Notes

6 fevrier, 2015

quiz

1. $\int_{[a,b]} f \; \mathrm{d}m = \inf\limits_{\psi \geq f \text{ and simple}} \psi \; \mathrm{d}m$
2. $\{f_n\} \to F$ pointwise if $f_n(x) = f(x)$ for all x

thm

if f is bounded and riemann integrable on $[a,b]$ then $\int_{[a,b]} f \; \mathrm{d}m$ exists and $\int_{[a,b]} f \; \mathrm{d}m = \int_a^b f(x) \; \mathrm{d}x$

proof

let $\varepsilon > 0$. there is a partition $P = \{x_0, \ldots, x_n\}$ such that $U(P,f) - L(P,f) < \varepsilon$. let

$$\psi(x) = \sum_{k=1}^{n} M_k \chi_{[x_{k-1}, x_k]}$$
$$\varphi = \sum_{k=1}^{n} m_k \chi_{[x_{k-1}, x_k]}$$

and $\varphi(x) \leq f(x) \leq \psi(x)$ for all x.

$$\int_{[a,b]} \psi(x) \; \mathrm{d}m = \sum_{k=1}^{n} M_k m * ([x_{k-1}, x_k = \sum_{k=1}^{n} M_k \Delta_k = U(P,f)$$
$$\int_{[a,b]} \varphi(x) \; \mathrm{d}m = \sum_{k=1}^{n} m_k m * ([x_{k-1}, x_k = \sum_{k=1}^{n} m_k \Delta_k = L(P,f)$$

and $\inf \int \theta \; \mathrm{d}m \leq \int_{[a,b]} \psi \; \mathrm{d}m = U(P,f)$ and $\sup \int \tau \; \mathrm{d}m \geq \int_{[a,b]} \varphi \; \mathrm{d}m = U(P,f)$

then $0 \leq \inf \int \theta - \sup \int \tau \leq U(P,f) - L(P,f) < \varepsilon$

let $\varepsilon \to 0$ we let $0 \leq \inf \int \theta - \sup \int \tau \leq 0$ so $\inf = \sup$. and they are all equal to $\int_a^b f(x) \; \mathrm{d}x$

we note that continuous implies measurable.

sequences of functions

motivation: we would love for all functions to be polynomials. not all are though. lets approximate. approximations are not created equally. sometimes can approximate by other “nice” functions like trig functions.

$\{f_n\}_{n=1}^{\infty}$ we say that $f_n \to f$ pointwise if for every x $\{f_n(x)\}_{n=1}^{\infty} \to f(x)$. (sequence convergence).

examples

1. $f_n : [0,1] \to [0,1]$ and $f_n(x) = x^n$. $\lim_{n\to\infty} x^n = f(x) = \begin{cases} 0 & x < 1 \\ 1 & x = 1 \end{cases}$.

   note that f_n is continuous, but f is not.

2. $f_n(x) = \frac{1}{n} \sin nx$ and $f_n \to 0$ but $f_n' = \cos nx$ which diverges for all x.

3. $n\chi_{(0,\frac{1}{n}]} \to 0$. but $\int n\chi_{(0,\frac{1}{n}]} = 1$

to fix this

$||f - g||_\infty = \sup_{k\in S}\{|f(x) - g(x)|\}$ which is the furthest apart f and g are.

define **uniform convergence** as $\lim_{n\to\infty} ||f_n - f||_\infty = 0$
$x^n \not\to \lim x^n$ uniformally

fact

$f_n \to f$ uniformally on S implies $f_n \to f$ pointwise on S.

proof

we fix $x \in S$. $|f_n(x) - f(x)| \le \sup\{|f_n(x) - f(x)|\} = ||f_n - f||_\infty \to 0$ and so pointwise convergence by squeeze theorem

dini's theorem

$\{f_n\}, f : [a,b] \to \mathbb{R}$ (it's compact) with $f_n \le f_{n+1}$ for all n and $f_n \to f$ pointwise, then $f_n \to f$ uniformally.

proof

$g_n = f - f_n$ and then $g_n \le g_{n+1}$ and $0 \le g_n \le g_1$ for all n. and $g_n \to 0$ pointwise. if g_n converges to 0 uniformally then f_n converges to f uniformally.

$||g_n - 0||_\infty = ||f - f_n||\infty$
to be continued