Notes

February 12, 2014

lesson 10

outside of class

sine transform

$$\begin{cases} \mathcal{F}_s[f] = F(\omega) = \dfrac{2}{\pi}\displaystyle\int_0^\infty \sin(\omega x) f(x)\,\mathrm{d}x \\ {\mathcal{F}_s}^{-1}[F] = f(t) = \displaystyle\int_0^\infty \sin \omega x) F(\omega)\,\mathrm{d}\omega \end{cases}$$

$$\begin{aligned} U(\omega,t) &= \frac{2A}{\pi\omega}\left(1 - e^{-\alpha^2\omega^2 t}\right) && \leftarrow \text{sine transform of u(x,t)} \\ u(x,t) &= A\ \mathit{erfc}\left(\frac{x}{2\alpha\sqrt{t}}\right) && \leftarrow \text{look up in a table} \end{aligned}$$

complementary error function

$$\begin{array}{rccc} PDE & u_t = \alpha^2 u_{xx} & 0 < x < \infty, & 0 < t < \infty \\ BC & u(0,t) = A & & 0 < t < \infty \\ IC & u(x,0) = 0 & 0 < x < \infty & \end{array}$$

converted PDE in x, t to ODE in t (with ω as a parameter)

now the outside of class bit

we want to find

$$\begin{aligned} u(x,t) &= \int_0^\infty \sin(\omega x) U(\omega,t)\,\mathrm{d}\omega \\ &= \int_0^\infty \sin(\omega x)\frac{2A}{\pi\omega}\left(1 - e^{-\alpha^2 t\omega^2}\right)\,\mathrm{d}\omega \end{aligned}$$

two pieces

$$I_1(x) - \int_0^\infty \frac{\sin(\omega x)}{\omega}\,\mathrm{d}w \qquad\qquad I_2 = \int_0^\infty \frac{\sin(\omega x)}{\omega} e^{-\alpha^2 t\omega^2}\,\mathrm{d}\omega$$

note:

$$\frac{\sin(\omega x}{\omega} e^{-\alpha^2 t\omega^2} \to x \text{ as } \omega \to 0$$

$$\int_0^\infty \frac{\sin(\omega x}{\omega}\,\mathrm{d}x = -\frac{1}{x}\cos(\omega x)\frac{1}{\omega}\,|_1^\infty \; + \int_1^\infty \frac{1}{x}\cos(\omega x)\frac{-\mathrm{d}\omega}{\omega^2}$$
$$\left|\frac{\sin(\omega)}{\omega}\right| \leq \frac{1}{\omega} \qquad \left|\frac{\cos(\omega x}{\omega^2}\right| \leq \frac{1}{\omega^2}$$

very slow convergence, and oscillating, difficult to do numerically. approaches from calc 2:

1) elementary antiderivatives - NO
2) series expansion - get answers that are infinite series
3) convert to a differential equation and solve that
4) contour integration (in complex analysis)

$$I_1(x) = \int\limits_{\omega=0}^{\omega=\infty} \frac{sin(\omega x}{\omega}\,\mathrm{d}\omega = \int_0^\infty \frac{\sin(\omega)}{\omega}\,\mathrm{d}\omega \text{ constant}$$

define

$$\begin{aligned}
f(s) &= \int_0^\infty e^{s\omega}\frac{\sin(\omega}{\omega}\,\mathrm{d}\omega \\
\lim_{s\to 0} f(0) &= I_1(const) \\
f'(s) &= \frac{\mathrm{d}}{\mathrm{d}s}\int_0^i nftye^{-s\omega}\frac{\sin(\omega)}{\omega}\,\mathrm{d}\omega \\
&= \int_0^\infty \frac{\partial}{\partial s}\overbrace{\underbrace{\left(e^{-s\omega}\frac{\sin(\omega)}{\omega}\right)}}\,\mathrm{d}\omega \\
&= -\int_0^\infty e^{-s\omega}\sin(\omega)\,\mathrm{d}\omega \text{ can be done by integrating by parts} \\
&= -\frac{1}{1+s^2} \\
f(s) &= -\arctan(s) + C \\
\lim_{s\to\infty} f(s) &= 0 \\
0 &= c - \lim_{s\to\infty}\arctan(s) \\
c &= \frac{\pi}{2}
\end{aligned}$$

back to the second piece

$$\begin{aligned}
\omega &= \frac{s}{\alpha\sqrt{t}} \\
I_2 &= \int\limits_{s=0}^{\infty} \frac{\sin\left(\frac{sx}{\alpha\sqrt{t}}\right)}{s} e^{-s^2}\,\mathrm{d}s \\
\beta &= \frac{x}{\alpha\sqrt{t}} > 0 \\
I_2(\beta) &= \int_0^\infty \frac{\sin(\beta s}{s} e^{-s^2}\,\mathrm{d}s
\end{aligned}$$