Notes

April 30, 2014

lesson 30

$$\begin{aligned}
\text{pde} \quad & u_{tt} = c^2\nabla^2 u = c^2(u_{rr} + \frac{1}{r}u_r + 1\frac{1}{r^2}u_{\theta\theta}) \\
\text{bc} \quad & u(1,\theta,t) = 0 \\
\text{ic} \quad & u(r,\theta,0) = f(r,\theta) \\
& u_t(r,\theta,0) = g(r,\theta)
\end{aligned}$$

$$\begin{aligned}
u = T(t)U(r,\theta) \to \frac{T''(t)}{c^2T(t)} = \frac{\nabla^2 U}{U} &= -\lambda^2 \leq 0 \\
\nabla^2 U + \lambda^2 U &= 0 \text{ Helmholtz equation} \\
U &= R(r)\Theta(\theta) \\
0 &= R''(r) + \frac{1}{r}R'(r) + \left(\frac{1}{r^2}\frac{\Theta''(\theta)}{\Theta(\theta)} + \lambda^2\right)R(r)
\end{aligned}$$

solutionsmust be 2π periodicin θ

$$\begin{aligned}
\Theta''(\theta) + n^2\Theta(\theta) &= 0 \\
n &= 0, 1, 2, \ldots \\
& \cos(n\theta), \sin(n\theta)
\end{aligned}$$

$$0 = R''(r) + \frac{1}{r}R'(r) + \left(\lambda^2 - \frac{n^2}{r^2}\right)R(r)$$

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eigenvalues λ will besuch that $R(1) = 0$. Condition $R(r)$ should be bounded at $r = 0$

$$\begin{aligned}
\lambda &= 0 \text{ not an eigenvalue} \\
0 &= R'' + \frac{1}{r}R' - \frac{n^2}{r^2}R \\
0 &= r^2R'' + rR' - n^2R \text{ Euler de} \\
R &= r^p \\
p(p-1)r^p + pr^p - n^2r^p &= 0 \\
p^2 - n^2 &= 0 \quad p = \pm n
\end{aligned}$$

if $n = 1, 2, \ldots$

$$R$$

if $n = 0$

bessel function website is at http://dlmf.nist.gov/10.2
$J_v(z)$ bessel functions of first kind, bounded at origin
$Y_v(z)$ bessel functions of second kind, unbounded.
also look at http://dlmf.nist.gov/10.8
blah....

λ must satisfy $J_n(\lambda) = 0$ mathematica function name for bessel: BesselJZero[n,k] is k^{th} zero of the Bessel function $J_n(x)$. add //N for numeric output

$$U(r, \theta) = J)n(k_{n,m} r)(a_n \cos(n\theta) + b_n \sin(n\theta))$$

for

$$n = 0, 1, 2, \ldots$$
$$m = 1, 2, 3, \ldots$$

here $\lambda = k_{n,m}$

$$T''(t) + c^2 {k_{n,m}}^2 T(t) = 0$$

has solutions $\cos(k_{n,m} ct), \sin(k_{n,m} ct)$

frequency $\frac{k_{n,m} c}{2\pi}$ figure 30.3 $n =$ the number of zeros of the trig part of $U(r, \theta) = J_n(k_{n,m} r)(a_n \cos(n\theta) + b_n \sin(n\theta))$