Graph Theory Homework

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February 20, 2015

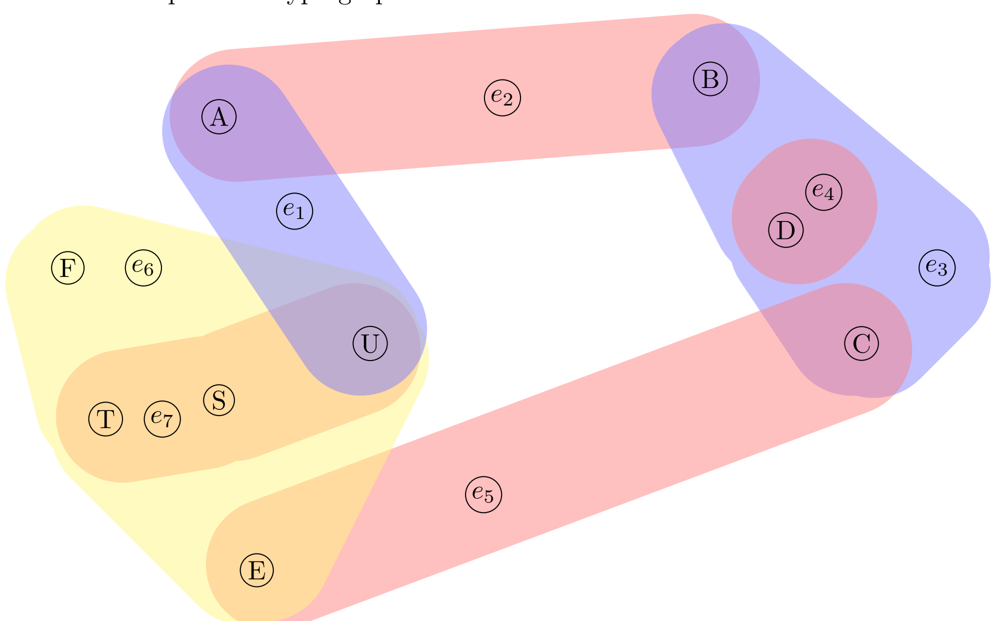
Worksheet

1.) Create a 3-regular, 3-uniform hypergraph. Is it possible to create an k-regular, k-uniform (simple) hypergraph? Prove or disprove.

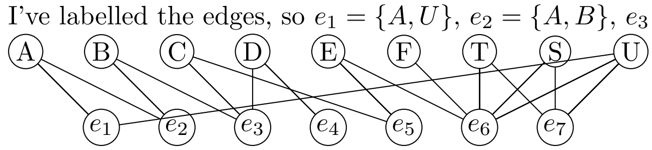
For the k=3 case, let the vertex set be $V = \{v_1, \ldots, v_9\}$ and the edge set be $E = \{e_1 = \{v_1, v_2, v_3\}, e_2 = \{v_4, v_5, v_6\}, e_3 = \{v_7, v_8, v_9\}\}$

For the more general case, let $V = \{v_1, \ldots, v_{k^2}\}$ and let $E = \{e_1, \ldots, e_k | e_i = \{v_{k(i-1)+1}, \ldots, v_{k(i-1)+k}\}\}$. This works as long as $0 < k < \infty$, but I think that's implicit anyhow.

2.) Let H be the pictured hypergraph.



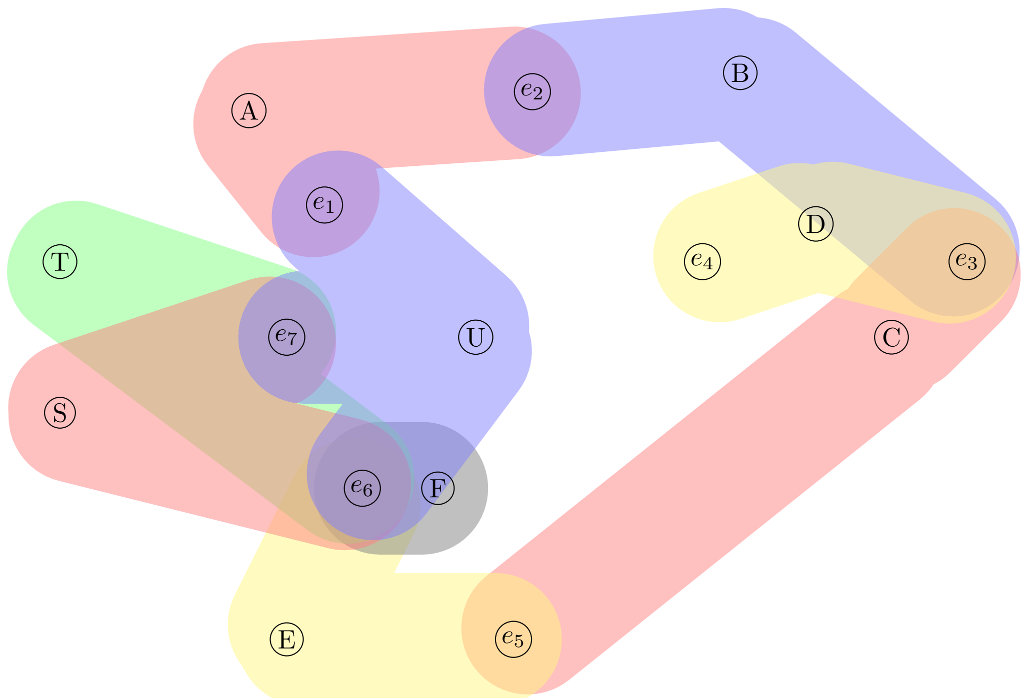
(a) Create the associated bipartite graph to H.
I've labelled the edges, so $e_1 = \{A, U\}$, $e_2 = \{A, B\}$, $e_3 = \{B, D, C\}$ and so on.



(b) What is the adjacency matrix of H?

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

(c) What is $H*$?



3.3

10. Prove that if v is any vertex of a connected graph G of order at least 4, then $G^3 - v$ is Hamiltonian.

    We have three possibilities for v. Either it is a leaf, it is a cut vertex, or it is neither. If it a leaf or it is not a cut vertex then removing v from G leaves a connected graph $G - v$. Now $(G - v)^3$ is a Hamiltonian subgraph of $G^3 - v$ where $V((G - v)^3) = V(G^3 - v)$ so in either of these cases we are done.

    Now we assume that v is a cut vertex. Then $G - v$ consists of k components. We will call them $G_i \forall 1 \leq i \leq k$. We know that all the G_i^3's are Hamiltonian-connected. In particular there exists some $u_i \in V(G_i)$ which is adjacent to v in G for every G_i. Now there exists some $w_i \in V(G_i)$ which is adjacent to u_i or else $|V(G_i)| = 1$ and we shall say $u_i = w_i$. Because G_i^3 is Hamiltonian-connected there exists a Hamiltonian path P_i from u_i to w_i in G_i^3.

    Furthermore, Because $d_G(u_i, w_{i+1}) = d_G(u_k, w_1) \leq 3$ when $i < k$ then $u_i w_{i+1}, u_k w_1 \in E(G^3 - v)$. Now taking P_i and $u_i w_{i+1}$ together, along with P_k and $u_k w_1$ we can form a Hamiltonian cycle in $G^3 - v$. And so $G^3 - v$ is Hamiltonian.

11. Determine a formula for the number of triangles in the line graph $L(G)$ in terms of quantities in G.

The hint is relatively helpful, but one has to think about exactly how to count the triangles from G and the $K_{1,3}$ subgraphs of G.

First we notice that triangles are formed from other triangles or claw graphs. The number of claws in a graph can be computed for every vertex v in G with a degree of at least 3. If we choose any three adjacent vertices of v then we have found a claw. The number of ways we can choose a claw is the number of ways we can choose three adjacent vertices to v. The formula for the number of claws in this vertex is $\binom{\deg(v)}{3}$. So if $W = \{v : v \in V(G) \text{ and } \deg v \geq 3\}$ then the number of $K_{1,3}$ in G is $\sum\limits_{v_i \in W} \binom{\deg(v_i)}{3}$. Add in the number of triangles in G and you have the number of triangles in $L(G)$

12. Prove that $L(G)$ is Eulerian if G is Eulerian.

    A graph is Eulerian if and only if all of it's vertices have an even degree. Let us choose any vertex $w \in V(L(G))$. Then w comes from some edge in G. Lets say this edge is incident to the vertices $u, v \in V(G)$. Now the degree of w is equal to the number of edges incident to u minus the uv edge and the number of edges incident to v minus the uv edge. That is to say $\deg w = \deg u + \deg v - 2$. Now if G is Eulerian, then $\deg u$ and $\deg v$ are both even. That is to say there exists some $k, l \in \mathbb{N}$ such that $\deg u = 2k$ and $\deg v = 2l$. Now we see that $\deg w = 2k + 2l + 2 = 2(k + l + 1)$ which is even. Because our choice of w was arbitrary we see that all the vertices of $L(G)$ have even degree, and therefore $L(G)$ is Eulerian.

4.1

2. Show that a digraph D is strong if and only if it's converse $\overrightarrow{D}$ is strong

   Let $W = (u_1, \ldots, u_k, u_1)$ be a closed spanning walk in D, then $u_i u_{i+1}$ is an arc in D and $u_{i+1} u_i$ is and arc in $\overrightarrow{D}$ when $i \leq 1 < k$. Also $u_k u_1$ is an arc in D and $u_1 u_k$ is an arc in $\overrightarrow{D}$. Obviously then we have a closed spanning walk in $\overrightarrow{D}$ in the form of $W' = (u_1, u_k, u_{k-1}, \ldots, u_1)$.

7. Prove theorem 4.4: *Let D be a nontrivial connected digraph. This D is Eulerian if and only if od v = id v for every vertex v of D.*

   Let us assume that D is Eulerian. Then it contains an Eulerian circuit C. Let v be a vertex of D. If v is the initial vertex of C then it is also the terminal vertex of C. The initial and terminal arcs of C contribute 1 or 0 to both the incoming and outgoing degrees of v depending on if we think of v as the initial vertex. Now if we have an incoming arc on v that is not the terminal arc, then we must also have an outgoing arc incident to v. Conversely, if we have an outgoing arc incident to v that is not the initial arc, then there must be a correlating incoming arc. This holds for each incoming and outgoing arc that is incident to v. Thus id v = od v.

   Now we assume that id v = od v for all $v \in D$. We choose an arbitrary v. We construct a trail T beginning at v that contains a maximal number of arcs of D. Suppose that T is a $v - u$ trail with $u \neq v$. Then because u terminates the trail and u has the same number of incoming and outgoing arc, then there must be an outgoing arc from u to match the terminal arc in T. But then if we add this arc to T to get a new trail T' then we have constructed a trail with more arcs that T which is a contradiction. Thus T must be a circuit. If T is Eulerian, then we are done. If not then there is a vertex x in T which is incident to an incoming (and therefore outgoing arc) in D not in T. Let us say $F = D - E(T)$. Where $E(T)$ are the arcs in T. Since every vertex in T is incident to the same number of incoming and outgoing arcs, then the vertices of F must also be. Let F' be the component of F which contains x. As in the above argument, F' contains some circuit T' with initial and terminal vertex x. By inserting F' as some occurrence of x in C, a $v - v$ circuit T'' in D is produced, having more edges than T. Again a contradiction. And so T is Eulerian and by extension, so is D.