PDE A.

$$
\begin{array}{lllll}
\text{PDE.} & \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} & \text{for} & 0 < x < 1, & 0 < t < \infty \\
\text{BC.} & u_x(0,t) = 0 = u_x(1,t) & \text{for} & & 0 < t < \infty \\
\text{IC.} & u(x,0) = f(x) & \text{for} & 0 < x < 1 &
\end{array}
$$

For PDE A, determine the full solution for the general initial condition $f(x)$. State how orthogonality is used and how coefficients in the series expansion are determined using $f(x)$

$$
\begin{aligned}
u_0(x,t) &= c_0 \\
u_n(x,t) &= c_n e^{-n^2\pi^2 t}\cos(n\pi x) && n = 1,2,3,\dots \\
u(x,t) &= c_0 + \sum_{n=1}^{\infty} c_n e^{-n\pi^2 t}\cos(n\pi x) \\
u(x,t) &= \sum_{n=0}^{\infty} c_n e^{-n\pi^2 t}\cos(n\pi x) \\
f(x) = u(x,0) &= \sum_{n=0}^{\infty} c_n \cos(n\pi x) && \text{let } m \in \mathbb{Z} \\
\int_0^1 f(x)\cos(m\pi x)\,\mathrm{d}x &= \int_0^1 \cos(m\pi x)\sum_{n=0}^{\infty} c_n\cos(n\pi x)\,\mathrm{d}x \\
\text{let } n &= m = 0 \\
\int_0^1 c_0\cos(0)^2\,\mathrm{d}x &= c_0 \\
\text{let } n &= m \neq 0 \\
\int_0^1 c_m\cos(m\pi x)^2\,\mathrm{d}x &= \frac{1}{2}c_m\int_0^1 2\cos(m\pi x)^2\,\mathrm{d}x \\
&= \frac{1}{2}c_m\int_0^1 1 + \cos(2m\pi x)\,\mathrm{d}x \\
&= \frac{1}{2}c_m\left|x + \frac{1}{2m\pi}\sin(2m\pi x)\right|_0^1 \\
&= \frac{1}{2}c_m + \frac{1}{2}c_m\frac{\sin(2m\pi)}{2m\pi} = \frac{1}{2}c_m && m \in \mathbb{Z} \to \sin(2m\pi) = 0
\end{aligned}
$$

and to establish orthogonality let $n \neq m$

$$
\begin{aligned}
\int_0^1 c_m\cos(n\pi x)\cos(m\pi x)\,\mathrm{d}x &= \frac{1}{2}c_m\int_0^1 2\cos(n\pi x)\cos(m\pi x)\,\mathrm{d}x \\
&= \frac{1}{2}c_m\int_0^1 \cos(n\pi x - m\pi x) + \cos(n\pi x + m\pi x)\,\mathrm{d}x \\
&= \frac{1}{2}c_m\left[\frac{1}{(n-m)\pi x}\sin((n-m)\pi x)\right. \\
&\qquad \left.+\frac{1}{(n+m)\pi x}\sin((n+m)\pi x)\right]_0^1 \\
&= \frac{1}{2}c_m\left[\frac{1}{(n-m)\pi}\sin((n-m)\pi)\right. \\
&\qquad \left.+\frac{1}{(n+m)\pi}\sin((n+m)\pi)\right] && n-m, n+m \in \mathbb{Z}
\end{aligned}
$$

$$= \frac{1}{2}c_m \cdot 0 = 0 \qquad \sin((m \pm n)\pi) = 0$$

And now we have everything we need to determine the full solution. Since we can find the coefficient to the mth term by multiplying $\cos(m\pi x)$ and then integrating, letting orthogonality kill off all the extra terms.

$$u(x,t) = \sum_{n=0}^{\infty} c_n e^{-n\pi^2 t} \cos(n\pi x)$$

$$c_0 = \int_0^1 f(x)\,\mathrm{d}x$$

$$c_m = 2\int_0^1 f(x)\cos(m\pi x)\,\mathrm{d}x \qquad \text{where } m = 1, 2, 3, \ldots$$