PDE D.2

PDE.	$\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$	for	$0 < x < \infty,$	$0 < t < \infty$
BC.	$u(0,t) = f(t)$	for		$0 < t < \infty$
IC.	$u(x,0) = 0$	for	$0 < x < \infty$	

Apply the Laplace transform with respect to t to PDE D.2 to obtain the relation $U(x,s) = sF(s)W(x,s)$, where $U(x,s)$ is the Laplace transform of the solution $u(x,t), F(s)$ is the transform of $f(t)$ and $W(x,s)$ is the transform of problem 5. Show that the result leads to the formula

$$u(x,t) = f(0)\text{erfc}\left(\frac{x}{2\sqrt{t}}\right) + t\int_0^1 \text{erfc}\left(\frac{x}{2\sqrt{t}}(1-u)^{-1/2}\right) f'(tu)\text{d}u$$

$$\begin{aligned}
sU(x) - 0 &= \frac{\text{d}^2 U}{\text{d}^2 x} \\
U(0) &= F(s) \\
0 &= \frac{\text{d}^2 U}{\text{d}^2 x} - sU(x) \\
U(x) &= c_1 e^{x\sqrt{s}} + c_2 e^{-x\sqrt{s}} \\
U(0) &= F(s) = c_1 + c_2 \\
c_1 = 0 \quad c_2 &= F(s) \\
U(x) &= F(s)e^{-x\sqrt{s}} = \frac{s}{s}F(s)e^{-x\sqrt{s}} \\
W(s) &= \frac{1}{s}e^{-x\sqrt{s}} \\
U(x) &= sF(s)W(x)
\end{aligned}$$

And now we do the reverse transform

$$\begin{aligned}
U(x) &= (sF(s) - f(0) + f(0))W(x) \\
&= (sF(s) - f(0))W(x) + f(0)W(x) \\
u(x,t) &= f(0)\text{erfc}\left(\frac{x}{2\sqrt{t}}\right) + \int_0^t f'(u)\,\text{erfc}\left(\frac{x}{2\sqrt{(t-u)}}\right)\text{d}u \\
&= f(0)\text{erfc}\left(\frac{x}{2\sqrt{t}}\right) + t\int_0^1 f'(tu)\,\text{erfc}\left(\frac{x}{2\sqrt{(t-tu)}}\right)\text{d}u \\
&= f(0)\text{erfc}\left(\frac{x}{2\sqrt{t}}\right) + t\int_0^1 f'(tu)\,\text{erfc}\left(\frac{x}{2\sqrt{t}\sqrt{1-u}}\right)\text{d}u
\end{aligned}$$