Homework 11

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8.1 D
8.2 D,F,H

8.1 D. Does the sequence $f_n(x) = \dfrac{x}{1+nx^2}$ converge uniformly on $\mathbb{R}$?

It's pretty obvious that $f_n(x)$ is continuous and $f_n(x)$ approaches 0 as x gets really large and really close to 0 and gets close to 0 as n gets large.

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x}\left(f_n(x)\right) &= \frac{1}{1+nx^2} - \frac{x}{(1+nx^2)^2}(2xn) \\
&= \frac{1+nx^2-2nx^2}{(1+nx^2)^2} \\
0 &= \frac{1-nx^2}{(1+nx^2)^2} \\
&= 1-nx^2 \\
x^2 &= \frac{1}{n} \\
x &= \pm\frac{1}{\sqrt{n}}
\end{aligned}$$

Because f_n is continuous and approaches 0 at 0 and infinity then $\pm\frac{1}{\sqrt{n}}$ must be minimum/maximum points and then

$$\begin{aligned}
\lim_{n\to\infty} ||f_n - f||_\infty &= \lim_{n\to\infty} ||f_n - 0||_\infty \\
&= \lim_{n\to\infty}\left(\frac{\frac{1}{\sqrt{n}}}{1+n\frac{1}{\sqrt{n}^2}}\right) \\
&= \lim_{n\to\infty}\left(\frac{1}{2\sqrt{n}}\right) \\
&= 0
\end{aligned}$$

Looks like it converges uniformly.

8.2 D. Let (f_n) and (g_n) be sequences of continuous functions on $[a,b]$. Suppose that (f_n) converges uniformly to f and (g_n) converges uniformly to g on $[a,b]$. Prove that f_ng_n converges uniformly to fg on $[a,b]$.

First we note that from the extreme value theorem, f_n and g both have attain a max and min in $[a,b]$ which means said critical points are finites and so we know that if we multiply $||f_n||_\infty$ or $||g||_\infty$ by zero we will get zero.

$$||f_ng_n - fg||_\infty = ||f_ng_n - f_ng + f_ng - fg||_\infty$$

$$\leq ||f_ng_n - f_ng||_\infty + ||f_ng - fg||_\infty$$
$$= |||f_n||_\infty||g_n - g||_\infty + ||g||_\infty||f_n - f||_\infty$$
$$= 0$$

And so f_ng_n converges uniformly.

F. Let $f_n(x) = \arctan(nx)/\sqrt{n}$.

(a) Find $f(x) = \lim_{n\to\infty} f_n(x)$, and show that (f_n) converges uniformly to f on R

We know that $\tan(x)$ has vertical asymptotes at $\pm\frac{\pi}{2}$ and so $\arctan(nx)/\sqrt{n}$ must have horizontal asymptotes at $\pm\frac{\pi}{2\sqrt{n}}$ (the n next to the x is just a horizontal contraction, not important). And so the limit of f_n as $n \to \infty$ is 0 from the $\sqrt{n}$ term on the bottom of the asymptote. Now $\lim_{n\to\infty} ||f_n - f||_\infty = \lim_{n\to\infty} ||f_n||_\infty = \lim_{n\to\infty} \frac{\pi}{2\sqrt{n}} = 0$ and because f_n is bounded then it is uniformly convergent.

(b) Compute $\lim_{n\to\infty} f'_n(x)$, and compare this with $f'(x)$

$\lim f'_n(x) = \lim \dfrac{n}{\sqrt{n}((nx)^2+1)} = \lim \dfrac{\sqrt{n}}{n^2x^2+1} = \lim \dfrac{1}{n^{(3/2)}(x^2+1/n^2)} = 0$ if $x \neq 0$. Obviously f' is also zero.

(c) Where is the convergence of f'_n uniform? Prove your answer.

For uniform convergence, we need the derivative to be bounded. If $x^2 \geq 1$ then the derivative is less than one for all n, and therefore bounded and fits the condition for theorem 8.1.4. So it is uniformally continuous on $[1, \infty)$ and $(-\infty, -1]$.

H. Suppose that f_n in $C[0, 1]$ all have Lipschitz constant L. Show that if (f_n) converges pointwise to f, then the convergence is uniform and f is Lischitz with constant L.

First off we know that $|f_n(x) - f_n(y)| \leq L|x - y|$. We also know that $\lim_{k\to\infty} f_k(x) = f(x)$ Note that because the function is Lipschitz then