1. Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

   Justify all responses

   (a) Is A invertible?
   No, by theorem 13

   (b) Write down the characteristic polynomial $\chi_A(x)$
   $\chi_A(x) = x(x-1)(x-2)(x-3)$ by definition.

   (c) Write down all eigenvalues of A.
   By theorem 14 the eigenvalues are $0, 1, 2, 3$

   (d) Write down the basis vectors for each eigenspace.
   $\lambda = 0$:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 2 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \alpha \\ \alpha \\ -2\alpha \\ 0 \end{bmatrix}$$

   (e) Is A diagonalizable? If so, find an invertible matrix P and a diagonal matrix D such that $D = PAP^{-1}$

2. Let

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

   Justify all responses

   (a) Is A invertible?
   Theorem 13 says, yes

   (b) Write down the characteristic polynomial $\chi_A(x)$
   $\chi_A(x) = (x-2)^3(x-3)^3$

   (c) Write down all eigenvalues of A.
   By theorem 14 we have eigenvalues of $2, 3$

   (d) Write down the basis vectors for each eigenspace.

(e) Is A diagonalizable? If so, find an invertible matrix P and a diagonal matrix D such that $D = PAP^{-1}$

3. Let $f(x) \in F[x]$ and let $\lambda \in F$ be an eigenvalue of $T \in \mathcal{A}(F^n)$. Prove the following statements.
   (a) $f(\lambda)$ is an eigenvalue of $f(T)$.
   (b) $f(T) = 0$ implies $f(\lambda) = 0$.
   (c) $f(T) = 0$ and $f(\mu) = 0$ does not imply μ is an eigenvalue of T. (Give a counterexample and a careful explanation.)

4. Upper triangularity is really neeeded in theorem 13! Give an example of a square matrix $A \in \mathcal{M}_n$ with the following properites.
   (a) $\text{ent}_{ii}(A) \neq 0$ for each $i \leq n$ but A is not invertible.

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

   (b) $\text{ent}_{ii}(A) = 0$ for each $i \leq n$ but A is invertible.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

5. Let $S, T \in \mathcal{A}(F^n)$ with S invertible. Given any polynomial $p(x) \in F[x]$, prove that $p(STS^{-1}) = Sp(T)S^{-1}$.

6. Let $T \in \mathcal{A}(\mathbb{C}^n)$ and $p(x) \in \mathbb{C}[x]$. Prove that $\lambda \in \mathbb{C}$ is an eigenvalue of $p(T)$ if and only if T has an eigenvalue $\mu \in \mathbb{C}$ such that $p(\mu) = \lambda$. Does the result hold if $\mathbb{C}$ is replaced by $\mathbb{R}$?

7. Let $T \in \mathcal{A}(F^n)$. Prove that for each $k \in \{1, 2, \ldots, n\}$, there is a T-invariant subspace $U_k \leq F^n$ such that $\dim(U_k) = k$.