Coordinates and Change of Basis

Definition 1. Let $B = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ be a basis for $\mathbb{R}^n$. If $\mathbf{v} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + ... + x_n\mathbf{v}_n$, then $x_1, x_2, ..., x_n$ are called the coordinates of $\mathbf{v}$ with respect to B. In this case, we write

$$[\mathbf{v}]_B = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Examples 2. Let $S_3 = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard basis in $\mathbb{R}^3$ and let $B = \{(1,1,0), (1,0,1), (0,1,1)\}$ For $\mathbf{v} = (1,2,3)$, we compute $[\mathbf{v}]_{S_3}$ and $[\mathbf{v}]_B$. It is clear that $\mathbf{v} = \mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3$ and so

$$[\mathbf{v}]_{S_3} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

To write

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

we must solve

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

We find that

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}^{-1} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

That is,

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

and so

$$[\mathbf{v}]_B = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

Theorem 3. Let $B = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ and $C = \{\mathbf{w}_1, \mathbf{w}_2 ..., \mathbf{w}_n\}$ be bases for $\mathbb{R}^n$. If

$$\begin{aligned}
\mathbf{v}_1 &= c_{11}\mathbf{w}_1 + c_{21}\mathbf{w}_2 + ... + c_{n1}\mathbf{w}_n \\
\mathbf{v}_2 &= c_{12}\mathbf{w}_1 + c_{22}\mathbf{w}_2 + ... + c_{n2}\mathbf{w}_n \\
&\vdots \\
\mathbf{v}_n &= c_{1n}\mathbf{w}_1 + c_{2n}\mathbf{w}_2 + ... + c_{nn}\mathbf{w}_n
\end{aligned}$$

and

$$Q = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix},$$

then

$$Q[\mathbf{x}]_B = [\mathbf{x}]_C.$$

Moreover, the matrix Q is invertible.

Proof. Choose any $\mathbf{x} \in \mathbb{R}^n$. Since $B = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ is a basis of $\mathbb{R}^n$, we can write

$$\mathbf{x} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + ... + x_n\mathbf{v}_n.$$

Hence,

$$[\mathbf{x}]_B = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Then

$$Q[\mathbf{x}]_B = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} c_{11}x_1 + c_{12}x_2 + ...c_{1n}x_n \\ c_{21}x_1 + c_{22}x_2 + ...c_{2n}x_n \\ \vdots \\ c_{n1}x_1 + c_{n2}x_2 + ...c_{nn}x_n \end{bmatrix}.$$

On the other hand,

$$\begin{aligned}
\mathbf{x} &= x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + ... + x_n\mathbf{v}_n \\
&= x_1 \sum_{i=1}^{n} c_{i1}\mathbf{w}_i + x_2 \sum_{i=1}^{n} c_{i2}\mathbf{w}_i + ... + x_n \sum_{i=1}^{n} c_{in}\mathbf{w}_i \\
&= \sum_{i=1}^{n} (c_{i1}x_1 + c_{i2}x_2 + ... + c_{in}x_n)\mathbf{w}_i
\end{aligned}$$

and so

$$[\mathbf{x}]_C = \begin{bmatrix} c_{11}x_1 + c_{12}x_2 + ...c_{1n}x_n \\ c_{21}x_1 + c_{22}x_2 + ...c_{2n}x_n \\ \vdots \\ c_{n1}x_1 + c_{n2}x_2 + ...c_{nn}x_n \end{bmatrix}.$$

Therefore, $Q[\mathbf{x}]_B = [\mathbf{x}]_C$ as needed. For invertibility, the same procedure as above (but writing the $\mathbf{w}$s in terms of the $\mathbf{v}$s) furnishes a matrix P such that $[\mathbf{x}]_B = P[\mathbf{x}]_C$. For every $\mathbf{x} \in \mathbb{R}^n$, we have $[\mathbf{x}]_B = P[\mathbf{x}]_C$ and so $PQ[\mathbf{x}]_B = P[\mathbf{x}]_C = [\mathbf{x}]_B$. Therefore, $PQ = I_n$ which is equivalent to the fact that $Q = P^{-1}$.

Example 4. Let $B = \{(1,1,0),(1,0,1),(0,1,1)\}$ and $C = \{(1,2,4),(-1,2,0),(2,4,0)\}$. We find the matrix Q that $Q[\mathbf{x}]_B = [\mathbf{x}]_C$ for all $\mathbf{x}$ in $\mathbb{R}^3$. To do this, we must write the basis B in terms of C. That is, we must solve

$$\begin{aligned}
c_{11}\begin{pmatrix}1\\2\\4\end{pmatrix} + c_{21}\begin{pmatrix}-1\\2\\0\end{pmatrix} + c_{31}\begin{pmatrix}2\\4\\0\end{pmatrix} &= \begin{pmatrix}1\\1\\0\end{pmatrix}\\
c_{12}\begin{pmatrix}1\\2\\4\end{pmatrix} + c_{22}\begin{pmatrix}-1\\2\\0\end{pmatrix} + c_{32}\begin{pmatrix}2\\4\\0\end{pmatrix} &= \begin{pmatrix}1\\0\\1\end{pmatrix}\\
c_{13}\begin{pmatrix}1\\2\\4\end{pmatrix} + c_{23}\begin{pmatrix}-1\\2\\0\end{pmatrix} + c_{33}\begin{pmatrix}2\\4\\0\end{pmatrix} &= \begin{pmatrix}0\\1\\1\end{pmatrix}.
\end{aligned}$$

We find that

$$Q = \begin{bmatrix} c_{11} & c_{12} & c_{13}\\ c_{21} & c_{22} & c_{23}\\ c_{31} & c_{32} & c_{33}\end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{4} & \frac{1}{4}\\ -\frac{1}{4} & -\frac{1}{2} & \frac{1}{4}\\ \frac{3}{8} & \frac{1}{8} & 0\end{bmatrix}.$$

Theorem 5. Let $B = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ be a basis for $\mathbb{R}^n$ and let $C = \{\mathbf{w}_1, \mathbf{w}_2, ..., \mathbf{w}_m\}$ be a basis for $\mathbb{R}^m$. If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, there exists a matrix $M(T,B,C) \in \mathcal{M}_{m\times n}$ such that $M(T,B,C)\,[\mathbf{x}]_B = [T(\mathbf{x})]_C$ for every $\mathbf{x} \in \mathbb{R}^n$.

Proof. Since $T(\mathbf{v}_j) \in \mathbb{R}^m$ and $C = \{\mathbf{w}_1, \mathbf{w}_2, ..., \mathbf{w}_m\}$ is a basis, we can write

$$\begin{aligned}
T(\mathbf{v}_1) &= a_{11}\mathbf{w}_1 + a_{21}\mathbf{w}_2 + ... + a_{m1}\mathbf{w}_m\\
T(\mathbf{v}_2) &= a_{12}\mathbf{w}_1 + a_{22}\mathbf{w}_2 + ... + a_{m2}\mathbf{w}_m\\
&\vdots\\
T(\mathbf{v}_n) &= a_{1n}\mathbf{w}_1 + a_{2n}\mathbf{w}_2 + ... + a_{mn}\mathbf{w}_m.
\end{aligned}$$

It follows that

$$[T(\mathbf{v}_1)]_C = \begin{bmatrix} a_{11}\\ a_{21}\\ \vdots\\ a_{m1}\end{bmatrix}, [T(\mathbf{v}_2)]_C = \begin{bmatrix} a_{12}\\ a_{22}\\ \vdots\\ a_{m2}\end{bmatrix}, ..., [T(\mathbf{v}_m)]_C = \begin{bmatrix} a_{1n}\\ a_{2n}\\ \vdots\\ a_{mn}\end{bmatrix}$$

Set

$$M(T,B,C) = \left[\ [T(\mathbf{v}_1)]_C \mid \ [T(\mathbf{v}_2)]_C \mid \ \cdots \mid \ [T(\mathbf{v}_n)]_C\ \right].$$

and write $\mathbf{x} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + ... + x_n\mathbf{v}_n$. Since

$$T(\mathbf{x}) = T(\sum_{j=1}^n x_j\mathbf{v}_j) = \sum_{j=1}^n x_j T(\mathbf{v}_j) = \sum_{j=1}^n x_j(\sum_{k=1}^m a_{kj}\mathbf{w}_k) = \sum_{k=1}^m(\sum_{j=1}^n x_j a_{kj})\mathbf{w}_k,$$

we have that

$$[T(\mathbf{x})]_C = \begin{bmatrix} \sum_{j=1}^n x_j a_{1j} \\ \sum_{j=1}^n x_j a_{2j} \\ \vdots \\ \sum_{j=1}^n x_j a_{mj} \end{bmatrix} = \sum_{j=1}^n x_j \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}.$$

On the other hand,

$$M(T,B,C)\,[\mathbf{x}]_B = M(T,B,C)\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \sum_{j=1}^n x_j\,[T(\mathbf{v}_j)]_C = \sum_{j=1}^n x_j \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}.$$

Therefore, $M(T,B,C)\,[\mathbf{x}]_B = [T(\mathbf{x})]_C$ as needed.

Definition 6. Let $B = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ be a basis for $\mathbb{R}^n$ and let $C = \{\mathbf{w}_1, \mathbf{w}_2, ..., \mathbf{w}_m\}$ be a basis for $\mathbb{R}^m$. If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then $M(T,B,C)$ is called the matrix of the transformation T with respect to the bases B, C.

Example 7. Let $B = \{(1,1,0),(1,0,1),(0,1,1)\}$ be a basis of $\mathbb{R}^3$ and let $C = \{(1,1),(-2,1)\}$ be a basis of $\mathbb{R}^2$. If $T : \mathbb{R}^3 \to \mathbb{R}^2$ is given by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x+3y+z \\ 3x+3y+z \end{pmatrix},$$

we compute $M(T,B,C)$.

$$T\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix} = c_{11}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_{21}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

and so

$$\begin{pmatrix} c_{11} \\ c_{21} \end{pmatrix} = \begin{pmatrix} \frac{17}{3} \\ \frac{1}{3} \end{pmatrix}.$$

Similarly

$$T\begin{pmatrix}1\\0\\1\end{pmatrix}=\begin{pmatrix}3\\4\end{pmatrix}=c_{12}\begin{pmatrix}1\\1\end{pmatrix}+c_{22}\begin{pmatrix}-2\\1\end{pmatrix}$$

and so

$$\begin{pmatrix}c_{12}\\c_{22}\end{pmatrix}=\begin{pmatrix}\frac{11}{3}\\\frac{1}{3}\end{pmatrix}.$$

Finally

$$T\begin{pmatrix}0\\1\\1\end{pmatrix}=\begin{pmatrix}4\\4\end{pmatrix}=c_{13}\begin{pmatrix}1\\1\end{pmatrix}+c_{23}\begin{pmatrix}-2\\1\end{pmatrix}$$

and so

$$\begin{pmatrix}c_{13}\\c_{23}\end{pmatrix}=\begin{pmatrix}4\\0\end{pmatrix}.$$

Therefore,

$$M(T,B,C)=\begin{bmatrix}\frac{17}{3}&\frac{11}{3}&4\\\frac{1}{3}&\frac{1}{3}&0\end{bmatrix}=\frac{1}{3}\begin{bmatrix}17&11&12\\1&1&0\end{bmatrix}$$

Theorem 8. Let $B=\{\mathbf{v}_1,\mathbf{v}_2,...,\mathbf{v}_n\}$ and $C=\{\mathbf{w}_1,\mathbf{w}_2,...,\mathbf{w}_n\}$ be bases for $\mathbb{R}^n$. If $Q\in\mathcal{M}_n$ is the matrix such that $Q[\mathbf{x}]_B=[\mathbf{x}]_C$, then $Q=M(1_{\mathbb{R}^n},B,C)$.

Proof. We have $M(1_{\mathbb{R}^n},B,C)\,[\mathbf{x}]_B=[1_{\mathbb{R}^n}(\mathbf{x})]_C=[\mathbf{x}]_C=Q[\mathbf{x}]_B$ for all $\mathbf{x}\in\mathbb{R}^n$. It follows that

Theorem 9. Let B,C,D be bases for $\mathbb{R}^n,\mathbb{R}^p,\mathbb{R}^m$ (resp) and let $T:\mathbb{R}^n\to\mathbb{R}^p$ and $S:\mathbb{R}^p\to\mathbb{R}^m$ be linear transformations. Then $M(S\circ T,B,D)=M(S,C,D)M(T,B,C)$.

Proof. Choose any $\mathbf{x}\in\mathbb{R}^n$ and consider the coordinate vector $[\mathbf{x}]_B$. Then

$$\begin{aligned}&M(S\circ T,B,D)\,[\mathbf{x}]_B\\=\;&[(S\circ T)(\mathbf{x})]_D\\=\;&[S(T(\mathbf{x}))]_D\\=\;&M(S,C,D)\,[T(\mathbf{x})]_C\\=\;&M(S,C,D)M(T,B,C)\,[\mathbf{x}]_B\,.\end{aligned}$$

It follows that $M(S\circ T,B,D)=M(S,C,D)M(S,B,C)$.

Example 10. Let $B=\{(1,1,0),(1,0,1),(0,1,1)\}$ be a basis of $\mathbb{R}^3$ and let $C=\{(1,1),(-2,1)\}$ be a basis of $\mathbb{R}^2$. If $T:\mathbb{R}^3\to\mathbb{R}^2$ is given by

$$T\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}2x+3y+z\\3x+3y+z\end{pmatrix},$$

we compute $M(T,B,C)$ using a different method from the one in Example 7. Let E_3 and E_2 be the standard bases of $\mathbb{R}^3$ and $\mathbb{R}^2$ (resp). We have

$$\begin{aligned}
& M(1_{\mathbb{R}^2}, E_2, C)M(T, E_3, E_2) \\
= \ & M(1_{\mathbb{R}^2} \circ T, E_3, C) \\
= \ & M(T, E_3, C) \\
= \ & M(T \circ 1_{\mathbb{R}^3}, E_3, C) \\
= \ & M(T, B, C)M(1_{\mathbb{R}^3}, E_3, B).
\end{aligned}$$

It follows that

$$M(T,B,C) = M(1_{\mathbb{R}^2}, E_2, C)M(T, E_3, E_2)M(1_{\mathbb{R}^3}, E_3, C)^{-1}.$$

It is easy to check that

$$M(1_{\mathbb{R}^2}, E_2, C) = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}^{-1}$$

and

$$M(T, E_3, E_2) = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \end{bmatrix}$$

and

$$M(1_{\mathbb{R}^3}, E_3, C)^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Therefore,

$$M(T,B,C) = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{17}{3} & \frac{11}{3} & 4 \\ \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix}$$

as expected. Here is a picture

$$\begin{tikzcd}
\mathbb{R}^3 \arrow[r, "{M(T,E_3,E_2)}"] \arrow[d, "{M(1_{\mathbb{R}^3},E_3,C)}"'] & \mathbb{R}^2 \arrow[d, "{M(1_{\mathbb{R}^2},E_2,C)}"] \\
\mathbb{R}^3 \arrow[r, "{M(T,B,C)}"'] & \mathbb{R}^2
\end{tikzcd}$$

Theorem 11. Let B_1, B_2 be bases for $\mathbb{R}^n$ and let C_1, C_2 be bases for $\mathbb{R}^m$. If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then $\operatorname{rank}(M(T, B_1, C_1)) = \operatorname{rank}(M(T, B_2, C_2))$.

Proof. Exercise.

Definition 12. Let B be any basis for $\mathbb{R}^n$ and let C be any basis for $\mathbb{R}^m$. If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, we define the rank of the transformation T to be $\operatorname{rank}(M(T, B, C))$. By Theorem 11, this number is well-defined. That is, $\operatorname{rank}(M(T, B, C))$ does not depend on the choice of bases B, C.

Definition 13.

Exercises

1. Let

$$B = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \end{pmatrix} \right\}\} \text{ and } C = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

be bases of $\mathbb{R}^2$ and $\mathbb{R}^3$ (resp). Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 2y \\ -x \\ y \end{pmatrix}$$

and let

$$\mathbf{x} = \begin{pmatrix} -7 \\ 7 \end{pmatrix}$$

(a) Find the image $T(\mathbf{x})$ of the vector $\mathbf{x}$ under the action of T.
(b) Find the change of basis matrices $P = M(1_{\mathbb{R}^2}, S_2, B)$ and $Q = M(1_{\mathbb{R}^3}, S_3, C)$.
(c) Use part (b) to compute $[\mathbf{x}]_B$ and $[T(\mathbf{x})]_C$.
(d Find $M(T, B, C)$ using the method of Example 7.
(e) Find $M(T, B, C)$ using the method of Example 10.
(f) Check your answer in part (c) by verifying that $M(T, B, C)\,[\mathbf{x}]_B = [T(\mathbf{x})]_C$.

2. Let

$$B = \left\{ \begin{pmatrix} -3 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ -2 \end{pmatrix} \right\} \text{ and } C = \left\{ \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \end{pmatrix} \right\}$$

be bases of $\mathbb{R}^2$. If

$$M(T, B, B) = \begin{bmatrix} -2 & 7 \\ -3 & 7 \end{bmatrix},$$

find $M(T, B, B)$.

3. Prove Theorem 11. Let B_1, B_2 be bases for $\mathbb{R}^n$ and let C_1, C_2 be bases for $\mathbb{R}^m$. If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then $\text{rank}(M(T, B_1, C_1)) = \text{rank}(M(T, B_2, C_2))$.

4. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and suppose that $V \leq \mathbb{R}^n$ is invariant under T (that is, $T(V) \subseteq V$). Prove that there exists a basis B such that

$$M(T, B, B) = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$$

where A is a $\dim(V) \times \dim(V)$ matrix.

5. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation of $\text{rank}(T) = r$. Prove that there exist bases B, C of $\mathbb{R}^n, \mathbb{R}^m$ (resp) such that

$$M(T, B, C) = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

Do you see how this establishes the Rank Nullity Theorem?

Similarity

Definition. Define a relation $\sim$ on $\mathcal{M}_n$ given by $A \sim B$ if and only if there exists an invertible matrix $P \in \mathcal{M}_n$ such that $B = PAP^{-1}$. If $A \sim B$, we say that A is similar to B. This relationis a special case of Example 10 where $B = PAQ^{-1}$.

6. Prove that $\sim$ is an equivalence relation on $\mathcal{M}_n$.

7. Prove that $A \sim A'$ if and only if A and A' represent the same linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$. That is, $A \sim A'$ if and only if $A = M(T, B, B)$ and $A' = M(T, B', B')$ for some bases B, B' of $\mathbb{R}^n$.

8. Prove that if $A \sim B$, then $A^{-1} \sim B^{-1}$ and $A^{\text{T}} \sim B^{\text{T}}$.