$$\frac{X''(x)}{X(x)} = \lambda \qquad \text{on} \qquad 0 < x < 1$$
$$X'(0) = 0$$
$$X'(1) - X(1) = 0$$

Show there is exactly one positive eigenvalue $\lambda = {\mu_1}^2$ with corresponding eigenfunction $X_1(x) = \cosh(\mu_1 x)$. Find $\int_0^1 X_1(x)^2\,dx$ as an *algebraic* function of μ_1 (eliminate hyperbolic functions by use of the eigenvalue equation). Find μ_1 numerically.

$$\begin{aligned}
X'' - \lambda X &= 0\\
r^2 - \lambda &= 0\\
r &= \frac{0 \pm \sqrt{0^2 - 4\cdot 1\cdot(-\lambda)}}{2} = \frac{\pm 2\sqrt{\lambda}}{2}\\
&= \pm\sqrt{\lambda} = \pm\sqrt{{\mu_1}^2} = \pm\mu_1\\
X(x) &= c_1 e^{\mu_1 x} + c_2 e^{-\mu_1 x}\\
&= \frac{c_1 e^{\mu_1 x} + c_1 e^{\mu_1 x} + c_2 e^{-\mu_1 x} + c_2 e^{-\mu_1 x}}{2} + \frac{c_1 e^{-\mu_1 x} - c_1 e^{-\mu_1 x} + c_2 e^{\mu_1 x} - c_2 e^{\mu_1 x}}{2}\\
&= \frac{c_1 e^{\mu_1 x} + c_1 e^{-\mu_1 x} + c_2 e^{\mu_1 x} + c_2 e^{-\mu_1 x}}{2} + \frac{c_1 e^{\mu_1 x} - c_1 e^{-\mu_1 x} - c_2 e^{\mu_1 x} + c_2 e^{-\mu_1 x}}{2}\\
&= (c_1 + c_2)\frac{e^{\mu_1 x} + e^{-\mu_1 x}}{2} + (c_1 - c_2)\frac{e^{\mu_1 x} - e^{-\mu_1 x}}{2}\\
&\Rightarrow c_1 \cosh(\mu_1 x) + c_2 \sinh(\mu_1 x)\\
X'(x) &= c_1\mu_1 \sinh(\mu_1 x) + c_2\mu_1 \cosh(\mu_1 x)\\
X'(0) = 0 &= c_1\mu_1 \sinh(0) + c_2\mu_1\cosh(0)\\
&= c_2\mu_1\\
\mu_1 \neq 0 \Rightarrow 0 &= c_2\\
X(x) &= c_1 \cosh(\mu_1 x)
\end{aligned}$$

Let's assume that μ_1 is not unique and see what happens.

$$\begin{aligned}
\cosh(\mu_1 x) &= \cosh(\mu_2 x)\\
\frac{e^{\mu_1 x} + e^{-\mu_1 x}}{2} &= \frac{e^{\mu_2 x} + e^{-\mu_2 x}}{2}\\
e^{\mu_1 x} = a, &\quad e^{\mu_2 x} = b\\
a + \frac{1}{a} &= b + \frac{1}{b}\\
a^2 + 1 &= a(b + \frac{1}{b})\\
a^2 - a(b + \frac{1}{b}) + 1 &= 0\\
a &= \frac{(b + \frac{1}{b}) \pm \sqrt{(b + \frac{1}{b})^2 - 4}}{2}\\
&= \frac{(b + \frac{1}{b}) \pm \sqrt{b^2 + 2 + \frac{1}{b^2} - 4}}{2}\\
&= \frac{b + \frac{1}{b} \pm \sqrt{b^2 - 2 + \frac{1}{b^2}}}{2}
\end{aligned}$$

$$
\begin{aligned}
a &= \frac{b+\frac{1}{b} \pm \sqrt{(b-\frac{1}{b})^2}}{2} = \frac{b+\frac{1}{b} \pm (b-\frac{1}{b})}{2} \\
&= \frac{1}{2}(2b) \text{ or } \frac{1}{2}\left(\frac{2}{b}\right) \\
e^{\mu_1 x} &= e^{\mu_2 x} \text{ or } \frac{1}{e^{\mu_2 x}} \\
\ln(e^{\mu_1 x}) &= \ln(e^{\mu_2 x}) \text{ or } \ln(e^{-\mu_2 x}) \\
\mu_1 &= \pm\mu_2 \Rightarrow (-\mu_1)^2 = (\mu_1)^2 = \lambda
\end{aligned}
$$

So we see λ is unique if it is positive. Now lets do our integral.

$$
\begin{aligned}
\int_0^1 X_1(x)^2 \,\mathrm{d}x &= \int_0^1 \cosh(\mu_1 x)^2 \,\mathrm{d}x \\
&= \int_0^1 \frac{(e^{\mu_1 x} + e^{-\mu_1 x})^2}{4} \,\mathrm{d}x \\
&= \frac{1}{4}\int_0^1 (e^{2\mu_1 x} + 2 + e^{-2\mu_1 x}) \,\mathrm{d}x \\
&= \frac{1}{4}\left[\frac{e^{2\mu_1 x}}{2\mu_1} + 2x + \frac{e^{-2\mu_1 x}}{-2\mu_1}\right]_0^1 \\
&= \frac{1}{4}\left[\frac{1}{\mu_1}\frac{e^{2\mu_1 x} - e^{-2\mu_1 x}}{2} + 2x\right]_0^1 \\
&= \frac{1}{2\mu_1}\left[\frac{(e^{\mu_1 x} + e^{-\mu_1 x})(e^{\mu_1 x} - e^{-\mu_1 x})}{4} + \mu_1 x\right]_0^1 \\
&= \frac{1}{2\mu_1}\left[\cosh(\mu_1 x)\sinh(\mu_1 x) + \mu_1 x\right]_0^1 \\
&= \frac{1}{2\mu_1}\left[\cosh(\mu_1)\sinh(\mu_1) + \mu_1 - \cosh(0)\sinh(0)\right] \\
&= \frac{1}{2\mu_1}\left[\cosh(\mu_1)\sinh(\mu_1) + \mu_1\right] \\
&= \frac{1}{2\mu_1}\cosh(\mu_1)\sinh(\mu_1) + \frac{1}{2} \\
&= \frac{1}{2{\mu_1}^2}\cosh(\mu_1)\mu_1\sinh(\mu_1) + \frac{1}{2} \\
&= \frac{1}{2{\mu_1}^2}X_1(1){X_1}'(1) + \frac{1}{2}
\end{aligned}
$$

And to find μ_1

$$
\begin{aligned}
X'(1) - X(1) &= 0 \\
\mu_1\sinh(\mu_1) - \cosh(\mu_1) &= 0 \\
\mu_1\frac{e^{\mu_1} - e^{-\mu_1}}{2} - \frac{e^{\mu_1} + e^{-\mu_1}}{2} &= 0 \\
\mu_1(e^{2\mu_1} - 1) - (e^{2\mu_1} + 1) &= 0 \\
e^{2\mu_1}(\mu_1 - 1) - \mu_1 - 1 = 0 &
\end{aligned}
$$

μ_1	$e^{2\mu_1}(\mu_1-1)-\mu_1-1$
0	-2
1	-2
2	$e^4-3\approx 51.5$
$\frac{3}{2}$	$\frac{1}{2}\cdot e^3-\frac{5}{2}\approx 7.5$
$\frac{5}{4}$	$\frac{1}{4}e^{5/2}-\frac{9}{4}\approx .8$
$\frac{9}{8}$	$\frac{1}{8}e^{9/4}-\frac{17}{8}\approx -.9$
$\frac{19}{16}$	$\frac{3}{16}e^{19/8}-\frac{35}{16}\approx -.17$
$\frac{39}{32}$	$\frac{7}{32}e^{39/16}-\frac{71}{32}\approx .28$
$\frac{77}{64}$	$\frac{13}{64}e^{77/32}-\frac{141}{64}\approx .05$
$\frac{153}{128}$	$\frac{25}{128}e^{153/64}-\frac{281}{128}\approx -.06$
$\frac{307}{256}$	$\frac{51}{256}e^{307/128}-\frac{563}{256}\approx -1.5$
$\frac{615}{512}$	$\frac{103}{512}e^{615/256}-\frac{1127}{512}\approx .02$

$$\mu_1\approx\frac{615}{512}\approx 1.2$$