Due at the beginning of class Wednesday, March 12.

Read the rest of Section 5.6, and also Sections 5.3, and 5.4 and work the following problems:

Chapter 5: #30, 33, 34, 37, 38, 40, 42 (Grad: #32, 34)

Read Section 6.1 and work the following problems:

Chapter 6: #1, 2, 3

Also, do the following exercise:

Project Prep #1: Investigate one of the combinatorial collections on findstat.org. (On the main page, click on 'wiki' next to the combinatorial collection of your choice, and see what's on the wiki page. Also, go to the 'statistics finder' and 'statistics database' tabs.) Write a paragraph stating some things you were able to learn about your combinatorial collection from this website. Be sure to mention at least one statistic. Write down at least one question you still have about these objects. *NOT* finite cartan types.

Chapter 5

30. Prove that the only antichain of $S = \{1, 2, 3, 4\}$ of size 6 is the antichain of all 2-subsets of S

    We can easily see that the only antichain that contains $\emptyset$ is $\{\emptyset\}$, similarly the only antichain containing S is $\{S\}$. Both of these are obviously of size one. We see then that any antichain that is to be bigger than 6 must contain a subset of size 1,2 or 3. Lets pick a subset of size one to be in our antichain. Without loss of generality we say that subset is $\{1\}$. Since any other subsets in our antichain must not be a superset of $\{1\}$ we can form the remaining subsets in our antichain from $\{2, 3, 4\}$. We know from theorem 5.3.3 that we can form an antichain of maximum size $\binom{3}{\lfloor\frac{3}{2}\rfloor} = 3$ from this set of size 3. The maximum size then of an antichain containing a subset of size 1 is then $3 + 1 = 4$ which is less than 6. The antichain we are looking for then must be made up entirely of subsets of size 2 and 3. Because there are only 6 subsets of size two and we wish to find an antichain with more than 6 subsets, there must be a subset of size 3 in our antichain.

    Let us assume without loss of generality that our antichain contains a subset of size 2.

33. Construct a partition of the subsets of $\{1, 2, 3, 4, 5\}$ into symmetric chains.

$$\begin{aligned}
\emptyset \subset \{1\} \subset \{1,2\} \subset \{1,2,3\} \subset \{1,2,3,4\} &\subset \{1,2,3,4,5\} \\
\{5\} \subset \{1,5\} \subset \{1,2,5\} &\subset \{1,2,3,5\} \\
\{4\} \subset \{1,4\} \subset \{1,2,4\} &\subset \{1,2,4,5\} \\
\{4,5\} &\subset \{1,4,5\} \\
\{2\} \subset \{2,3\} \subset \{2,3,4\} &\subset \{2,3,4,5\} \\
\{2,5\} &\subset \{2,3,5\} \\
\{2,4\} &\subset \{2,4,5\} \\
\{3\} \subset \{1,3\} \subset \{1,3,4\} &\subset \{1,3,4,5\} \\
\{3,5\} &\subset \{1,3,5\} \\
\{3,4\} &\subset \{3,4,5\}
\end{aligned}$$

37. Use the multinomial theorem to show that, for positive integers n and t,

$$t^n = \sum \binom{n}{n_1 n_2 \dots n_t},$$

where the summation extends over all nonnegative integral solutions $n_1, n_2, \dots, n_t$ of $n_1 + n_2 + \cdots + n_t = n$

$$\begin{aligned} t^n &= (1_1 + 1_2 + \dots 1_t)^n \\ &= \sum \binom{n}{n_1 n_2 \cdots n_t} {1_1}^{n_1} {1_2}^{n_2} \cdots {1_t}^{n_t} \\ &= \sum \binom{n}{n_1 n_2 \cdots n_t} \end{aligned}$$

38. Use the multinomial theorem to expand $(x_1 + x_2 + x_3)^4$

$$\begin{aligned} (x_1 + x_2 + x_3)^4 = &\sum_{n_1+n_2+n_3=4} \binom{4}{n_1 n_2 n_3} {x_1}^{n_1} {x_2}^{n_2} {x_3}^{n_3} \\ &= \binom{4}{4\,0\,0} {x_1}^4 {x_2}^0 {x_3}^0 + \binom{4}{0\,4\,0} {x_1}^0 {x_2}^4 {x_3}^0 + \binom{4}{0\,0\,4} {x_1}^0 {x_2}^0 {x_3}^4 \\ &+ \binom{4}{3\,1\,0} {x_1}^3 {x_2}^1 {x_3}^0 + \binom{4}{3\,0\,1} {x_1}^3 {x_2}^0 {x_3}^1 + \binom{4}{0\,3\,1} {x_1}^0 {x_2}^3 {x_3}^1 \\ &+ \binom{4}{1\,3\,0} {x_1}^1 {x_2}^3 {x_3}^0 + \binom{4}{0\,1\,3} {x_1}^0 {x_2}^1 {x_3}^3 + \binom{4}{1\,0\,3} {x_1}^1 {x_2}^0 {x_3}^3 \\ &+ \binom{4}{2\,0\,2} {x_1}^2 {x_2}^0 {x_3}^2 + \binom{4}{2\,2\,0} {x_1}^2 {x_2}^2 {x_3}^0 + \binom{4}{0\,2\,2} {x_1}^0 {x_2}^2 {x_3}^2 \\ &+ \binom{4}{2\,1\,1} {x_1}^2 {x_2}^1 {x_3}^1 + \binom{4}{1\,2\,1} {x_1}^1 {x_2}^2 {x_3}^1 + \binom{4}{1\,1\,2} {x_1}^1 {x_2}^1 {x_3}^2 \\ &= {x_1}^4 + {x_2}^4 + {x_3}^4 \\ &+ 4{x_1}^3 x_2 + 4{x_1}^3 x_3 + 4{x_2}^3 x_3 \\ &+ 4x_1 {x_2}^3 + 4x_2 {x_3}^3 + 4x_1 {x_3}^3 \\ &+ 6{x_1}^2 {x_3}^2 + 6{x_1}^2 {x_2}^2 + 6{x_2}^2 {x_3}^2 \\ &+ 12{x_1}^2 x_2 x_3 + 12 x_1 {x_2}^2 x_3 + 12 x_1 x_2 {x_3}^2 \end{aligned}$$

40. What is the coefficient of ${x_1}^3 {x_2}^3 x_3 {x_4}^2$ in the expansion of

$$(x_1 - x_2 + 2x_3 - 2x_4)^9?$$

$$\binom{9}{3\,3\,1\,2} \cdot 1^3 \cdot (-1)^3 \cdot 2^1 \cdot (-2)^2 = 5040 \cdot -1 \cdot 2 \cdot 4 = -40320$$

42. Prove the identity (5.21) by a combinatorial argument. (*Hint:* Consider the permutations of a multiset of objects of t different types with repetition numbers $n_1, n_2, \dots, n_t$, respectively. Partition these permutations according to what type of object is in the first position.)

Chapter 6

1. Find the number of integers between 1 and 10,000 inclusive that are not divisible by 4,5, or 6.

   I'll start by just saying that all the math I do will be integer math, e.g. $\frac{10}{2} = 2$. So we can just subtract from 10,000 the number of integers that are divisable by 4,5, or 6 to get the number of integers that are not. The number of integers divisable by 4 is $10000/4 = 2500$. Similarly for 5 we have $10000/5 = 2000$ and for 6 we have $10000/6 = 1666$. Since 20 is the least common multiple of 5 and 4 the number of integers that can be obtained by dividing 10000 by 4 or 5 is $10000/20 = 500$ and similarly for 4 and 6 we have $10000/12 = 833$. And for 5 and 6 $10000/30 = 333$. Finally the number of integers that can be obtained by dividing by 4,5, or 6 is $10000/60 = 166$. Putting it all together with the inclusion-exclusion principle (inverted since we are subtracting from the total) we have $10000 - 2500 - 2000 - 1666 + 500 + 833 + 333 - 166 = 5334$ integers.

2. Find the number of integers between 1 and 10,000 inclusive that are not divisible by 4,6,7, or 10.

   As in problem 1, all math will be integer math, not real. Logic is also the same as in problem 1.

$$\begin{array}{cccc}
\text{lcm}(4{,}6) = 12 & \text{lcm}(6,7) = 42 & \text{lcm}(7,10) = 70 & \text{lcm}(10,4) = 20 \\
\text{lcm}(4,6,7) = 84 & \text{lcm}(4,6,10) = 60 & \text{lcm}(6,7,10) = 210 & \text{lcm}(4,6,7,10) = 420 \\
10000/4 = 2500 & 10000/6 = 1666 & 10000/7 = 1428 & 10000/10 = 1000 \\
10000/12 = 833 & 10000/42 = 238 & 10000/70 = 142 & 10000/20 = 500 \\
10000/84 = 119 & 10000/60 = 166 & 10000/210 = 47 & 10000/420 = 23
\end{array}$$

   So we have $10000-2500-1666-1428-1000+833+238+142+500-119-166-47+23 = 4810$ integers.

3. Find the number of integers between 1 and 10,000 that are neither perfect squares nor perfect cubes.

   Because $\sqrt{10000} = 100$ the perfect squares between 1 and 10000 inclusive are $\{1^2, 2^2, \ldots, 100^2\}$. There are then 100 perfect squares. Similarly because $21^3 < 10000 < 22^3$ there are 21 perfect cubes which are $\{1^3, 2^3, \ldots, 21^3\}$. Now if something is a perfect cube *and* a perfect square, than it is a perfect sixth root. Because $4^6 < 10000 < 5^6$ there are 4 numbers which are perfect roots and perfect squares between 1 and 10000. Putting it all together we have $10000 - 100 - 21 + 4 = 9883$ integers that are neither perfect cubes, nor perfect squares.