Subspaces of $\mathbb{R}^n$

Definition 1. Let $A \in M_{m\times n}$.
(1) The *null space* of A is the subspace $\mathrm{Null}(A) = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$.
(2) The *column space* of A is the subspace $\mathrm{Col}(A) = \mathrm{Span}(\mathbf{c}_1(A), \mathbf{c}_2(A), ..., \mathbf{c}_n(A)\} \leq \mathbb{R}^m$.
(3) The *row space* of A is the subspace $\mathrm{Row}(A) = \mathrm{Span}(\mathbf{r}_1(A), \mathbf{r}_2(A), ..., \mathbf{r}_m(A)\} \leq \mathbb{R}^n$.
(4) The *left null space* of A is the subspace $\mathrm{Null}(A^{\mathrm{T}}) = \{\mathbf{x} \in \mathbb{R}^m : \mathbf{x}^{\mathrm{T}}A = \mathbf{0}^{\mathrm{T}}\}$.

Fact 2. Let $E \in \mathcal{M}_m$ be a product of elementary matrices and let $\mathbf{b} \in \mathbb{R}^m$. If

$$\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \text{ then } E\mathbf{b} = \begin{pmatrix} c_{11}b_1 + c_{12}b_2 + ... + c_{1m}b_m \\ c_{21}b_1 + c_{22}b_2 + ... + c_{2m}b_m \\ \vdots \\ c_{m1}b_1 + c_{m2}b_2 + ... + c_{mm}b_m \end{pmatrix} \text{ for some } c_{ij} \in \mathbb{R}.$$

Sketch of Proof. We have

$$E_{r\to r+cs}\mathbf{b} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_r + cy_s \\ \vdots \\ y_m \end{pmatrix} \quad \text{and} \quad E_{r\to cr}\mathbf{b} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ cy_r \\ \vdots \\ y_m \end{pmatrix}.$$

Products of such matrices gives the desired form.

Definition 3. Let $A \in \mathcal{M}_{m\times n}$ with $\mathrm{rank}(A) = r$ and let $A\mathbf{x} = \mathbf{b}$ be a system of linear equations. Suppose that E is a product of elementary matrices such that $rref(A) = EA$ and write

$$\begin{bmatrix} 1 & & & & \alpha_{1,r+1} & \alpha_{1,r+2} & \cdots & \alpha_{1n} \\ & 1 & & & \alpha_{2,r+1} & \alpha_{2,r+2} & \cdots & \alpha_{2n} \\ & & \ddots & & \vdots & & & \\ & & & 1 & \alpha_{r,r+1} & \alpha_{r,r+2} & & \alpha_{rn} \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \\ x_{r+1} \\ x_{r+2} \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} c_{11}b_1 + c_{12}b_2 + ... + c_{1m}b_m \\ c_{21}b_1 + c_{22}b_2 + ... + c_{2m}b_m \\ \vdots \\ c_{r1}b_1 + c_{r2}b_2 + ... + c_{rm}b_m \\ c_{r+1,1}b_1 + c_{r+1,2}b_2 + ... + c_{r+1,m}b_m \\ \vdots \\ c_{m1}b_1 + c_{m2}b_2 + ... + c_{mm}b_m \end{pmatrix}. \tag{R}$$

The equations

$$\begin{aligned} c_{r+1,1}b_1 + c_{r+1,2}b_2 + ... + c_{r+1,m}b_m &= 0 \\ &\vdots \\ c_{m1}b_1 + c_{m2}b_2 + ... + c_{mm}b_m &= 0 \end{aligned} \tag{S}$$

are called the *constraint* equations. We set

$$\begin{aligned} \mathbf{v}_1 &= (c_{r+1,1}, c_{r+1,2}, ..., c_{r+1,m}) \\ &\vdots \\ \mathbf{v}_{m-r} &= (c_{m1}, c_{m2}, ..., c_{mm}) \end{aligned}$$

Theorem 4. Let $A \in \mathcal{M}_{m\times n}$ with $\text{rank}(A) = r$. The following statements are equivalent for $b \in \mathbb{R}^m$.
(1) $A\mathbf{x} = \mathbf{b}$ is consistent (i.e. at least one solution).
(2) $\mathbf{b} \in \text{Col}(A) = \text{Span}(\mathbf{c}_1(A), \mathbf{c}_2(A), ..., \mathbf{c}_n(A))$.
(3) $\mathbf{v}_i \cdot \mathbf{b} = 0$ for each $i \leq m - r$.

Theorem 5. If $A \in \mathcal{M}_{m\times n}$, then $\text{Null}(A) = \text{Row}(A)^\perp$.

Proof. ($\subseteq$) Choose any $\mathbf{x} \in \text{Null}(A)$ so that $A\mathbf{x} = \mathbf{0}$. Since $A\mathbf{x}$ is an $m \times 1$ matrix, we can write all of its entries as $\text{ent}_{i1}(A\mathbf{x}) = \mathbf{r}_i(A) \cdot \mathbf{c}_1(\mathbf{x}) = 0$. But $\mathbf{x}$ is a single column vector and so $\mathbf{c}_1(\mathbf{x}) = \mathbf{x}$. It follows that $\mathbf{r}_i(A) \cdot \mathbf{x} = 0$ for each $i \leq m$. Since $\text{Row}(A) = \text{Span}(\mathbf{r}_1(A), \mathbf{r}_2(A), ..., \mathbf{r}_m(A))$, it must be the case that $\mathbf{x} \cdot \mathbf{v} = 0$ for all $\mathbf{v} \in \text{Row}(A)$. In other words, $\mathbf{x} \in \text{Row}(A)^\perp$.

($\supseteq$) Reverse the above argument.

Definition 6. Two subspaces $V, W \leq \mathbb{R}^n$ are called orthogonal subspaces if $\mathbf{v} \cdot \mathbf{w} = 0$ for all $\mathbf{v} \in V$ and $\mathbf{w} \in W$.

Lemma 7. Let $V, W \leq \mathbb{R}^n$.
(1) If V, W are orthogonal, then $V \subseteq W^\perp$ and $W \subseteq V^\perp$.
(2) If $V = W^\perp$ then V, W are orthogonal.
(3) If $V \subseteq W$, then $W^\perp \subseteq V^\perp$.

Proof.
(1) Choose any $\mathbf{v} \in V$. Since V, W are orthogonal, $\mathbf{v} \cdot \mathbf{w} = 0$ for all $\mathbf{w} \in W$. It follows from the definition of orthogonal complement that $\mathbf{v} \in W^\perp$. A symmetric argument shows that $W \subseteq V^\perp$.
(2) Choose any $\mathbf{v} \in V$ and $\mathbf{w} \in W$. Since $V = W^\perp$, we have $\mathbf{v} \in W^\perp$. It follows from the definition of orthogonal complement that $\mathbf{v} \cdot \mathbf{w} = 0$.
(3) Choose any $\mathbf{x} \in W^\perp$. Then $\mathbf{x} \cdot \mathbf{w} = 0$ for all $\mathbf{w} \in W$. Since $V \subseteq W$, it is certainly true that $\mathbf{x} \cdot \mathbf{v} = 0$ for all $\mathbf{v} \in V$. Therefore,

Theorem 8. If $A \in \mathcal{M}_{m\times n}$, then $\text{Col}(A) = \text{Null}(A^{\mathrm{T}})^\perp$.

Proof. Setting $A = A^{\mathrm{T}}$ in Theorem 4, we have that $\mathrm{Null}(A^{\mathrm{T}}) = \mathrm{Row}(A^{\mathrm{T}})^{\perp}$. One easily checks that $\mathrm{Row}(A^{\mathrm{T}}) = \mathrm{Col}(A)$ and so $\mathrm{Null}(A^{\mathrm{T}}) = \mathrm{Col}(A)^{\perp}$.

($\subseteq$) By Lemma 7(2), $\mathrm{Null}(A^{\mathrm{T}})$ and $\mathrm{Col}(A)$ are orthogonal. By Lemma 7(3), we have that $\mathrm{Col}(A) \subseteq \mathrm{Null}(A^{\mathrm{T}})^{\perp}$.

($\supseteq$) Bring A to $rref$ in (R) and set $V = \mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_{m-r})$ as in Theorem 4. Since $\mathbf{x} \in \mathrm{Col}(A)$ if and only if $\mathbf{v}_i \cdot \mathbf{x} = 0$ for each $i \leq m - r$ it follows that $\mathrm{Col}(A) = V^{\perp}$. Hence, $\mathrm{Col}(A)$ and V are orthogonal and Lemma 7 gives

$$\begin{aligned} V &\subseteq \mathrm{Col}(A)^{\perp} = \mathrm{Null}(A^{\mathrm{T}}) \\ &\implies \mathrm{Null}(A^{\mathrm{T}})^{\perp} \subseteq V^{\perp} \quad \text{(Lemma 7(3)} \\ &\implies \mathrm{Null}(A^{\mathrm{T}})^{\perp} \subseteq \mathrm{Col}(A) \end{aligned}$$

Example 9. We compute $\mathrm{Null}(A), \mathrm{Col}(A), \mathrm{Row}(A), \mathrm{Null}(A^{\mathrm{T}})$ of

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 0 & 1 \\ 1 & 2 \end{bmatrix}.$$

We have

$$rref(A) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

We can see immediately that

$$\mathrm{Null}(A) = \{\mathbf{0}\}$$

and so

$$\mathrm{Row}(A) = \mathrm{Null}(A)^{\perp} = \{\mathbf{0}\}^{\perp} = \mathbb{R}^2.$$

Now consider the transpose

$$A^{\mathrm{T}} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 2 \end{bmatrix}$$

and

$$rref(A^{\mathrm{T}}) = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix}.$$

To find $\mathrm{Null}(A^{\mathrm{T}})$ we solve the corresponding system

$$\begin{aligned} x + z + w &= 0 \\ y - z &= 0 \end{aligned}$$

and find that

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -z - w \\ z \\ z \\ w \end{pmatrix} = z \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + w \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

That is,

$$\mathrm{Null}(A^{\mathrm{T}}) = \mathrm{Span}\left(\begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right).$$

Incidently, we can see from $\mathrm{Null}(A^{\mathrm{T}})$ that the constraint equations are

$$\begin{aligned} -b_1 + b_2 + b_3 &= 0 \\ -b_1 + b_4 &= 0. \end{aligned}$$

Using the fact that $\mathrm{Col}(A) = \mathrm{Null}(A^{\mathrm{T}})^{\perp}$, we are led to solving

$$\begin{bmatrix} -1 & 0 & 0 & 1 \\ -1 & 1 & 1 & 0 \end{bmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

As usual, we find $rref$:

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & -1 \end{bmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and so

$$\mathrm{Col}(A) = \mathrm{Span}\left(\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}\right)$$

Exercises Section 3.2: Due 10/19/2015

1. Read and Understand Example 3 of this section.
2. Find $\mathrm{Null}(A), \mathrm{Row}(A), \mathrm{Null}(A^{\mathrm{T}}), \mathrm{Col}(A)$ for the matrices in 2(a), 3(c), 4.
3. Do Exercises 6-8 (Computational)
4. Do Exercise 2.5.15 (The section on Transpose)
5. Do Exercise 10-13 (Proofs).