1. If f_n and g_n are sequences of continuous functions converging uniformly to f and g respectively, prove that $f_n g_n$ converges uniformly to fg.

   First we note that functions like $f_n(x) = g_n(x) = x + \frac{1}{n}$ are problematic. I only have "continuous" added to the homework problem, but I am going to assume that these functions have a bounded domain S.

   We know that there exists some M such that $\sup f \leq M - 1$ and $\sup g \leq M - 1$. Now lets choose some $\varepsilon > 0$. Then we can find some $N \in \mathbb{N}$ such that $||f_n - f||_\infty < \frac{\varepsilon}{2M}$ and $||g_n - g||_\infty < \frac{\varepsilon}{2M}$ and $\sup ||f_n|| \leq M$ and $\sup ||g_n|| \leq M$.

   Now then we make the following calculations for all $n \geq N$:

$$\begin{aligned}
||f_n g_n - fg||_\infty &= ||f_n g_n - f g_n - fg + f g_n||_\infty \\
&= ||g_n(f_n - f) + f(g_n - g)||_\infty \\
&\leq ||g_n||_\infty ||f_n - f||_\infty + ||f||_\infty ||g_n - g||_\infty \\
&< M\frac{\varepsilon}{2M} + M\frac{\varepsilon}{2M} = \varepsilon
\end{aligned}$$

   Well it looks like $\lim\limits_{n\to\infty} ||f_n g_n - fg||_\infty = 0$ and so the product is uniformly convergent.

2. For which values of $x \geq 1$ does the expression $x^{x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}}$ make sense?

   Hint: Define $f_1(x) = x$ and $f_{n+1}(x) = x^{f_n(x)}$ for $n \geq 1$. Then

   (a) Show that $f_{n+1}(x) \geq f_n(x)$ for all $n \geq 1$.
   First we note that $x \geq 1$ and so $x^x \geq x$. In other words $f_2 \geq f_1$. Next we assume that $x^{f_n} \geq f_n \geq f_1 \geq 1$. It follows then that $x^{x^{f_n}} \geq x^{f_n}$. And so $f_{n+1}(x) \geq f_n(x)$

   (b) When $L(x) = \lim\limits_{n\to\infty} f_n(x)$ exists, find optimal upper bounds for x and L.
   If the limit exists, then raising x to the limit will give us the limit again. That is $L = x^L$. Now we are looking for optimal values, so lets solve and find the derivative:

$$\begin{aligned}
L &= x^L \\
x &= L^{1/L} \\
\ln x &= \ln L^{1/L} \\
\frac{\mathrm{d}x}{\mathrm{d}L}\frac{1}{x} &= \frac{1}{L^2} + \left(-\frac{1}{L^2}\right)\ln L \\
\frac{\mathrm{d}x}{\mathrm{d}L} &= x\frac{1}{L^2}(1 - \ln L) \\
&= \frac{L^{1/L}}{L^2}(1 - \ln L)
\end{aligned}$$

$$0 = L^{1/L-2}(1 - \ln L)$$

Now because $L \geq f_n \geq f_1 \geq 1$ we know $L^{1/L-2} > 0$ and so $1 - \ln L = 0$ or $L = e$ and $x = e^{1/e}$.

(c) For these values of x, show by induction that $f_n(x)$ is bounded above by e for all $n \geq 1$.
So obviously $e^{1/e} < e$ and so if $x \leq e^{1/e}$ then $f_1 < e$. Now then if we assume that $f_n < e$ and $x \leq e^{1/e}$ then $f_{n+1} = x^{f_n} < \left(e^{1/e}\right)^e = e$

(d) What happens for larger x?
We found that $\frac{\mathrm{d}x}{\mathrm{d}L} = 0$ when $x = e^{1/e}$. This means that $L(x)$ goes vertical at this point (it's inverse goes horizontal). And so when $x > e^{1/e}$ then $L(x)$ is not finite and f_n diverges.

References

1. http://en.wikipedia.org/wiki/Tetration
2. https://www.khanacademy.org/math/differential-calculus/taking-derivatives/derivatives-inverse-functions/v/calculus-derivative-of-x-x-x