdot\text{deg} &= \text{deg} \\
\alpha^2 &= \frac{\text{cm}^2}{\text{sec}} & \alpha_{1,2} = \beta_{1,2} = 1 &= \text{dimensionless} \\
L &= \text{cm} & \phi(X) &= \text{deg}
\end{aligned}$$

Define dimensionless variables x, t by $x = X/L$ and $t = \frac{\alpha^2}{L^2}T$. Define $w(x,t) = u(X,T)$

(b) Find $\frac{\partial u}{\partial T}, \frac{\partial u}{\partial X}, \frac{\partial^2 u}{\partial X^2}$ in terms of $\frac{\partial w}{\partial t}, \frac{\partial w}{\partial x}, \frac{\partial^2 w}{\partial x^2}$.

$$\begin{aligned}
\frac{\partial u}{\partial T} &= \frac{\partial}{\partial T}\left(w(x,t)\right) & \frac{\partial u}{\partial X} &= \frac{\partial}{\partial X}(w(x,t)) \\
T &= \frac{L^2}{\alpha^2}t & X &= xL \\
&= \frac{\partial w}{\partial\left(\frac{L^2}{\alpha^2}t\right)} & &= \frac{\partial}{\partial(xL)}(w(x,t)) \\
\frac{\partial u}{\partial T} &= \frac{\alpha^2}{L^2}\frac{\partial w}{\partial t} & &
\end{aligned}$$

(c) Show that the PDE can be written as

$$\text{PDE} \qquad \frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + f(x,t) \qquad 0 < x < 1, \qquad 0 < t < +\infty$$

What is $f(x,t)$ in terms of $F(X,T)$?

(d) Show hat the BC can be written as

$$\text{BC} \qquad \alpha_1 \frac{\partial w}{\partial x}(0,t) + \beta_1 w(0,t) = g_1(t) \qquad 0 < t < +\infty$$

$$\alpha_2 \frac{\partial w}{\partial x}(1,t) + \beta_2 w(1,t) = g_2(t) \qquad 0 < t < +\infty$$

What are $g_1(t)$ and $g_2(t)$ in terms of $G_1(T)$ and $G_2(T)$?

(e) Show that the IC can be written as

$$\text{IC} \qquad w(x,0) = \phi(x) \qquad 0 < x < 1$$

What is $\phi(x)$ in terms of $\phi(X)$?