Notes

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real numbers

we define a real number as $\sum_{i=0}^{\inf} a_i \cdot 10^{-i} = a_0.a_1a_2a_3a_4, 0 \leq a_i \leq 9, a_i \in \mathbb{Z}$

2.2e

$$x = 2.1357$$

example

$0.99999... = \sum_{i=0}^{\inf} 9 \cdot 10^{-i} = 9\left(\sum_{i=1}^{\inf} 10^{-1}\right) = 9\left(\frac{\frac{1}{10}}{1-\frac{1}{10}}\right) = \frac{9}{9} = 1$

not in book

the real line axiomatic definition

the real numbers are the set $\mathbb{R}$ with two operations $+, \cdot$ multiplication and addition that are closed in $\mathbb{R}$. if $a, b \in \mathbb{R}$ then $a + b \in \mathbb{R}$ and $a + b \in \mathbb{R}$

and verify the following axioms:

1. for all a,b in R, a+b=b+a and a*b=b*a
2. for all a,b,c in R, (a+b)+c=a+(b+c) and a(bc)=(ab)c
3. for all a,b,c in R, a(b+c)=a*b+a*c
4. give a,b in R, there exists c in R such that a+c=b (each c is denoted by (b-a)

   in particular the real number a-a is independent of a and is denoted by 0 and the real number 0-a is denoted by -a and called the negative of a
5. (a) there is a non-zero element in R

   (b) given x,y in R, x not 0 then there exists z in R such that xz=y. such z is denoted by y/x

   in particular, x/x is independant of x in R, x not 0 and denoted by 1, and 1/x is called the reciprocal of x (x^{-1}.

actually these are properties of fields, not just reals. *field axioms.* this includes rationals.

now for the things that differentiate from rationals

order axioms

no order in complex numbers, unlike reals and rationals

1. for all x,y in R, exactly one of these hold:

$$x < y, x = y, x > y$$

2. if $x < y$ then for all z in R $x + z < y + z$
3. if $x > 0$ and $y > 0$ then $xy > 0$
4. if $x > y$ and $y > z$ then $x > z$

now we have an ordered field defined, both reals and rationals (Q) are ordered fields

completeness axiom

every non-empty set $S \subset R$ (not proper subset) that is bounded above has a least upper bound

moving on

prove that if $x < y$ and $z < 0$ then $xz < yz$

lemma

if $z < 0$ then $-z > 0$

assume that $z < 0$. by 6 we know that $-z$ and 0 we get that exactly one of these happen:

$$-z < 0, -z = 0, -z > 0$$

we know that $z < 0$ and $z + (-z) = 0$. use 7. $z < 0$ so $z + (-z) < 0 + (-z)$ and $0 < -z$

case 1, x is positive

then by transitivity property (9) y is also positive and by 8 $x(-z) < y(-z)$. by 7 $x(-z) + xz < y(-z) + xz$ and $0 < y(-z) + xz$ and similarly $yz < xz$

2.3.1 definition

a set S contained by R is bounded above if there exists M in R such that $x \leq M$ for all $x \in S$. M is called an upper bound for S.

A set S contained by R is bounded below if there exists m in R such that x is greater than or equal to m for all x in S. m is called a lower bound for S

a set S contained by R is bounded if it is bounded above and below

Let S be a bounded subset of R

L is the supremum or least upper bound of S if 1) L is an upper bound of S and 2) if M is another upper bound of S then L is less than or equal to M

l is the infimum or greatest lower bound of S if 1) l is a lower bound of S and 2) if m is another lower bound of

examples

$$S = (1, 5]$$
$$sup(S) = 5$$
$$inf(S) = 1$$

6 is an upper bound and 0 or -4324 are lower bounds for s.

the sup and inf need not necessarily be elements of s

let s be contained in R and bounded. if sup(s) is in s then we call it the maximum of s. if inf(s) is in s then we call it the minimum of s.

examples

find sup, inf, max and min(if they exist)

if S=$\{$2/n,n in N, $n \geq 1\}$ then sup(S)=2=max S and inf(S)=0

if S=$\{\frac{(-1)^n n}{n+1}$,$n \in N\}$ then sup(S)=1, inf(S)=-1, no max or min

if S=$\{x \in Q : x^2 < 2\}$ then sup(S)=$\sqrt{2}$

if we view s as a subset only of q and forget about R, s has no supremum. any q in Q, q squared greater than 2 is an upper bound for s but there is not a smallest such q in Q