

Notes

October 27, 2014

if $\liminf x_n = L$ then there exists $\{x_{n_k}\}$ such that $\lim x_{n_k} = L$

$$l = \liminf x_n = \liminf \underbrace{\{x_{n_1}, x_{n_2}, x_{n_3}, \dots\}}_{c_n}$$

why not just let c_n be the subsequence? because c_n may not be equal to any of the x_k in the sequence

$c_n = \inf\{x_{n_1}, x_{n_2}, \dots\}$ give $\varepsilon = 2^{-n}$ there exists $x_{n_k} \in \{x_{n_1}, x_{n_2}, x_{n_3}, \dots\}$ such that $|c_n - x_{n_k}| < 2^{-n}$
by def of infimum

we have a sequence $\{c_n\}$ given $\varepsilon > 0$ there exists N such that $|c_n - L| < \varepsilon$ if $n \geq N$. we approximate each c_n by some x_{n_k} from the original sequence such that

convergence test for series

first we talk about series with positive terms $\sum_{k=1}^{\infty} a_k$, $s_n = \sum_{k=1}^n a_k$. So if s_n is bounded above then the series is convergent. and if not, it is divergent.

geometric series $\sum_{n=0}^{\infty} r^n$ is convergent if $|r| < 1$. $s_n = \sum_{k=0}^n nr^k = 1 + r + r^2 + \dots + r^n$, $rs_n = r + r^2 + r^3 + \dots + r^{n+1}$
 $s_n - rs_n = 1 - r^{n+1}$
 $s_n = \frac{1-r^{n+1}}{1-r} \rightarrow \frac{1}{1-r}$

comparison test

if $\forall n, |a_n| \leq b_n$

- if $\sum_{n=1}^{\infty} b_n$ is convergent then $\sum_{n=1}^{\infty} a_n$ is convergent,
- if $\sum a_n$ is divergent, so is $\sum b_n$.

3.2.b

show that if $(|a_n|)_{n=1}^{\infty}$ is summable then so is $(a_n)_{n=1}^{\infty}$.

$$\sum_{k=n+1}^m |a_k| < \varepsilon \text{ for all } N \leq n \leq m \text{ because } (|a_n|)_{n=1}^{\infty} \text{ is summable}$$
$$\left| \sum_{k=n+1}^m a_k \right| \leq \sum_{k=n+1}^m |a_k| < \varepsilon$$

so then $\sum a_k$ is also cauchy and summable

cauchy-schwartz inequality

$$\sum_{k=1}^n a_k b_k \leq \left(\sum_{k=1}^n a_k^2 \right)^{1/2} \left(\sum_{k=1}^n b_k^2 \right)^{1/2}$$

3.2.f

leibniz test for alternating series

if $\{a_n\}$ is a monotone decreasing sequence of positive terms with the $\lim a_n = 0$ then $\sum_{n=1}^{\infty} (-1)^n a_n$ is convergent

note!

a sequence may have the property $\lim |a_n - a_{n+1}| = 0$ but not be cauchy

3.2.h

Show that if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with $b_n \geq 0$ such that $\limsup_{n \rightarrow \infty} \frac{a_n}{b_n} < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then the series $\sum_{n=1}^{\infty} a_n$ converges.

$$\begin{aligned} \left| \left(\sup_{k \geq n} \frac{|a_k|}{b_k} \right) - L \right| &< \varepsilon \\ \left(\sup_{k \geq n} \frac{|a_k|}{b_k} \right) &< L + \varepsilon \\ \frac{|a_k|}{b_k} &< L\varepsilon \\ |a_k| &< (L + \varepsilon)b_k \end{aligned}$$

3.2.j

$$\liminf \frac{a_{n+1}}{a_n} \leq \liminf a_n^{\frac{1}{n}} \leq \limsup a_n^{\frac{1}{n}} \leq \limsup \frac{a_{n+1}}{a_n}.$$

step 1

if $x \geq r$ for all $r > b$ then x is a lower bound for the set $\{r \in \mathbb{R} : r > b\}$, $x \leq \inf\{r \in \mathbb{R} : r > b\} = b$

we will show that if $\limsup \frac{a_n}{b_n} < r$ then $\limsup a_n^{\frac{1}{n}} \leq r$ and then apply step one.

let $r > \limsup \frac{a_{n+1}}{a_n}$ then $\exists N$ such that $r > \frac{a_{n+1}}{a_n} \forall n \geq N$

$$\begin{aligned} a_{N+1} &< r a_N \\ a_{N+2} &< r a_{N+1} \leq r^2 a_N \\ a_{N+K} &< r^K a_N \\ a_{N+K}^{\frac{1}{N+K}} &< (r^K a_N)^{\frac{1}{N+K}} \end{aligned}$$

quiz from 10/1/2014

$L_k \rightarrow L$ then $\{x_n\}$ such that $\forall k, \exists$ a subsequence of $\{x_n\}$ converging to L_k . prove that $\{x_n\}$ has a subsequence converging to L .

given $\varepsilon > 0 \exists N_0$ such that $|L_k - L| < \varepsilon$ if $k \geq N_0$

$$|x_{N_k} - L| \leq |x_{N_k} - L_k| + |L_k - L| < 2\varepsilon$$

example

let $A, B \subseteq \mathbb{R}$, prove that $\sup A \leq \inf B$, if $\forall a \in A, b \in B, a \leq b$

3.3.5

any rearrangement of an absolutely convergent series converges to the same limit

proof

let $\sum a_n = L < \infty$. We know $\sum |a_n|$ is convergent (not necessarily to L). by the Cauchy criterion for series $\forall \varepsilon > 0 \exists N$ such that $\left(\sum_{n=N+1}^{\infty} |a_n| \right) < \varepsilon$

$\pi : \mathbb{N} \rightarrow \mathbb{N}$ is bijective, the rearranged series is $\sum_{n=1}^{\infty} a_{\pi(n)}$ and $\{a_1 \dots a_N\} \subseteq \{a_{\pi(1)} \dots a_{\pi(M)}\}$

3.3.7 rearrangement theorem

let $\sum a_n = L < \infty$ and define $b_n = (a_n \geq 0) ? a_n : 0$ and $c_n = (a_n < 0) ? a_n : 0$
consider the series $\sum b_n$ and $\sum |c_n|$

case 1

both convergent

$\sum |a_n| = \sum b_n + \sum |c_n|$ which is convergent, which contradicts the fact that a_n is conditionally convergent

case 2

one convergent, one divergent

assume $\sum |c_n| = A < \infty$ and $\sum b_n$ is divergent to $+\infty$

given any $R \in \mathbb{N}$ big, $\exists N$ such that $\sum_{n=1}^N b_n > R + A$, then we pick M big enough so that $\{b_1, \dots, b_N\} \subseteq \{b_1, \dots, b_M\}$

$\{a_1, a_2, \dots, a_M\}$ and $\sum_{n=1}^M a_n \geq \sum_{n=1}^M b_n - \sum_{n=1}^M |c_n| > R$ so $\sum a_n$ is divergent, which is a contradiction.

case 3

both divergent

chapter 4

$\mathbb{R}^n = \{(x_1, x_2, \dots, x_n), x_i \in \mathbb{R}\}$, vector space (or point in n -space).
with the coordinate wise sum and the product by real numbers (scalars).

$$\begin{aligned}(x_1, \dots, x_n) + (y_1, \dots, y_n) &= (x_1 + y_1, \dots, x_n + y_n) \\ \lambda(x_1, \dots, x_n) &= (\lambda x_1, \dots, \lambda x_n) \\ x^{\rightarrow} &= (x_1, \dots, x_n) = x\end{aligned}$$

euclidean norm

$$||x|| = \sqrt{x_1^2 + \dots + x_n^2}$$

distance from x to y

$$||x - y||$$

cauchy-schwarz

$$\left| \sum_{j=1}^n a_j b_j \right| \leq \left(\sum_{j=1}^n a_j^2 \right)^{1/2} \left(\sum_{j=1}^n b_j^2 \right)^{1/2}$$
$$|a \cdot b| \leq ||a|| ||b||$$

dot product

$$a \cdot b = \sum a_i b_i$$

triangle inequality

$$||x + y|| \leq ||x|| + ||y||$$

proof

$$\begin{aligned}||x + y||^2 &= \sum (x_i + y_i)^2 \\ &= (x + y) \cdot (x + y) \\ &= x \cdot x + 2x \cdot y + y \cdot y \\ &= ||x||^2 + 2x \cdot y + ||y||^2 \\ &\leq ||x||^2 + 2||x|| ||y|| + ||y||^2 \\ &= (||x|| + ||y||)^2\end{aligned}$$

standard orthogonal base of \mathbb{R}^n

$$\begin{aligned}e_1 &= \langle 1, 0, \dots, 0 \rangle \\ e_2 &= \langle 0, 1, \dots, 0 \rangle \\ &\vdots \\ e_n &= \langle 0, 0, \dots, 1 \rangle\end{aligned}$$

4.2 convergence in \mathbb{R}^n

definition: a sequence $\{x^i\}$ of points in \mathbb{R}^n converge to $c \in \mathbb{R}^n$ if $\forall \varepsilon > 0 \exists N = N(\varepsilon) \in \mathbb{N}$, such that $\|x^i - c\| < \varepsilon$ if $i \geq N$ we say $\lim x^i = c$.

4.2.2 lemma

$\lim x^i = a$ if and only if $\lim \|x^i - a\| = 0$.

4.2.3 lemma

$\lim x^i = a$ if and only if $\forall j = 1, \dots, n, \lim x_j^i = a_j$

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lemma 4.2.3****know this

a sequence $\{x^i\}$ of points in \mathbb{R}^n converges to $a \in \mathbb{R}^n$ if and only if for each coordinate $\lim x_j^i = a_j$

thm 4.2.5

every cauchy sequence of points in \mathbb{R}^n converges to a point in \mathbb{R}^n .

def

a sequence $\{x^i\}$ of points in \mathbb{R}^n is cauchy if $\forall \varepsilon > 0 \exists N \in \mathbb{N}$ such that $\|x^i - x^j\| < \varepsilon$ for all $i, j \geq N$

proof

we have a cauchy sequence. given $\varepsilon > 0, \exists N, \|x^i - x^j\| < \varepsilon$ if $i, j \geq N$

$$|x_k^i - x_k^j| \leq \|x^i - x^j\| = \sqrt{(x_1^i - x_1^j)^2 + (x_2^i - x_2^j)^2 + \dots + (x_n^i - x_n^j)^2}$$

so each of the coordinates for the sequence is cauchy ($\{x_k^i\}$ is cauchy). So it converges to some $a_k \in \mathbb{R}$ and $a = (a_1, a_2, \dots, a_n)$ and so by lemma 4.2.3 $\|x^i - a\| \rightarrow 0$

4.2.6

read, useful for next weeks hw

4.3 open, closed sets in \mathbb{R}^n

def: let $A \subseteq \mathbb{R}^n$. we say that x is a limit point of A if there exists a sequence $\{a_k\}$ with $a_k \in A$ such that the limit of the sequence is x .

def: a set $A \subseteq \mathbb{R}^n$ is closed if it contains all of its limit points.

example

is $[0, 1]$ closed? and is $(0, 1]$ not closed?

0 is a limit point because $0 = \lim \frac{1}{n}$ and $\frac{1}{n} \in (0, 1]$.

consider $[a, b]$. $\{x_n\} \subseteq [a, b]$. $a \leq x_n \leq b, \forall n = 1, 2, 3, \dots$

assume $\lim x_n$ exists, call it x . we will show $a \leq x \leq b$. then assume not and show wlog $x > b$.

take $\varepsilon = \frac{x-b}{2}$. then $\exists N$ such that $|x_N - x| < \varepsilon$. $x - x_N < \frac{x-b}{2}$ and $b < \frac{x}{2} + \frac{b}{2} < x_N$ and so we have a contradiction

special cases

\emptyset is closed. $[a, +\infty)$ and $(-\infty, a]$ are closed.

finite sets of \mathbb{R}^n are closed.

proof

let $A = \{a_1, \dots, a_M\}$. consider a sequence $\{x_j\}$ such that $x_j \in A, \forall j \in \mathbb{N}$. at least one of the points appears ∞ many times. if $\lim x_j$ exists then at some point the sequence is a single repeating point, which is the limit, which is in A .

proposition 4.3.3

the finite union of closed sets is closed, arbitrary intersections of closed sets are closed.

proof

let A, B be closed, we need to check that $A \cup B$ is closed, then by induction, if A_1, \dots, A_N is closed then $\bigcup_{i=1}^N A_i$ is closed.

pick a sequence $\{x_j\}$ of points in $A \cup B$. converging to some $x \in \mathbb{R}^n$. We need to show that $x \in A \cup B$. $x_j \in A \cup B \Rightarrow x_j \in A$ or $x_j \in B$. We have infinitely many points and so either A or B contains infinitely many of the points. but since the sequence has a limit $\exists N$ such that $x_j \in A \forall j \geq N$

infinitely many of the points are in one of the sets, but since the sequence has a limit, passing to a subsequence if necessary, we get that all points in the sequence are eventually in one of the sets, hence the limit is in that set because the set is closed $\Rightarrow x \in A \cup B$

and for the second part: let $\{A_i\}$ be a collection of closed sets. let $\{x_n\}$ be a sequence such that $x_n \in \bigcap_{i \in I} A_i$ and $\lim x_n = x$. we need to show $x \in \bigcap A_i$ since $x_n \in A_i \forall i$ and A_i is closed $\lim x_n = x \in A_i \forall i$

example

a countable union of closed sets may not be closed. $A_n = [\frac{1}{n}, 1]$ $\bigcup_{n=1}^{\infty} [\frac{1}{n}, 1] = (0, 1]$

def

let $A \subseteq \mathbb{R}^n$. The closure of A , \bar{A} is the set containing all the limit points of A . \bar{A} is the smallest closed set that contains A .

def

a set $U \subseteq \mathbb{R}^n$ is open if $\forall x \in U \exists B(x, \varepsilon) \subset U$. $B(x, \varepsilon) = \{y \in \mathbb{R}^2 : \|y - x\| < \varepsilon\}$

proposition

a set is open iff A^C is closed.

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$\overline{\mathbb{Q}} = \mathbb{R}$ and $\mathbb{Q}^\circ = \emptyset$

$\overline{\mathbb{R} - \mathbb{Q}} = \mathbb{R}$ and $(\mathbb{R} - \mathbb{Q})^\circ = \emptyset$

define: a set $A \subseteq \mathbb{R}^n$ is dense if $\overline{A} = \mathbb{R}^n$.

A set $A \subseteq B$ is dense in B if $\overline{A} = \overline{B}$

4.3M

Let A be dense in \mathbb{R}^n and let U be an open set

a) we need to show that $U \subseteq \overline{A \cap U}$. Pick $x \in U$ to show that $x \in \overline{A \cap U}$. we have to find a sequence $a^i \in A \cap U$ such that $\lim a^i = x$.

since A is dense in \mathbb{R}^n and $x \in U \subseteq \mathbb{R}^n \Rightarrow \{b^i\} \subseteq A$ such that $\lim b^i = x$

since $x \in U$ and U is open there is a ball $(B(x, r), r > 0)$ in U and $b^i \in B(x, r)$ so $\|x - b^i\| < r$. the sequence $\{b^i\}$ is in $A \cap U$ and converges to $x \in U$ hence $A \cap U$ is dense in U .

more notes

\overline{C} is closure

C° is interior

$C = \{(x, y) : y = x^2\}$ $\overline{C} = C$ $C^\circ = \emptyset$

$S = \{(x, y) : y = \sin \frac{1}{x}\}$ $\overline{S} = S \cup \{(0, a) : a \in [-1, 1]\}$ $S^\circ = \emptyset$

cantor set

start with $C_0 = [0, 1]$, $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1] \dots$ and continue on removing the middle thirds. $\mathcal{C} = \bigcap_{n=1}^{\infty} C_n$ and by the nested intervals theorem $\mathcal{C} \neq \emptyset$. this is actually an uncountable set.

what are we removing? $C_0 = \frac{1}{3}$, $C_1 = \frac{2}{9}$, $C_n = \frac{2^{n-1}}{3^n}$

length of the removed part is $\sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} = \frac{1}{3} \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^{n-1} = 1$ so \mathcal{C} has length 0.

if we look at the sets C_0, C_2, C_2 in base 3 then $x \in [0, 1] = \sum_{i=1}^{\infty} \frac{a_i}{3^i}$, $a_i \in \{0, 1, 2\}$.

in step n we are removing from $[0, 1]$ all the points whose base 3 expansion has a 1 in the n th position.

so $\mathcal{C} = \{x \in [0, 1] \text{ st } x\text{'s base 3 expansion does not contain 1}\}$. $F : \mathcal{C} \rightarrow [0, 1]$ we map all the twos to ones. $x = 0.2\ldots > 0.1 = y$. this is a surjection. so cardinality of \mathcal{C} is greater than or equal to $[0, 1]$ which is the continuum (cardinality of the real numbers. but $\mathcal{C} \subseteq [0, 1]$ so they have the same cardinality.

assume $(a, b) \subseteq \mathcal{C}$ then $(a, b) \subseteq C_n$. Every interval has length 3^{-n} and so we can find an N such that $|a - b| > 3^{-N}$.

definitions

def: a set $A \subseteq \mathbb{R}^n$ is compact if every sequence $\{a^k\}$ of elements of A has a convergent subsequence $\{a^{k_n}\}$ and $\lim a^{k_n} = a \in A$ any closed and bounded set in \mathbb{R}^n is compact by bolzano-weierstrass

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chap 5 functions

def

The limit of the function f as $\vec{x} \in S$ approaches $\vec{a} \in \mathbb{R}^n$ is $\vec{v} \in \mathbb{R}^m$ if $\forall \varepsilon > 0 \exists r > 0$ such that $\|f(\vec{x}) - \vec{v}\| < \varepsilon$ whenever $\|\vec{x} - \vec{a}\| < r$

def

f is continuous at \vec{a} if $\forall \varepsilon > 0 \exists r > 0$ such that $\|f(\vec{x}) - f(\vec{a})\| < \varepsilon$ whenever $\|\vec{x} - \vec{a}\| < r$.

example

prove that $\lim_{x \rightarrow 2} x^2 = 4$. using the definition of limit of a function.

given $\varepsilon > 0$ we have to find r depending on ε such that if $|x - 2| < r$ then $|x^2 - 4| < \varepsilon$.

since x is approaching 2 then $|x - 2|$ will eventually be less than 1. $1 \leq x \leq 3$. $3 \leq x + 2 \leq 5$

$$|x^2 - 4| = |x - 2||x + 2| < r \cdot 5 = \varepsilon$$

now take $r = \frac{\varepsilon}{5}$ we get $|x^2 - 4| < \varepsilon$. so given $\varepsilon > 0$, take $r = \frac{\varepsilon}{5}$ and this guarantees that if $|x - 2| < r$ then $|x^2 - 4| < \varepsilon$

5.1.C

$$\text{let } f(x) = \begin{cases} x - 2n & \text{if } 2n \leq x \leq 2n + 1 \\ 2n + 2 - x & \text{if } 2n + 1 \leq x \leq 2n + 2 \end{cases}$$

take $2n < a < 2n + 1$. Then $|f(x) - f(a)| = |x - 2n(a - 2n)| = |x - a|$. take $r = \varepsilon$ if $|x - a| < r$ then $|f(x) - f(a)| < \varepsilon$. Similar with $a \in (2n + 1, 2n + 2)$.

now $a = 2n$. Then $|f(x) - f(a)| = \begin{cases} x - 2n - 0 & \text{if } x > a \\ |2n - x - 0| & \text{if } x < a \end{cases}$ assuming $|x - a| < 1$. In either case $|f(x) - f(a)| = |x - 2n| = |x - a|$ again, take $r = \varepsilon$.

definition

a function is Lipschitz if $\exists C > 0$ such that $\|f(x) - f(y)\| \leq C\|x - y\| \forall x, y \in S$

by definition $\|x - y\| < \frac{\varepsilon}{C}$ then $\|f(x) - f(y)\| < \varepsilon$. so it is continuous, and $r = \frac{\varepsilon}{C}$ for every point so it is "uniformly" continuous because r does not depend on x, y .

$\lim_{x \rightarrow b} x^2 = b^2$ is continuous but r depends on both ε and b so x^2 is not uniformly continuous, but it is continuous.

let $f : \mathbb{R} \rightarrow \mathbb{R}$ be such that $|f'(x)| \leq A \forall x \in \mathbb{R}$ then f is Lipschitz, therefore uniformly continuous.

proof

mean value theorem $|f(x) - f(y)| = |f'(\zeta)|x - y| < A|x - y|$. $x < y \therefore \zeta \in (x, y)$.

5.1.k

find a bounded continuous function on \mathbb{R} that is not Lipschitz. note x^2 is not bounded.

$$f(x) = \sin \frac{1}{x} \text{ on } (0, +\infty)$$

$$y = \arctan x$$

example

if $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is linear then it is Lipschitz.

$$\text{let } A \text{ be the matrix of } f. A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix}$$

$$f(\vec{x}) = A\vec{x} = \left(\sum_{j=1}^n a_{1j}x_j, \dots, \sum_{j=1}^n a_{mj}x_j\right)$$

$$||f(x) - f(y)|| \leq C||x - y||$$

$$||F(x) - f(y)|| = ||f(x - y)|| = \sqrt{\sum_{k=1}^m \sum_{j=1}^n a_{kj}(x_j - y_j)} \leq \text{cauchy schwartz}$$