Notes

September 8, 2014

excercise

if $a_n \leq b_n \forall n$ then $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$

$$\begin{aligned}
a_n &\leq b_n \\
0 &\leq b_n - a_n \\
\lim_{n\to\infty} 0 &\leq \lim_{n\to\infty}(b_n - a_n) \\
0 &\leq \lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n \\
\lim_{n\to\infty} a_n &\leq \lim_{n\to\infty} b_n
\end{aligned}$$

monotone sequences

definition:

a sequence is increasing iff $a_{n+1} \geq a_n$ for every $n \in \mathbb{N}$ strictly increasing ...decreasing if $a_{n+1} \geq a_n$ for every $n \in \mathbb{N}$, strictly decreasing.... monotone if is it any of these types

theorem 2.6.1

an increasing sequence that is bounded above is convergent. a decreasing sequence that is bounded below is convergent.

proof

we are given $\{a_n\}_{n=1}^{\infty}$, increasing $a_n \leq a_n + 1 \forall n \in \mathbb{N}$ and bounded above. since it is bdd above it has a supremum L. we prove that $\lim_{n\to\infty} a_n = L$.

$$L = \sup\{a_n : n \in \mathbb{N}\}$$

L is the least upper bound. if $M < L$ M cannot be an upper bound.

Need: $\lim_{n\to\infty} a_n = L$ ie $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $|a_n - L| < \epsilon \forall n \geq N$.

$\forall a_n, a_n \leq L$

$\forall \epsilon > 0 \exists N\ L - \epsilon < a_N \leq L \Rightarrow |a_N - L| < \epsilon$. If $n \geq N, L \geq a_n \geq a_N > L - \epsilon$ because $\{a_n\}$ is increasing. hence $a_n - L| < \epsilon, \forall n \geq N \to \lim_{n\to\infty} = L\Box$

example

let $0 < x_1 < 1$ and define sequence x_n recursively by $x_{n+1} = 1 - \sqrt{1 - x_n}$. prove that $\{x_n\}$ has a limit and find its value

need to show it's monotone and bounded

if $0 < x_n < 1$ then $0 < 1 - \sqrt{1 - x_n} < 1$

$$\begin{aligned}
\sqrt{1 - x_n} &= 1 - x_{n+1} \\
1 - x_n &= (1 - x_{n+1})^2 \\
1 - x_n &= (1 - x_{n+1})^2 \leq 1 - x_{n+1} \\
x_{n+1} &\leq x_n
\end{aligned}$$

sequence is bounded and decreasing so it has a limit

$$\begin{aligned}
x_{n+1} = 1 - \sqrt{-x_n} \\
n \to \infty \\
L = 1 - \sqrt{1 - L} \\
\sqrt{1 - L} = 1 - L \\
1 - L = 1 - 2L + L^2 \\
L^2 - L = 0 \\
L = 0 \text{ or } 1
\end{aligned}$$

Limit is 0 since sequence is decreasing

example

let $7x_{n+1} = x_n^3 + 6, n \geq 1$. study whether the limit eists and find it's value if it doest for $x_1 = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$

$$x_{n+1} = \frac{x_n^3}{7} + \frac{6}{7}$$

if $0 < x_n^3 \leq 1$ then $0 < x_{n+1} \leq 1$

$$\begin{aligned}
\frac{x_n^3 + 6}{7} \leq \geq x_n \\
x_n^3 - 7x_n + 6 = 0 = (x_n - 1)(x_n^2 + x_n - 6) = (x_n - 1)(x_n - 2)(x_n - 3)
\end{aligned}$$

look at graph, between -3 and 1 equation is positive so $\frac{x_n^3+6}{7} \geq x_n$. between 1 and 2 $\frac{x_n^3+6}{7} \leq x_n$ and above 2 $\frac{x_n^3+6}{7} \geq x_n$

for $x_1 = 1/2$ the sequence is increasing and bounded above by 1, for $x_1 = 3/2$ the sequence is decreasing. for $x_1 = 5/2$ the sequence is increaasing

$$\begin{aligned}
x_1 &= 1/2 \\
7L &= L^3 + 6 \\
0 &= L^3 + 7L + 6
\end{aligned}$$

$$L = -3, 1, 2$$

$$L = 1 \text{ because it is bounded above by 1 and is increasing}$$

for $x_1 = 3/2$. possibilities are 1,2,-3, it's between 1 and 2 so it will be 1. assume $1 < x_n < 2$

$$1 < x_n^3 < 8$$
$$7 < x_n^3 + 6 < 14$$
$$1 < \frac{x_n^3 + 6}{7} < 2$$

so it's bounded therefore the limit exists and is one

for $x_1 = 5/2$ it is increasing and greater than 3 possible limits