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Homework 4
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2.5

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Solve $(t^2 - y^2)\mathrm{d}y + (y^2 + ty)\mathrm{d}t = 0$.
$(x^2t^2 - x^2y^2) = x^2(t^2 - y^2) = (x^2y^2 + xtxy)$
The equation is homogeneous so we substitute $t = vy$

$$\begin{aligned}
(v^2y^2 - y^2)\,\mathrm{d}y + (y^2 + vy^2)(y\,\mathrm{d}v + v\,\mathrm{d}y) &= 0 \\
(v^2y^2 - y^2)\,\mathrm{d}y + y^2y\,\mathrm{d}v + vy^2y\,\mathrm{d}v + y^2v\,\mathrm{d}y + vy^2v\,\mathrm{d}y &= 0 \\
(v^2y^2 - y^2 + y^2v + v^2y^2)\,\mathrm{d}y + (y^3 + vy^3)\,\mathrm{d}v &= 0 \\
y^2(v^2 - 1 + v + v^2)\,\mathrm{d}y + y^3(1+v)\,\mathrm{d}v &= 0 \\
y^2(2v^2 + v - 1)\,\mathrm{d}y + y^3(1+v)\,\mathrm{d}v &= 0
\end{aligned}$$

$$\begin{aligned}
y^2(2v^2 + v - 1)\,\mathrm{d}y &= -y^3(1+v)\,\mathrm{d}v \\
\frac{1}{y}\,\mathrm{d}y &= -\frac{1+v}{2v^2+v-1}\,\mathrm{d}v \\
\frac{1}{y}\,\mathrm{d}y &= -\frac{1+v}{(v+1)(2v-1)}\,\mathrm{d}v \\
\frac{1}{y}\,\mathrm{d}y &= -\frac{1}{2v-1}\,\mathrm{d}v && u = 2v - 1 \\
\int \frac{1}{y}\,\mathrm{d}y &= -\frac{1}{2}\int \frac{1}{u}\,\mathrm{d}u && \mathrm{d}u = 2\mathrm{d}v \\
\ln y &= \ln \frac{1}{\sqrt{2v-1}} + C_0 \\
y &= e^{C_0}\frac{1}{\sqrt{2v-1}} && e^{C_0} = C \\
y &= C\frac{1}{\sqrt{2\frac{t}{y} - 1}}
\end{aligned}$$

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Solve $(t + y)\,\mathrm{d}t - t\,\mathrm{d}y = 0, y(1) = 1$

$$(tx + yx) - tx = xM(x,y) + xN(x,y)$$

The equation is homogeneous and $N(x, y)$is simpler so set $y = wt$and $\mathrm{d}y = t\,\mathrm{d}w + w\,\mathrm{d}t$

$$\begin{aligned}
(t + wt)\,\mathrm{d}t - t(t\,\mathrm{d}w + w\,\mathrm{d}t) &= 0 \\
t\,\mathrm{d}t - t^2\,\mathrm{d}w &= 0 \\
\frac{1}{t}\,\mathrm{d}t &= \mathrm{d}w \\
\int \frac{1}{t}\,\mathrm{d}t &= \int \mathrm{d}w \\
\ln t &= \frac{y}{t} + C_0 \qquad\qquad C = -C_0 \\
y &= t(\ln t + C) \\
1 &= 1(0 + C) = C \\
y &= t \ln t + t
\end{aligned}$$

2.6

Do only the case for h=0.1

2

Approximate the solution to the IVP $y' = 4x - y + 1, y(0) = 0$at $x = 1$for $h = 0.1$

$$\begin{aligned}
y_0 &= 0 \\
y_1 &= 0.1 * (4 * 0 - 0 + 1) + 0 = 0.1 \\
y_2 &= 0.1 * (4 * 0.1 - 0.1 + 1) + 0.1 = 0.23 \\
y_3 &= 0.1 * (4 * 0.2 - 0.23 + 1) + 0.23 = 0.387 \\
y_4 &= 0.1 * (4 * 0.3 - 0.387 + 1) + 0.387 = 0.5683 \\
y_5 &= 0.1 * (4 * 0.4 - 0.5683 + 1) + 0.5683 = 0.77147 \\
y_6 &= 0.1 * (4 * 0.5 - 0.77147 + 1) + 0.77147 = .9943230000000001 \\
y_7 &= 0.1 * (4 * 0.6 - .994323 + 1) + .994323 = 1.2348907 \\
y_8 &= 0.1 * (4 * 0.7 - 1.2348907 + 1) + 1.2348907 = 1.49140163 \\
y_9 &= 0.1 * (4 * 0.8 - 1.49140163 + 1) + 1.49140163 = 1.762261467 \\
y_{10} &= 0.1 * (4 * 0.9 - 1.762261467 + 1) + 1.762261467 = 2.0460353203 \\
y(1) &\approx 2.0460353203
\end{aligned}$$

3

Approximate the solution to the IVP $y' - x = y^2 - 1, y(0) = 1$at $x = 1$for $h = 0.1$

$$\begin{aligned}
y_0 &= 1 \\
y_1 &= 0.1 * (y_0^2 - 1 + 0) + y_0 = 1 \\
y_2 &= 0.1 * (y_1^2 - 1 + 0.1) + y_1 = 1.01 \\
y_3 &= 0.1 * (y_2^2 - 1 + 0.2) + y_2 = 1.03201 \\
y_4 &= 0.1 * (y_3^2 - 1 + 0.3) + y_3 = 1.06851446401 \\
y_5 &= 0.1 * (y_4^2 - 1 + 0.4) + y_4 = 1.122686779989858 \\
y_6 &= 0.1 * (y_5^2 - 1 + 0.5) + y_5 = 1.198729340586257 \\
y_7 &= 0.1 * (y_6^2 - 1 + 0.6) + y_6 = 1.302424543784494 \\
y_8 &= 0.1 * (y_7^2 - 1 + 0.7) + y_7 = 1.442055513009718 \\
y_9 &= 0.1 * (y_8^2 - 1 + 0.8) + y_8 = 1.630007923269891 \\
y_{10} &= 0.1 * (y_9^2 - 1 + 0.9) + y_9 = 1.885700506262153 \\
y(1) &\approx 1.885700506262153
\end{aligned}$$

3.1

5

Suppose that the half-life of an element is 1000 h. If there are initially 100 g, how much remains after 1 h? How much remains after 500 h?

$$\begin{aligned}
y(0) &= 100 \\
y(1000) &= 50 \\
y(t) &= 100e^{kt} \\
50 &= 100e^{1000m} \\
\left(\frac{5}{10}\right)^{1/1000} &= e^m \\
\frac{\ln\frac{1}{2}}{1000} &= m \\
y(1) &= 100e^m = 99.93070929904525\text{g} \\
y(500) &= 100e^{500m} = 70.71067811865476\text{g}
\end{aligned}$$

6

Suppose that the population of a small town is initially 5000. Due to the construction of an interstate highway, the population doubles over the next year. If the rate of growth is proportional to the current population, when will the population reach 25,000? What is the population after 5 years?

$$\begin{aligned}
y(0) &= 5000 \\
y(1) &= 10000 \\
y(t) &= y_0 e^{mt} \\
10000 &= 5000 e^{m} \\
\ln 2 &= m \\
25000 &= 5000 e^{t \ln 2} \\
\ln 5 &= t \ln 2 \\
t &= \frac{\ln 5}{\ln 2} \approx 2.3 \text{years} \\
y(5) &= 5000 e^{5 \ln 2} = 5000 * 2^5 = 160000 \text{people}
\end{aligned}$$