Homework

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October 22, 2014

Section 3.5: #5, 15, 11 Section 3.6: #7, 20.

3.5 5. Find the cyclic subgroup of $\mathbb{C}^\times$ generated by $(\sqrt{2}+\sqrt{2}i)/2$.

$$\frac{\sqrt{2}+\sqrt{2}i}{2} = \frac{\sqrt{2}}{2}(1+i)$$

$$\left(\frac{\sqrt{2}}{2}(1+i)\right)^2 = \frac{2}{4}2i = i \qquad \left(\frac{\sqrt{2}}{2}(1+i)\right)^3 = \frac{\sqrt{2}}{2}(1+i)\,i = \frac{\sqrt{2}}{2}(i-1)$$

$$\left(\frac{\sqrt{2}}{2}(1+i)\right)^4 = i^2 = -1 \qquad \left(\frac{\sqrt{2}}{2}(1+i)\right)^5 = -\frac{\sqrt{2}}{2}(1+i)$$

$$\left(\frac{\sqrt{2}}{2}(1+i)\right)^6 = i^3 = -i \qquad \left(\frac{\sqrt{2}}{2}(1+i)\right)^7 = -\frac{\sqrt{2}}{2}(i-1) = \frac{\sqrt{2}}{2}(1-i)$$

$$\left(\frac{\sqrt{2}}{2}(1+i)\right)^8 = (-1)^2 = 1 \qquad \left(\frac{\sqrt{2}}{2}(1+i)\right)^9 = \frac{\sqrt{2}}{2}(1+i)$$

And to double check

$$\left(\frac{\sqrt{2}}{2}(1+i)\right)^{-1} = \sqrt{2}\frac{1}{1+i} \qquad \sqrt{2}\frac{1}{1+i} = \sqrt{2}\frac{1-i}{(1+i)(1-i)} = \frac{\sqrt{2}}{2}(1-i)$$

$$\left(\frac{\sqrt{2}}{2}(1+i)\right)^8 = \left(\frac{\sqrt{2}}{2}(1+i)\right)^0 \qquad \left(\frac{\sqrt{2}}{2}(1+i)\right)^7 = \left(\frac{\sqrt{2}}{2}(1+i)\right)^{-1}$$

And so the generated group is:

$$\langle(\sqrt{2}+\sqrt{2}i)/2\rangle = \{1, i, -1, -i, \frac{\sqrt{2}}{2}(1+i), i\frac{\sqrt{2}}{2}(1+i), -\frac{\sqrt{2}}{2}(1+i), -i\frac{\sqrt{2}}{2}(1+i)\}$$

11. Which of the multiplicative groups $\mathbb{Z}_7^\times, \mathbb{Z}_{10}^\times, \mathbb{Z}_{12}^\times, \mathbb{Z}_{14}^\times$ are isomorphic?

$$\mathbb{Z}_7^\times = \{[2^{\alpha_1}3^{\alpha_2}5^{\alpha_3}]_7 : \alpha_1, \alpha_2, \alpha_3 \in \mathbb{Z}\}$$

$$5^6 = 25\cdot 5^4 = 4\cdot 5^4 = 20\cdot 5^3 = 6\cdot 5^3 = 30\cdot 5^2 = 2\cdot 5^2 = 10\cdot 5 = 3\cdot 5 = 15 = 1$$

$$\mathbb{Z}_7^\times = \{[\left(5^4\right)^{\alpha_1}\left(5^5\right)^{\alpha_2}5^{\alpha_3}]_7 : \alpha_1, \alpha_2, \alpha_3 \in \mathbb{Z}\} = \langle 5\rangle \cong \mathbb{Z}_6$$

$$\mathbb{Z}_{10}^\times = \{[3^{\alpha_1}7^{\alpha_2}9^{\alpha_3}]_{10} : \alpha_1, \alpha_2, \alpha_3 \in \mathbb{Z}\}$$

$$
\begin{aligned}
&3^4 = 9 \cdot 3^2 = 27 \cdot 3 = 7 \cdot 3 = 21 = 1 \\
&\mathbb{Z}_{10}^{\times} = \{[3^{\alpha_1} \left(3^3\right)^{\alpha_2} \left(3^2\right)^{\alpha_3}]_7 : \alpha_1, \alpha_2, \alpha_3 \in \mathbb{Z}\} = \langle 3 \rangle \cong \mathbb{Z}_4 \\
&\mathbb{Z}_{12}^{\times} = \{5^{\alpha_1} 7^{\alpha_2} 11^{\alpha_3} : \alpha_1, \alpha_2, \alpha_3 \in \mathbb{Z}\} \\
&5^2 = 25 = 1 \qquad 7^2 = 49 = 1 \qquad 11^2 = 121 = 1 \\
&5 \cdot 7 = 35 = 11 \qquad 7 \cdot 11 = 77 = 5 \qquad 5 \cdot 11 = 55 = 7 \\
&Z_{12}^{\times} = \{1, 5\} \times \{1, 7\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \\
&Z_{14}^{\times} = \{[3^{\alpha_1} 5^{\alpha_2} 9^{\alpha_3} 11^{\alpha_4} 13^{\alpha_5}\} \\
&3^6 = 9 \cdot 3^4 = 27 \cdot 3^3 = 13 \cdot 3^3 = 39 \cdot 3^2 = 11 \cdot 3^2 = 33 \cdot 3 = 5 \cdot 3 = 15 = 1 \\
&Z_{14}^{\times} = \{[3^{\alpha_1} \left(3^5\right)^{\alpha_2} \left(3^2\right)^{\alpha_3} \left(3^4\right)^{\alpha_4} \left(3^3\right)^{\alpha_5}] : \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \in \mathbb{Z}\} = \langle 3 \rangle \cong \mathbb{Z}_6
\end{aligned}
$$

So $\mathbb{Z}_7^{\times}$ and $\mathbb{Z}_{14}^{\times}$ are isomorphic.

15.

3.6 7.

20.