Split-step balanced θ-method for SDEs with non-globally Lipschitz continuous coefficients[☆]

Yufen Liu[a], Wanrong Cao[a,*], Yuelin Li[b]

[a]School of Mathematics, Southeast University, Nanjing 210096, P.R.China.
[b]Institute for Social and Economic Research and Policy, Columbia University, New York 10027, USA

Abstract

In this paper, the split-step balanced θ-method (SSBT) has been presented for solving stochastic differential equations (SDEs) under non-global Lipschitz conditions, where $\theta \in [0, 1]$ is a parameter of the scheme. The boundedness and convergence of the numerical solution have been studied in mean-square sense. Furthermore, under some condition it is proved that the SSBT scheme can preserve the exponential mean-square stability of the exact solution when $\theta \in [1/2, 1]$ for every stepsize $h > 0$. Numerical examples verify the theoretical findings.

Key words: nonlinear problems, the balanced method, strong convergence, exponential stability, mean-square contraction
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1. Introduction

Stochastic differential equations (SDEs) have been widely used in many branches of science and industry. The convergence and stability of numerical methods are well studied for SDEs with globally Lipschitz continuous coefficients [1, 16, 22–24, 29, 31].

However, the coefficients of some reliable complex systems in many application fields, such as economics, physics, engineering, etc., are nonlinear and violate the global Lipschitz conditions. When solving such nonlinear systems, explicit schemes generally are neither stable nor convergent [11, 12, 14, 16]. To deal with the problems, some kinds of numerical schemes have been constructed, e.g., the tamed/balanced schemes [3, 13, 27, 32, 34], the truncated schemes [17, 20], the projected explicit Itô-Taylor method [4], etc.

Explicit schemes usually have a simple structure and cheap computational cost, but when solving strong nonlinear/stiff problems, implicit schemes are more efficient than explicit schemes due to less restriction on step-size. The semi-implicit Euler scheme, including the backward Euler scheme in the case of $\theta = 1$, is very popular, and its convergence and stability have been well studied for SDEs under non-global Lipschitz conditions. For example, considering the SDE with one-sided Lipschitz continuous drift coefficient and globally Lipschitz continuous diffusion coefficient [9] and the SDE under the local Lipschitz conditions and monotone condition [19]. In [18] and [33], it was proved that the backward Euler method is almost surely exponential stable for solving some highly

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*Corresponding author. Tel/Fax: +86 25 52090590
Email addresses: liuyf@seu.edu.cn (Yufen Liu), wrcao@seu.edu.cn (Wanrong Cao), yuelin.li@columbia.edu (Yuelin Li)

nonlinear SDEs. Moreover, the convergence and stability of θ-Milstein scheme and double-implicit Milstein schemes are studied in [30, 36]. A fully implicit scheme was also proposed in [28].

Both the balanced scheme and the splitting technique are powerful for solving stiff problems. For SDEs with globally Lipschitz continuous coefficients, some split-step schemes and split-step balanced schemes have been presented, which have shown good stability behavior, see [1, 7, 25, 26, 29]. There are only a few works on split-step schemes for SDEs under non-globally Lipschitz conditions. In [6, 15], convergence of the split-step Euler-type schemes were studied for SDEs under non-global Lipschitz conditions. Higham et al. proved that the backward Euler-Maruyama and the split-step backward Euler-Maruyama can preserve the exponential mean-square stability of nonlinear SDEs [8], and Huang showed that when $\theta \in (\frac{1}{2}, 1]$, the split-step θ scheme could preserve the exponential mean-square stability of exact solution for SDEs with a coupled non-global Lipschitz conditions [10].

In this work, we consider the following autonomous SDE in Itô sense:

$$\begin{cases} dX(t) = f(X(t))dt + g(X(t))dW(t), \quad 0 < t \leq T, \\ X(0) = X_0, \end{cases} \tag{1.1}$$

where $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ and $g : \mathbb{R}^n \mapsto \mathbb{R}^{n\times d}$, $W(t)$ is a d-dimensional standard Brownian motion, $X(0)$ is independent of $W(t)$.

In this work, based on the balanced scheme presented in [28] and the split-step technique, we construct a split-step balanced θ-scheme (SSBT) for the SDE (1.1) with some non-global Lipschitz conditions on coefficients of Eq. (1.1), which are given in Assumption 2.1, and to study its convergence and exponential stability in strong sense. The main contribution is to prove that under some non-global Lipschitz conditions the presented scheme is convergent with order 0.5 in mean-square sense and can preserve the exponential stability of the exact solution. The proposed SSBT scheme has better stability properties than the original counterpart. Moreover, the technique we presented in Lemma 4.2 and Theorem 4.3 can be generalized for proving stability of other schemes. Numerical examples verify the theoretical results and show that the proposed scheme is efficient.

We organize the paper as follows. In Section 2 we give necessary notations and some assumptions on the initial value and coefficients of the SDE (1.1). In Section 3, we present the SSBT scheme and prove that the numerical solution is bounded and convergent in strong sense. The contraction and exponential stability of the SSBT scheme are considered in Section 4. Some numerical examples are given in Section 5 to verify the theoretical findings on both convergence rate and exponential stability of the SSBT scheme in mean-square sense. Finally, we give some concluding remarks in Section 6.

2. Preliminary notations and assumptions

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, where $\mathcal{F}_t,\ t_0 \leq t \leq T$, is a nondecreasing family of σ-subalgebras of $\mathcal{F}$. The coefficients in Eq. (1.1) are Borel measurable. We will use $\langle x, y\rangle$ and $|\cdot|$ to denote the inner product and the Euclidean norm of vectors x, y in n-dimensional Euclidean space $\mathbb{R}^n$, respectively. Also, $|\cdot|$ denotes the norm of matrix in $\mathbb{R}^{n\times d}$. In the rest of the paper, We use a capital letter K to denote dynamic constants.

To study the strong convergence, we adopt the assumptions made in [28] as follows.

Assumption 2.1.

(i) For all $p \geq 1$, the initial condition satisfies

$$\mathbb{E}[|X(0)|^{2p}] \leq K < \infty. \tag{2.1}$$

(ii) For a sufficiently large $k_0 \geq 1$, there is a constant $c_1 \geq 0$ such that $\forall\ x, y \in \mathbb{R}^n$

$$\langle x-y, f(x)-f(y)\rangle + \frac{2k_0-1}{2}|g(x)-g(y)|^2 \leq c_1|x-y|^2. \tag{2.2}$$

(iii) There exist $c_2 \geq 0$ and $\nu \geq 1$ such that $\forall\ x, y \in \mathbb{R}^n$

$$|f(x)-f(y)|^2 \leq c_2(1+|x|^{2\nu-2}+|y|^{2\nu-2})|x-y|^2. \tag{2.3}$$

The following corollary can be derived from Assumption 2.1 easily.

Corollary 2.2. *([28]) Suppose that Assumption 2.1 holds, then there exist constants c_3, c_4, c_5 such that $\forall\ x, y \in \mathbb{R}^n$*

$$\langle x, f(x)\rangle + \frac{2k_0-3}{2}|g(x)|^2 \leq c_3(1+|x|^2), \tag{2.4}$$

$$|f(x)|^2 \leq c_4(1+|x|^{2\nu}), \tag{2.5}$$

$$|g(x)|^2 \leq c_5(1+|x|^{2\nu}). \tag{2.6}$$

Remark 2.3. *([5]) The inequalities (2.1) and (2.4) can guarantee the boundedness of the analytical solution of the SDE (1.1)*

$$\mathbb{E}[|X(t)|^{2p}] \leq K\left(1+\mathbb{E}[|X_0|^{2p}]\right), \qquad 1 \leq p \leq k_0-1. \tag{2.7}$$

3. Boundeness and strong convergence of the SSBT scheme

This section will introduce a split-step balanced θ-scheme (SSBT) for the SDE (1.1) and prove the boundedness and convergence of the numerical solution. Take some integer N and let $h = \frac{T}{N}$. Denote $t_k = kh,\ 0 \leq k \leq N$, X_k to be approximation of $X(t_k)$ for $1 \leq k \leq N$, and $X_0 = X(0)$.

By the balanced scheme given in [28] and the split-step technique, we give the following SSBT scheme of the SDE (1.1)

$$\begin{cases} X_k^* = X_k + \theta f(X_k^*)h, \\ X_{k+1} = X_k + \dfrac{f(X_k^*)h + g(X_k^*)\xi_k\sqrt{h}}{1+|f(X_k^*)|h+|g(X_k^*)\xi_k|\sqrt{h}}, \quad 0 \leq k \leq N-1, \end{cases} \tag{3.1}$$

where ξ_k are d-dimensional vectors and every component of ξ_k is Gaussian $\mathcal{N}(0,1)$ i.i.d random variable.

For convenience, we denote $B(X_k^*, \xi_k) := (1+|f(X_k^*)|h+|g(X_k^*)\xi_k|\sqrt{h})^{-1}$.

Lemma 3.1. *Suppose that Assumption 2.1 holds. For the conditional expectation*

$$A(X_k, X_k^*, \xi_k) := \mathbb{E}[\langle X_k, f(X_k^*)h\rangle\, B(X_k^*, \xi_k) + \frac{2p-1}{2}\left(|f(X_k^*)|^2\, h^2 + |g(X_k^*)\xi_k|^2\, h\right) B(X_k^*, \xi_k)|\mathcal{F}_{t_k}],$$

where $1 \leq p \leq k_0-1$, there exist a constant $K > 0$ independent of h and k, such that

$$A(X_k, X_k^*, \xi_k) \leq Kh\left(1+|X_k^*|^2+|X_k^*|^{3\nu}h^{\frac{1}{2}}\right).$$

Proof Noting that X_k, X_k^* are $\mathcal{F}_{t_k}$-measurable and $B(X_k^*, \xi_k) \le 1$, we have

$$\begin{aligned}
&A(X_k, X_k^*, \xi_k)\\
\le&\mathbb{E}[\langle X_k, f(X_k^*)h\rangle\, B(X_k^*, \xi_k) + \frac{2p-1}{2}\left|g(X_k^*)\right|^2 \xi_k^2 h B(X_k^*, \xi_k)\big|\mathcal{F}_{t_k}] + \frac{2p-1}{2}\left|f(X_k^*)\right|^2 h^2\\
=&\mathbb{E}[(\langle X_k - X_k^*, f(X_k^*)h\rangle + \langle X_k^*, f(X_k^*)h\rangle)\, B(X_k^*, \xi_k) + \frac{2p-1}{2}\left|g(X_k^*)\right|^2 \xi_k^2 h B(X_k^*, \xi_k)\big|\mathcal{F}_{t_k}]\\
&+ \frac{2p-1}{2}\left|f(X_k^*)\right|^2 h^2\\
\le&\mathbb{E}[\theta\left|f(X_k^*)\right|^2 h^2 B(X_k^*, \xi_k) + \left(\langle X_k^*, f(X_k^*)h\rangle + \frac{2p-1}{2}h\left|g(X_k^*)\right|^2\right) B(X_k^*, \xi_k)\\
&+ \frac{2p-1}{2}h\left|g(X_k^*)\right|^2 (\xi_k^2 - 1)B(X_k^*, \xi_k)\big|\mathcal{F}_{t_k}] + \frac{2p-1}{2}\left|f(X_k^*)\right|^2 h^2.
\end{aligned}$$

By using the estimation (2.4), we obtain

$$\begin{aligned}
A(X_k, X_k^*, \xi_k) \le&\frac{2p-1}{2}h\left|g(X_k^*)\right|^2 \mathbb{E}[(\xi_k^2 - 1)B(X_k^*, \xi_k)\big|\mathcal{F}_{t_k}] + \theta\left|f(X_k^*)\right|^2 h^2\\
&+ c_3(1 + |X_k^*|^2)h + \frac{2p-1}{2}\left|f(X_k^*)\right|^2 h^2.
\end{aligned}$$

Since $\mathbb{E}[\xi_k^2 - 1] = 0$ and ξ_k is independent of $\mathcal{F}_{t_k}$, it follows that

$$\begin{aligned}
\mathbb{E}[(\xi_k^2 - 1)B(X_k^*, \xi_k)\big|\mathcal{F}_{t_k}] &= \mathbb{E}[(\xi_k^2 - 1)B(X_k^*, \xi_k) - (\xi_k^2 - 1)\big|\mathcal{F}_{t_k}]\\
&= -\mathbb{E}[(\xi_k^2 - 1)(|f(X_k^*)|h + |g(X_k^*)\xi_k|\sqrt{h})B(X_k^*, \xi_k)\big|\mathcal{F}_{t_k}]\\
&\le \mathbb{E}[|\xi_k^2 - 1|(|f(X_k^*)|h + |g(X_k^*)\xi_k|\sqrt{h})\big|\mathcal{F}_{t_k}]\\
&\le K(|f(X_k^*)|h + |g(X_k^*)|\sqrt{h}).
\end{aligned}$$

According to (2.5) and (2.6), we have

$$\begin{aligned}
A(X_k, X_k^*, \xi_k) &\le Kh + K|X_k^*|^2 h + Kh|g(X_k^*)|^2(|f(X_k^*)|h + \sqrt{h}|g(X_k^*)|) + Kh^2 + K|X_k^*|^{2\nu}h^2\\
&\le Kh(1 + |X_k^*|^2 + |X_k^*|^{2\nu}h + |X_k^*|^{3\nu}h^{\frac{1}{2}})\\
&\le Kh(1 + |X_k^*|^2 + |X_k^*|^{3\nu}h^{\frac{1}{2}}).
\end{aligned}$$

□

Theorem 3.2. *Suppose that Assumption 2.1 holds. If $\theta h < \frac{1}{2c_3}$, then for the SSBT scheme (3.1) we have*

$$\mathbb{E}[|X_k^*|^{2p}] \le K(1 + \mathbb{E}[|X_0|^{2p\beta}]), \quad \mathbb{E}[|X_k|^{2p}] \le K(1 + \mathbb{E}[|X_0|^{2p\beta}]), \quad 0 \le k \le N, \tag{3.2}$$

where $K > 0,\ \beta \ge 1$ are constants independent of $h,\ k,$ and $1 \le p \le \frac{k_0-1}{4(3\nu-2)} - \frac{1}{2}$.

Proof We observe from the SSBT scheme (3.1) that

$$|X_{k+1}| \le |X_k| + 1 \le |X_0| + k + 1. \tag{3.3}$$

Let $L > 0$ be a sufficiently large number. Introduce the sets

$$\Delta_{L,k} := \{\omega : |X_l(\omega)| \le L,\ l = 0, 1, \ldots, k\}$$

and their complements $\widehat{\Delta}_{L,k}$. Denote E is an event set and $\chi_E(\omega) = 1$, when $\omega \in E$; $\chi_E(\omega) = 0$, otherwise.

We first prove the theorem for integer $p \geq 1$. We have

$$\begin{aligned}\mathbb{E}[\chi_{\Delta_{L,k+1}}(\omega)|X_{k+1}|^{2p}] \leq& \mathbb{E}[\chi_{\Delta_{L,k}}(\omega)|X_{k+1}|^{2p}] \\ =& \mathbb{E}[\chi_{\Delta_{L,k}}(\omega)|X_{k+1}-X_k+X_k|^{2p}] \\ \leq& \mathbb{E}[\chi_{\Delta_{L,k}}(\omega)|X_k|^{2p-2}\left(2p\langle X_k, X_{k+1}-X_k\rangle + p(2p-1)|X_{k+1}-X_k|^2\right)] \\ &+ K\sum_{l=3}^{2p}\mathbb{E}[\chi_{\Delta_{L,k}}(\omega)|X_k|^{2p-l}|X_{k+1}-X_k|^l] + \mathbb{E}[\chi_{\Delta_{L,k}}(\omega)|X_k|^{2p}]. \quad (3.4)\end{aligned}$$

Note that $\chi_{\Delta_{L,k}}(\omega)$ and X_k are $\mathcal{F}_{t_k}$-measurable. According to

$$\mathbb{E}[B(X_k^*,\xi_k)g(X_k^*)\xi_k\sqrt{h}|\mathcal{F}_{t_k}] = 0, \quad (3.5)$$

and Lemma 3.1, the first term in the right-hand side of (3.4) yields

$$\begin{aligned}&\mathbb{E}[\chi_{\Delta_{L,k}}(\omega)|X_k|^{2p-2}\left(2p\langle X_k, X_{k+1}-X_k\rangle + p(2p-1)|X_{k+1}-X_k|^2\right)] \\ &= 2p\mathbb{E}[\chi_{\Delta_{L,k}}(\omega)|X_k|^{2p-2}\mathbb{E}[\langle X_k, (f(X_k^*)h + g(X_k^*)\xi_k\sqrt{h})B(X_k^*,\xi_k)\rangle \\ &\quad + \frac{2p-1}{2}\left|(f(X_k^*)h + g(X_k^*)\xi_k\sqrt{h})\right|^2 B^2(X_k^*,\xi_k)|\mathcal{F}_{t_k}]] \\ &\leq K\mathbb{E}[\chi_{\Delta_{L,k}}(\omega)|X_k|^{2p-2}A(X_k,X_k^*,\xi_k)] \\ &\leq Kh\mathbb{E}[\chi_{\Delta_{L,k}}(\omega)|X_k|^{2p-2}(1+|X_k^*|^2+|X_k^*|^{3\nu}h^{\frac{1}{2}})]. \quad (3.6)\end{aligned}$$

Now consider the second term in the right-hand side of (3.4). Using (2.5), (2.6) and $B(X_k^*,\xi_k) \leq 1$, it follows that

$$\begin{aligned}\mathbb{E}[\chi_{\Delta_{L,k}}(\omega)|X_k|^{2p-l}|X_{k+1}-X_k|^l] &\leq K\mathbb{E}[\chi_{\Delta_{L,k}}(\omega)|X_k|^{2p-l}\left(h^l|f(X_k^*)|^l + h^{\frac{l}{2}}|g(X_k^*)|^l|\xi_k|^l\right)] \\ &\leq K\mathbb{E}[\chi_{\Delta_{L,k}}(\omega)|X_k|^{2p-l}h^{\frac{l}{2}}\left(1+|X_k^*|^{l\nu}\right)]. \quad (3.7)\end{aligned}$$

Substituting (3.6) and (3.7) into (3.4), we obtain

$$\begin{aligned}&\mathbb{E}[\chi_{\Delta_{L,k+1}}(\omega)|X_{k+1}|^{2p}] \\ \leq& \mathbb{E}[\chi_{\Delta_{L,k}}(\omega)|X_k|^{2p}] + Kh\mathbb{E}[\chi_{\Delta_{L,k}}(\omega)|X_k|^{2p-2}\left(1+|X_k^*|^2+|X_k^*|^{3\nu}h^{\frac{1}{2}}\right)] \\ &+ K\sum_{l=3}^{2p}\mathbb{E}[\chi_{\Delta_{L,k}}(\omega)|X_k|^{2p-l}h^{\frac{l}{2}}\left(1+|X_k^*|^{l\nu}\right)] \\ =& \mathbb{E}[\chi_{\Delta_{L,k}}(\omega)|X_k|^{2p}] + Kh\mathbb{E}[\chi_{\Delta_{L,k}}(\omega)|X_k|^{2p-2}] + Kh\mathbb{E}[\chi_{\Delta_{L,k}}(\omega)|X_k|^{2p-2}|X_k^*|^2] \\ &+ Kh\mathbb{E}[\chi_{\Delta_{L,k}}(\omega)|X_k|^{2p-2}|X_k^*|^{3\nu}h^{\frac{1}{2}}] + K\sum_{l=3}^{2p}\mathbb{E}[\chi_{\Delta_{L,k}}(\omega)|X_k|^{2p-l}h^{\frac{l}{2}}] \\ &+ Kh\sum_{l=3}^{2p}\mathbb{E}[\chi_{\Delta_{L,k}}(\omega)|X_k|^{2p-l}|X_k^*|^{l\nu}h^{\frac{l}{2}-1}]. \quad (3.8)\end{aligned}$$

By Young's inequality, we have

$$\mathbb{E}[\chi_{\Delta_{L,k}}(\omega)|X_k|^{2p-2}|X_k^*|^2] \leq \mathbb{E}[\chi_{\Delta_{L,k}}(\omega)\left(\frac{p-1}{p}|X_k|^{2p} + \frac{1}{p}|X_k^*|^{2p}\right)].$$

Then

$$\mathbb{E}[\chi_{\Delta_{L,k}}(\omega)|X_k|^{2p-2}|X_k^*|^2] \leq K\left(\mathbb{E}[\chi_{\Delta_{L,k}}(\omega)|X_k|^{2p}] + \mathbb{E}[\chi_{\Delta_{L,k}}(\omega)|X_k^*|^{2p}]\right).$$

Similarly, it can be obtained that

$$\mathbb{E}[\chi_{\Delta_{L,k}}(\omega)|X_k|^{2p-2}|X_k^*|^{3\nu}h^{\frac{1}{2}}] \le K\left(\mathbb{E}[\chi_{\Delta_{L,k}}(\omega)|X_k|^{2p-2+3\nu}h^{\frac{1}{2}}] + \mathbb{E}[\chi_{\Delta_{L,k}}(\omega)|X_k^*|^{2p-2+3\nu}h^{\frac{1}{2}}]\right),$$

$$\mathbb{E}[\chi_{\Delta_{L,k}}(\omega)|X_k|^{2p-l}|X_k^*|^{l\nu}h^{\frac{l}{2}-1}] \le K\left(\mathbb{E}[\chi_{\Delta_{L,k}}(\omega)|X_k|^{2p+l(\nu-1)}h^{\frac{l}{2}-1}] + \mathbb{E}[\chi_{\Delta_{L,k}}(\omega)|X_k^*|^{2p+l(\nu-1)}h^{\frac{l}{2}-1}]\right).$$

Therefore, the inequality (3.8) becomes

$$\begin{aligned}
\mathbb{E}[\chi_{\Delta_{L,k+1}}(\omega)|X_{k+1}|^{2p}] \le& \mathbb{E}[\chi_{\Delta_{L,k}}(\omega)|X_k|^{2p}] + K\sum_{l=2}^{2p}\mathbb{E}[\chi_{\Delta_{L,k}}(\omega)|X_k|^{2p-l}h^{\frac{l}{2}}] + Kh\mathbb{E}[\chi_{\Delta_{L,k}}(\omega)|X_k|^{2p}] \\
&+ Kh^{\frac{3}{2}}\mathbb{E}[\chi_{\Delta_{L,k}}(\omega)|X_k|^{2p-2+3\nu}] + Kh\sum_{l=3}^{2p}\mathbb{E}[\chi_{\Delta_{L,k}}(\omega)|X_k|^{2p+l(\nu-1)}h^{\frac{l}{2}-1}] \\
&+ Kh\mathbb{E}[\chi_{\Delta_{L,k}}(\omega)|X_k^*|^{2p}] + Kh^{\frac{3}{2}}\mathbb{E}[\chi_{\Delta_{L,k}}(\omega)|X_k^*|^{2p-2+3\nu}] \\
&+ Kh\sum_{l=3}^{2p}\mathbb{E}[\chi_{\Delta_{L,k}}(\omega)|X_k^*|^{2p+l(\nu-1)}h^{\frac{l}{2}-1}].
\end{aligned}$$

Using Young's inequality, it follows that

$$\sum_{l=2}^{2p}\mathbb{E}[\chi_{\Delta_{L,k}}(\omega)|X_k|^{2p-l}h^{\frac{l}{2}}] \le Kh(1+\mathbb{E}[\chi_{\Delta_{L,k}}(\omega)|X_k|^{2p}]). \tag{3.9}$$

Choosing $L = L(h) = h^{-\frac{1}{6\nu-4}}$, we can obtain that for $l = 3, \dots, 2p$

$$\chi_{\Delta_{L(h),k}}(\omega)|X_k|^{2p-2+3\nu}h^{\frac{l}{2}-1} \le \chi_{\Delta_{L(h),k}}(\omega)|X_k|^{2p}, \tag{3.10}$$

$$\chi_{\Delta_{L(h),k}}(\omega)|X_k|^{2p+l(\nu-1)}h^{\frac{l}{2}-1} \le \chi_{\Delta_{L(h),k}}(\omega)|X_k|^{2p}. \tag{3.11}$$

Thus, it follows from (3.9)-(3.11) that

$$\begin{aligned}
\mathbb{E}[\chi_{\Delta_{L(h),k+1}}(\omega)|X_{k+1}|^{2p}] \le& \mathbb{E}[\chi_{\Delta_{L(h),k}}(\omega)|X_k|^{2p}] + Kh\mathbb{E}[\chi_{\Delta_{L(h),k}}(\omega)|X_k|^{2p}] + Kh \\
&+ Kh\mathbb{E}[\chi_{\Delta_{L(h),k}}(\omega)|X_k^*|^{2p}] + Kh^{\frac{3}{2}}\mathbb{E}[\chi_{\Delta_{L(h),k}}(\omega)|X_k^*|^{2p-2+3\nu}] \\
&+ Kh\sum_{l=3}^{2p}\mathbb{E}[\chi_{\Delta_{L(h),k}}(\omega)|X_k^*|^{2p+l(\nu-1)}h^{\frac{l}{2}-1}].
\end{aligned} \tag{3.12}$$

By the SSBT scheme (3.1), we can have the following estimation.

$$\mathbb{E}[\chi_{\Delta_{L(h),k}}(\omega)|X_k^*|^{2p}] \le K\mathbb{E}[\chi_{\Delta_{L(h),k}}(\omega)|X_k|^{2p}] + Kh^p. \tag{3.13}$$

In fact, by $X_k = X_k^* - \theta f(X_k^*)h$ and $\theta h < \frac{1}{2c_3}$, we have

$$\begin{aligned}
|X_k|^2 &= |X_k^*|^2 - 2\theta h\langle X_k^*, f(X_k^*)\rangle + \theta^2h^2|f(X_k^*)|^2 \\
&\ge (1-2c_3\theta h)|X_k^*|^2 - 2\theta c_3 h.
\end{aligned}$$

It follows that

$$(1-2c_3\theta h)^p|X_k^*|^{2p} \le 2^{p-1}\left(|X_k|^{2p} + (2\theta c_3 h)^p\right),$$

i.e.

$$|X_k^*|^{2p} \le K\left(|X_k|^{2p} + h^p\right), \tag{3.14}$$

and (3.13) holds.

Substituting inequality (3.13) into (3.12) we can get

$$\begin{aligned}\mathbb{E}[\chi_{\Delta_{L(h),k+1}}(\omega)|X_{k+1}|^{2p}] \leq &\mathbb{E}[\chi_{\Delta_{L(h),k}}(\omega)|X_k|^{2p}] + Kh\mathbb{E}[\chi_{\Delta_{L(h),k}}(\omega)|X_k|^{2p}] + Kh \\ &+ Kh\mathbb{E}[\chi_{\Delta_{L(h),k}}(\omega)|X_k|^{2p}] + Kh^{\frac{3}{2}}\mathbb{E}[\chi_{\Delta_{L(h),k}}(\omega)|X_k|^{2p-2+3\nu}] \\ &+ Kh\sum_{l=3}^{2p}\mathbb{E}[\chi_{\Delta_{L(h),k}}(\omega)|X_k|^{2p+l(\nu-1)}h^{\frac{l}{2}-1}].\end{aligned}$$

Utilizing (3.10) and (3.11) again, it is obtained that

$$\mathbb{E}[\chi_{\Delta_{L(h),k+1}}(\omega)|X_{k+1}|^{2p}] \leq \mathbb{E}[\chi_{\Delta_{L(h),k}}(\omega)|X_k|^{2p}] + Kh\mathbb{E}[\chi_{\Delta_{L(h),k}}(\omega)|X_k|^{2p}] + Kh. \tag{3.15}$$

According to Gronwall's inequality, we can get

$$\mathbb{E}[\chi_{\Delta_{L(h),k}}(\omega)|X_k|^{2p}] \leq K(1 + \mathbb{E}[|X_0|^{2p}]). \tag{3.16}$$

To complete the proof, we next consider $\mathbb{E}[\chi_{\widehat{\Delta}_{L,k}}(\omega)|X_k|^{2p}]$. As the corresponding part given in [28], we can estimate it as follows.

$$\chi_{\widehat{\Delta}_{L,k}} = 1 - \chi_{\Delta_{L,k}} = 1 - \chi_{\Delta_{L,k-1}}\chi_{|X_k|\leq L} = \chi_{\widehat{\Delta}_{L,k-1}} + \chi_{\Delta_{L,k-1}}\chi_{|X_k|>L} = \cdots = \sum_{l=0}^{k}\chi_{\Delta_{L,l-1}}\chi_{|X_l|>L},$$

where $\chi_{\Delta_{L,-1}} = 1$.

According to Cauchy-Schwarz's inequality, Markov's inequality, (3.3) and (3.16), we have

$$\begin{aligned}\mathbb{E}[\chi_{\widehat{\Delta}_{L(h),k}}(\omega)|X_k|^{2p}] &= \sum_{l=0}^{k}\mathbb{E}[\chi_{\Delta_{L(h),l-1}}\chi_{|X_l|>L}|X_k|^{2p}] \\ &\leq \left(\mathbb{E}\left[|X_0| + k\right]^{4p}\right)^{\frac{1}{2}}\left(\sum_{l=0}^{k}\mathbb{E}[\chi_{\Delta_{L(h),l-1},|X_l|>L(h)}]\right)^{\frac{1}{2}} \\ &= \left(\mathbb{E}\left[|X_0| + k\right]^{4p}\right)^{\frac{1}{2}}\left(\sum_{l=0}^{k}P\big(\chi_{\Delta_{L(h),l-1}}|X_l| > L(h)\big)\right)^{\frac{1}{2}} \\ &\leq \left(\mathbb{E}\left[|X_0| + k\right]^{4p}\right)^{\frac{1}{2}}\sum_{l=0}^{k}\frac{\left(\mathbb{E}[\chi_{\Delta_{L(h),l-1}}|X_l|^{2(2p+1)(6\nu-4)}]\right)^{\frac{1}{2}}}{\big(L(h)\big)^{(2p+1)(6\nu-4)}} \\ &\leq K\left(\mathbb{E}\left[|X_0| + k\right]^{4p}\right)^{\frac{1}{2}}\left(\mathbb{E}[1 + |X_0|^{2(2p+1)(6\nu-4)}]\right)^{\frac{1}{2}}kh^{2p+1} \\ &\leq K\left(1 + \mathbb{E}[|X_0|^{4p+2(2p+1)(6\nu-4)}]\right)^{\frac{1}{2}}.\end{aligned}$$

Combining the above estimate with (3.16), the first inequality of (3.2) is proved for integer $p \geq 1$. As to non-integer p, it still holds by using Jensen's inequality. Furthermore, from (3.14), we can obtain the second inequality in (3.2)

$$\mathbb{E}[|X_k^*|^{2p}] \leq K\left(1 + \mathbb{E}[|X_0|^{2p\beta}]\right).$$

□

The following theorem states the strong convergence of the SSBT scheme.

Theorem 3.3. *Let Assumption 2.1 hold, and $X(t)$ be the exact solution of Eq. (1.1), and X_k, $0 \leq k \leq N$ be the numerical solution produced by the SSBT scheme (3.1) with parameter $\theta \in [0,1]$ and step size $h = T/N$. If $\theta h < \frac{1}{2c_3}$, then there exist constants $K > 0$, $\gamma \geq 1$ independent of h, k, such that*

$$\left(\mathbb{E}[|X(t_k) - X_k|^{2p}]\right)^{\frac{1}{2p}} \leq K\left(1 + \mathbb{E}[|X_0|^{2\gamma p}]\right)^{\frac{1}{2p}}h^{\frac{1}{2}}, 1 \leq k \leq N. \tag{3.17}$$

Proof We consider one-step approximations of the SDE (1.1) obtained by the SSBT scheme (3.1)

$$\begin{cases} x^* = x + \theta f(x^*)h, \\ X = x + \dfrac{f(x^*)h + g(x^*)\xi\sqrt{h}}{1 + |f(x^*)|h + |g(x^*)\xi|\sqrt{h}}, \end{cases} \tag{3.18}$$

and by the balanced method presented in [28]

$$\bar{X} = x + \frac{f(x)h + g(x)\xi\sqrt{h}}{1 + |f(x)|h + |g(x)\xi|\sqrt{h}},$$

where ξ denotes the Gaussian $\mathcal{N}(0,1)$ d-dimensional random vector with components of i.i.d. random variables.

Define

$$X_{t,x}(t+h) = x + \int_t^{t+h} f(X_{t,x}(s))ds + \int_t^{t+h} g(X_{t,x}(s))dW(s)$$

and

$$\rho(t,x) = X_{t,x}(t+h) - X.$$

Then

$$\rho(t,x) = X_{t,x}(t+h) - \bar{X} + \bar{X} - X =: \rho_1(t,x) + \rho_2(t,x). \tag{3.19}$$

The estimations of $\rho_1(t,x)$ are given in [28] as follows.

$$|\mathbb{E}[\rho_1(t,x)]| \le Kh^{\frac{3}{2}}(1 + |x|^{2\nu}), \tag{3.20}$$

$$\mathbb{E}[|\rho_1(t,x)|^{2p}] \le Kh^{2p}(1 + |x|^{4p\nu}). \tag{3.21}$$

Notice that $\mathbb{E}[\xi] = 0$ and

$$\rho_2(t,x) = \frac{f(x)h + g(x)\xi\sqrt{h}}{1 + |f(x)|h + |g(x)\xi|\sqrt{h}} - \frac{f(x^*)h + g(x^*)\xi\sqrt{h}}{1 + |f(x^*)|h + |g(x^*)\xi|\sqrt{h}},$$

then

$$\begin{aligned} &|\mathbb{E}[\rho_2(t,x)]| \\ &= \left|\mathbb{E}[\frac{f(x)h}{1 + |f(x)|h + |g(x)\xi|\sqrt{h}} - \frac{f(x^*)h}{1 + |f(x^*)|h + |g(x^*)\xi|\sqrt{h}}]\right| \\ &\le K\left|\mathbb{E}[h(f(x) - f(x^*)) + h^2(f(x)|f(x^*)| - |f(x)|f(x^*)) + h^{\frac{3}{2}}(f(x)|g(x^*)| - f(x^*)|g(x)|)]\right|. \end{aligned} \tag{3.22}$$

According to (2.3), (2.5), (2.6), (3.14) and (3.18), it follows that

$$|f(x) - f(x^*)| \le Kh(1 + |x|^{2\nu}), \tag{3.23}$$

$$\left|f(x)|f(x^*)| - |f(x)|f(x^*)\right| \le K(1 + |x|^{2\nu}), \tag{3.24}$$

$$\left|f(x)|g(x^*)| - f(x^*)|g(x)|\right| \le K(1 + |x|^{2\nu}). \tag{3.25}$$

Substituting (3.23)-(3.25) into (3.22), we have

$$|\mathbb{E}[\rho_2(t,x)]| \le Kh^{\frac{3}{2}}(1 + |x|^{2\nu}). \tag{3.26}$$

Combining it with (3.20) and (3.19), we obtain

$$|\mathbb{E}[\rho(t,x)]| \leq Kh^{\frac{3}{2}}(1+|x|^{2\nu}). \tag{3.27}$$

Similar to the prove of (3.26), we can obtain

$$\mathbb{E}[|\rho_2(t,x)|^{2p}] \leq Kh^{2p}(1+|x|^{4p\nu}), \tag{3.28}$$

where we need the inequality $|g(x)-g(x^*)|^2 \leq K(1+|x|^{2\nu})$, which can be readily derived from (2.6). Combining (3.28) with (3.21) and (3.19), we have

$$\mathbb{E}[|\rho(t,x)|^{2p}] \leq Kh^{2p}(1+|x|^{4p\nu}). \tag{3.29}$$

According to (3.27), (3.29) and the fundamental strong convergence theorem, see Theorem 2.1 in [28], we get that the strong convergence rate of the SSBT scheme (3.1) is $\frac{1}{2}$, that is, the estimation (3.17) holds. $\square$

4. Exponential stability of the SSBT scheme

In this section, we discuss the exponential stability of the SSBT scheme (3.1). We introduce the necessary definitions and assumptions given in [10], and consider the exponential mean-square stability of the SSBT scheme.To overcome the difficulty caused by the splitting, we present a new technique, see Theorem 4.3 and Lemma 4.2.

Definition 4.1 (mean-square contraction). *For a fixed step size h, a numerical method is said to be mean-square contractive if the numerical solutions X_k, $k=0,1,2,\ldots,$ satisfy*

$$\mathbb{E}[X_k^T X_k] \leq \mathbb{E}[X_0^T X_0], \quad \forall\ k \geq 0.$$

Definition 4.2 (exponential mean-square stability). *For a fixed step size h, a numerical method is said to be exponentially mean-square stable if the numerical solutions X_k, $k=0,1,2,\ldots,$ satisfy*

$$\mathbb{E}[X_k^T X_k] \leq C\mathrm{e}^{-\varsigma t_k}\mathbb{E}[X_0^T X_0], \quad \forall\ k \geq 0,$$

where C and ς are positive constants independent of k.

The following Lemma is a special case of Theorem 5.1 in [10], which provides a sufficient condition for exponential stability of the exact solution in mean-square sense.

Lemma 4.1. *If there exist a symmetric, positive-definite matrix $M_{n\times n}$ and a constant α such that*

$$x^T M f(x) + \frac{1}{2}\mathrm{trace}\left[g^T(x) M g(x)\right] \leq \alpha x^T M x, \tag{4.1}$$

then the solution $X(t)$ of the SDE (1.1) obeys

$$\mathbb{E}[X^T(t) M X(t)] \leq \mathrm{e}^{2\alpha t}\mathbb{E}[X^T(0) M X(0)].$$

Hence, the solution $X(t)$ is exponentially mean-square stable if $\alpha < 0$, and mean-square contractive if $\alpha = 0$.

To consider the exponential mean-square stability of the SSBT scheme (3.1), we need the following estimation.

Lemma 4.2. *Given $l \geq 1$, $\eta \sim N(0,\sigma^2)$, it holds that*

$$\mathbb{E}[\frac{1}{1+l^2\sigma^2+l^2|\eta|}] > \frac{1}{(1+\sigma^2)l^2\sqrt{\pi}}.$$

Proof

$$
\begin{aligned}
\mathbb{E}[\frac{1}{1+l^2\sigma^2+l^2|\eta|}] &= \int_{-\infty}^{\infty} \frac{1}{1+l^2\sigma^2+l^2|x|}\frac{1}{\sqrt{2\pi\sigma^2}}\mathrm{e}^{-\frac{x^2}{2\sigma^2}}dx \\
&= \frac{2}{\sqrt{2\pi\sigma^2}}\int_0^{\infty} \frac{1}{1+l^2\sigma^2+l^2x}\mathrm{e}^{-\frac{x^2}{2\sigma^2}}dx \\
&\xlongequal{\text{Let } x=\sqrt{2\sigma^2}t} \frac{2}{\sqrt{2\pi\sigma^2}}\int_0^{\infty} \frac{1}{1+l^2\sigma^2+l^2\sqrt{2\sigma^2}t}\mathrm{e}^{-t^2}\sqrt{2\sigma^2}dt \\
&\geq \frac{2}{\sqrt{2\pi\sigma^2}}\frac{1}{l^2}\int_0^{\infty} \frac{1}{1+\sigma^2+\sqrt{2\sigma^2}t}\mathrm{e}^{-t^2}\sqrt{2\sigma^2}dt \\
&= \frac{1}{1+\sigma^2}\frac{2}{\sqrt{\pi}}\frac{1}{l^2}\int_0^{\infty} \frac{1}{1+\frac{\sqrt{2\sigma^2}}{1+\sigma^2}t}\mathrm{e}^{-t^2}dt \\
&\geq \frac{2}{(1+\sigma^2)l^2\sqrt{\pi}}\int_0^{\infty} \frac{1}{1+t}\mathrm{e}^{-t^2}dt \\
&= \frac{2}{(1+\sigma^2)l^2\sqrt{\pi}}\frac{\pi\mathrm{erfi}(1)-\mathrm{Ei}(1)}{2\mathrm{e}} \\
&> \frac{1}{(1+\sigma^2)l^2\sqrt{\pi}}.
\end{aligned}
$$

In the above inequality we use the fact that

$$
\int_0^{\infty} \frac{1}{1+t}e^{-t^2}dt = \frac{\pi\mathrm{erfi}(1)-\mathrm{Ei}(1)}{2\mathrm{e}} > \frac{1}{2},
$$

where

$$
\mathrm{erfi}(z) = \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{z^{2n+1}}{n!(2n+1)},\ \mathrm{Ei}(z) = -\lim_{\varepsilon\to 0^+}\left[\int_{-z}^{-\varepsilon}\frac{\mathrm{e}^{-t}}{t}dt + \int_{\varepsilon}^{\infty}\frac{\mathrm{e}^{-t}}{t}dt\right] = \mathrm{P.V.}\int_{-\infty}^{z}\frac{e^t}{t}dt,\ z>0.
$$

□

Theorem 4.3. *Suppose that the inequality (4.1) holds. Then we have*

(i) If $\theta \in [\frac{1}{2}, 1]$ and $\alpha \leq 0$, then for every step size $h > 0$, the SSBT scheme (3.1) is mean-square contractive, i.e.,

$$
\mathbb{E}[X_k^T M X_k] \leq \mathbb{E}[X_0^T M X_0].
$$

Furthermore, if $\theta \in (\frac{1}{2}, 1]$ and $\alpha < 0$, for every step size $h > 0$, the SSBT scheme (3.1) is exponentially mean-square stable. That is, there exists a constant $\lambda > 0$ dependent on θ, α and h, such that

$$
\mathbb{E}[X_k^T M X_k] \leq \exp(-\lambda t_k)\mathbb{E}[X_0^T M X_0].
$$

(ii) If $\theta \in [0, \frac{1}{2}]$, $\alpha < 0$ and $f(x)$ satisfies

$$
f^T(x)Mf(x) \leq Kx^T Mx, \tag{4.2}
$$

then there exists a constant $h_0 > 0$ such that the SSBT scheme (3.1) is exponentially mean-square stable for fixed $h \in (0, h_0)$.

Proof Let us first prove the statement (i). According to (3.1), the definition of $B(X_k^*, \xi_k)$ and $X_k^T = X_k^{*T} - \theta h f^T(X_k^*)$, we can get

$$
\begin{aligned}
X_{k+1}^T M X_{k+1} =& X_k^T M X_k + h^2 f^T(X_k^*)Mf(X_k^*)B^2(X_k^*, \xi_k) + \xi_k^T\sqrt{h}g^T(X_k^*)Mg(X_k^*)\xi_k\sqrt{h}B^2(X_k^*, \xi_k) \\
&+ 2h\big(X_k^{*T}Mf(X_k^*) - \theta h f^T(X_k^*)Mf(X_k^*)\big)B(X_k^*, \xi_k) + 2X_k^T Mg(X_k^*)\xi_k\sqrt{h}B(X_k^*, \xi_k) \\
&+ 2hf^T(X_k^*)Mg(X_k^*)\xi_k\sqrt{h}B^2(X_k^*, \xi_k).
\end{aligned}
$$

Then

$$\begin{aligned}\mathbb{E}[X_{k+1}^T M X_{k+1}] \leq &\mathbb{E}[X_k^T M X_k] + (1-2\theta)h^2\mathbb{E}[f^T(X_k^*)Mf(X_k^*)B(X_k^*,\xi_k)] \\ &+ 2h\mathbb{E}[X_k^{*T}Mf(X_k^*)B(X_k^*,\xi_k)] + h\mathbb{E}[\text{trace}\left(g^T(X_k^*)Mg(X_k^*)B(X_k^*,\xi_k)\right)],\end{aligned}$$

where we have used $0 < B(X_k^*, \xi_k) < 1$ and

$$\mathbb{E}[\xi_k^T \sqrt{h} g^T(X_k^*)Mg(X_k^*)\xi_k\sqrt{h}B(X_k^*,\xi_k)] = h\mathbb{E}[\text{trace}\left(g^T(X_k^*)Mg(X_k^*)B(X_k^*,\xi_k)\right)].$$

By the condition (4.1) it follows

$$\begin{aligned}\mathbb{E}[X_{k+1}^T M X_{k+1}] \leq \mathbb{E}[X_k^T M X_k] &+ (1-2\theta)h^2\mathbb{E}[f^T(X_k^*)Mf(X_k^*)B(X_k^*,\xi_k)] \\ &+ 2h\alpha\mathbb{E}[X_k^{*T}MX_k^*B(X_k^*,\xi_k)].\end{aligned} \tag{4.3}$$

If $1-2\theta \leq 0$ and $\alpha \leq 0$, we obtain that

$$\mathbb{E}[X_{k+1}^T M X_{k+1}] \leq \mathbb{E}[X_k^T M X_k].$$

Furthermore, note that $\alpha < 0$ and $\theta \in (\frac{1}{2}, 1]$. Substituting $hf(X_k^*) = \frac{X_k^* - X_k}{\theta}$ into (4.3) and by Cauchy's inequality, we have

$$\begin{aligned}&\mathbb{E}[X_{k+1}^T M X_{k+1}] \\ &\leq \mathbb{E}[X_k^T M X_k] + \frac{1-2\theta}{\theta^2}\mathbb{E}[\left(X_k^{*T}MX_k^* + X_k^T M X_k - 2X_k^{*T}MX_k\right)B(X_k^*,\xi_k)] \\ &\quad + 2h\alpha\mathbb{E}[X_k^{*T}MX_k^*B(X_k^*,\xi_k)] \\ &= \mathbb{E}[X_k^T M X_k] + \frac{1-2\theta}{\theta^2}\mathbb{E}[X_k^T M X_k B(X_k^*,\xi_k)] + \left(\frac{1-2\theta}{\theta^2} + 2h\alpha\right)\mathbb{E}[X_k^{*T}MX_k^*B(X_k^*,\xi_k)] \\ &\quad + \frac{2\theta-1}{\theta^2}\mathbb{E}[2X_k^{*T}MX_kB(X_k^*,\xi_k)] \\ &\leq \mathbb{E}[X_k^T M X_k] + \frac{1-2\theta}{\theta^2}\mathbb{E}[X_k^T M X_k B(X_k^*,\xi_k)] + \left(\frac{1-2\theta}{\theta^2} + 2h\alpha\right)\mathbb{E}[X_k^{*T}MX_k^*B(X_k^*,\xi_k)] \\ &\quad + \frac{2\theta-1}{\theta^2}\mathbb{E}[\frac{2\theta-1-2\alpha h\theta^2}{2\theta-1}X_k^{*T}MX_k^*B(X_k^*,\xi_k)] + \frac{2\theta-1}{\theta^2}\mathbb{E}[\frac{2\theta-1}{2\theta-1-2\alpha h\theta^2}X_k^TMX_kB(X_k^*,\xi_k)] \\ &= \mathbb{E}[X_k^T M X_k] + \frac{2\alpha h(2\theta-1)}{2\theta-1-2\alpha h\theta^2}\mathbb{E}[X_k^T M X_k B(X_k^*,\xi_k)].\end{aligned} \tag{4.4}$$

Note that $\dfrac{2\theta-1-2\alpha h\theta^2}{2\theta-1} > 0$, which ensures the use of Cauchy's inequality in the above estimation.

Introduce the sets

$$\Omega_l = \{\omega: |f(X_k^*)| \leq l^2 \text{ and } |g(X_k^*)| \leq l^2\} \cap \{\omega: |f(X_k^*)| > (l-1)^2 \text{ or } |g(X_k^*)| > (l-1)^2\},\ l = 1, 2, \ldots$$

Then

$$\begin{aligned}\mathbb{E}[X_k^T M X_k B(X_k^*,\xi_k)] &= \sum_{l=1}^{\infty}\mathbb{E}[\chi_{\Omega_l}X_k^TMX_kB(X_k^*,\xi_k)] \\ &\geq \mathbb{E}[X_k^TMX_k]\sum_{l=1}^{\infty}\mathbb{E}[\frac{1}{1+l^2h+l^2|\xi_k|\sqrt{h}}].\end{aligned}$$

According to Lemma 4.2, we have

$$\mathbb{E}[X_k^T M X_k B(X_k^*,\xi_k)] \geq \mathbb{E}[X_k^T M X_k]\sum_{l=1}^{\infty}\frac{1}{(1+h)l^2\sqrt{\pi}} = \frac{\pi^{\frac{3}{2}}}{6(1+h)}\mathbb{E}[X_k^T M X_k]. \tag{4.5}$$

Substituting (4.5) into (4.4) and noticing that $\alpha < 0$ and $\theta \in (\frac{1}{2}, 1]$, it follows that

$$\begin{aligned}\mathbb{E}[X_{k+1}^T M X_{k+1}] &\leq \left(1 + \frac{2\alpha h(2\theta-1)}{2\theta - 1 - 2\alpha h\theta^2}\frac{\pi^{\frac{3}{2}}}{6(1+h)}\right)\mathbb{E}[X_k^T M X_k] \\ &\leq \exp\left(\frac{2\alpha h(2\theta-1)}{2\theta - 1 - 2\alpha h\theta^2}\frac{\pi^{\frac{3}{2}}}{6(1+h)}\right)\mathbb{E}[X_k^T M X_k].\end{aligned}$$

Recursively using the above inequality, we obtain

$$\begin{aligned}\mathbb{E}[X_{k+1}^T M X_{k+1}] &\leq \exp\left(\frac{\alpha(k+1)h(2\theta-1)}{2\theta - 1 - 2\alpha h\theta^2}\frac{\pi^{\frac{3}{2}}}{3(1+h)}\right)\mathbb{E}[X_0^T M X_0] \\ &\leq \mathrm{e}^{\alpha t_{k+1} C(h,\theta)}\mathbb{E}[X_0^T M X_0].\end{aligned}$$

Therefore the SSBT (3.1) is exponentially mean-square stable $\forall\ h > 0$.

Next we prove the statement (ii). Combining (4.2) and (4.3), it follows

$$\mathbb{E}[X_{k+1}^T M X_{k+1}] \leq \mathbb{E}[X_k^T M X_k] + h\left(K(1-2\theta)h + 2\alpha\right)\mathbb{E}[X_k^{*T} M X_k^* B(X_k^*, \xi_k)]. \tag{4.6}$$

Let $h_0 = -2\alpha/(K(1-2\theta))$, where $\theta \in [0, 1/2]$. It is obvious that $h_0 = +\infty$ when $\theta = 1/2$. By inequalities (4.2), (3.1) and Young's inequality, one has

$$X_k^{*T} M X_k^* \geq \frac{X_k^T M X_k}{2(1 + K\theta^2 h^2)}.$$

Therefore, when $h \in (0, h_0)$ the inequality (4.6) yields

$$\mathbb{E}[X_{k+1}^T M X_{k+1}] \leq \mathbb{E}[X_k^T M X_k] + h\left(K(1-2\theta)h + 2\alpha\right)\mathbb{E}[\frac{X_k^T M X_k}{2(1 + K\theta^2 h^2)} B(X_k^*, \xi_k)].$$

Similar to the proof of the first statement, it can be immediately gotten that

$$\begin{aligned}\mathbb{E}[X_{k+1}^T M X_{k+1}] &\leq \left(1 + \frac{h\left(K(1-2\theta)h + 2\alpha\right)}{2(1 + K\theta^2 h^2)}\frac{\pi^{\frac{3}{2}}}{6(1+h)}\right)\mathbb{E}[X_k^T M X_k] \\ &\leq \exp\left(\frac{h\left(K(1-2\theta)h + 2\alpha\right)}{2(1 + K\theta^2 h^2)}\frac{\pi^{\frac{3}{2}}}{6(1+h)}\right)\mathbb{E}[X_k^T M X_k].\end{aligned}$$

Therefore the scheme is exponentially mean-square stable when $h \in (0, h_0)$. $\square$

5. Numerical examples

In this section, two numerical examples are given to test errors, convergence orders and stability properties of the SSBT scheme in mean-square sense for solving nonlinear SDEs.

The semi-implicit Euler scheme

$$X_{k+1} = X_k + (1-\theta)hf(X_k) + \theta h f(X_{k+1}) + g(X_k)\xi_k\sqrt{h}, \quad \theta \in [0, 1],$$

has been introduced to make comparison. In order to compute errors in mean-square sense, we run M independent trajectories:

$$\mathrm{Err(h)} := \left(\frac{1}{M}\sum_{m=1}^{M}|X^{Ref}(T, \omega_m) - X_N^{(h)}(\omega_m)|^2\right)^{\frac{1}{2}}$$

where $X^{Ref}(T, \omega_i)$ denotes the reference solution calculated by the SSBT scheme with small step size $h_{small} = 2^{-12}$ for generally the exact solution of the nonlinear SDEs is hardly obtained.

Example 5.1. *Consider a nonlinear stochastic differential system*

$$d\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}-2x_1-3x_1^3\\-1.5x_2-5x_2^3\end{bmatrix}dt+\begin{bmatrix}x_1\sin x_1 & x_1x_2\\x_1x_2 & 3x_2^2\end{bmatrix}\begin{bmatrix}dW_1\\dW_2\end{bmatrix},\tag{5.1}$$

with initial vector
Case I: $X(0)=(x_1(0),x_2(0))^T=(1,1)^T$;
Case II: $X(0)=(x_1(0),x_2(0))^T=(10,10)^T$.

Table 5.1: Comparison of the mean-square errors and convergence rate of the SSBT scheme and the semi-implicit Euler scheme with different values of θ when solving the SDE (5.1). T=1, M=5000 (Example 5.1, **Case I**).

SSBT	$\theta=0.2$		$\theta=0.5$		$\theta=0.7$		$\theta=1$	
h	Err(h)	rate	Err(h)	rate	Err(h)	rate	Err(h)	rate
2^{-5}	3.80e-02	-	4.41e-02	-	4.99e-02	-	6.02e-02	-
2^{-6}	2.26e-02	0.75	2.46e-02	0.84	2.70e-02	0.89	3.13e-02	0.95
2^{-7}	1.38e-02	0.71	1.41e-02	0.80	1.49e-02	0.85	1.66e-02	0.91
2^{-8}	8.81e-03	0.65	8.64e-03	0.71	8.79e-03	0.76	9.39e-03	0.82
Semi-implicit Euler	$\theta=0.2$		$\theta=0.5$		$\theta=0.7$		$\theta=1$	
h	Err(h)	rate	Err(h)	rate	Err(h)	rate	Err(h)	rate
2^{-5}	Fail		7.33e-02	-	6.85e-02	-	6.42e-02	-
2^{-6}	Fail		5.33e-02	0.46	5.08e-02	0.43	4.64e-02	0.47
2^{-7}	Fail		3.56e-02	0.58	3.36e-02	0.60	2.91e-02	0.67
2^{-8}	Fail		2.70e-02	0.40	2.55e-02	0.39	2.27e-02	0.36

It is readily verified that the coefficients of the system (5.1) satisfies Assumption 2.1. Recall Theorem 3.3, which states that the convergence rate of the SSBT scheme should be 0.5. In practice, Table 5.1 suggests that when solving the system (5.1) under **Case I**, the convergence rate is greater than 0.5, especially when θ is close to one. In contrast, for $h=2^{-5}$, 2^{-6}, 2^{-7}, 2^{-8}, by taking the same iteration algorithm for solving the implicit difference scheme as that using for the SSBT scheme, the semi-implicit Euler scheme cannot obtain convergent numerical solution when $\theta=0.2$, and its accuracy and convergence rates are lower than the SSBT scheme when $\theta=0.5$, 0.7, and $\theta=1$, the case of the backward Euler scheme.

According to Lemma 4.1 and direct calculation, we get that the exact solution of system (5.1) is exponential mean-square stable with $\alpha=-1.5$. When $\theta=0$, the SSBT scheme reduces to the balanced method presented in [28]. Figs. 5.1 and 5.2 show that it is not exponentially stable when taking $h=1$ for both Cases I and II, while for $\theta=0.5$, 0.7 and 1, the SSBT scheme with $h=1$ can preserve the exponential stability of the exact solution of system (5.1) for the two cases. It is also shown by Fig. 5.1 that when taking the initial vector as $X(0)=(1,1)^T$ (Case I), the numerical solution of the semi-implicit Euler scheme with $h=1$ is exponential mean-square stable in the cases of $\theta=0.5$, $\theta=0.7$ and $\theta=1$. However, the situation is different when taking a larger scale initial vector $X(0)=(10,10)^T$ (Case II). Fig. 5.2 illustrates that the semi-implicit Euler scheme cannot preserve the exponential mean-square stability when $\theta=0.5$ if taking step size $h=1$.

It is worth to mention that the system (5.1) does not satisfy the condition (4.2) in Theorem 4.3, while we still observe the exponentially stable numerical solution when $\theta=0.5$ and $h=1$. It implies that the inequality (4.2) in the last statement of Theorem 4.3 may not be necessary for $\theta=\frac{1}{2}$.

To further verify the last statement of Theorem 4.3, we consider a toy model in Example 5.2, in which the parameters $\alpha=-8.5$ and $K=81$ can be calculated directly.

Example 5.2.

$$\begin{cases} dX(t) = -9X(t)dt + X(t)\sin X(t)dW(t), \quad 0 < t \le T, \\ X(0) = 1. \end{cases} \tag{5.2}$$

By Theorem 4.3, when $\theta = 0.2$, $h_0 = 0.35$. Fig. 5.3 shows that the numerical solution produced by the SSBT scheme is exponentially mean-square stable when $h = 0.2 < h_0$, while it is not exponentially stable when $h = 0.4 > h_0$. It is well consistent with the theoretical results.

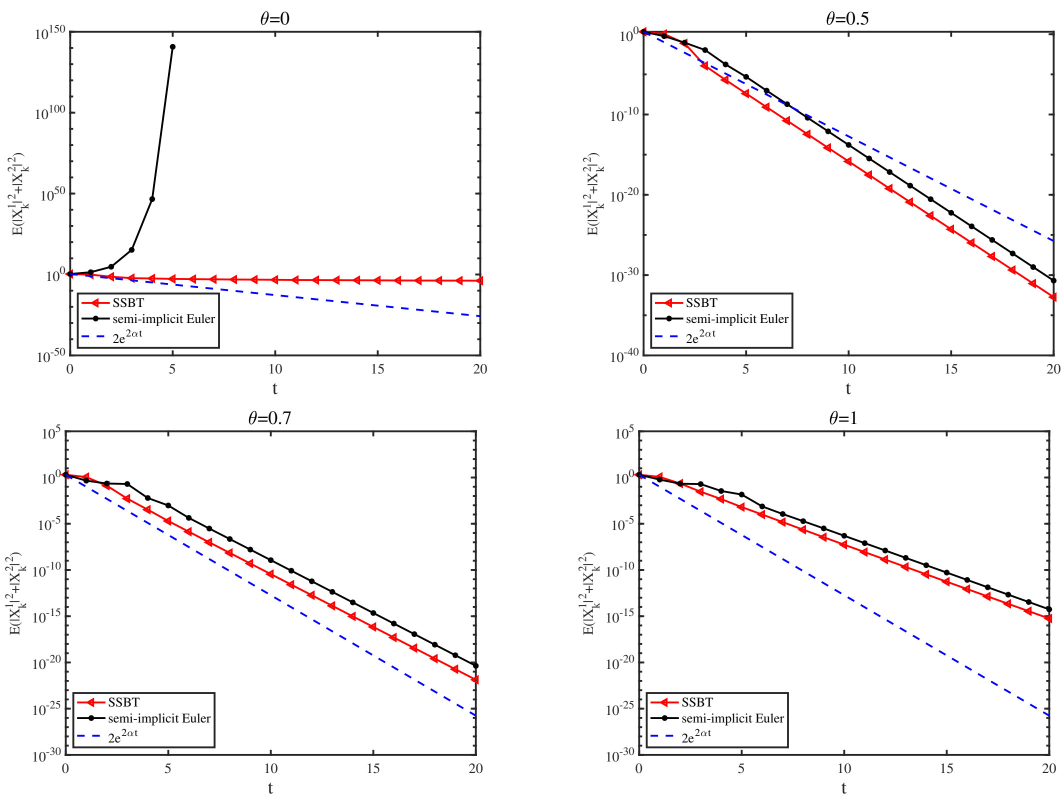

Figure 5.1: Comparison of $\mathbb{E}(|X_k^1|^2 + |X_k^2|^2)$ of the SSBT and semi-implicit Euler schemes for solving the system (5.1). $T = 20, h = 1, M = 2000$, the reference line $X_0^T X_0 e^{2\alpha t} = 2e^{-3t}$ (Example 5.1, **Case I**).

6. Conclusions

In this work, we combine the balanced method with the split-step algorithm to construct the SSBT scheme for nonlinear SDEs (1.1) under some non-globally Lipschitz conditions. The former method is often adopted to overcome the difficulty caused by nonlinear growth coefficients, and the latter is good at stability. The combination brings us an efficient SSBT scheme (3.1), which is convergent with order 0.5 in mean-square sense. It has been proved in Theorem 4.3 that the SSBT scheme is exponentially mean-square stable when $\theta \in (0.5, 1]$ for every step size h and $\theta \in [0, 0.5]$ for $h < h_0$.

It is numerically shown that compared to the semi-implicit Euler scheme, the SSBT scheme has higher accuracy and convergence rate, as well as better stability behavior for solving (5.1) with both small and large scale initial vectors, see Example 5.1, Case I and II.

The SSBT scheme and analysis of the convergence and stability presented in this work can be extended to stochastic delay differential equations (SDDEs) without any technical difficulties. We

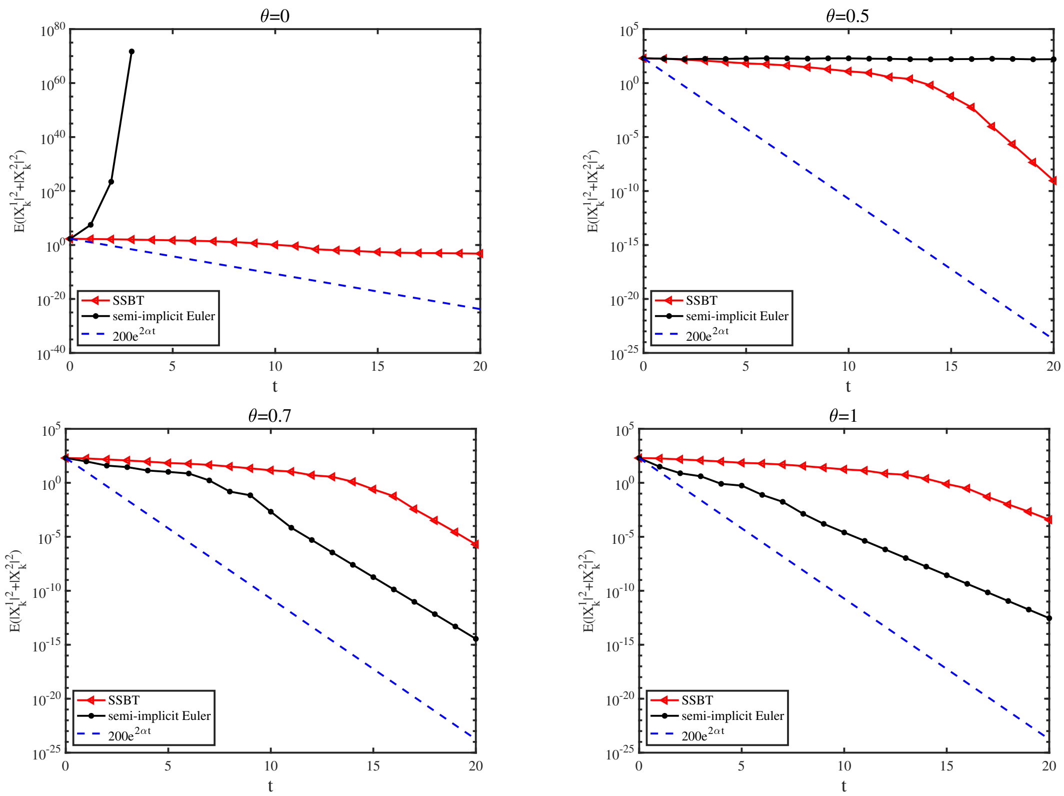

Figure 5.2: Comparison of $\mathbb{E}(|X_k^1|^2 + |X_k^2|^2)$ of the SSBT and semi-implicit Euler schemes for solving the system (5.1). $T = 20, h = 1, M = 2000$, the reference line $X_0^T X_0 e^{2\alpha t} = 200e^{-3t}$ (Example 5.1, **Case II**).

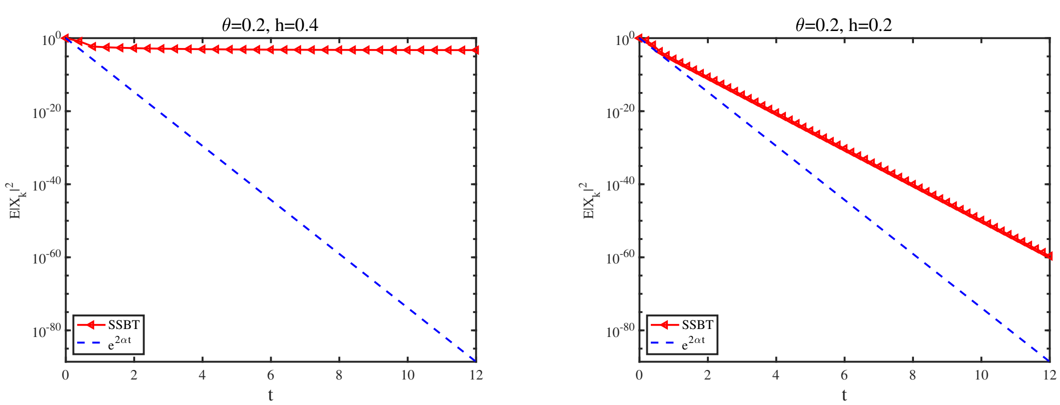

Figure 5.3: Comparison of exponential stability of the SSBT scheme with step size $h > h_0$ and $h < h_0$, $h_0 = 0.35$, $\theta = 0.2$, $T = 12$, $M = 2000$, the reference line is $e^{2\alpha t} = e^{-17t}$ (Example 5.2).

refer to [2] for the fundamental strong convergence theorem of SDDEs and [35] for the method to study convergence and mean-square stability of θ-methods that will be used when considering the SSBT scheme for SDDEs under non-global Lipschitz conditions.

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