9.1

B. Show that every subset of a discrete metric space is both open and closed.

e have a discrete metric d on a set X. Now we take $U \subset X$. For any $x \in U$ we have $B(x, r) \subset U$ if $r \leq 1$ because the ball will contain only the point x. Note that this is trivially true even if $U = \emptyset$ because there is no $x \in U$ that does not have a ball around it. Now because our choice of U was arbitrary we know that all subsets of X are open. And the complements of any subsets of X are themselves subsets of X, and so they are open. But they are the complement of an open set, and so they must be closed. Thus every subset of a discrete metric space is both open and closed.

D. Prove Theorem 9.1.7

Let f map a metric space (X, ρ) into a metric space (Y, ρ). The following are equivalent:

(1) f is continuous on X;

(2) for every sequence (x_n) with $\lim_{n\to\infty} x_n = a \in X$, we have $\lim_{n\to\infty} f(x_n) = f(x)$; and

(3) $f^{-1}(U) = \{x \in X : f(x) \in U\}$ is open in X for every open set U in Y.

We start by assuming that f is continuous on X. Now we know that for every $a \in X$ and $\varepsilon > 0$ there exists a $\delta > 0$ such that $\sigma(f(x), f(a)) < \varepsilon$ whenever $\rho(x, a) <$
$delta$. We also know that the limit of $\rho(x, a)$ is zero as $x \to a$. Now from the definition of limit, we know that we can find some $\delta > 0$ such that $\rho(x, a) < \delta$. But then from the definition of continuity we can have $\sigma(f(x), f(a)) < \varepsilon$ which means that the limit of $\sigma(f(x), f(a))$ is zero as $f(x) \to f(a)$ as required.

Now if we assume 1 is false, and f is discontinuous at some point a then we can find some number $\varepsilon > 0$

H. Two metrics ρ and σ on a set X are **equivalent** if there are constants $0 < c < C$ such that $c\rho(x, y) \leq \sigma(x, y) \leq C\rho(x, y)$ for all $x, y \in X$

(a) Prove that equivalent metrics are topologically equivalent

If we say $\sigma(x, y) = r$ and let $s = r/c$ then we have $B^\rho_s(x) \subset B^\sigma_r(x)$, straight from the inequality in the definition of equivalence. Now if $\rho(x, y) = r$ then $\sigma(x, y)/C \leq r/c = s$ because $C > c$ and so we see that $B^\sigma_s(x) \subset B^\rho_r$.

(b) Prove that equivalent metrics have the saame Cauchy sequences

We begin with some Cauchy sequence $(x_n) \in \rho$. Then for every $\varepsilon/C > 0$ there exists some N such that $\rho(x_i, x_j) < \varepsilon/C$. But $\sigma(x_i, x_j) \leq C\rho(x_i, x_j) < \varepsilon$ and so the sequence is Cauchy in σ. Now let us assume that our sequence is Cauchy in σ. Then for every $c\varepsilon > 0$ there exists some N such that $c\rho(x_i, x_j) \leq \sigma(x_i, x_j) < c\varepsilon$ and so certainly $\rho(x_i, x_j) < \varepsilon$.

(c) Give examples of topologically equivalent metrics that are not equivalent

If we let $\sigma(x, y) = \min\{1, \rho(x, y)\}$ and $\rho(x, y) = |x - y|$ then, no matter how small we make c, we can make $y = x + 1/c + 1$ and then no matter our choice of c we can make $y = x + 1/c + 1$ and $\sigma(x, x+1/c+1) = 1$ but $c\rho(x, x+1/c+1) = c+1 > \sigma(x, x+1/c+1)$ so they are not equivalent. But if we choose any r for $B_r^\sigma(x)$ we will have either all real numbers or all real numbers in $[-r, r]$. Either way, we can certainly can say that $s = \min(1/2, r/2)$ and then $B_s^\rho(x) \subset B_r^\sigma(x)$ and $B_s^\sigma(x) \subset B_r\rho(x)$

K. Recall the 2-adic metric of examples 9.1.2 (4) and 9.1.5 (4). Extend it to $\mathbb{Q}$ by setting $\rho_2(a/b, a/b) = 0$ and, if $a/b \neq c/d$, then $\rho_2(a/b, c/d) = 2^{-e}$, where e is the unique integer such that $a/b - c/d = 2^e(f/g)$ and both f and g are odd integers

(a) Prove that ρ_2 is a metric on $\mathbb{Q}$

if $a/b \neq c/d$ then $a/b - c/d = \frac{ad-cb}{db}$. Now $ad - cb = 2^i f$ for some odd f and $db = 2^j g$ for some odd g. Then $a/b - c/d = 2^{i-j}(f/g)$. Of course 2^{i-j} is non-zero and so $\rho_2(a/b, c/d) \neq 0$.

Now we assume that $a/b - c/b = 2^e \frac{f}{g}$. Then $c/d - a/b = 2^e(-f/g)$ and so $\rho_2(x, y) = \rho)2(y, x)$.

And finally, if $\rho_2(a/b, c/d) = 2^{-i+l}$, $\rho_2(a/b, e/f) = 2^{-k+l}$ and $\rho_2(c/d, e/f) = 2^{-j+l}$ then $a/b - c/d = (adf - bcf)/bdf$ and $c/d - e/f = (bcf - bde)/bdf$ while $a/b - e/f = (adf - bde)/bdf = (adf - bcf)/bdf + (bcf - bde)/bdf$. Now we see that $\rho_2(a/b, e/f) = 2^{-i-j+l} \leq 2^{-i-j+2l} = 2^{-i+l} + 2^{-j+l}$

(b) Show that the sequence of integers $a_n = (1 - (-2)^n)/3$ converges in $(\mathbb{Q}, \rho_2)$

$(1 - (-2)^n)/3 - 1/3 = -(-1)^n 2^n/3$ so $\rho_2((1 - (-2)^n)/3, 1/3) = 2^{-n}$ which converges to zero, so (a_n) converges to $\frac{1}{3}$

(c) Find the limit of $\dfrac{n!}{n! + 1}$ in this metric.

We know that $n!$ is even for $n \geq 2$, so $n! + 1$ is odd for $n \geq 2$. We also know that every other term of $n!$ adds at least one factor of 2 to $n!$. Thus $\rho_2(n!/(n! + 1), 0) \leq 2^{-n/2}$. And so we know that if we choose N large enough that $0 < 2^{N/2} \leq \varepsilon$ for any $\varepsilon > 0$ then $\rho_2(n!/(n! + 1), 0) \leq 2^{-n/2} \leq 2^{-N/2}$ for all $n > N$. We see that the limit must be 0.