Notes

January 23, 2015

E is measurable if ...

irrational numbers have a measure because rationals have measure 0 and the complement of a measurable set is 0

today

explain why any open interval is measurable

if E_1 and E_2 are disjoint and measurable then $m*(A \cap (E_1 \cup E_2) = m*(A \cap E_1) + m*(A \cap E_2)$ for any A

$E_1, E_2, E_3, \dots, E_n$ disjoint and measurable then $m*(A \cap (\bigcup\limits_{i=1}^{n} E_i)) = \sum\limits_{i=1}^{n} m*(A \cap E_i)$

theorem

if $\{E_i\}_{i=1}^{\infty}$ are measurable, countable then

1. $\bigcup\limits_{i=1}^{\infty}$ measurable
2. $\bigcap\limits_{i=1}^{\infty}$ measurable
3. $m*(\bigcup\limits_{i=1}^{\infty} E_i) \leq \sum\limits_{i=1}^{\infty} m*(E_i)$
4. if $E_i \cap E_j = \emptyset$ for all i, j

   $m*(\bigcup\limits_{i=1}^{\infty} E_i) = \sum\limits_{i=1}^{\infty} m*(E_i)$

proof

case 1

$E_i \cap E_j = \emptyset, i \neq j$

$F_n = \bigcup\limits_{i=1}^{n} E_i$

we know that F_n is measurable so $m*(A \cap F_n) = \sum\limits_{i=1}^{n} m*(A \cap E_i)$

$m*(A) = m*(A \cap F_n) + m*(A \cap F_n^C)$

for all n

$$m*(A) = m*(A \cap F_n) + m*(A \cap F_n^C)$$

$$\geq m*(A\cap F_n)+m*(A\cap(\bigcup_{i=1}^{\infty}E_i)^C)$$

$m*(A\cap F_n^C)$ contains $m*(A\cap(\bigcup_{i=1}^{\infty}E_i)^C)$

$$= \sum_{i=1}^{n} m*(A\cap E_i)+m*(A\cap(\bigcup_{i=1}^{\infty}E_i)^C)$$
$$m*(A)\geq \sum_{i=1}^{n} m*(A\cap E_i)+m*(A\cap(\bigcup_{i=1}^{\infty}E_i)^C)$$
$$\geq m*(A\cap\bigcup_{i=1}^{\infty}E_i)+m*(A\cap(\bigcup_{i=1}^{\infty}E_i)^C)$$

case 2 not disjoint
$E_1=\Omega_1$ and $E_2\setminus\Omega_1=\Omega_2$ and $E_3\setminus(\Omega_1\cup\Omega_2)=\Omega_3$
$\{E_i\}_{i=1}^{\infty}\to\{\Omega_i\}_{i=1}^{\infty}$ and $\Omega_i\cap\Omega_j=\emptyset\forall i\neq j$

1. $\bigcup\limits_{i=1}^{\infty}E_i=\bigcup\limits_{i=1}^{\infty}\Omega_i$
2. Ω_i is measurable $(A\setminus B=A\cap B^C)$
3. use case one to get $\bigcup\limits_{i=1}^{\infty}\Omega_i$ is measurable and imply that $\bigcup\limits_{i=1}^{\infty}E_i$ is measurable

1. union of countable measurable sets is measurable
2. $\{E_i\}_{i=1}^{\infty}$ with E_i measurable

   $\bigcap_{i=1}^{\infty}E_i=\left(\bigcup\limits_{i=1}^{\infty}(E_i)^C\right)^C$
3. we already knew that $m*(\bigcup\limits_{i=1}^{\infty}E_i\leq\sum\limits_{i=1}^{\infty}m*(E_i)$
4. $\{E_i\}_{i=1}^{\infty}$ disjoint and measurable

   $m*(\bigcup_{i=1}\infty E_i$

   (a) $m*(\bigcup\limits_{i=1}^{\infty}E_i\leq\sum\limits_{i=1}^{\infty}m*(E_i)$
   $m*(\bigcup\limits_{i=1}^{n}E_i=\sum\limits_{i=1}^{n}m*(E_i)$
   $m*(\bigcup_{i=1}^{\infty}E_i)\geq m*(\bigcup\limits_{i=1}^{n}E_i=\sum\limits_{i=1}^{n}m*(E_i)$
   and then
   $m*(\bigcup_{i=1}^{\infty}E_i)\geq=\sum\limits_{i=1}^{\infty}m*(E_i)$

theorem

any open set is measurable

proof

based on homework we know that (a, ∞) is measurable

1. need to show that $[b, \infty)$ is measurable for any b
   $[b, \infty) = \bigcap_{n=1}^{\infty}(b - \frac{1}{n}, \infty)$
2. $(-\infty, c)$ is measurable because it's the complement of $[c, \infty)$
3. (a, d) is measurable for any $a < d$
   $(a, d) = (a, \infty) \cap (-\infty, d)$
4. any open interval $\mathbb{O}$ is measurable
   $\mathbb{O} = \bigcup_{i=1}^{\infty}(a_i, b_i)$ Lindlfs theorem
   or any closed set is measurable

cantor set is the intersection of a countable number of closed sets

proposition

if $E_i \supseteq E_{i+1}$ in $\{E_i\}_{i=1}^{\infty}$ and $m*(E_1)$ is finite then $m*(\bigcap_{i=1}^{\infty} E_i) = \lim_{n\to\infty} m*(E_n)$
$\{n, n+1)\}_{n=1}^{\infty}$ will not work, why?
$\bigcap_{i=i}^{\infty} E_i \subseteq E_n$ for all n. $m*(\bigcap_{i=1}^{\infty} E_i \leq m*(E_n)$ for all n