Notes

March 3, 2014

lesson 13 laplace transform

sample problem

Get a transform $U(s,t)$ for a solution (last page of lesson 13) and text gives $u(x,t)$ and says "follows from tables". Turns out it's not in the texts, tables. Not available in Mathematica either.

We will take a couple of days on approaches to finding inverse laplace transforms. Started talking about this on the 28th.

elementary inversion based

Find laplace transforms for t^p and $t^n e^{at} \begin{cases} \cos(bt) \\ \sin(bt) \end{cases}$ that occur in circuit analysis

the general answer is

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} F(x) \, \mathrm{d}s$$

this is the general inversion formula for laplace transforms. Requires essentialuse of complex analysis. the typical result is an infinite series of an integral representation for $f(t)$

approach 1

expand $F(s)$ in reciprocal powers and invert termwise. $\frac{F(p+1)}{s^{p+1}} \to t^p$

example

$$\begin{aligned} F(s) &= (s^2+1)^{-1/2} \text{ in text table} \\ \text{Find } f(t) & \\ \text{Found } f(t) &= \sum_{n=0}^{\infty} \frac{(-t^2/4)^n}{(n!)^2} \end{aligned}$$

approach 1 give series for $f(t)$.

note this happens to be $J_0(t)$

approach 2

using $F(s)$, try to find an equation (typically differential equatin) for $f(t)$.

example

$$
\begin{aligned}
F(s) &= (s^2+1)^{-1/2} && = \mathcal{L}\{f(t)\} \\
F'(s) &= -\frac{1}{2}(s^2+1)^{-3/2}(2s) && = \mathcal{L}\{-t\cdot f(t)\} \\
&= -\frac{1}{s^2+1}(s^2+1)^{-1/2} \\
&= -\frac{s}{s^2+1}F(s) \\
(s^2+1)F'(s)+sF(s) &= 0 \\
sF(s)-f(0) & && = \mathcal{L}\{f'(t)\} \\
F'(s) & && = \mathcal{L}\{-t\cdot f(t)\} \\
s^2F'(s)-s(-t\cdot f(t))_{t=0}-\left(\frac{\mathrm{d}}{\mathrm{d}t}(-t\cdot f(t))\right) & && = \mathcal{L}\left\{\frac{\mathrm{d}^2}{\mathrm{d}t^2}(-t\cdot f(t)\right\} \\
s^2G(s)-sg(0)-g'(0) & && = \mathcal{L}\{g''(t)\} \\
\mathcal{L}\{f'(t)-tf(t)-\frac{\mathrm{d}^2}{\mathrm{d}t^2}(-tf(t))\} & \\
+f(0)\underbrace{-s(tf(t))_{t=0}}_{=0}-\frac{\mathrm{d}}{\mathrm{d}t}(tf(t))_{t=0}-(f(t)+tf'(t))_{t=0} &= 0 \\
-tf'(t) \text{ as } t\to 0 &
\end{aligned}
$$

we assume $f(0)$ exists, and that $\lim_{t\to 0^+} tf'(t)=0$. *AFTER* solving for $f(t)$ wee can check that these conditions hold.

$$
\begin{aligned}
\mathcal{L}\{f'(t)-tf(t)-\frac{\mathrm{d}^2}{\mathrm{d}t^2}(tf(t))-(tf''(t)+2f'(t)\} &= 0 \\
tf''(t)+f'(t)+tf(t) &= 0
\end{aligned}
$$

bessel de

$$
\begin{aligned}
z^2\frac{\mathrm{d}^2w}{\mathrm{d}z^2}+z\frac{\mathrm{d}w}{\mathrm{d}z}+(z^2+\mu^2)w &= 0 \\
\mu &= \text{order} \\
J_\mu(z) & \text{ Bessel function of first kind (order } \mu) \\
Y_\mu(z) & \text{ Bessel function of second kind (order } \mu)
\end{aligned}
$$

note dlmf.nist.gov is reference for standard functions.