Calculus 3: Project 3

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The aim of this report was to analyze the properties of a comet and its impacts on an orbiter and a lander. The comet's mass was found to be 7.727·10^{13} kg. The mass of the lander is 105 kg. The surface area of the comet is 9.620·10^{7}m^{2}. The lander followed a preset flight plan, using maneuvering thrusters to maintain the ideal trajectory. The lander was able to land safely in 42,690 seconds. However, the lander would not land successfully if the thrusters were not used. This mission also included an orbiter that collected dust within the coma of the comet. The period of orbit is 62,830 seconds. The orbiter collected 4.170g of material during one orbit. Computed values are presented to 4 significant figures.

I. Nomenclature

ρ_{comet}	=	mass density of comet
a, b, c	=	principal axes of the comet
I_x, I_y, I_z	=	moment of inertia about each coordinate axis
SA_{comet}	=	surface area of the comet's nucleus
v	=	gravitational potential around the nucleus
r_{lander}	=	trajectory of the lander
t_{land}	=	time at which the lander lands on the surface
m_{lander}	=	mass of lander
m_{comet}	=	mass of comet
G	=	gravitational constant ($6.674 \cdot 10^{-11}$)
kg	=	kilograms
g	=	grams
km	=	kilometers
m	=	meters
J	=	joules
PE	=	potential energy
KE	=	kinetic energy

II. Introduction

Comets, relics from the formation of star systems across the universe, have been an area of study and admiration for over 2200 years. Their importance is not limited to the scientific though, it is theorized that comets carried the vital water and complex molecules that started life on the primordial Earth.

Comets are comprised of a loosely packed nucleus, made of dust, ice and small rocks; a coma made of dust and other small debris; and occasionally dust and ion tails.

NASA has sent numerous probes on exploratory missions, such as the Deep Impact mission into the comet 9P/Temple's nucleus. This report will follow this trend by studying the various properties of a comet, such as its gravitational potential field. Further, this report will analyze the feasibility of sending a lander to the nucleus surface, and placing a probe in orbit around the nucleus to collect coma dust. For clarity, SI units will be used in this report.

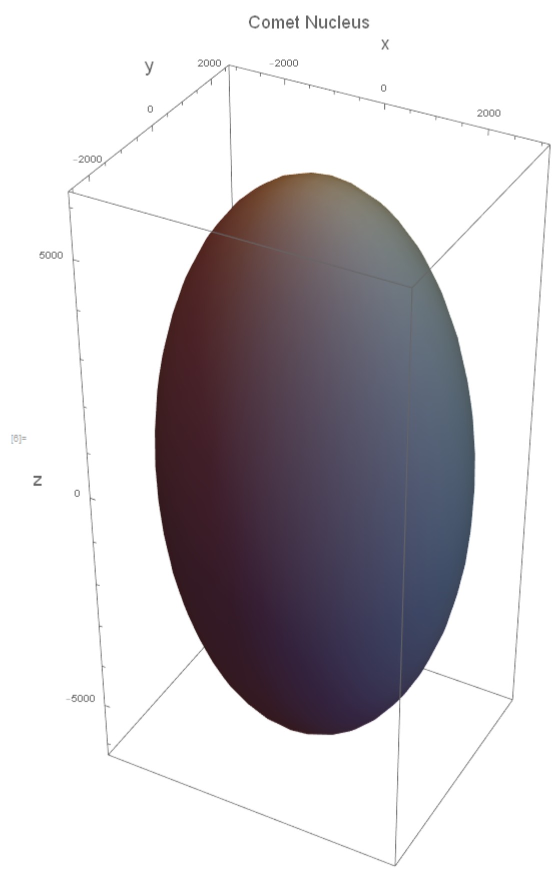
III. Comet Properties

Telescope observations indicate that the nucleus of comet 1998P/Willis is a solid ellipsoid with principal axes a = 3025 m, b = 2520 m, and c = 6050 m.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$$

The density of the comet, ρ_{comet}, is assumed to be constant as 400 $\frac{kg}{m^3}$.
The nucleus can be visualized using Mathematica, and is shown below.
It is clear that the comet nucleus is of uniform density, as no striations or gradients exist on the surface.

Fig. 1 Contour Plot of Nucleus



The mass of the comet can be found using the triple integral shown below

$$Mass = \int\int\int_R \rho(x, y, z) dV$$

This report evaluated the integral in Mathematica and found the mass of the comet to be $7.727 \cdot 10^{13}$ kg.
The moments of Inertia of a solid about the coordinate axes can be computed using the following formula

$$I_a = \int\int\int_U (b^2 + c^2) \rho(x, y, z) dV$$

Where I_a is the moment about axis **a**, **U** is the region, and **b** and **c** are the distances from the other axes. The surface area of the comet's nucleus was found using a double integral.

$$SA = \int\int_D \sqrt{1 + [f_x^2] + [f_y^2]}dA$$

Where D is the region in the xy plane, and f_x and f_y are the partials of function f with respect to x and y. The surface area was calculated as $9.620{\cdot}10^7 m^2$.

The moments about each axis were found as:

Table 1 Moments of Inertia

	Ix	**Iy**	**Iz**
m⁴	$6.638 \cdot 10^{20}$	$7.071 \cdot 10^{20}$	$2.396 \cdot 10^{20}$

IV. Multipole Expansion

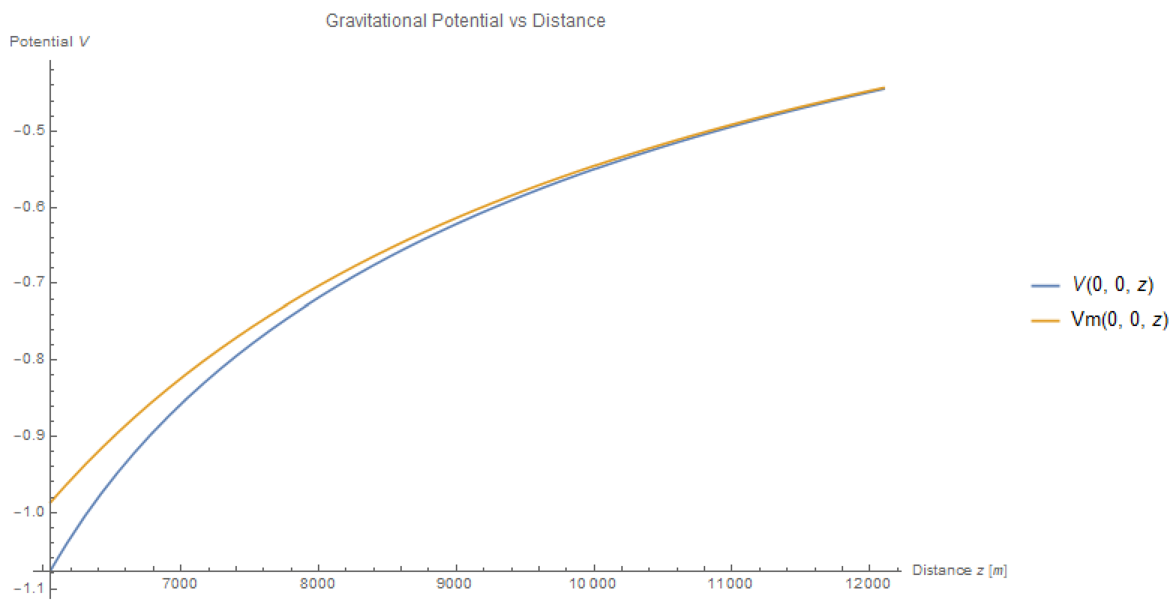
To safely land the lander, this report needed an accurate formula for the gravitational potential V around the nucleus. $\mathbf{x} = \langle x, y, z\rangle$ be the Cartesian coordinates of space and $\mathbf{r}$=$\langle \xi, \eta, \zeta\rangle$ be the Cartesian integration variables.

$$V(x) = \int\int\int_\Omega \frac{G}{|\mathbf{x} - \mathbf{r}|}dm(\mathbf{r}) = -\int\int\int_\Omega \frac{G}{|\mathbf{x} - \mathbf{r}|}\rho(\mathbf{r})dV(\mathbf{r})$$

This can be approximated using the integral below. The function V$_m$ is used to calculate the forces on and potential energies of the lander for this report.

$$V_m(x, y, z) = \frac{-G}{\sqrt{x^2 + y^2 + z^2}}(m_{comet} + \int\int\int_\Omega (\xi^2 + \eta^2 + \zeta^2)\frac{3\cos^2(\theta(x, y, z; \xi, \eta, \zeta)) - 1}{2}\rho_{comet}(\xi, \eta, \zeta)dV(\xi, \eta, \zeta))$$

Fig. 2 V$_m$ vs V



As seen in figure 2, V$_m$ is a close approximation to V for values above 10 000 m. V$_m$ is an accurate approximation outside of the nucleus, which is 10km across in the longest direction. Hence, the computed function V$_m$ is sufficiently accurate for further calculations within this report.

V. Landing on the Comet

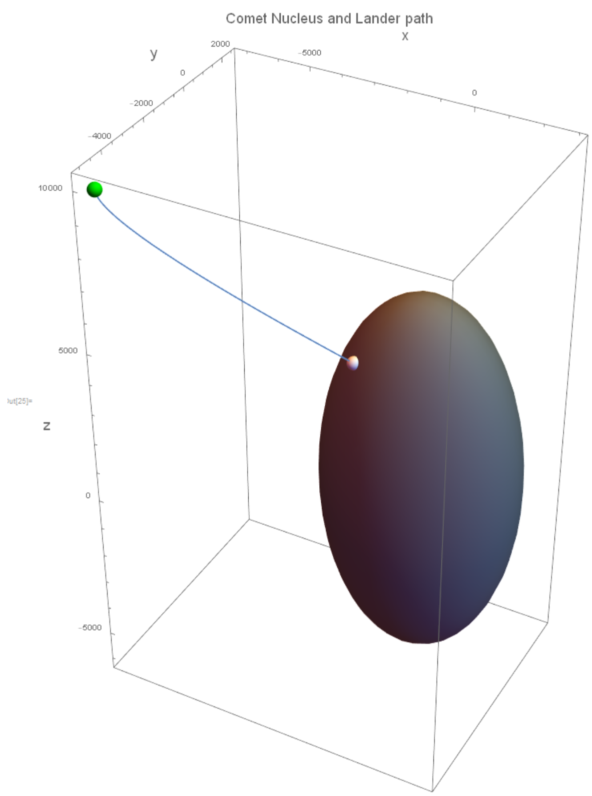
The lander trajectory was modeled as:

$$\mathbf{r}_{lander}(t) = \langle -7000 + 0.001t + 3 \cdot 10^{-6}t^2, -5000 + 0.001t + 2 \cdot 10^{-6}t^2, 10000 - 0.05t - 2 \cdot 10^{-6}t^2 \rangle$$

where t≥0 is the time in seconds from the orbiter's release of the lander and $\mathbf{r}_{lander}$ is measured in meters. The maneuvering thrusters on the lander maintain this trajectory.

The lander lands on the surface of the comet at t = 42 690 seconds. Using this information, this report plotted the path of the lander (shown below).

Fig. 3 Lander Path



The green sphere represents the initial position of the lander, and the white sphere represents the final position of the lander. Maneuvering thrusters were used to maintain the prescribed trajectory.

The initial velocity of the lander is $\langle 0.001, 0.001, -0.05 \rangle \frac{m}{s}$. The initial force on the lander from the comet's gravity is $3.231 \cdot 10^{-3}$ N. For a safe landing, the lander's landing speed must be less than $0.5 \frac{m}{s}$. If the lander impacts with a higher speed, the lander may bounce off the nucleus. To calculate the speed of the lander at time zero, the magnitude of the velocity was found using the equation shown below.

$$speed(t) = \sqrt{\mathbf{r}'(t) \cdot \mathbf{r}'(t)}$$

$\mathbf{r}'$(t) is the velocity vector of the orbiter at time t. Evaluating at time = zero, the initial speed of the orbiter was $0.05 \frac{m}{s}$.

The velocity of the lander at the time of impact while using boosters was computed to be $\langle 0.2572, 0.1718, -0.2208 \rangle \frac{m}{s}$ and the final speed was calculated to be $0.3800 \frac{m}{s}$. Hence, the lander will land safely.

Next, this report calculated the work done by gravity on the lander by using a line integral of force along the path (shown below).

$$Work = \int_C \mathbf{F} \cdot \mathbf{dr}$$

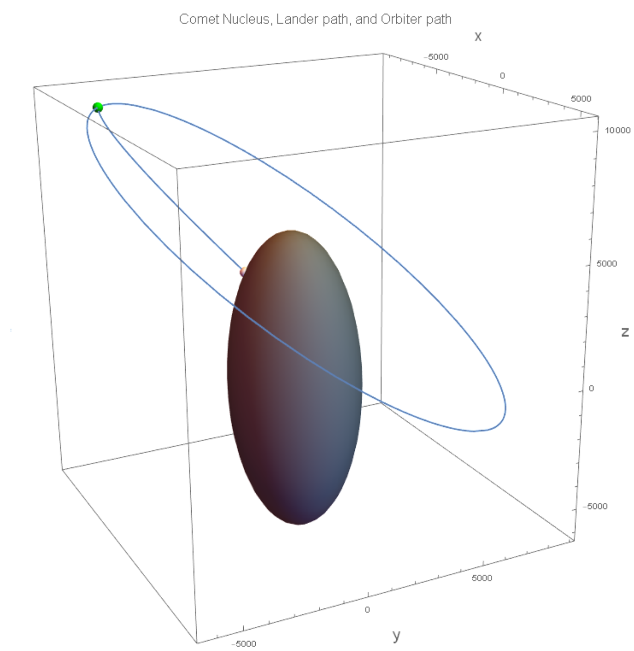
Where C is the curve **r**(t), **F** is the vector field, and dr is the derivative of the position function.
The line integral evaluated to 97.05 J. This number was confirmed using the theory of Conservation of Energy.

Conservation of Energy

$$PE_i + KE_i = PE_f + KE_f$$

The work done by the comet's potential field on the lander is the difference between the initial kinetic and potential energies and the final kinetic energy (conservation of energy). The work done was confirmed as 97.05 Joules. Extending this concept, the work done by the thrusters was the difference between lander kinetic energy and potential energy. Hence, the thrusters produced -89.52 Joules, as the thrusters acted against the force of gravity.
Assuming that all energy is converted to kinetic energy, the final energy of the lander is equal to the sum of the initial kinetic energy and the work done by the potential field. Hence, the final speed was calculated to be $1.361\frac{m}{s}$. At this speed the orbiter will definitely bounce off the surface of the comet.

Fig. 4 Orbiting in the Coma



As seen from the above figure, the orbiter orbits the comet through the coma. The lander is ejected from the green sphere and follows the preset trajectory to the white sphere.

The orbiter orbit is described as:

$$\mathbf{r}_{orbiter}(t) = \langle -1769.42 - 5230.58cos(0.0001t) - 5362.31sin(0.0001t),$$

$$1233.56 - 6233.56cos(0.0001t) + 4499.51sin(0.0001t),$$

$$4187.62 + 5812.38cos(0.0001t)\rangle$$

Where t is measured in seconds and $\mathbf{r}_{orbiter}$ is measured in meters. The period of one orbit, T_{orbit}, is 62,830 seconds.

The orbiter collected dust particles while in orbit within the coma. The mass of dust collected in one period was calculated using:

$$m_{collect} = A_{det} \int_0^{T_{orbit}} max\left\{0, (\mathbf{v}_{orbiter}(t) - \mathbf{v}_{dust}(\mathbf{r}_{orbiter}(t))) \cdot \mathbf{T}_{orbiter}(t)\right\} \rho_{dust}(\mathbf{r}_{orbiter}(t))dt$$

The computed mass of dust collected in a single orbit is 4.170 grams

VI. Conclusion

The aim of this report was to study various comet properties, such as the gravitational field potential, and relate them to the trajectory of a lander and orbiter system. This report found that the comet had a mass of $7.727 \cdot 10^{13}$ kg and a surface area of $9.620 \cdot 10^{7} m^{2}$. The lander would land safely at a speed of $0.3800 \frac{m}{s}$ if it used its maneuvering thrusters to follow the prescribed trajectory. The lander would not land safely without the thrusters, as the free fall speed of $1.361 \frac{m}{s}$ is greater than the maximum allowable speed of $0.5 \frac{m}{s}$. The orbiter collected 4.170 g of dust from one orbit of the comet. Overall, this mission was a success as the lander was able to land on the comet, and the orbiter was able to collect coma samples.