ON THE MARKOV BASIS

JON ALLEN

Abstract. In this article will study Markov bases. We will see how Markov bases are used in algebraic statistics and algebraic geometry. First we will start by looking at the relationship between Markov bases and numerical semigroups.

1. Introduction

Here we will sketch out what and how markov bases are used and give a heads up of the goodies coming up in the paper.

2. Prerequisites

Lets discuss the various things we will need to know moving forward.

The first few things we need to understand are some common maps that the literature takes for granted.

There is a straightforward map between $\mathbb{N}^n$ and monomials. We choose some $\alpha \in \mathbb{N}^n$ such that $\alpha = (a_1, \ldots, a_n)$. We map this vector to a monomial $x^\alpha = x_1^{a_1} \ldots x_n^{a_n}$. We can do the same thing with binomials. In the binomial case we start with some $\mathbf{z} \in \mathbb{Z}^n$. We define $\mathbf{z} = (z_1, \ldots, z_n)$. We also define $\alpha = (a_1, \ldots, a_n)$ and $\beta = (b_1, \ldots, b_n)$ where

$$a_i = \begin{cases} z_i & z_i \geq 0 \\ 1 & z_i < 0 \end{cases} \qquad\qquad b_i = \begin{cases} 1 & z > 0 \\ -z_i & z_i \leq 0 \end{cases}$$

If we let $x^\alpha = x_1^{a_1} \ldots x_n^{a_i}$ and $y^\beta = y_1^{b_1} \ldots y_n^{b_i}$ then we can map $\mathbf{z}$ to a binomial $x^\alpha - y^\beta$ (citation needed)

What are Markov bases? We are given a definition of *Markov basis* by [3].

Definition 1. Let $\mathcal{M}_A$ be the log-linear model associated with a matrix A whose integer kernel we denote by $\ker_{\mathbb{Z}}(A)$. A finite subset $\mathcal{B} \subset \ker_{\mathbb{Z}}(A)$ is a *Markov basis* for $\mathcal{M}_A$ if for all $u \in \mathcal{T}(n)$ and all pairs $v, v' \in \mathcal{F}(u)$ there exists a sequence $u_1, \ldots, u_L \in \mathcal{B}$ such that

$$v' = v + \sum_{k=1}^{L} u_k \text{ and } v + \sum_{k=1}^{l} u_k \geq 0 \text{ for all } l = 1, \ldots, L.$$

The literature often refers to the elements of a Markov basis as *moves*[3, p.16]

These bases are relevant because of the fundamental theorem of Markov bases which follows[2]

Theorem 1. *A finite set of moves $\mathcal{B}$ is a Markov basis for A if and only if the set of binomials $\{p^{\mathbf{z}^+} - p^{\mathbf{z}^-} | \mathbf{z} \in \mathcal{B}\}$ generates the toric ideal I_A.*

We will be dealing extensively with lattices. Informally, a lattice is what you think it is. For example $\mathbb{N}^n$ forms a lattice over $\mathbb{R}^n$.

Definition 2. A *lattice* is a partially ordered set in which every two elements have a unique least upper bound and a unique greatest lower bound. (citation needed)

Example 1. *Notice that* $(1, 2)$ *and* $(2, 1)$ *have a lower bound of* $(1, 1)$ *and an upper bound of* $(2, 2)$*.*

Example 2. *We can form a lattice if we order* $\mathbb{N}$ *by division. The least common multiple forms a least upper bound and an greatest lower bound is formed by the greatest common denominator.*

3. Numerical Semigroups

This is the section where we will go over what use Markov bases are to numerical semigroups.

4. And now for something completely different

And this is the section where, time willing, I will explore something that hasn't been looked at much up until now.

References

1. 4ti2 team, *4ti2—a software package for algebraic, geometric and combinatorial problems on linear spaces*, Available at www.4ti2.de.
2. S. Aoki, H. Hara, and A. Takemura, *Markov bases in algebraic statistics*, Springer Series in Statistics, Springer New York, 2012.
3. Mathias Drton, Bernd Sturmfels, and Seth Sullivant, *Lectures on algebraic statistics*, 2008.
4. J.C. Rosales and P.A. García-Sánchez, *Numerical semigroups*, Developments in Mathematics, Springer New York, 2009.