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HW 20
CASE 1. $\alpha_2\beta_1 - \alpha_1\beta_2 + \beta_1\beta_2 \neq 0$
Given the problem:

$$\begin{array}{llll} \text{PDE} & \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + f(x,t) & 0 < x < 1, & 0 < t < \infty \\ \text{BC} & g_1(t) = \alpha_1 \dfrac{\partial u}{\partial x}(0,t) + \beta_1 u(0,t) & & 0 < t < \infty \\ & g_2(t) = \alpha_2 \dfrac{\partial u}{\partial x}(1,t) + \beta_2 u(1,t) & {\alpha_1}^2 + {\beta_1}^2 \neq 0 & {\alpha_2}^2 + {\beta_2}^2 \neq 0 \\ \text{IC} & u(x,0) = \phi(x) & 0 < x < 1 & \end{array}$$

Introduce the change of variables

- $u(x,t) = w(x,t) + a(t)x + b(t)(1-x)$

where $a(t), b(t)$ are to be determined so that $w(x,t)$ satisfies the homogeneous BC:

$$\begin{array}{lll} \text{BC} & \alpha_1 \dfrac{\partial w}{\partial x}(0,t) + \beta_1 w(0,t) = 0 & 0 < t < \infty \\ & \alpha_2 \dfrac{\partial w}{\partial x}(1,t) + \beta_2 w(1,t) = 0 & \end{array}$$

(a) Assuming $a(t), b(t)$ can be found so that $w(x,t)$ satisfies homogeneous BC, give the resulting PDE and IC for $w(x,t)$. (State it in terms of $a(t), b(t)$ - solving for them is done next.)

$$\text{PDE} \qquad \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(x,t) \qquad 0 < x < 1, \quad 0 < t < \infty$$

$$\frac{\partial}{\partial t}\left(w(x,t) + a(t)x + b(t)(1-x)\right) = \frac{\partial^2}{\partial x^2}\left(w(x,t) + a(t)x + b(t)(1-x)\right) + f(x,t)$$

$$\frac{\partial w}{\partial t} + x\frac{\mathrm{d}a}{\mathrm{d}t} + (1-x)\frac{\mathrm{d}b}{\mathrm{d}t} = \frac{\partial^2 w}{\partial x^2} + \underbrace{\frac{\partial^2}{\partial x^2}(a(t)x)}_{\to 0} + \underbrace{\frac{\partial^2}{\partial x^2}(b(t)(1-x))}_{\to 0} + f(x,t)$$

$$\frac{\partial w}{\partial t} + x\frac{\mathrm{d}a}{\mathrm{d}t} + (1-x)\frac{\mathrm{d}b}{\mathrm{d}t} = \frac{\partial^2 w}{\partial x^2} + f(x,t)$$

$$\text{IC} \qquad u(x,0) = \phi(x) \qquad 0 < x < 1$$

$$\phi(x) = w(x,0) + a(0)x + b(0)(1-x)$$

(b) Show that homogeneous BC for $w(x,t)$ can be achieved (that is, a solution for $a(t), b(t)$ can be found) for *arbitrary* functions $g_1(t), g_2(t)$ in the original problem if and only if $\alpha_2\beta_1 - \alpha_1\beta_2 + \beta_1\beta_2 \neq 0$

$$g_1(t) = \alpha_1 \frac{\partial u}{\partial x}(0,t) + \beta_1 u(0,t)$$

$$g_2(t) = \alpha_2 \frac{\partial u}{\partial x}(1,t) + \beta_2 u(1,t)$$

$$g_1(t) = \alpha_1 \frac{\partial}{\partial x}\left(w(0,t) + a(t)\cdot 0 + b(t)(1-0)\right) + \beta_1\left(w(0,t) + a(t)\cdot 0 + b(t)(1-0)\right)$$

$$= \alpha_1 \frac{\partial w}{\partial x}(0,t) + \beta_1 w(0,t) + \beta_1 b(t)$$

$$g_1(t) - \beta_1 b(t) = \alpha_1 \frac{\partial w}{\partial x}(0,t) + \beta_1 w(0,t) = 0$$

$$
\begin{aligned}
g_2(t) &= \alpha_2 \frac{\partial}{\partial x}\left(w(1,t) + a(t)\cdot 1 + b(t)(1-1)\right) + \beta_2\left(w(1,t) + a(t)\cdot 1 + b(t)(1-1)\right) \\
&= \alpha_2 \frac{\partial w}{\partial x}(1,t) + \beta_2 w(1,t) + \beta_2 a(t) \\
g_2(t) - \beta_2 a(t) &= \alpha_2 \frac{\partial w}{\partial x}(1,t) + \beta_2 w(1,t) = 0
\end{aligned}
$$

(c) Assuming $\alpha_2\beta_1 - \alpha_1\beta_2 + \beta_1\beta_2 \neq 0$, give the solution for $a(t), b(t)$ in terms of $g_1(t), g_2(t)$.

$$
\begin{aligned}
g_1(t) &= \beta_1 b(t) \\
\frac{1}{\beta_1} g_1(t) &= b(t) \\
g_2(t) &= \beta_2 a(t) \\
\frac{1}{\beta_2} g_2(t) &= a(t)
\end{aligned}
$$