APPM 2360 Project 2: Loggerhead Turtle Population
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I. Nomenclature

F_s	=	fecundity for any given life stage
G_s	=	probability the species will survive and move onto the next stage
$\mathbf{L}$	=	transition matrix
$\mathbf{n}_t$	=	current population
P_s	=	probability the species will survive and stay in the same stage
$\mathbf{v}_i$	=	right eigenvector
$\mathbf{w}_i$	=	left eigenvector
λ	=	eigenvalue

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II. Introduction

The coastal bays and continental shelves of the Atlantic and Pacific oceans are ideal habitats for the loggerhead sea turtle. The increase in coastal development and pollution by humans has threatened the survival of the loggerheads which have been listed as a threated species 1978. Recently, some populations of turtles are nearing extinction so something must be done if the species is to survive. Luckily, with mathematics on our side we can determine which conservation methods would be most effective. In this study, matrices were used to calculate the population over time and a sensitivity and elasticity analysis was performed to see how to most effectively help the turtles. The life of loggerhead sea turtles can be broken into seven different life stage classes that each have their own probability of survival and reproduction. There are three main parameters that dictate the turtle population in the next time step: the probability that the turtles will survive and move onto the next life stage, G_s, the probability that the turtles will survive and stay in the same life stage, P_s, and the fecundity, F_s. Fecundity is the measure of the amount of offspring produced by every surviving individual in the breeding stages. Given the initial population and analyzing all of these parameters makes it possible to know the growth rate of the species and how to best protect it.

III. Bee Population

The example bee problem gives a simpler model to analyze than the sea turtle model because bees only go through four life stages instead of seven like the turtles. The transition matrix, **L**, is set up such that each row has the conditions for the population of a specific life stage with the first row corresponding to the first life stage. This is because the vector **n** is a column vector that contains the population of each life stage in order. Each column of **L** contains the data specific to a certain stage, once again, in order. Therefore, the diagonal contains the probability that the bees will survive and stay in the same life stage because the value P_s (where s is any life stage) affects the population in the same stage for the next time step so it must be in the corresponding row. The sub-diagonal is filled with the probability that the bees will survive and move on to the next stage G_s. This is the sub-diagonal because the following life stage depends on the value of G_s for the previous life stage. The structure of **L** can be visualized in a different way by looking at the equations for n_t.

$$P_1 n_t^{(1)} + F_4 n^{(4)} = n_{t+1}^{(1)} \tag{1}$$

$$G_1 n_t^{(1)} + P_2 n^{(2)} = n_{t+1}^{(2)} \tag{2}$$

$$G_2 n_t^{(2)} + P_3 n^{(3)} = n_{t+1}^{(3)} \tag{3}$$

$$G_3 n_t^{(3)} + P_4 n^{(4)} = n_{t+1}^{(4)} \tag{4}$$

There is no parameter for death because the values for G_s and P_s are probabilities and therefore less than one so death is still taken into account. If only a fraction of bees survive then the remaining percent died. When **L** is multiplied by **n** only the number of surviving bees is calculated for each stage so it is already known that the difference between the population at $t + 1$ and t is the amount of bees that died provided that the population decreases over the time step.

IV. Sea Turtle Population

The transition matrix for the loggerhead sea turtle population is a $7x7$ matrix that is set up the same way as the bee model. Besides size, the other main difference between the turtle and bee models is the turtles can produce offspring in the last three life stages instead of just the last like the bees. This makes the first row dependent on three fecundity values for the turtles. Figure 1 gives the turtle population for the first 100 time steps obtained by $\mathbf{n}_{t+1}$=$\mathbf{L}\mathbf{n}_t$.

The graph in Fig. 1 shows that initially the total population of turtles rapidly increases, reaches a peak, and then decreases steadily with time until the turtles eventually become extinct. From this information we can see that an intervention must be made if the loggerhead sea turtles are going to continue to exist. The next figure shows the same information as Fig. 1 on a logarithmic scale.

The logarithmic graph gives a better view of how the population of each life stage behaves over time. After an initial increase or decrease, the slopes for each life stage become parallel. This is because each life stage depends on the one before it with the exception of the first stage which depends on the fecundity of the last three stages, therefore, if the population of one life stage decreases at a certain rate, the other life stages will decrease at the same rate.

Changing the initial population of turtles in each life stage has little effect on the overall outcome. Testing a couple extreme cases where first, the initial population was set to 100 for each stage, and second, the order of populations

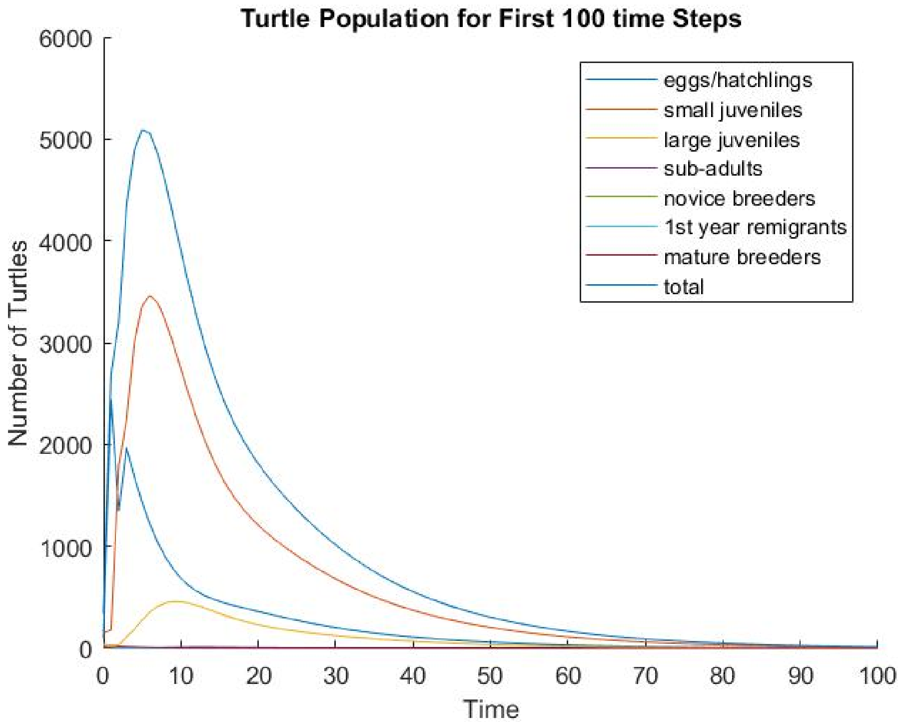

Fig. 1 Turtle Population for first 100 Time Steps

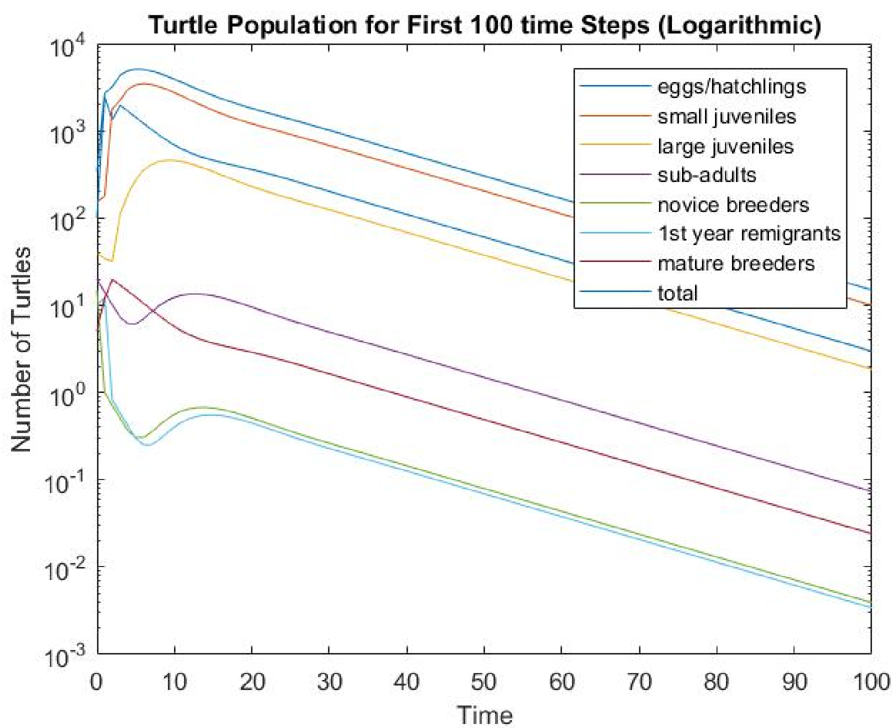

Fig. 2 Turtle Population on Logarithmic Scale

in each stage was reversed did not change the long term outcome of the total turtle population. However, changing conditions in the **L** matrix has a significant impact on the behavior over time. For example, changing the G_s for the second stage from 0.05 to 0.15 nearly stabilizes the population. The slope of the total population is still slightly negative but this shows that changing the probability of survival has the greatest impact on the turtle population.

A. Power Method

The Power Method which uses eigenvalues and eigenvectors allows for the calculation of the steady state distribution of the turtle population. The steady state distribution is the percent of the total population that is in a certain life stage. The steady state is reached when the fraction of turtles in a given stage does not change even when the total population does change.

If the eigenvectors are linearly independent, any linear combination of the vector elements will also be a solution to the system **Ax=B**. Equation 5 (Equation (4) from the project document) is the definition of an arbitrary linear combination.

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k \tag{5}$$

Therefore, if the determinant of the eigenvector matrix is not zero we know the system is linearly independent, and Eq. 5 must be true.

The value of λ is related to whether the population will grow, remain stable, or decline. If λ is greater than one, the population will increase, if λ is less than one, the population will decline. If the value of λ is exactly one, the population will not change over time. The asymptotic growth rate of the given population values is 0.9416, and the intrinsic growth rate is -0.0602. This means that the population of turtles will decrease over time and eventually become extinct.

Steady state population distribution	
Eggs/Hatchlings	19.881%
Small Juveniles	67.070%
Large Juveniles	12.349%
Sub-Adults	0.491%
Novice Breeder	0.026%
1st Year Remigrant	0.023%
Mature Breeder	0.161%

Changing the transition matrix to simulate the case where 100% of the hatchlings surviving results in an asymptotic growth rate of 1.0307 and an intrinsic growth rate of 0.0302. This shows that if an effort is made to ensure that all of the hatchlings survive, the turtle population can be brought above the equilibrium point and flourish.

B. Sensitivity and Elasticity Analysis

The built in MATLAB function "eig" returns a diagonal matrix of eigenvalues. This is a matrix whose columns correspond to right eigenvectors and also returns a full matrix W whose columns correspond to the left eigenvectors. To show that $\mathbf{x}_i$ is a right eigenvector of $\mathbf{A}^T$ and $\mathbf{x}_i^T$ is a left eigenvector of $\mathbf{A}$ the eigenvectors of both the matrix and its transpose were calculated.

The sensitivity and elasticity analysis gives information about which parameters have the greatest effect on the population growth rate. In order to perform these analyses the left eigenvector must be calculated. The right eigenvectors are column vectors of a matrix $\mathbf{A}^T$ designated as $\mathbf{x}_i$ and the left eigenvectors are row vectors of the matrix $\mathbf{A}$. This is because transposing a matrix switches the rows and columns.

The sensitivities of the graph can be seen below, but they do not necessarily correlate to effectiveness of preserving the long-term stability of the turtle population. This can be seen in the graph of the elasticity below. This is a normalized measure of the effect that any individual population factor will have on the long term population behaviors. Since the sensitivity is a normalized measure, we can use an equal y-span on each graph to get a true measure of which factors are most effective in assisting the population.

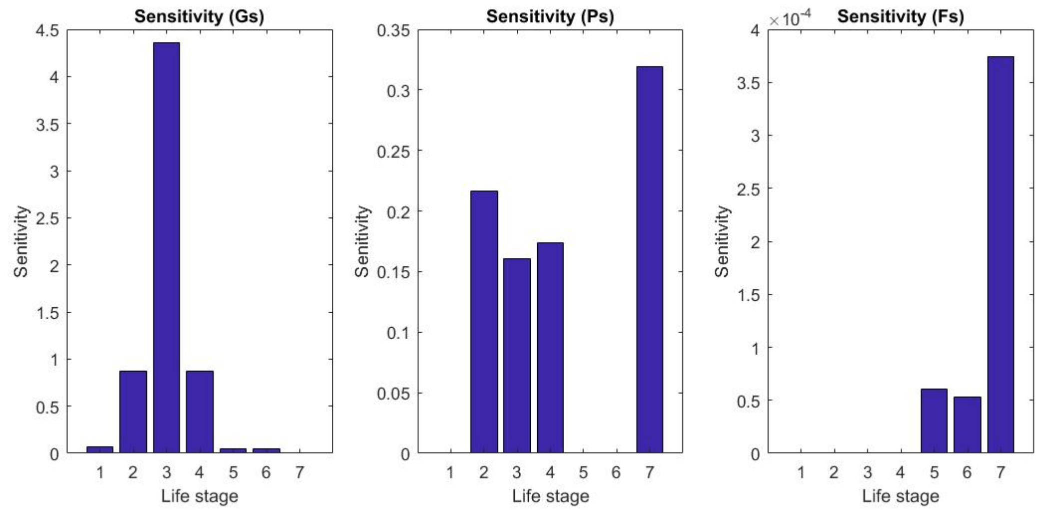

Fig. 3 Sensitivity of population factors

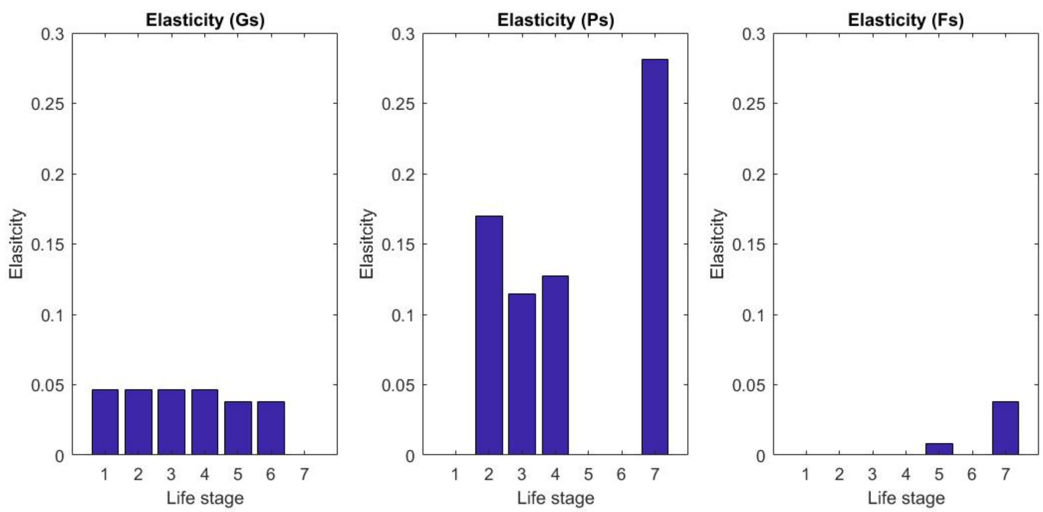

Fig. 4 Elasticity of population factors

Here it can be seen that the most effective factors in the survival of the loggerhead turtles are to ensure the probability of survival of the young juveniles and the mature adults. So long as the hatchling turtles make it to the ocean, they stand a strong chance of survival until the breeding period of their lives. The largest factor in the survivability rate of mature turtles are human influences such as hunting and fishing nets. These can be mitigated through spreading awareness and implementing methods and laws to reduce the number of turtles killed.

Efforts are already being made to assist the survival of loggerhead sea turtles. A Turtle Excluder Device (TED) is a device that keeps turtles out of trawl fishing nets so that turtles don't get harmed or killed by fishing operations. The TED consists of a grid of bars that has an opening located at either the top or bottom of the trawl net so that turtles and other large species cannot pass through the bars and exit the net through the hole. There are other types of TEDs that use different methods to sort the fish and TEDs can be made of soft or hard materials. According to [1], TEDs are 97% effective in reducing the amount of sea turtles killed by trawl nets. TED is one step that humanity has taken to save the turtles but more must be done if the species is to survive.

V. Conclusion

Loggerhead sea turltes are one of many engered species on Earth today. Invsigations such as analyzing the growth rate, steady state population distribution, sensitivity, and elasticity of a species can provide insight into how to best ensure their survial. The given data showed that if nothing is done the turtle population will eventually die out. However, the analysis showed that if efforts are made to help the small juvenile and mature aldult turtles the species could be saved.

References

[1] Mustain, P., "TEDs: A Net Positive for Fish and Sea Turtles," , 2016.

[2] NOAA, "Turtle Excluder Devices," , 2014.

[3] *APPM 2360 Project 2: Exploring Stage-Structured Population Dynamics with Loggerhead Sea Turtles*, APPM 2360, 2018.

Appendix

A. MATLAB Code

All calculations were performed with this MATLAB script.

```
close all
clear all
clc

% define initial population and L
n{1} = [100;150;40;20;15;10;5]; % initial amount of individuals in each life stage
%n{1} = [5;10;15;20;40;150;100]; %reversed population
%n{1} = [100;100;100;100;100;100;100]; %even start point
L =  [0 0 0 0 127 6 95;
      0.68 0.74 0 0 0 0 0;
      0 0.05 0.67 0 0 0 0;
      0 0 0.01 0.69 0 0 0;
      0 0 0 0.05 0 0 0;
      0 0 0 0 0.82 0 0;
      0 0 0 0 0 0.79 0.83];
t = [0:100];

% calculate the population for the first 100 time steps
for i = 1:100
    n{i+1} = L*n{i};
end


for i=1:101
    eggs(i) = n{i}(1);
    small(i) = n{i}(2);
    large(i) = n{i}(3);
    sub(i) = n{i}(4);
    nov(i) = n{i}(5);
    re(i) = n{i}(6);
    mature(i) = n{i}(7);
    total(i) = sum(n{i});
end
% plot data
```

```
figure
hold on
plot(t,eggs)
plot(t,small)
plot(t,large)
plot(t,sub)
plot(t,nov)
plot(t,re)
plot(t,mature)
plot(t,total)
title('Turtle Population for First 100 time Steps')
xlabel('Time')
ylabel('Number of Turtles')
legend('eggs/hatchlings','small juveniles','large juveniles','sub-adults','novice breeders', '1st year remigrants','mature breeders','total')

figure
% plot data on logarithmic scale
semilogy(t,eggs)
hold on
semilogy(t,small)
semilogy(t,large)
semilogy(t,sub)
semilogy(t,nov)
semilogy(t,re)
semilogy(t,mature)
semilogy(t,total)
title('Turtle Population for First 100 time Steps (Logarithmic)')
xlabel('Time')
ylabel('Number of Turtles')
legend('eggs/hatchlings','small juveniles','large juveniles','sub-adults','novice breeders', '1st year remigrants','mature breeders','total')


%eigenvector analysis
[V,D]=eig(L); %V = Right Eigenvectors, D = Eigenvalues, W = Left Eigenvectors
realV=real(V(:,7)); %real values of highest magnitude right eigenvector
realsD=real(D(:,:)); %real values of right eigenvalue result
[W,D]=eig(L.'); %W = Left Eigenvector, D = Eigenvalues
%realsD=real(D(:,:)); %real values of left eigenvalue result
%W=conj(W);
realW=real(W(:,4)); %real values of highest magnitude left eigenvector
detEig=det(V); %determinant of eigenvectors, ensure Linear Independence, should be ? 0

AsymGrowth=max(max(realsD)); %asymptotic growth rate (lambda)
IntrinGrowth=log(AsymGrowth); %intrinsic growth rate

%%%Normalize eigenvectors
Vnorm=sum(realV);
Wnorm=sum(realW);
for i=1:7
    normV(i,1)=realV(i)/Vnorm;
    normW(1,i)=realW(i)/Wnorm;
```

```
end


%%%Sensitivity/Elasticity Matrices
for i=1:7
    for j=1:7
        if L(i,j) ~= 0 %only perform analysis on nonzero elements
            sens(i,j)=(normW(i)*normV(j))/(dot(normV,normW)); %populate
                sensitivity matrix
            elas(i,j)=(L(i,j)/AsymGrowth)*sens(i,j); %populate elasticity
                matrix
            elseif L(i,j) == 0
            sens(i,j) = 0;

        end
    end
end

Ge(1)=elas(2,1);
Ge(2)=elas(3,2);
Ge(3)=elas(4,3);
Ge(4)=elas(5,4);
Ge(5)=elas(6,5);
Ge(6)=elas(7,6);
Ge(7)=0;

for i=1:7
    Pe(i)=elas(i,i);
end

Fe=zeros(7,1);
Fe(5)=elas(1,5);
Fe(6)=elas(1,6);
Fe(7)=elas(1,7);

figure
subplot(1,3,1)
bar(Ge);
title('Elasticity (Gs)')
xlabel('Life stage')
ylabel('Elasitcity')
axis([0 8 0 0.3])
subplot(1,3,2)
bar(Pe);
title('Elasticity (Ps)')
xlabel('Life stage')
ylabel('Elasitcity')
subplot(1,3,3)
bar(Fe);
title('Elasticity (Fs)')
axis([0 8 0 0.3])
xlabel('Life stage')
ylabel('Elasitcity')
```

```
Gs(1)=sens(2,1);
Gs(2)=sens(3,2);
Gs(3)=sens(4,3);
Gs(4)=sens(5,4);
Gs(5)=sens(6,5);
Gs(6)=sens(7,6);
Gs(7)=0;

for i=1:7
    Ps(i)=sens(i,i);
end

Fs=zeros(7,1);
Fs(5)=sens(1,5);
Fs(6)=sens(1,6);
Fs(7)=sens(1,7);

figure
subplot(1,3,1)
bar(Gs);
title('Sensitivity (Gs)')
xlabel('Life stage')
ylabel('Senitivity')
subplot(1,3,2)
bar(Ps);
title('Sensitivity (Ps)')
xlabel('Life stage')
ylabel('Senitivity')
subplot(1,3,3)
bar(Fs);
title('Sensitivity (Fs)')
xlabel('Life stage')
ylabel('Senitivity')
```