Homework 3

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2.6 F. Let a, b be positive real numbers. Set $x_0 = a$ and $x_{n+1} = ({x_n}^{-1} + b)^{-1}$ for $n \geq 0$.

(a) Prove that x_n is monotone decreasing.

proof

If x_n is monotone decreasing, then $x_n \geq x_{n+1}$ for all $n \geq 0$.

$$x_{n+1} = ({x_n}^{-1} + b)^{-1} = \frac{1}{\frac{1+bx_n}{x_n}} = \frac{x_n}{1 + bx_n}$$

Note that if x_n and b are positive, then so is x_{n+1}. Now we are told that x_0 and b are positive, so we know that all x_n are positive. This means of course that $1 + bx_n > 1$ which in turn means that $x_n > \frac{x_n}{1+bx_n} = x_n + 1$. Indeed it appears that not only is x_n monotone decreasing, it is strictly monotone decreasing. □

(b) Prove that the limit exists and find it.

proof

As we noted in the previous proof, x_n is positive for all $n \geq 0$. This implies that $x_n > 0$ and is therefore bounded from below. Because x_n is monotone decreasing and bounded from below, it has a limit. □

solution

$$L = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \left({x_n}^{-1} + b\right)^{-1} = \left(\left(\lim_{n\to\infty} x_n\right)^{-1} + b\right)^{-1} = \left(L^{-1} + b\right)^{-1}$$

$$\begin{aligned} L &= \frac{1}{\frac{1}{L} + b} \\ 1 &= 1 + bL \\ 0 &= bL \end{aligned}$$

So then $\lim_{n\to\infty} x_n = 0$.

G. Let $a_n = \left(\sum_{k=1}^{n} 1/k\right) - \log n$ for $n \geq 1$. **Euler's constant** is defined as $\gamma = \lim_{n\to\infty} a_n$. Show that $(a_n)_{n=1}^{\infty}$ is decreasing and bounded below by zero, and so this limit exists. HINT: Prove that $1/(n+1) \leq \log(n+1) - \log n \leq 1/n$

proof

$$\begin{aligned} a_{n+1} &= \left(\sum_{k=1}^{n+1} \frac{1}{k}\right) - \log(n+1) \\ &= \frac{1}{n+1} + \left(\sum_{k=1}^{n} \frac{1}{k}\right) - \log n - \log\left(1 + \frac{1}{n}\right) \end{aligned}$$

Now lets just take a step back here and recall what Euler's number is.

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$$

$$1 < 1 + \frac{1}{n} \quad \forall n > 0$$

therefore

$$\begin{aligned} \frac{1}{n+1} - \ln\left(1+\frac{1}{n}\right) &\leq 0 \\ \frac{1}{n+1} &\leq \ln\left(1+\frac{1}{n}\right) \\ 1 &\leq \ln\left(1+\frac{1}{n}\right)^{n+1} \\ e &\leq \left(1+\frac{1}{n}\right)^{n+1} \\ \lim_{k\to\infty}\left(1+\frac{1}{n+k}\right)^n \cdot \left(1+\frac{1}{n+k}\right)^{n+k} \end{aligned}$$

M. Suppose that $(a_n)_{n=1}^{\infty}$ has $a_n > 0$ for all n. Show that $\limsup {a_n}^{-1} = (\liminf a_n)^{-1}$.

proof

Lets take some i, j such that $a_i \geq a_j$. The fact that $a_n > 0$ implies that if $a_i \geq a_j$ then $\frac{1}{a_j} \geq \frac{1}{a_i}$.