# Numerical Semigroups, Lattice Ideals, and Markov Bases

Jon Allen

Department of Mathematics
North Dakota State University
Fargo, ND

Capstone Presentation, North Dakota State University, December 2015

# Overview

A numerical semigroup is a nonempty subset $S$ of $\mathbb{N}$ that is closed under addition, contains the zero element, and whose complement in $\mathbb{N}$ is finite.

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- It is closed under addition
- It is generated from positive (nonzero) integers
- The greatest common factor of its generators is 1

# Example

Let $S$ be the numerical semigroup generated by $\{n_1, \ldots, n_k\}$ with $n_i \in \mathbb{N} \setminus \{0\}$. Then the elements of $S$ are $a_1 n_1 + \ldots a_k n_k$ for all $a_i \in \mathbb{N}$.

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**Example**

*The semigroup generated by* $\{3, 4, 5\}$ *is* $\{3, 4, 5, 6, 7, 8, \ldots\}$

# Table

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| | 3 | 4 | 5 |
|---|---|---|---|
| 3 | 1 | 0 | 0 |
| 4 | 0 | 1 | 0 |
| 5 | 0 | 0 | 1 |
| 6 | 2 | 0 | 0 |
| 7 | 1 | 1 | 0 |
| 8 | 1 | 0 | 1 |

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| | 3 | 4 | 5 |
|---|---|---|---|
| 3 | 1 | 0 | 0 |
| 4 | 0 | 1 | 0 |
| 5 | 0 | 0 | 1 |
| 6 | 2 | 0 | 0 |
| 7 | 1 | 1 | 0 |
| 8 | 1 | 0 | 1 |
| 8 | 0 | 2 | 0 |

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| | 3 | 4 | 5 |
|---|---|---|---|
| 3 | 1 | 0 | 0 |
| 4 | 0 | 1 | 0 |
| 5 | 0 | 0 | 1 |
| 6 | 2 | 0 | 0 |
| 7 | 1 | 1 | 0 |
| 8 | 1 | 0 | 1 |
| 8 | 0 | 2 | 0 |

| | 3 | 4 | 5 |
|---|---|---|---|
| 9 | 3 | 0 | 0 |
| 9 | 0 | 1 | 1 |
| 10 | 2 | 1 | 0 |
| 10 | 0 | 0 | 2 |

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| | 3 | 4 | 5 |
|---|---|---|---|
| 3 | 1 | 0 | 0 |
| 4 | 0 | 1 | 0 |
| 5 | 0 | 0 | 1 |
| 6 | 2 | 0 | 0 |
| 7 | 1 | 1 | 0 |
| 8 | 1 | 0 | 1 |
| 8 | 0 | 2 | 0 |

| | 3 | 4 | 5 |
|---|---|---|---|
| 9 | 3 | 0 | 0 |
| 9 | 0 | 1 | 1 |
| 10 | 2 | 1 | 0 |
| 10 | 0 | 0 | 2 |
| 11 | 2 | 0 | 1 |
| 11 | 1 | 2 | 0 |

A fiber consists of the different linear combinations of generators that result in an element of our semigroup.

Moves happen when elements of fibers are 'disconnected'.

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**Example**

$$10 = 2 \cdot 3 + 1 \cdot 4 + 0 \cdot 5 \qquad 11 = 2 \cdot 3 + 0 \cdot 4 + 1 \cdot 5$$
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Moves are the elements of the Markov basis and are the difference of disconnected elements of fibers.

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## Example

| | 3 | 4 | 5 | | | |
|---|---|---|---|---|---|---|
| 8 | 1 | 0 | 1 | | | |
| 8 | 0 | 2 | 0 | -1 | 2 | -1 |
| 9 | 3 | 0 | 0 | | | |
| 9 | 0 | 1 | 1 | 3 | -1 | -1 |
| 10 | 2 | 1 | 0 | | | |
| 10 | 0 | 0 | 2 | -2 | -1 | 2 |

We have an easy bijection between our Markov basis and an integer lattice.

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**Example**

$$\begin{bmatrix} 3 & -1 & -1 \\ -1 & 2 & -1 \\ -2 & -1 & 2 \end{bmatrix} \Leftrightarrow \begin{cases} x^3 - yz \\ y^2 - xz \\ z^2 - x^2y \end{cases}$$

We have actually explicitly built our Markov basis to be the null space of the numerical semigroup basis.

We have actually explicitly built our Markov basis to be the null space of the numerical semigroup basis. If we can find some vector $x$ such that $\begin{bmatrix} 3 & -1 & -1 \\ -1 & 2 & -1 \\ -2 & -1 & 2 \end{bmatrix} x = 0$ Then we will have found our semigroup!

What we need is the Smith Normal Form.

$$UAV = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 2 & -1 \\ -2 & -1 & 2 \end{bmatrix} x = 0$$

We start with identity matrices on either side of our Markov matrix. The procedure is similar to finding and inverse matrix, (except the Markov matrix is singular).

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We reduce our Markov matrix, mirroring column and row operations in the adjacent matrices.
We can't use anything but integers for our row and column operations!

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 & -1 \\ -1 & 2 & -1 \\ -2 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Row operations on the left

$$\begin{bmatrix} 1 & 2 & 0 \\ 1 & 3 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & -3 \\ 0 & 5 & -4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Column operations on the right

$$\begin{bmatrix} 1 & 2 & 0 \\ 1 & 3 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \\ 0 & 1 & 5 \end{bmatrix}$$

# Thank You!