Notes

October 20, 2014

if $\liminf x_n = L$ then there exists $\{x_{n_k}\}$ such that $\lim x_{n_k} = L$

$l = \liminf x_n = \lim(inf\{\underbrace{x_{n_1}, x_{n_2}, x_{n_3}, \dots}_{c_n}\}$

why not just let c_n be the subseuence? because c_n may not be equal to any of the x_k in the sequence

$c_n = \inf\{x_{n_1}, x_{n_2}, \dots\}$ give $\varepsilon = 2^{-n}$ there exists $x_{n_k} \in \{x_{n_1}, x_{n+1}, x_{n+2}, \dots\}$ such that $|c_n - x_{n_k}| < 2^{-n}$ by def of infinum

we has a sequence $\{c_n\}$ given $\varepsilon > 0$ there exists N such that $|c_n - L| < \varepsilon$ if $n \geq N$. we approximate each c_n by some x_{n_k} from the oricinal sequence sutch that ....

convergence test for series

first we talk about series with positive terms $\sum_{k=1}^{\infty} a_k$, $s_n = \sum_{k=1}^{n} a_k$. So if s_n is bounded about then the series is convergent. and if not, it is divergent.

geometric series $\sum_{n=0}^{\infty} r^n$ is convergent if $|r| < 1$. $s_n = \sum_{k=0} nr^k = 1 + r + r^2 + \dots + r^n, rs_n = r + r^2 + r^3 + \dots, sn - rSn = 1 - r^{n+1}$

$s_n = \frac{1-r^{n+1}}{1-r} \to \frac{1}{1-r}$

comparison test

if $\forall n, |a_n| \leq b_n$

- if $\sum_{n=1}^{\infty} b_n$ is convergent then $\sum_{n=1}^{\infty} a_n$ is convergent,
- if $\sum a_n$ is divergent, so is $\sum b_n$.

3.2.b

show that if $(|a_n|)_{n=1}^{\infty}$ is summable then so is $(a_n)_{n=1}^{\infty}$.

$$\sum_{k=n+1}^{m} |a_k| < \varepsilon \text{ for all } N \leq n \leq m \text{ because is is summable}$$

$$\left|\sum_{k=n+1}^{m} a_k\right| \leq \sum_{k=n+1}^{m} |a_k| < \varepsilon$$

so then $\sum a_k$ is also cauchy and summable

cauchy-schwartz inequality

$\sum\limits_{k=1}^{n} a_k b_k \leq \left(\sum\limits_{k=1}^{n} a_k^2\right)^{1/2} \left(\sum\limits_{k=1}^{n} b_k^2\right)^{1/2}$

3.2.f

leibniz test for alternating series

if $\{a_n\}$ is a monotone decreasing sequence of positive terms with the $\lim a_n = 0$ then $\sum\limits_{n=1}^{\infty}(-1)^n a_n$ is convergent

note!

a sequence my have the property $\lim |a_n - a_{n+!}| = 0$ but not be cauchy

3.2.h

Show that if $\sum\limits_{n=1}^{\infty} a_n$ and $\sum\limits_{n=1}^{\infty} b_n$ are series with $b_n \geq 0$ such that $\limsup\limits_{n\to\infty} < \infty$ and $\sum\limits_{n=1}^{\infty} b_n < \infty$, then the series $\sum\limits_{n=1}^{\infty} a_n$ converges.

$$\left|\left(\sup_{k\geq n} \frac{|a_k|}{b_k}\right) - L\right| < \varepsilon$$
$$\left(\sup_{k\geq n} \frac{|a_k|}{b_k}\right) < L + \varepsilon$$
$$\frac{|a_k|}{b_k} < L\varepsilon$$
$$|a_k| < (L+\varepsilon)b_k$$

3.2.j

$\liminf \frac{a_n+1}{a_n} \leq \liminf a_n^{\frac{1}{n}} \leq \limsup a_n^{\frac{1}{n}} \leq \limsup \frac{a_n+1}{a_n}$.

step 1

if $x \geq r$ for all $r > b$ then x is a lower bound for the set $\{r \in \mathbb{R} : r > b\}$, $x \leq \inf\{r \in \mathbb{R} : r > b\} = b$

we will show that if $\limsup \frac{a_n}{b_n} < r$ then $\limsup a_n^{\frac{1}{n}} \leq r$ and then apply step one.

let $r > \limsup \frac{a_{n+1}}{a_n}$ then $\exists N$ such that $r > \frac{a_{n+!}}{a_n} \forall n \geq N$

$$a_{N+1} < ra_N$$
$$a_{N+2} < ra_{N+1} \leq r^2 a_N$$
$$a_{N+K} < r^k a_N$$
$$a_{N+k}^{\frac{1}{N+k}} < (r^k a_N)^{\frac{1}{N+k}}$$

quiz from 10/1/2014

$L_k \to L$ then $\{x_n\}$ such that $\forall k, \exists$ a subsequence of $\{x_n\}$ converging to L_k. prove that $\{x_n\}$ has a subsequence converging to L.

given $\varepsilon > 0 \exists N_0$ such that $|L_k - L| < \varepsilon$ if $k \geq N_0$

$|x_{N_k} - L| \leq |x_{N_k} - L_k| + |L_k - L| < 2\varepsilon$

example

let $A, B \subseteq \mathbb{R}$, prove that $\sup A \leq \inf B$, if $\forall a \in A, b \in B, a \leq b$

3.3.5

any rearrangement of an absolutely convergent series converges tothe same limit

proof

let $\sum a_n = L < \infty$. We know $\sum |a_n|$ is convergent (not necessarily to L). by th cauchy riterion for series $\forall \varepsilon > 0 \exists N$ such that $\left(\sum\limits_{n=N+1}^{\infty} |a_n| \right) < \varepsilon$

$\pi : \mathbb{N} \to \mathbb{N}$ is bijective, the rearranged series is $\sum\limits_{n=1}^{\infty} a_{\pi(n)}$ and $\{a_1 \ldots a_N\} \subseteq \{a_{\pi(1)1} \ldots a_{\pi(M)}\}$

3.3.7 rearrangement theorem

let $\sum a_n = L < \infty$ and define $b_n = (a_n \geq 0)?a_n : 0$ and $c_n = (a_n < 0)?a_n : 0$

consider the series $\sum b_n$ and $\sum |c_n|$

case 1

both convergent

$\sum |a_n| = \sum b_n + \sum |c_n|$ which is convergent, which contradicts the fact that a_n is conditionally convergent

case 2

one convergent, one divergent

assume $\sum |c_n| = A < \infty$ and $\sum b_n$ is divergent to $+\infty$

given any $R \in \mathbb{N}$ big, $\exists N$ such that $\sum\limits_{n=1}^{N} b_n > R + A$, then we pick M big enough so that $\{b_1, \ldots, b_N\} \subseteq \{a_1, a_2, \ldots, a_M\}$ and $\sum\limits_{n=1}^{M} a_n \geq \sum\limits_{n=1}^{N} b_n - \sum |c_n| > R$ so $\sum a_n$ is divergent, which is a contradiction.

case 3

both divergent

chapter 4

$\mathbb{R}^n = \{(x_1, x_2, \dots, x_n), x_i \in \mathbb{R}\}$, vector space (or point in n-space).
with the coordinate wise sum and the product by real numbers (scalars).

$$\begin{aligned}(x_1, \dots x_n) + (y_1, \dots, y_n) &= (x_1 + y_1, \dots, x_n + y_n)\\ \lambda(x_1, \dots, x_n) &= (\lambda x_1, \dots, \lambda x_n)\\ x^{\rightarrow} &= (x_1, \dots, x_n) = x\end{aligned}$$

euclidean norm

$$||x|| = \sqrt{x_1^2 + \dots + x_n^2}$$

distance from x to y

$$||x - y||$$

cauchy-schwarz

$\left|\sum_{i=1}^{n} a_j b_j\right| \leq \left(\sum_{i=1}^{n} a_j^2\right)^{1/2} \left(\sum_{i=1}^{n} b_j^2\right)^{1/2}$
$|a \cdot b| \leq ||a||||b||$

dot product

$a \cdot b = \sum a_i b_i$

triangle inequality

$$||x + y|| \leq ||x||| + ||y||$$

proof

$$\begin{aligned}||x + y||^2 &= \sum (x_i + y_i)^2\\ &= (x + y) \cdot (x + y)\\ &= x \cdot x + 2x \cdot y + y \cdot y\\ &= ||x||^2 + 2x \cdot y + ||y||^2\\ &\leq ||x||^2 + 2||x||||y|| + ||y||^2\end{aligned}$$
$$= (||x|| + ||y||)^2$$

standard orthogonal base of $\mathbb{R}^n$

$$\begin{aligned}e_1 &=< 1, 0, \dots, 0 >\\ e_2 &=< 0, 1, \dots, 0 >\\ &\vdots\\ e_n &=< 0.0, \dots, 1 >\end{aligned}$$

4.2 convergence in $\mathbb{R}^n$

definition: a sequence $\{x^i\}$ of parts in $\mathbb{R}^n$ converge to $c \in \mathbb{R}^n$ if $\forall \varepsilon > 0 \exists N = N(\varepsilon) \in \mathbb{N}$, such that $||x^i - c|| < \epsilon$ if $i \geq N$ we say $\lim x^i = c$.

4.2.2 lemma

$\lim x^i = a$ if and only if $\lim ||x^i - a|| = 0$.

4.2.3 lemma

$\lim x^i = a$ if and only if $\forall j = 1, \ldots, n, \lim x^i_j = a_j$

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lemma 4.2.3****know this

a sequence $\{x^i\}$ of points $\mathbb{R}^n$ converges to $a \in \mathbb{R}^n$ if and only if for each coordinate $\lim x^i_j = a_j$

thm 4.2.5

every cauchy sequence of points in $\mathbb{R}^n$ converges to a point in $\mathbb{R}^n$.

def

a sequence $\{x^i\}$ of points in $\mathbb{R}^n$ is cauchy if $\forall \varepsilon > 0 \exists N \in \mathbb{N}$ such that $||x^i - x^j|| < \varepsilon$ for all $i, j \geq N$

proof

we have a cauchy sequence. given $\varepsilon > 0, \exists N$, $||x^i - x^j|| < \varepsilon$ if $i, j \geq N$

$|x^i_k - x^j_k| \leq ||x^i - x^j|| = \sqrt{(x^i_1 - x^j_1)^2 + (x^i_2 - x^j_2)^2 + \cdots + (x^i_n - x^j_n)^2}$

so each of the coordinates for the sequence is cauchy ($\{x^i_k\}$ is cauchy). So it converges to some $a_k \in \mathbb{R}$ and $a = (a_1, a_2, \ldots, a_n)$ and so b lemma 4.2.3 $||x^i - a|| \to 0$

4.2.6

read, useful for next weeks hw

4.3 open, closed sets in $\mathbb{R}^n$

def: let $A \subseteq \mathbb{R}^n$. we say that x is a limit point of A if there exists a sequence $\{a_k\}$ with $a^k \in A$ such that the limit of the sequence is x.

def: a set $A \subseteq \mathbb{R}^n$ is closed if it contains all of it's limit points.

example

is $[0,1]$ closed? and is $(0,1]$ not closed?

0 is a limit point because $0 = \lim \frac{1}{n}$ an $\frac{1}{n} \in (0,1]$.

consider $[a,b]$. $\{x_n\} \subseteq [a,b]$. $a \leq x_n \leq b, \forall n = 1,2,3,\dots$

assume $\lim x_n$ exists, call it x. we will show $a \leq x \leq b$. then assume not and show wlog $x > b$.

take $\varepsilon = \frac{x-b}{2}$. then $\exists N$ such that $|x_N - x| < \epsilon$. $x - x_N < \frac{x-b}{2}$ and $b < \frac{x}{2} + \frac{b}{2} < x_n$ and so we have a contradiction

special cases

$\emptyset$ is closed. $[a,+\infty)$ and $(-\infty,a]$ are closed.

finite sets of $\mathbb{R}^n$ are closed.

proof

let $A = \{a_1,\dots,a_M\}$. considera sequence $\{x_j\}$ such that $x_j \in A, \forall j \in \mathbb{N}$. at least one of the points appears ∞ many times. if $\lim x_j$ exists then at some point the sequence is a single repeating point, which is the limit, which is in A.

proposition 4.3.3

the finite union of closed sets is closed, arbitrary intersections of closed sets are closed.

proof

let A,B be closed, we need to check that $A \cup B$ is closed, then by induction, if $A_1,\dots,A_N$ is closed then $\bigcup_{i=1}^{N} A_i$ is closed.

pick a sequence $\{x_j\}$ of points in $A \cup B$. converging to some $x \in \mathbb{R}^n$. We need to show that $x \in A \cup B$

$x_j \in A \cup B \Rightarrow x_j \in A$ or $x_j \in B$. We have infinitely many points and so either A or B contains infinitely many of the points. but since the sequence has a limit $\exists N$ such that $x_j \in A \forall j \geq N$

infinitely many of the points are in one of the sets, but since the sequence has a limit, passing to a subsequence oif necessary, we get that all points in the sequence are eventually in one of the seats, hence the limit is in that set because the set is closed $\Rightarrow x \in A \cup B$

and for the second part: let $\{A_i\}$ be a collection of closed sets. let $\{x_n\}$ be a sequence such that $x_n \in \bigcap_{i \in I} A_i$ and $\lim x_n = x$. we need to show $x \in \cap A_i$ since $x_n \in A_i \forall i$ and A_i is closed $\lim x_n = x \in A_i \forall i$

example

a countable union of closed sets may not be closed. $A_n = [\frac{1}{n}, 1]$ $\quad \bigcup_{n=1}^{\infty} [\frac{1}{n}, 1] = (0,1]$

def

let $A \subseteq \mathbb{R}^n$. The closure of A, $\bar{A}$ is the set containing all the limit points of A. $\bar{A}$ is the smallest closed set that contains A.

def

a set $U \subseteq \mathbb{R}^n$ is open if $\forall x \in U \exists B(x,\varepsilon) \subset U$. $B(x,\varepsilon) = \{y \in \mathbb{R}^2 : ||y - x|| < \varepsilon\}$

proposition

a set is open iff A^C is closed.

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$\overline{\mathbb{Q}} = \mathbb{R}$ and $\mathbb{Q}^\circ = \emptyset$

$\overline{\mathbb{R} - \mathbb{Q}} = \mathbb{R}$ and $(\mathbb{R} - \mathbb{Q})^\circ$

define: a set $A \subseteq \mathbb{R}^n$ is dense if $\overline{A} = \mathbb{R}^n$.

A set $A \subseteq B$ is dense in B if $B \subseteq \overline{A}$

4.3M

Let A be dense in $\mathbb{R}^n$ and let U be an open set

a) we need to show that $U \subseteq \overline{A \cap U}$. Pick $x \in U$ to show that $x \in \overline{A \cap U}$. we have to find a sequence $a^i \in A \cap U$ such that $\lim a^i = x$.

since A is dense in $\mathbb{R}^n$ and $x \in U \in \mathbb{R}^n \exists \{b^i\} \subseteq A$ such that $\lim b^i = x$

since $x \in U$ and U s open there is a ball $(B(x, r), r > 0)$ in U and $b^i \in B(x, r)$ so $||x - b^i|| < r$. the sequence $\{b^i\}$ is in $A \cap U$ and converges to $x \in U$ hence $A \cap U$ is dense in U.

more notes

$\overline{C}$ is closure

C° is interior

$C = \{(x, y) : y = x^2\}$ $\quad \overline{C} = C \quad C^\circ = \emptyset$

$S = \{(x, y) : y = \sin \frac{1}{x}\}$ $\quad \overline{S} = S \cup \{(0, a) : a \in [-1, 1]\} \quad S^\circ = \emptyset$

cantor set

start with $C_0 = [0, 1], C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1] \ldots$ and continue on removing the middle thirds. $\mathcal{C} = \bigcap_{n=1}^{\infty} C_n$ and by the nested intervals theorem $\mathcal{C} \neq \emptyset$. this is actually an uncountable set.

what are we removing? $C_0 = \frac{1}{3}, C_1 = \frac{2}{9}, C_n = \frac{2^{n-1}}{3(n)}$

length of the removed part is $\sum\limits_{n=1}^{\infty} \frac{2^{n-1}}{3^n} = \frac{1}{3} \sum\limits_{n=1}^{\infty} \left(\frac{2}{3}\right)^{n-1} = 1$ so $\mathcal{C}$ has length 0.

if we look at the sets C_0, C_2, C_2 in base 3 then $x \in [0, 1] = \sum\limits_{i=1}^{\infty} \frac{a_i}{3^i}, a_i \in \{0, 1, 2\}$.

in step n we are removing from $[0, 1]$ all the points whose base 3 expansion has a 1 in the nth position.

so $\mathcal{C} = \{x \in [0, 1]$ st x's base 3 expansion does not contain $1\}$. $F : \mathcal{C} \to [0, 1]$ we map all the twos to ones. $x = 0.2- > 0.1 = y$. this is a surjection. so cardinality of $\mathcal{C}$ is greater than or equal to $[0, 1]$ which is the continuum (cardinality of the real numbers. but $\mathcal{C} \subseteq [0, 1]$ so they have the same cardinality.

assume $(a, b) \subseteq \mathcal{C}$ then $(a, b) \subseteq C_n$. Every interval has lenght 3^{-n} and so we can find an N such that $|a - b| > 3^{-N}$.

defintions

def: a set $A \subseteq \mathbb{R}^n$ is compact if every sequence $\{a^k\}$ of elements of A has a convergent subsequence $\{a^{k_n}\}$ and $\lim a^k = a \in A$ any closed and bounded set in $\mathbb{R}^n$ is compact by bolzano-weierstrass

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definition

a set $A \subseteq \mathbb{R}^n$ is (sequentially) compact if every sequence $\{a_k\}$ such that $a_k \in A \forall k$ has a convergent subsequence $\{a_{k_l}\}$ and $\lim\limits_{l \to \infty} a_{k_l} = a \in A$.

observation

a closed bounded set $B \subseteq \mathbb{R}$ is compact by bolzno-weierstrass

goal

a set $A \subseteq \mathbb{R}^n$ is compat $\Leftrightarrow$ it is losed and bounded. In other topological spaces a closed and bouned set need not be compact.

definition

a set $A \subseteq \mathbb{R}^n$ is bounded if $\exists M > 0$ st $A \subseteq B(o, M)$

lemma 4.4.3

if A is ompact, then A is closed and bounded

proof

to prove that A is closed pick a sequenct $\{a_k\}$, $a_k \in A$ st $\lim a_k = x \in \mathbb{R}^n$. we have to show that $x \in A$. since A is compact and $a_k \in A \forall k$ the sequence $\{a_k\}$ has a convergent subsequence $\{a_{k_l}\}$ st $\lim a_{k_l} \in A$. but a subsequence of a convergent sequence has the same limit as the original sequence, so $x \in A$ as we wanted, so A is closed.

to prove that A is bounded, we assume that A is not bounded.

$\forall M \in \mathbb{N} \exists a_N \in A$ with $||a_N|| \geq N$. consider $\{a_N\}$. any susequence of this subsequence is unbounded also and cannot be convergent. this contradicts compactness, and so A must be bounded

lemma 4.4.4

if $K \in \mathbb{R}^n$ is compact and $C \in K$ and C is closed then C is compact.

proof

To prove the C is compact, we pick any sequence $\{a_k\} \in C$ and we need to show that $\{a_k\}$ has a subsequence convergent to a point in C. $\{a_k\}$ is also a sequence in K and K is compact. hence $\exists \{a_{k_l}\}$ st $\lim a_{k_l} = x \in K$. $\{a_{k_l}\}$ is a sequence in C which is closed. hence $\lim a_{k_l} = x \in C$

lemma 4.4.5

the cube $[-M, M]^n$ is compact in $\mathbb{R}^n$. (cartesian product notation)

proof

pick a sequence $\{a^k\}$ contained in our cube. $\forall k, j = 1, \ldots, n, |a^k_j| \geq M$. $\{a^k_1\} \in \mathbb{R}^1$ sequence of first coords. by BW $\exists \{a^{k_l}_1\}$ which is convergent. Then $\{a^{k_{l_m}}_2\}$ is a subsequence of the sequence of second coords, byt BW $\exists$ a subsequence $\{a^{k_{l_m}}_2\}$ convergent. keep going to n then we get $\{a^{k_j}_n\}$, a convergent subsequence of the nth coords. since each coordinate has a limit, then the original vector sequence has a limit which is in $[-M, M]^n$

finish proof interrupted by lemmas

if $A \in \mathbb{R}^n$ is closed and bounded then A is compact. Since A is bounded, $\exists M > 0$ st $A \in B(0, M) \in [-M, M]^n$ which is compact and so A is compact.

the cantor intersection theorem

let $\{A_n\}$ be a decreasing sequence of nonempty compact set in $\mathbb{R}^n$. $A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots$. then $\bigcap_{j=1}^{\infty} A_j$ nonempty.

proof

pick $a_k \in A_k$. $\{a_k\}$ is a sequence of points in A_1 and A_1 is compact. $\exists\{a_{k_l}\}$ convergent to some $x \in A_1$. for each $l, a_{k_l} \in \bigcap_{i=1}^{k_l} A_i$ for fixed l $a_{k_n} \in \bigcap_{i=1}^{k_l} A_i \forall n \geq l$. then $x \in \bigcap_{i=1}^{k_l} A_i \forall l$ so the same is true to ∞

$\mathcal{C}$ is closed, has empty interior, length 0 and contains uncountably many points, and is compact.

$\mathcal{C}$ has no isolated points. $\forall r > 0$ the inverval $(x-r, x+r)$ contains all the left, right endpoints of intervals in $\mathcal{C}_n$ for all $n \geq N$ and all these endpoints are in $\mathcal{C}$ so each $x \in \mathcal{C}$ is a cluster point of $\mathcal{C}$ because $x = \lim x_n, x_n \in \mathcal{C}, x_n \neq x$.