Notes

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$m*(A \cup B) \leq m*A + m*B$

we want

if $A_i \cap A_j = \emptyset$

$m*(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} m*(A_i)$

we don't always get this (sum of parts bigger than the whole)

this is a huge problem

measureable

a set E is measurable if for any set A $m*(A) = m*A \cap E) + m*(A \cap E^C)$

if we can chop A in half and the size of the pieces together is the same as the size of the pieces apart then E is measurable. this gives us what we want above.

two facts

1. if E is measurable, then so is E^C – definition is symmetric with respect to complements
2. E is measurable if and only if $m*A \geq m*(A \cap E) + m*(A \cap E^C)$ for any A

   proof

   $A = (A \cap E) \cup (A \cap E^C) \rightarrow m*(A) \leq m*(A \cap E) + m*(A \cap E^C)$ (less than comes free, only need to show the greater than part)

E measurable means that $m*A = m*(A \cap E) + m*(A \cap E^C)$ for any A

1. if $m*(E) = 0$ then E is measurable ($\emptyset$, singletons, cantor set)

   proof

   let A be a set. $A \cap E \subseteq E \rightarrow m*(A \cap E) \leq m*E = 0$ and $A \cap E^C \subset A \rightarrow m*(A \cap E^C \leq m*(A)$ and so $m*(A \cap E) + m*(A \cap E^C) \leq 0 + m*(A)$

intersections and unions

proposition

if E_1, E_2 are measurable, then so is $E_1 \cup E_2$.

proof

$m*A = m*A \cap E_1 + m*A \cap E_1^C$
$m*A = m*A \cap E_2 + m*A \cap E_2^C$
we want $m*A \geq m*A \cap (E_1 \cup E_2) + m*A \cap (E_1 \cup E_2)^C$

$$A \cap (E_1 \cup E_2)$$
$$(A \cap E_1) \cup (A \cap (E_2 \cap E_1^C))$$
$$A \cap (E_1 \cup E_2)^C$$
$$A \cap (E_1^C \cup E_2^C)$$
$$m*(A \cap E_1^C) = m*((A \cap E_1^C) \cap E_2) + m*((A \cap E_1^C) \cap E_2^C)$$
$$m*(A \cap (E_1 \cup E_2)) \leq m*(A \cap E_1) + m*(A \cap E_2 \cap E_1^2$$
$$m*(A \cap E_1^C) - m*((A \cap E_1^C) \cap E_2) = m*((A \cap E_1^C) \cap E_2^C)$$
$$m*(A \cap (E_1 \cup E_2)) + m*((A \cap E_1^C) \cap E_2) \leq m*(A \cap E_1) + m*(A \cap E_1^C) = m*(A)$$

proposition

if E_1, E_2 are measurable, so is $E_1 \cap E_2$.
$E_1 \cap E_2 = (E_1^C)^C \cap (E_2^C)^C$ demorgans law says $(E_1)^C \cup E_2^C)^C$ and so $(E_1^C \cup E_2^C)$ is measurable and so on

propostion

if A is a set and E_1, E_2 are measureable and $E_1 \cap E_2 = \emptyset$ then $m*(A \cap (E_1 \cup E_2)) = m*(A \cap E_1) + m*(A \cap E_2)$

proof

$m*A = m*A \cap E_1 + m*A \cap E_1^C$
$m*A \cap (E_1 \cup E_2) = m*A \cap (E_1 \cup E_2) \cap E_1 + m*A \cap (E_1 \cup E_2) \cap E_1^C = m*A \cap E_1 + m*A \cap E_2$