Notes

November 24, 2014

5.1#7

$1 = (1-a)^n = (1-a)(1+a+a^2+\cdots+a^(n-1)$

$u - a = a(1-u^{-1}a)$ and note that $u^{-1}a = b$ is nilpotent and so $(1-b)$ is invertible and then $u-a$ is invertible

5.1#8

$a + a = (a+a)^2 = a^2+a^2+a^2+a^2 = a+a+a+a \to 0 = a+a$

ring homomorphisms

for R, S commutative rings, $\varphi : R \to S$ is a ring homomorphism if

1. $\varphi(a+b) = \varphi(a) + \varphi(b)$
2. $\varphi(ab) = \varphi(a)\varphi(b)$
3. $\varphi(1_R) = 1_S$ (not always defined with this property, but are in this class)

prop

given that $\varphi : R \to S$ is a ring homomorphism then

1. $\varphi(0_R) = 0_S$
2. $\varphi(-a) = -\varphi(a)$
   because $\varphi(0 + (-1)a) = \varphi(0) + \varphi(-1)\varphi(a)$?

def

we say the φ is a isomorphism if it is bijective

exercises

if φ is an isomorphism then φ^{-1} is also an isomorphism

also composition of isomorphisms are isomorphisms (transitivity)

examples

$i : \mathbb{C} \to \mathbb{C}[x]$ where $i(a) = a$ is a ring homomorphism

$\mathbb{Z} \to \mathbb{Z}_n$ is not injective because it is an infinite set onto a finite set (pigeonhole principle)

$K \to K[x]/\langle f(x) \rangle$ where $\varphi(a) = [a]$

$K[x] \to K$. Fix $\alpha \in K$ and for each α we define $\varphi_\alpha : K[x] \to K$ and so $\varphi_\alpha(f(x)) = f(\alpha)$ is called evaluation function.

def

given a ring homomorphism where $\varphi : R \to S$ then $\ker \varphi = \{x \in R : \varphi(x) = 0\}$

proposition

if R and S are commutative rings and we have φ a ring homomorphism, then

1. for every $a, b \in \ker \varphi$ we have that $a - b$ and $a + b$ are also elements in $\ker \varphi$.
2. for every $r \in R$ and every $a \in \ker \varphi$ we have $ra \in \ker \varphi$

proof

1 follows because $\ker \varphi$ is an additive subgroup of the abelian group $(R, +)$.

2 folows because $\varphi(ra) = \varphi(r)\varphi(a) = \varphi(r) \cdot 0 = 0$

construction

two rings and a ring homomorphism $\varphi : R \to S$ and on R we define an equivalence relation $x \sim_\varphi y \Leftrightarrow \varphi(x) = \varphi(y)$ where $[x]$ is the equivalence class of x where $R/\ker \varphi = \{[x] : x \in R\}$. On $R/\ker \varphi$ we have the well defined operation $[x] + [y] = [x + y]$

now we define a new operation on the set $[x][y] = [xy]$

is this well defined?

$[x] = [x'] \to \varphi(x) = \varphi(x')$ and similarly with y and so $[x'][y'] = \varphi(x')\varphi(y') = \varphi(x)\varphi(y) = \varphi(xy) = [xy] = [x'y']$