Notes

August 29, 2014

given a set S contained in R bounded abouve, the supremum of S or least upper bound is a real number L such that

1. for all x in S, $x \leq L$
2. if there is a number M such that $x \leq M$ for all $x \in S$ then $L \leq M$

approximation property of the supremum theorem

let S be a subset of R, $S \neq 0$ bounded above, Let $b = \sup(S)$ then for all $a < b, \exists x \in S$ such that $a < x \leq b$

proof

if there is no x in S such that $a < x \leq b$ then a is an upper bound for S contradicting that b is the least upper bound$\square$

2.3.3 Least upper bound principle

tenth axiom from yesterday

every non-empty set of real numbers that is bounded above has a supremum

proof

required because book defines real numbers as decimal expansions, not axiomatic definition

1. observation: this is equivalent to proving that any non-empty set of reals that is bbounded below has an infimum. Why? homework: let S be a set of Reals, let -S=$\{-x : x \in S\}$. Then you will have to prove that sup(-S)=-(inf S)

   so we will prove infimum statement

S is bounded below. let m be a lower bound. $m = m_0.m_1m_2m_3m_4m_5m_5...$ wher $m_0 \in \mathbb{Z}, m_0 > 0$ without loss of generality and $m_i, i > 0$ is 0-9 digit. clearly m_0 is also a lower bound for S.

Consider all integers that are lower bounds for S, (there is at least m_0). Take the biggest of such integers (n_0).

n_0 is a lower bound for S, but $n_0 + 1$ is not. we build the infimum with $n_0.\_\_\_\_\_$. Now pick the gretest ineger n_1 such that $n_0 + \frac{n_1}{10}$ is a lower bound for S. Since n_0 is a lower bound, $0 \leq n_1$. Since $n_0 + 1$ is not a lower bound, $n_1 < 10$

Now pick the reatest integer n_2 such that $n_0 + \frac{n_1}{10} + \frac{n_2}{100}$ is still a lower bound for S. Claim $n_0.n_1n_2n_3n_4...$ is inf(S)$\square$

properties of the supremum

let A, B be subset of $\mathbb{R}$, nonempty, let $C = \{a + b : a \in A, b \in B\}$ if A, B have a supremum, then so does $A + B$ and sup$(A + B)$=sup$A$+sup$B$

proof

let $z \in C$, then $z = a + b$, where $a \in A$, $b \in B$

let $L_1 = \text{sup}A, L_2 = \text{sup}B$ then $a \leq L_1, b \leq L_2$ and then $z \leq L_1 + L_2$ for all $z \in C$. This shows that $L_1 + L_2$ is an upper bound for C. choose $\epsilon > 0$ and $x \in A, y \in B$ such that $L_1 - \epsilon < x, L_2 - \epsilon$ by important property of sup.

$L_1 + L_2 - 2\epsilon < x + x \leq L_1 + L_2$, $x + y \in C$, since for all tilde $\epsilon > 0$ there exists $z \in C$ such that $L_1 + L_2 - \epsilon < z \leq L_1 + L_2$ so $L_1 + L_2$ is the supremum of C

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Let S, T be subsets of R, nonempty, bounded above. if for all s in S and t in T, s is less than or equal to t then supremum of S is less than or equal to supremum of T (exercise)

proposition

$\mathbb{Z}^+$ is unbounded above.

proof

$\mathbb{Z}^+$ is a subset of R nonempty, if Z+ were bounded above it would have a supremum m. by the important property of supremum there exists som x in Z+ such that m-1 is less than x is less than or equal to m, but then m is less than x+1 which is in Z+ so we have a contradiction

corollary

for all x in R there exists an n in Z+ such that x is less than or equal to n.

proposition

archimedean property of R. page 12

for all x greater than 0, y in R there exists some n in Z+ such that nx is greater than y.

proof

apply previous corollary, with x replaced by $\frac{y}{x}\square$

defintion of absolut value

also on page 12 $|x| = \{x, x \geq 0, -x, x < 0$

properties

1. if $a \geq 0, |x| \leq a$ iff $-a \leq x \leq a$
2. for all $x, y \in \mathbb{R}, |x + y| \leq |x| + |y|$ (triangle inequality)
3. same as above but with more than two numbers
4. reverse triangle $||a| - |b|| \leq |a - b|$

5. $|xy| = |x|\,|y|,\ \left|x^{-1}\right| = |x|^{-1}$

proof 1

assume $|x| \leq a$
cases

1. $x \geq 0$ then $|x| = x$ so $0 \leq x \leq a$ since $x \geq 0$ and $-a < 0, -a \leq x$ so $-a \leq x \leq a$

2. $x < 0$ then $|x| = -x \leq a$ so $x \geq -a$ since $x < 0$ and $a > 0$ $x \leq a$ so $-a \leq x \leq a\square$

on the way back
assume $-a \leq x \leq a$ if $x \geq 0$ $x = |x|$ hence $-a \leq |x| \leq a$ then in particular $|x| \leq a$
if $x < 0$

cauchy-schwartz inequality

for every $a_k, b_k \in \mathbb{R}$

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \leq \left(\sum_{k=1}^{n} {a_k}^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{n} {b_k}^2\right)^{\frac{1}{2}}$$

with equality iff $\exists x \in \mathbb{R}$ such that $a_k x + b_k = 0$ for all $k = 1, ..., n$

proof

$$\sum_{k=1}^{n} (a_k x + b_k)^2 \geq 0, \forall x \in \mathbb{R}$$

$Ax^2 + Bx + C \geq 0,\ A = \sum_{k=1}^{n} a_k^2$