Jon Allen
HW 10
Lesson 7 problem 1. Find general series solution for PDE and BC's.

$$\begin{array}{llll} PDE & u_t = u_{xx} & 0 < x < 1 & 0 < t < \infty \\ BCs & \begin{cases} u(0,t) = 0 \\ u_x(1,t) = 0 \end{cases} & & 0 < t < \infty \\ IC & u(x,0) = x & 0 \le x \le 1 & \end{array}$$

$$\begin{aligned} u(x,t) &= X(x)T(t) & u_t = u_{xx} &= XT' = X''T \\ \frac{XT'}{XT} &= \frac{X''T}{XT} & \frac{T'}{T} &= \frac{X''}{X} = \mu \\ T' - \mu T &= 0 & X'' - \mu X &= 0 \end{aligned}$$

First μ is not positive as that would cause $T(t)$ to grow to infinity and therefore u to grow to infinity which doesn't make physical sense. Let's see if $\mu = 0$

$$\begin{aligned} X' &= 0 & X(x) &= A + Bx \\ u(0,t) &= X(0)T(t) & u_x(1,t) &= X'(1)T(t) \\ &= AT(t) = 0 & &= BT(t) = 0 \\ A &= 0 & B &= 0 \end{aligned}$$

Since this gives only the trivial solution ($u(x,t) = 0$) which is not interesting, we will just assume $\mu < 0$.

$$\begin{aligned} \mu &= -\lambda^2 \\ T' + \lambda^2 T &= 0 & X'' + \lambda^2 X &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\int \lambda^2 \, \mathrm{d}t} T\right) &= e^{\int \lambda^2 \, \mathrm{d}t} T' + \lambda^2 e^{\int \lambda^2 \, \mathrm{d}t} T & r^2 + 0r + \lambda^2 &= 0 \\ e^{\int \lambda^2 \, \mathrm{d}t} T &= \int e^{\int \lambda^2 \, \mathrm{d}t} \cdot 0 \, \mathrm{d}t = A & \frac{-0 \pm \sqrt{0^2 - 4\lambda^2}}{2} &= r \\ T &= Ae^{-\lambda^2 t} & 0 \pm \lambda i &= r \\ & & B\cos(\lambda x) + C\sin(\lambda x) &= X \end{aligned}$$

$$\begin{aligned} u(x,t) = XT &= e^{-\lambda^2 t}\left[A\sin(\lambda x) + B\cos(\lambda x)\right] & \lambda e^{-\lambda^2 t}\left[A\cos(\lambda x) - B\sin(\lambda x)\right] &= u_x \\ u(0,t) &= e^{-\lambda^2 t} B = 0 & \lambda e^{-\lambda^2 t}\left[A\cos(\lambda) - B\sin(\lambda)\right] &= u_x(1,t) = 0 \\ B &= 0 & A\lambda e^{-\lambda^2 t}\cos(\lambda) &= 0 \\ & & \cos(\lambda) &= 0 \\ \lambda_n &= \frac{2n-1}{2}\pi & u(x,t) &= \sum_{n=1}^{\infty} a_n e^{-{\lambda_n}^2 t}\sin(\lambda_n x) \end{aligned}$$