Basis and Dimension

1. Basis

Definition 1. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\} \subset \mathbb{R}^n$ be a *list* (as opposed to a set) of vectors.
(1) We say that S is linearly independent if $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_k\mathbf{v}_k = \mathbf{0}$ implies $c_1 = c_2 = ... = c_k = 0$.
(2) We say that S is linearly dependent if there exist $c_1, c_2, ..., c_k \in \mathbb{R}$ not all equal to 0 such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_k\mathbf{v}_k = \mathbf{0}$.

Remarks 2. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\} \subset \mathbb{R}^n$ be a list of vectors.
(1) By *list*, we mean that repitition is allowed. In that case, the set S is linearly dependent. For example, if $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_1\}$, then $1\mathbf{v}_1 + 0\mathbf{v}_2 + (-1)\mathbf{v}_1 = \mathbf{0}$ while not all of the scalars in the relation are 0.
(2) If $\mathbf{0} \in S$, then S is linearly dependent. Indeed, $(1)\mathbf{0} + 0\mathbf{v}_2 + ... + 0\mathbf{v}_k = \mathbf{0}$.

Theorem 3. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\} \subset \mathbb{R}^n$ be a list of vectors. Then S is linearly independent if and only if every $\mathbf{x} \in \mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k)$ can be written uniquely as $\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_k\mathbf{v}_k$.

Proof. $(\Rightarrow)$ Suppose that $\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_k\mathbf{v}_k$ and $\mathbf{x} = d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + ... + d_k\mathbf{v}_k$ for some $c_i, d_i \in \mathbb{R}$. Then

$$\begin{aligned}
\sum_{j=1}^{k} c_j\mathbf{v}_j &= \sum_{j=1}^{k} d_j\mathbf{v}_j \\
\sum_{j=1}^{k} c_j\mathbf{v}_j - \sum_{j=1}^{k} d_j\mathbf{v}_j &= \mathbf{0} \\
\sum_{j=1}^{k} (c_j - d_j)\mathbf{v}_j &= \mathbf{0} \\
(c_1 - d_1) &= 0 \text{ for each } i \leq n \quad (S \text{ is independent}) \\
c_i &= d_i \text{ for each } i \leq n
\end{aligned}$$

$(\Leftarrow)$ Suppose that $\mathbf{0} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_k\mathbf{v}_k$ for some $c_i \in \mathbb{R}$. It is certainly true that $\mathbf{0} = 0\mathbf{v}_1 + 0\mathbf{v}_2 + ... + 0\mathbf{v}_k$. By uniqueness, we have $c_1 = c_2 = ... = c_k = 0$ as needed.

Theorem 4. The list $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ of vectors is linearly independent in $\mathbb{R}^m$ if and only if the nullspace of

$$A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}$$

is trivial.

Proof. $(\Rightarrow)$ Suppose that $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ is linearly independent and let $A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}$ so that $\mathbf{v}_j = \mathbf{c}_j(A)$. We have

$$A \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \mathbf{0}$$

implies

$$x_1\mathbf{c}_1(A) + x_2\mathbf{c}_2(A) + ... + x_n\mathbf{c}_n(A) = \mathbf{0}$$

and so

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + ... + x_n\mathbf{v}_n = \mathbf{0}.$$

Since $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ is linearly independent, it must be the case that

$$x_1 = x_2 = ... = x_k = 0.$$

$(\Leftarrow)$ Suppose that $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + ... + x_k\mathbf{v}_k = \mathbf{0}$. If $A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}$, then

$$A \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \mathbf{0}.$$

Since $\text{Null}(A) = \{\mathbf{0}\}$, it follows that $x_1 = x_2 = ... = x_k = 0$.

Corollary 5. If $n > m$, then $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\} \subset \mathbb{R}^m$ is linearly dependent.

Proof. In the matrix $A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}$, we have $\text{rank}(A) \leq m < n$. By our Fundamental Theorem of Linear Systems, we have that $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions contradicting independence (Theorem 4).

Theorem 6. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\} \subset \mathbb{R}^n$ be a linearly independent set of vectors. The $S \cup \{\mathbf{v}\}$ is linearly independent if and only if $\mathbf{v} \notin \text{Span}(S)$.

Proof. We prove the contrapositive in both directions.

$(\Rightarrow)$ Suppose that $\mathbf{v} \in \text{Span}(S)$. If $\mathbf{v} = \mathbf{0}$, then Remarks 2 asserts that $S \cup \{\mathbf{v}\}$ is linearly dependent. Supose that $\mathbf{v} \neq \mathbf{0}$, and write $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_k\mathbf{v}_k$. Then $(-1)\mathbf{v} + c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_k\mathbf{v}_k = \mathbf{0}$. We have found sclars $d_1, d_2, ..., d_{k+1} \in \mathbb{R}$ not all 0 such that $d_1\mathbf{v} + d_2\mathbf{v}_1 + d_3\mathbf{v}_2 + ... + d_{k+1}\mathbf{v}_k = \mathbf{0}$. Therefore, $S \cup \{\mathbf{v}\}$ is linearly dependent.

$(\Leftarrow)$ If $S \cup \{\mathbf{v}\}$ is linearly dependent, then there exist $d_1, d_2, ..., d_{k+1} \in \mathbb{R}$ not all 0 such that $d_1\mathbf{v} + d_2\mathbf{v}_1 + d_3\mathbf{v}_2 + ... + d_{k+1}\mathbf{v}_k = \mathbf{0}$. If $d_1 = 0$, then we have $d_2\mathbf{v}_1 + d_3\mathbf{v}_2 + ... + d_{k+1}\mathbf{v}_k = \mathbf{0}$. Since $S = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\}$ is linearly independent, each of $d_2, ..., d_{k+1} = 0$ contradicting he fact that $d_1, d_2, ..., d_{k+1}$ are not all 0. Now that we have established the fact that $d_1 \neq 0$, we can write $\mathbf{v} = -\frac{d_2}{d_1}\mathbf{v}_1 - \frac{d_3}{d_1}\mathbf{v}_2 - ... - \frac{d_{k+1}}{d_1}\mathbf{v}_k$. Therefore, $\mathbf{v} \in \text{Span}(S)$.

Definition 7. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\} \subset V \leq \mathbb{R}^n$. We call S a basis for V if
(1) $\text{Span}(S) = V$.
(2) S is linearly independent.

Example 8. We show that

$$S = \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix} \right\}$$

is a basis of the vector space $\mathbb{R}^3$. Choose any $\mathbf{b} = (b_1, b_2, b_3)$. Using Theorem 4, we form the augmented matrix

$$[A \mid \mathbf{b}] = \begin{pmatrix} 1 & 1 & 0 & b_1 \\ 0 & 2 & -3 & b_2 \\ -1 & 1 & 2 & b_3 \end{pmatrix} \Longrightarrow rref[A \mid \mathbf{b}] = \begin{pmatrix} 1 & 0 & 0 & \frac{7b_1-2b_2-3b_3}{10} \\ 0 & 1 & 0 & \frac{3b_1+2b_2+3b_3}{10} \\ 0 & 0 & 1 & \frac{b_1-b_2+b_3}{5} \end{pmatrix}$$

This gives formulas for the unknowns x_i :

$$\begin{aligned} x_1 &= \frac{7b_1 - 2b_2 - 3b_3}{10} \\ x_2 &= \frac{3b_1 + 2b_2 + 3b_3}{10} \\ x_3 &= \frac{b_1 - b_2 + b_3}{5} \end{aligned}$$

and we have that

$$(b_1, b_2, b_3) = \frac{7b_1 - 2b_2 - 3b_3}{10} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \frac{3b_1 + 2b_2 + 3b_3}{10} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \frac{b_1 - b_2 + b_3}{5} \begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix}.$$

For example, if $\mathbf{b} = (-2, 3, 7)$, then

$$\begin{aligned} x_1 &= \frac{7(-2) - 2(3) - 3(7)}{10} = -\frac{41}{10} \\ x_2 &= \frac{3(-2) + 2(3) + 3(7)}{10} = \frac{21}{10} \\ x_3 &= \frac{-2 - 3 + 7}{5} = \frac{2}{5} \end{aligned}$$

and so

$$\mathbf{b} = -\frac{41}{10} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \frac{21}{10} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \frac{2}{5} \begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix}.$$

It is also easy to check linear independence: If $\mathbf{b} = \mathbf{0}$, then $(x_1, x_2, x_3) = (0, 0, 0)$.

Theorem 9. Every nontrivial subspace $V \leq \mathbb{R}^m$ has a basis.

Proof. Since $V \neq \{\mathbf{0}\}$, we can choose a nonzero vector $\mathbf{v}_1 \in V$. Our basic properties of vectors tells us that $S_1 = \{\mathbf{v}_1\}$ is linearly independent. If $\mathrm{Span}(S_1) = V$, then S_1 is a basis and we are done. If not, then $\mathrm{Span}(S_1) \subsetneq V$ and we can choose $\mathbf{v}_2 \in V - \mathrm{Span}(S_1)$. Since S_1 is linearly independent and $\mathbf{v}_2 \notin S_1$, Theorem 5 says that $S_2 = S_1 \cup \{\mathbf{v}_2\}$ is linearly independent. If $\mathrm{Span}(S_2) = V$, we are done, if not, we can choose $\mathbf{v}_3 \in V - \mathrm{Span}(S_2)$ such that $S_3 = S_2 \cup \{\mathbf{v}_3\}$ is linearly independent. We claim that this process stops at or before the mth step. Indeed, if we continue without reaching a basis, we have that $S_{m+1} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, ..., \mathbf{v}_{m+1}\}$ is linearly independent. But Corollary 5 says S_{m+1} must be dependent.

Corollary 10. A square matrix $A \in \mathcal{M}_n$ is nonsingular ($\mathrm{rank}(A) = n$) if and only if the set $\{\mathbf{c}_1(A), \mathbf{c}_2(A), ..., \mathbf{c}_n(A)\}$ of column vectors form a basis for $\mathbb{R}^n$.

Proof. ($\Rightarrow$) Since A is nonsingular, then $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. That is, $\mathrm{Null}(A) = \{\mathbf{0}\}$ and so Theorem 4 implies that $\{\mathbf{c}_1(A), \mathbf{c}_2(A), ..., \mathbf{c}_n(A)\}$ is linearly independent. To see that $\mathrm{Span}\left(\mathbf{c}_1(A), \mathbf{c}_2(A), ..., \mathbf{c}_n(A)\right) = \mathbb{R}^n$, choose any $\mathbf{b} \in \mathbb{R}^n$. Then by our Fundamental Theorem of Linear Systems, the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution, say $\mathbf{x} = (x_1, x_2, ..., x_n)$. But then $\mathbf{b} = A\mathbf{x} = x_1\mathbf{c}_1(A) + x_2\mathbf{c}_2(A) + ... + x_n\mathbf{c}_n(A)$ so that $\mathbf{b} \in \mathrm{Span}\left(\mathbf{c}_1(A), \mathbf{c}_2(A), ..., \mathbf{c}_n(A)\right)$.

($\Leftarrow$) If the set of column vectors $\{\mathbf{c}_1(A), \mathbf{c}_2(A), ..., \mathbf{c}_n(A)\}$ forms a basis of $\mathbb{R}^n$, then it is linearly independent. It follows that from Theorem 4 that $\mathrm{Null}(A) = \{\mathbf{0}\}$. That is, $A\mathbf{x} = \mathbf{0}$ has only the trivial solution which is equivalent to nonsingularity of A.

Corollary 11. If $V \leq \mathbb{R}^n$, then $V^{\perp\perp} = V$.

Proof. By Theorem 8, V has a basis $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m\}$. Build the $m \times n$ matrix

$$A = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_m \end{bmatrix}.$$

Then $\mathrm{Row}(A) = V$ and so $V^\perp = \mathrm{Row}(A)^\perp = \mathrm{Null}(A)$. It follows that $V^{\perp\perp} = \mathrm{Null}(A)^\perp = \mathrm{Row}(A)$. Therefore, $V^{\perp\perp} = V$.

2. Dimension

Lemma 12. Let $V \leq \mathbb{R}^n$ be a subspace and let $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\}$ be a basis. If $l > k$, then any set $\{\mathbf{w}_1, \mathbf{w}_2, ..., \mathbf{w}_l\}$ of vectors in V must be linearly dependent.

Proof. To show that $\{\mathbf{w}_1, \mathbf{w}_2, ..., \mathbf{w}_l\}$ is dependent, itis enough to show that $\mathbf{0}$ has more than one representation in $\mathrm{Span}(\mathbf{w}_1, \mathbf{w}_2, ..., \mathbf{w}_l)$. That is, we must show that there exist $x_1, x_2, ..., x_l \in \mathbb{R}$ not all 0 such that

$$x_1\mathbf{w}_1 + x_2\mathbf{w}_2 + ... + x_l\mathbf{w}_l = \sum_{j=1}^{l} x_j\mathbf{w}_j = \mathbf{0}$$

Since $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\}$ is a basis and $\mathbf{w}_1, \mathbf{w}_2, ..., \mathbf{w}_l \in V$, we can write

$$\begin{aligned}
\mathbf{w}_1 &= a_{11}\mathbf{v}_1 + a_{21}\mathbf{v}_2 + ... + a_{k1}\mathbf{v}_k \\
\mathbf{w}_2 &= a_{12}\mathbf{v}_1 + a_{22}\mathbf{v}_2 + ... + a_{k2}\mathbf{v}_k \\
&\vdots \\
\mathbf{w}_l &= a_{1l}\mathbf{v}_1 + a_{2l}\mathbf{v}_2 + ... + a_{kl}\mathbf{v}_k.
\end{aligned}$$

By a substitution, we have

$$\sum_{j=1}^{l} x_j(a_{1j}\mathbf{v}_1 + a_{2j}\mathbf{v}_2 + ... + a_{kj}\mathbf{v}_k) = \mathbf{0}.$$

The usual vector algebra gives

$$\left(\sum_{j=1}^{l} a_{1j}x_j\right)\mathbf{v}_1 + \left(\sum_{j=1}^{l} a_{2j}x_j\right)\mathbf{v}_2 + ... + \left(\sum_{j=1}^{l} a_{kj}x_j\right)\mathbf{v}_k = \mathbf{0}.$$

Since $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\}$ is a basis, each $\sum_{j=1}^{l} a_{ij}x_j = 0$. Expanding, we have a system of equations

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + ...a_{1l}x_l &= 0 \\
a_{21}x_1 + a_{22}x_2 + ...a_{2l}x_l &= 0 \\
&\vdots \\
a_{k1}x_1 + a_{k2}x_2 + ...a_{kl}x_l &= 0.
\end{aligned}$$

In short, $A\mathbf{x} = \mathbf{0}$ where $\mathrm{rank}(A) = \mathrm{rank}[A \mid \mathbf{0}] \leq k < l$. It follows that $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution and so not all of the x_js are 0.

Theorem 13. If B and C are bases of a subspace V. Then $|B| = |C|$.

Proof. The contrapositive of Lemma 12 says that if B is a basis and C is linearly independent, then $|C| \leq |B|$ (contrapositive). Since C isa basis, it is indeed independent and therefore, $|C| \leq |B|$. Similarly, $|B| \leq |C|$ and so $|B| = |C|$.

Definition 14. Let B be a basis for the subspace $V \leq \mathbb{R}^n$. The *dimension* of V is the positive integer $\dim(V) = |B|$. That is, $\dim(V)$ is the number of basis vectors for V.

Corollary 15. If $W \leq V \leq \mathbb{R}^n$ and $\dim(W) = \dim(V)$, then $W = V$.

Proof. Let $\dim(W) = k$ and write down a basis $B = \{\mathbf{w}_1, \mathbf{w}_2, ..., \mathbf{w}_k\}$. We claim that B is a basis for V. The set iscertainly linearly independent (anywhere in $\mathbb{R}^n$). If B is not a basis, then $\mathrm{Span}(\mathbf{w}_1, \mathbf{w}_2, ..., \mathbf{w}_k) \subsetneq V$. Choose $\mathbf{v} \in V - \mathrm{Span}(\mathbf{w}_1, \mathbf{w}_2, ..., \mathbf{w}_k)$. By Theorem 6, we have that $\{\mathbf{w}_1, \mathbf{w}_2, ..., \mathbf{w}_k, \mathbf{v}\} \subset V$ is linearly independent. But Lemma 12 says that $k+1$ many vectors in a k-dimensional spacemust be dependent. A contradiction is reached.

The Four Subspaces Again

Remarks 16. We will refer to the following matrix for the results that follow. We are assuming WOLOG that the leading entries occur in consecutive columns.

$$rref(A)\mathbf{x} = \begin{bmatrix} 1 & & & & c_{1,r+1} & c_{1,r+2} & \cdots & c_{1n} \\ & 1 & & & c_{2,r+1} & c_{2,r+2} & \cdots & c_{2n} \\ & & \ddots & & \vdots & & & \\ & & & 1 & c_{r,r+1} & c_{r,r+2} & & c_{rn} \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \\ x_{r+1} \\ x_{r+2} \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_r \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

The $n-r$ variables $x_{r+1}, x_{r+2}, ..., x_n$ are free and the r variables $x_1, x_2, ..., x_r$ are dependent. Indeed,

$$\begin{aligned} x_1 &= b_1 - c_{1,r+1}x_{r+1} - c_{1,r+2}x_{r+2} - ... - c_{1n}x_n \\ x_2 &= b_2 - c_{2,r+1}x_{r+1} - c_{2,r+2}x_{r+2} - ... - c_{2n}x_n \\ &\vdots \\ x_r &= b_r - c_{r,r+1}x_{r+1} - c_{r,r+2}x_{r+2} - ... - c_{rn}x_n \end{aligned}$$

Theorem 17. If $A \in M_{m\times n}$, then $\dim(\mathrm{Null}(A)) = n - \mathrm{rank}(A)$.

Proof. If $\mathbf{x} \in \mathrm{Null}(A)$,then as usual,

$$\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_r \\ x_{r+1} \\ x_{r+2} \\ \vdots \\ x_n \end{pmatrix} = x_{r+1} \begin{pmatrix} -c_{1,r+1} \\ \vdots \\ -c_{r,r+1} \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + x_{r+2} \begin{pmatrix} -c_{1,r+2} \\ \vdots \\ -c_{r,r+2} \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + ... + x_n \begin{pmatrix} -c_{1n} \\ \vdots \\ -c_{rn} \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

It follows that

$$\mathrm{Null}(A) = \mathrm{Span}\left(\begin{pmatrix} -c_{1,r+1} \\ \vdots \\ -c_{r,r+1} \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} -c_{1,r+2} \\ \vdots \\ -c_{r,r+2} \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, ..., \begin{pmatrix} -c_{1n} \\ \vdots \\ -c_{rn} \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \right)$$

To conclude, we must show that these $n-r$ many vectors are independent. But if

$$x_{r+1} \begin{pmatrix} -c_{1,r+1} \\ \vdots \\ -c_{r,r+1} \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + x_{r+2} \begin{pmatrix} -c_{1,r+2} \\ \vdots \\ -c_{r,r+2} \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + ... + x_n \begin{pmatrix} -c_{1n} \\ \vdots \\ -c_{rn} \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

then

$$\begin{pmatrix} * \\ * \\ \vdots \\ * \\ x_{r+1} \\ x_{r+2} \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

and so

$$x_{r+1} = x_{r+2} = ... = x_n = 0.$$

Lemma 18. Let $A \in M_{m\times n}$ and $E \in M_m$. If E is invertible, then $\mathrm{Row}(EA) = \mathrm{row}(A)$.

Proof. We have

$$\begin{array}{rl} & \mathbf{r}_i(EA) \\ = & e_{i1}\mathbf{r}_1(A) + e_{i2}\mathbf{r}_2(A) + ... + e_{im}\mathbf{r}_m(A) \\ \in & \mathrm{Span}(\mathbf{r}_1(A), \mathbf{r}_2(A), ..., \mathbf{r}_m(A)). \end{array}$$

It follows that $\mathrm{Span}(\mathbf{r}_1(EA), \mathbf{r}_2(EA), ..., \mathbf{r}_m(EA)) \le \mathrm{Span}(\mathbf{r}_1(A), \mathbf{r}_2(A), ..., \mathbf{r}_m(A))$. That is, $\mathrm{Row}(EA) \le \mathrm{Row}(A)$. If $EA = B$, then $A = E^{-1}B$, and so $\mathrm{Row}(A) = \mathrm{Row}(E^{-1}B) \le \mathrm{Row}(B) = \mathrm{row}(EA)$.

Theorem 19. Let $A \in M_{m\times n}$. Then $\mathrm{Row}(A) = \mathrm{Span}(\mathbf{r}_1(EA), \mathbf{r}_2(EA), ..., \mathbf{r}_r(EA))$ and $\dim(\mathrm{Row}(A)) = \mathrm{rank}(A)$.

Proof. Given A, we reduce to $rref(A)$. That is, the matrix above is just EA where E is a product of elementary matrices. It follows from Lemma 18 that $\mathrm{Row}(A) = \mathrm{Row}(EA)$. But

$$\mathrm{Row}(EA) = \mathrm{Span}\left(\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ c_{1,r+1} \\ \vdots \\ c_{1n} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ c_{2,r+1} \\ \vdots \\ c_{2n} \end{pmatrix}, ..., \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ c_{r,r+1} \\ \vdots \\ c_{rn} \end{pmatrix} \right).$$

It remains to show that these r vectors are linearly independent. But

$$x_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ c_{1,r+1} \\ \vdots \\ c_{1n} \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ c_{2,r+1} \\ \vdots \\ c_{2n} \end{pmatrix} + ... + x_r \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ c_{r,r+1} \\ \vdots \\ c_{rn} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \\ * \\ \vdots \\ * \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

implies

$$x_{r+1} = x_{r+2} = ... = x_n = 0.$$

(as in the proof of Theorem 17).

Lemma 20. Let $A \in M_{m\times n}$ and let $rref(A) = EA$ be the matrix in Remarks 16. If $A' = \left[\begin{array}{ccc} \mathbf{c}_1(A) & \mathbf{c}_2(A) & \quad \mathbf{c}_r(A) \end{array}\right]$, then $EA' = \left[\begin{array}{c} I_r \\ 0 \end{array}\right]$.

Proof. For $j \in \{1, 2, ..., r\}$ we have

$$\begin{aligned} \mathbf{c}_j(EA') &= e_{1j}\mathbf{c}_1(A') + e_{2j}\mathbf{c}_2(A') + ... + e_{rj}\mathbf{c}_r(A') \\ &= e_{1j}\mathbf{c}_1(A) + e_{2j}\mathbf{c}_2(A) + ... + e_{rj}\mathbf{c}_r(A) \\ &= \mathbf{c}_j(EA). \end{aligned}$$

It follows that

$$EA' = \left[\begin{array}{c} I_r \\ 0 \end{array}\right].$$

Theorem 21. Let $A \in M_{m\times n}$. Then $\{\mathbf{c}_1(A), \mathbf{c}_2(A), ..., \mathbf{c}_r(A)\}$ is a basis for $\mathrm{col}(A)$ and $\dim(\mathrm{Col}(A)) = \mathrm{rank}(A)$.

Proof. Suppose that $\alpha_1\mathbf{c}_1(A) + \alpha_2\mathbf{c}_2(A) + ... + \alpha_r\mathbf{c}_r(A) = \mathbf{0}$. Then $A'\mathbf{x} = \mathbf{0}$ has $\mathbf{x} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{pmatrix}$ as a solution. Using the matrix E to bring A to $rref(A)$, we have $E(A'\mathbf{x}) = E\mathbf{0}$. It follows that $EA' = \mathbf{0}$ and hence $\left[\begin{array}{c} I_r \\ 0 \end{array}\right]\mathbf{x} = \mathbf{0}$. Therefore, $\alpha_1 = \alpha_2 = ... = \alpha_r = 0$ and $\{\mathbf{c}_1(A), \mathbf{c}_2(A), ..., \mathbf{c}_r(A)\}$ is a linearly independent set. It remains to show that $\mathrm{Span}(\mathbf{c}_1(A), \mathbf{c}_2(A), ..., \mathbf{c}_r(A)) = \mathrm{Col}(A)$. Since $\mathrm{Col}(A) = \{\mathbf{c}_1(A), \mathbf{c}_2(A), ..., \mathbf{c}_r(A), \mathbf{c}_{r+1}(A), ..., \mathbf{c}_n(A)\}$, it is clear that $\mathrm{Span}(\mathbf{c}_1(A), \mathbf{c}_2(A), ..., \mathbf{c}_r(A)) \subseteq \mathrm{Col}(A)$. For the reverse containment, notice that

$$A \begin{pmatrix} -c_{1,r+1} \\ \vdots \\ -c_{r,r+1} \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \mathbf{0} \quad \text{(Theorem 17)}$$

and so

$$-c_{1,r+1}\mathbf{c}_1(A) - ... - c_{1,r+1}\mathbf{c}_r(A) + \mathbf{c}_{r+1}(A) = \mathbf{0}.$$

Therefore,

$$\mathbf{c}_{r+1}(A) = c_{1,r+1}\mathbf{c}_1(A) + ... + c_{1,r+1}\mathbf{c}_r(A) + \mathbf{c}_{r+1}(A) \in \mathrm{Span}(\mathbf{c}_1(A), \mathbf{c}_2(A), ..., \mathbf{c}_r(A)).$$

Similarly,

$$\mathbf{c}_j(A) \in \mathrm{Span}(\mathbf{c}_1(A), \mathbf{c}_2(A), ..., \mathbf{c}_r(A)) \text{ for } j \in \{r+1, r+2, ..., n\}.$$

Exercises
Section 3.3: 3, 5b, 7, 9, 10, 11, 14, 15, 21
Section 3.4: 1(b,d), 3(b,e), 8, 11, 17, 18, 19, 21, 22, 24