8.4

D. Does $\sum\limits_{n=1}^{\infty} \frac{1}{x^2+n^2}$ converge uniformly on the whole real line?

We know $0 \leq \sum\limits_{n=1}^{\infty} \frac{1}{x^2+n^2} < \sum\limits_{n=1}^{\infty} \frac{1}{n^2}$. Because $\frac{1}{n^2}$ is convergent, then $\frac{1}{x^2+n^2}$ must also be convergent.

This also gives us uniform convergence, because for every $\varepsilon > 0$ there exists an N such that $0 \leq || \sum\limits_{i=k+1}^{l} \frac{1}{x^2+i^2} \leq || \sum\limits_{i=k+1}^{l} \frac{1}{i^2} \leq \varepsilon$ for every $l > k \geq N$ regardless of our choice of x.

E. Show that if $\sum\limits_{n=1}^{\infty} |a_n| < \infty$, then $\sum\limits_{n=1}^{\infty} a_n \cos nx$ converges uniformly on $\mathbb{R}$.

Because $0 \leq |\cos nx| \leq 1$ then $|a_n \cos nx| \leq |a_n|$. Now we know that $|a_n|$ converges and so then for any $\varepsilon > 0$ there exists an N such that $\sum\limits_{i=k+1}^{l} |a_n| < \varepsilon$ for any $l > k \geq N$. But $\sum\limits_{i=k+1}^{l} |a_n \cos nx| \leq \sum\limits_{i=k+1}^{l} |a_n| < \varepsilon$ regardless of our choice of x. And since $|a_n \cos nx|$ converges uniformly, then we get $a_n \cos nx$ converging uniformly for free.

F. (a) Let $f_n(x) = \frac{x^2}{(1+x^2)^n}$ for $x \in \mathbb{R}$. Evaluate the sum $S(x) = \sum\limits_{n=0}^{\infty} f_n(x)$.

At $x = 0$ the sum is 0. At all other values we have a geometric series which converges to $\frac{x^2}{1-(\frac{1}{1+x^2})} = \frac{x^2}{\frac{x^2}{1+x^2}} = 1 + x^2$

(b) Is this convergence uniform? For which values $a < b$ does this series converge uniformly on $[a, b]$?

The convergence is not uniform. Our series converges to a discontinuous function ($0 < 1 < 1+x^2$), and so it is not uniformly continuous, by theorem 8.4.4.

We take the derivative

$$\begin{aligned}
\frac{\partial}{\partial x} \frac{x^2}{(1+x^2)^n} &= 2x(1+x^2)^{-n} - nx^2(1+x^2)^{-n-1}2x \\
&= \frac{2x(1+x^2)}{(1+x^2)(1+x^2)^n} - \frac{2nx^3}{(1+x^2)(1+x^2)^n} \\
&= \frac{2x(1+x^2-nx^2)}{(1+x^2)(1+x^2)^n} \\
&= \frac{2x(1+x^2(1-n))}{(1+x^2)(1+x^2)^n}
\end{aligned}$$

So the denominator of our derivative has no zeros, and our numerator has zeros at $x = 0$ at $x = \pm\frac{1}{\sqrt{n-1}}$. Zero is obviously a minimum because the function has no negative terms. And $\frac{1}{\sqrt{n-1}}$ is less than 1 for all $n > 2$. So if comparing $x = \frac{1}{\sqrt{n-1}}$ and $x = 1$ when $n = 3$ we see that

$$\frac{\frac{1}{n-1}}{(1+\frac{1}{n-1})^3} ? \frac{1}{(1+1)^3}$$
$$\frac{\frac{1}{n-1}}{(\frac{n}{n-1})^3} ? \frac{1}{2^3}$$
$$\frac{1}{n-1}\left(\frac{n-1}{n}\right)^3 ? \frac{1}{8}$$
$$\frac{(n-1)^2}{n^3} ? \frac{1}{8}$$
$$\frac{2^2}{3^3} ? \frac{1}{8}$$
$$0.\overline{148} > .125$$

And so $\frac{1}{\sqrt{n-1}}$ is a maximum. Observe that

$$\frac{\frac{1}{n-1}}{\left(1+\frac{1}{n-1}\right)^n} = \frac{1}{n-1}\left(\frac{n-1}{n}\right)^n = \frac{(n-1)^{n-1}}{n^n}$$

We have a higher degree on the bottom, so this will converge to zero. And so we have uniform convergence on $[a, \infty)$ for all $a > 0$. And of course $(-\infty, -a]$ or any subinterval of these.

H. Suppose that $a_k(x)$ are continuous functions on $[0, 1]$, and define $s_n(x) = \sum_{k=1}^{n} a_k(x)$. Show that if (s_n) converges uniformly on $[0, 1]$, then (a_n) converges uniformly to 0.

If we assume that (a_n) does not converge uniformly to 0. We know that (a_n) converges to zero at least pointwise, else (s_n) would not converge for some x. And so we assume that (a_n) converges but not uniformly. Now (s_n) must be uniformly Cauchy and so given any $\varepsilon > 0$ there exists some N large enough that $||\sum_{i=k+1}^{l} a_i(x)||_\infty \le \varepsilon$ for all $l > k \ge N$ We take $l = k+1$ and obtain $||a_l(x)||_\infty \le \varepsilon \forall l \ge N$. But we are assuming that (a_n) does not converge uniformly. Therefore $\lim_{k\to\infty} ||a_k||_\infty = L$ for some $L > 0$. If we choose $\varepsilon = \frac{L}{2}$, then we have $||a_l||_\infty > \varepsilon$ for some N and all $l > N$. Thus we have a contradiction, and (a_n) must converge uniformly.

J. Let (f_n) be a sequence of functions defined on $\mathbb{N}$ such that $\lim_{k\to\infty} f_n(k) = L_n$ exists for each $n \ge 0$. Suppose that $||f_n||_\infty \le M_n$, where $\sum_{n=0}^{\infty} M_n < \infty$.

Define a function $F(k) = \sum_{n=0}^{\infty} f_n(k)$. Prove that $\lim_{k\to\infty} F(k) = \sum_{n=0}^{\infty} L_n$. HINT: Think of f_n as a function g_n on $\{\frac{1}{k} : k \geq 1\} \cup 0$. How will you define $g_n(0)$?

We define $g_n(x) = f_n(\frac{1}{x})$. Because $\lim_{k\to\infty} \frac{1}{k} = 0$ and $\lim_{k\to\infty} f_n(k) = L_n$ then it makes sense to define $g_n(0) = L_n$. Further, the Weierstrass M-Test tells us that the series converges uniformly and so $G(x)$ is continuous. Thus $g_n(x)$ is defined for all $x \in [0,1]$.

Now we define $G(x) = \sum_{n=0}^{\infty} g_n(x)$ and $\lim_{k\to\infty} F(k) = G(0) = \sum_{n=0}^{\infty} L_n$

8.5

A. Determine the interval of convergence of the following power series:

(a) $\sum_{n=0}^{\infty} n^3 x^n$ We have $\lim_{n\to\infty} |\frac{(n+1)^3}{n^3}| = 1$. Obviously $\sum_{n=0}^{\infty} n^3$ and $\sum_{n=0}^{\infty} (-1)^n n^3$ diverge, and so our interval of convergence is $(-1,1)$

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} x^n$. So $\lim_{n\to\infty} \left|\frac{(-1)^{n+1} n^2}{(-1)^n (n+1)^2}\right| = 1$. But $\sum (-1)^n/n^2$ and $\sum_{n=0}^{\infty} 1/n^2$ both converge, and so our interval of convergence is $[-1,1]$.

(c) $\sum_{n=0}^{\infty} \frac{n^2}{2^n} x^n$

The limit of $\left|\frac{(n+1)^2 2^n}{2^{n+1} n^2}\right|$ as $n \to \infty$ is $1/2$. Now $\sum_{n=0}^{\infty} \frac{n^2}{2^n} (\pm 2)^n = \sum_{n=0}^{\infty} (-1)^n n^2$ or $\sum_{n=0}^{\infty} n^2$ and $\lim_{n\to\infty}$ which obvious diverge, so our interval of convergence is $(-2,2)$

(d) $\sum_{n=0}^{\infty} \sqrt{n} x^n$. Now $\lim_{n\to\infty} \frac{\sqrt{n+1}}{\sqrt{n}} = \sqrt{1} + \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 1$. Of course $\sum_{n=0}^{\infty} \sqrt{n}$ diverges along with $\sum_{n=0}^{\infty} (-1)^n \sqrt{n}$ and so our interval is $(-1,1)$

(e) $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$. So we define $a_{2k+1} = 0$ and $a_{2k} = (-1)^k/(2k)!$. And $\lim_{n\to\infty} \left|\frac{(2k)!}{(2k+2)!}\right| = \lim_{n\to\infty} \frac{1}{(2k+2)(2k+1)} = 0$. And so our sum only for any interval in $\mathbb{R}$.

(f) $\sum_{n=0}^{\infty} x^{n!}$.

Let us say $g(x) = x^{n!}$ and $g'(x) = n! x^{n!-1}$. Then $xg'(x) = n! g(x)$ and $g(0) = 0$. Now we assume there is a power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ that satisfies this DE. Then $x \sum_{n=1}^{\infty} n a_n x^{n-1} = n! \sum_{n=0}^{\infty} a_n x^n$.

(g) $\sum_{n=0}^{\infty} \frac{n!}{n^n}x^n$ Now $\lim_{n\to\infty} \frac{(n+1)!n^n}{n!(n+1)^{n+1}} = \lim_{n\to\infty} \frac{(n+1)n^n}{(n+1)(n+1)^n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{-n} = \frac{1}{e}$. And $\lim \left|(-1)^n \frac{n!}{e/n}^n\right| \neq 0$ because factorials grow faster than exponentials. And so our interval it $(-e, e)$

(h) $\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}x^n$

$\lim \frac{((n+1)!)^2(2n)!}{(2n+2)!(n!)^2} = \lim \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{1}{4}$. And $\sum \frac{(n!)^2}{(2n)!}4^n = \sum \frac{n!4^n}{(n+1)\ldots(n+n)}$. The computer claims this diverges, and so our interval of convergence is $(-4, 4)$

(i) $\sum_{n=0}^{\infty} \frac{1}{n}x^n$

$\lim \frac{n}{n+1} = 1$ but $\frac{1}{n}$ diverges while $\frac{(-1)^n}{n}$ converges, so our interval is $[-1, 1)$

B.