3.1

6. (a) Let U and V be subspaces of $\mathbb{R}^n$. Define the *intersection* of U and V to be

$$U \cap V = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \in U \text{ and } \mathbf{x} \in V\}$$

Show that $U \cap V$ is a subspace of $\mathbb{R}^n$. Give two examples.

We know that $\mathbf{0} \in U$ and $\mathbf{0} \in V$. And so $\mathbf{0} \in U \cap V$. Now if we take any $\mathbf{u}, \mathbf{v} \in U \cap V$ then $\mathbf{u}, \mathbf{v} \in U$ and $\mathbf{u}, \mathbf{v} \in V$ and so $\mathbf{u} + \mathbf{v} \in U$ and $\mathbf{u} + \mathbf{v} \in V$. Thus $\mathbf{u} + \mathbf{v} \in U \cap V$. Similarly $\alpha\mathbf{u} \in U$ and $\alpha\mathbf{u} \in V$ for any $\alpha \in \mathbb{R}$. Thus $\alpha\mathbf{u} \in U \cap V$.

(b) Is $U \cup V = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \in U \text{ or } \mathbf{x} \in V\}$ always a subspace of $\mathbb{R}^n$? Give a proof or counterexample.

Let $V = \{3n : \forall n \in \mathbb{Z}\}$ and $U = \{2n : \forall n \in \mathbb{Z}\}$. Then $3 \in V$ and $2 \in U$. Therefore $3, 2 \in U \cup V$. But $3 + 2 = 5 \notin V$ and $5 \notin U$. Therefore $5 \notin U \cup V$ and so we have a counterexample.

8. Let $\mathbf{v}_1, \dots, \mathbf{v}_k \in \mathbb{R}^n$ and let $\mathbf{v} \in \mathbb{R}^n$. Prove that $\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k) = \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{v})$ if and only if $\mathbf{v} \in \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$.

Define $A = \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$ and $B = \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{v})$.

We know that $\mathbf{v} \in B$ and so if we assume that $A = B$ then $\mathbf{v} \in A$ follows immediately. Now lets assume that $\mathbf{v} \in A$. Then for any $\mathbf{x} \in A$ we have $\mathbf{x} = \alpha_1\mathbf{v}_1 + \cdots + \alpha_k\mathbf{v}_k + 0\mathbf{v} \in B$. Now, because $\mathbf{v} = \alpha_1\mathbf{v}_1 + \cdots + \alpha_k\mathbf{v}_k$ Then any element $\mathbf{x} \in B$ we have $\mathbf{x} = \beta_1\mathbf{v}_1 + \cdots + \beta_k\mathbf{v}_k + \beta\mathbf{v} = \beta_1\mathbf{v}_1 + \cdots + \beta_k\mathbf{v}_k + \beta(\alpha_1\mathbf{v}_1 + \cdots + \alpha_k\mathbf{v}_k) = (\beta_1 + \beta\alpha_1)\mathbf{v}_1 + \cdots + (\beta_k + \beta\alpha_k)\mathbf{v}_k \in A$. $\square$

9. Determine the intersection of the subspaces $\mathcal{P}_1$ and $\mathcal{P}_2$ in each case:

(b) $\mathcal{P}_1 = \text{Span}\left((1,2,2),(0,1,1)\right), \mathcal{P}_2 = \text{Span}\left((2,1,1),(1,0,0)\right)$

$$a\begin{pmatrix}1\\2\\2\end{pmatrix} + b\begin{pmatrix}0\\1\\1\end{pmatrix} - c\begin{pmatrix}2\\1\\1\end{pmatrix} - d\begin{pmatrix}1\\0\\0\end{pmatrix} = 0 \Rightarrow \begin{bmatrix}1&0&2&1\\2&1&1&0\\2&1&1&0\end{bmatrix} \Rightarrow \begin{bmatrix}1&0&2&1\\0&1&-3&-2\\0&0&0&0\end{bmatrix}$$

And so we have two free variables. Which means given any element in $\mathcal{P}_1$ we can find that element in $\mathcal{P}_2$ and the other way around. That is $\mathcal{P}_1 = \mathcal{P}_2$

(c) $\mathcal{P}_1 = \text{Span}\left((1,0,-1),(1,2,3)\right), \mathcal{P}_2 = \{\mathbf{x} : x_1 - x_2 + x_3 = 0\}$ Converting $\mathcal{P}_2$ to standard form we have $\mathbf{x} = \begin{pmatrix}x_1\\x_1+x_3\\x_3\end{pmatrix}$ or $\mathcal{P}_2 = \text{Span}\{(1,1,0),(0,1,1)\}$. And so we put everything in an array as above

$$\begin{bmatrix}1&1&-1&0\\0&2&-1&-1\\-1&3&0&-1\end{bmatrix}\begin{bmatrix}1&1&-1&0\\0&2&-1&-1\\0&4&-1&-1\end{bmatrix}\begin{bmatrix}1&1&-1&0\\0&2&0&0\\0&2&-1&-1\end{bmatrix}\begin{bmatrix}1&1&-1&0\\0&1&0&0\\0&0&1&1\end{bmatrix}\begin{bmatrix}1&0&0&1\\0&1&0&0\\0&0&1&1\end{bmatrix}$$

Which gives us one free variable, and so the two planes intersect on a line. We notice that $(1,0,-1) = (1,1,0) - (0,1,1)$ and so since $(1,0,-1)$ is in P_1 and in $\mathcal{P}_2$ so any element in $\text{Span}(1,0,-1)$ is in the intersection, thus we have found our line.

11. Suppose V and W are orthogonal subspaces of $\mathbb{R}^n$, i.e., $\mathbf{v} \cdot \mathbf{w} = 0$ for every $\mathbf{v} \in V$ and every $\mathbf{w} \in W$. Prove that $V \cap W = \{\mathbf{0}\}$.

Start by assuming there exists some element $\mathbf{u} \in V \cap W$ such that $\mathbf{u} \neq \mathbf{0}$. Then because $\mathbf{u} \in V$ and $\mathbf{u} \in W$ we know that $\mathbf{u} \cdot \mathbf{u} = \mathbf{0}$. And of course $\mathbf{u} \cdot \mathbf{u} = ||\mathbf{u}||^2$. But $\mathbf{u} \neq 0$ and so $||\mathbf{u}||^2 > 0$ and $\mathbf{u} \cdot \mathbf{u} \neq 0$ which leaves us with a contradiction. We must assume then that the intersection of these sets contains only $\mathbf{0}$.

15. Let A be an $m \times n$ matrix. Let $V \subset \mathbb{R}^n$ and $W \subset \mathbb{R}^m$ be subspaces.

(a) Show that $E = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} \in W\}$ is a subspace of $\mathbb{R}^n$.

Of course $A\mathbf{0} = \mathbf{0} \in W$. Let us choose $\mathbf{x}, \mathbf{y} \in E$. Then we have $A\mathbf{x} \in W$ and $A\mathbf{y} \in W$ and so $A\mathbf{x} + A\mathbf{y} \in W$. But $A\mathbf{x} + A\mathbf{y} = A(\mathbf{x} + \mathbf{y})$ and so $\mathbf{x} + \mathbf{y} \in E$. And if $A\mathbf{x} \in W$ then $\alpha A\mathbf{x} \in W$ for any $\alpha \in \mathbb{R}$. So because $\alpha A\mathbf{x} = A(\alpha\mathbf{x})$ then we have $\alpha\mathbf{x} \in E$. And so E is a subspace.

(b) Show that $F = \{\mathbf{y} \in \mathbb{R}^m : \mathbf{y} = A\mathbf{x} \text{ for some } \mathbf{x} \in V\}$ is a subspace of $\mathbb{R}^m$

Observe that $\mathbf{0} = A\mathbf{0}$ and $\mathbf{0} \in V$. And so $\mathbf{0} \in F$. Now, let us take $\mathbf{u}, \mathbf{v} \in F$. Then $\mathbf{u} = A\mathbf{x}_1$ and $\mathbf{v} = A\mathbf{x}_2$ where $\mathbf{x}_1, \mathbf{x}_2 \in V$. Now $\mathbf{u} + \mathbf{v} = A\mathbf{x}_1 + A\mathbf{x}_2 = A(\mathbf{x}_1 + \mathbf{x}_2)$. But $\mathbf{x}_1 + \mathbf{x}_2 \in V$ and so $\mathbf{u} + \mathbf{v} \in F$. And if we choose $\alpha \in \mathbb{R}$ then $\alpha\mathbf{u} = \alpha A\mathbf{x} = A(\alpha\mathbf{x})$ for some $\mathbf{x} \in V$. And since $\mathbf{x} \in V$ then $\alpha\mathbf{x} \in V$ and so $\alpha\mathbf{u} \in F$. Thus F is a subspace.

16. Suppose A is a symmetric $n \times n$ matrix. Let $V \subset \mathbb{R}^n$ be a subspace with the property that $A\mathbf{x} \in V$ for every $\mathbf{x} \in V$. Show that $A\mathbf{y} \in V^\perp$ for all $\mathbf{y} \in V^\perp$

We choose some $\mathbf{y} \in V^\perp$ and $\mathbf{x} \in V$. Then $\mathbf{x} \cdot \mathbf{y} = \mathbf{0}$. Now if $\mathbf{x}A\mathbf{y} = \mathbf{0}$ then $A\mathbf{y} \in V^\perp$. Of course $\mathbf{x}A\mathbf{y} = A^T\mathbf{x}^T\mathbf{y} = (A\mathbf{x})\mathbf{y}$. But $A\mathbf{x} \in V$ and so $(A\mathbf{x})\mathbf{y} = 0$ therefore $A\mathbf{y} \in V^\perp$