10.5

G. Recall that a norm is strictly convex if $||x|| = ||y|| = ||(x+y)/2||$ implies that $x = y$.

(a) Suppose that V is a vector space with a strictly convex norm and M is a finite-dimensional subspace of V. Prove that each $v \in V$ has a unique closest point in M.

We choose two points $u, w \in M$ such that $||u-v|| = ||w-v|| \le ||z-v||$ for all $z \in M$. In particular $||\frac{u+w}{2} - v|| \ge ||u-v||$. Some algebraic manipulation gives us $||\frac{u+w}{2} - v|| = ||\frac{u-v+w-v}{2}|| = \frac{1}{2}||(u-v)+(w-v)|| \le \frac{1}{2}||u-v|| + \frac{1}{2}||w-v|| = ||u-v||$. And so $||\frac{u-v+w-v}{2}|| = ||u-v|| = ||w-v||$ and because V is strictly convex then we know that $u - v = w - v$ or $u = w$. And so we know there is only one closest point to v in M

(b) Prove that an inner product norm is strictly convex.

First, observe the following identity:

$$\begin{aligned}
||x+y||^2 &= \langle x+y, x+y\rangle = \langle x,x\rangle + 2\langle x,y\rangle + \langle y,y\rangle \\
||x-y||^2 &= \langle x-y, x-y\rangle = \langle x,x\rangle - 2\langle x,y\rangle + \langle y,y\rangle \\
||x+y||^2 + ||x-y||^2 &= 2\langle x,x\rangle + 2\langle y,y\rangle = 2(||x||^2 + ||y||^2) \\
||x+y||^2 &= 2(||x||^2 + ||y||^2) - ||x-y||^2
\end{aligned}$$

Now if we assume that $||x|| = ||y|| = ||(x+y)/2|| = c$ then we can use the above identity to obtain:

$$\begin{aligned}
c^2 &= \frac{1}{4}||x+y||^2 \\
4c^2 &= ||x+y||^2 \\
2c^2 + 2c^2 &= 2||x||^2 + 2||y||^2 - ||x-y||^2 0 \qquad &= ||x-y||^2 \\
0 &= \langle x-y, x-y\rangle
\end{aligned}$$

But $\langle x-y, x-y\rangle = 0$ if and only if $x - y = 0$ so $x = y$

(c) Show by example that $C[0,1]$ is not strictly convex.

$\cos x, 1 \in C[0,1]$ and $||\cos x||_\infty = ||1||_\infty = ||(1+\cos x)/2||_\infty = 1$ but $\cos x \ne 1$