1. Suppose that $f, g \in C^n[a-\delta, a+\delta]$ and $f^{(k)}(a) = g^{(k)} = 0$ for $0 \leq k < n$ and $g^{(n)}(a) \neq 0$ then use Taylor polynomials to prove that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f^{(n)}(a)}{g^{(n)}(a)}$$

Let $F_n(x)$ be the Taylor polynomial for $f$ and $G_n(x)$ be the Taylor polynomial for $g$. Further note that because $f, g, F_n, G_n$ are all continuous on $[a-\delta, a+\delta]$ and $f(a) = F_n(a)$ and $g(a) = G_n(a)$ then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{F_n(x)}{G_n(x)}$.

$$\begin{aligned}
\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{F_n(x)}{G_n(x)} &= \lim_{x\to a} \frac{\sum\limits_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k}{\sum\limits_{k=0}^{n} \frac{g^{(k)}(a)}{k!}(x-a)^k} \\
&= \lim_{x\to a} \frac{\frac{f^{(n)}(a)}{k!}(x-a)^n + \sum\limits_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k}{\frac{g^{(n)}(a)}{k!}(x-a)^n + \sum\limits_{k=0}^{n-1} \frac{g^{(k)}(a)}{k!}(x-a)^k} \\
&= \lim_{x\to a} \frac{\frac{f^{(n)}(a)}{k!}(x-a)^n + \sum\limits_{k=0}^{n-1} \frac{0}{k!}(x-a)^k}{\frac{g^{(n)}(a)}{k!}(x-a)^n + \sum\limits_{k=0}^{n-1} \frac{0}{k!}(x-a)^k} \\
&= \lim_{x\to a} \frac{\frac{f^{(n)}(a)}{k!}(x-a)^n + 0}{\frac{g^{(n)}(a)}{k!}(x-a)^n + 0} = \lim_{x\to a} \frac{f^{(n)}(a)}{g^{(n)}(a)} = \frac{f^{(n)}(a)}{g^{(n)}(a)}
\end{aligned}$$

2. Find the Taylor polynomial of order 3 for each of the following functions at the given point $a$, and estimate the error at the point $b$

   (a) $f(x) = \sqrt{1+x^2}$ about $a = 0$ and $b = 0.1$.
   First we need the first three derivatives:

$$f(x) = \sqrt{1+x^2} \qquad f''(x) = \frac{1}{(1+x^2)^{3/2}}$$

$$f'(x) = \frac{x}{\sqrt{1+x^2}} \qquad f'''(x) = -\frac{3x}{(1+x^2)^{5/2}}$$

   And about $a$:

$$f(a) = 1 \qquad f'(a) = 0 \qquad f''(a) = 1 \qquad f'''(a) = 0$$

And finally the polynomial:

$$P_3(x) = 1 + \frac{1}{2}x^2$$

Now we need to find the bounds of the fourth derivative, so we will need the fourth and fifth derivatives also.

$$f^{(4)}(x) = \frac{3(4x^2-1)}{(1+x^2)^{7/2}} \quad f^{(5)}(x) = \frac{45x - 60x^3}{(1+x^2)^{9/2}} = \frac{15x(-4x^2+3)}{(1+x^2)^{9/2}}$$

And using the quadratic formula: $x = \frac{\pm\sqrt{16\cdot 3}}{-8} = \sqrt{3}/2$.
So we are interested in the points 0 and $\sqrt{3}/2$. Now $f^{(4)}(0) = -3$ and $f^{(4)}(\frac{\sqrt{3}}{2}) = \frac{3(4\cdot 3/4-1)}{\text{something positive}} = \frac{6}{\text{something positive}} > -3$
Because $f^{(4)}$ is increasing in the interval $[0, \frac{\sqrt{3}}{2}]$ we can just focus on $[0, 0.1]$. Now $4x^2 - 1 = 4(x^2 - \frac{1}{4}) = 4(x + \frac{1}{2})(x - \frac{1}{2})$ and so $0 > f^{(4)}(0.1) > -3$ and so $|f^{(4)}| \leq 3 = M$. Using Taylor's Theorem we know that the error of our $P_3$ estimate is at least as close to zero as $\frac{3|0.1|^4}{4!} = \frac{0.0003}{24} = 0.125 \cdot 0.0001 = 0.0000125$

(b) $g(x) = \tan x$ about $a = \frac{\pi}{4}$ and $b = 0.75$.
Get the first several derivatives:

$$g(x) = \frac{\sin x}{\cos x} \qquad g'(x) = \frac{1}{(\cos x)^2} \qquad g''(x) = \frac{2\sin x}{(\cos x)^3}$$

$$g'''(x) = \frac{2}{\cos^2 x} + \frac{6\sin^2}{\cos^4 x} \qquad g^{(4)}(x) = \frac{16\sin x}{\cos^3 x} + \frac{24\sin^3 x}{\cos^5 x}$$

$$g^{(5)}(x) = 8\left(\frac{2}{\cos^2 x} + \frac{6\sin^2 x}{\cos^4 x}\right) + 24\left(\frac{3\sin^4 x}{\cos^4 x} + \frac{5\sin^4 x}{\cos^6 x}\right)$$

And now the derivatives at $\frac{\pi}{4}$:

$$g(a) = 1 \quad g'(a) = 2 \quad g''(x) = 2g'(a) = 4 \quad g'''(a) = 4 + 12 = 16$$

And so our $P_3(x) = 1 + 2(x - \frac{\pi}{4}) + 2(x - \frac{\pi}{4})^2 + \frac{8}{3}(x - \frac{\pi}{4})^3$
Now we are interested in the range $[0.75, \frac{pi}{4}]$. Looking at $g^{(5)}$ we see no zeros and critical points at $n\pi - \frac{\pi}{2}$. Further $g^{(4)}(0) = 0$ and $g^{(4)}(\frac{\pi}{4}) = 16g'(a) + 24g'(a) = 80$ and so that is our bound. Now then with Taylor's Theorem we have an error of less than $\frac{80|\frac{\pi}{4} - \frac{3}{4}|^4}{4!} = \frac{5}{6}\frac{(\pi-3)^4}{4^3} \approx 5 \cdot 10^{-6}$

## References

1. I used www.wolphramalpha.com to do the derivatives in 1a and to calculate decimal approximation in 1b.