1. Let $g_n = n\chi_{[0,\frac{1}{n}]}$. Prove that for every x and $\epsilon > 0$ there is an $N \geq 0$ such that $|g_n(x)| < \varepsilon$ for all $n \geq N$. Prove that

$$\int_{[0,1]} g_n \,\mathrm{d}m = 1$$

for all n.

First we look at what happens when $N = n = 0$. Then we have $g_0 = 0\chi_{[0,\frac{1}{0}]}$. We could just define $g_0 = 0$ which is fine I guess, but then observe that $g_1 = 1$ and so we never really want to set $N = 0$ and so we will just say $N > 0$. Notice that $g_n(0) = n$ for all $n > 0$. So clearly, no matter our choice of ε we can find some $n \geq N > \varepsilon$. Let us just be clear and define $x > 0$. So, now that I've changed the problem to what I want it to be, lets restate:

Let $g_n(x) = n\chi_{[0,\frac{1}{n}]}(x)$.

(a) Prove that for every $x > 0$ and $\varepsilon > 0$ there is an $N > 0$ such that $|g_n(x)| < \varepsilon$ for all $n \geq N$.

proof

First we observe that for any $x > 0$ we can find $N > 0$ such that $\frac{1}{N} < x$. Of course then for any $n \geq N$ we have $\frac{1}{n} \leq \frac{1}{N} < x$. And because $\frac{1}{n} < x$ then $x \notin [0, \frac{1}{n}]$ and so $g_n(x) = 0 < \varepsilon$ and we are done. $\square$

(b) Prove that

$$\int_{[0,1]} g_n \,\mathrm{d}m = 1$$

for all n.

proof

By definition we have that

$$\begin{aligned}\int_{[0,1]} g_n \,\mathrm{d}m &= n \cdot m * \left[0, \frac{1}{n}\right] + 0 \cdot m * \left(\frac{1}{n}, 1\right] \\ &= n \cdot \frac{1}{n} + 0 \cdot \left(1 - \frac{1}{n}\right) \\ &= 1\end{aligned}$$

A little light on words but not so heavy on math either, so maybe it's okay. $\square$

2. Prove that ψ is simple if and only if $a\psi$ is simple for every $a \in \mathbb{R}$

proof

Let's assume that ψ is simple. Then by definition $\psi = \{\alpha_1, \dots, \alpha_n\}$. We say that $E_i = \psi^{-1}(\{a_i\})$ and then $\psi = \sum\limits_{i=1}^{n} \alpha_i \chi_{E_i}$. We note that all our E_i's are disjoint ($E_i \cap E_j = \emptyset \forall i \neq j$).

Now then $a\psi = a \sum\limits_{i=1}^{n} \alpha_i \chi_{E_i} = \sum\limits_{i=1}^{n} a\alpha_i \chi_{E_i}$.

Furthermore, because our E_i's are disjoint, then $a\psi(E_i) = a\alpha_i$. And so $a\psi = \{a\alpha_1, \dots, a\alpha_n\}$. Which is the definition of simple.

Now if we assume that $a\psi$ is simple, then we have a nearly identical argument. Let $a\psi = \{\alpha_1, \dots, \alpha_n\}$ and for $E_i = (a\psi)^{-1}(\{a_i\})$ then $a\psi = \sum\limits_{i=1}^{n} \alpha_i \chi_{E_i}$. Now $\psi = \frac{1}{a} a\psi = \frac{1}{a} \sum\limits_{i=1}^{n} \alpha_i \chi_{E_i} = \sum\limits_{i=1}^{n} \frac{1}{a} \alpha_i \chi_{E_i}$. Again we note that our E_i's are disjoint and so $\psi = \{\frac{1}{a}\alpha_1, \dots, a\alpha_n\}$. Then ψ fits the definition of simple, and we are done. $\square$

References

None