Notes

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if $\liminf x_n = L$ then there exists $\{x_{n_k}\}$ such that $\lim x_{n_k} = L$
$l = \liminf x_n = \lim(inf\{\underbrace{x_{n_1}, x_{n_2}, x_{n_3}, \dots}_{c_n}\}$

why not just let c_n be the subseuence? because c_n may not be equal to any of the x_k in the sequence

$c_n = \inf\{x_{n_1}, x_{n_2}, \dots\}$ give $\varepsilon = 2^{-n}$ there exists $x_{n_k} \in \{x_{n_1}, x_{n+1}, x_{n+2}, \dots\}$ such that $|c_n - x_{n_k}| < 2^{-n}$ by def of infinum

we has a sequence $\{c_n\}$ given $\varepsilon > 0$ there exists N such that $|c_n - L| < \varepsilon$ if $n \geq N$. we approximate each c_n by some x_{n_k} from the oricinal sequence sutch that ....

convergence test for series

first we talk about series with positive terms $\sum\limits_{k=1}^{\infty} a_k$, $s_n = \sum\limits_{k=1}^{n} a_k$. So if s_n is bounded about then the series is convergent. and if not, it is divergent.

geometric series $\sum\limits_{n=0}^{\infty} r^n$ is convergent if $|r| < 1$. $s_n = \sum\limits_{k=0} nr^k = 1 + r + r^2 + \cdots + r^n, rs_n = r + r^2 + r^3 + \dots, sn - rSn = 1 - r^{n+1}$

$s_n = \frac{1-r^{n+1}}{1-r} \to \frac{1}{1-r}$

comparison test

if $\forall n, |a_n| \leq b_n$

- if $\sum\limits_{n=1}^{\infty} b_n$ is convergent then $\sum\limits_{n=1}^{\infty} a_n$ is convergent,
- if $\sum a_n$ is divergent, so is $\sum b_n$.

3.2.b

show that if $(|a_n|)_{n=1}^{\infty}$ is summable then so is $(a_n)_{n=1}^{\infty}$.

$$\sum_{k=n+1}^{m} |a_k| < \varepsilon \text{ for all } N \leq n \leq m \text{ because is is summable}$$

$$\left|\sum_{k=n+1}^{m} a_k\right| \leq \sum_{k=n+1}^{m} |a_k| < \varepsilon$$

so then $\sum a_k$ is also cauchy and summable

cauchy-schwartz inequality

$\sum_{k=1}^{n} a_k b_k \leq \left(\sum_{k=1}^{n} a_k^2\right)^{1/2} \left(\sum_{k=1}^{n} b_k^2\right)^{1/2}$

3.2.f

leibniz test for alternating series

if $\{a_n\}$ is a monotone decreasing sequence of positive terms with the $\lim a_n = 0$ then $\sum_{n=1}^{\infty} (-1)^n a_n$ is convergent

note!

a sequence my have the property $\lim |a_n - a_{n+!}| = 0$ but not be cauchy

3.2.h

Show that if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with $b_n \geq 0$ such that $\limsup_{n\to\infty} < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then the series $\sum_{n=1}^{\infty} a_n$ converges.

$$\left|\left(\sup_{k\geq n} \frac{|a_k|}{b_k}\right) - L\right| < \varepsilon$$
$$\left(\sup_{k\geq n} \frac{|a_k|}{b_k}\right) < L + \varepsilon$$
$$\frac{|a_k|}{b_k} < L\varepsilon$$
$$|a_k| < (L+\varepsilon)b_k$$

3.2.j

$\liminf \frac{a_n+1}{a_n} \leq \liminf a_n^{\frac{1}{n}} \leq \limsup a_n^{\frac{1}{n}} \leq \limsup \frac{a_n+1}{a_n}$.

step 1

if $x \geq r$ for all $r > b$ then x is a lower bound for the set $\{r \in \mathbb{R} : r > b\}$, $x \leq \inf\{r \in \mathbb{R} : r > b\} = b$

we will show that if $\limsup \frac{a_n}{b_n} < r$ then $\limsup a_n^{\frac{1}{n}} \leq r$ and then apply step one.

let $r > \limsup \frac{a_{n+1}}{a_n}$ then $\exists N$ such that $r > \frac{a_{n+!}}{a_n} \forall n \geq N$

$$a_{N+1} < ra_N$$
$$a_{N+2} < ra_{N+1} \leq r^2 a_N$$
$$a_{N+K} < r^k a_N$$
$$a_{N+k}^{\frac{1}{N+k}} < (r^k a_N)^{\frac{1}{N+k}}$$

quiz from 10/1/2014

$L_k \to L$ then $\{x_n\}$ such that $\forall k, \exists$ a subsequence of $\{x_n\}$ converging to L_k. prove that $\{x_n\}$ has a subsequence converging to L.

given $\varepsilon > 0 \exists N_0$ such that $|L_k - L| < \varepsilon$ if $k \geq N_0$

$|x_{N_k} - L| \leq |x_{N_k} - L_k| + |L_k - L| < 2\varepsilon$

example

let $A, B \subseteq \mathbb{R}$, prove that $\sup A \leq \inf B$, if $\forall a \in A, b \in B, a \leq b$

3.3.5

any rearrangement of an absolutely convergent series converges tothe same limit

proof

let $\sum a_n = L < \infty$. We know $\sum |a_n|$ is convergent (not necessarily to L). by th cauchy riterion for series $\forall \varepsilon > 0 \exists N$ such that $\left(\sum_{n=N+1}^{\infty} |a_n| \right) < \varepsilon$

$\pi : \mathbb{N} \to \mathbb{N}$ is bijective, the rearranged series is $\sum_{n=1}^{\infty} a_{\pi(n)}$ and $\{a_1 \dots a_N\} \subseteq \{a_{\pi(1)1} \dots a_{\pi(M)}\}$

3.3.7 rearrangement theorem

let $\sum a_n = L < \infty$ and define $b_n = (a_n \geq 0)?a_n : 0$ and $c_n = (a_n < 0)?a_n : 0$

consider the series $\sum b_n$ and $\sum |c_n|$

case 1

both convergent

$\sum |a_n| = \sum b_n + \sum |c_n|$ which is convergent, which contradicts the fact that a_n is conditionally convergent

case 2

one convergent, one divergent

assume $\sum |c_n| = A < \infty$ and $\sum b_n$ is divergent to $+\infty$

given any $R \in \mathbb{N}$ big, $\exists N$ such that $\sum_{n=1}^{N} b_n > R + A$, then we pick M big enough so that $\{b_1, \dots, b_N\} \subseteq \{a_1, a_2, \dots, a_M\}$ and $\sum_{n=1}^{M} a_n \geq \sum_{n=1}^{N} b_n - \sum |c_n| > R$ so $\sum a_n$ is divergent, which is a contradiction.

case 3

both divergent

chapter 4

$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n), x_i \in \mathbb{R}\}$, vector space (or point in n-space).
with the coordinate wise sum and the product by real numbers (scalars).

$$\begin{aligned}
(x_1, \ldots x_n) + (y_1, \ldots, y_n) &= (x_1 + y_1, \ldots, x_n + y_n) \\
\lambda(x_1, \ldots, x_n) &= (\lambda x_1, \ldots, \lambda x_n) \\
x^{\rightarrow} &= (x_1, \ldots, x_n) = x
\end{aligned}$$

euclidean norm

$$||x|| = \sqrt{x_1^2 + \cdots + x_n^2}$$

distance from x to y

$$||x - y||$$

cauchy-schwarz

$\left|\sum_{i=1}^{n} a_j b_j\right| \leq \left(\sum_{i=1}^{n} a_j^2\right)^{1/2} \left(\sum_{i=1}^{n} b_j^2\right)^{1/2}$
$|a \cdot b| \leq ||a||||b||$

dot product

$a \cdot b = \sum a_i b_i$

triangle inequality

$$||x + y|| \leq ||x||| + ||y||$$

proof

$$\begin{aligned}
||x + y||^2 &= \sum (x_i + y_i)^2 \\
&= (x + y) \cdot (x + y) \\
&= x \cdot x + 2x \cdot y + y \cdot y \\
&= ||x||^2 + 2x \cdot y + ||y||^2 \\
&\leq ||x||^2 + 2||x||||y|| + ||y||^2
\end{aligned}$$
$$= (||x|| + ||y||)^2$$

standard orthogonal base of $\mathbb{R}^n$

$$\begin{aligned}
e_1 &=< 1, 0, \ldots, 0 > \\
e_2 &=< 0, 1, \ldots, 0 > \\
&\vdots \\
e_n &=< 0.0, \ldots, 1 >
\end{aligned}$$

4.2 convergence in $\mathbb{R}^n$

definition: a sequence $\{x^i\}$ of parts in $\mathbb{R}^n$ converge to $c \in \mathbb{R}^n$ if $\forall \varepsilon > 0 \exists N = N(\varepsilon) \in \mathbb{N}$, such that $||x^i - c|| < \epsilon$ if $i \geq N$ we say $\lim x^i = c$.

4.2.2 lemma

$\lim x^i = a$ if and only if $\lim ||x^i - a|| = 0$.

4.2.3 lemma

$\lim x^i = a$ if and only if $\forall j = 1, \ldots, n, \lim x^i_j = a_j$