Notes

September 22, 2014

cont'd example from last time

$a_{n+1} = 1 + \frac{1}{a_n}$ and is fibanocci sequence. terms are back and forth but converging. it is clear that $a_n > 1 \forall n$.

$$\begin{aligned}
|a_{n+1} - a_n| &= \left|1 + \frac{1}{a_n} - \left(1 + \frac{1}{a_{n-1}}\right)\right| \\
&= \left|\frac{1}{a_n} - \frac{1}{a_{n-1}}\right| \\
&= \frac{|a_n - a_{n-1}|}{a_n \cdot a_{n-1}} \text{with } a_n > 1 \text{ so} \\
|a_{n+1} - a_n| &< |a_n - a_{n-1}|
\end{aligned}$$

does this mean we have a limit? contractive sequence has the property $|a_{n+1} - a_n| < r|a_n - a_{n-1}|$ wher $r \in (0, 1)$

no, it is possible in principle that $\lim |a_{n+1} - a_n| = b > 0$ and that would mean that $\{a_n\}$ is not convergent

$$\begin{aligned}
a_n a_{n+1} = a_n(1 + \frac{1}{a_n} &= a_n + 1 > 2 \\
|a_{n+1} - a_n| &< \frac{1}{2}\,|a_n - a_{n-1}|
\end{aligned}$$

so it is convergent because $\frac{1}{2} \in (0, 1)$

2.8.D

pick a sequence of ε, $\varepsilon_n = \{\frac{1}{2^n}\}$. given $\varepsilon_1 = \frac{1}{2}$ there exists $N + 1 \in \mathbb{N}$ such that $|a_m - a_n < \varepsilon_1$ if $m, n \geq N_1$, $\varepsilon_2 = \frac{1}{4}, \exists N_2$ st $|a_m - a_n| < \varepsilon_2$ and so on.

so $|a_{\mathbb{N}_{n+1}} - a_{N_n}| < \frac{1}{2^n}$ and so the sum is less than 1 and we win.

convergent series

given (a_n) we consider the series $\sum\limits_{n=1}^{\infty} a_n$ let $s_n = \sum\limits_{k=1}^{n} a_k$ be the nth partial sum of the series. if $\lim s_n$ exists, we say that the series $\sum\limits_{k=1}^{\infty} a_k$is convergent

the following are equivalent

- $\sum a_n$ is convergent
- $\forall \varepsilon > 0 \exists N$ st if $n \geq N$, $\left| \sum_{k=n+1}^{\infty} a_k \right| < \varepsilon$
- $\forall \varepsilon > 0 \exists N$ st if $m, n \geq N$, $\left| \sum_{k=n+1}^{m} a_k \right| < \varepsilon$

proofs are in the book

note

if $\sum a_k < \infty$ then $\lim a_k = 0$ but the converse is false, as shown by the harmonic series.

telescoping and geometric series are basically the only ones where we know how to find the sums

3.1.c

if $\sum t_k$ is a convergent series of positive terms and $p > 1$ show that $\sum t_k^p$ is convergent

the necessary condition is that $\lim t_k = 0$. by the necessary condition $\exists N$ st $0 \leq t_k \leq 1 \forall k \geq N$. Therefore $\sum_{k=N}^{\infty} t_k^p \leq \sum_{k=N}^{\infty} t_k < \infty$

3.1.d

if $\lim |a_n| = 0$ then there exists $\sum a_{n_k} < \infty$

example argument: harmonic series