Homework

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Section 3.6: 10, 21
Section 3.7: 16, 1 (a)–you may want to read carefully examples 3.7.6 and 3.7.7 first

3.6 10. Show that the following matrices form a subgroup of $GL_2(\mathbb{C})$ isomorphic to D_4:

$$\pm\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \pm\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad \pm\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \pm\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}.$$

Because D_n has two generators: a of order n and b of order 2 then $D_n \cong \mathbb{Z}_2 \times \mathbb{Z}_n$. So $D_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_4$.

$$\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}^4 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}^2 = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} 0 & b \\ a & 0 \end{bmatrix}$$

So our subgroup has two generators, $\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ of order 4 and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ of order 2 and so is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_4$ which is isomorphic to D_4 and so this subgroup is also isomorphic to D_4

21. Find the center of the dihedral group D_n.
Hint: Consider two cases, depending on whether n is odd or even.
First, we notice that every element of D_n is of the form a^k or ba^k. Second, we notice that $\langle b\rangle$ and $\langle a\rangle$ are abelian. Lets take it in four cases. First, notice that $a^z a^g = a^g a^z$ and so if both elements have the form a^k we have no restrictions. Now lets take $a^z \cdot ba^g$. Recall that $ba^k = a^{-k}b = a^{n-k}b$. And so

$$a^z ba^g = a^z a^{-g} b = a^{-g} a^z b = a^{-g} ba^{-z} = ba^g a^{-z}$$

Then $a^{-z} = a^z$ is a restriction on our center.
Now we look at elements of the form $ba^z a^g$

$$ba^z a^g = a^g ba^z$$
$$ba^z a^g = ba^g a^= a^{-g} ba^z = a^g ba^z$$
$$a^{-g} = a^g$$

This equality depends on a^g and so we will never find an element of the form ba^z that will commute with every element of the form a^g.
Lets take the last case. Note that the above already eliminated this case, but we will examine it for completeness.

$$ba^z ba^g = ba^g ba^z$$

$$
\begin{aligned}
a^{-z}bba^g &= a^{-g}bba^z \\
a^{-z}a^g &= a^{-g}a^z \\
e &= a^{-2g}a^{2z} \\
e &= a^{-g}a^z \\
a^g &= a^z
\end{aligned}
$$

So in this case our choice of a^z depends on a^g and so we can not find some ba^z that commutes with every ba^g.
This means that $Z(D_n) = \{a^k \in D_n : a^k = a^{-k}\}$.

$$
\begin{aligned}
a^k &= a^{-k} \\
a^{2k} &= e = a^n \\
2k &\equiv 0 \mod n \\
2k + nm &= 0 \\
k &= \frac{-nm}{2}
\end{aligned}
$$

Now lets assume n is even.

$$k = -\frac{2jm}{2} = -jm = -\frac{n}{2}m$$

Of course $a^{-\frac{n}{2}m} = (a^{n-\frac{n}{2}})^m = a^{\frac{n}{2}m}$. Now $(a^{\frac{n}{2}})^2 = a^n = e$. This means $a^{\frac{n}{2}}$ has order 2 (which we already know, as it is it's own inverse). So if n is even then the center of our group is $\{e, a^{\frac{n}{2}}\}$.
Now lets assume n is odd.

$$k = -\frac{(2j+1)m}{2} = -jm - \frac{m}{2}$$

Now k is an integer so $-jm - \frac{m}{2}$ must be an integer, so $2|m$. Say $2l = m$.

$$k = -2jl - \frac{2l}{2} = -2\frac{n-1}{2}l - l = -l(n-1+1) = -ln$$

Now notice that $a^k = a^{-ln} = (a^n)^{-l} = e^{-l} = e$. And so if n is odd then the center is $\{e\}$.

3.7 1. (a) Write down the formulas for all homomorphisms from $\mathbb{Z}_6$ into $\mathbb{Z}_9$. All homomorphisms will be completely determined by $\phi([x]_6) = [mx]_9$ when $9|6m$ according to example 3.7.7.

$9\|0$	$9\not|6$	$9\not|12$
$9\|18$	$9\not|24$	$9\not|30$
$9\|36$	$9\not|42$	$9\not|48$

So $\phi([x]_6) = [0]_9, \phi([x]_6) = [3x]_9$, or $\phi([x]_6) = [6x]_9$ are all the formulas that produce a homomorphism from $\mathbb{Z}_6$ into $\mathbb{Z}_9$

16. Let G be a finite group of even order, with n elements, and let H be a subgroup with $n/2$ elements. Prove that H must be normal.

Hint: Define $\phi : G \to \mathbb{R}^\times$ by $\phi(x) = 1$ if $x \in H$ and $\phi(x) = -1$ if $x \notin H$ and show that ϕ is a homomorphism with kernel H. To show that ϕ preserves products, show that if $g \notin H$ then $\{x : gx \in H\} = G - H$.

As the hint suggests, we define $\phi(x) = \begin{cases} 1 & \text{if } x \in H \\ -1 & \text{if } x \notin H \end{cases}$.

Now if $x_1, x_2 \in H$ then $x_1 x_2 \in H$ and $\phi(x_1 x_2) = 1 = 1 \cdot 1 = \phi(x_1) \cdot \phi(x_2)$.

Now if $g \notin H$ and $gx \in H$ then note that $gxx^{-1} = g \notin H$. Similarly, if $g \notin H$ and $xg \in H$ then $x^{-1}xg = g \notin H$. And so $x^{-1} \notin H$ because if it were then closure says that $g \in H$ and we would have a contradiction. And because if $x \in H$ then $x^{-1} \in H$ we know that if $x^{-1} \notin H$ then $x \notin H$. So if an element is not in H and it's product with another element is in H then the other element is not in H. This leads us to $\phi(gx) = 1 = -1 \cdot -1 = \phi(g)\phi(x)$ and $\phi(xg) = 1 = -1 \cdot -1 = \phi(x)\phi(g)$.

And the contrapositive says that if an element is in H then it's product with another element is not in H or the other element is in H.

So for $x \in H, g \notin H$ we know that $gx \notin H$ and $xg \notin H$. This leads us to $\phi(gx) = \phi(xg) = -1 = -1 \cdot 1 = \phi(g)\phi(x) = 1 \cdot -1 = \phi(x)\phi(g)$. And so we have established that ϕ preserves products and is therefore a homomorphism. Notice that $\ker \phi = H$. By proposition 3.7.4 we know that $ghg^{-1} \in H$ for all $h \in H$ and $g \in G$. This is actually the definition of a normal subgroup, so we are done.