Notes

2 mars, 2015

when does taylor series converge?
better question is when is $f(x) = \sum f^{(n)}(a)/n! \cdot (x-a)^n$

taylor's thm

$f \in C^\infty[a,b]$ and $f^{(n+1)}$ is defined on $[A,B]$ with $|f^{(n+1)}(x)| \leq M$ for $x \in [A,B]$ then $R_n(x) = f(x) - \sum \frac{f^k}{k!}(x-a)^k$ satisfies $|R_n(x)| \leq M|x-a|^{n+1}/(n+1)!$

1. $f(x) = \lim \sum \frac{f^{(k)}}{k!}(x-a)^k$ if and only if $\lim R_n(x) = 0$

   the basic issue is that M needs to not get too big to fast

2. if $f \in C^\infty[A,B]$ then these hypotheses happen automatically (it's infinitely differentiable), although there is no guarantee that the taylor series converges to f.

we want to use induction (what are we inducting on?)
we will show that $|R_n^{(n-k)}(x)| \leq \frac{M|x-a|^{k+1}}{(k+1)!}$.
base case is $k=0$. $R_n(x) = f(x) - \sum \frac{f^{(k)}}{k!}(x-a)^k = f^{(n)} - \frac{n!}{n!}f^{(n)}x$ and so $|R_n(x)| = |f^{(n)}(x) - f^{(n)}(a)|$. by MVT we know $M|x-a| \geq f^{(n+1)}(c)|x-a| = |R_n(x)$. So base case is done
assume $|R_n^{(n-k)}(x)| \leq M|x-a|^{k+1}/(k+1)!$. consider $|R_n^{(n-(k+1))}(x)| = |R_n^{(n-(k+1))}(a) - \int_a^x R^{(n-k)}(t)\,\mathrm{d}t|$

example

$f(x) = \sin x$ and $a = \frac{\pi}{2}$.
$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(\frac{\pi}{2}(x-\frac{\pi}{2})^k}{k!}$ and then we have $P_n = \sum_{k=0}^{n}(-1)^k(x-\frac{\pi}{2})^{2k}/(2k)!$. Now $M_n = 1$ (it is bounded by 1). Now then $R_n(x) \leq \frac{1 \cdot |x-a|^{n+1}}{(n+1)!}$ and because factorials are bigger than powers, the limit is $R = 0$.
and so the power series gives the same value as the function all the time.
$f(x) = \log x$ (natural log). taylor series at 1. note that zero is a problem. one is nice because it's symmetric and easy to compute.
$f'(x) = \frac{1}{x}, f''(x) = -\frac{1}{x^2}, f'''(x) = \frac{2}{x^3}, \ldots$
$P_n = \sum_{k=1}^{n}(-1)^{k+1}\frac{(k-1)!}{k!}(x-1)^k$ now use ratio test
$\frac{\frac{(-1)^{k+2}}{k+1}}{\frac{(-1)^{k+1}}{k}} \to$