3.2

2. Find Null(A), Row(A), Null(A^T), Col(A) for

   (a)

$$A = \begin{bmatrix} 3 & -1 \\ 6 & -2 \\ -9 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

   So Null(A)=Span((1,3)) and Row(A)=Span((3,-1))

$$A^T = \begin{bmatrix} 3 & 6 & -9 \\ -1 & -2 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

   So Null(A^T)=Span((-2,1,0),(3,0,1)) and Col(A)=Span((1,2,-3))

3. Find Null(A), Row(A), Null(A^T), Col(A) for

   (c)

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

   So Row(A)=Span((1,0,2),(0,1,-1)) and Null(A)=Span((-2,1,1))

$$A^T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 0 \\ 1 & 0 & 1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

   So Col(A)=Span((1,0,1,2),(0,1,0,-1)) and Null(A^T)=Span((-1,0,1,0),(-2,1,0,1))

6. (a) Construct a matrix whose column space contains [1,1,1] and [0,1,1] and whose nullspace contains [1,0,1]and [0,1,0], or explain why none can exist.

   If we know the matrix is 3×3 and because $[0, 1, 0]$ is in the nullspace then the center column must be all zeroes. And because $[1, 0, 1]$ is in the nullspace then the first column is equal to the negative of the last column. So the column space of our vector is Span((1,1,1)). But $[0, 1, 1]$ is not in this form so we cannot construct such a matrix.

   (b) Construct a matrix whose column space contains $[1, 1, 1]$ and $[0, 1, 1]$ and whose nullspace contains $[1, 0, 1, 0]$ and $[1, 0, 0, 1]$, or explain why none can exist.

$$\begin{bmatrix} 1 & 0 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix}$$

7. Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & -1 & -1 \end{bmatrix}$

   (a) Give $\mathbf{C}(A)$ and $\mathbf{C}(B)$. Are they lines, planes or all of $\mathbb{R}^3$?

$$A^T = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B^T = \begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix} \qquad \Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So $\mathbf{C}(A) = \text{Span}([1,1,0],[0,0,1])$ and $\mathbf{C}(B) = \text{Span}([1,-1,0],[0,0,1])$ which are both planes.

(b) Describe $\mathbf{C}(A+B)$ and $\mathbf{C}(A)+\mathbf{C}(B)$. Compare your answers.

This question is confusing to me. It is asking me to compare a set with two sets under a binary operation, but doesn't really define what the operation is. I don't think it's a direct sum. I guess it's either a union, or the sum of any two elements from each set.

First I guess we will figure out what $\mathbf{C}(A+B)$ is.

$$(A+B)^T = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So $\mathbf{C}(A+B) = \text{Span}([1,0,0])$, which is a line.

Now $a[1,1,0] \neq b[1,-1,0] \forall a,b \in \mathbb{R}$ and so $\mathbf{C}(A)$ and $\mathbf{C}(B)$ are not the same planes, but they both contain $[0,0,1]$ and so they are nonparallel and intersecting. So $\mathbf{C}(A) \cup \mathbf{C}B)$ is two nonparallel planes.

Now as we have noted, $[1,1,0]$ and $[1,-1,0]$ are linearly independent. Obviously these are both idependant to $[0,0,1]$ and so these three vectors form a basis for $\mathbb{R}^3$. Thus we can represent any element of $\mathbb{R}^3$ as a linear combination of these vectors, which in turn means that we can represent any $x \in \mathbb{R}^3$ as $x = u + v$ where $u \in \mathbf{C}(A)$ and $v \in \mathbf{C}(B)$.

8. (a) Construct a 3×3 matrix A with $\mathbf{C}(A) \subset \mathbf{N}(A)$.

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(b) Construct a 3×3 matrix A with $\mathbf{N}(A) \subset \mathbf{C}(A)$.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(c) Do you think there can be a 3×3 matrix A with $\mathbf{N}(A) = \mathbf{C}(A)$? Why or why not?

There can't. $\dim \mathbf{N}(A) = 3 - \text{rank } A = 3 - \mathbf{R}(A) = 3 - \mathbf{C}(A)$. And so because 3 is odd then the nullspace and the column space can't have the same dimension, and so can't be the same.

(d) Construct a 4×4 matrix A with $\mathbf{C}(A) = \mathbf{N}(A)$.

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

10. Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. Prove that

(a) $\mathbf{N}(B) \subset \mathbf{N}(AB)$

We take any $\mathbf{x} \in \mathbf{N}(B)$. Then $B\mathbf{x} = \mathbf{0}$ and $AB\mathbf{x} = A\mathbf{0} = \mathbf{0}$. And so $\mathbf{x} \in \mathbf{N}(AB)$

(b) $\mathbf{C}(AB) \subset \mathbf{C}(A)$

We choose some $\mathbf{b} \in \mathbf{C}(AB)$. We know that $\mathbf{b} = (AB)\mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^p$. But then $(AB)\mathbf{x} = A(B\mathbf{x})$ and so $\mathbf{b} \in \mathbf{C}(A)$ also by proposition 2.1 and so we are done.

(c) $\mathbf{N}(B) = \mathbf{N}(AB)$ when A is $n \times n$ and nonsingular

If A is nonsingular than it is invertible. And so if we have $AB\mathbf{x} = \mathbf{0}$ then we also have $A^{-1}AB\mathbf{x} = A^{-1}\mathbf{0}$ or $B\mathbf{x} = \mathbf{0}$. Thus if $\mathbf{x} \in \mathbf{N}(AB)$ then $\mathbf{x} \in \mathbf{N}(B)$. We already did the reverse containment in part a

(d) $\mathbf{C}(AB) = \mathbf{C}(A)$ when B is $n \times n$ and nonsingular

We choose some $\mathbf{b} \in \mathbf{C}(A)$. Then $\mathbf{b} = A\mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^n$. And so $\mathbf{b} = ABB^{-1} = AB(B^{-1}x)$. Because $B^{-1}\mathbf{x}$ exists, then $\mathbf{b}$ is in $\mathbf{C}(AB)$. The reverse containment was done above.

11. Let A be an $m \times n$ matrix. Prove that $\mathbf{N}(A^TA) = N(A)$

If $A\mathbf{x} = \mathbf{0}$ then $A^TA\mathbf{x} = A^T\mathbf{0} = \mathbf{0}$ and so $\mathbf{N}(A) \subset \mathbf{N}(A^TA)$. Now from 2.5.15 we know that if $A^TA\mathbf{x} = \mathbf{0}$ then $A\mathbf{x} = \mathbf{0}$ and so $\mathbf{N}(A^TA) \subset \mathbf{N}(A)$.

12. Suppose A and B are $m \times n$ matrices. Prove that $\mathbf{C}(A)$ and $\mathbf{C}(B)$ are orthogonal subspaces of $\mathbb{R}^m$ if and only if $A^TB = O$

If $\mathbf{C}(A)$ and $\mathbf{C}(B)$ are orthogonal subspaces, then $\text{row}_i(A^TB) = \text{row}_i(A^T)B = \text{col}_i(A)B = [\text{col}_i(A) \cdot \text{col}_1(B), \ldots, \text{col}_i(A) \cdot \text{col}_n] = \mathbf{0}$. Similarly, if they are not orthogonal subspaces, then there must exist some k, l such that $\text{col}_k(A) \cdot \text{col}_l(B) \neq 0$. But then $\text{row}_k(A^T) \cdot \text{col}_l(B) = \text{elem}_k l(A^TB) \neq 0$. And so then $A^TB \neq O$.

13. Suppose A is an $n \times n$ matrix with the property that $A^2 = A$.

(a) Prove that $\mathbf{C}(A) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = A\mathbf{x}\}$.

If $\mathbf{x} = A\mathbf{x}$ then $\mathbf{x} \in \mathbf{C}(A)$ by Theorem 4. Let us choose $\mathbf{b} \in \mathbf{C}(A)$. Then we know that $\mathbf{b} = A\mathbf{x}$ for some $\mathbf{x}$. Now because $A^2 = A$ then $A\mathbf{b} = A^2\mathbf{x} = A\mathbf{x}$. But $A\mathbf{x} = \mathbf{b}$ and so $\mathbf{b} = A\mathbf{b}$. Thus if $\mathbf{b} \in \mathbf{C}(A)$ then $\mathbf{b} \in \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = A\mathbf{x}\}$ and so we have equality.

(b) Prove that $\mathbf{N}(A) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \mathbf{u} - A\mathbf{u} \text{ for some } \mathbf{u} \in \mathbb{R}^n\}$.

If $\mathbf{x} = \mathbf{u} - A\mathbf{u}$ then $A\mathbf{x} = A(\mathbf{u} - A\mathbf{u}) = A\mathbf{u} - A^2\mathbf{u} = A\mathbf{u} - A\mathbf{u} = \mathbf{0}$ and so $\mathbf{x} \in \mathbf{N}(A)$. Now if $\mathbf{x} \in \mathbf{N}(A)$ then $A\mathbf{x} = \mathbf{0}$. Obviously $A\mathbf{0} = \mathbf{0}$ and so $A\mathbf{x} = \mathbf{0} - A\mathbf{0}$ and so $\mathbf{x} \in \{x \in \mathbb{R}^n : \mathbf{x} = \mathbf{u} - A\mathbf{u}\}$. Thus we have equality.

(c) Prove that $\mathbf{C}(A) \cap \mathbf{N}(A) = \{\mathbf{0}\}$.

Let us choose some $\mathbf{b} \in \mathbf{C}(A)$ such that $\mathbf{b} \neq \mathbf{0}$. Now then we know that $\mathbf{b} = A\mathbf{x}$ for some $\mathbf{x}$. And because $A = A^2$ then $A\mathbf{b} = A\mathbf{x}$. But because $\mathbf{b} \neq \mathbf{0}$ then $A\mathbf{x} \neq 0$ and $A\mathbf{b} \neq 0$. Thus $\mathbf{b} \neq 0$ and so $\mathbf{b} \notin \mathbf{N}(A)$. It is obvious that $\mathbf{0}$ is in the span of any set of vectors, including $\mathbf{C}(A)$. Just as obviously $A\mathbf{0} = \mathbf{0}$ and so $\mathbf{0} \in \mathbf{C}(A) \cap \mathbf{N}(A)$. And so we have our proof.

(d) Prove that $\mathbf{C}(A) + \mathbf{N}(A) = \mathbb{R}^n$.

We choose any $\mathbf{x} \in \mathbb{R}^n$. Then $A\mathbf{x} \in \mathbf{C}(A)$ and $\mathbf{x} - A\mathbf{x} \in \mathbf{N}(A)$. And so $\mathbf{x} - A\mathbf{x} + A\mathbf{x} = \mathbf{x}$. Thus any element of $\mathbb{R}^n$ can be written as the sum of elements in $\mathbf{N}(A)$ and $\mathbf{C}(A)$.

2.5

15. Suppose A is an $m \times n$ matrix and $\mathbf{x} \in \mathbb{R}^n$ satisfies $(A^TA)\mathbf{x} = \mathbf{0}$. Prove that $A\mathbf{x} = \mathbf{0}$.

If $A^TA\mathbf{x} = \mathbf{0}$ then $(A^TA\mathbf{x}) \cdot \mathbf{x} = \mathbf{0}$. This leads to $A^T(A\mathbf{x}) \cdot \mathbf{x} = (A\mathbf{x}) \cdot A\mathbf{x} = ||A\mathbf{x}||^2 = 0$. This means that $A\mathbf{x}$ must be $\mathbf{0}$