Notes

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7.2 generating functions

we have fibonacci #'s $0, 1, 1, 2, 3, 5, 8, \ldots$. We wish to construct an algebraic function which encodes this sequence (or any sequence of numbers)

Given a sequence $h_0, h_1, h_2, h_3, \ldots$ define it's generating function to be the infinite series $g(x) = h_0 + h_1x + h_2x^2 + h_3x^3 + \ldots$

example

Generating function for the fibanocci numbers is $f(x) = 0 + 1 \cdot x + 1 \cdot x^2 + 2 \cdot x^3 + 3 \cdot x^4 + \ldots$.

goal

express generating functions in closed form.

example

find the generating function for the sequence $1, 1, 1, \ldots$. Generating function is $f(x) = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$. this is the geometric series formula.

example

Find the generating function for the sequence $1, 1, 1, \ldots, 1, 0, 0, \ldots$. This is a finite series.

$$f(x) = 1 + x + x^2 + \cdots + x^{n-1} = \frac{1 - x^n}{1 - x}$$

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example

what is the generating function for the sequence for $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \binom{n}{3}, \ldots, \binom{n}{n}, 0, 0, \ldots$. binomial theorem

$$f(x) = (1 + x)^n$$

or if $\alpha \in \mathbb{R}$ the generating function for $\binom{\alpha}{0}, \binom{\alpha}{1}, \binom{\alpha}{2}, \binom{\alpha}{3}, \ldots, \binom{\alpha}{n}, 0, 0, \ldots$

13. (a) $\frac{1}{1-cx}$
    (b) $\frac{1}{1+x}$
    (d) e^x

example

the generating function for the fibonacci numbers is $f(x) = 0 + x + x^2 + 2x^3 + 3x^4 + \dots$

$$\begin{aligned}
\sum_{n=1}^{\infty} f_n x^n &= f_0 + f_1 x + \sum_{n=2}^{\infty} f_n x^n \\
&= 0 + x + \sum_{n=0}^{\infty} f_{n+2} x^{n+2} \\
&= x + x^2 \sum_{n=0}^{\infty} (f_{n+1} + f_n) x^n \\
&= x + x^2 \sum_{n=0}^{\infty} f_{n+1} x^n + x^2 \sum_{n=0}^{\infty} f_n x^n \\
&= x + x \sum_{n=0}^{\infty} f_n x^n + x^2 \sum_{n=0}^{\infty} f_n x^n \\
(1 - x - x^2) \sum_{n=0}^{\infty} f_n x^n &= x \\
f(x) &= \frac{x}{1 - x - x^2}
\end{aligned}$$

example

find the generating function for $h_0, h_1, h_2, h_3, \dots$ where $h_n = \#$ n-combinations fo $\{\infty \cdot e_1, \infty \cdot e_2, \infty \cdot e_3, \infty \cdot e_4\}$

find $g(x) = \sum\limits_{k=0}^{\infty} h_k x^k$ so h_n is the coefficient of x^n. $h_n = \#$ non-negative solutions: $e_1 + e_2 + e_3 + e_4 = n = \binom{n+4-1}{n}$ so $x^n = x^{e_1} x^{e_2} x^{e_3} x^{e_4}$ so consider the product $(1 + x + x^2 + \dots)(1 + x + x^2 + \dots)(1 + x + x^2 + \dots)(1 + x + x^2 + \dots)$ with each term coming from x^{e_n}. Generating function is $\frac{1}{(1-x)^4}$