Notes

20 avril, 2015

Heawood Map-coloring theorem

Sylvester converted maps into planar graphs
we are biased towards planar graphs because we live on a sphere
suppose we lived on a toroidal planet in addition to strange gravity we would color things differently

question?

is there a four color theorem for toroidal graphs?
heawood was interested in the "chromatic number of a surface".

definition

let S_k be the surface of genus k (ie the k-holed torus) $\chi(S_k)$ is the largest chromatic number of any graph embeddable on the surface S_k.

note

we are not assuming 2-cell embedding, in general we don't care about the embedding (it will happen naturally that we focus on more complicated embeddings)

example

four color theorem: $\chi(S_0) = 4$

task

lower bounds on $\chi(S_1)$, ie find graphs of various chromatic numbers embeddable on S_1

thrm 7.9?

$\tau(k_n) = \lceil \frac{(n-3)(n-4)}{12} \rceil$
we have that K_7 embeds on S_1 but k_8 does not. this shows $\chi(S_1) \geq 7$ but does not show $\chi(S_1) \leq 8$.

thm

$\chi(S_1) = 7$

proof

k_7 embeds in S_1 so $\chi(S_1) \geq 7$. we will use $\chi(G) \leq 1 + \delta(G)$ (1 plus minimal degree) to show that $\chi(S_1) \leq 7$
we need to show that for every graph embeddable on the torus has a degree of no more than six.
let G be a toroidal graph and $\delta(G)$ is maximal among all toroidal graphs.
if $|G| \leq 7$ then $\delta(G) \leq 6$. this finishes the claim, so we have $|G| > 7$.
since $\tau(G) \leq 1$ we have $1 \geq \tau(G) \geq \frac{m}{6} - \frac{n}{2} + 1$
solving for m we have $m \leq 3n$.
further back $\sum \deg_G(v_i) = 2m$ (twice the edges)
now $\sum \deg_G(v_i) = 2m \leq 6n$ therefore average degree is 6 and minimum degree is less than or equal to average degree.

another theorem

for $k > 0$ $\chi(S_k) \leq \left\lfloor \frac{7+\sqrt{1+48k}}{2} \right\rfloor$
what about equality?
recall thm 7.9: $\tau(k_n) = \lceil \frac{(n-3)(n-4)}{12} \rceil$
$\tau(k_n) \geq \frac{(n-3)(n-4)}{12}$
let $n = \left\lfloor \frac{7+\sqrt{1+48k}}{2} \right\rfloor$ so $n \leq \frac{7+\sqrt{1+48k}}{2}$
solving we have $\frac{(2n-7)^2-1}{48} \leq k$
$\frac{4n^2-28n+48}{48} = \frac{n^2-7n+12}{12} \leq k$
$\frac{(n-3)(n-4)}{12} \leq k$
$\tau(k_n) = \lceil \frac{(n-3)(n-4)}{12} \rceil \leq k$
hence the chromatic number of the complete graph on $\chi(k_{\tau(k_n)}) \leq \chi(S_k)$
since $k_{\tau(k_n)}$ is trivially embeddable (one vertex for each hole) on $S_{\tau(k_n)}$
so it follow that $\chi(S_k) \geq n =$ 1+48 root mess