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# 8.3

In Exercises 115, use Laplace transforms to solve the IVPs. Briefly describe how each equation could be solved using other methods such as undetermined coefficients or variation of parameters.

## #12

$$y''' + y' = 0, \quad y(0) = y'(0) = 0, \quad y''(0) = 1$$

this could be solved by integrating to get $y'' + y = C$ and then proceeding with the method of undetermined coefficients

### solution

$$\begin{aligned}
y''' + y' &= 0 \\
\mathcal{L}\{y(t)\} &= Y(s) \\
\mathcal{L}\{y'''\} + \mathcal{L}\{y'\} &= \mathcal{L}\{0\} \\
s^3\mathcal{L}\{y(t)\} - s^2y(0) - sy'(0) - y''(0) + s\mathcal{L}\{y(t)\} - y(0) &= 0 \\
Y(s)\left(s^3 + s\right) - 1 &= 0 \\
Y(s)\left(s^3 + s\right) &= 1 \\
Y(s) &= \frac{1}{s(s^2+1)} \\
\frac{1}{s(s^2+1)} &= \frac{A}{s} + \frac{B}{s^2+1} \\
1 &= A(s^2+1) + Bs \\
A = 1, \quad s^2 + Bs = 0, \quad B &= -s \\
Y(s) &= \frac{1}{s} - \frac{s}{s^2+1} \\
y(t) &= 1 - \cos t
\end{aligned}$$

## #13

$$y''' - y' = e^t + e^{-t}, \quad y(0) = y'(0) = y''(0) = 0$$

This could be solved essentially the same way that I describe for #10, it's just the right hand side of the equation will be a little more complicated

### solution

$$\begin{aligned}
y''' - y' &= e^t + e^{-t} \\
\mathcal{L}\{y(t)\} &= Y(s) \\
\mathcal{L}\{y'''\} - \mathcal{L}\{y'\} &= \mathcal{L}\{e^t + e^{-t}\} \\
s^3\mathcal{L}\{y(t)\} - s^2y(0) - sy'(0) - y''(0) - s\mathcal{L}\{y(t)\} + y(0) &= \frac{1}{s-1} + \frac{1}{s+1}
\end{aligned}$$

$$
\begin{aligned}
Y(s)(s^3 - s) &= \frac{1}{s-1} + \frac{1}{s+1} \\
Y(s)s(s^2-1) &= \frac{s+1+s-1}{s^2-1} \\
Y(s) &= \frac{2}{(s-1)^2(s+1)^2} \\
\frac{A}{(s-1)} + \frac{B}{(s-1)^2} + \frac{C}{(s+1)} + \frac{D}{(s+1)^2} &= \frac{1}{(s-1)^2(s+1)^2} \\
A(s-1)(s+1)^2 + B(s+1)^2 + C(s+1)(s-1)^2 + D(s-1)^2 &= 1 \\
As^3 + As^2 - As - A + Bs^2 + B2s + B + Cs^3 - Cs^2 - Cs + C + Ds^2 - D2s + D &= 1 \\
s^3(A+C) + s^2(A+B-C+D) + s(-A+2B-C-2D) + (-A+B+C+D) &= 1, \quad A = -C \\
s^2(B-2C+D) + s(2B-2D) + (B+2C+D) &= 1, \quad B = D \\
s^2(2D-2C) + (2D+2C) &= 1, \quad C = D \\
4D &= 1, \quad D = \frac{1}{4} \\
\frac{2}{4}\left(-\frac{1}{s-1} + \frac{1}{(s-1)^2} + \frac{1}{s+1} + \frac{1}{(s+1)^2}\right) &= Y(s) \\
y(t) &= \frac{1}{2}\left(-e^t + te^t + e^{-t} + te^{-t}\right)
\end{aligned}
$$

# 8.5

## #10

find the laplace transform of $h$ where $h(t) = \int_0^t v\cos(t-v)\,\mathrm{d}v$

**solution**

$$
\begin{aligned}
f(t) &= \cos t, \quad g(t) = t \\
\mathcal{L}\{h(t)\} &= \mathcal{L}\{f * g\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} \\
\mathcal{L}\{h(t)\} &= \frac{s}{s^2+1} \cdot \frac{1}{s^2} = \frac{1}{s^3+s}
\end{aligned}
$$

## #18

find the inverse laplace of $\frac{s}{(s^2+1)^2}$

**solution**

$$
\begin{aligned}
\frac{s}{(s^2+1)^2} &= \frac{s}{s^2+1} \cdot \frac{1}{s^2+1} \\
\mathcal{L}\{\sin t\} &= \frac{1}{s^2+1}, \quad \mathcal{L}\{\cos t\} = \frac{s}{s^2+1} \\
\frac{s}{(s^2+1)^2} &= \mathcal{L}\{\cos t\}\mathcal{L}\{\sin t\} = \mathcal{L}\{\cos t * \sin t\} \\
\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} &= \int_0^t \cos(t-v)\sin v\,\mathrm{d}v
\end{aligned}
$$

## #27 extra credit

finish this to get +2 points on exam solve $\frac{\mathrm{d}y}{\mathrm{d}t} - 4y + 4\int_0^t y(v)\,\mathrm{d}v = t^3e^{2t}, y(0) = 0$

### solution

$$
\begin{aligned}
g(t) &= 1 \\
sY(s) - 4Y(s) + 4\frac{1}{s}Y(s) &= \frac{6}{(s-2)^4} \\
Y(s)\left[s - 4 + \frac{4}{s}\right] &= \frac{6}{(s-2)^4} \\
Y(s)\left[\frac{s^2 - 4s + 4}{s}\right] &= \frac{6}{(s-2)^4} \\
Y(s)\left[(s-2)^2\right] &= \frac{6s}{(s-2)^4} \\
Y(s) &= \frac{6s}{(s-2)^6} = \frac{6(s-2)+12}{(s-2)^6} = \frac{6}{(s-2)^5} + \frac{12}{(s-2)^6} \\
&= \frac{1}{4}\cdot\frac{24}{(s-2)^5} + \frac{1}{10}\cdot\frac{120}{(s-2)^6} = \frac{1}{4}\cdot\frac{4!}{(s-2)^5} + \frac{1}{10}\cdot\frac{5!}{(s-2)^6} \\
\mathcal{L}\{e^{at}t^n\} &= \frac{n!}{(s-a)^{n+1}} \\
y(t) &= \frac{1}{4}e^{2t}t^4 + \frac{1}{10}e^{2t}t^5
\end{aligned}
$$