Notes

20 avril, 2015

quiz

1. $D : \pi \to \pi$ with $D(e^{2\pi it}) = e^{2\pi i(2t)}$
2. $T : [0,1] \to [0,1]$ $T(x) = \begin{cases} 2x & 0 \le x \le \frac{1}{2} \\ 2-2x & \frac{1}{2} \le x \le 1 \end{cases}$

   dense

last time

$T_\alpha : [0,1] \to [0,1]$
  $T_\alpha(x) = x \oplus \alpha$
  α irrational
  $x, y \in [0,1]$ and $\epsilon > 0$ find M such that $|T^m(x) - y| < \epsilon$

1. given any $\epsilon > 0$ there is N such that $|T^n(0)| < \epsilon$

   proof

   choose N' such that $\frac{1}{N'} < \epsilon$
   now $[0,1] = [0, \frac{1}{N'}] \cup [\frac{1}{N'}, \frac{2}{N'}] \cup \cdots \cup [\frac{N'-1}{N'}, 1]$
   $T_\alpha^0(0), \ldots, T_\alpha^{N'}(0)$ find j, k such that $|T_\alpha^j(0) - T_\alpha^k(0)| < \epsilon$
   if $m = k - j$ $|T_\alpha^m(0)| = |T^{k-j} - T^0| = |T^j(T^{k-j}) - T^j(T^0)| = |T^k - T^j|$
2. given any point in $[0,1]$ there is M' such that $T_0^{M'}(0)$ is arbitrarily close to the point we care about

$$\begin{aligned} 0 &< |T_\alpha^m(0)| < \epsilon \\ \epsilon &< |T_0^{2m}| = |T^m(0)| + |T^m(0)| < 2\epsilon \\ 2\epsilon &< |T^{3m}| < 3\epsilon \\ &\vdots \\ 1 - \epsilon &< |T^{nm}(0)| < 1 \end{aligned}$$

   given $x, y \in [0,1]$ consider $x \ominus y$ then there is k such that $|(x \ominus y) - T^{km}(0)| < \epsilon$

note

because alpha is irrational you will always get an irrational back out, not 0

3. finish let $\epsilon > 0$ and $x, y \in [0,1]$

$$\begin{aligned} |y - T_\alpha^{km}(x)| &= |y \ominus x \oplus km\alpha| \\ &= |y \ominus x \ominus 0 \oplus km\alpha| \\ &= |y \ominus x - T_\alpha^{km}(0)| < \epsilon \end{aligned}$$

lemma

$T : [a,b] \to [a,b]$ is continuous then T has a fixed point

proof

let $f(x) = Tx - x$ is continuous then $f(a)$
$epsilon 0$ and $f(b) \leq 0$. there is $c \in [a,b]$ such that $f(c) = 0$ and $0 = T(c) - c \Rightarrow T(c) = c$

lemma

$T : [a,b] \to \mathbb{R}$ is continuous such that $T([a,b]) \supseteq [a,b]$then T has a fixed point

proof

$$\begin{aligned} &f(x) = T(x) - x \\ &c, d \in [a,b] \text{ where } T(c) = a, T(d) = b \\ &f(c) = T(c) - c = a - c \leq 0 \\ &f(d) = T(d) - d = b - d \geq 0 \end{aligned}$$

there is e such that $f(e) = 0 \Rightarrow T(e) = e$

fact

if $T : [a,b] \to \mathbb{R}$ and is continuous then image is compact and connected so $T([a,b]) = [x,y]$ for some x, y.

lemma

if $[c,d] \subseteq [x,y]$ there $a', b' \in [a,b]$ such that $T([a',b'] = [c,d]$ and $T(a',b') = c, d$
note that we don't know whether $a' \to c$ or $a' \to d$ and same for b'

proof

1. notice $T^{-1}(c)$ is closed and $T^{-1}(\{d\})$ is closed and both are nonempty.

   let $a_0 \in T^{-1}(c)$ and $b_0 \in T^{-1}(d)$. Say $T(a_0) = c$ and $T(b_0) = d$

   recall that $[x,y]$ is range. now for any closed interval in the range we want to find a region in our domain that hits closed interval but never leaves it

   choose $a_0 < b_0$ for first case

   $a' = \sup\{x \in [a, b_0] : Tx = c\}$ which is nonempty and $a_0 \leq a'$

   let $b' = \inf\{x \in [a', b_0] : T(x) = d\}$ nonempty and notice $a' < b'$