MATH 450: Homework 1

Problems marked with * will be presented in class by students.

1. Prove the following using Axioms 1-9. Indicate which axiom you use at each step.
   (a) * If $x > 1$, then $x^2 > x$. If $0 < x < 1$, then $x^2 < x$.
   (b) If $x > 0$, then $x^{-1} > 0$.
   (c) If $0 < x < y$, then $0 < y^{-1} < x^{-1}$.

2. Find the supremum and infimum of the following sets (if they exist). Indicate also if the set has a maximum or minimum element.
   (a) $\{\frac{1}{n} + (-1)^n : n \in \mathbb{Z}^+\}$
   (b) $\{x : x^2 + x + 1 \geq 0\}$

3. * Let $A \subseteq \mathbb{R}$ be bounded below. Let $-A = \{-x : x \in A\}$. Show that $-A$ is bounded above, and that $\inf(A) = -\sup(-A)$.

Graduate Problem (for students enrolled in Math 650):
Existence of the n-th root: Prove that for any positive real number x and integer $n \geq 1$ there exists a unique real number y such that $y^n = x$.

- *Hint 1:* Consider the set $\{z > 0 : z^n \leq x\}$. Show that this set is bounded above.
- *Hint 2:* Use Hint 1 to find a candidate for y. Now you have to show that your candidate is actually the root we are looking for. What method of proof seems appropriate?
- *Hint 3:* For $0 < h \leq 1$, use the binomial formula to show that $(y + h)^n \leq y^n + h((1 + y)^n - y^n)$.