10.4

G. Show that if $\lim\limits_{\delta\to 0^+} \frac{\omega(f;\delta)}{\delta} = 0$, then f is constant.

We assume that f is not constant. Then there exists some c such that $\lim\limits_{\delta\to 0^+} \frac{|f(c+\delta)-f(c)|}{\delta} > 0$. Of course $\sup\{|f(x_1) - f(x_2)| : |x_1 - x_2| < \delta, x_1, x_2 \in [a,b]\} \geq |f(c+\delta) - f(c)|$. Thus $\lim\limits_{\delta\to 0^+} \frac{\omega(f;\delta)}{\delta} \geq \lim\limits_{\delta\to 0^+} \frac{|f(c+\delta)-f(c)|}{\delta} > 0$. But the limit is zero, so by contradiction, f must be constant.

10.5

C. Find *all* closest lines $p(x) = ax + b$ to $f(x) = x^2$ in the $C^1[0,1]$ norm. Note that the best approximation is not unique.

We are looking for the values of a, b which will give us $E_1(f) = \inf \left\{ \max\limits_{0\leq i\leq 1} ||\frac{\mathrm{d}^i}{\mathrm{d}x^i} x^2 - ax - b||_\infty \right\}$. Now we note that the first derivative is $2x - a$. As x varies in $[0,1]$ it is clear that if $a = 1$ then $||2x-a||_\infty = 1$, but if $a \neq 1$ then $||2x-a||_\infty > 1$ therefore $E_1(f) = 1$. As long as $||x^2 - x - b||_\infty \leq 1$ then $p(x) = x - b$. Now we start with the functions $f(x) = x^2$ and $q(x) = x$. On our interval of $[0,1]$ these two functions intersect at their endpoints $x = 0, 1$, have the property that $x \geq x^2$ and are farthest apart at $\frac{\mathrm{d}}{\mathrm{d}x} x - x^2 = 0$ or $x = 1/2 \to q(1/2) = 1/4$. Now moving $q(x) = x$ up or down by any value will give $p(x) = x + b$. Notice that because $x \geq x^2$, as b grows negatively, the first moment in which $||x^2 - x - b||_\infty > 1$ is when $b < -1$. And of course as $p(x) = x + b$ moves up, the first moment when $||x^2 - x - b||_\infty > 1$ is when $(1/2) + b = 1 + 1/4$ or $b = 3/4$. Thus $p(x) = x + b$ where $-1 \leq b \leq 3/4$

D. Find the closest polynomial to $\sin x$ on $\mathbb{R}$.

Let $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ be the closest polynomial to $\sin x$. Now if any $a_i \neq 0$ where $i > 0$ then $||\sin x - p(x)||_\infty = \infty$ but if $p(x) = a_0$ then $||\sin x - p(x)||_\infty = 1 + |a_0| \geq 1$. And so we see that $||\sin x - p(x)||_\infty \geq ||\sin x - 0||_\infty$. Therefore the closest polynomial to $\sin x$ on $\mathbb{R}$ is $p(x) = 0$

G. Recall that a norm is strictly convex if $||x|| = ||y|| = ||(x+y)/2||$ implies that $x = y$.

(a) Suppose that V is a vector space with a strictly convex norm and M is a finite-dimensional subspace of V. Prove that each $v \in V$ has a unique closest point in M.

We choose two points $u, w \in M$ such that $||u-v|| = ||w-v|| \leq ||z-v||$ for all $z \in M$. In particular $||\frac{u+w}{2} - v|| \geq ||u - v||$. Some algebraic manipulation gives us $||\frac{u+w}{2} - v|| = ||\frac{u-v+w-v}{2}|| = \frac{1}{2}||(u-v) + (w - v)|| \leq \frac{1}{2}||u-v|| + \frac{1}{2}||w-v|| = ||u-v||$. And so $||\frac{u-v+w-v}{2}|| = ||u-v|| = ||w-v||$ and because V is strictly convex then we know that $u-v=w-v$ or $u=w$. And so we know there is only one closest point to v in M

(b) Prove that an inner product norm is strictly convex.

First we observe that if $\langle a,x\rangle=0$ for any x then in particular $\langle a,a\rangle = 0$ and so $a=0$.

Now if we start with $||x||=||y||=||(x+y)/2||$ and apply the definition of an inner product norm, then we can do some manipulations to achieve the result we are looking for.

$$\begin{aligned}
||x|| = ||y|| &= ||(x+y)/2|| \\
\sqrt{\langle x,x\rangle} = \sqrt{\langle y,y\rangle} &= \sqrt{\langle (x+y)/2,(x+y)/2\rangle} \\
\langle x,x\rangle = \langle y,y\rangle &= 1/2\langle x,(x+y)/2\rangle + 1/2\langle y,(x+y)/2\rangle \\
\langle x,x\rangle = \langle y,y\rangle &= 1/2\langle (x+y)/2,x\rangle + 1/2\langle (x+y)/2,y\rangle \\
\langle x,x\rangle = \langle y,y\rangle &= 1/4\langle x,x\rangle + 1/2\langle x,y\rangle + 1/4\langle y,y\rangle \\
\langle x,x\rangle &= 1/2\langle x,x\rangle + 1/2\langle x,y\rangle \\
0 &= -1/2\langle x,x\rangle + 1/2\langle x,y\rangle = \langle y,x\rangle - \langle x,x\rangle \\
0 &= \langle y-x,x\rangle
\end{aligned}$$

As we observed at the start, $y-x=0$ and so $y=x$

(c) Show by example that $C[0,1]$ is not strictly convex.

$f(x)=(x-1/2)^3\in C[0,1]$