# Graph Theory Homework

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January 23, 2015

## Definitions

**path** A graph of order $n$ and size $n-1$ whose vertices can be labeled by $v_1, v_2, \ldots, v_n$ and whose edges are $v_1 v_{i+1}$ for $i = 1, 2, \ldots, n-1$.

**cycle** A graph of order $n$ and size $n$ whose vertices can be labeled by $v_1, v_2, \ldots, v_n$ and whose edges are $v_1 v_n$ and $v_1 v_{i+1}$ for $i = 1, 2, \ldots, n-1$.

**isomorphism** If $G$ and $H$ are graphs and $\phi : V(G) \to V(H)$ is a bijective function such that two vertices $u$ and $v$ are adjacent in $G$ if and only if $\phi(u)$ and $\phi(v)$ are adjacent in $H$. The function $\phi$ is an isomorphism.

**subgraph** Let $G$ and $H$ be graphs. Then if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ then $H$ is a subgraph of $G$. That is to say, $H$ is a subgraph of $G$ if $G$ contains all the vertices and edges of $H$.

**regular graph** A graph whose vertices all have the same degree.

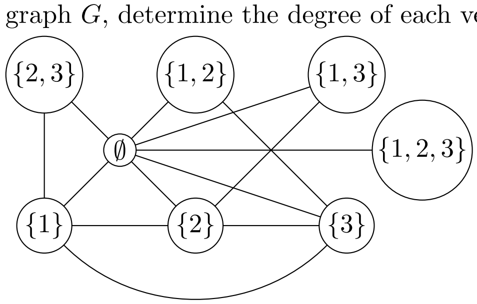
**bipartate graph** A graph whose vertices can be partitioned into two sets in such a way that every edge of the graph joins vertices from both sets.

**complement** A complement of a graph $G$ is the graph $\overline{G}$ which has the same vertex set as $G$ and where any two vertices are adjacent if and only if these vertices are not adjacent in $G$.

## Exercises

1.1

2. A graph $G = (V, E)$ of order 8 has the power set of the set $S = \{1, 2, 3\}$ as its vertex set, that is $V$ is the set of subsets of $S$. Two vertices $A$ and $B$ of $V$ are adjacent if $A \cap B = \emptyset$. Draw the graph $G$, determine the degree of each vertex of $G$ and determine the size of $G$.




$$\deg \emptyset = 7 \qquad \deg\{1\} = \deg\{2\} = \deg\{3\} = 3$$
$$\deg\{1,2,3\} = 1 \qquad \deg\{1,2\} = \deg\{2,3\} = \deg\{1,3\} = 2$$

The size $|E(G)|$ of $G$ is $7 + 1 + 9 + 6 = 23$

3. A graph $G$ of order 26 and size 58 has 5 vertices of degree 4, 6 vertices of degree 5 and 7 vertices of degree 6. The remaining vertices of $G$ all have the same degree. What is this degree?

$$26 - 5 - 6 - 7 = 8$$
$$116 - 5 \cdot 4 - 6 \cdot 5 - 7 \cdot 6 = 24$$
$$24 \div 8 = 3$$

The remaining 8 vertices have degree 3.

4. A graph of $G$ has order $n = 3k + 3$ for some positive integer $k$. Every vertex of $G$ has degree $k + 1, k + 2$ or $k + 3$. Prove that $G$ has at least $k + 3$ vertices of degree $k + 1$ or at least $k + 1$ vertices of degree $k+2$ or at least $k+2$ vertices of degree $k+3$. Hint: you can (probably should) use Corollary 1.5 to reach a contradiction in a few separate cases. The primary cases are for k even, and k odd.

11. Prove for every graph $G$ and every integer $r \geq \Delta(G)$ that there exists an $r$-regular graph containing $G$ as an induced subgraph.

    Hint: For number 11, the problem CAN be solved using simple graphs, but it can also be solved with multigraphs. The simple graph case is harder, but doable.

13. Determine all bipartite graphs $G$ such that $\overline{G}$ is bipartite

    Hint: For the bipartite graph question, first examine what happens if you have three vertices in one of your partites.

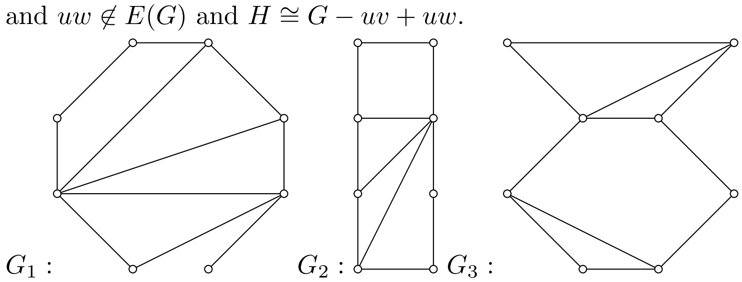
    If $n < 3$ then $\overline{G}$ is bipartite. If $n = 3$ and $G$ is not empty or complete then $\overline{G}$ is bipartite.

    Now we assume $n > 3$

$$|G| \leq \left\lfloor \frac{n^2}{4} \right\rfloor$$
$$|\overline{G}$$

18. Let $G$ be a self-complementary graph of order $n$, where $n \equiv 1 \mod 4$. Prove that $G$ contains an odd number of vertices of degree $(n-1)/2$.

1.2 6. Let $G$ and $H$ be two graphs that are neither empty nor complete. The graph $H$ is said to be obtained from $G$ by an **edge rotation** if $G$ contains three vertices $u, v$, and $w$ where $uv \in E(G)$ and $uw \notin E(G)$ and $H \cong G - uv + uw$.




   (a) Show that the graph $G_2$of figure 1.33 is obtained from $G_1$ by an edge rotation.

   (b) Show that $G_3$ of figure 1.33 cannot be obtained from $G_1$ by an edge rotation.

7. Determine whether the following sequences are graphical. If so, construct a graph with the appropriate degree sequence.

   (a) 4,4,3,2,1

   (b) 3,3,2,2,2,2,1,1

(c) 7,7,6,5,4,4,3,2

(d) 7,6,6,5,4,3,2,1

(e) 7,4,3,3,2,2,2,1,1,1

10. For which integers $x (0 \leq x \leq 7)$, if any, is the sequence $7, 6, 4, 3, 2, 1, x$ graphical?

15. Two finite sequences $s_1$ and $s_2$ of nonnegative integers are called **bigraphical** if there exists a bipartite graph $G$ with partite sets $V_1$ and $V_2$ such that $s_i$ lists the degrees of the vertices of $G$ in $V_i$ for $i = 1, 2$. Prove that the sequences $s_1 : a_1, a_2, \ldots, a_r$ and $s_2 : b_1, b_2, \ldots, b_t$ of nonnegative integers with $r \geq 2, a_1 \geq a_2 \geq \cdots \geq a_r, b_1 \geq b_2 \geq \cdots \geq b_t, 0 < a_1 \leq t$ and $0 < b_1 \leq r$ are bigraphical if and only if the sequences $s_1' : a_2, a_3, \ldots, a_r$ and $s_2' : b_1 - 1, b_2 - 1, \ldots, b_{a_1} - 1, b_{a_1+1}, \ldots, b_t$ are bigraphical