$$\frac{X''(x)}{X(x)} = \lambda \qquad \text{on} \qquad 0 < x < 1$$
$$X'(0) = 0$$
$$X'(1) - X(1) = 0$$

Show there is exactly one positive eigenvalue $\lambda = {\mu_1}^2$ with corresponding eigenfunction $X_1(x) = \cosh(\mu_1 x)$. Find $\int_0^1 X_1(x)^2 \, dx$ as an *algebraic* function of μ_1 (eliminate hyperbolic functions by use of the eigenvalue equation). Find μ_1 numerically.

$$\begin{aligned}
X'' - \lambda X &= 0 \\
r^2 - \lambda &= 0 \\
r &= \frac{0 \pm \sqrt{0^2 - 4 \cdot 1 \cdot (-\lambda)}}{2} = \frac{\pm 2\sqrt{\lambda}}{2} \\
&= \pm\sqrt{\lambda} = \pm\sqrt{{\mu_1}^2} = \pm\mu_1 \\
X(x) &= c_1 e^{\mu_1 x} + c_2 e^{-\mu_1 x} \\
&= \frac{c_1 e^{\mu_1 x} + c_1 e^{\mu_1 x} + c_2 e^{-\mu_1 x} + c_2 e^{-\mu_1 x}}{2} + \frac{c_1 e^{-\mu_1 x} - c_1 e^{-\mu_1 x} + c_2 e^{\mu_1 x} - c_2 e^{\mu_1 x}}{2} \\
&= \frac{c_1 e^{\mu_1 x} + c_1 e^{-\mu_1 x} + c_2 e^{\mu_1 x} + c_2 e^{-\mu_1 x}}{2} + \frac{c_1 e^{\mu_1 x} - c_1 e^{-\mu_1 x} - c_2 e^{\mu_1 x} + c_2 e^{-\mu_1 x}}{2} \\
&= (c_1 + c_2)\frac{e^{\mu_1 x} + e^{-\mu_1 x}}{2} + (c_1 - c_2)\frac{e^{\mu_1 x} - e^{-\mu_1 x}}{2} \\
&\Rightarrow c_1 \cosh(\mu_1 x) + c_2 \sinh(\mu_1 x) \\
X'(x) &= c_1 \mu_1 \sinh(\mu_1 x) + c_2 \mu_1 \cosh(\mu_1 x) \\
X'(0) = 0 &= c_1 \mu_1 \sinh(0) + c_2 \mu_1 \cosh(0) \\
&= c_2 \mu_1 \\
\mu_1 \neq 0 \Rightarrow 0 &= c_2 \\
X(x) &= c_1 \cosh(\mu_1 x)
\end{aligned}$$

Let's assume that μ_1 is not unique and see what happens.

$$\begin{aligned}
\cosh(\mu_1 x) &= \cosh(\mu_2 x) \\
\frac{e^{\mu_1 x} + e^{-\mu_1 x}}{2} &= \frac{e^{\mu_2 x} + e^{-\mu_2 x}}{2} \\
e^{\mu_1 x} = a, &\quad e^{\mu_2 x} = b \\
a + \frac{1}{a} &= b + \frac{1}{b} \\
a^2 + 1 &= a(b + \frac{1}{b}) \\
a^2 - a(b + \frac{1}{b}) + 1 &= 0 \\
a &= \frac{(b + \frac{1}{b}) \pm \sqrt{(b + \frac{1}{b})^2 - 4}}{2} \\
&= \frac{(b + \frac{1}{b}) \pm \sqrt{b^2 + 2 + \frac{1}{b^2} - 4}}{2} \\
&= \frac{b + \frac{1}{b} \pm \sqrt{b^2 - 2 + \frac{1}{b^2}}}{2}
\end{aligned}$$

$$
\begin{aligned}
a &= \frac{b+\frac{1}{b} \pm \sqrt{(b-\frac{1}{b})^2}}{2} = \frac{b+\frac{1}{b} \pm (b-\frac{1}{b})}{2} \\
&= \frac{1}{2}(2b) \text{ or } \frac{1}{2}\left(\frac{2}{b}\right) \\
e^{\mu_1 x} &= e^{\mu_2 x} \text{ or } \frac{1}{e^{\mu_2 x}} \\
\ln(e^{\mu_1 x}) &= \ln(e^{\mu_2 x}) \text{ or } \ln(e^{-\mu_2 x}) \\
\mu_1 &= \pm\mu_2 \Rightarrow (-\mu_1)^2 = (\mu_1)^2 = \lambda
\end{aligned}
$$

So we see λ is unique if it is positive. Now lets do our integral.

$$
\begin{aligned}
\int_0^1 X_1(x)^2 \,\mathrm{d}x &= \int_0^1 \cosh(\mu_1 x)^2 \,\mathrm{d}x \\
&= \int_0^1 \frac{(e^{\mu_1 x} + e^{-\mu_1 x})^2}{4} \,\mathrm{d}x \\
&= \frac{1}{4}\int_0^1 (e^{2\mu_1 x} + 2 + e^{-2\mu_1 x}) \,\mathrm{d}x \\
&= \frac{1}{4}\left[\frac{e^{2\mu_1 x}}{2\mu_1} + 2x + \frac{e^{-2\mu_1 x}}{-2\mu_1}\right]_0^1 \\
&= \frac{1}{4}\left[\frac{1}{\mu_1}\frac{e^{2\mu_1 x} - e^{-2\mu_1 x}}{2} + 2x\right]_0^1 \\
&= \frac{1}{2\mu_1}\left[\frac{(e^{\mu_1 x} + e^{-\mu_1 x})(e^{\mu_1 x} - e^{-\mu_1 x})}{4} + \mu_1 x\right]_0^1 \\
&= \frac{1}{2\mu_1}\left[\cosh(\mu_1 x)\sinh(\mu_1 x) + \mu_1 x\right]_0^1 \\
&= \frac{1}{2\mu_1}\left[\cosh(\mu_1)\sinh(\mu_1) + \mu_1 - \cosh(0)\sinh(0)\right] \\
&= \frac{1}{2\mu_1}\left[\cosh(\mu_1)\sinh(\mu_1) + \mu_1\right] \\
&= \frac{1}{2\mu_1}\cosh(\mu_1)\sinh(\mu_1) + \frac{1}{2} \\
&= \frac{1}{2{\mu_1}^2}\cosh(\mu_1)\mu_1\sinh(\mu_1) + \frac{1}{2} \\
&= \frac{1}{2{\mu_1}^2}X_1(1){X_1}'(1) + \frac{1}{2}
\end{aligned}
$$

And to find μ_1

$$
\begin{aligned}
X'(1) - X(1) &= 0 \\
\mu_1\sinh(\mu_1) - \cosh(\mu_1) &= 0 \\
\mu_1\frac{e^{\mu_1} - e^{-\mu_1}}{2} - \frac{e^{\mu_1} + e^{-\mu_1}}{2} &= 0 \\
\mu_1(e^{2\mu_1} - 1) - (e^{2\mu_1} + 1) &= 0 \\
e^{2\mu_1}(\mu_1 - 1) - \mu_1 - 1 = 0
\end{aligned}
$$

μ_1	$e^{2\mu_1}(\mu_1-1)-\mu_1-1$
0	-2
1	-2
2	$e^4-3\approx 51.5$
$\frac{3}{2}$	$\frac{1}{2}\cdot e^3-\frac{5}{2}\approx 7.5$
$\frac{5}{4}$	$\frac{1}{4}e^{5/2}-\frac{9}{4}\approx .8$
$\frac{9}{8}$	$\frac{1}{8}e^{9/4}-\frac{17}{8}\approx -.9$
$\frac{19}{16}$	$\frac{3}{16}e^{19/8}-\frac{35}{16}\approx -.17$
$\frac{39}{32}$	$\frac{7}{32}e^{39/16}-\frac{71}{32}\approx .28$
$\frac{77}{64}$	$\frac{13}{64}e^{77/32}-\frac{141}{64}\approx .05$
$\frac{153}{128}$	$\frac{25}{128}e^{153/64}-\frac{281}{128}\approx -.06$
$\frac{307}{256}$	$\frac{51}{256}e^{307/128}-\frac{563}{256}\approx -1.5$
$\frac{615}{512}$	$\frac{103}{512}e^{615/256}-\frac{1127}{512}\approx .02$

$\mu_1\approx\frac{615}{512}\approx 1.2$

$$\begin{aligned} \frac{X''(x)}{X(x)} &= \lambda \qquad \text{on} \qquad 0 < x < 1 \\ X'(0) &= 0 \\ X'(1) - X(1) &= 0 \end{aligned}$$

Show there are infinitely many distinct eigenvalues $\lambda = -{\mu_n}^2$ with corresponding eigenfunctions $X_n(x) = \cos(\mu_n x)$ for $n = 2, 3, 4, \ldots$. Find $\int_0^1 X_n(x)^2 \,\mathrm{d}x$ as an *algebraic* function of μ_n. Find μ_2, μ_3 numerically.

$$\begin{aligned} X'' - \lambda X &= 0 \\ r^2 - \lambda &= 0 \\ r &= \frac{0 \pm \sqrt{0^2 - 4 \cdot 1 \cdot (-\lambda)}}{2} = \frac{\pm 2\sqrt{\lambda}}{2} \\ &= \pm\sqrt{\lambda} = \pm\sqrt{-{\mu_n}^2} = \pm \mu_n i \\ X_n(x) &= c_1 \cos(\mu_n x) + c_2 \sin(\mu_n x) \\ {X_n}'(x) &= -c_1 \mu_n \sin(\mu_n x) + c_2 \mu_n \cos(\mu_n x) \\ {X_n}'(0) = 0 &= -c_1 \mu_n \sin(\mu_n 0) + c_2 \mu_n \cos(\mu_n 0) \\ &= c_2 \mu_n \\ 0 \neq \mu_n &\Rightarrow c_2 = 0 \\ X_n(x) &= c_1 \cos(\mu_n x) \end{aligned}$$

Lets see if we can find a $\mu_n \neq \mu_m$

$$\begin{aligned} X_n(x) &= \cos(\mu_n x) \\ k &\in \mathbb{Z} \\ \cos(\mu_n x) &= \cos(2\pi k + \mu_n x) \\ \mu_m &= 2\pi k + \mu_n \\ \cos(\mu_n x) &= \cos(\mu_m x) \\ \text{but } \mu_n &\neq \mu_m \end{aligned}$$

Also note that $|\mathbb{Z}| = \infty$ so there are infinitely many possibilities for k and by extension μ_m.

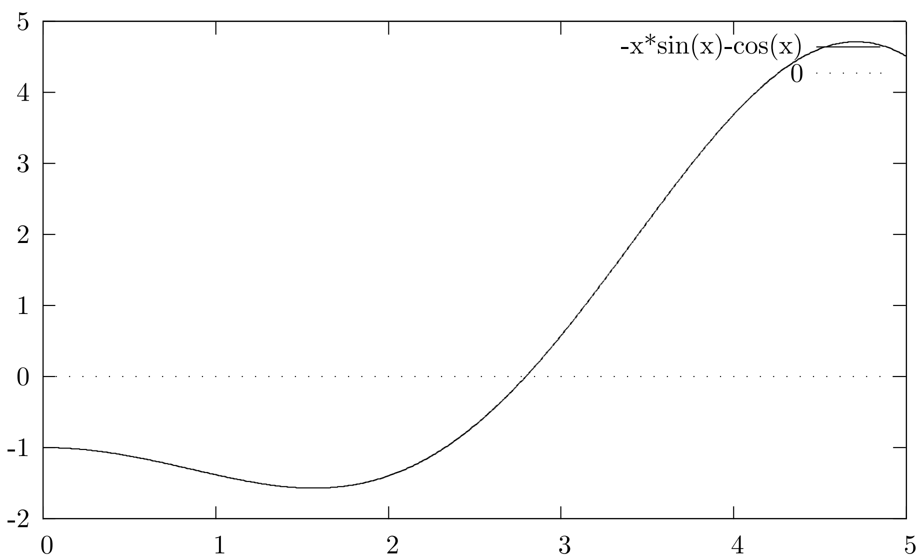
Let's do the integral

$$\begin{aligned} \int_0^1 X_n(x)^2 \,\mathrm{d}x &= \int_0^1 \cos(\mu_n x)^2 \,\mathrm{d}x \\ u = \cos(\mu_n x) &\quad \mathrm{d}v = \cos(\mu_n x)\mathrm{d}x \\ \mathrm{d}u = -\mu_n \sin(\mu_n x) &\quad v = \frac{1}{\mu_n}\sin(\mu_n x) \\ \int \cos(\mu_n x)^2 \,\mathrm{d}x &= \frac{1}{\mu_n}\cos(\mu_n x)\sin(\mu_n x) + \int \sin(\mu_n x)^2 \,\mathrm{d}x \\ &= \frac{1}{\mu_n}\cos(\mu_n x)\sin(\mu_n x) + \int 1 - \cos(\mu_n x)^2 \,\mathrm{d}x \\ 2\int \cos(\mu_n x)^2 \,\mathrm{d}x &= \frac{1}{\mu_n}\cos(\mu_n x)\sin(\mu_n x) + \int \mathrm{d}x \\ \int \cos(\mu_n x)^2 \,\mathrm{d}x &= \frac{1}{2\mu_n}\cos(\mu_n x)\sin(\mu_n x) + \frac{x}{2} \\ \int_0^1 \cos(\mu_n x)^2 \,\mathrm{d}x &= \left(\frac{1}{2\mu_n}\cos(\mu_n)\sin(\mu_n) + \frac{1}{2}\right) - \left(\frac{1}{2\mu_n}\cos(\mu_n 0)\sin(\mu_n 0) + \frac{0}{2}\right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{\cos(\mu_n)\mu_n\sin(\mu_n)}{2{\mu_n}^2} + \frac{1}{2} \\
&= \frac{X_n(1)^2}{2{\mu_n}^2} + \frac{1}{2}
\end{aligned}
$$

And now we attempt to find μ_2, μ_3 numerically. First we have to figure out what μ_1 is. And setup Newton's method.

$$
\begin{aligned}
X'(1) - X(1) &= 0 \\
-\mu_1\sin(\mu_1) - \cos(\mu_1) &= 0 \\
x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\
x_{n+1} &= x_n - \frac{-x_n\sin(x_n) - \cos(x_n)}{-x_n\cos(x_n) - \sin(x_n) + \sin(x_n)} \\
x_{n+1} &= x_n - \frac{x_n\sin(x_n) + \cos(x_n)}{x_n\cos(x_n)} \\
x_{n+1} &= x_n - \tan(x_n) - \frac{1}{x_n}
\end{aligned}
$$



Three looks like a good place to start. $x_1 = 3$

n	$x_n - \tan(x_n) - \frac{1}{x_n}$
1	2.809
2	2.798427
3	2.798386
4	2.798386

$$
\begin{aligned}
\mu_1 &\approx 2.798386 \\
\mu_1 - 2\pi &\approx -3.484799 \\
\mu_n &\approx 2\pi n - 3.484799 \\
\mu_2 &\approx 9.081571 \\
\mu_3 &\approx 15.364757
\end{aligned}
$$

Show that $\int_0^1 X_m(x)X_n(x)\,\mathrm{d}x = 0$ for ALL $m \neq n$ by integrating by parts. This calculation should make no explicit reference to trigonometric or hyperbolic functions.

$$\begin{aligned}
\frac{X''(x)}{X(x)} &= \lambda \\
X'(0) &= 0 \\
X'(1) - X(1) &= 0 \\
\int_0^1 X_m(x)X_n(x)\,\mathrm{d}x &= \int_0^1 X_m(x)\frac{1}{\lambda_n}X_n{}''(x)\,\mathrm{d}x \\
u = X_m(x) &\quad \mathrm{d}v = X_n{}''(x)\mathrm{d}x \\
\mathrm{d}u = X_m{}'(x)\mathrm{d}x &\quad v = X_n{}'(x) \\
\int_0^1 X_m(x)X_n(x)\,\mathrm{d}x &= \frac{1}{\lambda_n}\left[X_m(x)X_n{}'(x) - \int X_m{}'(x)X_n{}'(x)\,\mathrm{d}x\right]_0^1 \\
u = X_m{}'(x) &\quad \mathrm{d}v = X_n{}'(x)\mathrm{d}x \\
\mathrm{d}u = X_m{}''(x)\mathrm{d}x &\quad v = X_n(x) \\
\lambda_n \cdot \int_0^1 X_m(x)X_n(x)\,\mathrm{d}x &= \left[X_m(x)X_n{}'(x) - X_m{}'(x)X_n(x) + \int X_m{}''(x)X_n(x)\,\mathrm{d}x\right]_0^1 \\
&= \left[X_m(x)X_n{}'(x) - X_m{}'(x)X_n(x)\right]_0^1 + \int_0^1 \lambda_m X_m(x)X_n(x)\,\mathrm{d}x \\
(\lambda_n - \lambda_m)\cdot \int_0^1 X_m(x)X_n(x)\,\mathrm{d}x &= \left[X_m(1)X_n{}'(1) - X_m{}'(1)X_n(1)\right] - \left[X_m(0)X_n{}'(0) - X_m{}'(0)X_n(0)\right] \\
X'(1) - X(1) = 0 &\to X'(1) = X(1) \\
\int_0^1 X_m(x)X_n(x)\,\mathrm{d}x &= \frac{[X_m(1)X_n(1) - X_m(1)X_n(1)] - [X_m(0)\cdot 0 - 0\cdot X_n(0)]}{\lambda_n - \lambda_m} \\
\int_0^1 X_m(x)X_n(x)\,\mathrm{d}x &= 0
\end{aligned}$$

Verify that the equation

$$3u_{xx} + 7u_{xy} + 2u_{yy} = 0$$

is hyperbolic for all x and y and find the new *characteristic coordinates.*

$$B^2 - 4AC = 7^2 - 4 \cdot 3 \cdot 2 = 49 - 24 = 25 > 0$$

Since $B^2 - 4AC > 0$ the equation is hyperbolic. Now for the characteristic coordinates.

$$\begin{aligned}
\xi &= \xi(x,y) \\
\eta &= \eta(x,y) \\
u(x,y) \to u(\xi,\eta) &= u(\xi(x,y),\eta(x,y)) \\
u_{xx} &= u_{\xi\xi}{\xi_x}^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}{\eta_x}^2 + u_\xi\xi_{xx} + u_\eta\eta_{xx} \\
u_{yy} &= u_{\xi\xi}{\xi_y}^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}{\eta_y}^2 + u_\xi\xi_{yy} + u_\eta\eta_{yy} \\
u_{xy} &= u_{\xi\xi}\xi_x\xi_y + u_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + u_{\eta\eta}\eta_x\eta_y + u_\xi\xi_{xy} + u_\eta\eta_{xy} \\
\overline{A} = 0 &= A{\xi_x}^2 + B\xi_x\xi_y + C{\xi_y}^2 \\
\overline{C} = 0 &= C{\eta_x}^2 + B\eta_x\eta_y + C{\eta_y}^2 \\
\overline{A} = 0 &= A\left[\frac{\xi_x}{\xi_y}\right]^2 + B\left[\frac{\xi_x}{\xi_y}\right] + C \\
\overline{C} = 0 &= A\left[\frac{\eta_x}{\eta_y}\right]^2 + B\left[\frac{\eta_x}{\eta_y}\right] + C \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\xi_x}{\xi_y} = \frac{-B - \sqrt{B^2 - 4AC}}{2A} = \frac{-7 - \sqrt{25}}{6} = -2 \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\eta_x}{\eta_y} = \frac{-B + \sqrt{B^2 - 4AC}}{2A} = \frac{-7 + \sqrt{25}}{6} = -\frac{1}{3} \\
y &= -2x + c_1 \quad \xi = y + 2x = c_1 \\
y &= -\frac{1}{3}x + c_2 \quad \eta = y + \frac{1}{3}x = c_2
\end{aligned}$$

Continue with problem 3 by finding the new canonical equation.

$$u_{\xi\eta} = \Phi(\xi, \eta, u, u_\xi, u_\eta)$$

$$\begin{aligned}
\xi &= y + 2x \\
\eta &= y + \frac{1}{3}x \\
A = 3, \qquad B &= 7, \qquad C = 2 \\
D = E &= F = G = 0 \\
\xi_x = 2, &\qquad \xi_y = 1 \\
\xi_{xx} = \xi_{xy} &= \xi_{yy} = 0 \\
\eta_x = \frac{1}{3} &\qquad \eta_y = 1 \\
\eta_{xx} = \eta_{xy} &= \eta_{yy} = 0 \\
\overline{A} &= \overline{C} = 0
\end{aligned}$$

We solved for $\overline{A} = \overline{C} = 0$ to get ξ, η. Also, because all the second derivatives are zero along with D, E, F we quickly see that $\overline{D} = \overline{E} = 0$. And of course $\overline{F} = F = \overline{G} = G = 0$.

$$\begin{aligned}
\overline{B} &= 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y \\
&= 2 \cdot 3 \cdot 2 \cdot \frac{1}{3} + 7\left(2 \cdot 1 + 1 \cdot \frac{1}{3}\right) + 4 \\
&= \frac{12}{3} + 14 + \frac{7}{3} + 4 = \frac{19 + 18 \cdot 3}{3} \\
\overline{B}u_{\xi\eta} &= 0 \\
\overline{B} \neq 0 \rightarrow u_{\xi\eta} &= 0
\end{aligned}$$

Continue with problem 4 by finding the *alternative* canonical form

$$u_{\alpha\alpha} - u_{\beta\beta} = \Psi(\alpha, \beta, u, u_\alpha, u_\beta)$$

$$\begin{aligned}
\alpha(\xi,\eta) &= \xi + \eta \\
\beta(\xi,\eta) &= \xi - \eta \\
\alpha_\xi &= 1 \quad \alpha_\eta = 1 \\
\beta_\xi &= 1 \quad \beta_\eta = -1 \\
u_\xi &= u_\alpha \alpha_\xi + u_\beta \beta_\xi = u_\alpha + u_\beta \\
u_{\xi\eta} &= u_{\alpha\alpha}\alpha_\eta + u_{\alpha\beta}\beta_\eta + u_{\beta\alpha}\alpha_\eta + u_{\beta\beta}\beta_\eta = u_{\alpha\alpha} - u_{\beta\beta}
\end{aligned}$$

Since $u_{\xi\eta} = 0$ we know $u_{\alpha\alpha} - u_{\beta\beta} = 0$