Notes

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5.7 monotone functions

a function $f : \mathbb{R} \to \mathbb{R}$ is monotonic (strictly) increasing (decreasing) if $\forall x < y, f(x) \leq f(y)$ changing $f(x)$ and $f(y)$ relation as necessary

note $\mathbb{R}^n$ has no natural order, so this is only defined in $\mathbb{R}$

proposition 5.7.2

if $f : (a, b) \to \mathbb{R}$ is increasing then $\alpha = \lim_{x\to a^+} f(x)$ and $\lim_{x\to b^-} f(x)$ exists and $\forall x \in (a, b)$ we have $\alpha \leq f(x) \leq \beta$. and every element in (a, b) has a left and right limit

proof

let $c \in (a, b)$. let $F : \{f(x) : x \in (a, c)\}$. because f is increasing and $x < c$, then $f(x) \leq f(c) \forall x \in (a, c)$. therefore $f(c)$ is an upper bound for F. so F has a supremum L. also $L \leq f(c)$ because $f(c)$ is an upper bound and L is the least upper bound. since $L - \varepsilon$ is not an upper bound for F then $\exists y \in (a, c)$ such that $L - \varepsilon \leq f(y) \leq L$. take $\varepsilon = \frac{1}{n}$, for each $\varepsilon = \frac{1}{n}$, a corresponding $y_n \in (a, c)$. $L - \frac{1}{n} < f(y_n) \leq L \to |f(y_n) - L| < \frac{1}{n}$ and so the limit exists.

corollary

monotonic functions have only jump disccontinuities. and the number of these continuities is countable

proof

by 5.7.2 if f has a discontinuity at a point $c \in (a, b)$ since $\lim_{x\to c^+} f(x)$ and $\lim_{x\to c^-} f(x)$ exists, they must be different. thus the discontinuity is a jump discontinuity. (if the limits were equal, then it would be continuous at that point)

let c be a point where f has a discontinuity. then wlog $f(x)$ is increasing. $\gamma_1 \lim_{x\to c^-} f(x) < \lim_{x\to c^+} = \gamma_2$. to the discontinuity point c we can associate the interval (γ_1, γ_2) which is not in the image of f. if $d \neq c$ is another point of discontinuity, it's corresponding interval (σ_1, σ_2) can not intersect (γ_1, γ_2). if wlog $c \leq d$ then $\gamma_2 = \lim_{x\to c^+} \leq \lim_{x\to d^-} f(x) = \sigma_1$ and so $\gamma_2 < \sigma_1$ and they don't intersect.

let $F : \{\text{discontinuities}\} \to \mathbb{Q}$. then $c \to c' \in (\gamma_1, \gamma_2)$. F is injective because the intervals are disjoint, $|F| \leq |\mathbb{Q}|$

example 5.7.8 cantor function

in general the limit of a sequence of continous functions is not continuous. $f_n(x) = \begin{cases} 0 & x \leq 0 \\ x^n & x \in [0, 1] \\ 1 & x \geq 1 \end{cases}$

in the limt if $x \in [0, 1)$ then $f_n(x) = x^n \to 0$ but $f_n(1) = 1$.