8.5

A. Determine the interval of convergence of the following power series:

(f) $\sum\limits_{n=0}^{\infty} x^{n!}$.

We first compare x^n to $x^{n!}$. If $|x| < 1$ then $|x^{n!}| < |x^n|$ and if $|x| > 1$ then $|x^{n!}| > |x^n|$. Of course if $|x| = 1$ then $|x^n| = 1 = |x^{n!}|$.

Now examining $\sum\limits_{n=0}^{\infty} x^n$ we see that $\lim\limits_{n\to\infty} |1|^{1/n} = 1$ and so our radius of convergence is 1.

Now $\sum\limits_{n=0}^{\infty} x^n$ is a geometric series, and so it converges only if $|x| < 1$. And so by comparison $\sum\limits_{n=0}^{\infty} x^{n!}$ has an interval of convergence of $(-1, 1)$

B. Find a power series $\sum\limits_{n=0}^{\infty} a_n x^n$ that has a different *interval* of convergence than $\sum\limits_{n=0}^{\infty} n a_n x^{n-1}$.

We choose $a_n = \frac{1}{n+1}$ and $\lim\limits_{n\to\infty} \frac{n+1}{n+2} = 1$. Our radius of convergence then is 1. $\sum\limits_{n=0}^{\infty} \frac{x^n}{n+1}$ converges at -1 by the alternating series test. Now $\sum\limits_{n=0}^{\infty} \frac{1}{2n} < \sum\limits_{n=0}^{\infty} \frac{1}{n+1}$. But $\frac{1}{2}\sum\limits_{n=0}^{\infty} \frac{1}{n}$ diverges and so $\sum\limits_{n=0}^{\infty} \frac{1}{n+1}$ diverges. And so our interval of convergence is $[-1, 1)$. Now $\sum\limits_{n=0}^{\infty} \frac{n}{n+1} x^{n-1}$ has the same radius of convergence. Now $\sum\limits_{n=0}^{\infty} \frac{n}{n+1} = \sum\limits_{n=0}^{\infty} 1 - \frac{1}{n+1}$. But $\lim\limits_{n\to\infty} 1 - \frac{1}{n+1} = 1$ and so this series diverges at 1. And similarly $(-1)^{n-1} - \frac{(-1)^{n-1}}{n+1}$ alternately approaches 1 and -1 as n goes to infinity. And so because $(-1)^{n-1}\frac{n}{n+1}$ has no limit, the series can not converge. Thus our interval of convergence is $(-1, 1)$

10.1

C. Let f satisfy the hypotheses of Taylor's Theorem at $x = a$.

(a) Show that $\lim\limits_{x\to a} \frac{f(x)-P_n(x)}{(x-a)^n} = 0$.

$$\begin{aligned}
\lim_{x\to a} \left| \frac{f(x) - P_n(x)}{(x-a)^n} \right| &= \lim_{x\to a} \left| \frac{R_n(x)}{(x-a)^n} \right| \\
&\le \lim_{x\to a} \left| \frac{M(x-a)^{n+1}}{(n+1)!(x-a)^n} \right|
\end{aligned}$$

$$= \frac{M}{(n+1)!} \lim_{x \to a} |(x-a)|$$
$$= \frac{M}{(n+1)!} 0 = 0$$

(b) If $Q(x) \in \mathbb{P}_n$ and $\lim_{x \to a} \frac{f(x)-Q(x)}{(x-a)^n} = 0$, prove that $Q = P_n$.

Because $\lim_{x \to a} \frac{f(x)-Q(x)}{(x-a)^n} = 0$ and $\lim_{x \to a} \frac{f(x)-P_n(x)}{(x-a)^n} = 0$ it follows that

$$\lim_{x \to a} \frac{f(x)-Q(x)}{(x-a)^n} - \lim_{x \to a} \frac{f(x)-P_n(x)}{(x-a)^n} = 0$$
$$\lim_{x \to a} \frac{f(x)-Q(x)-(f(x)-P_n(x))}{(x-a)^n} = 0$$
$$\lim_{x \to a} \frac{P_n(x)-Q(x)}{(x-a)^n} = 0$$

Recalling that $P_n(X), Q(x) \in \mathbb{P}_n$

$$\lim_{x \to a} \frac{P_n(x)-Q(x)}{(x-a)^n} = \lim_{x \to a} \sum_{i=0}^{n} \frac{a_i x^i}{(x-a)^n}$$
$$\lim_{x \to a} \sum_{i=0}^{n} \frac{a_i x^i}{(x-a)^n} = \sum_{i=0}^{n} \lim_{x \to a} \frac{a_i x^i}{(x-a)^n}$$

Now if we assume $P_n(x) \neq Q(x)$ then there exists some $a_i \neq 0$. $\frac{a_i x^i}{(x-a)^n}$ does not converge as $x \to a$, and so neither does $\frac{P_n(x)-Q(x)}{(x-a)^n}$, which is contrary to our assumption.

F. Let $f(x) = \log x$.

(a) Find the Taylor series of f about $x = 1$.

$$f'(x) = \frac{1}{x} \qquad f''(x) = -\frac{1}{x^2}$$
$$f^{(3)}(x) = \frac{2}{x^3} \qquad f^{(k)}(x) = \frac{(-1)^{k+1}(k-1)!}{x^k}$$
$$P_n(x) = \sum_{k=1}^{n} \frac{(-1)^{k+1}(k-1)!}{1^k k!}(x-1)^k \qquad P_n(x) = \sum_{k=1}^{n} \frac{(-1)^{k+1}(x-1)^k}{k}$$

(b) What is the radius of convergence of this series?

$$\lim_{k \to \infty} \left| \frac{(-1)^{k+2} k}{(-1)^{k+1}(k+1)} \right| = \lim_{k \to \infty} \frac{k}{(k+1)} = 1 = R$$

(c) What happens at the two endpoints of the interval of convergence? Hence find a series converging to $\log 2$.

$$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}(-1)^k}{k}=\sum_{k=1}^{\infty}\frac{-1}{k}=\infty$$
$$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}(2-1)^k}{k}=-\sum_{k=1}^{\infty}\frac{(-1)^k}{k}$$

So the series does not converge at 0, but it does at 2, and the series is above.

(d) By observing that $\log 2=\log 4/3-\log 2/3$, find another series converging to $\log 2$. Why is this series more useful?

$$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}(\frac{4}{3}-1)^k}{k}-\sum_{k=1}^{\infty}\frac{(-1)^{k+1}(\frac{2}{3}-1)^k}{k}$$
$$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{3^k k}+\sum_{k=1}^{\infty}\frac{1}{3^k k}$$

We know that our error $(R_n(x))$ is not more than $\frac{M|x-1|^{n+1}}{(n+1)!}$ where $M\geq|f^{(n+1)}(x)|=\left|\frac{(-1)^{k+2}k!}{x^{k+1}}\right|$. And swapping out M we have

$$\begin{aligned}R_n(x)&\leq\left|\frac{(-1)^{k+2}k!}{x^{k+1}}\right|\cdot\frac{|x-1|^{k+1}}{(k+1)!}\\&=\frac{|x-1|^{k+1}}{x^{k+1}(k+1)}\\&\simeq\frac{|x-1|^k}{kx^k}\end{aligned}$$

And so $R_n(2)\simeq\frac{1}{k2^k}$ and $R_n(4/3)\simeq\frac{1}{3^k k\frac{4}{3}^k}=\frac{1}{k4^k}$ and $R_n(2/3)\simeq\frac{1}{3^k k\frac{2}{3}^k}=\frac{1}{k2^k}$. So we are using the $\log 4/3$ term to improve the accuracy of our estimate because $R_n(4/3)\leq R_n(2)$.

I. Let $f(x)=(1+x)^{-1/2}$

(a) Find a formula for $f^{(k)}(x)$. Hence show that

$$f^{(k)}(0)=\binom{-\frac{1}{2}}{k}:=\frac{-\frac{1}{2}(-\frac{1}{2}-1)\cdots(-\frac{1}{2}+1-k)}{k!}=\frac{(-1)^k(2k)!}{2^{2k}(k!)^2}=\left(\frac{-1}{4}\right)^k\binom{2k}{k}.$$

$$f^{(k)}(x)=(1+x)^{-1/2-k}\prod_{i=1}^{k}\frac{1}{2}-i$$

$$
\begin{aligned}
&= (1+x)^{-1/2-k}\binom{-1/2}{k}k! \\
f^{(k)}(0) &= \binom{-1/2}{k}k! = \frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)\cdots\left(-\frac{1}{2}-(k-1)\right)}{k!}k! \\
&= \left(-\frac{1}{2}\right)^k (1+0)(1+2)\cdots(2k-1) \\
&= \left(-\frac{1}{2}\right)^k \frac{1\cdot 3\cdots(2k-1)\cdot 2\cdot 4\cdots 2k}{2\cdot 4\cdots 2k} \\
&= \left(-\frac{1}{2}\right)^k \frac{(2k)!}{2^k(1\cdot 2\cdots k)} \\
&= \left(-\frac{1}{2}\right)^k \frac{(2k)!}{2^k k!} = \frac{(-1)^k(2k)!}{2^{2k}k!} = \left(\frac{-1}{4}\right)^k \binom{2k}{k}k!
\end{aligned}
$$

(b) Show that the Taylor series for f about $x=0$ is $\sum_{k=0}^{\infty}\binom{2k}{k}\left(\frac{-x}{4}\right)^k$, and compute the radius of convergence.

By definition the Taylor series is $\sum_{k=0}^{\infty}\left(-\frac{1}{4}\right)^k\binom{2k}{k}\frac{k!}{k!}(x-0)^k = \sum_{k=0}^{\infty}\left(-\frac{x}{4}\right)^k\frac{(2k)!}{(k!)^2}$. And

$$
\begin{aligned}
\lim_{k\to\infty}\left|\frac{(2(k+1))!2^{2k}(k!)^2}{(2k)!2^{2(k+1)}((k+1)!)^2}\right| &= \lim_{k\to\infty}\frac{(2k+1)(2k+2)}{2^2(k+1)^2} \\
&= \lim_{k\to\infty}\frac{(2k+1)}{2k+2} = 1
\end{aligned}
$$

So the series has a radius of convergence of 1 by Hadamard's theorem.

(c) Show that $\sqrt{2} = 1.4f(-0.02)$. Hence compute $\sqrt{2}$ to 8 decimal places.

$$
1.4f(-0.02) = \frac{\sqrt{1.4^2}}{\sqrt{1-0.02}} = \frac{\sqrt{1.96}}{\sqrt{.98}} = \sqrt{2}
$$

To compute to 8 decimal places we need $|f(-.02) - \sum| \leq \dfrac{|f^{(k+1)}(-.02)|(.02)^{k+1}}{(k+1)!} < 0.5\cdot 10^{-8}$ or $\dfrac{(.98)^{-1/2-k-1}(2k+2)!\cdot .02^{k+1}}{2^{2k+2}(k+1)!^2} - 0.5\cdot 10^{-8} < 0$. Using the computer we find that $k=4$ is sufficient to make said expression smaller than zero. And again using the computer to evaluate the taylor series for the first five terms we have $\sqrt{2}\approx 1.41421356$

(d) Express $\sqrt{2} = 1.415f(\varepsilon)$, where ε is expressed as a fraction in lowest terms. Use this to obtain an alternating series for $\sqrt{2}$. How many terms are needed to estimate $\sqrt{2}$ to 100 decimal places?

$$\begin{aligned}
\sqrt{2} = 1.415\frac{1}{\sqrt{1+\varepsilon}} &= \frac{283}{200\sqrt{1+\varepsilon}} \\
2 &= \frac{80089}{40000\sqrt{1+\varepsilon}} \\
\frac{80089}{80000} &= 1+\varepsilon \\
\varepsilon &= \frac{89}{80000}
\end{aligned}$$

To compute to 100 decimal places we need $|f(89/80000) - \sum| \leq \frac{|f^{(k+1)}(89/80000)|(89/80000)^{k+1}}{(k+1)!} < 0.5{\cdot}10^{-100}$ or $\frac{(80089/80000)^{-1/2-k-1}(2k+2)! \cdot (89/80000)^{k+1}}{2^{2k+2}(k+1)!^2} - 0.5 \cdot 10^{-100} < 0$. Using a computer we find that $k = 33$ gives a negative value, so we would need 34 terms.

10.2

D. Suppose that f is a continuous function on $[0,1]$ such that $\int_0^1 f(x)x^n \, \mathrm{d}x = 0$ for all $n \geq 0$. Prove that $f = 0$. HINT: Use the Weierstrass Theorem to show that $\int_0^1 |f(x)|^2 \, \mathrm{d}x = 0$

From the Weierstrass Theorem, we know there there is a sequence of polynomials p_n that converge uniformly to f on $[0,1]$. This means that $\lim_{n\to\infty} \int_0^1 p_n(x) \, \mathrm{d}x = \int_0^1 f(x) \, \mathrm{d}x$. Thus $\int_0^1 f(x)^2 \, \mathrm{d}x = \lim_{n\to\infty} \int_0^1 p_n(x)f(x) \, \mathrm{d}x$. But we know that $\int_0^1 f(x)x^n \, \mathrm{d}x = 0$ and every term of $f(x)p_n(x)$ will go to zero under the integral. Thus $\int_0^1 |f(x)|^2 \, \mathrm{d}x = \int_0^1 f(x)^2 \, \mathrm{d}x = 0$. Now if $\int_0^1 |f(x)|^2 \, \mathrm{d}x = 0$ then surely $\int_0^1 |f(x)| \, \mathrm{d}x = 0$ as well. Now we are told that f is continuous on $[0,1]$. And so if we assume that $\exists x$ such that $|f(x)| > 0$ then by continuity we must have $\int_0^1 |f(x)| \, \mathrm{d}x > 0$. And so $|f(x)| = 0$. Thus $f(x) = 0$.