$$\frac{X''(x)}{X(x)} = \lambda \qquad \text{on} \qquad 0 < x < 1$$
$$X'(0) = 0$$
$$X'(1) - X(1) = 0$$

Show there are infinitely many distinct eigenvalues $\lambda = -{\mu_n}^2$ with corresponding eigenfunctions $X_n(x) = \cos(\mu_n x)$ for $n = 2, 3, 4, \ldots$. Find $\int_0^1 X_n(x)^2 \,\mathrm{d}x$ as an *algebraic* function of μ_n. Find μ_2, μ_3 numerically.

$$\begin{aligned}
X'' - \lambda X &= 0 \\
r^2 - \lambda &= 0 \\
r &= \frac{0 \pm \sqrt{0^2 - 4 \cdot 1 \cdot (-\lambda)}}{2} = \frac{\pm 2\sqrt{\lambda}}{2} \\
&= \pm\sqrt{\lambda} = \pm\sqrt{-{\mu_n}^2} = \pm \mu_n i \\
X_n(x) &= c_1 \cos(\mu_n x) + c_2 \sin(\mu_n x) \\
{X_n}'(x) &= -c_1 \mu_n \sin(\mu_n x) + c_2 \mu_n \cos(\mu_n x) \\
{X_n}'(0) = 0 &= -c_1 \mu_n \sin(\mu_n 0) + c_2 \mu_n \cos(\mu_n 0) \\
&= c_2 \mu_n \\
0 \neq \mu_n &\Rightarrow c_2 = 0 \\
X_n(x) &= c_1 \cos(\mu_n x)
\end{aligned}$$

Lets see if we can find a $\mu_n \neq \mu_m$

$$\begin{aligned}
X_n(x) &= \cos(\mu_n x) \\
k &\in \mathbb{Z} \\
\cos(\mu_n x) &= \cos(2\pi k + \mu_n x) \\
\mu_m &= 2\pi k + \mu_n \\
\cos(\mu_n x) &= \cos(\mu_m x) \\
\text{but } \mu_n &\neq \mu_m
\end{aligned}$$

Also note that $|\mathbb{Z}| = \infty$ so there are infinitely many possibilities for k and by extension μ_m.

Let's do the integral

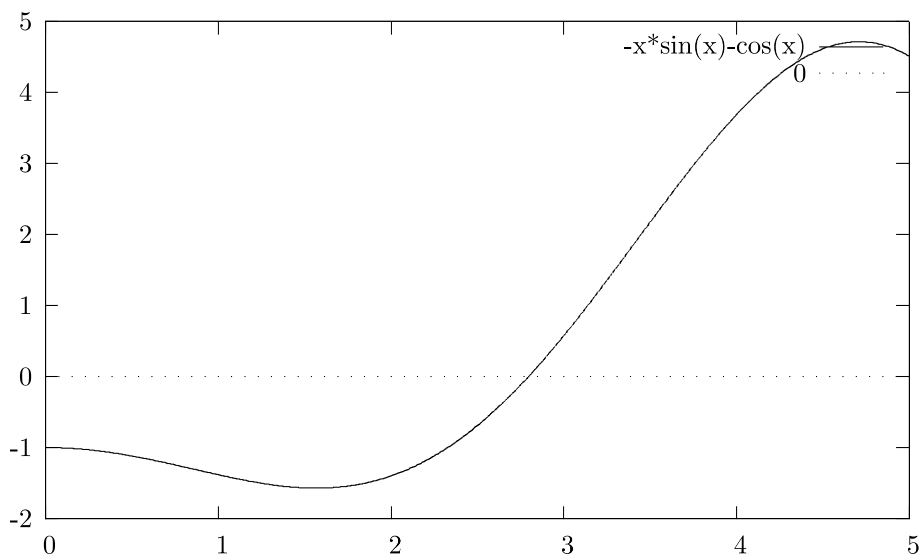
$$\begin{aligned}
\int_0^1 X_n(x)^2 \,\mathrm{d}x &= \int_0^1 \cos(\mu_n x)^2 \,\mathrm{d}x \\
u = \cos(\mu_n x) &\quad \mathrm{d}v = \cos(\mu_n x)\mathrm{d}x \\
\mathrm{d}u = -\mu_n \sin(\mu_n x) &\quad v = \frac{1}{\mu_n}\sin(\mu_n x) \\
\int \cos(\mu_n x)^2 \,\mathrm{d}x &= \frac{1}{\mu_n}\cos(\mu_n x)\sin(\mu_n x) + \int \sin(\mu_n x)^2 \,\mathrm{d}x \\
&= \frac{1}{\mu_n}\cos(\mu_n x)\sin(\mu_n x) + \int 1 - \cos(\mu_n x)^2 \,\mathrm{d}x \\
2\int \cos(\mu_n x)^2 \,\mathrm{d}x &= \frac{1}{\mu_n}\cos(\mu_n x)\sin(\mu_n x) + \int \mathrm{d}x \\
\int \cos(\mu_n x)^2 \,\mathrm{d}x &= \frac{1}{2\mu_n}\cos(\mu_n x)\sin(\mu_n x) + \frac{x}{2} \\
\int_0^1 \cos(\mu_n x)^2 \,\mathrm{d}x &= \left(\frac{1}{2\mu_n}\cos(\mu_n)\sin(\mu_n) + \frac{1}{2}\right) - \left(\frac{1}{2\mu_n}\cos(\mu_n 0)\sin(\mu_n 0) + \frac{0}{2}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\cos(\mu_n)\mu_n\sin(\mu_n)}{2{\mu_n}^2} + \frac{1}{2} \\
&= \frac{X_n(1)^2}{2{\mu_n}^2} + \frac{1}{2}
\end{aligned}
$$

And now we attempt to find μ_2, μ_3 numerically. First we have to figure out what μ_1 is. And setup Newton's method.

$$
\begin{aligned}
X'(1) - X(1) &= 0 \\
-\mu_1\sin(\mu_1) - \cos(\mu_1) &= 0
\end{aligned}
$$

$$
\begin{aligned}
x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\
x_{n+1} &= x_n - \frac{-x_n\sin(x_n) - \cos(x_n)}{-x_n\cos(x_n) - \sin(x_n) + \sin(x_n)} \\
x_{n+1} &= x_n - \frac{x_n\sin(x_n) + \cos(x_n)}{x_n\cos(x_n)} \\
x_{n+1} &= x_n - \tan(x_n) - \frac{1}{x_n}
\end{aligned}
$$



Three looks like a good place to start. $x_1 = 3$

n	$x_n - \tan(x_n) - \frac{1}{x_n}$
1	2.809
2	2.798427
3	2.798386
4	2.798386

$$
\begin{aligned}
\mu_1 &\approx 2.798386 \\
\mu_1 - 2\pi &\approx -3.484799 \\
\mu_n &\approx 2\pi n - 3.484799 \\
\mu_2 &\approx 9.081571 \\
\mu_3 &\approx 15.364757
\end{aligned}
$$