**Theorem.** The dot product has the following properties.
(1) $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.
(2) $\mathbf{x} \cdot \mathbf{x} = \|\mathbf{x}\|^2 \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$. Moreover $\mathbf{x} \cdot \mathbf{x} = 0$ if and only if $\mathbf{x} = \mathbf{0}$.
(3) $(c\mathbf{x}) \cdot \mathbf{y} = c(\mathbf{x} \cdot \mathbf{y}) = \mathbf{x} \cdot (c\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $c \in \mathbb{R}$.
(4) $\mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) = \mathbf{x} \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{z}$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$.
(5) If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, then $|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\| \|\mathbf{y}\|$.
(6) We define the angle $\theta$ between the vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ to be the real number $\theta = \cos^{-1}(\frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|})$.
(7) Let $\mathbf{a} \in \mathbb{R}^n$ be a nonzero vector. The *hyperplane* in $\mathbb{R}^n$ with normal vector $\mathbf{a}$ through the point $\mathbf{x}_0$ is the set

$$H(\mathbf{x}_0, \mathbf{a}) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a} \cdot \mathbf{x} = \mathbf{a} \cdot \mathbf{x}_0\}.$$

**Theorem.** If $A \in \mathcal{M}_{m \times p}$ and $B \in \mathcal{M}_{p \times n}$, then $\mathrm{ent}_{ij}(AB) = \mathbf{r}_i(A) \cdot \mathbf{c}_j(B)$.
**Theorem.** If $A \in \mathcal{M}_{m \times p}$ and $B \in \mathcal{M}_{p \times n}$, then $\mathbf{c}_j(AB) = A\mathbf{c}_j(B)$. In particular, we have that $\mathbf{c}_j(AB) = b_{1j}\mathbf{c}_1(A) + b_{2j}\mathbf{c}_2(A) + ... + b_{pj}\mathbf{c}_p(A)$ so that $\mathbf{c}_j(AB) \in \mathrm{Span}(\mathbf{c}_1(A), \mathbf{c}_2(A), ..., \mathbf{c}_p(A))$.
**Theorem .** If $A \in \mathcal{M}_{m \times p}$ and $B \in \mathcal{M}_{p \times n}$, then $\mathbf{r}_i(AB) = \mathbf{r}_i(A)B$. In particular, we have that $\mathbf{r}_i(AB) = a_{i1}\mathbf{r}_1(B) + a_{i2}\mathbf{r}_2(B) + .... + a_{ip}\mathbf{r}_p(B)$. so that $\mathbf{r}_i(AB) \in \mathrm{Span}(\mathbf{r}_1(B), \mathbf{r}_2(B), ...., \mathbf{r}_p(B))$.
**Theorem.** Matrix multiplication is is an associative operation in the sense that $A \in \mathcal{M}_{m \times p}$, $B \in \mathcal{M}_{p \times q}$, $C \in \mathcal{M}_{q \times n}$ implies $(AB)C = A(BC)$.
**Theorem.** Let $A, A' \in \mathcal{M}_{m \times p}$, $B, B' \in \mathcal{M}_{p \times n}$. Then
(1) $I_m A = A = A I_n$
(2) $(A + A')B = AB + A'B$
(3) $A(B + B') = AB + AB'$
(4) $(cA)B = cAB = A(cB)$ for all $c \in \mathbb{R}$.
**Theorem.** Let $A, B \in \mathcal{M}_n$ be invertible matrices. Then, (1) $A^{-1}$ is invertible and $(A^{-1})^{-1} = A$, (2) $AB$ is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.
**Theorem.** Elementary matrices are invertible and their inverses are elementary.
**Theorem** Let $A \in \mathcal{M}_{m \times n}$ and let $A\mathbf{x} = \mathbf{b}$ be a linear system. Assume that this system is in $rref$.
(1) If $\mathrm{rank}(A) < \mathrm{rank}[A \mid \mathbf{b}]$, then $A\mathbf{x} = \mathbf{b}$ has no solution.
(2) If $\mathrm{rank}(A) = \mathrm{rank}[A \mid \mathbf{b}] = n$, then $A\mathbf{x} = \mathbf{b}$ has exactly one solution.
(3) If $\mathrm{rank}(A) = \mathrm{rank}[A \mid \mathbf{b}] < n$, then $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions. In particular, if $A\mathbf{x} = \mathbf{0}$ has more unknowns than equations, then it has infinitely many solutions
**Theorem.** The following statements are equivalent for a *square* matrix $A \in \mathcal{M}_n$. (1) $A$ is nonsingular. (2) $rref(A) = I_n$ (3) $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b} \in \mathbb{R}^n$. (4) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. (5) $A$ is invertible. (6) $A$ is a product of elementary matrices.
**Theorem.** Let $A, A' \in \mathcal{M}_{m \times p}$, $B \in \mathcal{M}_{p \times n}$, and $c \in \mathbb{R}$. Then (1) $(A^{\mathrm{T}})^{\mathrm{T}} = A$ (2) $cA^{\mathrm{T}} = (cA)^{\mathrm{T}}$ (3) $(A + A')^{\mathrm{T}} = A^{\mathrm{T}} + A'^{\mathrm{T}}$ (4) $(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}$.
**Theorem.** Let $A \in \mathcal{M}_{m \times n}$ and $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^m$. Then $A\mathbf{x} \cdot \mathbf{y} = \mathbf{x} \cdot A^{\mathrm{T}}\mathbf{y}$.