Polynomials with Many Roots - the Mean Field Limit of Differentiation

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Feb 2020

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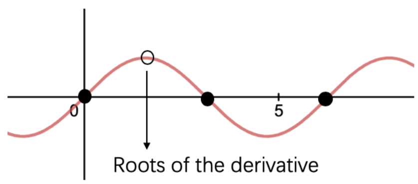
1 Abstract

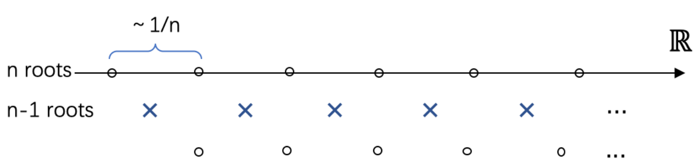
Let Pn(x) be a polynomial of degree n and all its roots distributed according to a smooth function u(0, x) (e.g. probability density function of Gaussian distribution) on the real line. If we describe it in n distinct roots:

$$Pn(x) = \prod_{k=1}^{n}(x - x_k)$$

where $\mathrm{x_k}$ are independent and identically distributed random variables picked from this distribution, then we can sketch n roots and roots of the derivative as follows:



The k-th derivative of Pn(x) is a polynomial with degree $n-k$ and $n-k$ distinct roots. If $n \to \infty$, Pn'(x) also has its roots distributed according to the same function u(0, x). By mathematical induction, k-th derivative of Pn(x) is distributed following the same function for every fixed k as $n \to \infty, \forall k < n$. Thus, it forms an interlacing structure:



In this seminar, we discuss how the distribution of roots behaves under iterated differentiation of the function, i.e. how the density of roots of Pn(x) evolves. We will derive and introduce the nonlinear equation

$$u_t + \frac{1}{\pi}(\arctan(\frac{Hu}{u}))_x = 0$$

on $u(x) > 0$, where Hu is the Hilbert transformation. [4]

We will discuss two supported closed-form solutions contained in the Wigner law and the Marchenko-Pastur law. We also show that these solutions satisfy an infinite number of conservation laws. [4]

2 Problem

Let Pn(x) be a polynomial with degree n and only have real roots on $\mathbb{R}$ whose distribution approximates a given nice distribution function. If we differentiate Pn(x) at least a number of times t, $0 < t < 1$, such that is proportional to n, how are the roots of $P_n^{(n,t)}(x)$ distributed?

3 Background

The study of distribution of roots has been an active field. [4] The transport equation is nonlinear but somewhat similar to a series of recently derived one-dimensional transport equations with nonlocal flux given by the Hilbert transform or the fractional Laplacian. These were introduced as models for the quasi-geostrophic equation and one-dimensional analogues of the three-dimensional Navier-Stokes and Euler equations. [4]

4 Analysis

Suppose the function u(t, x) satisfies

$$u_t + \frac{1}{\pi}(\arctan(\frac{Hu}{u}))_x = 0$$

on supp $u = \{x : u(x) > 0\}$, where Hu is the Hilbert transformation:

$$(Hf)(x) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{f(y)}{x-y}dy$$

This equation has three properties such that:
a) Symmetry $u(x) \rightarrow\ u(x-\lambda), \lambda \in \mathbb{R}$
b) Reflection $u(x) \rightarrow\ u(-x)$
c) Scaling $u(t, x) \rightarrow\ \lambda u(t, \frac{x}{\lambda}), \lambda > 0$. Assuming sufficient regularity, we have

$$\frac{d}{dt}\int_{\mathbb{R}} u(x)dx = \int_{\mathbb{R}} u_t(x)dx = -\frac{1}{\pi}\int_{supp\,u}\frac{d}{dx}(\arctan(\frac{Hu}{u}))dx = -1$$

$$\int_{\mathbb{R}} u(t,x)dx = 1 - t$$

This indicates that the solution vanishes at $t = 1$. Since $Pn^{(t \cdot n)}$ has $(1-t)n$ roots, there should be a constant loss of mass. [4] We derive two explicit solutions:

4.1 The semicircle distribution

There are two propositions regarding the Hermite polynomial H_n:
a) the roots of H_n are approximately given by the measure

$$\mu = \frac{1}{\pi}\sqrt{2n - x^2}dx$$

b) the derivatives of Hermite polynomials are also Hermite polynomials

$$\frac{d^m}{dx^m}H_n(x) = C_{n,m}H_{n-m}(x)$$

where $C_{n,m} = \dfrac{2^n n!}{(n-m)!}$ is a constant depends on n and m. This implies that if the transport equation approximates the flow of roots, then semicircle solution should turn into a self-similar one parameter family of solutions. [4]

$$u(t,x) = \frac{2}{\pi}\sqrt{(T-t) - x^2} \cdot \chi_{|x| \leqslant \sqrt{T-t}}$$

for $T - t \geqslant 0$, should be a solution. We verify by computing u_t and the Hilbert transform Hu.

$$u_t = -\frac{1}{\pi\sqrt{T-t-x^2}}$$

We scale the function by $\sqrt{T-t}$ to reduce it to the computation of the Hilbert transform of $(1-x^2)_+^{1/2}$ on (-1, 1). This reduces to an identity for Chebyshev polynomials of the second kind U_k

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\sqrt{1-y^2}U_{n-1}(y)}{x-y}dy = T_n(x)$$

for n=1, $U_0(x) = 1, T_1(x) = x$, $Hu(t, x) = \frac{2x}{\pi}\chi(-\sqrt{T-t}, \sqrt{T-t})$, then

$$\frac{1}{\pi}(\arctan(\frac{Hu}{u}))_x = \frac{1}{\pi}(\arctan(\frac{x}{\sqrt{T-t-x^2}}))_x = \frac{1}{\pi\sqrt{T-t-x^2}}$$

which verified that the semicircle solution solves the transport equation. [2–5] The semicircle solution behaves like shrinking circles.

4.2 The Marchenko-Pastur distribution

Laguerre polynomials $L_n^{(\alpha)}$ and their differentiation are given by the formula respectively:

$$L_n^{(\alpha)}(x) = \frac{x^{-\alpha}}{n!}(\frac{d}{dx} - 1)^n x^{n+\alpha}$$

$$\frac{d^k}{dx^k}L_n^{(\alpha)}(x) = (-1)^k L_{n-k}^{(\alpha+k)}(x)$$

If α_n is a sequence such that $\alpha_n/n \to c \in (-1, \infty)$, then the distribution of the roots of $L_n^{(\alpha_n)}$ rescaled by n converges weakly to the Marchenko-Pastur distribution. [1, 4]For $c > 0$,

$$v(c, x) = \frac{\sqrt{(x_+ - x)(x - x_-)}}{2\pi x}\chi_{(x_-, x_+)}dx$$

where $x_\pm = (\sqrt{c+1} \pm 1)^2$. If the roots of $L_n^{(c)} \sim v(c, x)$ then the roots of $L_{n(1-\epsilon)}^{((c+\epsilon))} \sim v(\frac{c+\epsilon}{1-\epsilon}, \frac{x}{1-\epsilon})$, and the solution of the transport equation is

$$u_c(t, x) = v(\frac{c+t}{1-t}, \frac{x}{1-t}), 0 < t < 1$$

The solutions have a shape of inclined semicircle distribution, while for large value of c, the distribution is close to semicircle.

4.3 Derivation

One fact is the Gauss electrostatic law that for any polynomial Pn having roots in $\{x_1, \cdots, x_n\}$

$$\frac{p_n'(x)}{p_n(x)} = \sum_{i=1}^{n} \frac{1}{x - x_i} \tag{1}$$

and we should ask, is it true that in a few steps of derivation, we get regularity in the distribution of roots? We track the revolution of the derivatives and when it is true, it deforms in an arithmetic progression with a small error. By the discussion of Polya (30s), Farmer & Rhoades (00s), the space between the roots becomes more regular and the change in spacing evens out. Another important fact is Euler's cotangent identity:

$$\pi \cot \pi x = \frac{1}{x} + \sum_{n=1}^{\infty} \left(\frac{1}{x+n} + \frac{1}{x-n}\right)$$

We rewrite the right-hand side of (1) in a way that splitting around y into far-field and near field:

$$\sum_{i=1}^{n} \frac{1}{x - x_i} = \sum_{|x_i - y| small}^{n} \frac{1}{x - x_i} + \sum_{|x_i - y| large}^{n} \frac{1}{x - x_i} \tag{2}$$

By the discussion (Farmer and Rhoades), the first term of the right-hand side will be regularly spaced, and the second term of the right-hand side will be approximately $n\pi(Hu_0)(y)$, where H is the Hilbert transform. Along the solutions that correctly modeling the roots of polynomials, an infinite number of conservation laws is satisfied, for instance, $\int_{\mathbb{R}} u(t,x)dx = 1 - t$ indicates there exists constant loss of mass to shrink things down, $\int_{\mathbb{R}} u(t,x)xdx = (1-t)\int_{\mathbb{R}} u(0,x)xdx$ indicates that the weight against x also shrinks down by $1 - t$.

5 Further directions

There are many questions about the properties of the transport equation itself, and whether there is a rigorous derivation of the equation from the polynomial dynamics in the small scale limit. A rigorous understanding of the dynamics at the boundary of the support is needed to make the derivation more rigorous. [4]

References

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