PDE A.

PDE.	$\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$	for	$0 < x < 1,$	$0 < t < \infty$
BC.	$u_x(0,t) = 0 = u_x(1,t)$	for		$0 < t < \infty$
IC.	$u(x,0) = f(x)$	for	$0 < x < 1$	

For PDE A, apply separation of variables and, for separated solutions $u = T(t)X(x)$, analyze the associateed eigenvalue problem $X''(x) = \lambda X(x)$ and determine the eigenfunctions (or their nonexistence) for the cases:

$$\begin{aligned}
u &= T(t)X(x) \\
\frac{\partial u}{\partial t} &= T'(t)X(x) \\
\frac{\partial u}{\partial x} &= X'(x)T(t) \\
\frac{\partial^2 u}{\partial x^2} &= X''(x)T(t) \\
T'(t)X(x) &= X''(x)T(t)
\end{aligned}$$

t, and x are independant of eachother, therefore:

$$\begin{aligned}
\frac{T'(t)}{T(t)} &= \frac{X''(x)}{X(x)} = \lambda \\
T'(t) - \lambda T(t) &= 0 \\
\omega(t) &= e^{\int -\lambda \, \mathrm{d}t} \\
\omega(t)T(t) &= \int 0 \, \mathrm{d}t = c_3 \\
T(t) &= c_3 e^{\lambda t} \\
X''(x) - \lambda X(x) &= 0 \\
X'' - \lambda X &= 0 \\
r^2 + 0r - \lambda &= 0 \\
r &= \frac{-0 \pm \sqrt{0^2 - 4(-\lambda)}}{2} \\
&= \pm\sqrt{\lambda}
\end{aligned}$$

(a) $\lambda = +\mu^2 > 0$

$$\begin{aligned}
r &= \pm\mu \\
X(x) &= c_1 e^{\mu x} + c_2 e^{-\mu x} \\
u_x &= X'(x)T(t) = \left(c_1 \mu e^{\mu x} - c_2 \mu e^{-\mu x}\right) T(t) \\
u_x(0,t) &= 0 = u_x(1,t) \\
\left(c_1\mu - c_2\mu\right) T(t) &= 0 = \left(c_1 \mu e^{\mu} - c_2 \mu e^{-\mu}\right) T(t)
\end{aligned}$$

note that if $T(t) = 0$ then we are dealing with the trivial case $u(x,t) = 0$ which is not what we are looking for, so we say that $T(t) \neq 0$

$$\begin{aligned}
c_1\mu - c_2\mu &= 0 = c_1 \mu e^{\mu} - c_2 \mu e^{-\mu} && \mu \neq 0 \\
c_1 - c_2 &= 0 && c_1 = c_2
\end{aligned}$$

$$\begin{aligned}
c_1e^{\mu} - c_1e^{-\mu} &= 0\\
e^{\mu} &= e^{-\mu}\\
e^{2\mu} &= 1\\
\ln(e^{2\mu}) &= \ln(1) = 2\mu = 0\\
\mu &= 0
\end{aligned}$$

But we have defined $\mu^2 > 0$ so we have no solutions.

(b) $\lambda = 0$

$$\begin{aligned}
r &= \pm\sqrt{0} = 0\\
X(x) &= (c_1 + c_2x)e^{0x} = c_1 + c_2x\\
u_x(0,t) &= 0 = u_x(1,t)\\
c_2T(t) &= 0 = c_2T(t)
\end{aligned}$$

Again we take $T(t) \neq 0$

$$\begin{aligned}
c_2 &= 0\\
X(x) &= c_1 \qquad\qquad T(t) = c_3e^{0t} = c_3\\
u(x,t) &= c_1 \cdot c_3 = c_4
\end{aligned}$$

So we have one eigenfunction, $u(x,t) = c_0$

(c) $\lambda = -\mu^2 < 0$

$$\begin{aligned}
r &= \pm\sqrt{-\mu^2} = \pm\mu i\\
X(x) &= c_1\cos(\mu x) + c_2\sin(\mu x)\\
X'(x) &= -c_1\mu\sin(\mu x) + c_2\mu\cos(\mu x)\\
u_x(0,t) &= 0 = u_x(1,t)\\
[c_2\mu\cos(0) - c_1\mu\sin(0)]T(t) &= 0 = [c_2\mu\cos(\mu) - c_1\mu\sin(\mu)]T(t)
\end{aligned}$$

Taking $T(t) \neq 0$

$$\begin{aligned}
c_2\mu &= 0 = c_2\mu\cos(\mu) - c_1\mu\sin(\mu) \qquad\qquad \mu > 0 \to c_2 = 0\\
-c_1\mu\sin(\mu) &= 0
\end{aligned}$$

Avoiding the trivial solution requires $\sin(\mu) = 0$

$$\begin{aligned}
\mu &= n\pi \qquad\qquad n = 1, 2, 3, \ldots\\
T(t) &= c_3e^{-\mu^2t} = c_3e^{-n^2\pi^2t}\\
u_n(x,t) &= c_ne^{-n^2\pi^2t}\cos(n\pi x)
\end{aligned}$$