APPM 2360 Project 1: Black Holes

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I. Nomenclature

$x(t)$	=	astronaut's distance from black hole center as a function of time
$v(t)$	=	astronaut's velocity
dx/dt	=	change in astronaut's position with respect to time
t	=	time, in seconds
t_0	=	initial time
x	=	astronaut's distance from center of black hole, units of half Schwartzschild radii
x_0	=	astronaut's initial position

II. Introduction

This report investigates the physical phenomena near a black hole. An astronaut is mercilessly pushed from his space station as it orbits the black hole and is pulled towards it. Eq. (1) models the rate of change of the x distance of the astronaut from the center of the black hole. Solving this equation gives the x position of the astronaut, $x(t)$, as a function of time measured in seconds. The x position is measured in units of half-Schwartzschild radii where the Schwartzschild radius is the distance from the center of the black hole to the event horizon. The event horizon is located at $x = 2$.

$$\frac{dx}{dt} = \left(\frac{2}{x} - 1\right)\frac{1}{\sqrt{x}} \quad ; \quad x(0) = 5 \tag{1}$$

III. Analysis of the Model of the Astronaut's Position

A. Classify the Differential Equation

Eq. (1) is a non-linear, autonomous, first degree differential equation. Because the differential equation is autonomous it does not depend on time, only distance. Therefore the velocity of the astronaut is solely a function of the astronaut's x position. The concept of homogeneity is not applicable to Eq. (1) because it is non-linear.

B. Initial Velocity

The velocity of the astronaut is the rate of change of his position meaning it can be found without solving Eq (1) for $x(t)$ because $v(t) = \frac{dx}{dt}$. Plugging the x position into Eq. (1) will give the velocity of the astronaut. His initial velocity is $\frac{-3\sqrt{5}}{25} \approx -0.2683 \frac{halfradius}{sec}$ and was obtained using the initial position $x(0) = 5$.

C. Existence and Uniqueness

Solutions for Eq. (1) exist when the first condition of Picard's Theorem is satisfied, when $x > 0$. The differential equation is discontinuous for $x \leqslant 0$ and therefore the theorem does not apply for $x \leqslant 0$. To test the uniquness we need to look at the partial derivative of the differential equation with respect to x.

$$\frac{\partial x}{\partial t} = \frac{-3}{x^{5/2}} + \frac{1}{2x^{3/2}} \tag{2}$$

Unique solutions exist when Eq. (2) is continuous, at $x > 0$. Both $\frac{dx}{dt}$ and $\frac{\partial x}{\partial t}$ are discontinuous for $x \leqslant 0$ so nothing can be said about the existence or uniqueness of $x(t)$ when $x \leqslant 0$.

*Recitation: 274
†Recitation: 291
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D. Equilibrium Solutions

The equilibrium solution to Eq. (1) is at $x = 2$ and was found by setting $\frac{dx}{dt}$ equal to zero and solving for x.

E. Long Term Behavior

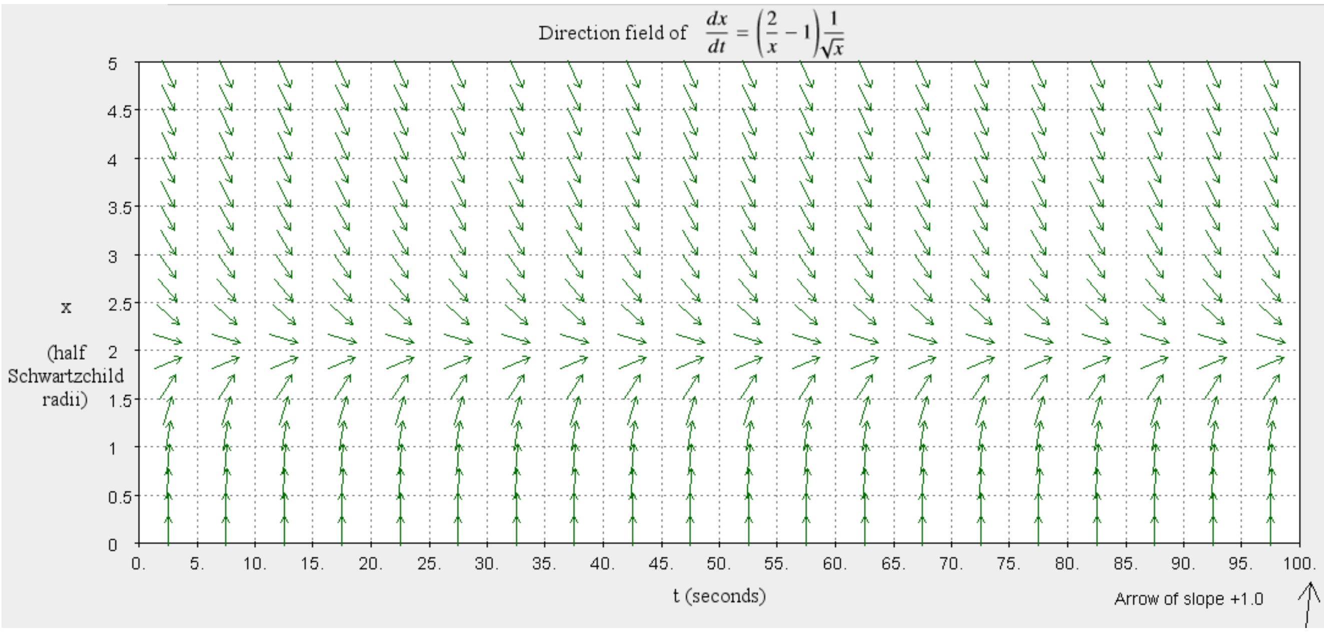

Fig. 1 Direction field of Eq. (1)

As t increases, the direction vectors of Eq. (1) continue to point towards $x = 2$ from above and below. This shows that $x = 2$ is a stable equilibrium solution to the differential equation and all solutions will converge to this value regardless of t_0 or x_0. This is modeled in the direction field in Figure 1.

IV. Numerical Methods

A. Test Problem

Eq. (3) is a test initial value problem that was given to ensure our Euler's method code works properly. To test the code we compared the Euler's method approximation to the actual solution.

Separation of Variables

Separation of variables was used to find the exact solution to Eq. (3). The solution process can be found in the Appendix.

$$\frac{dx}{dt} = \frac{1}{x}, \quad x(0) = 5 \tag{3}$$

The exact solution to Eq. (3) is

$$x(t) = \sqrt{2t + 25}$$

The next step is to use Euler's Method to find an approximate solution.

Euler's Method

Euler's method uses the tangent line to approximate the solution of a differential equation.

$$x_{n+1} = x_n + f(t_n, x_n) * h \tag{4}$$

Where $f(t_n, x_n)$ represents the differential equation and h is the step size. Eq. (3) was approximated by creating a MATLAB script to implement Euler's method using a step size of $h = 0.1$. The figure below shows the actual solution plotted with the Euler approximation for $0 \leq t \leq 20$.

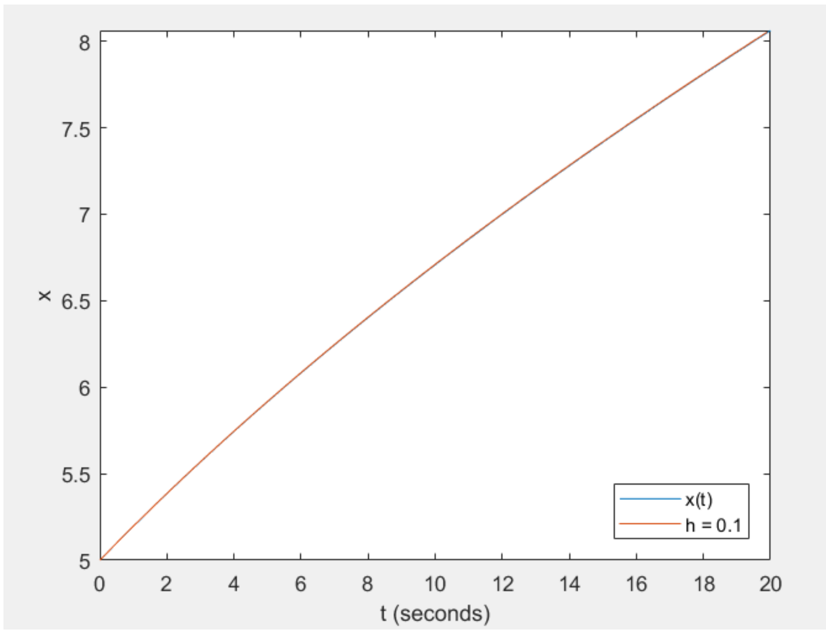

Fig. 2 Actual solution plotted with Euler Approximation

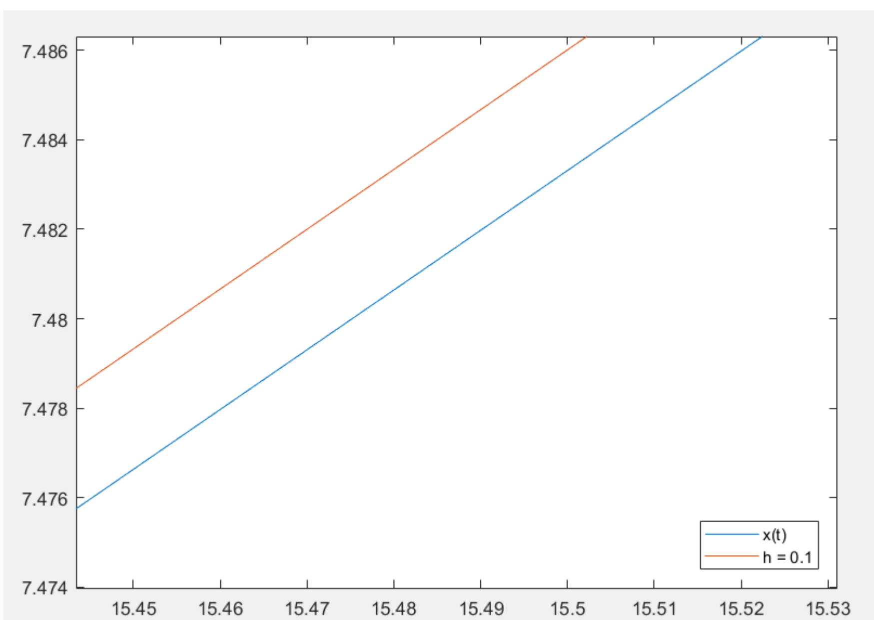

Fig. 3 Detail of previous graph

B. Applying Euler's Method to Eq. (1) With Varied Step Size

Once it was determined that the MATLAB code sufficiently approximates the differential equation using Euler's method, it was modified to approximate a solution to Eq. (1). Different step sizes were used to explore the effect step size has on the accuracy of the approximation. Solutions with step sizes of $h = 2$, $h = 1$, $h = 0.1$, and $h = 0.01$ are plotted over $0 < t < 100$ in Figure 4.

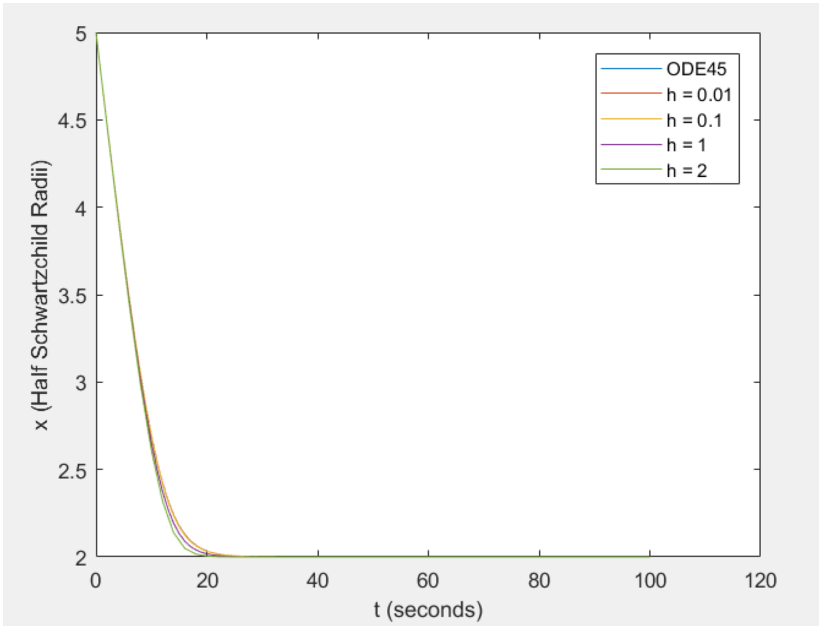

Fig. 4 Euler's Approximations

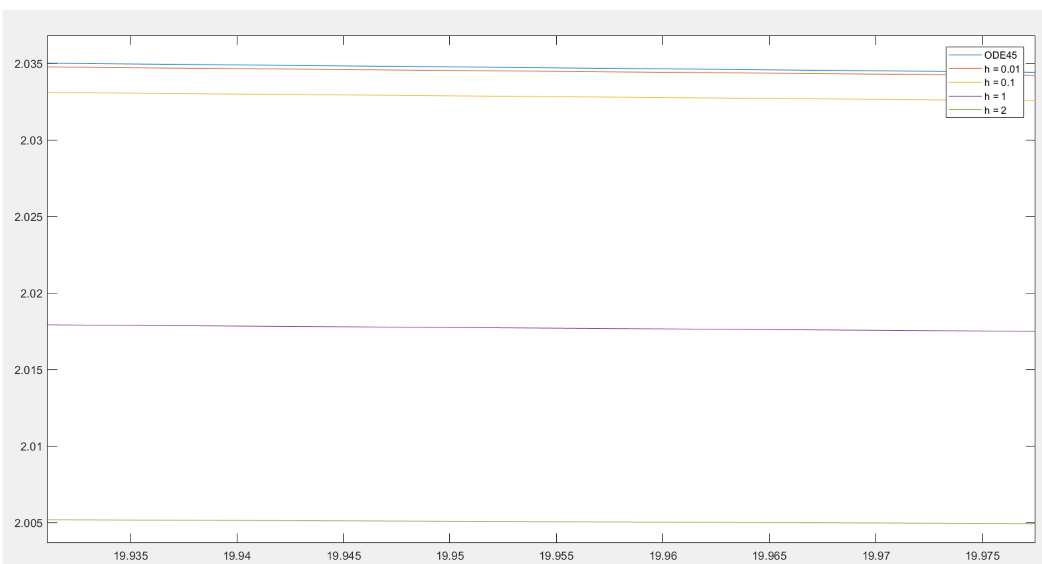

Fig. 5 Detail of previous graph

These plots all have a very similar curvature, and only diverge slightly from each other. This shows that even a small step size can allow you to get a general understanding of the equation's behavior over time. If the long-term behavior of the equation is being studied, a direction field gives a better intuition, since it is independent of specific solutions.

C. Time vs. Accuracy

A normalized value for comparison was determined by $time \cdot error$. This was used to find the optimal step size of 0.01, since the lowest product of these two numbers would be the best balance between computational time and error value. This results in an approximation that is neither too inaccurate, nor takes a large amount of time to compute. Because Eq. (1) is too messy to solve by hand we will use the ODE45 calculated value to compare the results of the different step sizes. ODE45 is one of the most commonly used DE solvers due to its accuracy.

h value	Computational Time (ms)	Absolute error	Error · Time
0.01	1.611	$1.8414 \cdot 10^{-04}$	$2.966 \cdot 10^{-4}$
0.1	1.368	0.0018	0.00246
1	1.175	0.0168	0.0197
2	0.714	0.0293	0.0209

Looking at the table above the most sensible step size is 0.01. The loss of time for each step is marginal compared to the orders of magnitude decrease in absolute error. Upon further decreasing h, the Error · Time value would eventually begin to increase as the computational time gained begins to outweigh the increase in accuracy of the smaller step size.

D. Varying Initial Conditions

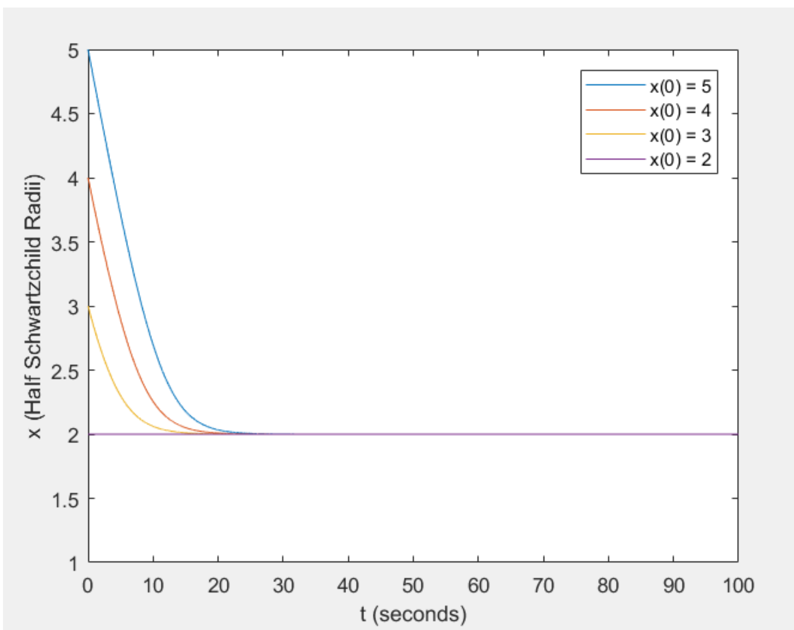

Fig. 6 Euler's Approximations

This plot visualizes how the initial conditions of a stable equation will not affect what the system tends towards as $t \longrightarrow \infty$. Since the effects of time are distorted within a certain distance from the black hole, from the perspective of the station, the astronaut would never cross the boundary and appear to remain there infinitely. This can be modeled by Eq.(1), which converges to $x = 2$.

E. Comparison of Approximation and Numerical Results

After examining the results of our direction field analysis, the system would tend towards $x = 2$ for all values of t, quickly converging regardless of the initial condition. The numerical analysis method agrees, showing a convergence towards $x = 2$ as t increases. An unstable equation could result in different solutions depending on the initial condition chosen. Because this equation has a stable equilibrium solution, it makes sense that all methods of analysis would result in the same outcome.

V. Conclusion

In this project, we have explored some of the various methods available for approximating and exactly solving differential equations. One "practical" example of how a DE can be used to model motion was given in the form of an astronaut floating through space towards a black hole. The effects of varied step sizes on incremental approximations of differential equations was studied, weighing computational time against the accuracy of the approximation. Claims that the astronaut would never visibly cross the event horizon of the black hole were investigated by way of analysis of the long-term behavior of the DE used to model the motion of the astronaut. These analyses were performed using directional fields and classification of the stability of the model equation's equilibrium solution. The claims were found to be valid, as the DE reaches equilibrium at the event horizon, independent of the passing of time.

Appendix

Solving the test initial value problem using separation of variables

- Separate Variables

$$x dx = 1 dt$$

- Integrate both sides

$$\int x dx = \int 1 dt$$

$$\frac{x^2}{2} + c_1 = t + c_2$$

$$c_1 \ and \ c_2 \ are \ the \ constants \ of \ integration$$

- Solve for x

$$x(t) = \sqrt{2t + C}$$

$$where \ C = c_1 - c_2$$

- Solve for C using the initial value

$$x(0) = 5$$

$$5 = \sqrt{C}$$

$$C = 25$$

$$x(t) = \sqrt{2t + 25}$$

Acknowledgments

Leonhard Euler, the man so good at mathematics that they had to stop abbreviating things with the letter E.