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HW 11
Lesson 7 problem 1. Solve the IC. Explain how orthogonality is used.

$$\begin{array}{llll} PDE & u_t = u_{xx} & 0 < x < 1 & 0 < t < \infty \\ BCs & \begin{cases} u(0,t) = 0 \\ u_x(1,t) = 0 \end{cases} & & 0 < t < \infty \\ IC & u(x,0) = x & 0 \le x \le 1 & \end{array}$$

$$\lambda_n = \frac{2n-1}{2}\pi \qquad u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n{}^2 t} \sin(\lambda_n x)$$

$$u(x,0) = x = \sum_{n=1}^{\infty} a_n \sin(\lambda_n x) \qquad \int_0^1 \xi \sin(\lambda_m \xi)\,\mathrm{d}\xi = \sum_{n=1}^{\infty} a_n \int_0^1 \sin(\lambda_n \xi)\sin(\lambda_m \xi)\,\mathrm{d}\xi$$

Because $\{\sin(\lambda_i x)\}_{0\le i\le n}$ are orthogonal functions we can convert the above equation into the following.

$$\begin{aligned} \int_0^1 \xi \sin(\lambda_m \xi)\,\mathrm{d}\xi &= a_m \int_0^1 \sin(\lambda_m \xi)^2\,\mathrm{d}\xi \\ &= a_m \cdot -\frac{\sin(2\lambda_m) - 2\lambda_m}{4\lambda_m} \\ &= a_m \frac{\lambda_m - \sin(\lambda_m)\cos(\lambda_m)}{2\lambda_m} \end{aligned}$$

Recall that we discovered in HW 10 that $\cos(\lambda) = 0$

$$\begin{aligned} &= \frac{a_m}{2} \\ a_n &= 2\int_0^1 \xi \sin(\lambda_n \xi)\,\mathrm{d}\xi \\ &= 2\left[\frac{\sin(\lambda_n x) - \lambda_n x\cos(\lambda_n x)}{\lambda_n{}^2}\right]_0^1 \\ &= 2\left[\frac{\sin(\lambda_n 1) - \lambda_n 1\cos(\lambda_n 1)}{\lambda_n{}^2} - \frac{\sin(\lambda_n 0) - \lambda_n 0\cos(\lambda_n 0)}{\lambda_n{}^2}\right] \\ &= 2\left[\frac{\sin(\lambda_n)}{\lambda_n{}^2} - \frac{0}{\lambda_n{}^2}\right] = 2\left[\frac{\sin(\lambda_n)}{\lambda_n{}^2}\right] \\ &= 2\left[\frac{\sin\left(\frac{2n-1}{2}\pi\right)}{\left(\frac{2n-1}{2}\pi\right)^2}\right] \\ &= \frac{8\sin\left(\frac{2n-1}{2}\pi\right)}{(2n-1)^2\pi^2} \\ &= -1^{(n+1)}\frac{8}{(2n-1)^2\pi^2} \\ u(x,t) &= \sum_{n=1}^{\infty} -1^{(n+1)}\frac{8}{(2n-1)^2\pi^2} e^{-\lambda_n{}^2 t}\sin(\lambda_n x) \\ &= \sum_{n=1}^{\infty} -1^{(n+1)}\frac{8}{(2n-1)^2\pi^2} e^{-\left(\frac{2n-1}{2}\pi\right)^2 t}\sin\left[\left(\frac{2n-1}{2}\pi\right)x\right] \end{aligned}$$