Notes

January 26, 2015

non measurable set example

$x, y \in [0,1)$

$$x \oplus y : \begin{cases} x+y & \text{if } x+y<1 \\ x+y-1 & \text{if } x+y \geq 1 \end{cases}$$

facts

1. $x \oplus y = y \oplus x$
2. $(x \oplus y) \oplus z = x \oplus (y \oplus z)$

if $E \subseteq [0,1)$ and $x \in [0,1)$ then define $E \oplus x : \{z : y \oplus x = z \text{ with } y \in E\}$

lemma

if $E \subseteq [0,1)$ is measurable and $x \in [0,1)$ then $E \oplus x$ is measurable and $m*(E \oplus x) = m*(E)$

proof

$E_1 = E \cap [0, 1-x), E_2 = E \cap [1-x, 1)$ where E_1 is regular addition, and E_2 is overflow.

notice E_1 and E_2 are measurable. Now $E_1 \cup E_2 = E$ and because they are disjoint and measurable then $m*(E_1) + m*(E_2) = m*(E)$

$E_1 \oplus x = E_1 + x$. Now $m*(E_1 \oplus x) = m*(E_1 + x) = m*(E_1)$ (from homework)

on the other hand $E_2 \oplus x = E_2 + x - 1$ and so $m*(E_2 \oplus x) = m*(E_2 + x - 1) = m*(E_2)$.

$$\begin{aligned} m*(E) = m*(E_1) + m*(E_2) &= m*(E_1 \oplus x) + m*(E_2 \oplus x) \\ &= m*((E_1 \oplus x) \cup (E_2 \oplus x)) \\ y &\in E_1 \oplus x \\ y &\in E_2 \oplus x \\ y &= x + z_1, z_1 \in [0, 1-x) \\ y &= x + z_2 - 1, z_2 \in [1-x, 1) \\ z_1 &= z_2 - 1 \\ (E_1 \oplus x) \cap (E_2 \oplus x) &= \emptyset \qquad = m*(E \oplus x) \end{aligned}$$

take $x, y \in [0,1)$ where $x \sim y$ if $x = y + q$ where q is rational. claim is that this is equivalence relation.

1. symmetry

   $x = y + q \to y = x + (-q)$

2. reflexivity

   $x = x + 0$ and $0 \in \mathbb{Q}$

3. transitivity

   $x = y + q_1, y = z + q_2 \rightarrow x = (z + q_2) + q_1 = z + (q_1 + q_2)$

$[0, 1)/\sim=$ equivalence classes (space modulo equivalence classes)
$[0, 1) = \cup[x]_\sim \leftarrow$ disjoint

axiom of choice

if you have a collection of sets, then you can choose something from each set.

now for each equivalence class, choose one element. the collection of these choices is P

every rational number is equivalent to every other rational and so P contains one and only one rational number.

one can show that for example $\pi/4$ will create it's own equivalence class. and $e/3$ will also have it's own class. It is difficult to show that e and π are in their own equivalence classes. P has at least 3 elements. Actually it has an uncountable number of elements.

$\{r_i\}_{i=0}^{\infty}$ is an enumeration of $\mathbb{Q} \cap [0, 1)$ with $r_0 = 0$.This list is the rationals in any order, but 0 is at the beginning

let $P_i = P \oplus r_i$

1. $m * (P_i) = m * (P)$

2. $P_0 = P$

3. $P_i \cap P_j = \emptyset$ if $i \neq j$.

   if $x \in P_i \cap P_j$ then $x = x_1 \oplus r_i = x_2 \oplus r_j$ where $x_1, x_2 \in P$.

   $x_1 + r_1 = x_2 + r_2 \rightarrow x_1 = x_2 + (r_2 - r_1)$

   $x_1 + r_1 - 1 = x_2 + r_2 \rightarrow x_1 = x_2 + (r_2 - r_1 + 1)$

   $x_1 + r_1 = x_2 + r_2 - 1 \rightarrow x_1 = x_2 + (r_2 - 1 - r_1)$

   $x_1 + r_1 - 1 = x_2 + r_2 - 1 \rightarrow x_1 = x_2 + (r_2 - r_1)$

   and so $x_1 \sim x_2$ and so x is not in both and we have a contradiction

4. $\bigcap_{i=1}^{\infty} P_i = [0, 1)$

   $x \in [0, 1)$ and $x \sim y$ with $y \in P$ and $x = y + q$ with $q \in \mathbb{Q}$

   $q \in \mathbb{Q}$ implies $q = r_j$ for some j so $x \in P_j$.

if P is measurable then so is P_i for all i and so $1 = m * ([0, 1)) = m * (\bigcup_{i=1}^{i} nfty P_i = \sum_{i=1}^{\infty} m * (P_i) = \sum_{i=1}^{\infty} m * (P)$. Now either the sum is 0 and we can't get to one, or it's not 0 and we can get bigger than one. contradiction