Stability of Time Discretizations for Semi-discrete High Order Schemes for Time-dependent PDEs

Chi-Wang Shu
Division of Applied Mathematics, Brown University

Typed by Yuelin Li

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1 Abstract

We are interested in solving an evolutionary PDE, such as a hyperbolic conservation law

$$u_t + f(u)_x = 0$$

or a convection-diffusion equation

$$u_t + f(u)_x = (a(u)u_x)_x$$

where $a(u) \geq 0$, as well as the multi-dimensional cases. It is common that We discretize the spatial derivatives to obtain an ODE system

$$\frac{d}{dt}u = L(u) \tag{1}$$

where L is the spatial discretization operator. The spatial discretization could be a finite difference method, a finite element method, a spectral method, etc. For problems with smooth solutions, a linear stability analysis is adequate; for problems with discontinuous solutions, a stronger measure of stability is required. We will discuss several classes of high order time discretization.

2 Objective

We assume that the spatial discertization is stable, the solution of the method of lines ODE (1) satisfies

$$||u(t)|| \leq C(t)||u(0)|| \tag{2}$$

where $C(t)$ is a constant depending on t (stability). The objective is to maintain the strong stability (2) property with a high order accurate time discretization. We will consider both linear and nonlinear problems with three types of high order time discretizations. [12]

3 Literature

The class of high order SSP time discretization method was first developed in [11] and [10], and was called total variation diminishing (TVD) time discretization. It was further studied in [3,4,7–9] for nonlinear SSP Runge-Kutta methods. More extensions of the methods are studied in [2, 5, 6].

4 Discussions

4.1 SSP time discretization

The strong stability preserving (SSP) high order time discretization were developed to ensure strong stability properties for nonlinear problems. The idea was to assume that the first order forward Euler time discretization of the method of lines ODE is strongly under a certain norm, when the time step is retricted. Then try to find a higher order time discretization that maintains strong stability for the same norm. The SSP framework is as follows.

We assume the Euler forward time discretization for the method of lines ODE is strongly stable

$$||u + \Delta t L(u)|| \leq ||u|| \tag{3}$$

for a certain norm, semi-norm or convex functional $||\cdot||$ under a suitable condition $\Delta t \leq \Delta t_0$, then the SSP high order Runge-Kutta time discretization satisfies the strong stability property

$$||u^{n+1}|| \leq ||u^n|| \tag{4}$$

for the same norm, semi-norm or convex functional $||\cdot||$, under a modified condition $\Delta t \leq c\Delta t_0$, $c > 0$. Similar definitions can be made for high order multi-step methods or hybrid multi-step Runge-Kutta methods. Every stage in the SSP Runge-Kutta method is a convex combination of forward Euler operators. For example, a second order SSP Runge-Kutta method is

$$u^{(1)} = u^n + \Delta t L(u^n) \tag{5}$$

$$u^{n+1} = \frac{1}{2}u^n + \frac{1}{2}u^{(1)} + \frac{1}{2}\Delta t L(u^{(1)})$$

If (3) is satisfied under the CFL condition $\Delta t \leq \Delta t_0$, then under the same condition we have

$$||u^{(1)}|| \leq ||u^n||$$

The third order SSP Runge-Kutta method is derived as [11],

$$\begin{aligned} u^{(1)} &= u^n + \Delta t L(u^n) \\ u^{(2)} &= \frac{3}{4}u^n + \frac{1}{4}u^{(1)} + \frac{1}{4}\Delta t L(u^{(1)}) \\ u^{n+1} &= \frac{1}{3}u^n + \frac{2}{3}u^{(2)} + \frac{2}{3}\Delta t L(u^{(2)}) \end{aligned} \tag{6}$$

There is no four-stage, fourth order SSP Runge-Kutta method. The following is five stage, fourth order SSP Runge-Kutta method. We could obtain SSP multi-step methods: a second order SSP three-step method is given by

$$u^{n+1} = \frac{3}{4}u^n + \frac{1}{4}u^{n-2} + \frac{3}{2}\Delta t L(u^n)$$

which is SSP with the SSP coefficient $c = \frac{1}{2}$. Similarly, a third order four-step method is given by

$$u^{n+1} = \frac{16}{27}u^n + \frac{16}{9}\Delta t L(u^n) + \frac{11}{27}u^{n-3} + \frac{4}{9}\Delta t L(u^{n-3})$$

which is SSP with the SSP coefficient $c = \frac{1}{3}$. A practical application of SSP is the recent framework in obtaining positivity-preserving high order discontinuous Galerkin methods or finite volumn schemes for solving Euler equations. [14]

A simple scaling limiter with the SSP Runge-Kutta or multistep time discretization can lead to provably high order positivity-preserving results. A simulation of Mach 2000 astrophysical jet flow can be computed by this method. [12]

Consider

$$||u + \Delta t L(u)|| \leq ||u||$$

$$||u - \Delta t \widetilde{L}(u)|| \leq ||u||$$

The issues related to negative coefficient in front of Δt require introducing an associated operator $\widetilde{L}$ corresponding to stepping backward in time.

4.2 IMEX time discretization

If we are solving a convection-diffusion PDE

$$u_t + f(u)_x = du_{xx} \tag{7}$$

where $d > 0$, by explicit time discretizations such as the SSP methods would require a very small time step. However, a fully implicit method would be costly. IMEX would be useful in this case. The first order IMEX scheme is

$$u^{n+1} = u^n + \Delta t N(u^n) + \Delta t L(u^{n+1}) \tag{8}$$

We have the following stability result:

Proposition 1. There exists a positive constant τ_0 independent of the spatial mesh size h, s.t. if $\Delta t \leq \tau_0$ (the only condition), then the solution of the first order IMEX LDG scheme satisfies the strong stability property $||u^n|| \leq ||u^0||, \forall n$.

The second order IMEX scheme we consider is [1]:

$$u^{(1)} = u^n + \gamma \Delta t N(u^n) + \gamma \Delta t L(u^{(1)})$$

$$u^{n+1} = u^n + \delta \Delta t N(u^n) + (1 - \delta)\Delta t N(u^{(1)}) + (1 - \gamma)\Delta t L(u^{(1)}) + \gamma \Delta t L(u^{n+1})$$

where $\gamma = 1 - \frac{\sqrt{2}}{2}, \delta = 1 - \frac{1}{2\gamma}$. Thus we prove:

Proposition 2. There exists a positive constant τ_0 independent of the spatial mesh size h, s.t. if $\Delta t \le \tau_0$ (the only condition), then the solution of the second order IMEX LDG scheme satisfies the strong stability property. The third order IMEX scheme we consider is

$$u^{(1)} = u^n + \gamma \Delta t N(u^n) + \gamma \Delta t L(u^{(1)})$$

$$u^{(2)} = u^n + (\frac{1+\gamma}{2} - \alpha_1)\Delta t N(u^n) + \alpha_1 \Delta t N(u^{(1)})) +$$
$$\frac{1-\gamma}{2}\Delta t L(u^{(1)}) + \gamma \Delta t L(u^{(2)})$$
$$u^{(3)} = u^n + (1-\alpha_2)N(u^{(1)}) + \alpha_2 N(u^{(2)})) +$$
$$\beta_1 \Delta t L(u^{(1)}) + \beta_2 \Delta t L(u^{(2)}) + \gamma \Delta t L(u^{(3)})$$

$$u^{n+1} = u^n + \beta_1 \Delta t N(u^{(1)}) + \beta_2 \Delta t N(u^{(2)}) + \gamma \Delta t N(u^{(3)})$$
$$+\beta_1 \Delta t L(u^{(1)}) + \beta_2 \Delta t L(u^{(2)}) + \gamma \Delta t L(u^{(3)})$$

where γ is the middle root of $6x^3 - 18x^2 + 9x - 1 = 0$, $\gamma = 0.4359$, $\beta_1 = -\frac{3}{2}\gamma^2 + 4\gamma - \frac{1}{4}, \beta_2 = \frac{3}{2}\gamma^2 - 5\gamma + \frac{5}{4}$. $\alpha_1 = -0.35$ and $\alpha_2 = \frac{\frac{1}{3} - 2\gamma^2 - 2\beta_2\alpha_1\gamma}{\gamma(1-\gamma)}$. Then We prove the following proposition:

Proposition 3. There exists a positive constant τ_0 independent of the spatial mesh size h, s.t. if $\Delta t \le \tau_0$, then the solution of the third order IMEX LDG scheme satisfies the strong stability property. [12] If both the convection and the diffusion terms are nonlinear, then we write the PDE as

$$u_t + f(u)_x - (a(u)u_x)_x + a_0 u_{xx} = a_0 u_{xx}$$

where a_0 is a constant. We treat the left-hand side explicitly and the right-hand side implicitly.

4.3 Strongly stable Runge-Kutta methods for linear systems

The SSP time discretization framework cannot be applied if the Euler forward time discretization is unstable, or only stable under very restrictive CFL time step restrictions. In such cases, we consider high order Runge-Kutta methods directly. Consider autonomous linear ordinary differential equation systems

$$\frac{d}{dt}u = Lu \tag{9}$$

where $u \in \mathbb{R}^N$ and L is an $N \times N$ real matrix. We make an assumption that if the semidiscrete scheme honors the L^2 stability of the original PDE, then for certain symmetric and positive definite matrix H,

$$L^T H + HL \leq 0 \tag{10}$$

is a semi-negative definite matrix. If (10) holds, then L is semi-negative and the ODE satisfies the energy decay law

$$\frac{d}{dt}||u||_H^2 = \langle \frac{d}{dt}u, u\rangle_H + \langle u, \frac{d}{dt}u\rangle_H \tag{11}$$

$$= \langle Lu, Hu\rangle + \langle u, HLu\rangle = \langle u, (L^T H + HL)u\rangle \leq 0$$

The explicit Runge-Kutta method is strongly stable if $||u^{n+1}||_H \leq ||u^n||_H$ is hold when discretizing the ODE under (10). [12] If the problem is coercive,

$$L^T H + HL \leq -\eta L^T HL \tag{12}$$

for $\eta > 0$, then the Euler forward time discretization is strongly stable under a suitable CFL condition. All SSP high order Runge-Kutta or multistep time discretizations are strongly stable under similar CFL condition. [4] However, there are cases where semi-discrete schemes are not coercive then the framework based on coercive properties does not work.

4.4 General framework for stability analysis

Consider explicit Runge-Kutta time discretization for the linear autonomous system in (1), the scheme,

$$u^{n+1} = R_s u \tag{13}$$

where $R_s = \sum_{k=0}^{s} \alpha_k (\tau L)^k, \alpha_0 = 1, \alpha_s \neq 0$.
For an s-stage method, it is of linear order p if and only if the first p+1 terms in the summation coincide with the truncated taylor series of $e^{\tau L}$. We check if there exists a constant λ such that

$$||R_s^u||_H^2 \leq ||u||_H^2 \tag{14}$$

for all $\tau||L||_H \leq \lambda$ and u. One method is comparing $||R_s^u||_H^2$ with $||u||_H^2$, however, there is difficulty in deciding the sign of the difference term in the expansion. [12]Thus we convert $\langle L^i u, L^j u\rangle_H$ into linear combinations of terms and arrive at

Lemma 4. (Energy equality) Given H and $R_s = \sum_{k=0}^{s} \alpha_k (\tau L)^k$ with $\alpha_0 = 1$. There exists a unique set of coefficients $\beta_k^s{}_{k=0} \cup \gamma_{i,j}^{s-1}{}_{i,j=0}$ such that $\forall u$ and L satisfying $L^T H + HL \leq 0$

$$||R_s^u||_H^2 = \sum_{k=0}^{s} \beta_k \tau^{2k} ||L^k u||_H^2 + \sum_{i,j=0}^{s-1} \gamma_{i,j} \tau^{i+j+1} [L^i u, L^j u]_H \tag{15}$$

5 Conclusion

Upshot: the characterization of strongly stable methods [12]:

Theorem 5. Consider a linear Runge-Kutta method of order p with p stages. i) The method is not strongly stable if $p \equiv 1 \pmod 4$ or $p \equiv 2 \pmod 4$. ii) The method is strongly stable if $p \equiv 3 \pmod 4$.

Theorem 6. An Runge-Kutta method of odd linear order p is strongly stable if and only if

$$(-1)^{\frac{p+1}{2}} (\alpha_{p+1} - \frac{1}{(p+1)!} < 0 \tag{16}$$

Theorem 7. An Runge-Kutta method of even linear order p is strongly stable if

$$(-1)^{\frac{p}{2}+1} (\alpha_{p+2} - \alpha_{p+1} + \frac{1}{p!(p+2)} < 0 \tag{17}$$

$$(-1)^{\frac{p}{2}+1} (\frac{p}{2}!)^2 (\alpha_{p+1} - \frac{1}{(p+1)!} < \epsilon \tag{18}$$

where ϵ is the smallest eigenvalue of the Hilbert matrix of order $\frac{p}{2} + 1$. [13] For method of line numerical schemes for PDE, we have discussed SSP time discretization methods which preserves strong stability of forward Euler or backward Euler first order time discretization, Runge-Kutta, and multi-step methods.

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