Using Duhamel's principle, what is the solution of the IBVP. Take $g(t) = \sin(t)$. Take $\alpha^2 = 1$.

PDE	$u_t = \alpha^2 u_{xx}$	$0 < x < 1$	$0 < t < \infty$
BCs	$\begin{cases} u(0,t) & = 0 \\ u(1,t) & = \sin t \end{cases}$		$0 < t < \infty$
IC	$u(x,0) = 0$	$0 \le x \le 1$	

Lets do a side by side as in the text.

Easy Problem

$$
\begin{aligned}
w_t &= \alpha^2 w_{xx} \\
w(0,t) &= 0 \\
w(1,t) &= 1 \\
w(x,0) &= 0 \\
\mathscr{L}\{w(x,t)\} &= W(x,s) \\
\frac{\mathrm{d}}{\mathrm{d}s}W(x,s) &= sW(x,s) - w(x,0) \\
&= sW(x,s) \\
0 &= \alpha^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2}W(x,s) - sW(x,s) \\
0 &= \alpha^2 r^2 + 0r - s \\
r &= \frac{\pm\sqrt{4\alpha^2 s}}{2\alpha^2} \\
r &= \pm\frac{\sqrt{s}}{\alpha} \\
W(x,s) &= c_1 e^{\frac{\sqrt{s}}{\alpha}x} + c_2 e^{-\frac{\sqrt{s}}{\alpha}x} \\
W(0,s) &= 0 = c_1 + c_2 \\
W(1,s) &= \frac{1}{s} = c_1 e^{\frac{\sqrt{s}}{\alpha}} - c_1 e^{-\frac{\sqrt{s}}{\alpha}} \\
\frac{1}{2s} &= c_1 \sinh\frac{\sqrt{s}}{\alpha} \\
W(x,s) &= \frac{1}{s}\frac{e^{\frac{\sqrt{s}}{\alpha}x} - e^{-\frac{\sqrt{s}}{\alpha}x}}{2\sinh\frac{\sqrt{s}}{\alpha}} \\
&= \frac{1}{s}\left[\frac{\sinh\left(\frac{\sqrt{s}}{\alpha}x\right)}{\sinh\frac{\sqrt{s}}{\alpha}}\right] \\
\mathscr{L}^{-1}\left\{\frac{1}{s}\left[\frac{\sinh(\sqrt{s}x)}{\sinh\sqrt{s}}\right]\right\} &= x + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-(n\pi)^2 t}\sin(n\pi x)
\end{aligned}
$$

Hard Problem

$$
\begin{aligned}
u_t &= \alpha^2 u_{xx} \\
u(0,t) &= 0 \\
u(1,t) &= \sin t \\
u(x,0) &= 0 \\
\mathscr{L}\{u(x,t)\} &= U(x,s) \\
\frac{\mathrm{d}}{\mathrm{d}s}U(x,s) &= sU(x,s) - u(x,0) \\
&= sU(x,s) \\
0 &= \alpha^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2}U(x,s) - sU(x,s) \\
&\vdots \\
U(x,s) &= c_1 e^{\frac{\sqrt{s}}{\alpha}x} + c_2 e^{-\frac{\sqrt{s}}{\alpha}x} \\
U(0,s) &= 0 = c_1 + c_2 \\
U(1,s) &= F(s) = \frac{1}{s^2+1} \\
&= c_1 e^{\frac{\sqrt{s}}{\alpha}} - c_1 e^{-\frac{\sqrt{s}}{\alpha}} \\
\frac{1}{2}F(s) &= c_1 \sinh\frac{\sqrt{s}}{\alpha} \\
U(x,s) &= F(s)\frac{e^{\frac{\sqrt{s}}{\alpha}x} - e^{-\frac{\sqrt{s}}{\alpha}x}}{2\sinh\frac{\sqrt{s}}{\alpha}} \\
&= F(s)\left[\frac{\sinh\left(\frac{\sqrt{s}}{\alpha}x\right)}{\sinh\frac{\sqrt{s}}{\alpha}}\right] \\
&= F(s)\left\{s\left[\frac{\sinh\left(\frac{\sqrt{s}}{\alpha}x\right)}{s\sinh\frac{\sqrt{s}}{\alpha}}\right]\right\} \\
\mathscr{L}\{w_t\} &= sW - w(x,0) \quad w(x,0) = 0 \\
U(x,s) &= F(s)\mathscr{L}\{w_t\}
\end{aligned}
$$

$$\mathscr{L}^{-1}\left\{W\left(x,\frac{s}{\alpha}\right)\right\} = \alpha w(x,\alpha t) = \alpha x + \frac{2\alpha}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-(n\pi)^2\frac{t}{\alpha}}\sin(n\pi x)$$

$$\begin{aligned}u(x,t) &= \mathscr{L}^{-1}\{F(s)\mathscr{L}\{w_t\}\}\\ &= f(t)*w_t(t)\\ &= \int_0^t f(\tau)w_\tau(x,t-\tau)\mathrm{d}\tau\\ &= \int_0^t w(x,t-\tau)f'(\tau)\mathrm{d}\tau + f(0)w(x,t)\end{aligned}$$

$$\begin{aligned}u(x,t) &= \int_0^t \alpha x\cos\tau + \frac{2\alpha}{\pi}\cos\tau\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-(n\pi)^2\frac{t-\tau}{\alpha}}\sin(n\pi x)\,\mathrm{d}\tau\\
&= \alpha x\int_0^t\cos\tau\,\mathrm{d}\tau + \frac{2\alpha}{\pi}\sum_{n=1}^{\infty}\left(\frac{(-1)^n}{n}\sin(n\pi x)\int_0^t e^{-(n\pi)^2\frac{t-\tau}{\alpha}}\cos\tau\,\mathrm{d}\tau\right)\\
&= \alpha x\int_0^t\cos\tau\,\mathrm{d}\tau + \frac{2\alpha}{\pi}\sum_{n=1}^{\infty}\left(\frac{(-1)^n}{n}\sin(n\pi x)\int_0^t e^{-(n\pi)^2\frac{t}{\alpha}+(n\pi)^2\frac{\tau}{\alpha}}\cos\tau\,\mathrm{d}\tau\right)\\
&= \alpha x\int_0^t\cos\tau\,\mathrm{d}\tau + \frac{2\alpha}{\pi}\sum_{n=1}^{\infty}\left(\frac{(-1)^n}{n}\sin(n\pi x)e^{-(n\pi)^2\frac{t}{\alpha}}\int_0^t e^{(n\pi)^2\frac{\tau}{\alpha}}\cos\tau\,\mathrm{d}\tau\right)\text{ use maxima to do second integral}\\
&= \alpha x\sin t + \frac{2\alpha}{\pi}\sum_{n=1}^{\infty}\left(\frac{(-1)^n}{n}\sin(n\pi x)e^{-(n\pi)^2\frac{t}{\alpha}}\left[\frac{e^{\frac{n^2\pi^2\tau}{\alpha}}\left(\sin\tau+\frac{n^2\pi^2\cos\tau}{\alpha}\right)}{\frac{n^4\pi^4}{\alpha^2}+1}\right]_0^t\right)\\
&= \alpha x\sin t + \frac{2\alpha}{\pi}\sum_{n=1}^{\infty}\left(\frac{(-1)^n}{n}\sin(n\pi x)e^{-(n\pi)^2\frac{t}{\alpha}}\left[\frac{\alpha^2e^{\frac{n^2\pi^2\tau}{\alpha}}\left(\sin\tau+\frac{n^2\pi^2\cos\tau}{\alpha}\right)}{n^4\pi^4+\alpha^2}\right]_0^t\right)\\
&= \alpha x\sin t + \frac{2\alpha^3}{\pi}\sum_{n=1}^{\infty}\left(\frac{(-1)^n}{n^5\pi^4+n\alpha^2}\sin(n\pi x)e^{-(n\pi)^2\frac{t}{\alpha}}\left[e^{\frac{n^2\pi^2\tau}{\alpha}}\left(\sin\tau+\frac{n^2\pi^2\cos\tau}{\alpha}\right)\right]_0^t\right)\\
&= \alpha x\sin t + \frac{2\alpha^3}{\pi}\sum_{n=1}^{\infty}\left(\frac{(-1)^n}{n^5\pi^4+n\alpha^2}\sin(n\pi x)e^{-(n\pi)^2\frac{t}{\alpha}}\left[\begin{array}{l}e^{\frac{n^2\pi^2t}{\alpha}}\left(\sin t+\frac{n^2\pi^2\cos t}{\alpha}\right)\\ -e^{\frac{n^2\pi^20}{\alpha}}\left(\sin 0+\frac{n^2\pi^2\cos 0}{\alpha}\right)\end{array}\right]\right)\\
&= \alpha x\sin t + \frac{2\alpha^3}{\pi}\sum_{n=1}^{\infty}\left(\frac{(-1)^n}{n^5\pi^4+n\alpha^2}\sin(n\pi x)e^{-(n\pi)^2\frac{t}{\alpha}}\left[e^{\frac{n^2\pi^2t}{\alpha}}\left(\sin t+\frac{n^2\pi^2\cos t}{\alpha}\right)-\frac{n^2\pi^2}{\alpha}\right]\right)\\
&= \alpha x\sin t + \frac{2\alpha^3}{\pi}\sum_{n=1}^{\infty}\left(\frac{(-1)^n}{n^5\pi^4+n\alpha^2}\sin(n\pi x)\left[\sin t+\frac{n^2\pi^2\cos t}{\alpha}-e^{-(n\pi)^2\frac{t}{\alpha}}\frac{n^2\pi^2}{\alpha}\right]\right)\end{aligned}$$

$\alpha^2 = 1$ If I had put done this substitution at the beginning there would be no $\pm\alpha$ worries, so I assume $\alpha = 1$

$$\begin{aligned}u(x,t) &= \pm x\sin t + \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{(-1)^n}{n^5\pi^4+n}\sin(n\pi x)\left[\pm\sin t+n^2\pi^2\cos t-e^{\mp(n\pi)^2t}n^2\pi^2\right]\right)\\
&= x\sin t + \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{(-1)^n}{n^5\pi^4+n}\sin(n\pi x)\left[\sin t+n^2\pi^2\cos t-e^{-(n\pi)^2t}n^2\pi^2\right]\right)\end{aligned}$$