Homework 1

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2.4 a(d), b, d, f, g*
2.5 c,d,i*

2.4 A. In each of the following, compute the limit. Then, using $\varepsilon = 10^{-6}$, find an integer N that satisfies the limit definition.

(d) $\lim_{n\to\infty} \frac{n^2+2n+1}{2n^2-n+2}$

$$\lim_{n\to\infty} \frac{n^2+2n+1}{2n^2-n+2} = \lim_{n\to\infty} \frac{1}{2}\cdot\frac{n^2+2n+1}{n^2-\frac{1}{2}n+1}$$

$$\frac{1}{2}\cdot\frac{n^2+2n+1}{n^2-\frac{1}{2}n+1+\frac{5}{2}n} \le \frac{1}{2}\cdot\frac{n^2+2n+1}{n^2-\frac{1}{2}n+1} \le \frac{1}{2}\cdot\frac{n^2+2n+1}{n^2-\frac{1}{2}n+1-\frac{3}{2}n}$$

$$\frac{1}{2}\cdot\frac{n^2+2n+1}{n^2+2n+1} \le \frac{1}{2}\cdot\frac{n^2+2n+1}{n^2-\frac{1}{2}n+1} \le \frac{1}{2}\cdot\frac{n^2+2n+1}{n^2-2n+1}$$

$$\frac{1}{2} \le \frac{n^2+2n+1}{2n^2-n+2} \le \frac{1}{2}\cdot\frac{(n+1)^2}{(n-1)^2}$$

$$\left|\frac{1}{2}\cdot\frac{(n+1)^2}{(n-1)^2}-\frac{1}{2}\right| = \frac{1}{2}\cdot\frac{(n+1)^2-(n-1)^2}{(n-1)^2}$$

$$\left|\frac{1}{2}\cdot\frac{(n+1)^2}{(n-1)^2}-\frac{1}{2}\right| = \frac{1}{2}\cdot\frac{(n+1)^2-(n-1)^2}{(n-1)^2}$$

$$= \frac{1}{2}\cdot\frac{n^2+2n+1-(n^2-2n+1)}{(n-1)^2}$$

$$= \frac{1}{2}\cdot\frac{4n}{(n-1)^2} = \frac{1}{2}\cdot\frac{4n}{n^2-2n+1}$$

$$= \frac{1}{2}\cdot\frac{1}{\frac{n}{4}-\frac{1}{2}+\frac{1}{4n}} = \frac{1}{\frac{n}{2}-1+\frac{1}{2n}}$$

$$1 < \frac{n}{2}-1+\frac{1}{2n} \quad \forall n \ge 4$$

$$\frac{N}{2}-1+\frac{1}{2N} > \frac{1}{2}\cdot 10^k$$

$$N-2+\frac{1}{N} > 10^k$$

$$(10^k+2)-2+\frac{1}{10^k+2} > 10^k$$

We choose $N = 10^k+2$ and $\varepsilon = 2\cdot 10^-k$

$$\left|\frac{1}{2}\cdot\frac{(n+1)^2}{(n-1)^2}-\frac{1}{2}\right| \le \frac{1}{\frac{10^k+2}{2}-1+\frac{1}{2(10^k+2)}} < 2\cdot 10^-k$$

$$\lim_{n\to\infty} \frac{(n+1)^2}{2(n-1)^2} = \frac{1}{2}$$

Using the squeeze theorem we can conclude that $\lim_{n\to\infty} \frac{n^2+2n+1}{2n^2-n+2} = \frac{1}{2}$

Now we find an appropriate value of N for $\varepsilon = 10^{-6}$

$$\begin{aligned}
\left|\frac{n^2+2n+1}{2n^2-n+2} - \frac{1}{2}\right| &= \left|\frac{n^2+2n+1}{2(n^2-\frac{1}{2}n+1)} - \frac{n^2-\frac{1}{2}n+1}{2(n^2-\frac{1}{2}n+1)}\right| \\
&= \left|\frac{\frac{3}{2}n}{2(n^2-\frac{1}{2}n+1)}\right| = \frac{3n}{4[n(n-\frac{1}{2})+1]}
\end{aligned}$$

$$\frac{3}{4}\cdot\frac{1}{n-\frac{1}{2}+\frac{1}{n}} < \frac{1}{10^6}$$

$$n - \frac{1}{2} + \frac{1}{n} > \frac{3}{4}10^6$$

$$(\frac{3}{4}10^6+1) - \frac{1}{2} + \frac{1}{\frac{3}{4}10^6+1} > \frac{3}{4}10^6$$

Looks like a good value for N is $\frac{3}{4}10^6+1$ or 750001.

B. Show that $\lim_{n\to\infty} \sin\frac{n\pi}{2}$ does not exist using the definition of limit.

Note:

$$\sin\frac{(n+4)\pi}{2} = \sin\left(\frac{n\pi}{2}+2\pi\right) = \sin\frac{n\pi}{2}\cos 2\pi + \cos\frac{n\pi}{2}\sin 2\pi = \sin\frac{n\pi}{2}$$

$$\sin\frac{(4k+a)\pi}{2} = \sin\left(2\pi k + \frac{a\pi}{2}\right) = \sin(2\pi k)\cos\frac{a\pi}{2} + \cos(2\pi k)\sin\frac{a\pi}{2} = \sin\frac{a\pi}{2}$$

We only need to look at four cases: $n=0, n=1, n=2, n=3$. I forget whether we defined $\mathbb{N}$ to include zero on that first day, but lets just include it here today. So the four values that $\left|\sin\frac{n\pi}{2} - \sin\frac{(n+1)\pi}{2}\right|$ can be are:

$$\begin{aligned}
n=0 &\to \left|\sin 0 - \sin\frac{\pi}{2}\right| = 1 \\
n=1 &\to \left|\sin\frac{\pi}{2} - \sin\pi\right| = 1 \\
n=2 &\to \left|\sin\pi - \sin\frac{3\pi}{2}\right| = 1 \\
n=3 &\to \left|\sin\frac{3\pi}{2} - \sin 2\pi\right| = 1
\end{aligned}$$

Now we notice that

$$|a_n - L| + |a_{n+1} - L| \geq |(a_n - L) - (a_{n+1} - L)| = |a_n - a_{n+1}| = 1$$

Lets choose $\varepsilon = \frac{1}{2}$. Then $|a_n - L| < \frac{1}{2}$

$$\begin{aligned}
|a_n - L| + |a_{n+1} - L| &< \frac{1}{2} + |a_{n+1} - L| \\
|a_n - a_{n+1}| &< \frac{1}{2} + |a_{n+1} - L|
\end{aligned}$$

$$1 < \frac{1}{2} + |a_{n+1} - L|$$
$$\frac{1}{2} < |a_{n+1} - L|$$

Which is a problem because if $|a_n - L| < \frac{1}{2}$ then since $n+1 > n \geq N$ we should have $|a_{n+1} - L| < \frac{1}{2}$ not the other way around. We must not have a limit. $\square$

D. Prove that if $L = \lim_{n\to\infty} a_n$, then $L = \lim_{n\to\infty} a_{2n}$ and $L = \lim_{n\to\infty} a_{n^2}$.

$$|a_n - L| < \varepsilon \quad \forall n \geq N$$

because $2n \geq n \quad \forall n \in \mathbb{N}$ we have $2n \geq n \geq N$ and therefore $|a_{2n} - L| < \varepsilon$. The argument is exactly the same for n^2 because $n^2 \geq n \geq N$

F. Define a sequence $(a_n)_{n=1}^{\infty}$ such that $\lim_{n\to\infty} a_{n^2}$ exists but $\lim_{n\to\infty} a_n$ does not exist.

$$a_n = \left\lfloor \frac{\lfloor \sqrt{n} \rfloor}{\sqrt{n}} \right\rfloor$$

G. Suppose that $\lim_{n\to\infty} a_n = L$ and $L \neq 0$. Prove there is some N such that $a_n \neq 0$ for all $n \geq N$.

proof

We know that there is some N such that $|a_n - L| < \varepsilon$ for all $0 < \varepsilon, n \geq N$. This is equivalent to $L - \varepsilon < a_n < L + \varepsilon$. We have two cases. $L > 0$ and $L < 0$. If $L > 0$ then we choose $\varepsilon = L$ and $0 = L - L < a_n < L + L$. Because $0 < a_n$ it is safe to say $a_n \neq 0$ I think. If $L < 0$ then we choos $\varepsilon = -L$ which leads to $L + L < a_n < L - L = 0$. Now again, because $a_n < 0$ we can say $a_n \neq 0$. $\square$

2.5 C. If $\lim_{n\to\infty} a_n = L > 0$, prove that $\lim_{n\to\infty} \sqrt{a_n} = \sqrt{L}$. Be sure to discuss the issue of when $\sqrt{a_n}$ makes sense. HINT: Express $|\sqrt{a_n} - \sqrt{L}|$ in terms of $|a_n - L|$

proof

We must specify that $a_n \geq 0$. This is after all *real* analysis.

$$|a_n - L| < \varepsilon$$
$$\left|\sqrt{a_n}^2 - \sqrt{L}^2\right| < \varepsilon$$
$$\left|(\sqrt{a_n} + \sqrt{L})(\sqrt{a_n} - \sqrt{L})\right| < \varepsilon$$
$$\sqrt{a_n} + \sqrt{L} > \sqrt{L} > 0$$
$$(\sqrt{L})\left|(\sqrt{a_n} - \sqrt{L})\right| < (\sqrt{a_n} + \sqrt{L})\left|(\sqrt{a_n} - \sqrt{L})\right| < \varepsilon$$
$$\left|(\sqrt{a_n} - \sqrt{L})\right| < \frac{\varepsilon}{\sqrt{L}}$$

Now we can write an arbitrary $\gamma > 0, \gamma \in \mathbb{R}$ as $\frac{\varepsilon}{\sqrt{L}}$ where $\epsilon > 0, \epsilon \in \mathbb{R}$ and so we have the inequality $\left|(\sqrt{a_n} - \sqrt{L})\right| < \gamma$ which fits the definition of a limit and proves that $\lim_{n\to\infty} \sqrt{a_n} = \sqrt{L}$. $\square$

D. Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be two sequences of real numbers such that $|a_n - b_n| < \frac{1}{n}$. Suppose that $L = \lim_{n\to\infty} a_n$ exists. Show that $(b_n)_{n=1}^{\infty}$ converges to L also.

$$\begin{aligned}
b_n - \frac{1}{n} < a_n < b_n + \frac{1}{n} \\
-\frac{1}{n} - a_n < -b_n < \frac{1}{n} - a_n \\
\frac{1}{n} + a_n > b_n > a_n - \frac{1}{n} \\
\lim_{n\to\infty} \left(\frac{1}{n} + a_n\right) &= 0 + L \\
\lim_{n\to\infty} \left(a_n - \frac{1}{n}\right) &= L - 0 \\
\lim_{n\to\infty} b_n &= L
\end{aligned}$$

Using theorem 2.5.2 and the squeeze theorem. □

I. Suppose that $\lim_{n\to\infty} a_n = L$. Show that $\lim_{n\to\infty} \dfrac{a_1 + a_2 + \cdots + a_n}{n} = L$.