Notes

February 21, 2014

last time, we derived Pascal's equation:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

combinatorial proof:

$\binom{n}{k}$ counts the number of k-elt subsets S of $[n]$. 2 cases.

case 1

$n \in S$ then the remaining elts of S are chosen from $[n-1]$ in $\binom{n-1}{k-1}$ ways

case 2

$n \notin S$ then S is made up of k-elts from $[n-1]$. There are $\binom{n-1}{k}$ such subsets.

Therefore $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ $\square$

fibanacci in pascal's triangle

neat

combinatorial proof of binomial theorem

$\overbrace{(x+y)(x+y)\ldots(x+y)}^{n\text{-factors}} = (x+1)^n$. Each term in the sum arises from picking x or y in each of the n factors. The coefficients of $x^k y^{n-k}$ counts the number of ways to pick k x's. This is counted by $\binom{n}{k}$

moving on

notice if you set $x - y = 1$, you obtain:

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

if you set $y = 1$ you get

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

wksht 6

$x = 3, y = 1$ in binomial theorem.

$$(3+1)^{10} = \sum_{k=0}^{10} \binom{10}{k} 3^k 1^{10-k} \qquad = 4^{10}$$

wksht 7

wksht 8

Vandermonde convolution:

$$\begin{aligned} &000\dots0 && m \text{ things} \\ &000\dots0 && n \text{ things} \end{aligned}$$

choose k things

wksht 9

calculus proof

$$\begin{aligned} \sum_{k=0}^{n} k \binom{n}{k} x^{k-1} &= \frac{\mathrm{d}}{\mathrm{d}x} \sum_{k=0}^{n} \binom{n}{k} x^k \\ &= \frac{\mathrm{d}}{\mathrm{d}x} (1+n)^n \\ &= n(1+x)^{n-1} \end{aligned}$$

set $x = 1$ $\sum_{k=0}^{n} k\binom{n}{k} = n(1+1)^{n-1} = n2^{n-1}$

question

What does this sum equal: $\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2$

$$\begin{aligned} \sum_{k=0}^{n} \binom{n}{k}^2 &= \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} \\ &= \binom{2n}{n} \end{aligned}$$

sidebar

question:

is there any meaning to something like $\binom{12/7}{3}$?

answer

for and $n \in \mathbb{R}, k \in \mathbb{Z}$ let $\binom{n}{k} = \begin{cases} \dfrac{\overbrace{n(n-1)(n-2)\dots(n-k+1)}^{k \text{ factors}}}{k!} & \text{if } k \geq 1 \\ 1 & \text{if } k = 0 \\ 0 & \text{if } k \leq -1 \end{cases}$

$$\binom{12/7}{3} = \frac{12/7(12/7-1)(12/7-2)}{3!} = \frac{-20}{7^3}$$

Is it a good definition? yes. e.g. it is true that

$$\binom{12/7}{3} = \binom{5/7}{2} + \binom{5/7}{3}$$

iterate pascal's identity:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$
$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} + \binom{n-2}{k-2}$$