Homework

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1.1

5. Let ℓ be the line given parametrically by $\mathbf{x} = (1,3) + t(-2,1), t \in \mathbb{R}$. Which of the following points lie on ℓ? Give your reasoning.

   No magic, just algebra, if we can work out a true equation it's on the line. If we work out a false equation, it's not.

   (a) $\mathbf{x} = (-1,4)$

   $$(-1,4) = (1,3) + t(-2,1) \qquad (-1-1, 4-3) = (-2,1) = t(-2,1) \qquad t = 1$$

   lies on the line

   (b) $\mathbf{x} = (7,0)$

   $$(7-1, 0-3) = (6,-3) = t(-2,1) \qquad t = -3$$

   also lies on the line

   (c) $\mathbf{x} = (6,2)$

   $$(6-1, 2-3) = (5,-1) \neq t(-2,1)$$

6. Find a parametric equation of each of the following lines:

   (a) $3x_1 + 4x_2 = 6$

   $$\begin{aligned} x_2 &= -\frac{3}{4}x_1 + \frac{6}{4} \\ (x_1, x_2) &= (0, \frac{6}{4}) + t(-3,4) \\ \mathbf{x} &= (2,0) + t(-3,4) \end{aligned}$$

   (c) the line with the slope $2/5$ that passes through $A = (3,1)$

   $$\mathbf{x} = (3,1) + t(5,2)$$

   (d) the line through $A = (-2,1)$ parallel to $\mathbf{x} = (1,4) + t(3,5)$

   $$\mathbf{x} = (-2,1) + t(3,5)$$

   (h) the line through $(1,1,0,-1)$ parallel to $\mathbf{x} = (2+t, 1-2t, 3t, 4-t)$

   $$\begin{aligned} \mathbf{x} &= (2+t, 1-2t, 3t, 4-t) \\ &= (2,1,0,4) + t(1,-2,3,-1) \\ \mathbf{x}' &= (1,1,0,-1) + t(1,-2,3,-1) \end{aligned}$$

7. Suppose $\mathbf{x} = \mathbf{x}_0 + t\mathbf{v}$ and $\mathbf{y} = \mathbf{y}_0 + s\mathbf{w}$ are two parametric representations of the same line ℓ in $\mathbb{R}$.

(a) Show that there is a scalar t_0 so that $\mathbf{y}_0 = \mathbf{x}_0 + t_0\mathbf{v}$
By definition 2.2 the line goes through $\mathbf{y}_0$ and $\mathbf{x}_0$. Because $\mathbf{y}_0 \in \ell = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \mathbf{x}_0 + t\mathbf{v}$ for some $t \in \mathbb{R}\}$ then there is some $t_0 \in \mathbb{R}$ such that $\mathbf{y}_0 = \mathbf{x} = \mathbf{x}_0 + t_0\mathbf{v}$

(b) Show that $\mathbf{v}$ and $\mathbf{w}$ are parallel.
Let us choose some point $\mathbf{z} \in \ell$ other than $\mathbf{y}_0$. Then there exists some $t_1, s_1 \in \mathbb{R}$ such that $\mathbf{y}_0 + s_1\mathbf{w} = \mathbf{z} = \mathbf{x}_0 + t_1\mathbf{v}$. We just saw that there exists some $t_0 \in \mathbb{R}$ such that $\mathbf{y}_0 = \mathbf{x}_0 + t_0\mathbf{v}$. So then letting the algebra work itself out:

$$\begin{aligned}
\mathbf{y}_0 + s_1\mathbf{w} &= \mathbf{x}_0 + t_1\mathbf{v} && \\
(\mathbf{x}_0 + t_0\mathbf{v}) + s_1\mathbf{w} &= \mathbf{x}_0 + t_1\mathbf{v} && \text{A1 and A4} \\
s_1\mathbf{w} &= t_1\mathbf{v} - t_0\mathbf{v} && \text{S1, S3, and S4} \\
\mathbf{w} &= \frac{t_1 - t_0}{s_1}\mathbf{v} &&
\end{aligned}$$

Now obviously $\frac{t_1 - t_0}{s_1} \in \mathbb{R}$ and so by definition 1.7 we know that $\mathbf{v}$ and $\mathbf{w}$ are parallel.

10. Find a parametric equation of each of the following planes:
   (a) the plane containing the point $(-1, 0, 1)$ and the line $\mathbf{x} = (1, 1, 1) + t(1, 7, -1)$

$$\begin{aligned}
(-1, 0, 1) &= (1, 1, 1) + t(1, 7, -1) + \mathbf{u} && \text{let } t = 0 \\
(-2, -1, -2) &= \mathbf{u} && \text{By A3 and Theorem 11} \\
\mathcal{P}(\mathbf{x}_0, \mathbf{u}, \mathbf{v}) &= (1, 1, 1) + t(1, 7, -1) + s(-2, 1, -2) &&
\end{aligned}$$

   (d) the plane in $\mathbb{R}^4$ containing the points $(1, 1, -1, 4), (2, 3, 0, 1)$ and $(1, 2, 2, 3)$

20. Assume that $\mathbf{u}$ and $\mathbf{v}$ are parallel vectors in $\mathbb{R}^n$. Prove that $\text{Span}(\mathbf{u}, \mathbf{v})$ is a line.

21. Suppose $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and c is a scalar. Prove that $\text{Span}(\mathbf{v} + c\mathbf{w}, \mathbf{w}) = \text{Span}(\mathbf{v}, \mathbf{w})$. (See the blue box on p. 12.)

22. Suppose the vectors $\mathbf{v}$ and $\mathbf{w}$ are both linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_k$.
   (a) Prove that for any scalar $c, c\mathbf{v}$ is a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_k$.
   (b) Prove that $\mathbf{v} + \mathbf{w}$ is a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_k$

23. Consider the line $\ell : \mathbf{x} = \mathbf{x}_0 + r\mathbf{v}(r \in \mathbb{R})$ and the plane $\mathcal{P} : \mathbf{x} = s\mathbf{u} + t\mathbf{v}(s, t \in \mathbb{R})$. Show that if ℓ and $\mathcal{P}$ intersect, then $\mathbf{x}_0 \in \mathcal{P}$

24. Consider the lines $\ell : \mathbf{x} = \mathbf{x}_0 + t\mathbf{v}$ and $m : \mathbf{x} = \mathbf{x}_1 + s\mathbf{u}$. Show that ℓ and m intersect if and only if $\mathbf{x}_0 - \mathbf{x}_1$ lies in $\text{Span}(\mathbf{u}, \mathbf{v})$.

25. Suppose $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ are nonparallel vectors. (Recall definition on p.3.)
   (a) Prove that if $s\mathbf{x} + t\mathbf{y} = \mathbf{0}$ then $s = t = 0$. (*Hint:* Show that neither $s \neq 0$ nor $t \neq 0$ is possible.)
   (b) Prove that if $a\mathbf{x} + b\mathbf{y} = c\mathbf{x} + d\mathbf{y}$, then $a = c$ and $b = d$

28. Verify algebraically that the following properties of vector arithmetic hold. (Do so for $n = 2$ if the general case is too intimidating.) Give the geometric interpretation of each property.
   (d) For each $\mathbf{x} \in \mathbb{R}^n$, there is a vector $-\mathbf{x}$ so that $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$