1.4

3. For each of the following matrices A, determine its reduced echelon form and give the general solution of $A\mathbf{x} = \mathbf{0}$ in standard form.

(c)

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 3 & 1 \\ 2 & 4 & 3 \\ -1 & 1 & 6 \end{bmatrix} \to \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 3 & 5 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 3 & 5 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(d)

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 \\ 1 & 3 & 2 & 4 \\ 1 & 2 & 2 & 3 \end{bmatrix} \to \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 3 & 0 \\ 0 & 1 & 1 & 2 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\mathbf{x} = x_4 \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

(f)

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 & -1 \\ -1 & -3 & 1 & 2 & 3 \\ 1 & -1 & 3 & 1 & 1 \\ 2 & -3 & 7 & 3 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & -1 & -1 \\ 0 & -1 & 1 & 1 & 2 \\ 0 & -3 & 3 & 2 & 2 \\ 0 & -7 & 7 & 5 & 6 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 2 & 0 & -1 & -1 \\ 0 & 1 & -1 & -1 & -2 \\ 0 & 0 & 0 & -1 & -4 \\ 0 & 0 & 0 & -2 & -8 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & -1 & -1 \\ 0 & 1 & -1 & -1 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & -1 & -1 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} -2x_3 & +x_5 \\ x_3 & -2x_5 \\ x_3 & \\ & -4x_5 \\ & x_5 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ -2 \\ 0 \\ -4 \\ 1 \end{bmatrix}$$

(h)

$$A = \begin{bmatrix} 1 & 1 & 0 & 5 & 0 & -1 \\ 0 & 1 & 1 & 3 & -2 & 0 \\ -1 & 2 & 3 & 4 & 1 & -6 \\ 0 & 4 & 4 & 12 & -1 & -7 \end{bmatrix} \to \left[\begin{array}{cccccc|c} 1 & 1 & 0 & 5 & 0 & -1 & 0 \\ 0 & 1 & 1 & 3 & -2 & 0 & 0 \\ 0 & 3 & 3 & 9 & 1 & -7 & 0 \\ 0 & 0 & 0 & 0 & 7 & -7 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cccccc|c} 1 & 1 & 0 & 5 & 0 & -1 & 0 \\ 0 & 1 & 1 & 3 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 7 & -7 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccccc|c} 1 & 0 & -1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 3 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\mathbf{x} = \begin{bmatrix} x_3 & -2x_4 & -x_6 \\ -x_3 & -3x_4 & +2x_6 \\ x_3 & & \\ & x_4 & \\ & & x_6 \\ & & x_6 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 2 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

4. (b)

$$\left[\begin{array}{cc|c} 2 & -1 & -4 \\ 2 & 1 & 0 \\ -1 & 1 & 3 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 2 & -1 & -4 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 2 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

$$\mathbf{x} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

(d)

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 1 \\ 2 & 1 & 3 & -1 \\ 1 & 1 & 2 & -1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & -3 & -3 & 3 \\ 0 & -1 & -1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\mathbf{x} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

(e)

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 6 \\ 3 & 3 & 2 & 0 & 17 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 6 \\ 0 & 0 & -1 & -3 & -1 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & -2 & 5 \\ 0 & 0 & 1 & 3 & 1 \end{array}\right]$$

$$\mathbf{x} = \begin{bmatrix} 5 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 0 \\ -3 \\ 1 \end{bmatrix}$$

14. Let A be an $m \times n$ matrix and let $b \in \mathbb{R}^m$.

(a) Show that if $\mathbf{u}$ and $\mathbf{v} \in \mathbb{R}^n$ are both solutions of $A\mathbf{x} = \mathbf{b}$ then $\mathbf{u} - \mathbf{v}$ is a solution of $A\mathbf{x} = \mathbf{0}$.
If $A\mathbf{u} = \mathbf{b} = A\mathbf{v}$ then $A\mathbf{u} - A\mathbf{v} = \mathbf{0}$ and by the distributive property $A(\mathbf{u} - \mathbf{v}) = \mathbf{0}$ as required.

(b) Suppose $\mathbf{u}$ is a solution of $A\mathbf{x} = \mathbf{0}$ and $\mathbf{p}$ is a solution of $A\mathbf{x} = \mathbf{b}$. Show that $\mathbf{u} + \mathbf{p}$ is a solutioon of $A\mathbf{x} = \mathbf{b}$.
We know that $A\mathbf{u} = \mathbf{0}$ and $A\mathbf{p} = \mathbf{b}$ and so we know that $A(\mathbf{u} + \mathbf{p}) = A\mathbf{u} + A\mathbf{p} = \mathbf{0} + \mathbf{b} = \mathbf{b}$

15. Prove or give a counterexample:

(a) If A is an $m \times n$ matrix and $\mathbf{x} \in \mathbb{R}^n$ satisfies $A\mathbf{x} = \mathbf{0}$, then either every entry of A is 0 or $\mathbf{x} = \mathbf{0}$

(b) If A is an $m \times n$ matrix and $A\mathbf{x} = \mathbf{0}$ for every vector $\mathbf{x} \in \mathbb{R}^n$, then every entry of A is 0.
Let $A = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}$ and let $\mathbf{x} = (1, 1, 10^{100})$. Then $A\mathbf{x} = 0$
Assume $A \neq O$. Then $\exists \mathbf{r}_i(A)$ such that $\mathbf{r}_i(A) \neq \mathbf{0}$. Choose $\mathbf{x} = \mathbf{r}_i(A)$. Now $\text{ent}_i(A\mathbf{x}) = \sum_{k=1}^{n} \text{ent}_k(\mathbf{r}_i(A))^2$. Note that $\text{ent}_k(\mathbf{r}_i(A))^2 \geq 0$. And because $\mathbf{r}_i(A) \neq 0$ then there is at least one non-zero element in that row and so $\text{ent}_i(A\mathbf{x}) > 0$. Thus we have a contradiction, and so every entry of A is 0.

1.5

13. Suppose A is an $m \times n$ matrix with rank m and $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^n$ are vectors with $\text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_k) = \mathbb{R}^n$. Prove that $\text{Span}(A\mathbf{v}_1, \ldots, A\mathbf{v}_k) = \mathbb{R}^m$

    First we note that $m \leq n$ or else A could not have rank m. We choose an arbitrary $\mathbf{b} \in \mathbb{R}^m$. Because the rank of A is no larger than n we know that we can find some $x \in \mathbb{R}^n$ such that $A\mathbf{x} = \mathbf{b}$. Because $x \in \text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$ then $A\mathbf{x} = A(c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k) = c_1A\mathbf{v}_1 + \cdots + c_kA\mathbf{v}_k \in \text{Span}(A\mathbf{v}_1, \ldots, A\mathbf{v}_k)$.

    Thus all elements of $\mathbb{R}^m$ are contained in the Span. Usually to show equality one needs to show the reverse. But I claim that it is obvious that any element of the Span must be in $\mathbb{R}^m$.

14. Let A be an $m \times n$ matrix with row vectors $\mathbf{A}_1, \ldots, \mathbf{A}_m \in \mathbb{R}^n$.

    (a) Suppose $\mathbf{A}_1 + \cdots + \mathbf{A}_m = \mathbf{0}$. Deduce that $\text{rank}(A) < m$.

    If $\text{rank}A = m$ then $\text{rref}(A) = \begin{bmatrix} 1 & \ldots & 0 & \ldots \\ \vdots & \ddots & \vdots & \ldots \\ 0 & \ldots & 1 & \ldots \end{bmatrix}$. Now it is clear that adding all the rows of this matrix will give us $(1_1, \ldots, 1_m, \ldots) \neq 0$. But the sum of all the rows in A is in fact $\mathbf{0}$ and so the rank of A is less than m.

    (b) More generally, suppose there is some linear combination $c_1\mathbf{A}_1 + \cdots + c_m\mathbf{A}_m = \mathbf{0}$ where some $c_i \neq 0$. Show that $\text{rank}(A) < m$.

    If $\text{rank}A = m$ then $\text{rref}(A) = \begin{bmatrix} 1 & \ldots & 0 & \ldots \\ \vdots & \ddots & \vdots & \ldots \\ 0 & \ldots & 1 & \ldots \end{bmatrix}$. Now it is clear that adding some combination of the rows of this matrix will give us $(c_1, \ldots, c_m, \ldots) \neq 0$. Now if there exists some $c_i \neq 0$ then we have a non-zero vector. But the combination of the rows in A is in fact $\mathbf{0}$ and so the rank of A is less than m.

15. Let A be an $m \times n$ matrix with column vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathbb{R}^m$.

    (a) Suppose $\mathbf{a}_1 + \cdots + \mathbf{a}_n = \mathbf{0}$. Prove that $\text{rank}(A) < n$.

    Recall that $A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n$. Since $\mathbf{a}_1 + \cdots + \mathbf{a}_n = \mathbf{0}$ we have $A\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \mathbf{0}$ and so we have more than one solution for $A\mathbf{x} = \mathbf{0}$ and so $\text{rank}(A) < n$.

    (b) More generally, suppose there is some linear combination $c_1\mathbf{a}_1 + \cdots + c_n\mathbf{a}_n = \mathbf{0}$ where some $c_i \neq 0$. Prove that $\text{rank}(A) < n$.

    Recall that $A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n$. Since $c_1\mathbf{a}_1 + \cdots + c_n\mathbf{a}_n = \mathbf{0}$ we have $A\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \mathbf{0}$.

    Because $\exists c_i \neq 0$ then $\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \neq \mathbf{0}$ and so we have more than one solution for $A\mathbf{x} = \mathbf{0}$ and so $\text{rank}(A) < n$.