Notes

April 25, 2014

riemanns method

pde $L[u] = u_{\xi\eta} + a(\xi,\eta)u_\xi + b(\xi,\eta)u_\eta + c(\xi,\eta)u = F(\xi,\eta)$. Boundary conditions $u(\xi, phi(\xi)) = f(\xi), u_\xi(\xi,\phi(\xi)) = g(\xi)$ for curve $C : \eta = \phi(\xi)$ with $\phi'(\xi) < 0$.

auxiliary problem, we obtained conditions defining $v(x,y)$. (ξ_0,η_0) lies above curve $C = C_1$, C_2 is vertical line from C_1 to (ξ_0,η_0) and C_3 is horizontal line.

pde $M[v] = 0$ in (at least in region $C_1C_2C_3$) ($M[v] = v_{xy} - (av)_x - (bv)_y + cv$. Boundary conditions $v_y - av = 0$ on C_2 and $v_x - bv = 0$ on C_3 and $v(\xi_0,\eta_0) = 1$.

last time $\int\int_{C_1C_2C_3} v(x,y)F(x,y)\,\mathrm{d}x\mathrm{d}y = \int_{C_1} [(v_x - bv)\,\mathrm{d}x - u(v_y - av)\,\mathrm{d}y] - \frac{1}{2}\left(u(Q)v(Q) + u(R)v(R)\right) + u(\xi_0,\eta_0$. Representation for $u(\xi_0,\eta_0)$ in terms of boundary data and $v(x,y)$

$v(x,y)$ from this problem is the riemann function $R(\xi_0,\eta_0;x,y)$.

assume a,b,c are constants, the riemann function for $L[u] = u_{\xi\eta} + au_\xi + bu_\eta + cu$ can be found explicitly. $M[v] = v_{xy} - av_x - bv_y + cv = 0$. $v = e^{bx+ay}\cdot w$, $v_x = (w_x + bw)e^{bx+ay}$, $v_y = (wy + aw)e^{bx+ay}$ and $v_{xy} = (w_{xy} + aw_x + bw_y + baw)e^{bx+ay}$.

$$
\begin{aligned}
(w_{xy} + aw_x + bw_y + abw) - a(w_x + bw) - b(w_y + aw) + cw = 0 \\
w_{xy} - abw + cw = 0 = w_{xy} + (c - ab)w \\
(w_x + bw) - bw = 0 \\
w_x(x,\eta_0) = 0 \text{ for } x \le \xi_0 \\
w(\xi_0,\eta_0) = e^{-(b\xi_0 + a\eta_0)} \\
w(x,\eta_0) = e^{-(b\xi_0 + a\eta_0)} \text{ for } x \le \xi_0 \\
v_y - av = 0 \text{ for } x = \xi_0 \\
(w_y + aw) - aw = 0 \text{ for } y \le \eta_0 \\
w(\xi_0,\eta_0) = e^{-(b\xi_0 + a\eta_0)} \\
w(\xi_0,y) = e^{-(b\xi_0 + a\eta_0)}
\end{aligned}
$$

w is a constant along C_2 and C_3 so divide off the constant to get $w = 1$. wait! $v = e^{bx+ay}\frac{w}{e^{b\xi_0 + a\eta_0}}$ so this change gives $w = 1$ on the boundary.

Idea: maybe there is a solution of one symmetric variable. $z = (\xi_0 - x)(\eta_0 - y) \ge 0$. Try $w = h(z)$

$$
\begin{aligned}
w_x &= h'(z)z_x & z_x &= -(\eta_0 - y) \\
w_y &= h'(z)z_y & z_y &= -(\xi_0 - x) \\
w_{xy} &= h'(z)z_{xy} + h''(x)z_xz_y & x_xz_y &= z, z_{xy=1}
\end{aligned}
$$