Graph Theory Homework

Jon Allen

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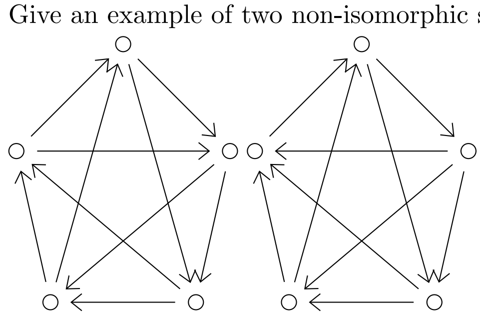
The assignment this week is 4.2 1,7 and 4.3 1, 3, 7.

The only theoretical problem is 4.3 7a. The hardest part of that problem will be succinct writing, as it is possible to ramble easily. Just be sure to label things, and that will make the proof flow (ha!).

For the drawing questions, don't worry so much about prettiness; building tournaments from score sequences can be an ugly affair. Just do your best.

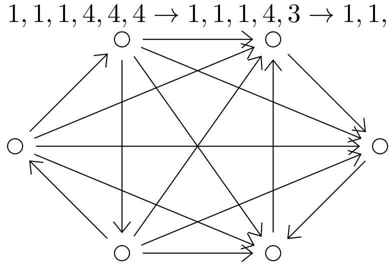
4.2 1. Give an example of two non-isomorphic strong tournaments of order 5.



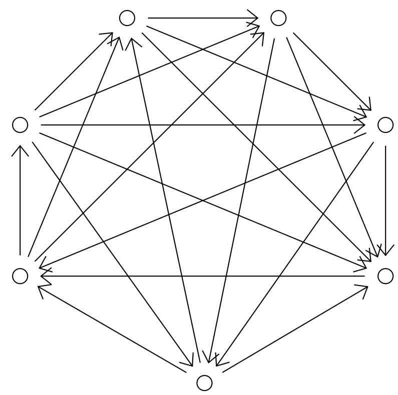
7. Which of the following sequences are score sequences of tournaments? For each sequence that is a score sequence, construct a tournament having the given sequence as a score sequence.

   (a) $0, 1, 1, 4, 4$
   The last two vertices have an outgoing degree of 4. Since there are only 5vertices, then they each must have an outgoing edge to every other vertex, including one another. This violates the definition of a tournament.

   (b) $1, 1, 1, 4, 4, 4 \rightarrow 1, 1, 1, 4, 3 \rightarrow 1, 1, 1, 3 \rightarrow 1, 1, 1$ This is a tournament:



   (c) $1, 3, 3, 3, 3, 3, 5 \rightarrow 1, 3, 3, 3, 3, 2 \rightarrow 1, 2, 3, 2, 2 \rightarrow 1, 2, 2, 1 \rightarrow 1, 1, 1$
   So it is a tournament



(d) $2,3,3,4,4,4,4,5 \to 2,3,3,4,4,3,3 \to 2,3,3,3,2,3 \to 2,2,3,2,2 \to 2,2,2,1 \to 1,2,1 \to 1,1$ which is not a tournament

4.3 1. Let N be the network with source u and sink v shown in Figure 4.24, where each arc is labeled with its capacity. A function f is defined on the arcs of N as follows:

$$\begin{array}{llll} f(u,s)=3 & f(s,t)=3 & f(t,v)=4 & f(u,x)=3 \\ f(x,y)=3 & f(y,v)=1 & f(x,t)=1 & f(w,u)=0 \\ f(y,w)=2 & f(w,v)=2. & & \end{array}$$

Is f a flow?

No it is not, $N^+(s)=1$, namely the st arc. And so $f(s,t)=2\neq 3$

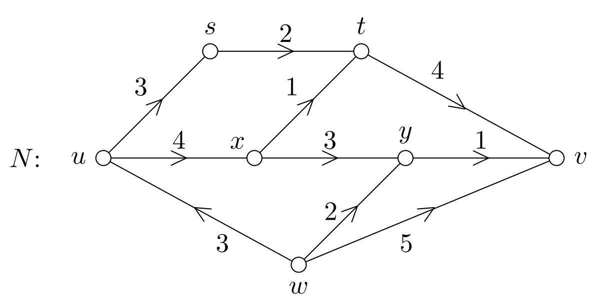


Figure 4.24: The network N in Exercise 1

3. For the network N shown in Figure 4.26 with source u and sink v, each arc has unlimited capacity. A flow f in the network is indicated by the labels on the arcs.

(a) Determine the missing flows a, b and c.

$$\begin{aligned} f^+(x)=7=f^-(x)&=3+3+c & \therefore c=1 \\ f^+(u)=4+1-a=f^-(v)&=6-1-1 & \therefore a=1 \\ f^+(w)=1+3=f^-(w)&=2+b & \therefore b=2 \end{aligned}$$

(b) Determine $\mathrm{val}(f)$

$\mathrm{val}(f)=f^+(u)=f^-(v)=4$

N:

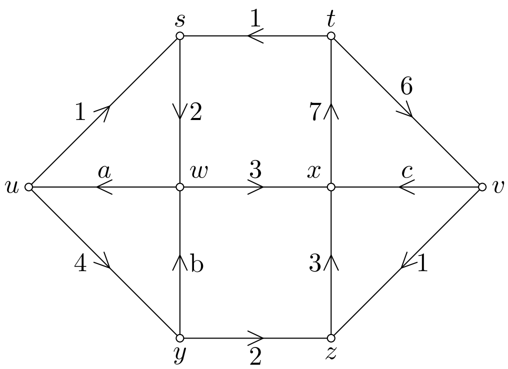
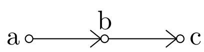


Figure 4.26: The network N in Exercise 3

7. Let u and v be two vertices of a digraph D and let A be a set of arcs of D such that every $u - v$ path in D contains at least one arc of A.

   (a) Show that there exists a set of arcs of the form $[X, \overline{X}]$ where $u \in X$ and $v \in \overline{X}$ and $[X, \overline{X}] \subseteq A$.
   Let $\mathcal{U}$ be the set of all maximal paths P in D whose initial vertex is u and such that no arc of P belongs to A. Consider the set $X = \bigcap_{P \in \mathcal{U}} V(P)$. Note that $\mathcal{U}$ are maximal paths, and so the subpath of any $u - v$ path in $\mathcal{U}$ will be in X until precisely the moment it reaches an arc in A. Because all $u - v$ paths contain an arc in A we know that $v \notin X$ or that $v \in \overline{X}$. And because all paths in $\mathcal{U}$ are maximal the only arcs in $[X, \overline{X}]$ are also in A.

   (b) Show that $[X, \overline{X}]$ may be a proper subset of A.

   Let $X = \{a\}$ and $A = \{(a, b), (b, c)\}$. Then $[X, \overline{X}] = \{(a, b)\} \subset A$