Notes

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if G is a group, and $A \subseteq G$ and $B \subseteq G$ then $AB = \{ab | a \in A, b \in B\} \subseteq G$.

proposition

let G be a group, then H, K subgroups of G. Assume that $h^{-1}kh \in K$ for all $h \in H$, $k \in K$ then HK is a subgroup of G that contain s both H and K, in fact, HK is the smallest subgroup of G that contains both H and K. Assumption only important if we are not dealing with abelian groups.

proof

$a, b \in HK$. Write $a = h_1k_1, b = h_2k_2$ with $h_i \in H, k_i \in K$ then $a \cdot b = h_1k_1h_2k_2 = h_1h_2(h_2^{-1}k_1h_2)k_2 \in HK$

$a = hk, a^{-1} = (hk)^{-1} = k^{-1}h^{-1} = h^{-1}(hk^{-1}h^{-1}) \in HK$

examples

$S_3, H = \{(1), (12)\}, K = \{(1), (123), (132)\}, (12)(123) = (23) \in HK, (12)(132) = (13) \in HK$ so $HK = G$ and is therefore contained by G

$(\mathbb{Z}, +)$, $H = a\mathbb{Z}, k = b\mathbb{Z}$, let $d = (a, b)$

claim: $a\mathbb{Z} + b\mathbb{Z} = d\mathbb{Z}$. clearly $a\mathbb{Z} \subseteq d\mathbb{Z}$, $b\mathbb{Z} \subseteq d\mathbb{Z}$.

$a\mathbb{Z} + b\mathbb{Z}$ is the smallest subgroup that contains both $a\mathbb{Z}$ and $b\mathbb{Z}$. so $a\mathbb{Z} + b\mathbb{Z} \subseteq d\mathbb{Z}$.

$d = \gcd(a, b)$ so we can write $d = ma + nb$. let $\alpha \in d\mathbb{Z}$ and write $\alpha = dt, t \in \mathbb{Z}$ then $\alpha = dt = mat + nbt \in a\mathbb{Z} + b\mathbb{Z}$. so $d\mathbb{Z} \subseteq a\mathbb{Z} + b\mathbb{Z}$