Jon Allen
HW 19

Let X and T be physical variables for distance and time. Consider the following general diffusion problem for $u(X,T)$:

$$\begin{array}{llll}
\text{PDE} & \dfrac{\partial u}{\partial t} = \alpha^2 \dfrac{\partial^2 u}{\alpha x^2} + F(X,T) & 0 < X < L, & 0 < T < +\infty \\
\text{BC} & G_1(T) = \alpha_1 L \dfrac{\partial u}{\partial X}(0,T) + \beta_1 u(0,T) & & 0 < T < +\infty \\
 & G_2(T) = \alpha_2 L \dfrac{\partial u}{\partial X}(L,T) + \beta_2 u(L,T) & & \\
\text{IC} & u(X,0) = \Phi(X) & 0 < X < L &
\end{array}$$

Note:

$$\alpha_1{}^2 + \beta_1{}^2 \neq 0 \qquad \alpha_2{}^2 + \beta_2{}^2 \neq 0$$

(a) If the units for X and T are [cm] and [sec] respectively (and u is taken as temerature with units [deg]), what are the units for L, α^2, F, ϕ, and for $\alpha_1, \beta_2, \alpha_2, \beta_2$?

$$\begin{aligned}
\frac{\text{deg}}{\text{sec}} &= \alpha^2 \frac{\text{deg}}{\text{cm}^2} + F \\
F &= \frac{\text{deg}}{\text{sec}} \\
\alpha^2 &= \frac{\text{cm}^2}{\text{sec}} \\
L &= \text{cm}
\end{aligned}
\qquad
\begin{aligned}
\alpha \cdot \text{cm} \frac{\text{deg}}{\text{cm}} + \beta \cdot \text{deg} &= \text{deg} \\
\alpha \cdot \text{deg} = \beta \cdot \text{deg} &= \text{deg} \\
\alpha_{1,2} = \beta_{1,2} = 1 &= \text{dimensionless} \\
\phi(X) &= \text{deg}
\end{aligned}$$

Define dimensionless variables x, t by $x = X/L$ and $t = \frac{\alpha^2}{L^2} T$. Define $w(x,t) = u(X,T)$

(b) Find $\frac{\partial u}{\partial T}, \frac{\partial u}{\partial X}, \frac{\partial^2 u}{\partial X^2}$ in terms of $\frac{\partial w}{\partial t}, \frac{\partial w}{\partial x}, \frac{\partial^2 w}{\partial x^2}$.

$$\begin{aligned}
T &= \frac{L^2}{\alpha^2} t \\
\frac{\partial u}{\partial T} &= \frac{\partial}{\partial T}(w(x,t)) \\
&= \frac{\partial w}{\partial \left(\frac{L^2}{\alpha^2} t\right)} \\
\frac{\partial u}{\partial T} &= \frac{\alpha^2}{L^2} \frac{\partial w}{\partial t} \\
\frac{\partial^2 u}{\partial X^2} &= \frac{\partial}{\partial X}\left(\frac{1}{L}\frac{\partial w}{\partial x}\right) \\
&= \frac{1}{L^2} \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right)
\end{aligned}
\qquad
\begin{aligned}
X &= xL \\
\frac{\partial u}{\partial X} &= \frac{\partial}{\partial X}(w(x,t)) \\
&= \frac{\partial}{\partial (xL)}(w(x,t)) \\
\frac{\partial u}{\partial X} &= \frac{1}{L}\frac{\partial w}{\partial x} \\
\frac{\partial^2 u}{\partial X^2} &= \frac{1}{L}\frac{\partial}{\partial (xL)}\left(\frac{\partial w}{\partial x}\right) \\
&= \frac{1}{L^2}\frac{\partial^2 w}{\partial x^2}
\end{aligned}$$

(c) Show that the PDE can be written as

$$\text{PDE} \qquad \frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + f(x,t) \qquad 0 < x < 1, \qquad 0 < t < +\infty$$

$$
\begin{array}{llll}
\text{PDE} & \dfrac{\partial u}{\partial t} = \alpha^2 \dfrac{\partial^2 u}{\alpha x^2} + F(X,T) & 0 < X < L, & 0 < T < +\infty \\
& \dfrac{\alpha^2}{L^2}\dfrac{\partial w}{\partial t} = \alpha^2 \dfrac{1}{L^2}\dfrac{\partial^2 w}{\partial x^2} + F(X,T) & 0 < xL < L, & 0 < \dfrac{L^2}{\alpha^2}t < +\infty \\
& \dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + \dfrac{L^2}{\alpha^2}F(X,T) & 0 < x < 1, & 0 < t < +\infty \\
& \dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + \dfrac{L^2}{\alpha^2}F\left(Lx, \dfrac{L^2}{\alpha^2}t\right) & 0 < x < 1, & 0 < t < +\infty \\
& \dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + f(x,t) & 0 < x < 1, & 0 < t < +\infty
\end{array}
$$

What is $f(x,t)$ in terms of $F(X,T)$?

$$f(x,t) = \frac{L^2}{\alpha^2}F\left(Lx, \frac{L^2}{\alpha^2}t\right)$$

(d) Show hat the BC can be written as

$$
\begin{array}{lll}
\text{BC} & \alpha_1 \dfrac{\partial w}{\partial x}(0,t) + \beta_1 w(0,t) = g_1(t) & 0 < t < +\infty \\
& \alpha_2 \dfrac{\partial w}{\partial x}(1,t) + \beta_2 w(1,t) = g_2(t) & 0 < t < +\infty \\
\text{BC} & \alpha_1 L \dfrac{\partial u}{\partial X}(0,T) + \beta_1 u(0,t) = G_1(T) & 0 < T < +\infty \\
& \alpha_1 L \dfrac{1}{L}\dfrac{\partial w}{\partial x}(0,t) + \beta_1 w(0,t) = G_1\left(\dfrac{L^2}{\alpha^2}t\right) & 0 < \dfrac{L^2}{\alpha^2}t < +\infty \\
& \alpha_1 \dfrac{\partial w}{\partial x}(0,t) + \beta_1 w(0,t) = g_1(t) & 0 < t < +\infty \\
& \alpha_2 L \dfrac{\partial u}{\partial X}(L,T) + \beta_2 u(L,t) = G_2(T) & \\
& \alpha_2 L \dfrac{1}{L}\dfrac{\partial w}{\partial x}\left(\dfrac{L}{L}, t\right) + \beta_2 w\left(\dfrac{L}{L}, t\right) = G_2\left(\dfrac{L^2}{\alpha^2}t\right) & 0 < \dfrac{L^2}{\alpha^2}t < +\infty \\
& \alpha_2 \dfrac{\partial w}{\partial x}\left(1,t\right) + \beta_2 w\left(1,t\right) = g_2\left(t\right) & 0 < t < +\infty
\end{array}
$$

What are $g_1(t)$ and $g_2(t)$ in terms of $G_1(T)$ and $G_2(T)$?

$$g_1(t) = G_1\left(\frac{L^2}{\alpha^2}t\right) \qquad\qquad g_2(t) = G_2\left(\frac{L^2}{\alpha^2}t\right)$$

(e) Show that the IC can be written as

$$
\begin{array}{lll}
\text{IC} & w(x,0) = \phi(x) & 0 < x < 1 \\
\text{IC} & u(X,0) = \Phi(X) & 0 < X < L \\
& u(X,0) = w(x,0) = \Phi(Lx) & 0 < Lx < L
\end{array}
$$

$$w(x,0) = \phi(x) \qquad\qquad 0 < x < 1$$

What is $\phi(x)$ in terms of $\Phi(X)$?

$$\phi(x) = \Phi(Lx) = \Phi(X)$$

Jon Allen
HW 20
CASE 1. $\alpha_2\beta_1 - \alpha_1\beta_2 + \beta_1\beta_2 \neq 0$
Given the problem:

$$
\begin{array}{llll}
\text{PDE} & \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + f(x,t) & 0 < x < 1, & 0 < t < \infty \\
\text{BC} & g_1(t) = \alpha_1 \dfrac{\partial u}{\partial x}(0,t) + \beta_1 u(0,t) & & 0 < t < \infty \\
 & g_2(t) = \alpha_2 \dfrac{\partial u}{\partial x}(1,t) + \beta_2 u(1,t) & {\alpha_1}^2 + {\beta_1}^2 \neq 0 & {\alpha_2}^2 + {\beta_2}^2 \neq 0 \\
\text{IC} & u(x,0) = \phi(x) & 0 < x < 1 &
\end{array}
$$

Introduce the change of variables

- $u(x,t) = w(x,t) + a(t)x + b(t)(1-x)$

where $a(t), b(t)$ are to be determined so that $w(x,t)$ satisfies the homogeneous BC:

$$
\begin{array}{lll}
\text{BC} & \alpha_1 \dfrac{\partial w}{\partial x}(0,t) + \beta_1 w(0,t) = 0 & 0 < t < \infty \\
 & \alpha_2 \dfrac{\partial w}{\partial x}(1,t) + \beta_2 w(1,t) = 0 &
\end{array}
$$

(a) Assuming $a(t), b(t)$ can be found so that $w(x,t)$ satisfies homogeneous BC, give the resulting PDE and IC for $w(x,t)$. (State it in terms of $a(t), b(t)$ - solving for them is done next.)

$$
\begin{aligned}
\text{PDE} \qquad \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + f(x,t) \qquad 0 < x < 1, \quad 0 < t < \infty \\
\frac{\partial}{\partial t}\left(w(x,t) + a(t)x + b(t)(1-x)\right) &= \frac{\partial^2}{\partial x^2}\left(w(x,t) + a(t)x + b(t)(1-x)\right) + f(x,t) \\
\frac{\partial w}{\partial t} + x\frac{\mathrm{d}a}{\mathrm{d}t} + (1-x)\frac{\mathrm{d}b}{\mathrm{d}t} &= \frac{\partial^2 w}{\partial x^2} + \underbrace{\frac{\partial^2}{\partial x^2}(a(t)x)}_{\to 0} + \underbrace{\frac{\partial^2}{\partial x^2}(b(t)(1-x))}_{\to 0} + f(x,t) \\
\frac{\partial w}{\partial t} + x\frac{\mathrm{d}a}{\mathrm{d}t} + (1-x)\frac{\mathrm{d}b}{\mathrm{d}t} &= \frac{\partial^2 w}{\partial x^2} + f(x,t) \\
\text{IC} \qquad u(x,0) &= \phi(x) \qquad 0 < x < 1 \\
\phi(x) &= w(x,0) + a(0)x + b(0)(1-x)
\end{aligned}
$$

(b) Show that homogeneous BC for $w(x,t)$ can be achieved (that is, a solution for $a(t), b(t)$ can be found) for *arbitrary* functions $g_1(t), g_2(t)$ in the original problem if and only if $\alpha_2\beta_1 - \alpha_1\beta_2 + \beta_1\beta_2 \neq 0$

$$
\begin{aligned}
g_1(t) &= \alpha_1 \frac{\partial u}{\partial x}(0,t) + \beta_1 u(0,t) \\
g_2(t) &= \alpha_2 \frac{\partial u}{\partial x}(1,t) + \beta_2 u(1,t) \\
g_1(t) &= \alpha_1 \frac{\partial}{\partial x}\left(w(0,t) + a(t)\cdot 0 + b(t)(1-0)\right) + \beta_1\left(w(0,t) + a(t)\cdot 0 + b(t)(1-0)\right) \\
&= \alpha_1 \frac{\partial w}{\partial x}(0,t) + \beta_1 w(0,t) + \beta_1 b(t) \\
g_1(t) - \beta_1 b(t) &= \alpha_1 \frac{\partial w}{\partial x}(0,t) + \beta_1 w(0,t) = 0
\end{aligned}
$$

$$\begin{aligned}
g_2(t) &= \alpha_2 \frac{\partial}{\partial x}\left(w(1,t) + a(t)\cdot 1 + b(t)(1-1)\right) + \beta_2\left(w(1,t) + a(t)\cdot 1 + b(t)(1-1)\right) \\
&= \alpha_2 \frac{\partial w}{\partial x}(1,t) + \beta_2 w(1,t) + \beta_2 a(t) \\
g_2(t) - \beta_2 a(t) &= \alpha_2 \frac{\partial w}{\partial x}(1,t) + \beta_2 w(1,t) = 0
\end{aligned}$$

(c) Assuming $\alpha_2\beta_1 - \alpha_1\beta_2 + \beta_1\beta_2 \neq 0$, give the solution for $a(t), b(t)$ in terms of $g_1(t), g_2(t)$.

$$\begin{aligned}
g_1(t) &= \beta_1 b(t) \\
\frac{1}{\beta_1} g_1(t) &= b(t) \\
g_2(t) &= \beta_2 a(t) \\
\frac{1}{\beta_2} g_2(t) &= a(t)
\end{aligned}$$

Jon Allen
HW 21
CASE 2: $\alpha_2\beta_1 - \alpha_1\beta_2 + \beta_1\beta_2 = 0$
Given the problem:

PDE	$\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + f(x,t)$	$0 < x < 1,$	$0 < t < \infty$
BC	$g_1(t) = \alpha_1 \dfrac{\partial u}{\partial x}(0,t) + \beta_1 u(0,t)$		$0 < t < \infty$
	$g_2(t) = \alpha_2 \dfrac{\partial u}{\partial x}(1,t) + \beta_2 u(1,t)$	${\alpha_1}^2 + {\beta_1}^2 \neq 0$	${\alpha_2}^2 + {\beta_2}^2 \neq 0$
IC	$u(x,0) = \phi(x)$	$0 < x < 1$	

Assume $\alpha_2\beta_1 - \alpha_1\beta_2 + \beta_1\beta_2 = 0$. A change of vriables of the form $u = w + a(t)x + b(t)(1-x)$ cannot convert the problem to homogeneous BC for w for arbitrary $g_1(t), g_2(t)$. Consider the change of variables

- $u(x,t) = w(x,t) + a(t)x^p + b(t)(1-x)^p$ with $p > 1$

Here $a(t), b(t)$ and p are to be determined so that $w(x,t)$ satisfies the homogeneous BC:

BC	$\alpha_1 \dfrac{\partial w}{\partial x}(0,t) + \beta_1 w(0,t) = 0$	$0 < t < \infty$
	$\alpha_2 \dfrac{\partial w}{\partial x}(1,t) + \beta_2 w(1,t) = 0$	

(a) Show that there always exist values $p > 1$ such that the homogeneous BC for $w(x,t)$ can be achieved (that is, a solution for $a(t), b(t)$ can be found) for *arbitrary* functions $g_1(t), g_2(t)$ in the original problem. This will involve conditions on p in terms of $\alpha_1, \alpha_2, \beta_1, \beta_2$.

(b) :

Jon Allen
HW 22
Given $F(s) = \frac{1}{1+\sqrt{s}}$

(a) Find $f(t)$ by expanding $F(s)$ in reciprocal powers and inverting termwise.

$$\begin{aligned}
\mathcal{L}^{-}1\{F(s)\} &= \mathcal{L}^{-}1\left\{\frac{1}{1+\sqrt{s}}\right\} \\
&= \mathcal{L}^{-}1\left\{\sum_{n=0}^{\infty}(-1)^n s^{n/2}\right\} \\
&= \sum_{n=0}^{\infty}(-1)^n \mathcal{L}^{-}1\left\{s^{n/2}\right\} \\
\mathcal{L}^{-1}\{s^n\} &= \frac{t^{-n-1}}{\Gamma(-n)} \\
&= \sum_{n=0}^{\infty}(-1)^n \frac{t^{-n/2-1}}{\Gamma(-n/2)}
\end{aligned}$$

(b) Reduce all occurrences of the gamma function to ordinary factorials

$$= \sum_{n=0}^{\infty}(-1)^n \frac{t^{-n/2-1}}{(-1-n/2)!} \quad \text{used computer to get this}$$

This result isn't really sane, but it seems to be closest to what you are looking for. I also have this, but it's not right either I think.

$$\begin{aligned}
&= \mathcal{L}^{-}1\left\{\frac{1-\sqrt{s}}{1-s}\right\} \\
&= e^t - \mathcal{L}^{-}1\left\{\frac{\sqrt{s}}{1-s}\right\} \\
&= e^t + \frac{1}{2}\mathcal{L}^{-}1\left\{\frac{1}{\sqrt{s}+1}\right\} + \frac{1}{2}\mathcal{L}^{-}1\left\{\frac{1}{\sqrt{s}-1}\right\} \\
\frac{1}{2}\mathcal{L}^{-}1\left\{\frac{1}{1+\sqrt{s}}\right\} &= e^t + \frac{1}{2}\mathcal{L}^{-}1\left\{\frac{1}{\sqrt{s}-1}\right\} \\
\mathcal{L}^{-}1\left\{\frac{1}{1+\sqrt{s}}\right\} &= 2e^t + \mathcal{L}^{-}1\left\{\frac{\sqrt{s}+1}{s-1}\right\} \\
&= 2e^t + e^t + \mathcal{L}^{-}1\left\{\frac{\sqrt{s}}{s-1}\right\} \\
&= 2e^t + e^t + \frac{1}{2}\mathcal{L}^{-}1\left\{\frac{1}{\sqrt{s}+1}\right\} + \frac{1}{2}\mathcal{L}^{-}1\left\{\frac{1}{\sqrt{s}-1}\right\} \\
&= 4e^t + 2e^t + \mathcal{L}^{-}1\left\{\frac{\sqrt{s}+1}{s-1}\right\} \\
f(t) &= \sum_{n=1}^{\infty} 2^n e^t
\end{aligned}$$

Jon Allen
HW 23
Given $F(s) = \frac{1}{1+\sqrt{s}}$

(a) Introduce $G(s) = \frac{1}{s}F(s)$ and show that $sG(s) - G(s) = \frac{1}{\sqrt{s}} - \frac{1}{s}$

$$\begin{aligned}
sG(s) - G(s) &= G(s)(s-1) \\
&= (s-1)\frac{1}{s}\frac{1}{1+\sqrt{s}} \\
&= \frac{1}{s}\frac{(1+\sqrt{s})(1-\sqrt{s})}{1+\sqrt{s}} \\
&= \frac{1}{s} - \frac{\sqrt{s}}{s} \\
&= \frac{1}{s} - \frac{1}{\sqrt{s}}
\end{aligned}$$

(b) Now obtain a first order DE for $g(t)$. You may assume $g(0) = 0$, but show where this assumption is used.

$$\begin{aligned}
\mathcal{L}^{-1}\{sG(s) - G(s)\} &= \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{\sqrt{s}}\right\} \\
\mathcal{L}^{-1}\{sG(s) - 0\} - g(t) &= 1 - \mathcal{L}^{-1}\left\{\frac{\sqrt{\pi}}{\sqrt{\pi}}\frac{1}{\sqrt{s}}\right\} \\
\mathcal{L}^{-1}\{sG(s) - g(0)\} - g(t) &= 1 - \frac{1}{\sqrt{\pi}}\mathcal{L}^{-1}\left\{\frac{\Gamma(1/2)}{s^{1/2}}\right\} \\
g'(t) - g(t) &= 1 - \frac{1}{\sqrt{\pi}}t^{-1/2} = 1 - \frac{1}{\sqrt{t\pi}}
\end{aligned}$$

(c) Solve for $g(t)$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\int -1\,\mathrm{d}t}g(t)\right) &= e^{-\int \mathrm{d}t}\left(1 - \frac{1}{\sqrt{t\pi}}\right) \\
e^{-t}g(t) &= \int e^{-t} - \frac{e^{-}t}{\sqrt{t\pi}}\,\mathrm{d}t \\
e^{-t}g(t) &= -e^{-t} - \operatorname{erfc}(\sqrt{t}) + c_1 \quad \text{used maxima here} \\
g(t) &= -1 - e^t\operatorname{erfc}(\sqrt{t}) + e^t c_1 \\
g(t) &= e^t c_1 - e^t\operatorname{erfc}(\sqrt{t}) - 1
\end{aligned}$$

(d) The relation $G(s) = \frac{1}{s}F(s)$ implies a relation between $g(t)$ and $f(t)$. What is the relation?

$$\begin{aligned}
G(s) &= \frac{1}{s}F(s) \\
\mathcal{L}^{-1}\{G(s)\} &= \mathcal{L}^{-1}\{\frac{1}{s}F(s)\} \\
g(t) &= \int_0^t f(u)\,\mathrm{d}u
\end{aligned}$$

Use it to find $f(t)$