1.2

1. For each of the following pairs of vectors $\mathbf{x}$ and $\mathbf{y}$, calculate $\mathbf{x}\cdot\mathbf{y}$ and the angle θ between the vectors.

   d. $\mathbf{x} = (1,4,-3), \mathbf{y} = (5,1,3)$

$$||\mathbf{x}||||\mathbf{y}||\cos\theta = \mathbf{x}\cdot\mathbf{y} \qquad \cos\theta = \frac{\mathbf{x}\cdot\mathbf{y}}{||\mathbf{x}||||\mathbf{y}||} \qquad \theta = \cos^{-1}\frac{\mathbf{x}\cdot\mathbf{y}}{||\mathbf{x}||||\mathbf{y}||}$$

$$\theta = \cos^{-1}\frac{5+4-9}{||\mathbf{x}||||\mathbf{y}||} \qquad \theta = \cos^{-1}0 \qquad \theta = \frac{\pi}{2}$$

2. For each pair in exercise 1, calculate $\text{proj}_{\mathbf{y}}\mathbf{x}$ and $\text{proj}_{\mathbf{x}}\mathbf{y}$

   d. The vectors are orthogonal. The projection of either onto the other is the zero vector.

7. Suppose $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n$, $||\mathbf{x}|| = \sqrt{2}, ||\mathbf{y}|| = 1$, and the angle between x and y is $3\pi/4$. Show that the vectors $2\mathbf{x}+3\mathbf{y}$ and $\mathbf{x}-\mathbf{y}$ are orthogonal.

   Using proposition 2.1 we see that $(2\mathbf{x}+3\mathbf{y})\cdot(\mathbf{x}-\mathbf{y}) = 2\mathbf{x}\cdot\mathbf{x} - 2\mathbf{x}\cdot\mathbf{y} + 3\mathbf{y}\cdot\mathbf{x} - 3\mathbf{y}\cdot\mathbf{y} = 2||\mathbf{x}||^2 + \mathbf{x}\cdot\mathbf{y} - 3||\mathbf{y}||^2$. From the definition of the angle between two vectors we can further simplify this expression to $2\cdot 2 + ||\mathbf{x}||||\mathbf{y}||\cos\frac{3\pi}{4} + 3 = 4 + \sqrt{2}\cdot(-\frac{\sqrt{2}}{2}) - 3 = 0$ and so the vectors are orthogonal.

10. Let $\mathbf{x} = (1,1,1,\ldots,1) \in \mathbb{R}^n$ and $\mathbf{y} = (1,2,3,\ldots,n) \in \mathbb{R}^n$. Let θ_n be the angle between $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$. Find $\lim\limits_{n\to\infty}\theta_n$. (The formulas $1+2+\cdots+n = n(n+1)/2$ and $1^2+2^2+\cdots+n^2 = n(n+1)(2n+1)/6$ may be useful.)

    We know that by definition $\theta_n = \cos^{-1}\frac{\mathbf{x}\cdot\mathbf{y}}{||\mathbf{x}||||\mathbf{y}||}$ and so

$$\begin{aligned}
\lim_{n\to\infty}\theta_n &= \lim_{n\to\infty}\cos^{-1}\frac{\mathbf{x}\cdot\mathbf{y}}{||\mathbf{x}||||\mathbf{y}||} \\
&= \lim_{n\to\infty}\cos^{-1}\frac{1\cdot 1+1\cdot 2+\cdots+1\cdot n}{\sqrt{n}\sqrt{1^2+2^2+\cdots+n^2}} \\
&= \lim_{n\to\infty}\cos^{-1}\frac{\sqrt{6}n(n+1)}{2\sqrt{n}\sqrt{n(n+1)(2n+1)}} \\
&= \lim_{n\to\infty}\cos^{-1}\sqrt{\frac{3(n+1)}{2(2n+1)}} = \lim_{n\to\infty}\cos^{-1}\sqrt{\frac{3(n+1/2+1/2)}{4(n+1/2)}} \\
&= \lim_{n\to\infty}\cos^{-1}\sqrt{\frac{3}{4}\left(1+\frac{1/2}{n+1/2}\right)} = \cos^{-1}\sqrt{\frac{3}{4}\left(1+\lim_{n\to\infty}\frac{1/2}{n+1/2}\right)} \\
&= \cos^{-1}\sqrt{\frac{3}{4}(1+0)} = \frac{\pi}{6}
\end{aligned}$$

11. Suppose $\mathbf{x},\mathbf{v}_1,\ldots,\mathbf{v}_k \in \mathbb{R}^n$ and $\mathbf{x}$ is orthogonal to each of the vectors $\mathbf{v}_1,\ldots\mathbf{v}_k$. Show that $\mathbf{x}$ is orthogonal to any linear combination $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$.

    We know that $\mathbf{x}$ is orthogonal if and only if the dot product is zero. So lets just find it.

$$\begin{aligned}
\mathbf{x}\cdot(c_1\mathbf{v}_1+\cdots+c_k\mathbf{v}_k) &= \mathbf{x}\cdot(c_1\mathbf{v}_1)+\cdots+\mathbf{x}\cdot(c_k\mathbf{v}_k) \\
&= c_1(\mathbf{x}\cdot\mathbf{v}_1)+\cdots+c_k(\mathbf{x}\cdot\mathbf{v}_k)
\end{aligned}$$

    But then $\mathbf{x}$ is orthogonal to $\mathbf{v}_i$ for all $0 < i \le k$. Which leads us to $\mathbf{x}\cdot\mathbf{v}_i = 0$ and $\mathbf{x}\cdot(c_1\mathbf{v}_1+\cdots+c_k\mathbf{v}_k) = c_1\cdot 0+\cdots+c_k\cdot 0 = 0$. And we have our result.

13. Use the algebraic properties of the dot product to show that

$$||\mathbf{x}+\mathbf{y}||^2 + ||\mathbf{x}-\mathbf{y}||^2 = 2(||\mathbf{x}||^2+||\mathbf{y}||^2).$$

Interpret the result geometrically.

$$\begin{aligned} ||\mathbf{x}+\mathbf{y}||^2 + ||\mathbf{x}-\mathbf{y}||^2 &= ||\mathbf{x}+\mathbf{y}||^2 + ||\mathbf{x}+(-\mathbf{y})||^2 \\ &= (||\mathbf{x}||^2 + 2\mathbf{x}\cdot\mathbf{y} + ||\mathbf{y}||^2) + (||\mathbf{x}||^2 + 2\mathbf{x}\cdot(-\mathbf{y}) + ||-\mathbf{y}||^2) \\ &= ||\mathbf{x}||^2 + 2\mathbf{x}\cdot\mathbf{y} + ||\mathbf{y}||^2 + ||\mathbf{x}||^2 - 2\mathbf{x}\cdot\mathbf{y} + ||\mathbf{y}||^2 \\ &= ||\mathbf{x}||^2 + ||\mathbf{y}||^2 + ||\mathbf{x}||^2 + ||\mathbf{y}||^2 \\ &= 2(||\mathbf{x}||^2 + ||\mathbf{y}||^2) \end{aligned}$$

14. Use the dot product to prove the law of cosines: As shown in Figure 2.8.

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

Let $\overline{CB} = \mathbf{a}, \overline{CA} = \mathbf{b}$, and $\overline{BA} = \mathbf{c}$. Notice that $\mathbf{c} = \mathbf{b} - \mathbf{a}$. And so $c = ||\mathbf{b}-\mathbf{a}||$, $a = ||\mathbf{a}||$, and $b = ||\mathbf{b}||$. Using corollary 2.3 from the notes and definition 2.9 from the notes we have

$$\begin{aligned} c^2 &= ||\mathbf{b}-\mathbf{a}||^2 \\ &= ||\mathbf{b}||^2 - 2\mathbf{b}\cdot\mathbf{a} + ||\mathbf{a}||^2 \\ &= ||\mathbf{a}||^2 + ||\mathbf{b}||^2 - 2\mathbf{b}\cdot\mathbf{a}\frac{||\mathbf{a}||||\mathbf{b}||}{||\mathbf{a}||||\mathbf{b}||} \\ &= a^2 + b^2 - 2ab\frac{\mathbf{a}\cdot\mathbf{b}}{||\mathbf{a}||||\mathbf{b}||} \\ &= a^2 + b^2 - 2ab\cos\theta \end{aligned}$$

Boom.$\square$

17. If $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, set $\rho(\mathbf{x}) = (-x_2, x_1)$.

   a. Check that $\rho(\mathbf{x})$ is orthogonal to $\mathbf{x}$. (Indeed, $\rho(\mathbf{x})$ is obtained by rotating $\mathbf{x}$ an angle $\pi/2$ counterclockwise.)

   $$\mathbf{x}\cdot\rho(\mathbf{x}) = (x_1, x_2)\cdot(-x_2, x_1) = -x_1x_2 + x_1x_2 = 0$$

   b. Given $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$,show that $\mathbf{x}\cdot\rho(\mathbf{y}) = -\rho(\mathbf{x})\cdot\mathbf{y}$. Interpret this statement geometrically.
   Let $\mathbf{y} = (y_1, y_2)$. Then

   $$\mathbf{x}\cdot\rho(\mathbf{y}) = (x_1, x_2)\cdot(-y_2, y_1) = -x_1y_2 + x_2y_1 = -(-x_2y_1 + x_1y_2) = -\rho(\mathbf{x})\cdot\mathbf{y}$$

18. Prove the *triangle inequality*: For any vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $||\mathbf{x}+\mathbf{y}|| \le ||\mathbf{x}|| + ||\mathbf{y}||$. (*Hint:* Use the dot product to calculate $||\mathbf{x}+\mathbf{y}||^2$.)
We know from Cauchy-Schwartz that $|\mathbf{x}\cdot\mathbf{y}| \le ||\mathbf{x}||||\mathbf{y}|||$ and that $\mathbf{x}\cdot\mathbf{y} \le |\mathbf{x}\cdot\mathbf{y}|$. Double both sides and we have $2\mathbf{x}\cdot\mathbf{y} \le 2||\mathbf{x}||||\mathbf{y}||$. Of course $\mathbf{x}\cdot\mathbf{x} = ||\mathbf{x}||^2 \ge 0$ and $\mathbf{y}\cdot\mathbf{y} = ||\mathbf{y}||^2 \ge 0$ which leads us to $\mathbf{x}\cdot\mathbf{x} + 2\mathbf{x}\cdot\mathbf{y} + \mathbf{y}\cdot\mathbf{y} \le ||\mathbf{x}||^2 + 2||\mathbf{x}||||\mathbf{y}|| + ||\mathbf{y}||^2$. Factoring we get $(\mathbf{x}+\mathbf{y})\cdot(\mathbf{x}+\mathbf{y}) = ||\mathbf{x}+\mathbf{y}||^2 \le (||\mathbf{x}|| + ||\mathbf{y}||)^2$. And finally, because $||\mathbf{x}+\mathbf{y}|| \ge 0$ and $||\mathbf{x}|| + ||\mathbf{y}|| \ge 0$ it is safe to say that $||\mathbf{x}+\mathbf{y}|| \le ||\mathbf{x}|| + ||\mathbf{y}||$

1.3

4. Find a normal vector to the given hyperplane and use it to find the distance from the origin to the hyperplane.

   a. $\mathbf{x} = (-1, 2) + t(3, 2)$

   $$\begin{aligned} &(-2, 3)\cdot\mathbf{x} = (-2, 3)\cdot(-1, 2) + t(3, 2)\cdot(-2, 3) \\ &(-2, 3)\cdot\mathbf{x} = 7 \end{aligned}$$

$$||\text{proj}_{\mathbf{a}}\mathbf{x}_0|| = \frac{|c|}{||\mathbf{a}||} = \frac{7}{\sqrt{4+9}} = \frac{7\sqrt{13}}{13}$$

And so we have a normal vector of $(-2,3)$ and a distance of $\frac{7\sqrt{13}}{13}$

b. The plane in $\mathbb{R}^3$ given by the equation $2x_1 + x_2 - x_3 = 5$

$$(2,1,-1)\cdot(x_1,x_2,x_3) = 5$$
$$||\text{proj}_{\mathbf{a}}\mathbf{x}_0|| = \frac{5}{\sqrt{4+1+1}}$$

And so we have a normal of $(2,1,-1)$ and a distance of $\frac{5\sqrt{6}}{6}$

c. The plane passing through $(1,2,2)$ and orthogonal to the line $\mathbf{x} = (3,1,-1) + t(-1,1,-1)$

$$(-1,1,-1)\cdot\mathbf{x} = (-1,1,-1)\cdot(1,2,2) = -1$$
$$||\text{proj}_{\mathbf{a}}\mathbf{x}_0|| = \frac{1}{\sqrt{1+1+1}}$$

Looks like a normal of $(-1,1,-1)$ and a distance of $\frac{\sqrt{3}}{3}$

d. The plane passing through $(2,-1,1)$ and orthogonal to the line $\mathbf{x} = (3,1,1) + t(-1,2,1)$
The normal is $(-1,2,1)$ and has a distance of $\frac{|-2-2+1|}{\sqrt{4+1+1}} = \frac{3\sqrt{6}}{6}$

e. The plane spanned by $(1,1,4)$ and $(2,1,0)$ and passing through $(1,1,2)$

$$\begin{aligned} a_1 + a_2 + 4a_3 &= 0 & 2a_1 + a_2 &= 0 \\ a_1 - 2a_1 + 4a_3 &= 0 & 4a_3 &= a_1 \\ a_3 &= 1 & a_1 &= 4 \\ a_2 &= -8 & \mathbf{a} &= (4,-8,1) \end{aligned}$$

So our normal vector is $(4,-8,1)$ and our distance is $\frac{|4-8+2|}{\sqrt{16+64+1}} = \frac{2}{9}$

f. The plane spanned by $(1,1,1)$ and $(2,1,0)$ and passing through $(3,0,2)$

$$\begin{aligned} a_1 + a_2 + a_3 &= 0 & 2a_1 + a_2 &= 0 \\ a_1 &= 1 & a_2 &= -2 \\ a_3 &= 1 \end{aligned}$$

Normal is $(1,-2,1)$ and distance is $\frac{3+2}{\sqrt{1+4+1}} = \frac{5\sqrt{6}}{6}$.

g. The hyperplane in $\mathbb{R}^4$ spanned by $(1,-1,1,-1),(1,1,-1,-1)$ and $(1,-1,-1,1)$ and passing through $(2,1,0,1)$

$$\begin{aligned} a_1 - a_2 - a_3 + a_4 &= 0 & a_1 - a_2 + a_3 - a_4 &= 0 & a_1 + a_2 - a_3 - a_4 &= 0 \\ a_1 - a_2 + a_3 &= a_4 & a_1 + a_2 - a_3 &= a_4 & 2a_3 &= 2a_2 \\ a_1 - a_2 - a_2 + (a_1 - a_2 + a_2) &= 0 & 2a_1 &= 2a_2 & a4 &= a_1 + a_1 - a_1 \end{aligned}$$

So we let $(1,1,1,1)$ be the normal vector and $\frac{|2+1+0+1|}{\sqrt{4}} = 2$

6. a. Give the general solution of the equation $x_1 + 5x_2 - 2x_3 = 0$ in $\mathbb{R}^3$ (as a linear combination of two vectors, as in the text).
   Note that $\mathbf{0}$ is on the plane and so we just need to find two vectors that aren't parallel and are on the plane, and we have our solution. Lets take $(-3, 1, 1)$ and $(2, 0, 1)$. Then our equation is $\mathbf{x} = x_2(-3, 1, 1) + x_3(2, 0, 1)$

   b. Find a specific solution of the equation $x_1 + 5x_2 - 2x_3 = 3$ in $\mathbb{R}^3$; give the general solution.
   $\mathbf{x} = (0, 1, 1)$ is a specific solution to the equation. Combining the specific solution with part a we have $\mathbf{x} = (0, 1, 1) + x_2(-3, 1, 1) + x_3(2, 0, 1)$

   c. Give the general solution of the equation $x_1 + 5x_2 - 2x_3 + x_4 = 0$ in $\mathbb{R}^4$. Now give the general solution of the equation $x_1 + 5x_2 - 2x_3 + x_4 = 3$
   As before we notice that we are going through $\mathbf{0}$, so we find three vectors that are independent and we are golden. We can even steal two of them from the previous part. Take $(-3, 1, 1, 0), (2, 0, 1, 0)$ and $(0, 1, 1, -3)$. These are obviously all on our hyperplane, and we can easily see that they are independent. So for the first part we have $\mathbf{x} = x_2(-3, 1, 1, 0) + x_3(2, 0, 1, 0) + x_4(0, 1, 1, -3)$. For the second part we just need a point on the plane. we see that $3+5\cdot 0-2\cdot 0+0 = 3$ so we use $(3, 0, 0, 0)$ and this gives us $\mathbf{x} = (3, 0, 0, 0)+x_2(-3, 1, 1, 0)+ x_3(2, 0, 1, 0) + x_4(0, 1, 1, -3)$

7. The equation $2x_1 - 3x_2 = 5$ defines a line in $\mathbb{R}^2$.

   a. Give a normal vector $\mathbf{a}$ to the line.
   $2x_1 - 3x_2 = (2, -3) \cdot (x_1, x_2)$ so our normal vector is $(2, -3)$

   b. Find the distance from the origin to the line by using projection.
   $||\text{proj}_{\mathbf{a}}\mathbf{x}|| = \frac{5}{\sqrt{4+9}} = \frac{5\sqrt{13}}{13}$

   c. Find the point on the line closest to the origin by using the parametric equation of the line through $\mathbf{0}$ with direction vector $\mathbf{a}$. Double-check your anser to part b.

$$\mathbf{x} = t(2, -3) \qquad 2(2t) - 3(-3t) = 5$$

$$13t = 5 \qquad ||\mathbf{x}|| = \sqrt{\left(\frac{2\cdot 5}{13}\right)^2 + \left(\frac{-3\cdot 5}{13}\right)^2} = \frac{5}{13}\sqrt{13}$$

   d. Find the distance from the point $\mathbf{w} = (3, 1)$ to the line by using projection.

$$\begin{aligned} |||\text{proj}_{\mathbf{a}}\mathbf{w}|| - ||\text{proj}_{\mathbf{a}}\mathbf{x}||| &= \left| \left|\left| \frac{\mathbf{w}\cdot\mathbf{a}}{||\mathbf{a}||^2}\mathbf{a} \right|\right| - \frac{5\sqrt{13}}{13} \right| \\ &= \left| \frac{|6-3|}{||(2,-3)||} - \frac{5\sqrt{13}}{13} \right| \\ &= \left| \frac{3}{\sqrt{13}} - \frac{5\sqrt{13}}{13} \right| = \frac{2\sqrt{13}}{13} \end{aligned}$$

   e. Find the point on the line closest to $\mathbf{w}$ by using the parametric equation of the line through $\mathbf{w}$ with direction vector $\mathbf{a}$. Double-check your answer to part d

$$(3, 1) + t(2, -3) = (2t + 3, -3t + 1) = \mathbf{x} \qquad 2(2t + 3) - 3(-3t + 1) = 5$$

$$13t = 2$$

$$\left|\left| \left(2\frac{2}{13} + 3, -3\frac{2}{13} + 1\right) - \mathbf{w} \right|\right| = \sqrt{\frac{4^2 + (-6)^2}{13^2}}$$

$$= \frac{\sqrt{2^2(2^2+3^2)}}{13}$$
$$= \frac{2\sqrt{13}}{13}$$

9. The equation $2x_1 + 2x_2 - 3x_3 + 8x_4 = 6$ defines a hyperplane in $\mathbb{R}^4$.
   a. Give a normal vector **a** to the hyperplane.
   $2x_1 + 2x_2 - 3x_3 + 8x_4 = (2, 2, -3, 8) \cdot (x_1, x_2, x_3, x_4)$ and so $\mathbf{a} = (2, 2, -3, 8)$
   b. Find the distance from the origin to the hyperplane using projection.
   $||\text{proj}_{\mathbf{a}}\mathbf{x}|| = \frac{6}{\sqrt{4+4+9+64}} = \frac{2}{3}$
   c. Find the point on the plane closest to the origin by using the parametric equation of the line through **0** with direction vector **a**. Double-check your answer to part *b*.

$$\mathbf{x} = t(2, 2, -3, 8)$$

$$2(2t) + 2(2t) - 3(-3t) + 8(8t) = 6$$
$$81t = 6$$

$$\mathbf{x}_0 = \left|\left|\left(\frac{12}{81}, \frac{12}{81}, \frac{-18}{81}, \frac{48}{81}\right)\right|\right|$$
$$= \frac{2\sqrt{18 + 18 + 81 + 9 \cdot 64}}{81}$$
$$= \frac{2}{3}$$

$$= \sqrt{\frac{3^2 4^2 + 3^2 4^2 + 6^2 3^2 + 6^2 8^2}{81^2}}$$
$$= \frac{2 \cdot 3\sqrt{81}}{81}$$

   d. Find the distance from the point $\mathbf{w} = (1, 1, 1, 1)$ to the hyperplane by using dot products.
   $\left|\frac{|\mathbf{w}\cdot\mathbf{a}|}{||\mathbf{a}||} - \frac{2}{3}\right| = \left|\frac{|2+2-3+8|}{\sqrt{4+4+9+64}} - \frac{2}{3}\right| = \left|\frac{9-6}{9}\right| = \frac{1}{3}$
   e. Find the point on the plane closest to **w** by using the parametric equation of the line through **w** with direction vector **a**. Double-check your answer to part *d*

$$\mathbf{x} = (1, 1, 1, 1) + t(2, 2, -3, 8) = (2t + 1, 2t + 1, -3t + 1, 8t + 1)$$
$$6 = 2(2t + 1) + 2(2t + 1) - 3(-3t + 1) + 8(8t + 1) = (4 + 4 + 9 + 64)t + (2 + 2 - 3 + 8)$$
$$t = -\frac{3}{81} = -\frac{1}{27}$$
$$||\mathbf{x} - \mathbf{w}|| = || - \frac{1}{27}(2, 2, -3, 8)|| = \frac{\sqrt{81}}{27} = \frac{1}{3}$$

10. a. The equations $x_1 = 0$ and $x_2 = 0$ describe planes in $\mathbb{R}^3$ that contain the x_3-axis. Write down the Cartesian equation of a general such plane.
   Such a plane would be spanned by a vector that lies on the x_3 axis and a non-zero vector that lies on the $x_1 \times x_2$ plane. Lets describe the plane by saying $(x_1, x_2, x_3) = \mathbf{x} = s(0, 0, 1) + t(a, b, 0)$. Taking the dot product of both sides with $(-b, a, 0)$ gives us $(-b, a, 0)\cdot(x_1, x_2, x_3) = 0 = -bx_1 + ax_2$. Of course since our original choices of a and b were arbitrary, we will just rewrite our equation to say $ax_1 + b_x 2 = 0$
   b. The equations $x_1 - x_2 = 0$ and $x_1 - x_3 = 0$ describe planes in $\mathbb{R}^3$ that contain the line through the origin with direction vector $(1, 1, 1)$. Write down the cartesian equation of a general such plane.
   Such a plane must contain a non-zero vector on the $x_1 \times x_2$ plane. Lets say $(a, b, 0)$. This gives us $\mathbf{x} = s(1, 1, 1) + t(a, b, 0)$. Now if we multiply both sides by the vector $(-b, a, b - a)$ then we get $-bx_1 + ax_2 + (b - a)x_3 = 0$. As before, our choice of a and b was arbitrary, and so we rewrite our equation to be $ax_1 + bx_1 - (b + a)x_3 = 0$

11. Don't have to do after all
12. Suppose $\mathbf{a} \neq \mathbf{0}$ and $\mathcal{P} \subset \mathbb{R}^3$ is the plane through the origin with normal vector $\mathbf{a}$. Suppose $\mathcal{P}$ is spanned by $\mathbf{u}$ and $\mathbf{v}$

    a. Suppose $\mathbf{u} \cdot \mathbf{v} = 0$. Show that for every $\mathbf{x} \in \mathcal{P}$, we have

    $$\mathbf{x} = \text{proj}_{\mathbf{u}}\mathbf{x} + \text{proj}_{\mathbf{v}}\mathbf{x}$$

    Using the definition of a plane and the definition of projection we have the following:

    $$\text{proj}_{\mathbf{u}}\mathbf{x} + \text{proj}_{\mathbf{v}}\mathbf{x} = \frac{\mathbf{x}\mathbf{u}}{||\mathbf{u}||^2}\mathbf{u} + \frac{\mathbf{x}\mathbf{v}}{||\mathbf{v}||^2}\mathbf{v}$$
    $$\exists s, t \in \mathbb{R} \text{ such that } \mathbf{x} = s\mathbf{u} + t\mathbf{v}$$

    Making appropriate substitution gives us

    $$\begin{aligned} \text{proj}_{\mathbf{u}}\mathbf{x} + \text{proj}_{\mathbf{v}}\mathbf{x} &= \frac{(s\mathbf{u} + t\mathbf{v}) \cdot \mathbf{u}}{||\mathbf{u}||^2}\mathbf{u} + \frac{(s\mathbf{u} + t\mathbf{v}) \cdot \mathbf{v}}{||\mathbf{v}||^2}\mathbf{v} \\ &= \frac{s\mathbf{u} \cdot \mathbf{u} + t\mathbf{v} \cdot \mathbf{u}}{||\mathbf{u}||^2}\mathbf{u} + \frac{s\mathbf{u} \cdot \mathbf{v} + t\mathbf{v} \cdot \mathbf{v}}{||\mathbf{v}||^2}\mathbf{v} \\ &= \frac{s||\mathbf{u}||^2 + t\mathbf{0}}{||\mathbf{u}||^2}\mathbf{u} + \frac{s\mathbf{0} + t||\mathbf{v}||^2}{||\mathbf{v}||^2}\mathbf{v} \\ &= s\mathbf{u} + s\mathbf{v} = \mathbf{x} \end{aligned}$$

    b. Suppose $\mathbf{u} \cdot \mathbf{v} = 0$. Show that for every $\mathbf{x} \in \mathbb{R}^3$, we have

    $$\mathbf{x} = \text{proj}_{\mathbf{a}}\mathbf{x} + \text{proj}_{\mathbf{u}}\mathbf{x} + \text{proj}_{\mathbf{v}}\mathbf{x}$$

    (*Hint:* Apply part a to the vector $\mathbf{x} - \text{proj}_{\mathbf{a}}\mathbf{x}$)
    First we note that $\mathbf{x} - \text{proj}_{\mathbf{a}}\mathbf{x} \in \mathcal{P}$ and so

    $$\begin{aligned} \mathbf{x} - \text{proj}_{\mathbf{a}}\mathbf{x} &= \text{proj}_{\mathbf{u}}(\mathbf{x} - \text{proj}_{\mathbf{a}}\mathbf{x}) + \text{proj}_{\mathbf{v}}(\mathbf{x} - \text{proj}_{\mathbf{a}}\mathbf{x}) \\ \text{proj}_{\mathbf{u}}(\mathbf{x} - \text{proj}_{\mathbf{a}}\mathbf{x}) &= \frac{(\mathbf{x} - \text{proj}_{\mathbf{a}}\mathbf{x}) \cdot \mathbf{u}}{||\mathbf{u}||^2}\mathbf{u} \\ &= \frac{(\mathbf{x} - \frac{\mathbf{x} \cdot \mathbf{a}}{||||\mathbf{a}||^2}\mathbf{a}) \cdot \mathbf{u}}{||\mathbf{u}||^2}\mathbf{u} \end{aligned}$$

    Now of course $\mathbf{a}$ is orthogonal to every vector in $\mathcal{P}$ including $\mathbf{u}$ and so $\mathbf{a} \cdot \mathbf{u} = 0$ which leads us to conclude that

    $$\begin{aligned} \text{proj}_{\mathbf{u}}(\mathbf{x} - \text{proj}_{\mathbf{a}}\mathbf{x}) &= \frac{\mathbf{x} \cdot \mathbf{u}}{||\mathbf{u}||^2}\mathbf{u} \\ &= \text{proj}_{\mathbf{u}}\mathbf{x} \end{aligned}$$

    In exactly the same way it can be shown that $\text{proj}_{\mathbf{v}}(\mathbf{x} - \text{proj}_{\mathbf{a}}\mathbf{x}) = \text{proj}_{\mathbf{v}}\mathbf{x}$. Thus $\mathbf{x} - \text{proj}_{\mathbf{a}}\mathbf{x} = \text{proj}_{\mathbf{u}}\mathbf{x} + \text{proj}_{\mathbf{v}}\mathbf{x}$ or $\mathbf{x} = \text{proj}_{\mathbf{a}}\mathbf{x} + \text{proj}_{\mathbf{u}}\mathbf{x} + \text{proj}_{\mathbf{v}}\mathbf{x}$

    c. Give an example to show the result of part a is false when $\mathbf{u}$ and $\mathbf{v}$ are not orthogonal
    Let $\mathbf{a} = (0, 0, 1), \mathbf{x} = \mathbf{u} = (1, 1, 0)$, and $\mathbf{v} = (1, 0, 0)$. Then $\frac{\mathbf{x} \cdot \mathbf{u}}{||\mathbf{u}||^2}\mathbf{u} + \frac{\mathbf{x} \cdot \mathbf{v}}{||\mathbf{v}||^2}\mathbf{v} = \mathbf{u} + \mathbf{v} = (2, 1, 0) \neq \mathbf{x}$
13. Consider the line ℓ in $\mathbb{R}^3$ given parametrically by $\mathbf{x} = \mathbf{x}_0 + t\mathbf{a}$. Let $\mathcal{P}_0$ denote the plane through the origin with normal vector $\mathbf{a}$ (so it is orthogonal to ℓ).

a. Show that ℓ and $\mathcal{P}_0$ intersect in the point $\mathbf{x}_0 - \text{proj}_{\mathbf{a}}\mathbf{x}_0$
Well we know that $\mathbf{a} \cdot \mathbf{x} = 0$ defines the plane. And so $\mathbf{a} \cdot \mathbf{x}_0 + t\mathbf{a} \cdot \mathbf{a} = 0$ and $-t = \frac{\mathbf{a} \cdot \mathbf{x}_0}{||\mathbf{a}||^2}$. Substituting t back in we get $\mathbf{x} = \mathbf{x}_0 - \frac{\mathbf{a} \cdot \mathbf{x}_0}{||\mathbf{a}||^2}\mathbf{a} = \mathbf{x}_0 - \text{proj}_{\mathbf{a}}\mathbf{x}_0$.

b. Conclude that the distance from the origin to ℓ is $||\mathbf{x}_0 - \text{proj}_{\mathbf{a}}\mathbf{x}_0||$
Well the origin is on the plane, and so is our intersection point, so the vector from the origin to the intersection point is on the plane. Further, ℓ is orthogonal to the plane, and so ℓ must be orthogonal to the $\mathbf{x}_0 - \text{proj}_{\mathbf{a}}\mathbf{x}_0$ vector. Thus the closest point to the origin on the line is where it intersects with the plane.