Chapter 7

33. Solve the recurrence relation $h_n = h_{n-1} + 9h_{n-2} - 9h_{n-3}, \quad (n \geq 3)$ with initial values $h_0 = 0, h_1 = 1$, and $h_2 = 2$.

$$\begin{aligned}
0 &= h_n - h_{n-1} - 9h_{n-2} + 9h_{n-3} \\
0 &= q^3 - q^2 - 9q + 9 \\
0 &= (q+a)(q^2+bq+c) = q^3 + (b+a)q^2 + (ab+c)q + ac \\
&= (q-3)(q^2+bq-3) = q^3 + (b-3)q^2 + (-3b-3)q + 9 \\
&= (q-3)(q^2+2q-3) = (q-3)(q-1)(q+3) \\
h_n &= c_1(3)^n + c_2(1)^n + c_3(-3)^n \\
h_0 = 0 &= c_1 + c_2 + c_3 \\
h_1 = 1 &= 3c_1 + c_2 - 3c_3 = 3c_1 + c_2 + 3c_1 + 3c_2 = 6c_1 + 4c_2 \\
h_2 = 2 &= 9c_1 + c_2 + 9c_3 = 9c_1 + c_2 - 9c_1 - 9c_2 = -8c_2 \\
c_2 &= -\frac{1}{4}, \quad c_1 = \frac{1}{3}, \quad c_3 = -\frac{1}{12} \\
h_n &= \frac{4 \cdot 3^n - 3 - (-3)^n}{12}
\end{aligned}$$

34. Solve the recurrence relation $h_n = 8h_{n-1} - 16h_{n-2}, \quad (n \geq 2)$ with initial values $h_0 = -1$ and $h_1 = 0$.

$$\begin{aligned}
0 &= h_n - 8h_{n-1} + 16h_{n-2} \\
0 &= q^2 - 8q + 16 \\
0 &= (q-4)^2 \\
h_n &= c_1 4^n + c_2 n 4^n \\
h_0 = -1 &= c_1 \\
h_1 = 0 &= -4 + c_2 4, \quad c_2 = 1 \\
h_n &= -4^n + n4^n
\end{aligned}$$

38. Solve the following recurrence relations by examining the first few values for a formula andd then proving your conjectured formula by induction.

(b) $h_n = h_{n-1} - n + 3, \quad (n \geq 1); \ h_0 = 2$

$h_0 = 2$	$h_1 = 2 - 1 + 3 = 4$	$h_2 = 4 - 2 + 3 = 5$	$h_3 = 5 - 3 + 3 = 5$
$h_4 = 5 - 4 + 3 = 4$	$h_5 = 4 - 5 + 3 = 2$	$h_6 = 2 - 6 + 3 = -1$	$h_7 = -1 - 7 + 3 = -5$

I have no intuition whatsoever.

$$\begin{aligned}
0 &= h_n - h_{n-1} \\
0 &= q - 1, \quad q = 1 \\
h_n &= c
\end{aligned}$$

$rn + s$ give the impossible condition $r = r - 1$

$$\begin{aligned}
rn^2 + sn + t &= r(n-1)^2 + s(n-1) + t - n + 3 = r(n^2 - 2n + 1) + sn - s + t - n + 3 \\
&= rn^2 - 2rn + sn - n + r - s + t + 3 = rn^2 + (-2r + s - 1)n + r - s + t + 3 \\
s &= -2r + s - 1, \quad r = -\frac{1}{2} \\
t &= r - s + t + 3 = -\frac{1}{2} - s + 3 + t, \quad s = \frac{5}{2} \\
t &= 0 \\
h_n &= c - \frac{1}{2}n^2 + \frac{5}{2}n \\
2 &= c \\
h_n &= 2 - \frac{1}{2}n^2 + \frac{5}{2}n
\end{aligned}$$

We don't exactly have a conjectured formula, but we will prove it anyhow. First by showing that $h_0 = 2$ and then by showing that $h_{n+1} = h_n - (n+1) + 3$ which is the same as $h_n = h_{n-1} - n + 3$

$$\begin{aligned}
h_0 &= 2 - \frac{1}{2}0^2 + \frac{5}{2}0 = 2 \\
h_{n+1} &= 2 - \frac{1}{2}(n+1)^2 + \frac{5}{2}(n+1) \\
&= 2 - \frac{1}{2}n^2 - n - \frac{1}{2} + \frac{5}{2}n + \frac{5}{2} \\
&= 2 - \frac{1}{2}n^2 + \frac{5}{2}n - n + 2 \\
&= 2 - \frac{1}{2}n^2 + \frac{5}{2}n - (n+1) + 3 \\
h_n &= 2 - \frac{1}{2}n^2 + \frac{5}{2}n \\
h_{n+1} &= h_n - (n+1) + 3
\end{aligned}$$

And scene. □

40. Let a_n equal the number of ternary strings of length n made up of 0s, 1s, and 2s, such that the substrings 00, 01, 10, and 11 never occur. Prove that

$$a_n = a_{n-1} + 2a_{n-2}, \quad (n \geq 2),$$

with $a_0 = 1$ and $a_1 = 3$. Then find a formula for a_n.

proof

So let us take a sequence of length n. There are a_n possibilities for the entire sequence, but only three possibilities for the last digit of the sequence. This number is the sum of the possible sequences that end in each possible digit. We will label the number of these possibilities with ${}_0a_{n-1},{}_1 a_{n-1}$, and ${}_2a_{n-1}$. Now if the nth digit in our sequence is 0 then we only have one choice (two) in the digit for $n-1$ and so ${}_0a_{n-1} = a_{n-2}$. Similarly ${}_1a_{n-1} = a_{n-2}$. Now if our nth digit is 2, then the possibilities for the previous digit are wide open, which is to say ${}_2a_{n-1} = a_{n-1}$. Now putting it all together we see that $a_n = {}_0a_{n-1} + {}_1a_{n-1} + {}_2a_{n-1} = a_{n-2} + a_{n-2} + a_{n-1} = a_{n-1} + 2a_{n-2}$. And we have our result. □

formula

$$\begin{aligned}
a_n &= a_{n-1} + 2a_{n-2} \\
0 &= a_n - a_{n-1} - 2a_{n-2} \\
0 &= q^2 - q - 2 \\
0 &= (q+1)(q-2) \\
a_n &= c_1(-1)^n + c_2 2^n \\
a_0 &= 1 = c_1 + c_2 \\
a_1 &= 3 = -c_1 + 2c_2 \\
c_1 &= 1 - c_2 \\
3 &= -(1-c_2) + 2c_2 = 3c_2 - 1 \\
c_2 &= \frac{4}{3}, \quad c_1 = -\frac{1}{3} \\
a_n &= -\frac{1}{3} \cdot (-1)^n + \frac{4}{3} \cdot 2^n
\end{aligned}$$

44. Solve the nonhomogeneous recurrence relation

$$\begin{aligned}
h_n &= 3h_{n-1} - 2, \quad (n \geq 1) \\
h_0 &= 1.
\end{aligned}$$

First find homogeneous solutions

$$\begin{aligned}
h_n &= 3h_{n-1} \\
0 &= h_n - 3h_{n-1} \\
0 &= q - 3 \\
h_n &= c3^n \\
h_n &= rn + s \\
rn + s &= 3(r(n-1) + s) - 2 \\
&= 3rn - 3r + 3s - 2 \\
r &= 3r = 0 \\
s &= -3r + 3s - 2 \\
2s &= 3r + 2 \\
1 &= 3rn + \frac{1}{2}(3r+2) \\
s &= \frac{1}{2}(3r+2) = 1
\end{aligned}$$

$$\begin{aligned}h_n &= 1 \to h_n = c3^n + 1\\ 1 &= c3^0 + 1\\ 0 &= c\\ h_n &= 1\end{aligned}$$

45. Solve the nonhomogeneous recurrence relation

$$\begin{aligned}h_n &= 2h_{n-1} + n, \quad (n \geq 1)\\ h_0 &= 1.\end{aligned}$$

$$\begin{aligned}h_n &= 2h_{n-1}\\ 0 &= h_n - 2h_{n-1} = q - 2\\ h_n &= c2^n\\ rn + s &= 2(r(n-1) + s) + n = 2rn - 2r + 2s + n = (2r+1)n - 2r + 2s\\ r &= 2r + 1 \to r = -1\\ s &= -2r + 2s = 2 + 2s \to s = -2\\ h_n &= -n - 2\\ h_n &= c2^n - n - 2\\ 1 &= c - 0 - 2 \to c = 3\\ h_n &= 3 \cdot 2^n - n - 2\end{aligned}$$

Chapter 8

Do two of 1,2 or 36

1. Let $2n$ (equally spaced) points on a circle be chosen. Show that the number of ways to join these points in pairs, so that the resulting n line segments do not intersect, equals the nth Catalan number C_n.

2. Prove that the number of 2-by-n arrays

$$\begin{bmatrix} x_{11} & x_{12} & \dots & x_{1n}\\ x_{21} & x_{22} & \dots & x_{2n}\end{bmatrix}$$

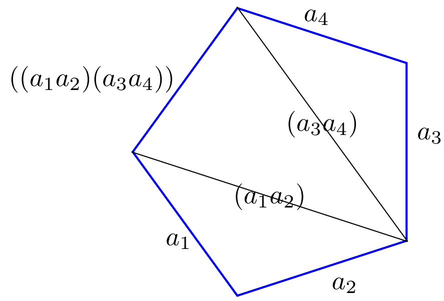
that can be made from the numbers $1, 2, \dots, 2n$ such that

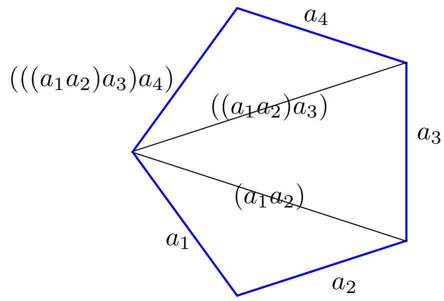
$$\begin{aligned}x_{11} &< x_{12} < \dots < x_{1n},\\ x_{21} &< x_{22} < \dots < x_{2n}\end{aligned}$$

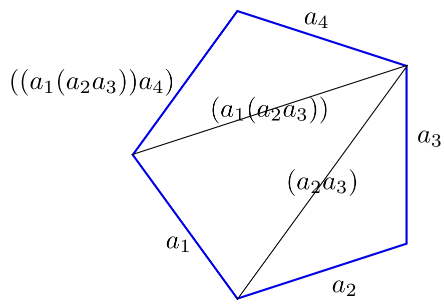
$$x_{11} < x_{21}, x_{12} < x_{22}, \dots, x_{1n} < x_{2n},$$

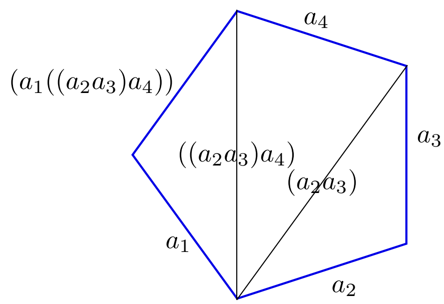
equalsthe nth Catalan number, C_n.

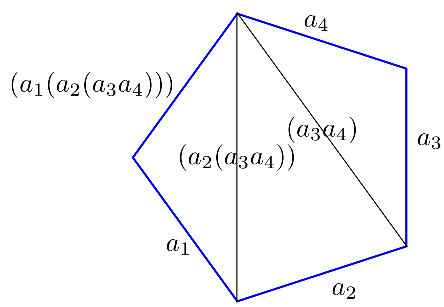
3. Write out all of the multiplication schemes for four numbers and the triangularization of a convex polygonal region of five sides corresponding to them.





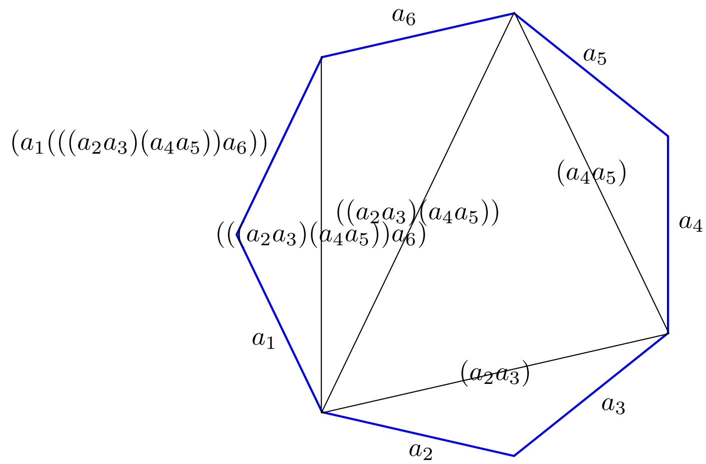




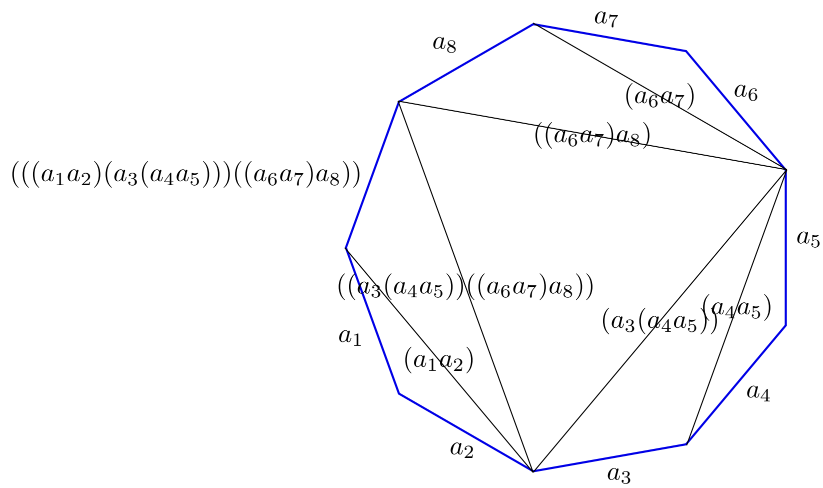


4. Determine the triangularization of a convex polygonal region corresponding to the following multiplication schemes:

   (a) $(a_1 \times (((a_2 \times a_3) \times (a_4 \times a_5)) \times a_6))$



(b) $(((a_1 \times a_2) \times (a_3 \times (a_4 \times a_5))) \times ((a_6 \times a_7) \times a_8))$



36. Prove that the Catalan number C_n equals the number of lattice paths from $(0, 0)$ to $(2n, 0)$ using only upsteps $(1, 1)$ and downsteps $(1, -1)$ that never go above the horizontal axis (so there are as many upsteps as there are downsteps). (These are sometimes call *Dyck paths*.)

proof

The number a_{2n} of paths from (0,0) to $(0, 2n)$ is the number of sequences of upsteps $(1, 1)$ and downsteps $(1, -1)$ where the sum of the second coordinates is always negative. Because the first coordinate is always 1 this is the same as a sequence of n ones and n -1's whose partial sums are always negative. We can multiply each element in each sequence by -1. This gives us the smae number of sequences, however the partial sums of these sequences is always positive. Now we see from Theorem 8.1.1 that the number of these sequences is the nth Catalan number. □