Free Boundary Problems

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1 Introduction

The term Free Boundary Problem refers to a problem in which we must solve a partial differential equation and along the way find out the region in which the PDE holds. An unknown boundary demarcates this region and on this border we are given over-determined side condition that are derived from certain physical laws or other constraints governing the phase transition.

$$Lu = 0 \text{ in } \Omega$$
$$u = g \text{ on } \partial\Omega$$

where L is an elliptic, parabolic or hyperbolic operator. The origin of the modern discipline owes much to the famous Stefan problem, named after the physicist Stefan who during the 19th century studied ice formations in the polar seas with a free boundary model. If we fast forward to the present, a large variety of problems are modeled using free boundary methods. Examples arise in flame propagation, image reconstructions, jet flows, optimal stopping problems in financial mathematics, tumor growth (obstacle problem), among many others. The theory of "Free Boundary Problems" by highlighting the main questions/results, literature, and possibly open questions, so as to give a general idea of this research area. As a model example, this talk will use the so-called "Bernoulli" one-phase free boundary problem which appears naturally in two-dimensional fluid-dynamics.

2 Bernoulli Free Boundary Problems

We describe the main question (literature) and open questions for the so-called Bernoulli (one phase) Free Boundary Problems: we seek $u \geq o$ s.t.

$$\begin{aligned} \Delta u = 0 \text{ in } \Omega^+(u) = \{u > 0\} \cap \Omega \\ \|\nabla u\| = 1 \text{ on } F(u) = \partial\Omega^+(u) \cap \Omega \end{aligned} \tag{1}$$

with a given fixed boundary condition.

2.1 Motivation for the one-phase FBP

The motion of a two dimensional incompressible fluid with constant density is governed by the Euler equations (in absence of gravity)

$$v_t + (Dv)v = -\|\nabla p\|$$

$v = (v_1, v_t)$ flow velocity vector, p is pressure. Since $divv = 0$ in 2-d, $v = \|\nabla u\|$ and the function u is known as the stream function. Applying the operator to the equations from the previous part:

$$\begin{aligned} w_t + \nabla wv = 0 \\ w = \Delta u \end{aligned}$$

In particular w is constant along the flow. If there is no vorticity w = 0 i.e.

$$\Delta u = 0$$

Otherwise,

$$\Delta u = r(u)$$

If the flow is steedy, $v_t = 0$, hence multiplying $v_t + (Dv)v = -\nabla_p$ by v^T and using that $v^t(Dv)v = \nabla(|v|^2/2)v$, we get, $\nabla(|v|^2/2 + p)v = 0$ As stated by Bernoulli law, $|v|^2/2 + p$ is constant along the streamlines. In particular on the surface that separates fluid and air, u must be constant, say u=0. If we assume that p = constant on the surface of separation, we get $|\Delta u|^2 =$ constant on the zero.... Bernoulli problem in equation (1) can be formulated variationally, that is as the Euler-Lagrange equation associated to the energy functional

$$J(u) = \int_\Omega (|\nabla u|^2 + \chi_{u>0})dx$$

2.2 1-D Computation

$$J(u) = \int_{-1}^{1} (|u'|^2 + \chi_{u>0})dx$$

If $supp\phi \subset u > 0$, then $\chi_u > 0 = \chi_{u+\epsilon\phi>0}$ hence,

$$0 = \lim_{\epsilon \to 0} \frac{J(u+\epsilon\phi) - J(u)}{\epsilon} = 2\int_{-1}^{1} u'\phi' dx$$

By integration by parts,

$$\int_{-1}^{1} u''\phi dx = 0$$

For all such ϕ i.e. $u'' = 0$ in $u > 0$. Let $u = \alpha x^+$ Find α: If $v = \frac{\alpha}{1-\epsilon}(x-\epsilon)^+$ then $J(u) \leq J(v) \implies \alpha^2 + \epsilon \leq (\frac{\alpha}{1-\epsilon})^2(x-\epsilon)\ \alpha^2(1+\epsilon)$
$\implies \alpha \geq 1$. Similarly, $\alpha \leq 1 \implies \alpha = 1$.

2.3 Existence of Minimizers

$$J(u) = \int_{\Omega} (|\nabla u|^2 + \chi_{u>0})dx$$

The set $K := \{u \in H(\Omega) : u = u_0 \geq 0 \text{ on } \partial\Omega\}$. Existence can be shown via the direct method of the calculus of variations. Weak derivatives: $u_1 v \in L(\Omega)$ $V = D^\alpha u \implies \int_\Omega (D^\alpha \phi dx - (-1)^{|\alpha|} \int_\Omega v\phi dx$ for all $\phi \in C_c^\infty(\Omega)$. v is unique up to a set of measure zero.

$$H^1(\Omega) = W^{1,2}(\Omega) := \{u \in L(\Omega) : \int_\Omega |u|^2 dx, \int_\Omega (|\nabla u|^2 dx < \infty\}$$

with ∇u = weak gradient. $H^1(\Omega)$ is a Hilbert space with inner product: $< f, g >= \int_\Omega f \cdot g + \int_\Omega \nabla f \cdot \nabla g$.

Baby problem: existence of minimizers of $I(u) = \int_\Omega |\nabla u|^2$ in A:= $\{u \in H^1(\Omega) : u = u_0 \text{ on } \partial\Omega, u_0 \in H^1(\Omega)\}$

Proof: let m = inf u, $0 \leq m \leq I(u_0) < \infty$. $\exists u_k : infI(u) = m = lim_k I((u_k) \leq I(u_0)$.

Back to Free Boundary Problem: The main questions include: are minimizers classical solutions? How regular are they? How smooth is the free

boundary? The optimal regularity is Lipschitz. Key regularity properties: Lipschitz and non-degenerate, i.e. $dist(x, F(u)) < u(x) < Cdist(x, F(u))$ near F(u).
Scale invariance...
Blow up analysis: $x_0 \in FB$, u_p rescaling around x_0, equilipschitz $\implies \exists$ a subsequence $u_p \to u_0$ when $p \to 0$. u_0 =global minimizer (by non-degenerator). $F(u_p) \to F(u_0)$ in the hausdorff distance.

Classification of global minimizers: Weiss monotonicity formula: for u, minimizer of F:

$$\phi_u(r) = r^{-u} J(u, B_r) - r^{-u-1} \int_{\partial B_r} u^2$$

is increasing in r. ϕ_u is constant if and only if u is homogeneous of degree 1. Blow-ups are homogeneous of degree 1! $\phi_{uk}(r) = \phi_u(\rho_k r) \to \phi(O)$. For n=2,3,4 the only such homogeneous minimizer is xr and for n=7, example of singular minimizer.
A free boundary that is trapped in... In particular n=2,3,4 $\implies$ FB are $C^{1,\alpha}$. Higher regularity: $C^{1,\alpha} \implies$ Analytic. Partial regularity: GMT, F(u) has finite perimeter, $H^{n-1}(F(u), F^*(u)) = 0$, $F^*(u)$ is $C^{1,\alpha}$, dim $\Sigma \leq n-5$.