APPM 2360 Project 3: Population Model
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I. Nomenclature

x_1	=	Predator Population
x_2	=	Prey population
a	=	Mortality rate of predator
b	=	Reproduction rate of predator per prey
c	=	Reproduction rate of prey
d	=	Mortality rate of predator per prey
t	=	time
k	=	Intensity constant

II. Introduction

Predator-prey relationships can be modeled multiple ways depending on the species being studied, their habitat, and their behavior. Here, three predator-prey models are investigated: Lotka-Volterra, logistic, and migration. The Lotka-Volterra model is the simplest and it provides an ideal scenario where no carrying capacity is included. In this model, the prey population grows exponentially in the absence of predators. The logistic model provides a more realistic prediction of how the populations will behave because it includes a carrying capacity for the prey based on the available resources necessary for the prey to live. If the carrying capacity is exceeded, the prey population will decrease and once the population is small it will increase at a rate proportional to its size. The third model includes periodic forcing to represent migration of the prey. Each system can be analyzed to show the long term behavior of the populations and the stability of the systems' equilibrium points.

III. Lotka-Volterra Qualitative Analysis

The nullclines and equilibrium points were found by setting the differential equations x_1' and x_2' equal to zero. The horizontal nullclines are at $x_2 = 0$ and $x_1 = \frac{c}{d}$ obtained by solving $cx_2 - dx_1x_2 = 0$. The vertical nullclines are $x_1 = 0$ and $x_2 = \frac{a}{b}$ found by solving $-ax_1 + bx_1x_2 = 0$. The equilibrium points are where both x_1' and x_2' equal zero and were found graphically at the intersection of the horizontal and vertical nullclines. The location of the equilibrium points are the origin and the point $(\frac{c}{d}, \frac{a}{b})$.

The eigenvalues of the Jacobian were evaluated at each equilibrium point. At $(0, 0)$ $\lambda_1 = -a$ and $\lambda_2 = c$. Here, one eigenvalue is positive and the other is negative, meaning that the origin is a saddle point and therefore an unstable equilibrium. In the physical model this means that with initial populations near the origin, the prey population will grow infinitely and the predator population will decrease to zero. At $(\frac{c}{d}, \frac{a}{b})$ the eigenvalues are $\lambda_3 = i\sqrt{ac}$ and $\lambda_4 = -i\sqrt{ac}$. These eigenvalues have a zero real part, therefore, the equilibrium point is neutrally stable.

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IV. Lotka-Volterra Numerical Analysis

The nullclines of the system of DE's show where the vertical and horizontal slopes will be zero. Upon crossing these lines, the direction of all slopes will switch sign according to which nullcline it crossed.

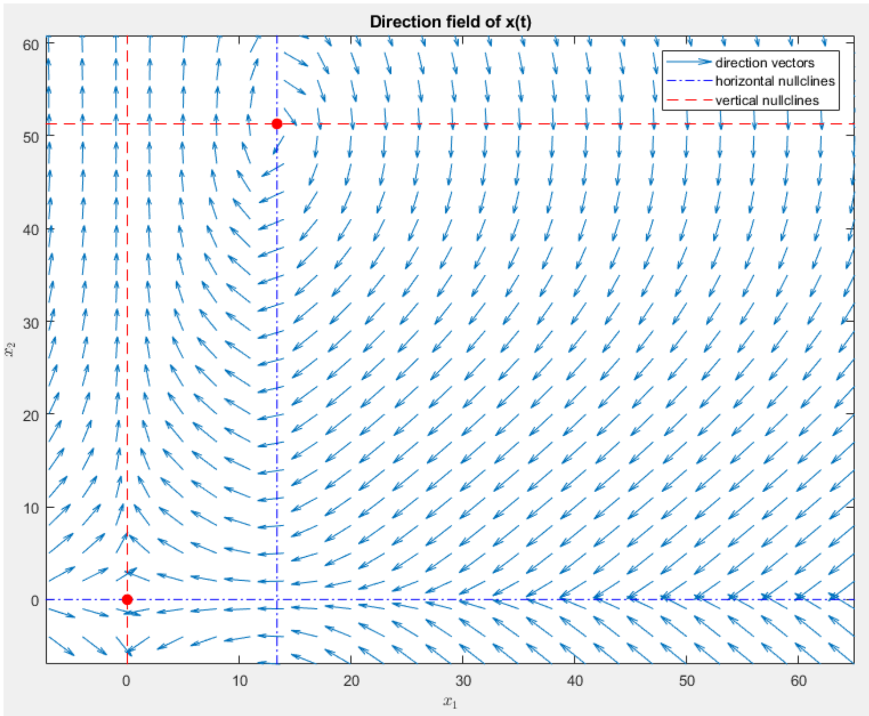

Fig. 1 Lotka-Volterra direction field

Figure 2 is the Lotka-Volterra predator prey model with the prey population plotted against the predator population (left) and the populations plotted against time (right).

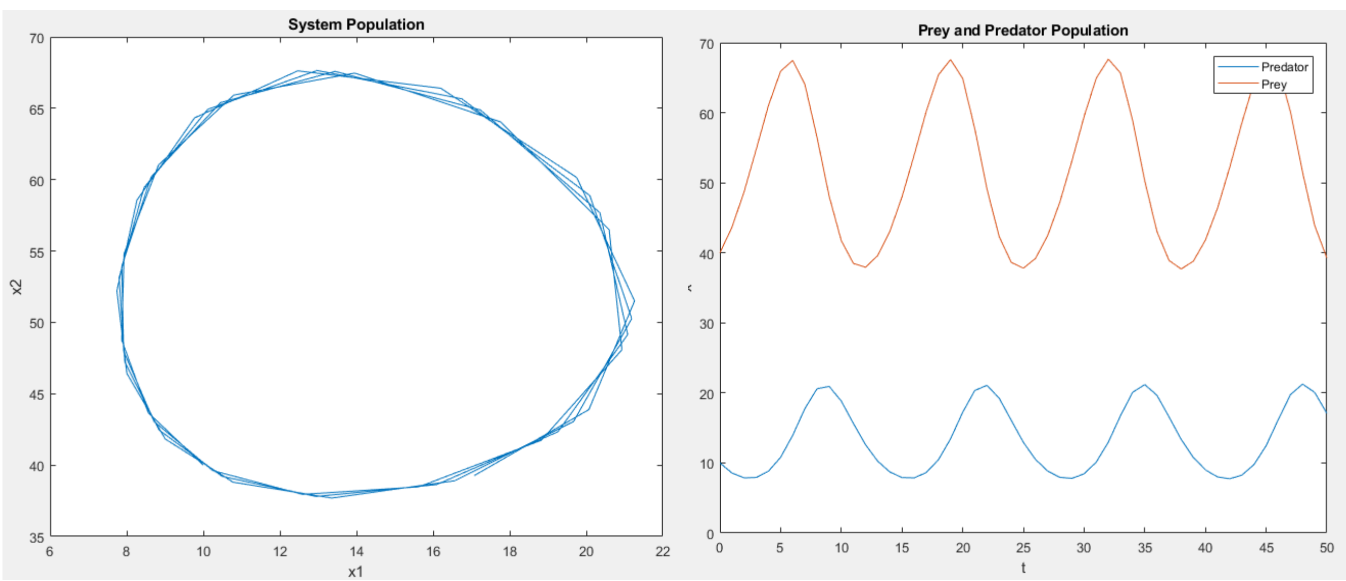

Fig. 2 Lotka-Volterra Model

The left-right motion of the phase plane is proportional to the slope of the predator component curve. Similarly, the up-down motion of the phase plane is proportional to the slope of the prey component curve. The period is approximately 13 time steps for both component curves, determined by finding the difference in x values from peak to peak. The Lotka-Volterra model shows that the prey and predators will survive indefinitely, fluctuating between high and low points in overall population.

V. Logistic Model

In the logistic model, a factor of k is taken into account where k is some "intensity" constant that is proportional to the rate of regression to the long-term behavior. As it is in the same term of the DE as c, k would have the same units as c: time^{-1}.

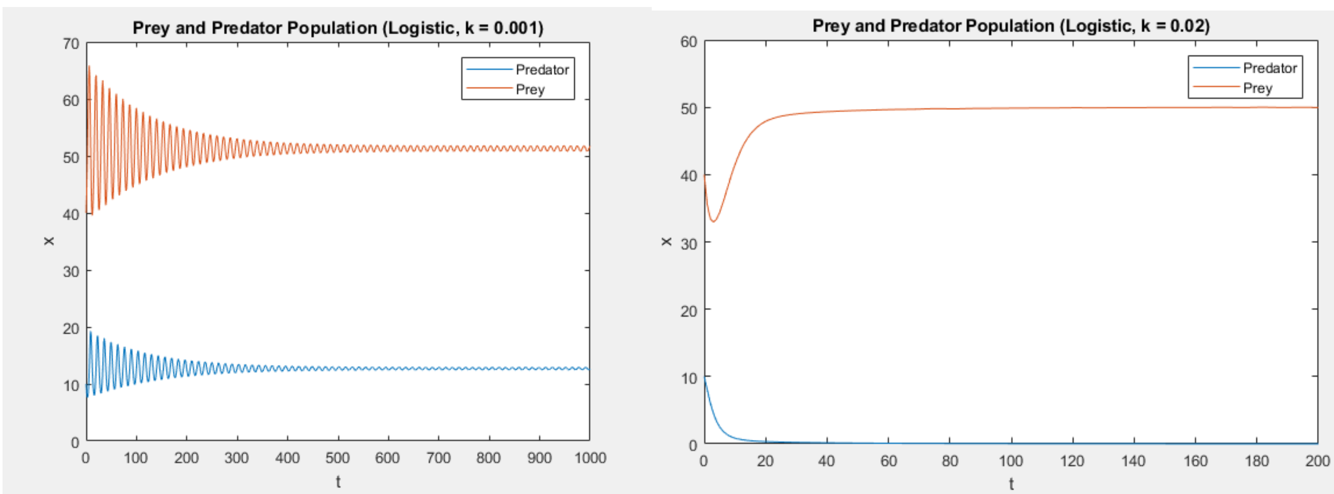

Fig. 3 Logistic models with varying k values

The Equilibrium solutions exist along both axes, with the only restriction being that both populations initial conditions must be in the first quadrant because it is impossible to have a negative population. The equilibrium points were found to be $(0, 0)$, $(0, \frac{1}{k})$, and $((\frac{a}{b} - \frac{1}{k})(\frac{-ck}{d}), \frac{a}{b})$. The solutions are periodic but have decreasing amplitude as t increases towards infinity. The solutions are non-asymptotic, and will eventually converge to their carrying capacities and begin oscillating about their stable state.

The predator and prey populations of the logistic model with $k = 0.001$ oscillates as t -> ∞ and quickly reaches the carrying capacity of prey which is around 50 individuals. The predator population declines from the start and prior to $t = 20$, the predator population goes extinct. The prey population reaches its carrying capacity in the absence of predators as t -> ∞.

VI. Migration Model

In the model that accounts for migration, by looking at smaller timescales the effects of the prey migration on the predator population become clear. The predator population continues to grow after the migration of the prey has begun. With each migration season, the prey population grows larger than before, and the predator population reactively grows as well.

Eventually, the prey reach a peak capacity as indicated by the largest amplitude of the wave that bounds the population curve. After this capacity is reached, the prey population can no longer grow fast enough to outpace the size of the predator population. Both populations then begin to decrease with each migration season until there are few enough predators that the prey population can begin to expand again.

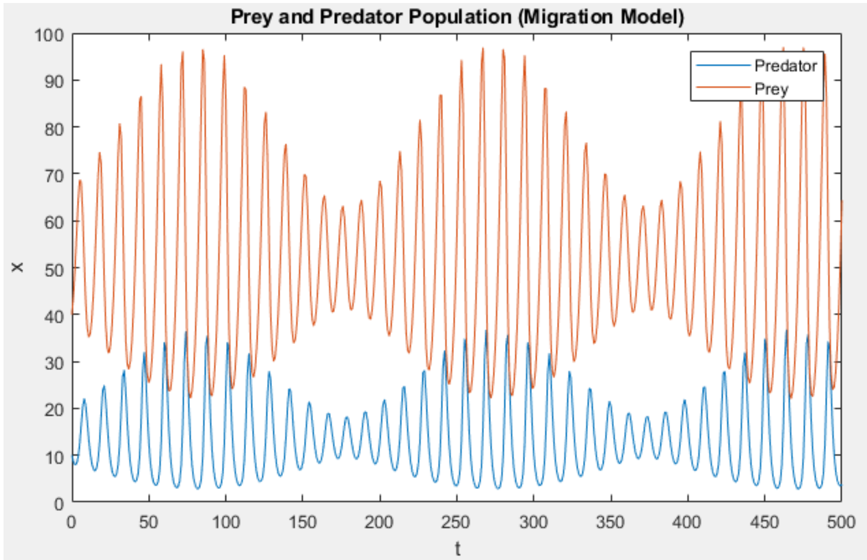

Fig. 4   Migration model

VII. Conclusion

In this project, models corresponding to various mechanisms were explored via systems of differential equations. An ideal model is represented by an infinitely oscillating pair of equations, but is not representative of any system that exists in the real world. The logistic model introduces a carrying capacity; the rate at which the system equilibrates is proportional to the value of k. In the migration model, the prey and predators alternate between being the ‘driving force’ in the system, causing the carrying capacity for the prey to oscillate over many cycles.

More complex systems of differential equations can be used to simulate other ecological factors such as larger numbers of species, limited resources, and seasonal effects. Perhaps a method similar to the matrix method of modeling sea turtle populations used in project 2 could be utilized. Systems of differential equations would serve as the matrix entries, with parameterized equations being the solutions. This would allow for modeling of numerous factors interacting with each other simultaneously. We can already see a similar effect by incorporating the logistic equation into the migration model. The populations undergo a transient initial state, before stabilizing to an equilibrium similar to the logistic model without migration.

Systems of differential equations can be used to model the behaviors of populations while accounting for any number of factors. As the number of factors affecting each element increase, the system becomes more complex with less noticeable trends developing. Despite the increased complexity, a population’s behavior can be reasonably predicted with an accurate enough set of equations.

Appendix

Matlab Code

```
%find nullclines
clear all
close all
clc

```

```
a = 0.831;
b = 0.0162;
c = 0.2824;
d = 0.0211;
t1 = 0:0.1:100;
t2 = 0:0.1:100;
x1Inish = 10;
x2Inish = 40;
tspan = 0:50;
f = @(t,x) [(-a*x(1)) + (b * x(1) * x(2));(c* x(2)) - (d*x(1)*x(2))];
[t,x1] = ode45(f,tspan,[10 40]);
figure(1)
plot(x1(:,1),x1(:,2))
xlabel('x1')
ylabel('x2')
title('System Population')

figure(2)
plot(t,x1(:,1))
hold on
plot(t,x1(:,2))
xlabel('t')
ylabel('x')
title('Prey and Predator Population')
legend('Predator','Prey')

%%%Logistic Model
k=0.001;
tspan=0:200;
g = @(t,x) [(-a*x(1)) + (b * x(1) * x(2));(c* x(2)*(1-k*x(2))) - (d*x(1)*x(2))];
[t2, x2] = ode45(g,tspan,[10 40]);
figure(3)
plot(t2,x2(:,1))
hold on
plot(t2,x2(:,2))
xlabel('t')
ylabel('x')
title('Prey and Predator Population (Logistic, k = 0.001)')
legend('Predator','Prey')


k=0.02;
tspan=0:200;
g = @(t,x) [(-a*x(1)) + (b * x(1) * x(2));(c* x(2)*(1-k*x(2))) - (d*x(1)*x(2))];
[t3, x3] = ode45(g,tspan,[10 40]);
figure(4)
plot(t3,x3(:,1))
hold on
plot(t3,x3(:,2))
xlabel('t')
ylabel('x')
title('Prey and Predator Population (Logistic, k = 0.02)')
```

```
legend('Predator','Prey')

%%%Migration Model
M = 0.02;
w=(2*pi)/13;
tspan=0:500;
g = @(t,x) [(-a*x(1)) + (b * x(1) * x(2));(c* x(2)) - (d*x(1)*x(2))+(M*x(2)*
    sin(w*t))];
[t4, x4] = ode45(g,tspan,[10 40]);
figure(5)
plot(t4,x4(:,1))
hold on
plot(t4,x4(:,2))
xlabel('t')
ylabel('x')
title('Prey and Predator Population (Migration Model)')
legend('Predator','Prey')
%find nullclines
clear all
close all
clc

a = 0.831;
b = 0.0162;
c = 0.2824;
d = 0.0211;
t1 = 0:0.1:100;
t2 = 0:0.1:100;
x1Inish = 10;
x2Inish = 40;
tspan = 0:50;
f = @(t,x) [(-a*x(1)) + (b * x(1) * x(2));(c* x(2)) - (d*x(1)*x(2))];
[t,x1] = ode45(f,tspan,[10 40]);
figure(1)
plot(x1(:,1),x1(:,2))
xlabel('x1')
ylabel('x2')
title('System Population')

figure(2)
plot(t,x1(:,1))
hold on
plot(t,x1(:,2))
xlabel('t')
ylabel('x')
title('Prey and Predator Population')
legend('Predator','Prey')

%%%Logistic Model
k=0.001;
tspan=0:200;
g = @(t,x) [(-a*x(1)) + (b * x(1) * x(2));(c* x(2)*(1-k*x(2))) - (d*x(1)*x(2))
    ];
[t2, x2] = ode45(g,tspan,[10 40]);
```

```
figure(3)
plot(t2,x2(:,1))
hold on
plot(t2,x2(:,2))
xlabel('t')
ylabel('x')
title('Prey and Predator Population (Logistic, k = 0.001)')
legend('Predator','Prey')


k=0.02;
tspan=0:200;
g = @(t,x) [(-a*x(1)) + (b * x(1) * x(2));(c* x(2)*(1-k*x(2))) - (d*x(1)*x(2))
    ];
[t3, x3] = ode45(g,tspan,[10 40]);
figure(4)
plot(t3,x3(:,1))
hold on
plot(t3,x3(:,2))
xlabel('t')
ylabel('x')
title('Prey and Predator Population (Logistic, k = 0.02)')
legend('Predator','Prey')

%%%Migration Model
M = 0.02;
w=(2*pi)/13;
tspan=0:500;
g = @(t,x) [(-a*x(1)) + (b * x(1) * x(2));(c* x(2)) - (d*x(1)*x(2))+(M*x(2)*
    sin(w*t))];
[t4, x4] = ode45(g,tspan,[10 40]);
figure(5)
plot(t4,x4(:,1))
hold on
plot(t4,x4(:,2))
xlabel('t')
ylabel('x')
title('Prey and Predator Population (Migration Model)')
legend('Predator','Prey')
```

```


%%%%flow.m
close all; clear all;
% Set the axis limits
x1min = -25; x1max = 100; x2min = -25; x2max = 100;
%set step size for x1 and x2;
x1step = 1; x2step = 1;
%generate mesh for plotting
[x1, x2] = meshgrid(x1min:3:x2max, x2min:3:x2max);

%set parameter values
a = 0.831;
b = 0.0162;
```

```
c = 0.2824;
d = 0.0211;
k = 0.001;
x1_0=10;
x2_0=40;

% Define the system of equations  -- Lotka-Volterra
%for this example we consider the equation
%               dx1/dt = a*x2
%               dx2/dt = -x1

dx1 = -(a*x1)+b*(x1.*x2);
dx2 = (c*x2)-d*(x1.*x2);
x1nullA=0;
x1nullB=c/d;
x2nullA=0;
x2nullB=a/b;

%normalize vectors (to help plotting)
dx1 = dx1./sqrt(dx1.^2 + dx2.^2);
dx2 = dx2./sqrt(dx1.^2 + dx2.^2);
% Generate the vector field
figure
quiver(x1, x2, dx1,dx2,'AutoScaleFactor',0.8)

%change axes limits, add labels
axis([x1min x1max x2min x2max])
xlabel('$x_1$','Interpreter','latex')
ylabel('$x_2$','Interpreter','latex')

vnull=refline([0,x1nullA]);
vnull.Color='blue';
vnull.LineStyle='-.';
hnull = refline([0,x2nullB]');
hnull.Color='r';
hnull.LineStyle='--';
line([0,0],ylim,'Color','red','LineStyle','--')
line([x1nullB,x1nullB],ylim,'Color','blue','LineStyle','-.')
hold on
plot(0, 0, '.r', 'MarkerSize',25)
plot(x1nullB, x2nullB, '.r', 'MarkerSize',25)
title('Direction field of x(t)')
legend('direction vectors','horizontal nullclines','vertical nullclines')
Google docs is disgusting
```