

Notes

October 10, 2014

if $\liminf x_n = L$ then there exists $\{x_{n_k}\}$ such that $\lim x_{n_k} = L$

$$l = \liminf x_n = \lim(\underbrace{\inf\{x_{n_1}, x_{n_2}, x_{n_3}, \dots\}}_{c_n})$$

why not just let c_n be the subsequence? because c_n may not be equal to any of the x_k in the sequence

$c_n = \inf\{x_{n_1}, x_{n_2}, \dots\}$ give $\varepsilon = 2^{-n}$ there exists $x_{n_k} \in \{x_{n_1}, x_{n+1}, x_{n+2}, \dots\}$ such that $|c_n - x_{n_k}| < 2^{-n}$
by def of infimum

we have a sequence $\{c_n\}$ given $\varepsilon > 0$ there exists N such that $|c_n - L| < \varepsilon$ if $n \geq N$. we approximate each c_n by some x_{n_k} from the original sequence such that

convergence test for series

first we talk about series with positive terms $\sum_{k=1}^{\infty} a_k$, $s_n = \sum_{k=1}^n a_k$. So if s_n is bounded above then the series is convergent. and if not, it is divergent.

geometric series $\sum_{n=0}^{\infty} r^n$ is convergent if $|r| < 1$. $s_n = \sum_{k=0}^n nr^k = 1 + r + r^2 + \dots + r^n$, $rs_n = r + r^2 + r^3 + \dots$, $s_n - rs_n = 1 - r^{n+1}$
 $s_n = \frac{1-r^{n+1}}{1-r} \rightarrow \frac{1}{1-r}$

comparison test

if $\forall n, |a_n| \leq b_n$

- if $\sum_{n=1}^{\infty} b_n$ is convergent then $\sum_{n=1}^{\infty} a_n$ is convergent,
- if $\sum a_n$ is divergent, so is $\sum b_n$.

3.2.b

show that if $(|a_n|)_{n=1}^{\infty}$ is summable then so is $(a_n)_{n=1}^{\infty}$.

$$\sum_{k=n+1}^m |a_k| < \varepsilon \text{ for all } N \leq n \leq m \text{ because } (|a_n|)_{n=1}^{\infty} \text{ is summable}$$
$$\left| \sum_{k=n+1}^m a_k \right| \leq \sum_{k=n+1}^m |a_k| < \varepsilon$$

so then $\sum a_k$ is also cauchy and summable

cauchy-schwartz inequality

$$\sum_{k=1}^n a_k b_k \leq \left(\sum_{k=1}^n a_k^2 \right)^{1/2} \left(\sum_{k=1}^n b_k^2 \right)^{1/2}$$

3.2.f

leibniz test for alternating series

if $\{a_n\}$ is a monotone decreasing sequence of positive terms with the $\lim a_n = 0$ then $\sum_{n=1}^{\infty} (-1)^n a_n$ is convergent

note!

a sequence may have the property $\lim |a_n - a_{n+1}| = 0$ but not be cauchy

3.2.h

Show that if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with $b_n \geq 0$ such that $\limsup_{n \rightarrow \infty} \frac{a_n}{b_n} < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then the series $\sum_{n=1}^{\infty} a_n$ converges.

$$\begin{aligned} \left| \left(\sup_{k \geq n} \frac{|a_k|}{b_k} \right) - L \right| &< \varepsilon \\ \left(\sup_{k \geq n} \frac{|a_k|}{b_k} \right) &< L + \varepsilon \\ \frac{|a_k|}{b_k} &< L\varepsilon \\ |a_k| &< (L + \varepsilon)b_k \end{aligned}$$

3.2.j

$$\liminf \frac{a_{n+1}}{a_n} \leq \liminf a_n^{\frac{1}{n}} \leq \limsup a_n^{\frac{1}{n}} \leq \limsup \frac{a_{n+1}}{a_n}.$$

step 1

if $x \geq r$ for all $r > b$ then x is a lower bound for the set $\{r \in \mathbb{R} : r > b\}$, $x \leq \inf\{r \in \mathbb{R} : r > b\} = b$

we will show that if $\limsup \frac{a_n}{b_n} < r$ then $\limsup a_n^{\frac{1}{n}} \leq r$ and then apply step one.

let $r > \limsup \frac{a_{n+1}}{a_n}$ then $\exists N$ such that $r > \frac{a_{n+1}}{a_n} \forall n \geq N$

$$\begin{aligned} a_{N+1} &< r a_N \\ a_{N+2} &< r a_{N+1} \leq r^2 a_N \\ a_{N+K} &< r^K a_N \\ a_{N+K}^{\frac{1}{N+K}} &< (r^K a_N)^{\frac{1}{N+K}} \end{aligned}$$

quiz from 10/1/2014

$L_k \rightarrow L$ then $\{x_n\}$ such that $\forall k, \exists$ a subsequence of $\{x_n\}$ converging to L_k . prove that $\{x_n\}$ has a subsequence converging to L .

given $\varepsilon > 0 \exists N_0$ such that $|L_k - L| < \varepsilon$ if $k \geq N_0$
 $|x_{N_k} - L| \leq |x_{N_k} - L_k| + |L_k - L| < 2\varepsilon$

example

let $A, B \subseteq \mathbb{R}$, prove that $\sup A \leq \inf B$, if $\forall a \in A, b \in B, a \leq b$

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