Notes

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lesson 16 wave equation

obtain $u_{tt} = \alpha^2 u_{xx}$ and other forms.

1740s "vibrating string"

equilibrium position

violin string. tension T is grams cm/sec^2. density ρ is grams/cm. we are looking at segemnt of string $x_0 - \frac{\Delta x}{2} < x_0 < x_0 + \frac{\Delta x}{2}$. mass is $\rho \Delta x$ and $u(x,t)$ is vertical displacement from equilibrium. Assumptions, difference in length from Δx causes no change in mass. horizontal shifts are negligible.

this is page 125(141) in text.

Newton says $F = ma$. So $\rho \Delta x u_{tt}$ =vertical force on segment arising from string forces+etc.

Tension acts tangentially to $x_0 - \frac{\Delta x}{2}$ and $x_0 + \frac{\Delta x}{2}$. θ is angle from tangential line to horizontal with θ_1 being on the left. Tension= $T \sin\theta_2 - T \sin\theta_2$. notice that $\tan\theta = u_x$ so

$$\sin\theta = \frac{u_x}{\sqrt{1 + {u_x}^2}}$$

.

So we have

$$\rho \Delta x u_{tt} = T \left[\left(\frac{u_x}{\sqrt{1 + {u_x}^2}} \right)_{x_0 + \frac{\Delta x}{2}} - \left(\frac{u_x}{\sqrt{1 + {u_x}^2}} \right)_{x_0 - \frac{\Delta x}{2}} \right]$$

$$\rho u_{tt} = \frac{T}{\Delta x} \Delta \left(\frac{u_x}{\sqrt{1 + {u_x}^2}} \right) + \frac{\text{etc}}{\Delta x}$$

$$\rho u_{tt} = T \frac{\partial}{\partial x} \left(\frac{u_x}{\sqrt{1 + {u_x}^2}} \right) + \lim_{\Delta x \to 0} \frac{\text{etc}}{\Delta x}$$

note

$$\frac{\mathrm{d}}{\mathrm{d}x} \left[u_x (1 + {u_x}^2)^{-1/2} \right] = u_{xx}(1 + u_x^2) + u_x(-1/2)...$$

$$\rho u_{tt} = T \frac{u_{xx}}{(1 + {u_x}^2)^{3/2}} + \lim_{\Delta x \to 0} (\text{etc}/\Delta x)$$

more assumptions:tension remains essentially constant and $|u_x|$ is small

Since $|u_x|$ is small we have

$$\rho u_{tt} = T u_{xx} + \lim_{\Delta x \to 0} \frac{\text{etc}}{\Delta x}$$

other forces on segment might be

1. $F(x,t)$ is vertical force per unit length (gravity)

2. $-\gamma u$ is elastic restoring force per unit length
3. $-\beta u_t$ is frictional force (of medium) per unit length
4. etc is additional forces on segment of length Δx

$= F(x_0, t)\Delta x - \gamma u \Delta x - \beta u_t \Delta x$
α="velocity"

PDE	$u_{tt} = \alpha^2 u_{xx}$	$0 < t < \infty$ various x-internal
BC		
IC	$u(x,0) = \phi(x)$ $u_t(x,0) = \psi(x)$	

question: are there travelling wave solutions? this means a function that depends on x and t such that $u(x,t) = f(x-ct)$. So wave is being translated through time with speed c.

$$\begin{aligned} u(x,t) &= f(x-ct) \\ u_x &= f'(x-ct) \\ u_{xx} &= f''(x-ct) \\ u_t &= f'(x-ct)(-c) \\ u_{tt} &= f''(x-ct)(-c)^2 \end{aligned}$$

back substitute into pde

$$\begin{aligned} f''(x-ct)c^2 &= \alpha^2 f''(x-ct) \\ c &= \pm\alpha \end{aligned}$$