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HW 23

Given $F(s) = \frac{1}{1+\sqrt{s}}$

(a) Introduce $G(s) = \frac{1}{s}F(s)$ and show that $sG(s) - G(s) = \frac{1}{\sqrt{s}} - \frac{1}{s}$

$$\begin{aligned}
sG(s) - G(s) &= G(s)(s-1) \\
&= (s-1)\frac{1}{s}\frac{1}{1+\sqrt{s}} \\
&= \frac{1}{s}\frac{(1+\sqrt{s})(1-\sqrt{s})}{1+\sqrt{s}} \\
&= \frac{1}{s} - \frac{\sqrt{s}}{s} \\
&= \frac{1}{s} - \frac{1}{\sqrt{s}}
\end{aligned}$$

(b) Now obtain a first order DE for $g(t)$. You may assume $g(0) = 0$, but show where this assumption is used.

$$\begin{aligned}
\mathcal{L}^{-1}\{sG(s) - G(s)\} &= \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{\sqrt{s}}\right\} \\
\mathcal{L}^{-1}\{sG(s) - 0\} - g(t) &= 1 - \mathcal{L}^{-1}\left\{\frac{\sqrt{\pi}}{\sqrt{\pi}}\frac{1}{\sqrt{s}}\right\} \\
\mathcal{L}^{-1}\{sG(s) - g(0)\} - g(t) &= 1 - \frac{1}{\sqrt{\pi}}\mathcal{L}^{-1}\left\{\frac{\Gamma(1/2)}{s^{1/2}}\right\} \\
g'(t) - g(t) &= 1 - \frac{1}{\sqrt{\pi}}t^{-1/2} = 1 - \frac{1}{\sqrt{t\pi}}
\end{aligned}$$

(c) Solve for $g(t)$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\int -1\,\mathrm{d}t}g(t)\right) &= e^{-\int \mathrm{d}t}\left(1 - \frac{1}{\sqrt{t\pi}}\right) \\
e^{-t}g(t) &= \int e^{-t} - \frac{e^{-}t}{\sqrt{t\pi}}\,\mathrm{d}t \\
e^{-t}g(t) &= -e^{-t} - \mathrm{erfc}(\sqrt{t}) + c_1 \quad \text{used maxima here} \\
g(t) &= -1 - e^t\mathrm{erfc}(\sqrt{t}) + e^t c_1 \\
g(t) &= e^t c_1 - e^t\mathrm{erfc}(\sqrt{t}) - 1
\end{aligned}$$

(d) The relation $G(s) = \frac{1}{s}F(s)$ implies a relation between $g(t)$ and $f(t)$. What is the relation?

$$\begin{aligned}
G(s) &= \frac{1}{s}F(s) \\
\mathcal{L}^{-1}\{G(s)\} &= \mathcal{L}^{-1}\{\frac{1}{s}F(s)\} \\
g(t) &= \int_0^t f(u)\,\mathrm{d}u
\end{aligned}$$

Use it to find $f(t)$