Homework 9

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Section 6.1 # E, F, N, O*, S*

6.1 E. Show that the derivative of an even function is odd, and the derivative of an odd function is even. Recall that a function f is **even** if $f(-x) = f(x)$ and is **odd** if $f(-x) = -f(x)$.

We first note that $f(-x) = f \circ g$ where $g(x) = -x$. Now if we wish to find the derivative of $f \circ g$ then we simply apply the chain rules and we find that the derivative of $f \circ g$ is $(f' \circ g)g'$. We know that g' is -1 but lets just do it for the practice and completeness.

$$g'(x) = \lim_{x \to x_0} g(x) = \lim_{x \to x_0} \frac{-x + x_0}{x - x_0} = \lim_{x \to x_0} -\frac{x - x_0}{x - x_0} = -1$$

And so $\frac{\mathrm{d}}{\mathrm{d}x} f(-x) = -f'(-x)$ Now reversing the composition, we see that the derivative of $-f(x) = g \circ f$ is $(g' \circ f)f'$ and so $\frac{\mathrm{d}}{\mathrm{d}x} -f(x) = -f'(x)$. I don't think we need to do any work to show that $\frac{\mathrm{d}}{\mathrm{d}x} f(x) = f'(x)$. Now putting it all together we see that if $f(-x) = f(x)$ then $-f'(-x) = f'(x)$ or $f'(-x) = -f'(x)$ and so the derivative of an even function is odd. And if $f(-x) = -f(x)$ then $-f'(-x) = -f'(x)$ or $f'(-x) = f'(x)$ and we have that the derivative of an odd function is even.

F. Prove that the product rule for functions f and g on $[a, b]$ that are differentiable at x_0. Hint: $f(x_0 + h)g(x_0 + h) - f(x_0)g(x_0) = (f(x_0 + h) - f(x_0))g(x_0 + h) + f(x_0)(g(x_0 + h) - g(x_0))$

First, because f and g are both differentiable at x_0, then we can rewrite them as follows:

$$\begin{aligned}
f(x_0 + h) &= f(x_0) + f'(x_0 + h)h \\
g(x_0 + h) &= g(x_0) + g'(x_0 + h)h \\
(fg)'(x) &= \lim_{h \to 0} \frac{f(x_0 + h)g(x_0 + h) - f(x_0)g(x_0)}{h} \\
&= \lim_{h \to 0} \frac{(f(x_0 + h) - f(x_0))g(x_0 + h) + f(x_0)(g(x_0 + h) - g(x_0))}{h} \\
&= \lim_{h \to 0} \frac{(f(x_0) + f'(x_0 + h)h - f(x_0))(g(x_0) + g'(x_0 + h)h) + f(x_0)((g(x_0) + g'(x_0 + h)h) - g(x_0))}{h} \\
&= \lim_{h \to 0} \frac{f'(x_0 + h)h(g(x_0) + g'(x_0 + h)h) + f(x_0)g'(x_0 + h)h)}{h} \\
&= \lim_{h \to 0} f'(x_0 + h)g(x_0) + f'(x_0 + h)g'(x_0 + h)h + f(x_0)g'(x_0 + h)) \\
&= f'(x_0)g(x_0) + f(x_0)g'(x_0)
\end{aligned}$$

N. If f is periodic with period T, show that f' is also T-periodic.

We can rewrite this as $f(x) = f(x + \alpha T)$ where $\alpha \in \mathbb{Z}$. Lets define $g(x) = x + \alpha T$. And now we see that $f(x) = (f \circ g)(x)$ and so $f' = (f' \circ g)g'$.

$$g'(x) = \lim_{h \to 0} \frac{g(x_0 + h) - g(x_0)}{h}$$

$$
\begin{aligned}
&= \lim_{h\to 0} \frac{x_0 + h + \alpha T - x_0 - \alpha T}{h} \\
&= 1
\end{aligned}
$$

And so $f'(x) = f' \circ g = f'(x + \alpha T)$, which means that the derivative is also periodic.

O. A function $f(x)$ is asymptotic to a curve $c(x)$ as $x \to +\infty$ if $\lim\limits_{x\to+\infty} |f(x) - c(x)| = 0$.

(a) Show that if $f(x)$ is asymptotic to a line $L(x) = ax + b$ as $x \to +\infty$ then $a = \lim\limits_{x\to+\infty} \frac{f(x)}{x}$ and $b = \lim\limits_{x\to+\infty} f(x) - ax$. (As usual, this includes showing that the limits exist.)

We are given that $\lim\limits_{x\to+\infty} |f(x) - L(x)| = \lim\limits_{x\to+\infty} |f(x) - ax - b| = 0$. That is to say that as for every $\varepsilon > 0$ we can find some M such that for all $x > M$ we have $||f(x) - ax - b| - 0| = |f(x) - ax - b| < \varepsilon$. And so by the definition of limit we see that $\lim\limits_{x\to+\infty} f(x) - ax = b$. This is just a restatement of what we are given, and so I think it is fair to say that this limit exists. Now we need to solve for a.

$$
\begin{aligned}
\lim_{x\to+\infty} f(x) - ax &= b \\
\lim_{x\to+\infty} \frac{1}{x} \cdot \lim_{x\to+\infty} f(x) - ax &= b \cdot \lim_{x\to+\infty} \frac{1}{x} \\
\lim_{x\to+\infty} \frac{f(x)}{x} - a &= \lim_{x\to+\infty} \frac{b}{x} \\
\lim_{x\to+\infty} \frac{f(x)}{x} &= a
\end{aligned}
$$

We know that if we put $\varepsilon = \frac{1}{M}$ then if $x > M$ we have $|\frac{1}{x}| < \frac{1}{M} < \varepsilon$ and so $\lim_{x\to+\infty} \frac{1}{x}$ exists and so the product of these limits exists and so we have our result.

(b) Find all of the asymptotes (including horizontal and vertical ones) for $f(x) = \frac{(x-2)^3}{(x+1)^2}$

$$
\begin{aligned}
a &= \lim_{x\to+\infty} \frac{x^3 - 6x^2 + 12x + 8}{x^3 + 2x^2 + x} \\
&= \frac{x^3\left(1 - \frac{6}{x} + \frac{12}{x^2} + \frac{8}{x^3}\right)}{x^3\left(1 + \frac{2}{x} + \frac{1}{x^2}\right)} \\
&= 1 \\
b &= \lim_{x\to+\infty} \frac{x^3 - 6x^2 + 12x + 8}{x^2 + 2x + 1} - x \\
&= \lim_{x\to+\infty} \frac{x^3 - 6x^2 + 12x + 8 - x^3 - 2x^2 - x}{x^2 + 2x + 1} \\
&= \lim_{x\to+\infty} \frac{-8x^2 + 11x + 8}{x^2 + 2x + 1} \\
&= -8
\end{aligned}
$$

Note also that the limit as x approaches -1 is negative infinity. So we have a vertical asymptote at $x = -1$ and another asymptote at $y = x - 8$

S. (a) Suppose that g is continuous at $x = 0$. Prove that $f(x) = xg(x)$ is differentiable at $x = 0$.

We know that $\lim\limits_{x\to 0} g(x) = g(0)$. Now lets check for differentiability

$$
\lim_{h\to 0} \frac{(0+h)g(0+h) - 0g(0)}{h} = \lim_{h\to 0} g(0+h) = \lim_{(x-0)\to 0} g(0 + x - 0) = \lim_{x\to 0} g(x)
$$

Now we know that this limit exists because $g(x)$ is continuous at $x = 0$ and so the function is differentiable at $x = 0$

(b) Conversely, suppose that $f(0) = 0$ and f is differentiable at $x = 0$. Prove that there is a function g that is continuos at $x = 0$ and satisfies $f(x) = xg(x)$.
We know from corollary 6.1.4 that there exists a function $g(x)$ which is continuous as 0 such that $f(x) = f(0) + g(x)(x - 0) = xg(x)$