Homework

August 29, 2014

Read Theorems 1.1.4, 1.1.6 and their proofs from Section 1.1
Exercises: 2(d), 7, 8, 16, 17.

2. Find the quotient and remainder when a is divided by b

   (d) $a = -1017, b = 99$

$$\begin{aligned}
\gcd(-1017, 99) &= \gcd(1017, 99) \\
1017 &= 99 * 10 + 27 \\
\gcd(-1017, 99) &= \gcd(99, 27) \\
99 = 27 * 3 + 18 \\
\gcd(-1017, 99) &= \gcd(27, 18) \\
27 = 18 * 1 + 9 \\
\gcd(-1017, 99) &= \gcd(18, 9) \\
9|18 &\to \gcd(-1017, 99) = 9
\end{aligned}$$

7. Let $a, b, c \in \mathbb{Z}$. Prove these facts about divisors:

   (a) if $b|a$, then $b|ac$
   if $b|a$ then $\exists q \in \mathbb{Z}$ such that $a = bq$. Then $ac = bqc$ and so b divides ac $\square$

   (b) if $b|a$ and $c|b$, then $c|a$
   if $b|a$ then $\exists q_1 \in \mathbb{Z}$ such that $a = bq_1$. If $c|b$ then $\exists q_2 \in \mathbb{Z}$ such that $b = cq_2$. We see then that $a = bq_1 = (cq_2)q_1 = c(q_1q_2)$ and therefore $c|a$ $\square$

   (c) if $c|a$, and $c|b$, then $c|(ma + nb)$ for any integers m, n
   if $c|a$ and $c|b$ then $\exists q_1, q_2 \in \mathbb{Z}$ such that $a = cq_1$ and $b = cq_2$. So we see that $(ma + nb) = (mcq_1 + ncq_2) = c(mq_1 + nq_2)$ and $c|(ma + nb)$

8. Let a, b, c be integers such that $a + b + c = 0$. Show that if n is an integer which is a divisor of two of the three integers, then it is also a divisor of the third.

   Because $a + b + c = 0$ we can say that $a + b = (-1)c$. Now let us say that $n|a$ and $n|b$. Then there exists $q_1, q_2 \in \mathbb{Z}$ such that $a + b = n(q_1 + q_2) = (-1)c$.

16. Let $a, b, c \in \mathbb{Z}$ with $b > 0, c > 0$, and let q be the quotient and r the remainder when a is divided by b.

    (a) Show that q is the qotient and rc is the remainder when ac is divided by bc

    (b) Show that if q' is the quotient when q is divided by c, then q' is the quotient when a is divided by bc. (Do not assume that the remainders are 0.)

17. Let a, b, n be integers with $n > 1$. Suppose that $a = nq_1 + r_1$ with $0 \leq r_1 < n$ and $b = nq_2 + r_2$ with $0 \leq r_2 < n$. Prove that $n|(a-b)$ if and only if $r_1 = r_2$

$$\begin{aligned} a - b &= nq_1 + r_1 - (nq_2 + r_2) \\ &= n(q_1 - q_2) + (r_1 - r_2) \end{aligned}$$

$0 \leq r_1 < n$ and $0 \leq r_2 < n$ therefore $|r1 - r2| < n$. So by the division algorithm $n|(a-b)$ iff $r_1 - r_2 = 0$

class proof

$$\begin{aligned} a &= nq_1 + r_1, 0 \leq r_1 < n \\ b &= nq_2 + r_2, 0 \leq r_2 < n \\ a - b &= nq \end{aligned}$$

q=some integer

$$\begin{aligned} nq_1 + r_1 - (nq_2 + r_2) &= nq \\ n(q_1 - q_2) + (r_1 - r_2) &= nq \\ r_1 - r_2 &= nq - n(q_1 - q_2) \\ n|(r_1 - r_2) \\ |r_1 - r_2| < n \end{aligned}$$

since $r_i < n, r_1 - r_2 = 0$

other direction if $r_1 = r_2 = r_0$ $a = nq_1 + r_1$ and $b = nq_2 + r_2$

$$\begin{aligned} a - nq_1 &= r_0 = r - nq_2 \\ a - b &= nq_1 - nq_2 \\ a - b &= n(q_1 - q_2) \\ n&|(a - b) \end{aligned}$$