8.2

B. Show that $h_n(x) = \frac{n+x}{4n+x}$ converges uniformly on $[0, N]$ for any $N > \infty$ but not uniformly on $[0, \infty)$.

First we note that $\lim\limits_{n\to\infty} h_n$ is $h(x) = \frac{1}{4}$.

Our discontinuity at $4n + x = 0$ can not happen when $n \geq 1$ and $x \geq 0$. This will not affect us then.

Lets find $||h_n - h||_\infty$. The partial derivative of $h_n - h$ with respect to x is $\dfrac{3n}{(4n+x)^2}$ with a second derivative of $-\dfrac{6n}{(4n+x)^3}$. Notice that $3n > 0, 6n > 0$, and $4n + x > 0$ when $n \geq 1$ and $x \geq 0$. And so we have no critical points and our function is concave over the entire domain. So then $||h_n - \frac{1}{4}||_\infty = \max(||h_n(0) - \frac{1}{4}||, ||h_n(N) - \frac{1}{4}||)$. Because both $h_n(0)$ and $h_n(N)$ both converge to $\frac{1}{4}$ then $||h_n - \frac{1}{4}||_\infty$ must converge to $\frac{1}{4}$. It then follows that $\lim\limits_{k\to\infty} ||h_k - h||_\infty = 0$ on $[0, N]$.

Now let us broaden our view a little and look at what happens when $x \in [0, \infty)$. Let's choose $\varepsilon = \frac{1}{2}$ and n arbitrarily large and see what happens when $x \geq 8n$. Keep in mind that we never have to deal with negative numbers.

$$\begin{aligned}
x &\geq 8n \\
\frac{1}{4}x &\geq 2n \\
n + x &\geq 3n + \frac{3}{4}x \\
\frac{n+x}{4n+x} &\geq \frac{3n + \frac{3}{4}x}{4n+x} = \frac{3}{4} \cdot \frac{4n+x}{4n+x} \\
\frac{n+x}{4n+x} - \frac{1}{4} &\geq \frac{3}{4} - \frac{1}{4} = \varepsilon
\end{aligned}$$

And so regardless of our choice of n we have $||h_n - h|| \geq \varepsilon$ for all $x \geq 8n$. This breaks the definition of uniform convergence, thus this function does not converge uniformly over $[0, \infty)$

D. Let (f_n) and (g_n) be sequences of continuous functions on $[a, b]$. Suppose that (f_n) converges uniformly to f and (g_n) converges uniformly to g on $[a, b]$. Prove that $(f_n g_n)$ converges uniformly to fg on $[a, b]$

First we observe that because f_n and g_n are both continuous and both converge uniformly, then f and g must also be continuous. From the extreme value theorem we know then that all of f_n, g_n, f, and g are all bounded. First we need to find a bound for all the g_ns.

We know that for all $\varepsilon > 0$ there exists some $N \in \mathbb{N}$ such that for all $n \geq N$ we have $||g_n - g||_\infty < \varepsilon$. If we choose $\varepsilon = 1$ then we can find some $N \in \mathbb{N}$ such that $||g_n - g||_\infty < \varepsilon$ for all $n \geq \mathbb{N}$. Lets set $M_g =$

$\max(||g_1||_\infty, \ldots, ||g_N||_\infty, ||g||_\infty)+1$. Now obviously $M_g > g_n$ when $n < N$ and if $n \geq N$ then for all $x \in [a,b]$ we have $||g_n(x)|| = ||g_n(x) - g(x) + g(x)|| \leq ||g_n(x) - g(x)|| + ||g(x)|| \leq ||g_n(x) - g(x)|| + (M_g - 1) \leq \varepsilon + M_g - 1$. Remembering that $\varepsilon = 1$ we have $||g_n(x)|| \leq M_g$ for all x and so $||g_n||_\infty \leq M_g$.

Okay now we have found our bound of M_g for g_n. Keeping in mind that f is also bounded, we can find some M that bounds both g_n and f. Lets say $M = \max(M_g, ||f||_\infty)$. Now we choose some arbitrary $\varepsilon > 0$. We know that we can find some $N \in \mathbb{N}$ large enough that $||g_n - g||_\infty \leq \frac{\varepsilon}{2M}$ and $||f_n - f||_\infty < \frac{\varepsilon}{2M}$ for all $n > N$.

Back to where we started, we can do some algebra on $||f_n(x)g_n(x) - f(x)g(x)||$ to make things a little more manageable. Lets to that. The following is true for all $x \in [a,b]$ so will leave the xs out of the notation.

$$\begin{aligned}
||f_n g_n - fg|| &= ||f_n g_n - f g_n + f g_n - fg|| \\
&\leq ||f_n g_n - f g_n|| + ||f g_n - fg|| \\
&= ||g_n|| \cdot ||f_n - f|| + ||f|| \cdot ||g_n - g||
\end{aligned}$$

If we remember that $||g_n||$ and $||f||$ are both no bigger than M while $||g_n - g||$ and $||f_n - f||$ are smaller than $\frac{\varepsilon}{2M}$ then we have $||f_n g_n - fg|| < M\frac{\varepsilon}{2M} + M\frac{\varepsilon}{2M} = \varepsilon$. And because this is true for every $x \in [a,b]$ then we have $||f_n g_n - fg||_\infty < \varepsilon$. Thus $\lim_{n\to\infty} ||f_n g_n - fg||_\infty = 0$ as required.$\square$

F. Let $f_n(x) = \arctan(nx)/\sqrt{n}$.

(a) Find $f(x) = \lim_{n\to\infty} f_n(x)$, and show that (f_n) converges uniformly to f on $\mathbb{R}$

Our function is bounded by $\left[-\frac{\pi}{2\sqrt{n}}, \frac{\pi}{2\sqrt{n}}\right]$. The limit of $\pm\frac{\pi}{2\sqrt{n}}$ as n goes to infinity is 0 and this function must converge uniformly to $f(x) = 0$

(b) Compute $\lim_{n\to\infty} f_n'(x)$, and compare this with $f'(x)$.

The derivative is $f_n'(x) = \frac{\sqrt{n}}{n^2x^2+1}$ and when we take the limit as n goes to infinity of that thing we get

$$\begin{aligned}
\lim_{n\to\infty} f_n' &= \begin{cases} \lim_{n\to\infty} \sqrt{n} & x = 0 \\ \frac{1}{x^2} \lim_{n\to\infty} \frac{\sqrt{n}}{n^2+1/x^2} & x \neq 0 \end{cases} \\
&= \begin{cases} \infty & x = 0 \\ \frac{1}{x^2} \lim_{n\to\infty} \frac{\sqrt{n}}{n^2+1/x^2} & x \neq 0 \end{cases} \\
&= \begin{cases} \infty & x = 0 \\ \frac{1}{x^2} \lim_{n\to\infty} \frac{1}{n^{3/2}+1/(\sqrt{n}x^2)} & x \neq 0 \end{cases}
\end{aligned}$$

$$= \begin{cases} \infty & x = 0 \\ 0 & x \neq 0 \end{cases}$$

And $f' = 0$ so the derivative of the limit and the limit of the derivative are the same everywhere except $x = 0$

(c) Where is the convergence of f'_n uniform? Prove your answer.

Look at $f''_n(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sqrt{n}}{n^2x^2+1}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(n^{3/2}x^2 + n^{-1/2}\right)^{-1} = -(n^{3/2}x^2 + n^{-1/2})^{-2}2n^{3/2}x$. We see immediately that because $n \geq 1$ then our only zero term is $x = 0$. If we think about the graph of arctan, this matches our intuition. The slope increases as we approach zero from either direction and decreases as we get farther away. Compressing the graph vertically and horizontally by any factor will not change this. Since for any $[a, \pm\infty)$ with $|a| > 0$ we have an upper bound at $f'_n(a)$ which goes to zero, then we know that we converge uniformly in either of these domains. However, taking the contrapositive of theorem 8.2.1 we find that when we include zero in our domain, then our limit is discontinuous, and so either our sequence is not continuous, or we do not converge uniformly. We know our sequence is continuous, and so we must not converge uniformly.

G. Suppose that functions f_k defined on $\mathbb{R}^n$ converge uniformly to a function f. Suppose that each f_n is bounded, say by A_k. Prove that f is bounded.

We know that for any given $\varepsilon > 0$ we can find some N such that if $k \geq N$ then $||f_k - f||_\infty < \varepsilon$. If we pick $1 = \varepsilon$ then we get $1 > ||f_k - f||_\infty \geq |||f_k|||_\infty - ||f||_\infty| = |A_k - ||f||_\infty|$ and so $-1 < A_k - ||f||_\infty < 1$ or $A_k + 1 > ||f||_\infty > A_k - 1$ and so f is bounded.

I. Give an example of a sequence of discontinuous functions f_n that converges uniformly to a continuous function.

$$f_n(x) = \begin{cases} 0 & x = 0 \\ \frac{|x|}{xn} & x \neq 0 \end{cases}$$

This function is $\pm\frac{1}{n}$ when $x \neq 0$ and 0 when $x = 0$. Obviously this is discontinuous, but $\pm\frac{1}{n}$ approaches zero as n goes to infinity and so it must converge uniformly since we can find $\frac{1}{n} < \varepsilon$ for any $\varepsilon > 0$