Jon Allen
HW 16
Lesson 10 exercise 3. Find the cosine transform $U(\omega, t)$ of the solution $u(x,t)$
Solve by means of the sine *or* cosine transform

$$\begin{array}{lcr} \text{PDE} & u_t = \alpha^2 u_{xx} & 0 < x < \infty \\ \text{BC} & u_x(0,t) = 0 & 0 < t < \infty \\ \text{IC} & u(x,0) = H(1-x) & 0 \le x < \infty \end{array}$$

where $H(x)$ is the *Heaviside function*:

$$H(x) = \begin{cases} 0 & 0 \le x < 1 \\ 1 & 1 \le x \end{cases}$$

Note: The text is wrong. The above is actually the function $H(1-x)$ not the function $H(x)$.

$$\begin{aligned}
\mathcal{F}_c[u_t] &= \alpha^2 \mathcal{F}_c[u_{xx}] \\
\mathcal{F}_c[u_t] &= \frac{2}{\pi}\int_0^\infty u_t \cos(\omega x)\,\mathrm{d}x \\
&= \frac{\partial}{\partial t}\left[\frac{2}{\pi}\int_0^\infty u \cos(\omega x)\,\mathrm{d}x\right] \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_c[u] = \frac{\mathrm{d}}{\mathrm{d}t}U(t) \\
\frac{\mathrm{d}U}{\mathrm{d}t} &= \alpha^2\left[-\omega^2 U(t)\right] \\
U' + \omega^2\alpha^2 U &= 0 \\
\mu &= e^{\int (\omega\alpha)^2\,\mathrm{d}t} \\
&= e^{(\omega\alpha)^2 t} \\
e^{\omega^2\alpha^2 t} U &= \int 0\,\mathrm{d}t \\
U(0) = c_1 e^0 &= c_1 = \frac{2}{\pi}\frac{\sin(\omega)}{\omega}
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}_c[u_{xx}] &= \frac{2}{\pi}\int_0^\infty u_{xx}\cos(\omega x)\,\mathrm{d}x \\
&= -\frac{2}{\pi}{u_x}'(0,t) - \omega^2\mathcal{F}_c[u] \\
&= -\omega^2 U(t) \\
\mathcal{F}_c[u(x,0)] &= \frac{2}{\pi}\int_0^\infty H(1-x)\cos(\omega x)\,\mathrm{d}x \\
&= \frac{2}{\pi}\int_0^1 \cos(\omega x)\,\mathrm{d}x \\
&= \frac{2}{\pi}\left[\frac{\sin(\omega x)}{\omega}\right]_0^1 \\
U(0) &= \frac{2}{\pi}\frac{\sin(\omega)}{\omega} \\
U(t) &= c_1 e^{-\omega^2\alpha^2 t} \\
U(t) &= \frac{2}{\pi}\frac{\sin(\omega)}{\omega}e^{-\omega^2\alpha^2 t}
\end{aligned}$$

Jon Allen
HW 17
Find the inverse transform u(x,t). Hint use mathematica.

$$
\begin{aligned}
U(t) &= \frac{2}{\pi}\frac{\sin(\omega)}{\omega}e^{-(\omega\alpha)^2 t} \\
{\mathcal{F}_c}^{-1}[U] &= u(x,t) \\
&= \int_0^\infty U(t)\cos(\omega x)\,\mathrm{d}\omega \\
&= \frac{2}{\pi}\int_0^\infty \frac{\sin(\omega)\cos(\omega x)}{\omega e^{(\omega\alpha)^2 t}}\,\mathrm{d}\omega \\
&= \frac{2}{\pi}\int_0^\infty \frac{1}{\omega}e^{-\omega^2\alpha^2 t}\sin(\omega)\cos(\omega x)\mathrm{d}\omega
\end{aligned}
$$

Punching the above integral into Mathematica we get the following:

$$
u(x,t) = \frac{1}{2}\left[\operatorname{erf}\left(\frac{x+1}{2\sqrt{\alpha^2 t}}\right) - \operatorname{erf}\left(\frac{x-1}{2\sqrt{\alpha^2 t}}\right)\right]
$$

Jon Allen
HW 18
Plot the solution $u(x,t)$ for $t = 0.01, 0.1, 1.0$ with $\alpha^2 = 1$

$$u(x,t) = \frac{1}{2}\left[\mathrm{erf}\left(\frac{x+1}{2\sqrt{t}}\right) - \mathrm{erf}\left(\frac{x-1}{2\sqrt{t}}\right)\right]$$