Notes

4 fevrier, 2015

reading

before friday, read 8.1 (quiz may be on this)

quiz

1

a function f is measurable if $\{x : f(x) \geq a\}$ is measurable $\forall a \in \mathbb{R}$. that is $f^{-1}((a, \infty))$ is measurable for all a

N non measurable. $\chi_n^{-1}((\frac{1}{2}, \infty)) = N$ is the standard non measurable example. notice that $\chi_N^{-1}(\alpha, \infty) = \emptyset, \alpha \geq 1$ but $\chi_N^{-1}(\alpha, \infty) = \mathbb{R}, \alpha < 0$

2

$\pi\chi_{E_1} + e\chi_{E_2} + 1.1\chi_{E_3}$ with $E_1 = \mathbb{Q} \cap [0, 1], E_2 = (1, 3], E_3 = (0, 1)$ and so integral is $\pi \cdot 0 + 2e + 1.1$

proposition

f is bounded on measurable E with $m * (E) < \infty$ then f is measurable if and only if $\inf \int_E \psi; \mathrm{d}m = \sup \int_E \varphi; \mathrm{d}m$.

the first is $\{f \leq \psi$ with ψ simple$\}$ and second is $\{\varphi \leq f$ with φ simple$\}$

proof

backwards direction is a technical mess. four lemmas etc, not doing it

forwards:

let f be bounded by $[-M, M]$. let $E_k = \{x : \frac{kM}{n} \geq f(x) > \frac{(k-1)M}{n}\}$ with $-n \leq k \leq n$. So we are chopping our range up into $2n$ pieces, and throwing the slices into disjoint E sets. but just when $f(x)$ is in thee slice, not when it's above the slice

note that $E_k \cap E_j = \emptyset$ if $k \neq j$. also note that $E = \bigcup\limits_{k=-n}^{n} E_k$

and so $m * \bigcup\limits_{k=-n}^{n} E_k = \sum\limits_{k=-n}^{n} m * E_k$

1. $\psi_n(x) = \frac{M}{n} \sum\limits_{k=-n}^{n} k\chi_{E_k}$
2. $\varphi_n(x) = \frac{M}{n} \sum\limits_{k=-n}^{n} (k-1)\chi_{E_k}$

Notice that ψ_n and φ_n are simple.

ψ is making riemann like blocks above $f(x)$ and φ is doing so under the curve.

notice that $\varphi_n \leq f \leq \psi_n$ for any n

and so $\inf \int_E \psi ;\mathrm{d}m \leq \int_E \psi_n ;\mathrm{d}m = \frac{M}{n} \sum_{k=-n}^{n} km * E_k \forall n$

$\{\psi : \psi \geq f, \psi \text{ simple}\}$

$\sup \int_E \varphi ;\mathrm{d}m \geq \int_E \varphi_n ;\mathrm{d}m = \frac{M}{n} \sum_{k=-n}^{n} (k-1)m * E_k$ so $0 \leq \inf - \sup \leq \int_E \psi_n - \int_E \varphi_n = \frac{M}{n} \sum_{k=-n}^{n} km * E_k - (k-1)m *$
$\frac{M}{n} \sum_{k=-n}^{n} m * E_k = \frac{M}{n} m * (E)$ as $n \to \infty$ this goes to zero.