# Notes

November 5, 2014

## 4.2 2a,8,9

### 9

show that the remainder when $f(x)$ is divided by $(x-a)^2$ is $f'(a)(x-a)+f(a)$

$f(x) = (x-a)^2 q(x) + r(x)$ and $\deg r < 2$ $r(x) = \alpha x + \beta$. $f(a) = r(a) = \alpha a + \beta$. $f'(x) = 2(x-a)q(x) + (x-a)^2 q'(x) + \alpha$ $f'(a) = \alpha$

$f(a) = f'(a)a + \beta \to \beta = f(a) - af'(a)$.o

$r(x) = f'(a)x + f(a) - af'(a) = f'(a)(x-a) + f(a)$

## proposition

let$I \subseteq K[x]$ such that

1. $I$ contains a non-zero polynomial
2. $f(x), g(x) \in I \Rightarrow f(x) + g(x) \in I$
3. $f(x) \in I, g(x) \in K[x] \Rightarrow f(x)g(x) \in I$

this is the ideal of $K[x]$ let $d(x) \in I$ of minimal degree. then $I = \{a(x)f(x) : f(x) \in K[x]\}$

### proof

let $h(x) \in I$ write $h(x) = d(x)q(x) + r(x)$ with $r(x) = 0$ or $\deg r > \deg d$

then $r(x) = h(x) + d(x)[-q(x)] \in I$ by 2 above. by choice of $d(x)$ we have $r(x) = 0$ and so $h(x) \in I$

## def

$f(x), g(x) \in K[x]$ where $K$ is a field. a monic polynomial $a(x) \in K[x]$ is called gcd of $f(x), g(x)$ if

1. $a(x)|f(x)$ and $a(x)|g(x)$
2. if $t(x)|f(x)$ and $t(x)|g(x)$ then $t(x)|a(x)$

monic means that the leading coefficient is 1

## thm

if we have $f(x), g(x)$ non-zero, then $\exists \gcd(f(x), g(x))$ and $\gcd(f(x), g(x))$ can be expressed in the form $\alpha(x)f(x) + \beta(x)g(x)$.

### proof

let ideal $I = \{\alpha(x)f(x) + \beta(x)g(x)\}$. Check that $I$ satisfies all conditions of earlier proposition.

let $d(x) \in I$ of minimal degree and without loss of generality assume $d(x)$ is monic. we can do this because multiplying by a constant is multiplying by a polynomial, so it's still in $I$.

claim $d(x)$ is a gcd of $f(x), g(x)$

$I = \{d(x)h(x) : h(x) \in K[x]\}$ in particular $d(x)f(x)$ and $d(x)g(x)$ are both in $I$. now if $t(x)|f(x)$ and $t(x)|g(x)$. $\exists \alpha(x), \beta(x)$ such that $d(x) = \alpha(x)f(x) + \beta(x)g(x)$. then $t(x)|d(x)$.

## thm

the gcd is unique. lets assume that $d_1(x)$ and $d_2(x)$ are gcd of $f(x)$ and $g(x)$. $d_2(x)|f(x)$ and $d_2(x)|g(x)$. $d_1$ is gcd so $d_1|d_2$ and $d_1(x)|f(x)$ and $d_1(x)|g(x)$ so because $d_2$ is gcd then $d_2|d_1$. we said our gcd was monic.

$d_1(x) = d_2(x)\alpha_1(x)$ and $d_2(x) = d_1(x)\alpha_2(x)$. now $d_1(x)\alpha_1(x)\alpha_2(x) \to d_2 = d_1(x)\alpha_1(x)\alpha_2(x)$

## thm

if $p(x)|f(x)g(x)$ and $\gcd(p(x), f(x)) = 1$ then $p(x)|g(x)$.

## def

$f(x) \in K[x]$. we say that $f(x)$ is irreducible over the field $K$ if $f(x)$ cannot be factored into a product of two polynomials of degree lower than $\deg f(x)$.

### example

$x^2 + 1$ is irreducible over $\mathbb{R}$

### example

$x^2 + 1 \in \mathbb{C}[x]$ is reducible (not irreducible) over $\mathbb{C}$

## prop

$f(x) \in K[x]$, $\deg f(x)$ is 2 or 3 and $f(x)$ has no roots in $K$. then $f(x)$ is irreducible.

### proof

we choose $\deg 2, 3$ because at least one of the factors is $\deg 1$

we assume the factors exist. then $f(x) = (\alpha x + \beta)h(x)$ and $f(-\beta\alpha^{-1}) = 0$ and it has a root.

## thm

every non-zero polynomial in $K[x]$ can be written uniquely as a product of irreducible polynomials.

proof is inductive argument

## def

$f(x) \in K[x], c \in K$ we say that $c$ is a root of multiplicity $m$ of $f(x)$ if $(x-c)^m$ divides $f(x)$ and $(x-c)^{m+1} \not| f(x)$

## prop

$f(x) \in \mathbb{R}[x]$, $\deg f(x) \geq 1$ then $f(x)$ has no repeatable factors iff the $gcd$ of $f(x)$ and $f'(x)$ is one