Week 1: Vectors

1. Vectors in $\mathbb{R}^2$

Definition 1.1. A *vector in* $\mathbb{R}^2$ is an ordered pair $\mathbf{v} = (x, y)$ where $x, y \in \mathbb{R}$. The real numbers x, y are called the *coordinates* of $\mathbf{v}$. Let $\mathbf{v} = (x, y)$ and $\mathbf{w} = (r, s)$ be two vectors in $\mathbb{R}^2$. We say write $\mathbf{v} = \mathbf{w}$ if $x = r$ and $y = s$. The zero vector is $\mathbf{0} = (0, 0)$.

Example 1.2.
(1) $\mathbf{v} = (\pi^2, e)$ is a vector in $\mathbb{R}^2$. In short math-speak, $(\pi^2, e) \in \mathbb{R}^2$. The coordinates of $\mathbf{v}$ are π^2 and e.
(2) $\mathbf{w} = (1 + i, 3)$ is not a vector in $\mathbb{R}^2$ since $1 + i \notin \mathbb{R}$ (here $i = \sqrt{-1}$).
(3) If $\mathbf{x} = (2, 5)$ and $\mathbf{y} = (3, 5)$, then $\mathbf{x} \neq \mathbf{y}$ since it is false that both coordinates are equal. That is, $2 \neq 3$.

Definition 1.3. Let $\mathbf{v} = (x, y)$ be any vector in $\mathbb{R}^2$.
(1) The *magnitude* of $\mathbf{v}$ is defined to be the non-negative real number $\|\mathbf{v}\| = \sqrt{x^2 + y^2}$.
(2) The vector $\mathbf{v}$ is called a *unit vector* if $\|\mathbf{v}\| = 1$.

Example 1.4.
(1) The vector $\mathbf{v} = (3, 4)$ is not a unit vector since $\|\mathbf{v}\| = \sqrt{3^2 + 4^2} = 5 \neq 1$.
(2) The vector $\mathbf{w} = (\frac{3}{5}, \frac{4}{5})$ is a unit vector since $\|\mathbf{w}\| = \sqrt{(\frac{3}{5})^2 + (\frac{4}{5})^2} = 1$.

Definition 1.5. Let $\mathbf{v} = (x, y)$ and $\mathbf{w} = (r, s)$ be two vectors in $\mathbb{R}^2$ and let $\alpha \in \mathbb{R}$.
(1) We define *vector addition* $\boxplus$ on $\mathbb{R}^2$ by the rule $\mathbf{v} \boxplus \mathbf{w} = (x + r, y + s)$ where $+$ is the usual addition in $\mathbb{R}$. Well-definedness of $\boxplus$ allows the implication

$$\mathbf{v}_1 = \mathbf{v}_2 \Longrightarrow \mathbf{v}_1 \boxplus \mathbf{w} = \mathbf{v}_2 \boxplus \mathbf{w}$$

(2) We define *scalar multiplication* $\mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ by the rule $\alpha\mathbf{v} = (\alpha x, \alpha y)$. Well-definedness of scalar multiplication allows the implication

$$\mathbf{v}_1 = \mathbf{v}_2 \Longrightarrow \alpha\mathbf{v}_1 = \alpha\mathbf{v}_2$$

Example 1.6. If $\mathbf{v} = (3, 5)$ and $\mathbf{w} = (\frac{1}{3}, 2)$, then $\mathbf{v} \boxplus \mathbf{w} = (3 + \frac{1}{3}, 5 + 2) = (\frac{10}{3}, 7)$. If $\alpha = \frac{3}{4}$,then $\alpha\mathbf{v} = ((\frac{3}{4})(3), (\frac{3}{4})(5)) = (\frac{9}{4}, \frac{15}{4})$.

Definition 1.7. Let $\mathbf{v}, \mathbf{w}$ be two nonzero vectors in $\mathbb{R}^2$.
(1) We say that $\mathbf{v}$ is *parallel* to $\mathbf{w}$ if there exists a nonzero real number $\alpha \in \mathbb{R}$ such that $\mathbf{v} = \alpha\mathbf{w}$.

(2) We say that $\mathbf{v}$ and $\mathbf{w}$ have the *same direction* if there exists a positive real number $\alpha \in \mathbb{R}$ such that $\mathbf{v} = \alpha\mathbf{w}$.

Theorem 1.8.
(1) $\|\alpha\mathbf{v}\| = |\alpha|\,\|\mathbf{v}\|$ for all $\mathbf{v} \in \mathbb{R}^2$ and all $\alpha \in \mathbb{R}$.
(2) For every nonzero vector $\mathbf{v} \in \mathbb{R}^2$, there exists a unit vector $\mathbf{u} \in \mathbb{R}^2$ in the same direction as $\mathbf{v}$.

Proof.
(1) Choose any arbitrary $\mathbf{v} \in \mathbb{R}^2$ and any arbitrary real number (scalar) $\alpha \in \mathbb{R}$. By Definition 1.1, $\mathbf{v} = (x, y)$ for some $x, y \in \mathbb{R}$. By Definition 1.5, we have $\alpha\mathbf{v} = \alpha(x, y) = (\alpha x, \alpha y)$. It follows that

$$\begin{aligned}
& \|\alpha\mathbf{v}\| \\
= \;& \|(\alpha x, \alpha y)\| \\
= \;& \sqrt{(\alpha x)^2 + (\alpha y)^2} \quad \text{(Definition 3(1))} \\
= \;& \sqrt{\alpha^2 x^2 + \alpha^2 y^2} \quad \text{(Property of } \mathbb{R}) \\
= \;& \sqrt{\alpha^2 (x^2 + y^2)} \quad \text{(Property of } \mathbb{R}) \\
= \;& |\alpha| \sqrt{x^2 + y^2} \quad \text{(Property of } \mathbb{R}) \\
= \;& |\alpha|\,\|\mathbf{v}\| \quad \text{(Definition 3(1))}
\end{aligned}$$

(2) Choose any vector $\mathbf{v} \in \mathbb{R}^2$. By Definition 1, we can write $\mathbf{v} = (x, y)$ for some $x, y \in \mathbb{R}$. Since $\mathbf{v}$ is a nonzero vector, it must be the case that $\|\mathbf{v}\|$ is a nonzero (and hence positive) real number. We put $\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v}$ and verify that $\mathbf{u}$ is a unit vector and that $\mathbf{u}$ is in the same direction as $\mathbf{v}$. By (1) above, we have the equality $\|\mathbf{u}\| = \left\|\frac{1}{\|\mathbf{v}\|}\mathbf{v}\right\| = \left|\frac{1}{\|\mathbf{v}\|}\right|\,\|\mathbf{v}\| =^{\text{why?}} \frac{1}{\|\mathbf{v}\|}\,\|\mathbf{v}\| = 1$. This proves that $\mathbf{u}$ is a unit vector. Since $\frac{1}{\|\mathbf{v}\|} > 0$ and $\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v}$, it follows from Definition 1.7(2) that $\mathbf{u}$ is in the same direction as $\mathbf{v}$.

Theorem 1.9. The following properties hold in the Euclidean plane $\mathbb{R}^2$.
(A1) $\mathbf{v} \boxplus (\mathbf{w} \boxplus \mathbf{z}) = (\mathbf{v} \boxplus \mathbf{w}) \boxplus \mathbf{z}$ for all $\mathbf{v}, \mathbf{w}, \mathbf{z} \in \mathbb{R}^2$ (Associative Law of Addition)
(A2) $\mathbf{v} \boxplus \mathbf{w} = \mathbf{w} \boxplus \mathbf{v}$ for all $\mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ (Commutative Law of Addition)
(A3) The vector $\mathbf{0}$ is the unique vector in $\mathbb{R}^2$ such that $\mathbf{v} \boxplus \mathbf{0} = \mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^2$ (Additive Identity)
(A4) For each $\mathbf{v} \in \mathbb{R}^2$, there exists a unique $\mathbf{w} \in \mathbb{R}^2$ such that $\mathbf{v} \boxplus \mathbf{w} = \mathbf{0}$ (Additive inverse)
(S1) $1\mathbf{v} = \mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^2$
(S2) $\alpha(\mathbf{v} \boxplus \mathbf{w}) = \alpha\mathbf{v} \boxplus \alpha\mathbf{w}$ for all $\alpha \in \mathbb{R}$ and for all $\mathbf{v}, \mathbf{w} \in \mathbb{R}^2$
(S3) $(\alpha + \beta)\mathbf{v} = \alpha\mathbf{v} \boxplus \beta\mathbf{v}$ for all $\alpha, \beta \in \mathbb{R}$ and for all $\mathbf{v} \in \mathbb{R}^2$
(S4) $\alpha(\beta\mathbf{v}) = (\alpha\beta)\mathbf{v}$ for all $\alpha, \beta \in \mathbb{R}$ and for all $\mathbf{v} \in \mathbb{R}^2$

Proof. We prove (A3) and (S2). The rest are proved similarly.

(A3) We need to prove that the vector $\mathbf{0}$ has the property that $\mathbf{v} \boxplus \mathbf{0} = \mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^2$ and that $\mathbf{0}$ is the only vector with said property. Choose any arbitrary vector $\mathbf{v} \in \mathbb{R}^2$. By Definition (1), we can write $\mathbf{v} = (x, y)$ for some $x, y \in \mathbb{R}$ and $\mathbf{0} = (0, 0)$. It follows that $\mathbf{v} \boxplus \mathbf{0} = (x, y) \boxplus (0, 0) = (x + 0, y + 0) = (x, y) = \mathbf{v}$ (notice that the second equality follows from Definition 1.5(1)). Now suppose that there exists another vector $\mathbf{z} \in \mathbb{R}^2$ such that $\mathbf{v} \boxplus \mathbf{z} = \mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^2$. Then it is certainly true that $\mathbf{0} \boxplus \mathbf{z} = \mathbf{0}$ (just set $\mathbf{v} = \mathbf{0}$). Since $\mathbf{v} \boxplus \mathbf{0} = \mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^2$,itis certainly true that $\mathbf{z} \boxplus \mathbf{0} = \mathbf{z}$. It nowfollows fromthe Commutative Law that $\mathbf{0} = \mathbf{0} \boxplus \mathbf{z} = \mathbf{z} \boxplus \mathbf{0} = \mathbf{z}$.
(S2) Choose any $\mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ and any $\alpha \in \mathbb{R}$. By Definition 1.1, we can write $\mathbf{v} = (x_1, y_1)$ and $\mathbf{w} = (x_2, y_2)$. Now,

$$
\begin{aligned}
& \alpha(\mathbf{v} \boxplus \mathbf{w}) \\
= \;& \alpha((x_1, y_1) \boxplus (x_2, y_2)) \\
= \;& \alpha(x_1 + x_2, y_1 + y_2) \quad \text{(Definition 1.5(1))} \\
= \;& (\alpha(x_1 + x_2), \alpha(y_1 + y_2)) \quad \text{(Definition 1.5(2))} \\
= \;& (\alpha x_1 + \alpha x_2, \alpha y_1 + \alpha y_2) \quad \text{(Distributive Property in } \mathbb{R}) \\
= \;& (\alpha x_1, \alpha y_1) \boxplus (\alpha x_2, \alpha y_2) \quad \text{(Definition 1.5(1))} \\
= \;& \alpha(x_1, y_1) \boxplus \alpha(x_2, y_2) \quad \text{(Definition 1.5(2))} \\
= \;& \alpha\mathbf{v} \boxplus \alpha\mathbf{w}
\end{aligned}
$$

Remark 10.
(1) From now on, we will simply write $\mathbf{v} + \mathbf{w}$ instead of $\mathbf{v} \boxplus \mathbf{w}$ for vector addition in $\mathbb{R}^2$.
(2) We will denote the unique additive inverse of a vector $\mathbf{v} \in \mathbb{R}^2$ by $-\mathbf{v}$. We write $\mathbf{v} - \mathbf{w}$ instead of $\mathbf{v} + (-\mathbf{w})$.

Theorem 11. The following properties hold in the Euclidean plane $\mathbb{R}^2$.
(1) $\alpha\mathbf{0} = \mathbf{0}$ for all $\alpha \in \mathbb{R}$.
(2) $0\mathbf{v} = \mathbf{0}$ for all $\mathbf{v} \in \mathbb{R}^2$.
(3) $(-\alpha)\mathbf{v} = \alpha(-\mathbf{v}) = -\alpha\mathbf{v}$ for all $\alpha \in \mathbb{R}$ and for all $\mathbf{v} \in \mathbb{R}^2$.
(4) $\alpha\mathbf{v} = \mathbf{0}$ implies $\alpha = 0$ or $\mathbf{v} = \mathbf{0}$.

Proof. We prove (1), (3), and (4)
(1) We have $\alpha\mathbf{0} = \alpha(\mathbf{0} + \mathbf{0}) = \alpha\mathbf{0} + \alpha\mathbf{0}$. Indeed, the first equality is (A3) and the second is (S2). Now

$$
\begin{aligned}
\alpha\mathbf{0} \;=\;& \alpha\mathbf{0} + \alpha\mathbf{0} \\
\Rightarrow\;& -\alpha\mathbf{0} + \alpha\mathbf{0} = -\alpha\mathbf{0} + (\alpha\mathbf{0} + \alpha\mathbf{0}) \quad \text{(Definition 5(1))} \\
\Rightarrow\;& -\alpha\mathbf{0} + \alpha\mathbf{0} = (-\alpha\mathbf{0} + \alpha\mathbf{0}) + \alpha\mathbf{0} \quad \text{(Property A1)} \\
\Rightarrow\;& \mathbf{0} = \mathbf{0} + \alpha\mathbf{0} \quad \text{(Property A4)} \\
\Rightarrow\;& \mathbf{0} = \alpha\mathbf{0} \quad \text{(Property A1)}
\end{aligned}
$$

(3) We have $\alpha(-\mathbf{v})+\alpha\mathbf{v} = \alpha((-\mathbf{v})+\mathbf{v}) = \alpha\mathbf{0} = \mathbf{0}$. Indeed, the first equality is (S2), the second is (A4), and the last is (1) above. The equation $\alpha(-\mathbf{v})+\alpha\mathbf{v} = \mathbf{0}$ says that the additive inverse of $\alpha\mathbf{v}$ is $\alpha(-\mathbf{v})$. But the *unique* additive inverse of $\alpha\mathbf{v}$ is $-\alpha\mathbf{v}$. Therefore, $\alpha(-\mathbf{v}) = -\alpha\mathbf{v}$ as needed. The equality $(-\alpha)\mathbf{v} = -\alpha\mathbf{v}$ is proved similarly.
(4) The statement to be proved is of the form "If P, then Q or R is true." The standard way to prove this is to assume that P is true, Q is not true and then to show that R must be true. So suppose that $\alpha\mathbf{v} = \mathbf{0}$ and that $\alpha \neq 0$. Then $\alpha^{-1} \in \mathbb{R}$ and we have

$$\begin{array}{rcc} & \alpha\mathbf{v} = \mathbf{0} & \\ \Rightarrow & \alpha^{-1}(\alpha\mathbf{v}) = \alpha^{-1}\mathbf{0} & \text{(Definition 5(2))} \\ \Rightarrow & (\alpha^{-1}\alpha)\mathbf{v} = \alpha^{-1}\mathbf{0} & \text{(S4 on the left)} \\ \Rightarrow & 1\mathbf{v} = \alpha^{-1}\mathbf{0} & \text{(Property in } \mathbb{R}\text{)} \\ \Rightarrow & \mathbf{v} = \alpha^{-1}\mathbf{0} & \text{(S1 on the left)} \\ \Rightarrow & \mathbf{v} = \mathbf{0} & \text{(1 above)} \end{array}$$

2. Lines in $\mathbb{R}^n$

Definition 2.1. Let $n \geq 1$ be any positive integer. A vector $\mathbf{v} \in \mathbb{R}^n$ is and n-tuple of the form $\mathbf{v} = (x_1, x_2, ..., x_n)$. The zero vector in $\mathbb{R}^n$ is the n-tuple $\mathbf{0} = (0, 0, ..., 0)$. All of the previous definitions and theorems can be generalized in natural way from $\mathbb{R}^2$ to $\mathbb{R}^n$.

Definition 2.2. Let $\mathbf{x}_0, \mathbf{v} \in \mathbb{R}^n$ with $\mathbf{v} \neq \mathbf{0}$. A *line* in $\mathbb{R}^n$ through the point $\mathbf{x}_0$ in the direction of $\mathbf{v}$ is the set

$$L = L(\mathbf{x}_0, \mathbf{v}) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \mathbf{x}_0 + t\mathbf{v} \text{ for some } t \in \mathbb{R}\}.$$

Remark 2.3. Although there exists a unique line L through $\mathbf{x}_0$ in the direction of $\mathbf{v}$, it is not true that L is uniquely represented by the parametric equation $\mathbf{x} = \mathbf{x}_0 + t\mathbf{v}$.

Example 2.4. Here is an example in $\mathbb{R}^2$. Let

$$L = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{v} = (2, 1) + t(-4, 2) \text{ for some } t \in \mathbb{R}\}$$

and let

$$N = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{v} = (-2, 3) + t(2, -1) \text{ for some } t \in \mathbb{R}\}.$$

We show that L and N represent the same line. In other words, we prove the equality of sets $L = N$. The standard way to do this is to prove that $L \subseteq N$ (every vector of L must belong to the set N) and that $N \subseteq L$ (every vector of N must belong to the set L).

($\subseteq$) Choose any vector $\mathbf{x} \in L$. What does $\mathbf{v}$ look like? Well, according to the way we define L, it must be the case that $\mathbf{x} = (2,1) + t(-4,2)$ for some $t \in \mathbb{R}$. Using our vector algebra, we have

$$\begin{aligned} \mathbf{x} &= (2,1) + t(-4,2) \\ &= (2,1) + (-4,2) - (-4,2) + t(-4,2) \\ &= (-2,3) - (-4,2) + t(-4,2) \\ &= (-2,3) + (t-1)(-4,2) \\ &= (-2,3) + 2(1-t)(2,-1). \end{aligned}$$

Since $2(1-t)$ is a real number, it follows that $\mathbf{x} \in N$.

($\subseteq$) Choose any vector $\mathbf{x} \in N$. It must be the case that $\mathbf{x} = (-2,3)+t(2,-1)$ for some $t \in \mathbb{R}$. Using our vector algebra, we have

$$\begin{aligned} \mathbf{x} &= (-2,3) + t(2,-1) \\ &= (-2,3) + (4,-2) - (4,-2) + t(2,-1) \\ &= (2,1) - (4,-2) + t(2,-1) \\ &= (2,1) - (4,-2) + \frac{t}{2}2(2,-1) \\ &= (2,1) - (4,-2) + \frac{t}{2}(4,-2) \\ &= (2,1) + (\frac{t}{2} - 1)(4,-2) \end{aligned}$$

Since $\frac{t}{2} - 1$ is a real number, it follows that $\mathbf{x} \in L$.

3. Spanning Sets and Planes in $\mathbb{R}^n$

Definition 3.1. Let $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m \in \mathbb{R}^n$. We define the *span* of the m vectors to be the set

$$\text{Span}(\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_n\mathbf{v}_m\}.$$

An element of $\text{Span}(\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m)$ is called a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m$.

Example 3.2. Consider the vectors $\mathbf{u} = (1,1)$ and $\mathbf{v} = (-2,1)$. Then $3\mathbf{u} + \frac{1}{2}\mathbf{v} = 3(1,1) + \frac{1}{2}(-2,1) = (2, \frac{7}{2})$ belongs to $\text{Span}(\mathbf{u}, \mathbf{v})$. So does $25\mathbf{u} + \pi\mathbf{v} = (25,25) + \pi(-2,1) = (25 - 2\pi, 25 + \pi)$. It is reasonable to ask which vectors in $\mathbb{R}^2$ do or don't belong to $\text{Span}(\mathbf{u}, \mathbf{v})$. As it turns out, every vector in $\mathbb{R}^2$ belongs to $\text{Span}(\mathbf{u}, \mathbf{v})$. That is, $\text{Span}(\mathbf{u}, \mathbf{v}) = \mathbb{R}^2$ as we will prove now.

($\subseteq$) Choose any $\mathbf{x} \in \text{Span}(\mathbf{u}, \mathbf{v})$. ByDefinition 3.1, there exist real numbers $r, s \in \mathbb{R}$ such that $\mathbf{x} = r\mathbf{u} + s\mathbf{v}$. It follows that $\mathbf{x} = r(1,1) + s(-2,1) = (r,r) + (-2s + s) = (r - 2s, r + s)$. Since r, s are real numbers, it must be the case that $r - 2s$ and $r + s$ are real numbers. It follows that $\mathbf{x} \in \mathbb{R}^2$.

($\supseteq$) Choose any $\mathbf{x} \in \mathbb{R}^2$ and write $\mathbf{x} = (x, y)$ where $x, y \in \mathbb{R}$. The question is, can we find $r, s \in \mathbb{R}$ such that $\mathbf{x} = r\mathbf{u} + s\mathbf{v}$? This is equivalent to asking if we can find $r, s \in \mathbb{R}$ such that $(x, y) = r(1, 1) + s(-2, 1)$? This in turn is the same thing as asking if we can find $r, s \in \mathbb{R}$ such that $(x, y) = (r - 2s, r + s)$. We can answer this question if we can solve the system

$$\begin{aligned} x &= r - 2s \\ y &= r + s \end{aligned}$$

for the variables r, s. Using the usual methods from high school we find that

$$r = \frac{2y + x}{3} \text{ and } s = \frac{y - x}{3}.$$

We now have a formula for writing every (x, y) as a linear combination of $\mathbf{u}$ and $\mathbf{v}$. For example, take $x = (2, 5)$ so that $r = \frac{(2)(5)+2}{3} = \frac{12}{3} = 4$ and $s = \frac{5-2}{3} = 1$. Now check that $4(1, 1) + 1(-2, 1) = (2, 5)$. More generally, we find that if (x, y) is any vector in $\mathbb{R}^2$, then

$$\begin{aligned} r\mathbf{u} + s\mathbf{v} &= \frac{2y + x}{3}\mathbf{u} + \frac{y - x}{3}\mathbf{v} \\ &= \frac{2y + x}{3}(1, 1) + \frac{y - x}{3}(-2, 1) \\ &= (\frac{2y + x}{3}, \frac{2y + x}{3}) + (\frac{2x - 2y}{3}, \frac{y - x}{3}) \\ &= (\frac{2y + x + 2x - 2y}{3}, \frac{2y + x + y - x}{3}) \\ &= (\frac{3x}{3}, \frac{3y}{3}) \\ &= (x, y). \end{aligned}$$

We have verified that every vector (x, y) can be writtenas a linear combination of $\mathbf{u}$ and $\mathbf{v}$ as needed.

Definition 3.3. Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ be nonparallel vectors. A *plane* in $\mathbb{R}^n$ through the point $\mathbf{x}_0$ spanned by $\mathbf{u}, \mathbf{v}$ is the set

$$P = P(\mathbf{x}_0, \mathbf{u}, \mathbf{v}) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \mathbf{x}_0 + s\mathbf{u} + t\mathbf{v} \text{ for some } s, t \in \mathbb{R}\}.$$

Example 3.4. Suppose that P is the plane spanned by $\mathbf{u} = (1, 1, 0)$ and $\mathbf{v} = (1, -1, 1)$ passing through the point $(3, 0, -2)$. We determine whether or not $(7, -2, 1)$ lies on the plane P. That is, we ask if $(7, -2, 1) \in P$. Well, by the definition of P, we know that $(7, -2, 1) \in P$ if and only if we can find $s, t \in \mathbb{R}$ such that $(7, -2, 1) = (3, 0, -2) + s(1, 1, 0) + t(1, -1, 1)$. After some vector algebra, we realize that we are trying to solve

$$\begin{aligned} 3 + s + t &= 7 \\ s - t &= -2 \\ -2 + t &= 1. \end{aligned}$$

Considering the first two equations, we add to arrive at $3 + 2s = 5$ or $s = 1$. It follows that $t = 3$. Now take these two solutions and plug them into the third. Since $-2+3 = 1$ is a true statement, we have found our solutions! We conclude that

$$(7, -2, 1) = (3, 0, -2) + 1(1, 1, 0) + 3(1, -1, 1)$$

and so $(7, -2, 1) \in P$ as needed.

Exercises Section 1.1: 5, 6(a,c,d,h), 7, 10(a,d), 20-25, 28(d). Due next Wednesday!

5. If $\mathbf{x}_0 = (1, 3)$ and $\mathbf{v} = (-2, 1)$, determine which of the following points lie on $L(\mathbf{x}_0, \mathbf{v})$.

(a) $\mathbf{x} = (-1, 4)$ (b) $\mathbf{x} = (7, 0)$ (c) $\mathbf{x} = (6, 2)$

6. Find a parametric representation $L(\mathbf{x}_0, \mathbf{v})$ for each of the following lines. Observe that it is enough to determine $\mathbf{x}_0$ and $\mathbf{v}$.
(a) $\{(x, y) \in \mathbb{R}^2 : 3x + 4y = 6\}$.
(c) Slope $m = \frac{2}{5}$ passing through the point $(3, 1)$.
(d) Passing through the point $(-2, 1)$ parallel to $\mathbf{x} = (1, 4) + t(3, 5)$.
(h) Passing through the point $(1, 1, 0, -1)$ parallel to $\mathbf{x} = (2+t, 1-2t, 3t, 4-t)$.

7. In $\mathbb{R}^n$, suppose that $L(\mathbf{x}_0, \mathbf{v}) = L(\mathbf{y}_0, \mathbf{w})$. Prove the following statements.
(a) There exists $t_0 \in \mathbb{R}$ such that $\mathbf{y}_0 = \mathbf{x}_0 + t_0\mathbf{v}$.
(b) The vectors $\mathbf{v}$ and $\mathbf{w}$ are parallel. **Hint:** It might help to consider the two cases $\mathbf{x}_0 \neq \mathbf{y}_0$ and $\mathbf{x}_0 = \mathbf{y}_0$.

10. Find a parametric representation $P(\mathbf{x}_0, \mathbf{u}, \mathbf{v})$ for each of the following planes. Observe that it is enough to determine $\mathbf{x}_0$,$\mathbf{u}$, and $\mathbf{v}$.
(a) Containing the point $(-1, 01)$ and the line $\mathbf{x} = (1, 1, 1) + t(1, 7, -1)$.
(d) Containing the points $(1, 1, -1, 2)$ and $(2, 3, 0, 1)$ and $(1, 2, 2, 3)$.

20. Assume that $\mathbf{u}$ and $\mathbf{v}$ are parallel vectors in $\mathbb{R}^n$. Prove that $\text{Span}(\mathbf{u}, \mathbf{v})$ is a line. That is, find vectors $\mathbf{x}_0, \mathbf{y} \in \mathbb{R}^n$ such that $\text{Span}(\mathbf{u}, \mathbf{v}) = L(\mathbf{x}_0, \mathbf{y})$ and then prove that the sets are actually equal.

21. Suppose that $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and fix any $c \in \mathbb{R}$. Prove that $\text{Span}(\mathbf{v} + c\mathbf{w}, \mathbf{w}) = \text{Span}(\mathbf{v}, \mathbf{w})$.

22. Suppose that $\mathbf{v}, \mathbf{w} \in \text{Span}(\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k)$ and fix any $c \in \mathbb{R}$.
(a) Prove that $c\mathbf{v} \in \text{Span}(\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k)$.
(b) Prove that $\mathbf{v} + \mathbf{w} \in \text{Span}(\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k)$.

23. Consider the line $L(\mathbf{x}_0, \mathbf{v})$ and the plane $P(\mathbf{0}, \mathbf{u}, \mathbf{v})$. Prove that if $L(\mathbf{x}_0, \mathbf{v}) \cap P(\mathbf{0}, \mathbf{u}, \mathbf{v})$ is nonempty, then $\mathbf{x}_0 \in P(\mathbf{0}, \mathbf{u}, \mathbf{v})$.

24. Consider the lines $L(\mathbf{x}_0, \mathbf{v})$ and $L(\mathbf{x}_1, \mathbf{u})$. Prove that $L(\mathbf{x}_0, \mathbf{v}) \cap L(\mathbf{x}_1, \mathbf{u})$ is nonempty if and only if $\mathbf{x}_0 - \mathbf{x}_1 \in \text{Span}(\mathbf{u}, \mathbf{v})$.

25. Suppose that $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ are nonparallel vectors.
(a) Prove that if $s\mathbf{x} + t\mathbf{y} = \mathbf{0}$, then $s = t = 0$.
(a) Prove that if $a\mathbf{x} + b\mathbf{y} = c\mathbf{x} + d\mathbf{y}$, then $a = c$ and $b = d$.

28(d). Prove that for each $\mathbf{v} \in \mathbb{R}^n$, there exists a unique $\mathbf{w} \in \mathbb{R}^n$ such that $\mathbf{v} + \mathbf{w} = \mathbf{0}$. In other words, prove that every vector in $\mathbb{R}^n$ has a unique additive inverse.