8.3

A. For $x \in [-1,1]$ let $F(x) = \int_0^1 x(1-x^2y^2)^{-1/2}\mathrm{d}y$. Show that $F'(x) = (1-x^2)^{-1/2}$ and deduce that $F(x) = \arcsin(x)$.

We let $f(x,y) = \dfrac{x}{\sqrt{1-x^2y^2}}$. Then $\dfrac{\partial}{\partial x}f(x,t) = (1-x^2y^2)^{-1/2} - \dfrac{1}{2}x(1-x^2y^2)^{-3/2}(-2xy^2) = \dfrac{1-x^2y^2+x^2y^2}{(1-x^2y^2)^{3/2}} = (1-x^2y^2)^{-3/2}$. We note that we can find critical points when $x \in \{-1,0,1\}$ and $y \in \{0,1\}$. As $x \to 0$ and $y \to 0$ we see that $1-x^2y^2 \to 1$, and so as we approach zero on the numerator of $f(x,y)$ then the denominator approaches 1. As we go away from zero, then the numerator increases, and the denominator decreases, and so $f(x)$ increases. Similarly, $\frac{\partial}{\partial x}f(x,y)$ is at a minimum when $(x,y) = (0,0)$ and increases as we go away from zero. Further, as $y \to 1$ and $x^2 \to 1$ then $1-x^2y^2 \to 0$. Thus we have a discontinuous point when $y = 1$ and $x^2 = 1$. This problem occurs in both the function and it's partial derivative. The exercise is practically begging us to use Leibniz's Rule, but this rule is only defined on continuous functions. Let us reformulate the question as follows:

$$F(x) = \lim_{n\to 1}\int_0^n \frac{x}{\sqrt{1-x^2y^2}}\mathrm{d}y$$

Now, because $0 \le n < 1$ we have a continuous function we can work with and so by Leibniz's Rule we have

$$F'(x) = \lim_{n\to 1}\int_0^n (1-x^2y^2)^{-3/2}\mathrm{d}y$$

The trick here is to notice that $-1 < xy < 1$ and so we can make the substitution $xy = \sin u$ and $\mathrm{d}y = \frac{\cos u}{x}\mathrm{d}u$. Of course $u = \arcsin xy$ and $\arcsin x \cdot 0 = 0$. Let us assign $m = \arcsin xn$

$$\begin{aligned}
\lim_{n\to 1}\int_0^n (1-x^2y^2)^{-3/2}\ \mathrm{d}y &= \frac{1}{x}\lim_{n\to 1}\int_0^m (1-\sin^2 u)^{-3/2} cosu\ \mathrm{d}u \\
&= \frac{1}{x}\lim_{n\to 1}\int_0^m \frac{1}{\cos^2 u}\ \mathrm{d}u \\
&= \frac{1}{x}\lim_{n\to 1}\int_0^m \frac{\sin^2 u + \cos^2 u}{\cos^2 u}\ \mathrm{d}u \\
&= \frac{1}{x}\lim_{n\to 1}\left[\int_0^m \frac{\sin^2 u}{\cos^2 u}\ \mathrm{d}u + \int_0^m 1\ \mathrm{d}u\right]
\end{aligned}$$

And continuing with substitution by parts:

$$w = \sin u \qquad\qquad \mathrm{d}w = \cos u\ \mathrm{d}u$$

$$\mathrm{d}v = \frac{\sin u}{\cos^2 u}\mathrm{d}u \qquad\qquad v = \frac{1}{\cos u}$$

$$\begin{aligned}
\lim_{n\to 1}\int_0^n (1-x^2y^2)^{-3/2}\,\mathrm{d}y &= \frac{1}{x}\lim_{n\to 1}\left[\left.\frac{\sin u}{\cos u}\right|_0^m - \int_0^m \frac{\cos u}{\cos u}\,\mathrm{d}u + \int_0^m 1\,\mathrm{d}u\right] \\
&= \lim_{n\to 1}\left.\frac{\sin u}{x\cos u}\right|_0^m = \lim_{n\to 1}\left.\frac{\sin u}{x\sqrt{1-\sin^2 u}}\right|_0^m \\
&= \lim_{n\to 1}\left.\frac{y}{\sqrt{1-x^2y^2}}\right|_0^n \\
F'(x) &= \lim_{n\to 1}\left(\frac{n}{\sqrt{1-x^2n^2}} - \frac{0}{\sqrt{1-x^2 0^2}}\right) \\
&= \frac{1}{\sqrt{1-x^2}}
\end{aligned}$$

Integrating $F'(x)$ by using the trig substitution of $x = \sin u$ gives us

$$\begin{aligned}
\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x &= \int \frac{1}{\sqrt{1-\sin^2 u}}\cos u\,\mathrm{d}u \\
&= \int \frac{\cos u}{\sqrt{\cos^2 u}}\,\mathrm{d}u = \int \mathrm{d}u \\
&= u + C = \arcsin x + C
\end{aligned}$$

Thus we have $F(x) = \arcsin x + C$. Substituting in $x = 0$ for both versions of $F(x)$ gives us $F(0) = \int_0^1 0\cdot(1-0)^{-1/2}\,\mathrm{d}y = 0 = \arcsin 0 + C = 0 + C$. And so we have $F(x) = \arcsin x$

B. For $n \geq 1$, define function f_n on $[0,\infty)$ by

$$f_n(x) = \begin{cases} e^{-x} & \text{for} \quad 0 \leq x \leq n, \\ e^{-2n}(e^n + n - x) & \text{for} \quad n \leq x \leq n + e^n, \\ 0 & \text{for} \quad x \geq n + e^n, \end{cases}$$

(a) Find the pointwise limit f of (f_n). Show that the convergence is uniform on $[0,\infty)$

Obviously for any x we know that for all $k \in \mathbb{N}$ where $k > x$ we have $f_k(x) = e^{-x}$ and so our pointwise limit is just e^{-x}.

And just as obvious, if we raise any number greater than zero to any power we must get a number greater than zero. Thus $e^{-x} > 0$. And we can immediately conclude that $\frac{\mathrm{d}}{\mathrm{d}x}e^{-x} = -e^{-x} < 0$. Because the derivative is always smaller than zero and the function is greater than zero we have a monotonically decreasing function that is bounded below by 0.

Let us move onto the second piece of our function. Note that $e^{-2n}(e^n-n+x)=\frac{1}{e^n}+\frac{n}{e^{2n}}-\frac{x}{e^{2n}}$. Notice that this is a linear function of x with a negative slope of $-e^{-2n}$. It is clear that $e^{-2n}(e^n-n+x)=e^{-x}$ when $x=n$. And when $x=n+e^n$ then we have $e^{-2n}(e^n+n-(n+e^n))=0$. And so, this piece of the function is bounded above by $\frac{1}{e^n}$ and below by 0.

Now let us choose some $\varepsilon>0$ and some $N\in\mathbb{N}$ such that $e^{-N}<\varepsilon$. We let $k\geq N$. Now if $x\leq N$ then $||f_k(x)-f(x)||=0$ and if $x>N$ then we know that $0\leq f(x)\leq\frac{1}{e^N}$ and $0\leq f_k(x)\leq\frac{1}{e^N}$. And so $||f_k(x)-f(x)||\leq\left|\frac{1}{e^n}-0\right|<\varepsilon$.

And so we have uniform continuity.

(b) Compute $\displaystyle\int_0^\infty f(x)\mathrm{d}x$ and $\displaystyle\lim_{n\to\infty}\int_0^\infty f_n(x)\mathrm{d}x$.

$$\begin{aligned}
\int_0^\infty f(x)\,\mathrm{d}x &= \lim_{k\to\infty}\int_0^k e^{-x}\,\mathrm{d}x\\
&= \lim_{k\to\infty} -\int_0^k -e^{-x}\,\mathrm{d}x\\
&= \lim_{k\to\infty} -e^{-x}\Big|_0^k\\
&= \lim_{k\to\infty}\left(-e^{-k}+e^0\right)=1\\
\lim_{n\to\infty}\int_0^\infty f_n(x)\mathrm{d}x &= \lim_{n\to\infty}\left[\int_0^n e^{-x}\mathrm{d}x+\int_n^{n+e^n} e^{-2n}(e^n+n-x)\,\mathrm{d}x+\int_{n+e^n}^\infty 0\,\mathrm{d}x\right]\\
&= \lim_{n\to\infty}\left(-e^{-x}\big|_0^n+e^{-2n}\left[x(e^n+n)-\frac{x^2}{2}\right]_n^{n+e^n}+0\right)\\
&= \lim_{n\to\infty}\left(e^{-2n}\left[(n+e^n)(e^n+n)-\frac{(n+e^n)^2}{2}\right.\right.\\
&\qquad\left.\left.-n(e^n+n)+\frac{n^2}{2}\right]-e^{-n}+1\right)\\
&= \lim_{n\to\infty}\left(e^{-2n}\left[n^2+2ne^n+e^{2n}-\frac{n^2}{2}-ne^n-\frac{e^{2n}}{2}\right.\right.\\
&\qquad\left.\left.-ne^n-n^2+\frac{n^2}{2}\right]-e^{-n}+1\right)\\
&= \lim_{n\to\infty}\left(e^{-2n}e^{2n}-e^{-2n}\frac{e^{2n}}{2}-e^{-n}+1\right)\\
&= \lim_{n\to\infty}\left(1-\frac{1}{2}-e^{-n}+1\right)=\frac{3}{2}
\end{aligned}$$

(c) Why does this not contradict theorem 8.3.1?

Because the theorem is for closed intervals. $[0,\infty)$ is not closed.

D. Find $\lim_{n\to\infty}\int_0^\pi \frac{\sin nx}{nx}\mathrm{d}x$ HINT: Find the limit of the integral over $[\varepsilon,\pi]$ and estimate the rest.

First lets just rewrite this expression with $u = nx$ and replace 0 with $\varepsilon \to 0$.

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{n}\int_\varepsilon^{\pi n}\frac{\sin u}{u}\mathrm{d}u$$

Of course $-1 \le \sin u \le 1$ and so $-\frac{1}{u} \le \frac{\sin u}{u} \le \frac{1}{u}$. Looking for the first zero of the integrand we obtain $\sin u = 0$ or $u = k\pi$ where $k \in \mathbb{N}$. Thus our functions first zero is at $u = \pi$. Now we have an upper bound at ε. Using L'hopital, we have $\lim_{u\to 0}\frac{\sin u}{u} = 1$ and so we have $\int_0^\pi 1\ \mathrm{d}u \ge \int_0^\pi \frac{\sin u}{u}\mathrm{d}u$. We also know that $\left|\int_\pi^{n\pi}\frac{\sin u}{u}\right| \le \int_\pi^{n\pi}\frac{1}{u}\ \mathrm{d}u$.

We are going to assume without loss of generality that $\int_\pi^{n\pi}\frac{\sin u}{u} \le 0$ and so we have

$$\begin{aligned}\lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{n}\int_\varepsilon^{\pi n}\frac{\sin u}{u}\mathrm{d}u &\le \lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{n}\left[\int_\varepsilon^{\pi}\mathrm{d}u + \int_\pi^{n\pi}\frac{1}{u}\ \mathrm{d}u\right]\\ &= \lim_{n\to\infty}\frac{1}{n}\left[\lim_{\varepsilon\to 0}(\pi-\varepsilon) + \log n\pi - \log\pi\right]\\ &= \lim_{n\to\infty}\left[\frac{\pi}{n} + \frac{\log n}{n}\right]\\ &= 0 + \lim_{n\to\infty}\frac{1}{n}\text{ from L'Hôpital} \qquad\qquad = 0\end{aligned}$$

And so we see that the limit of the integral is zero.

E. Define $f(x) = \int_0^\pi \frac{\sin xt}{t}\mathrm{d}t$.

(a) Prove that this integral is defined.
First notice that $\lim_{t\to 0}\sin xt = 0$ and $\lim_{t\to 0} t = 0$. Applying L'Hôpitals rule we get $\lim_{t\to 0}\frac{\sin xt}{t} = \lim_{t\to 0}\frac{x\cos xt}{1} = x\cos 0 = x$.
Now if we look at this integrand, then we see the sin function which is being scaled vertically by a factor of $1/t$. Now $\sin xt$ gets bigger as $t \to 0$ and so does $1/t$. Thus this function will be farthest from the t axis as $t \to 0$. Because this limit is bounded, then our integrand is bounded and so we will be able to find an integral.

(b) Compute $f'(x)$ explicitly.

The integrand is continuous on $[\varepsilon, \pi]$ for all $\epsilon > 0$. And so we can compute the integrand with a limit and Leibniz's Rule.

$$\begin{aligned} f'(x) &= \lim_{\varepsilon\to 0}\int_{\varepsilon}^{\pi} \frac{\partial}{\partial x}\frac{\sin xt}{t}\ \mathrm{d}t \\ &= \lim_{\varepsilon\to 0}\int_{\varepsilon}^{\pi} \cos xt\ \mathrm{d}t \\ &= \int_0^{\pi} \cos xt\ \mathrm{d}t \end{aligned}$$

we must be careful about what happens at zero

$$\begin{aligned} &= \begin{cases} \frac{\sin \pi x}{x} - \frac{\sin 0\cdot x}{x} & x \neq 0 \\ \int_0^{\pi} \cos 0\ \mathrm{d}t & x = 0 \end{cases} \\ &= \begin{cases} \frac{\sin \pi x}{x} & x \neq 0 \\ \pi & x = 0 \end{cases} \end{aligned}$$

(c) Prove that f' is continuous at 0.

Our function is clearly continuous at all points other than zero.

We need to show that $\lim_{x\to 0} f'(0) = f'(0) = \pi$. Now $\sin \pi 0 = 0$ so we have the $\frac{0}{0}$ case needed for L'Hôpitals Rule.

$$\begin{aligned} \lim_{x\to 0}\frac{\sin \pi x}{x} &= \lim_{x\to 0}\frac{\pi\cos \pi x}{1} \\ &= \pi\cos 0 = \pi \end{aligned}$$

And so our function is continuous.

H. Suppose that $f \in \mathbb{C}^2[0,1]$ such that $f''(x) + bf'(x) + cf(x) = 0, f(0) = 0$, and $f'(0) = 1$. Let $d(x)$ be continuous on $[0,1]$ and define $g(x) = \int_0^x f(x-t)d(t)\mathrm{d}t$. Prove that $g(0) = g'(0) = 0$ and $g''(x)+bg'(x)+cg(x) = d(x)$

We use the result from part G to obtain $g'(x) = \int_0^x f'(x-t)d(t)\ \mathrm{d}t + f(0)d(x)$ and $g''(x) = \int_0^x f''(x-t)d(t)\ \mathrm{d}t + f'(0)d(x)$ It doesn't take much to see that $g(0) = g'(0) = 0$

$$\begin{aligned} g(0) &= g'(0) \\ \int_0^0 f(-t)d(t)\mathrm{d}t &= \int_0^0 f'(-t)d(t)\ \mathrm{d}t \end{aligned}$$

This seems like a problem at first glance because we have $f(-t)$ and $f'(-t)$ as terms in both functions. But remember that in this case, $t \in [0,0]$. And

$f(-0) = 0$ while $f'(-0) = 1$. Similarly $d(x)$ is continuous on $[0, 1]$ so it must be defined for $d(0)$. Thus we have

$$\int_0^0 0 \, \mathrm{d}t = \int_0^0 d(t) \, \mathrm{d}t$$

We know that the integral for $d(x)$ is defined because it is continuous. Let us say the $s(x) = \int d(x) \, \mathrm{d}x$. Then

$$\int_0^0 0 \, \mathrm{d}t = 0$$
$$\int_0^0 d(t) \, \mathrm{d}t = s(0) - s(0) = 0$$

As for the second part:

$$\begin{aligned}
g''(x) + bg'(x) + cg(x) &= \int_0^x f''(x-t)d(t) \, \mathrm{d}t + d(x) + b\int_0^x f'(x-t)d(t) \, \mathrm{d}t + \\
&\qquad c\int_0^x f(x-t)d(t) \, \mathrm{d}t \\
&= \int_0^x (f''(x-t) + bf'(x-t) + cf(x-t))d(t) \, \mathrm{d}t + d(x) \\
&= \int_0^x 0 \cdot d(t) \, \mathrm{d}t + d(x) \\
&= d(x)
\end{aligned}$$