Notes

11 février, 2015

quiz

1. boundedness, continuity, integrable
2. 0

theorem

if $f_n \to f$ uniformly on $[a, b]$ and f_n is integrable (riemann, although it's true for lebesgue also) then so is f and $F_n = \int_x^x f_n(x)\, \mathrm{d}x \to \int_a^x f(x)\, \mathrm{d}x = F(x)$

proof

$|F_n(x) - F(x)| = \int_a^x (f_n - f)\, \mathrm{d}t \leq \int_a^x |f_n - f|\, \mathrm{d}t \leq \int_a^x ||f_n - f||_\infty\, \mathrm{d}t = ||f_n - f||_\infty \int_a^x\, \mathrm{d}t = ||f_n - f||_\infty(x - a) \leq ||f_n - f||_\infty(b - a)$. given $\epsilon > 0$ find N such that $||f_n - f||_\infty < \epsilon/(b-a) \forall n \geq N$ then $|F_n(x) - F(x)| \leq \epsilon/(b-a) \cdot (b-a) = \epsilon$ so $||F_n - F||_\infty \leq \epsilon \forall n \geq N$

corollary

if $f_n \to f$ uniformaly on $[a, b]$ then $\lim\limits_{n\to\infty} \int_a^b f_n(t)\mathrm{d}t = \int_a^b f(t)\mathrm{d}t$

corollary

if $\{f_n\}$ are continuously differentiable on $[a, b]$, $f'_n \to g$ uniformly on $[a, b]$ ant there is $c \in [a, b]$ such that $f_n(c) = \gamma$ for some γ then $f_n \to f$ uniformly with f diff func s.t. $f' = g$ and $f(c) = \gamma$

proof

notice $f_n(x) = f_n(0) + \int_c^x f'_n(t)\mathrm{d}t \to \gamma + \int_c^x g(t)\mathrm{d}t = f(x)$. f is differentiable by the fundamental theorem of calculus on $\int_c^x g(t)\, \mathrm{d}t$. $f_n \to f$ uniformly because $\int_c^x f'_n \to \int_c^x g$ uniformally. $f' = g(f'(x) = 0 + g(x))$ by FTC. and $f(c) = \gamma + \int_c^c g(t)\mathrm{d}t = \gamma$

example

$f_n(x) = \frac{e^{-n^2x^2}}{n}$

1. $f_n \to 0$ uniformly

$$|\frac{e^{-n^2x^2}}{n} - 0|$$

$$
\begin{aligned}
&= |\frac{e^{-n^2x^2}}{n}| \\
&= |\frac{1}{e^{n^2x^2}}\frac{1}{n}| \\
&\leq \frac{1}{n} \to 0
\end{aligned}
$$

2.

$$
f_n' = \frac{-2n^2xe^{-n^2x^2}}{n} = -2nxe^{-n^2x^2}
$$

which is continuously differentiable

3. $\lim\limits_{n\to\infty} -2nxe^{-n^2x^2} = \lim\limits_{n\to\infty} \frac{-2nx}{e^{n^2x^2}}$ and with l'hopitals rule
$\lim_{n\to\infty} \frac{-2n}{2xn^2e^{n^2x^2}} = \frac{-1}{xne^{n^2x^2}} = 0$

4. $f_n' \not\to 0$ uniformly on any neighborhood of 0.

$$
\begin{aligned}
||-2nxe^{-n^2x^2-0} &= ||-2nxe^{-n^2x^2} \\
&= 2nx||e^{-n^2x^2} \\
&< \epsilon \\
||x||||e^{-n^2x^2}|| < \frac{\epsilon}{2n}
\end{aligned}
$$

x close enough to zero

$$
\begin{aligned}
||e^{-n^2x^2}|| \geq \frac{1}{2} \\
|||x||\frac{1}{2} \leq ||x||||e^{-n^2x^2}|| < \frac{\epsilon}{2n} \\
||x|| < \frac{\epsilon}{n}
\end{aligned}
$$

when x is close to zero.

point is that it does depend on x and so not uniform.

does this break our theorem? $f_n(0) \to 0 = f(0)$, $f' = 0 = \lim_{n\to\infty} f_n'$. so we don't need all the hypotheses all the time.

commentary

the whole point is:

derivatives and integrals are limits. integral is a limit of closer and closer approximations. uniform convergence means you can interchange limits. that is pull limits into integrals and derivates. this is what uinformity gives us. switching limits in various contexts.

lebesgue integral

lebesgue integral and limits of functions:

thrm

if $f_n \to f$ uniformally on E then $\int_E f_n \, \mathrm{d}m \to \int_E f \, \mathrm{d}m$

thrm

dominated convergence thm

if $f_n \to f$ pointwise and there is a g with $\int_E g \, \mathrm{d}m > \infty$ with $|f_n(x)| < g(x)$ for all x then $\lim_{n\to\infty} \int_E f_n \, \mathrm{d}m = \int_E f \, \mathrm{d}m$

if we have one function that bounds f_n then you can get away with pointwise limits