Notes

November 10, 2014

4.3 #21a,c

Find multiplicative inverses of the given elements in the given fields

a

$[a+bx]$ in $\mathbb{R}[x]/\langle x^2+1\rangle$

$$\begin{aligned}
[a+bx][c+dx] &\equiv 1 \mod (x^2+1) \\
(a+bx)(c+dx)-1 &= x^2+1 \\
x^2+1 &= q(a+bx)+r \\
x^2+1 &= (bx+a)(\frac{1}{b}x-\frac{a}{b^2})+1+\frac{a^2}{b^2}
\end{aligned}$$

c

$[x^2-2x+1]$ in $\mathbb{R}[x]/\langle x^3-2\rangle$

$x^3-2=(x^2-2x+1)(x+2)+3x-4$
$x^2-2x+1=(\frac{1}{3}x-\frac{2}{9})(3x-4)+\frac{1}{9}$
$1=9(x^2-2x+1)-9(\frac{1}{3}x-\frac{2}{9})(3x-4)$
$1=(x^2-2x+1)(9+(3x-2)(x+2))-(x^3-2)(3x-2)$
$[1]=[x^2-2x+1][3x^2+4x+5]-[x^3-2][3x-2]$
$[0]=[x^3-2]$
$[x^2-2x+1]^{-1}=[3x^2+4x+5]$

htrm

f has no repeated factors iff $\gcd(f(x), f^{-1}(x))=1$

example

$K=Z_p(T)=\{\frac{f(t)}{g(t)}: f(t), g(t)\in\mathbb{Z}_p[T], g(t)\neq 0\}$ check that K is a field
let $f(x)=x^p-T\in K[x]$. claim that $f(x)$ is irreducible.

proposition

let $p(x) in K[x]\setminus\{0\}$ be irreducible with $\deg p\geq 1$. then $K[x]/\langle p(x)\rangle$ is a field.

proof

sind $p(x)$ is irreducible then $\gcd(f(x), p(x)) = 1$ and so $[f(x)]$ has mult inverse.

thm

take $f(x) \in K[x]$ with $\deg f \geq 1$ then there exists an extension of K called L such that $f(x)$ has a root in L

proof

let $p(x)$ be an irreducible factor of $f(x)$. now let $L = K[x]/\langle p(x)\rangle$. now $K \to \{[a] : a \in K\} \subseteq L$ with $x \to [x]$ is an isomorphism. let $u = [x] = L$ and $f(u) = p(u)g(u) = p([x])g(u) = 0g(u) = 0$