1. Prove that any countable set is measurable.

   proof

   Saying that a set is countable implies that we can index the elements of the set. And so we say that our countable set $E = \{e_1, e_2, \dots\}$. This is equivalent to saying $E = \bigcup\limits_{i=1}^{n} \{e_i\}$ with $n \in \mathbb{N}$ or $E = \bigcup\limits_{i=1}^{\infty} \{e_i\}$. We know that the measure of a singleton is 0 and we know that the union of some sets has a measure no larger than the sum of the measures of those sets. And so $m * (\bigcup\limits_{i=1}^{\infty} \{e_i\}) \leq \sum\limits_{i=1}^{\infty} m * (\{e_i\}) = \sum\limits_{i=1}^{\infty} 0 = 0$. And because we touched on countable but non-infinite sets, for completeness sake, $m * (\bigcup\limits_{i=1}^{n} \{e_i\}) \leq \sum\limits_{i=1}^{n} m * (\{e_i\}) \sum\limits_{i=1}^{n} 0 = 0$. Now we have proved not only that any countable set is measurable, but that any countable set has measure 0. $\square$

2. Prove that the Cantor set has measure 0.

   proof

   We'll call the Cantor set C and observe that $C \subset [0, 1]$. I'll also say $C' = C^C \cap [0, 1]$. That is, all the elements in $[0, 1]$ that aren't in the Cantor set are in C'. Obviously these two sets span $[0, 1]$ and are disjoint. That is to say $C \cup C' = [0, 1]$ and $C \cap C' = \emptyset$. We also know that $m * ([0, 1]) = 1$. So Cantor showed us that if we spend too long thinking about his set, then we will go mad. Lets try to avoid this.

$$\begin{aligned} m * ([0,1]) &= 1 \\ m * ([0,1]) &= m * (C \cup C') \\ m * ([C \cup C']) &= m * (C) + m * (C') \text{ because they are disjoint} \\ m * (C) + m * (C') &= 1 \\ m * (C) &= 1 - m * (C') \end{aligned}$$

   Yes, we can ignore C(razy). Now I know that $(\frac{1}{3}, \frac{2}{3}) \subset C'$ and that $(\frac{1}{9}, \frac{2}{9}) \cup (\frac{7}{9}, \frac{8}{9}) \subset C'$ and so on. Now notice that all the parts we cut out of $[0, 1]$ to make the Cantor set are disjoint. And so the measure of their unions is the same as the sum of their measures. Now every time we take a chunk out, we leave behind two chunks that are a third of the original chunk. And so

$$m * (C') = \sum_{i=1}^{\infty} 2^{i-1} m * \left(\left(\frac{1}{3^i}, \frac{2}{3^i} \right) \right)$$

$$= \frac{1}{2}\sum_{i=1}^{\infty} 2^i \left(\frac{2}{3^i} - \frac{1}{3^i}\right)$$
$$= \frac{1}{2}\sum_{i=1}^{\infty} \left(\frac{2}{3}\right)^i$$

using the formula for geometric series we get

$$m*(C') = \frac{1}{2} \cdot \frac{\frac{2}{3}}{1-\frac{2}{3}} = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{1} = 1$$

Well that is way better than going the way of Cantor. Just to be clear $m*(C) = 1 - m*(C') = 1 - 1 = 0$. $\square$

References

https://theoremoftheweek.wordpress.com/2010/09/30/theorem-36-the-cantor-set-is-an-uncountable-set-with-zero-measure/