Notes

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let $\epsilon > 0$ and $f \in C[0,1]$ Then there is N such that $||f - B_n(f)||_\infty < \epsilon$ for all $n > N$.
$B_n(f) = \sum_{k=0}^{n} f(\frac{k}{n}\binom{n}{k}x^k(1-x)^{n-k}$

1. since $[0,1]$ is compact and $f \in C[0,1]$ f is uniformly continuous on $[0,1]$ so there is δ such that $|f(x) - f(y)| < \frac{\epsilon}{2}$ when $|x - y| < \delta$
2. f is bounded on $[0,1]$ so $|f(x)| \leq M = \sup\{|f(x)|\}$ for all $x \in [0,1]$
3. now for $a \in [0,1]$, if $|x - a| < \delta$ then $|f(x) - f(a)| < \epsilon/2 + \frac{2M}{\delta^2}(x-a)^2$

   if $|x - a|\epsilon\delta$ then $|f(x) - f(a) \leq |f(x)| + |f(a)| \leq 2M \leq 2M/\left(x-a)/\delta\right)^2 \leq \frac{2M}{\delta^2}(x-a)^2 + \frac{\epsilon}{2}$

   so for a fixed a, no matter what x we let $|f(x) - f(a)| < \epsilon/2 + \frac{2M}{\delta^2}(x-a)^2$.

   this is not a good estimate, but it's an estimate that works no matter what.
4. $B_n(f(x) - f(a)\cdot 1) = B_n(f(x)) - B_n(f(a)\cdot 1) = B_n(f(x)) - f(a)B_n(1) = B_n(f(x)) - f(a)\cdot 1$ where 1 is the function that is one for all x and $f(a)$ is a constant.

so $|B_n(f(x)) - f(a)| = |B_n(f(x) - f(a)\cdot 1)| \leq B_n(|f(x) - f(a)\cdot 1|) \leq B_n(\epsilon/2 + 2M/\delta^2(x-a)^2) = B_n(\epsilon/2) + 2M/\delta^2 B_n((x-a)^2) = \epsilon/2 + 2M/\delta^2[B_n(x^2 - 2ax + a^2] = \frac{\epsilon}{2} + 2M/\delta^2(B_n(x^2) - 2aB_n(x) + a^2) = \frac{\epsilon}{2} + \frac{2M}{\delta^2}((x^2 + \frac{x-x^2}{n}) - 2a(x) + a^2$
recall that $B_n(1) = 1, B_n(x) = x$, and $B_n(x^2) = \frac{x+(n-1)x^2}{n} = x^2 + \frac{x-x^2}{n}$
if $g(x) = x - x^2$ then it's max in $[0,1]$ is $g(\frac{1}{2}) = \frac{1}{4}$ and so $|(B_n(f))(a) - f(a)\cdot 1| < \epsilon/2 + 2M/\delta^2$
so $|(B_n(f))(a) - f(a)\cdot 1| < \epsilon/2 + 2M/\delta^2\frac{a-a^2}{n}$ so $|(B_n(f))(a) - f(a)\cdot 1| < \epsilon/2 + 2M/\delta^2(\frac{1}{4n})$

finish

this inequality does not depend on a. $||B_n(f) - f||_\infty < \epsilon/2 + \frac{2M}{\delta^2}\frac{1}{4n}$. Choos N such that $N \geq \frac{M}{\delta^2\epsilon}$
so $\frac{M}{2\delta^2 N} < \epsilon/2$ and $||B_n(f) - f||_\infty < \epsilon/2 + \epsilon/2 = \epsilon$ for any $n > N$ in other words, $\{B_n(f)\}$ converges to f uniformly on $[0,1]$