MATH 420/620–Exam II

November 17th, 2008

Name: ______________________

For full credit you must **show all the details** of your work and **justify all the answers**. The test has extra credit problems worth a total of 40 points.

1. (10p) Let H be a subgroup of G of index 2. Prove that H is a normal subgroup of G.

2. (20p) Let $f : G_1 \to G_2$ be a group homomorphism. Prove that $\operatorname{Ker} f$ is a normal subgroup of G_1.

3. (a) (20p) Let G be a group, H a subgroup of G, and $a \in G$. Prove that aHa^{-1} is a subgroup of G and $H \cong aHa^{-1}$.

(b) (5p) Let G be a group and H a subgroup of G with n elements. Assume that there are no other subgroups of G with n elements. Prove that H is a normal subgroup of G.

4. (10p) Prove that the multiplicative groups $\mathbb{R} - \{0\}$ and $\mathbb{C} - \{0\}$ are not isomorphic.

5. (10p) Let p be a prime number and let G be a group with p^n elements (n positive integer). Prove that G contains an element of order p.

6. Let p be a prime number.

(a) (10p) Write the polynomial $X^{p-1} - 1 \in \mathbb{Z}_p[X]$ as a product of irreducible polynomials in $\mathbb{Z}_p[X]$.

(b) (5p) Prove that p divides $(p-1)! + 1$.

7. (a) (10p) Let H be a normal subgroup of index n in G. Prove that $a^n \in H$ for all $a \in G$.

(b) **Extra Credit (10p)** Is the above conclusion true if we only assume that H is a subgroup of G (not necessarily normal)? Justify your answer.

8. **Extra Credit (10p)** Let $G = \mathbb{Z} \times \mathbb{Z}$ and let H be the cyclic subgroup of G generated by $(0,1)$. Prove that $G/H \cong \mathbb{Z}$.

9. **Extra Credit (10p)** Give an example of a group G and a subgroup H of G such that $H \cong G$ but $H \neq G$.

10. **Extra Credit (10p)** Prove that the fields $\mathbb{Q}[X]/\langle x^2+1\rangle$ and $\mathbb{Q}[X]/\langle x^2+2\rangle$ are not isomorphic.