Notes

May 5, 2014

pde

$$u_{tt} = c^2\nabla^2 u \text{ on } 0 \leq x \leq, 0 < \theta < 2\pi, 0 < t < \infty$$

bc

$$u(1,\theta,t) = 0$$

ic

$$u(r,\theta,0) = f(r,\theta)$$
$$u_t(r,\theta,0) = g(r,\theta)$$

sep of var

$$u = T(t)U(t,\theta)$$

helmholtz

$\nabla^2 U + \lambda^2 U = 0$ has solutions

$$\nabla^2 U_{n,m} + {\lambda_{n,m}}^2 U_{n,m} = 0$$
where $\lambda_{n,m} = k_{n,m} = m^{\text{th}}$ positive zero of $J_n(x)$
$$U_{n,m}(t,\theta) = J_n(k_{nm}r)\cos(n\theta), J_n(k_{nm}r)\sin(n\theta$$

$$u_{tt} = c^2\nabla^2 u \text{ wave eqnfor 2d and 3d}$$

$$\begin{cases} 2d & \nabla^2 u = u_{xx} + u_{yy} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} \\ 3d & \nabla^2 u = u_{xx} + u_{yy} + u_{zz} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz} \end{cases}$$

bessel functions called zylinderfunktionen in german because of usefullness in cylinders (and he's german), measured distance to a star, funktionen introduced by dirichlet in 1829.

fact

for $(n_1,m_1) \neq (n_2,m_2)$, $k_{n_1,m_1} \neq k_{n_2,m_2}$ (a bit deep)

fact

for $(n_1,m_1) \neq (n_2,m_2)$, $\int\int_{x^2+y^2+z^2} U_{n_1,m_1}(r,\theta)U_{n_2,m_2}(r,\theta)r\,\mathrm{d}r\mathrm{d}\theta = 0$
eigenfunction for $k_{n_1,m_1} \neq k_{n_2,m_2}$are orthogonal
consequence of divergence theorem

general solution

$$u(r,\theta,t) = \sum_{n\geq 0, m\geq 1} J_n(k_{nm}r)\left[\cos(k_{nm}ct)(a_{nm}\cos(n\theta)+b_{nm}\sin(n\theta))+\sin(k_{nm}ct)(c_{nm}\cos(n\theta)+d_{nm}\sin(n\theta))\right]$$

satisfies pde and bc. how to satisfy ic?

at $t=0$ $u(t,\theta,0) = f(r,\theta) = \sum_{n\geq 0, m\geq 1} J_n(k_{nm}r)(a_{nm}\cos(n\theta)+b_{nm}\sin(n\theta))$ and $u_r(t,\theta,0) = g(r,\theta)$ is similar for d_{nm} and c_{nm}

to find a_{NM}

$$\begin{aligned}\int_{\theta=0}^{2\pi}\int_{r=0}^{1} f(r,\theta)\cdot J_N(k_{NM}r)\cos(N\theta)r\,\mathrm{d}r\mathrm{d}\theta &= a_{NM}\int_{\theta=0}^{2\pi}\int_{r=0}^{1} J_N(k_{NM})^2\cos(N\theta)^2 r\,\mathrm{d}r\mathrm{d}\theta \\ &= \int_0^{2\pi}\cos(N\theta)^2\,\mathrm{d}\theta\int_0^1 J_N(k_{NM}r)^2 r\,\mathrm{d}r\end{aligned}$$

to find b_{NM}

$$\begin{aligned}\int_{\theta=0}^{2\pi}\int_{r=0}^{1} f(r,\theta)\cdot J_N(k_{NM}r)\sin(N\theta)r\,\mathrm{d}r\mathrm{d}\theta &= a_{NM}\int_{\theta=0}^{2\pi}\int_{r=0}^{1} J_N(k_{NM})^2\sin(N\theta)^2 r\,\mathrm{d}r\mathrm{d}\theta \\ &= \int_0^{2\pi}\sin(N\theta)^2\,\mathrm{d}\theta\int_0^1 J_N(k_{NM}r)^2 r\,\mathrm{d}r\end{aligned}$$

note

$$\begin{aligned}\int_0^{2\pi}\cos(N\theta)^2\,\mathrm{d}\theta &= \int_0^{2\pi}\cos(N\theta)^2\,\mathrm{d}\theta = \pi \text{ where } n=1,2,3,\ldots \\ \int_0^{2\pi}\cos(N\theta)^2\,\mathrm{d}\theta &= \int_0^{2\pi}\cos(N\theta)^2\,\mathrm{d}\theta = 0 \text{ where } n=0\end{aligned}$$

page 239

$$\int_0^1 J_0(k_{0i}r)J_0(k_{0j}r)r\,\mathrm{d}r = \begin{cases} 0 & \text{for } i\neq j \\ \frac{1}{2}J_{1j}(k_{0i})^2 & \text{for } i=j\end{cases}$$

will obtain ageneralformula for

$$\int_0^1 J_N(k_{NM}r)^2 r\,\mathrm{d}r \text{ text gives } N=0$$

$J_v(z)$ satisfies $z^2\frac{\mathrm{d}^2w}{\mathrm{d}z^2}+z\frac{\mathrm{d}w}{\mathrm{d}z}+(z^2-v^2)w=0$. antiderivative for $\int J_v(ar)J_v(br)r\,\mathrm{d}r$ for $a\neq b$ $(a,b>0)$

write $y_a(r)=J_v(ar)$ then $y_z(r)$ satisfies $r^2\frac{\mathrm{d}^2y_a}{\mathrm{d}r^2}+r\frac{\mathrm{d}y_a}{\mathrm{d}r}+(a^2r^2-v^2)y_a=0$. note $z=ar$. want

$$\begin{aligned}I &= \int y_a(r)y_b(r)r\,\mathrm{d}r \\ b^2I &= \int y_a(r)(b^2ry_b(r))\,\mathrm{d}r \\ &= \int y_a(r)\left[\frac{v^2}{r}y_b(r)-(ry_b')'\right]i\,\mathrm{d}r\end{aligned}$$

note

$$
\begin{aligned}
\int y_a (r y_b')' \mathrm{d}r &= y_a r y_b' - \int (y_a' r) y_b' \mathrm{d}r \\
&= y_a r y_b' - y_a' r y_b + \int y_b (r y_a')' \mathrm{d}r \\
b^2 I &= \int \left[y_a \frac{v^2}{r} y_b - y_b (r y_a')' \right] \mathrm{d}r + r(y_a' y_b - y_a y_b') \\
y_b(\frac{v^2}{r} y_a - (r y_a')') &= a^2 r y_a
\end{aligned}
$$

note

$$
\begin{aligned}
r(r y_b')' - v^2 y_b + b^2 r^2 y_b &= 0 \\
b^2 \int y_a y_b r \, \mathrm{d}r &= a^2 \int y_a y_b r \, \mathrm{d}r + r(y_a' y_b - y_a y_b') \leftarrow \frac{\mathrm{d} y_a}{\mathrm{d}r}
\end{aligned}
$$