Notes

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$\epsilon\Delta$ definition of limit

let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. We say that L is the limit of the sequence $\{a_n\}_{n=1}^{\infty}$ if $\forall\epsilon > 0$ $\exists N \in \mathbb{N}$ such that $|a_n - L| < \epsilon \forall n \geq N$

example 1

$a_n = \frac{1}{n}, n \in \mathbb{N}$

prove that

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

by the definition fix $\epsilon > 0$. we need to find N such that $\left|\frac{1}{n} - 0\right| < \epsilon$ if $n \geq N$. find N such that $1 \leq \epsilon n, \forall n \geq N$.

archimedean property

given $x > 0, y \in \mathbb{R}, \exists N \in \mathbb{N}, N > 0$ such that $Nx > y$.

apply this to $x = \epsilon, y = 1$, $\exists N$ such that $N\epsilon > 1$ if $n \geq N, n\epsilon \geq N\epsilon > 1$ so the archimedean property gives us the N for the given ϵ

example 2

find

$$\begin{aligned}
\lim_{n\to\infty}\left(\sqrt{n+1} - \sqrt{n}\right) &= \lim_{n\to\infty}\left(\frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n+1} + \sqrt{n}}\sqrt{n+1} + \sqrt{n}\right) \\
&= \lim_{n\to\infty}\left(\frac{(n+1) - (n)}{\sqrt{n+1} + \sqrt{n}}\right) \\
&= \lim_{n\to\infty}\left(\frac{1}{\sqrt{n+1} + \sqrt{n}}\right) \\
&= 0
\end{aligned}$$

given $\epsilon > 0$ we need to find N such that $\left|(\sqrt{n+1} - \sqrt{n} - 0\right| < \epsilon \forall n \geq N \to \frac{1}{\sqrt{n+1}+\sqrt{n}} < \epsilon \forall n \geq N$.

enough to find N such that $\frac{1}{2\sqrt{n}} < \epsilon \forall n \geq N$

given ϵ and $\frac{1}{2} \exists N \in \mathbb{N}$ such that $\frac{1}{2} < N\epsilon$. for any $n \in N$ such that $\sqrt{n} > N$ (or $n > N^2$). $\frac{1}{2} < \sqrt{n}\epsilon \to \frac{1}{2\sqrt{n}} < \epsilon \forall n > N^2$

example 3

not convergent let $a_n = 2^n$.

proof

assum $\exists L \in \mathbb{N}$ such that $\forall \epsilon > 0 \exists N \in \mathbb{N}$ such that $|2^n - L| < \epsilon \forall n \geq N$. Take any $n > m > 1, n, m \in \mathbb{N}$

$$|2^n - 2^m| = 2^m \left(2^{n-m} - 1\right) > 1 \text{ or } 2$$

consider $|a_n - L| + |a_m - L| \geq |(a_n - L) + (L - a_m)|$ (triangle inequality) leads to $|a_n - a_m| > 1$. so either $|a_n - L| > \frac{1}{2}$ or $|a_m - L| > \frac{1}{2}$ (pigeonhole). take $\epsilon > \frac{1}{2}$. Since n, m are arbitrary natural numbers, we cannot find N such that $|a_n - L| < \epsilon \forall n \geq N$, and hence the sequence does not converge.

can use this argument with $|a_n - a_m| > \alpha, \alpha > 0$

squeeze theorem (2.4.6)

let $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}, \{c_n\}_{n=1}^{\infty}$ be sequences of reals such that $a_n \leq b_n \leq c_n$ for each $n \in \mathbb{N}$. Assume

$$\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$$

Then

$$\lim_{n\to\infty} b_n = L$$

proof

Given $\epsilon > 0$ we need to find $N\mathbb{N}$ such that $|b_n - L| < \epsilon \forall n \geq N$

have: an $\epsilon > 0$. hopothesis: $\lim a_n = L, \lim c_n = L$

since we have $\epsilon > 0$ and $\lim a_n = L$, aby definition of limit $\exists N_1$ such that $\forall n \geq N_1$, $|L - a_n| < \epsilon$ for the same $\epsilon > 0, \exists N_2$ such that $\forall n \geq N_2$ $|L - c_n| < \epsilon$

$$\begin{aligned}
L - \epsilon < a_n < L + \epsilon \\
L - \epsilon < c_n < L + \epsilon \\
L - \epsilon < a_n \leq b_n \leq c_n < L + \epsilon
\end{aligned}$$

if $n \geq \max\{N_1, N_2\}$. hence if $n \geq \tilde{N} = \max N_1, N_2, |b_n - L| < \epsilon$

example 2.4.7

variation of squeeze theorem. read it please

properties of limits

assume $\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n$ exist

1.

$$\begin{aligned}
\lim_{n\to\infty}(a_n + b_n) &= \lim a_n + \lim b_n \\
\lim r a_n &= r \lim a_n \forall r \in \mathbb{R} \\
\lim a_n b_n = \lim a_n \lim b_n \lim \frac{a_n}{b_n} &= \frac{\lim a_n}{\lim a_n}
\end{aligned}$$

let A be limit of a_n B be limit of b_n given $\epsilon > 0$ we need to find $N \in \mathbb{N}$ such that $\forall n \geq N$ $|a_n b_n - AB| < \epsilon$. for this given ϵ

$\exists N_1$ such that $|a_n - A| < \epsilon$ if $n \geq N$

$\exists N_2$ such that $|b_n - B| < \epsilon$ if $n \geq N$

$|a_n b_n - AB| = |a_n b_n - a_n B + a_n B - AB| \leq |a_n||b_n - B| + |a_n - A||B| < \epsilon\{\max\{|A+1|, |A-1|\} + |B|\}$ provided $n \geq \max\{N_1, N_2, M\}$