Numerical Semigroups, Lattice Ideals, and Markov Bases

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Overview

A numerical semigroup is a nonempty subset S of $\mathbb{N}$ that is closed under addition, contains the zero element, and whose complement in $\mathbb{N}$ is finite.

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- It is closed under addition
- It is generated from positive (nonzero) integers
- The greatest common divisor of its generators is 1

Example

Let S be the numerical semigroup generated by $\{n_1, \ldots, n_k\}$ with $n_i \in \mathbb{N} \setminus \{0\}$. Then the elements of S are $a_1 n_1 + \ldots a_k n_k$ for all $a_i \in \mathbb{N}$.

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- *The numerical semigroup generated by* $\{5, 7, 9\}$ *is* $\{0, 5, 7, 9, 10, 12, 14, 15, 16, 17, 18, \ldots\}$

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- *The numerical semigroup generated by* $\{5, 7, 9\}$ *is* $\{0, 5, 7, 9, 10, 12, 14, 15, 16, 17, 18, \ldots\}$
- *The complement of* $\langle 5, 7, 9 \rangle$ *in* $\mathbb{N}$ *is* $\{1,2,3,4,6,8,11,13\}$

Dot product

Each element of $\langle 5, 7, 9\rangle$ is the dot product of the vector $(5, 7, 9)$ and an element of $\mathbb{N}^3$.

Example

$$(5, 7, 9) \cdot (1, 0, 0) = 5$$
$$(5, 7, 9) \cdot (1, 1, 0) = 12$$

Table

We can make a where each row is the vector in $\mathbb{N}^3$ that corresponds to an element in $\langle 5, 7, 9 \rangle$.

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	5	7	9
5	1	0	0

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	5	*7*	*9*
5	*1*	*0*	*0*
7	*0*	*1*	*0*

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Example

	5	7	9
5	1	0	0
7	0	1	0
9	0	0	1

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Example

	5	7	9
5	1	0	0
7	0	1	0
9	0	0	1
10	2	0	0

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	5	7	9
5	1	0	0
7	0	1	0
9	0	0	1
10	2	0	0
12	1	1	0

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Example

	5	7	9
5	1	0	0
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9	0	0	1
10	2	0	0
12	1	1	0
14	1	0	1

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Example

	5	7	9
5	1	0	0
7	0	1	0
9	0	0	1
10	2	0	0
12	1	1	0
14	1	0	1
14	0	2	0

A fiber is the set of vectors which forms the preimage of each element of our NSG.

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Example

$$\mathcal{F}(12) = \{(1,1,0)\}$$
$$\mathcal{F}(14) = \{(1,0,1),(0,2,0)\}$$

Fibers can be disconnected or connected.

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Example

	5	7	9
5	1	0	0
7	0	1	0
9	0	0	1
10	2	0	0
12	1	1	0
14	1	0	1
14	0	2	0
15	3	0	0

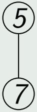
	5	7	9
16	0	1	1
17	2	1	0
19	2	0	1
19	1	2	0
20	4	0	0
21	1	1	1
21	0	3	0
22	3	1	0

It is useful to think of a fiber as a graph.

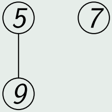
Example

$$\mathcal{F}(5) = \{(1,0,0)\}$$

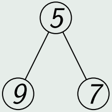
$$\mathcal{F}(12) = \{(1,1,0)\}$$



$$\mathcal{F}(14) = \{(1,0,1),(0,2,0)\}$$



$$\mathcal{F}(19) = \{(2,0,1),(1,2,0)\}$$



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Example

	5	7	5			
14	1	0	1			
14	0	2	0	-1	2	-1
25	5	0	0			
25	0	1	2	5	-1	-2
27	0	0	3			
27	4	1	0	-4	-1	3

We have an easy bijection between our Markov basis and a lattice ideal.

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Example

$$\begin{bmatrix} 1 & -2 & 1 \\ 5 & -1 & -2 \\ -4 & -1 & 3 \end{bmatrix} \Leftrightarrow \begin{cases} xz - y^2 \\ x^5 - yz^2 \\ z^3 - x^4y \end{cases}$$

- We have actually explicitly built our Markov basis to be the null space of the numerical semigroup basis.

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- If we can find some vector $\vec{x}$ such that

$$\begin{bmatrix} 1 & -2 & 1 \\ 5 & -1 & -2 \\ -4 & -1 & 3 \end{bmatrix} \vec{x} = 0$$

Then we will have found our semigroup!

What we need is the Smith Normal Form.

$$\begin{bmatrix} 1 & -2 & 1 \\ 5 & -1 & -2 \\ -4 & -1 & 3 \end{bmatrix} = UAV$$

We start with identity matrices on either side of our Markov matrix. The procedure is similar to finding an inverse matrix, (except the Markov matrix is singular).

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We reduce our Markov matrix, mirroring column and row operations in the adjacent matrices.
We can't use anything but integers for our row and column operations!

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ 5 & -1 & -2 \\ -4 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Row operations on the left

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ 5 & -1 & -2 \\ -4 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 9 & -7 \\ 0 & -9 & 7 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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Column operations on the right

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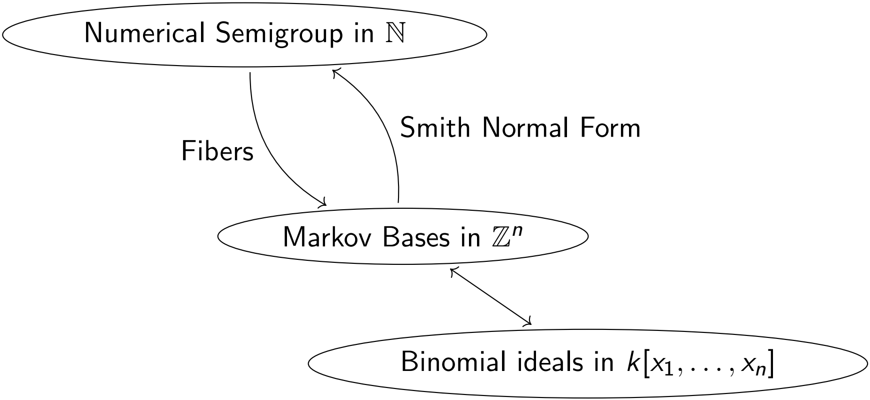
Numerical Semigroup in $\mathbb{N}$
Fibers
Smith Normal Form
Markov Bases in $\mathbb{Z}^n$
Binomial ideals in $k[x_1, \ldots, x_n]$

Thank You!

Questions?