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CSCI 3104, Algorithms **Charlie Carlson & Ewan Davies**
Problem Set 2 – Due Sept. 10, 2020 **Fall 2020, CU-Boulder**

Advice 1: For every problem in this class, you must justify your answer: show how you arrived at it and why it is correct. If there are assumptions you need to make along the way, state those clearly.

Advice 2: Informal reasoning is typically insufficient for full credit. Instead, write a logical argument, in the style of a mathematical proof.

Instructions for submitting your solutions:

- The solutions **should be typed using** LaTeXand we cannot accept hand-written solutions. Here's a short intro to LaTeX.
- You should submit your work through the **class Canvas page** only.
- You may not need a full page for your solutions; pagebreaks are there to help Gradescope automatically find where each problem is. Even if you do not attempt every problem, please submit this template of at least 6 pages (or Gradescope has issues with it). **We will not accept submissions with fewer than 5 pages**.
- **You must CITE any outside sources you use, including websites and other people with whom you have collaborated. You do not need to cite a CA, TA, or course instructor.**
- **Posting questions to message boards or tutoring services including, but not limited to, Chegg, StackExchange, etc., is STRICTLY PROHIBITED. Doing so is a violation of the Honor Code.**

Quicklinks: 1 (2a) (2b) (2c) (2d) (2e) (2f)

Problem 1. Name (a) one advantage, (b) one disadvantage, and (c) one alternative to worst-case analysis. For (a) and (b) you should use full sentences.

Answer:

- One advantage of using worst case analysis, if any changes are made to the algorithm, the new worst case analysis could be compared to the old to see how it affects the compl
- One disadvantage of worst case analysis is it is not very descriptive of the time complexity of your algorithm. It gives no description towards how fast it could run, only the upper bound.
- An alternative to worst-case analysis could be to use an average-case analysis.

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Problem 2. Put the growth rates in order, from slowest-growing to fastest. That is, if your answer is $f_1(n), f_2(n), \ldots, f_k(n)$, then $f_i(n) \leq O(f_{i+1}(n))$ for all i. If two adjacent ones have the same order of growth (that is, $f_i(n) = \Theta(f_{i+1}(n))$), you must specify this as well. Justify your answer (show your work).

- You may assume transitivity: if $f(n) \leq O(g(n))$ and $g(n) \leq O(h(n))$, then $f(n) \leq O(h(n))$, and similarly for little-oh, etc. Note that the goal is to order the growth rates, so transitivity is very helpful. We encourage you to make use of transitivity rather than comparing all possible pairs of functions, as using transitivity will make your life easier.
- You may also use the Limit Comparison Test (see Michael's Calculus Notes on Canvas). However, you **MUST** show all limit computations at the same level of detail as in Calculus I-II. Should you choose to use Calculus tools, whether you use them correctly will count towards your mastery score.
- You may **NOT** use heuristic arguments, such as comparing degrees of polynomials or identifying the "high order term" in the function.
- If it is the case that $g(n) = c \cdot f(n)$ for some constant c, you may conclude that $f(n) = \Theta(g(n))$ without using Calculus tools. You must clearly identify the constant c (with any supporting work necessary to identify the constant- such as exponent or logarithm rules) and include a sentence to justify your reasoning.

(2a) *Polynomials.*

$$3n+1, \qquad n^6, \qquad \frac{1}{n}, \qquad 1, \qquad n^2+3n-5, \qquad n^2, \qquad \sqrt{n}, \qquad 10^{100}.$$

Using the Limit Comparisson test, Let $f(n) = n^2 + 3n - 5$ and $g(n) = n^2$

$$L := \lim_{n \to \infty} \frac{n^2+3n-5}{n}$$

$$L := \lim_{n \to \infty} \frac{n^2}{n} + \lim_{n \to \infty} \frac{3n}{n} - \lim_{n \to \infty} \frac{5}{n}$$

$$L := \infty + 3 - 0 = \infty$$

Proving that $n^2 \leq O(n^2 + 3n - 5)$

Let $f(n) = n^2 + 3n - 5$ and $g(n) = n^6$

$$L := \lim_{n \to \infty} \frac{n^2+3n-5}{n^6}$$

$$L := \lim_{n\to\infty}\frac{1}{n^4} + \lim_{n\to\infty}\frac{3}{n^5} - \lim_{n\to\infty}\frac{5}{n^6}$$
$$L := 0+0+0 = 0$$

$L := 0, n^2 + 3(n) - 5 \leq (n^6)$ assuming transivity $n^2 + 3n - 5 \leq O(n^6)$ then $n^2 \leq O(n^6)$

Let $f(n) = 3n+1$ and $g(n) = n^2$

$$L := \lim_{n\to\infty}\frac{3n+1}{n^2}$$

$$L := \lim_{n\to\infty}\frac{3}{n} + \lim_{n\to\infty}\frac{1}{n^2}$$
$$L := 0+0 = 0$$

then $3n+1 \leq O(n^2)$ assuming transivity $3n+1 \leq O(n^2) \leq O(n^2+3n-5) \leq O(n^6)$

let $f(n) = 3n+1$ and $g(n) = \sqrt{n}$

$$L := \lim_{n\to\infty}\frac{3n+1}{\sqrt{n}}$$

$$L := \lim_{n\to\infty} 3\sqrt{n} + \lim_{n\to\infty}\frac{1}{\sqrt{n}}$$
$$L := \infty + 0 = \infty$$

Then $3n+1 \geq \Omega(\sqrt{n})$ or $\sqrt{n} \leq O(3n+1)$
Assuming Transivity $\sqrt{n} \leq O(3n+1) \leq O(n^2) \leq O(n^2+3n-5) \leq O(n^6)$

let $f(n) = \sqrt{n}$ and $g(n) = \frac{1}{n}$

$$L := \lim_{n\to\infty}\frac{\sqrt{n}}{\frac{1}{n}}$$

$$L := \lim_{n\to\infty} n\sqrt{n}$$

$$L := \infty$$

Because $L := \infty$ $\frac{1}{n} \leq O(\sqrt{n})$

Assuming Transivity, $\frac{1}{n} \leq O(\sqrt{n}) \leq O(3n+1) \leq O(n^2) \leq O(n^2+3n-5) \leq O(n^6)$
Let $f(n) = 1$ and $g(n) = \sqrt{n}$

$$L := \lim_{n \to \infty} \frac{1}{\sqrt{n}}$$

$$L := 0$$

, then $1 \leq O(\sqrt{n})$
Let $f(n) = 1$ and $g(n) = \frac{1}{n}$ so that

$$L := \lim_{n \to \infty} \frac{1}{\frac{1}{n}}$$

$$L := \lim_{n \to \infty} n = \infty$$

beacuse $L := \infty$ then $1 \geq \Omega(\frac{1}{n})$

assuming transivity $\frac{1}{n} \leq O(1) \leq O(\sqrt{n}) \leq O(3n+1) \leq O(n^2) \leq O(n^2+3n-5) \leq O(n^6)$

$1 < 10^{100}$ becasue $1 < C * 10^{100}$ for all $n \geq 0$
Let $f(n) = \sqrt{n}$ and $g(n) = 10^{100}$, $L := \lim_{n \to \infty} \frac{\sqrt{n}}{10^{100}} = \infty$ we have $10^{100} \leq O(\sqrt{n})$

Finally assuming transivity we reach the final answer

$$\frac{1}{n} \leq O(1) \leq O(10^{100}) \leq O(\sqrt{n}) \leq O(3n+1) \leq O(n^2) \leq O(n^2+3n-5) \leq O(n^6)$$

(2b) Prove that for any $a, b > 0$ where $a \neq 1$ and $b \neq 1$, that $\log_a(n) = \Theta(\log_b(n))$. Here, a and b do not depend on n. [**Hint:** Review the change of base formula.]

Proof. Using limit comparisson:

$$L := \lim_{n \to \infty} \frac{f(n)}{g(n)}$$

where $f(n) = log_a(n)$ and $g(n) = log_b(n)$

$$L := \lim_{n \to \infty} \frac{log_a(n)}{log_b(n)}$$

using the change of base formula we obtain the following

$$L := \lim_{n \to \infty} \frac{\frac{log(n)}{log(a)}}{\frac{log(n)}{log(b)}}$$

simplifying to

$$L := \lim_{n \to \infty} \frac{log(b)}{log(a)}$$

Because a and b are greater than zero and not equal to one we know the resulting limit will be a constant C between the bounds $0 < C < \infty$, the limit comparison test tells us if $0 < L < 100$ then $f(n) = \Theta(g(n))$ where

$$log_a(n) = \Theta(log_b(n))$$

□

(2c) *Logarithms and related functions. [**Hint** Use part (2b).]*

$$(\log_3(n))^3 \qquad \log_5(n) \qquad \log_3(n) \qquad \sqrt[3]{n} \qquad \log_{2.5}(n) \qquad \log_5(n^2)$$

Using the above proof that $log_a(n) = \Theta(log_b(n))$for any $a, b > 1$ we can then state that $log_{2.5}(n) = \Theta(log_3(n))$ and $log_3(n) = \Theta(log_5(n))$ and therefore $log_{2.5}(n) = \Theta(log_5(n))$

Using the limit comparison test, we let $f(n) = (log_3(n))^3$ and $g(n) = log_3(n)$

$$L := \lim_{n\to\infty} \frac{(log_3(n))^3}{log_3(n)}$$

$$L := \lim_{n\to\infty} (log_3(n))^2$$

$$L := \infty$$

therefore $log_3(n) \leq O((log_3(n))^3)$
asuming transivity, $log_{2.5}(n) = \Theta(log_3(n)) = \Theta(log_5(n)) \leq O((log_3(n))^3)$

running a limit test again letting $f(n) = log_5(n^2)$ and $g(n) = log_5(n)$

$$L := \lim_{n\to\infty} \frac{log_5(n^2)}{log_5(n)}$$

$$L := \lim_{n\to\infty} \frac{2 * log_5(n)}{log_5(n)}$$

$$L := \lim_{n\to\infty} 2 = 2$$

Becasue $L := 2$, $Log_5(n^2) = \Theta(log_5(n))$
Assuming transivity, $log_{2.5}(n) = \Theta(log_3(n)) = \Theta(log_5(n)) = \Theta(log_5(n^2)) \leq O((log_3(n))^3)$

Lastly run a limit test letting $f(n) = log_5(n^2)$ and $g(n) = \sqrt[3]{n}$

$$L := \lim_{n\to\infty} \frac{log_5(n^2)}{\sqrt[3]{n}}$$

Using **L'Hopitals**

$$L := \lim_{n\to\infty} \frac{\frac{ln(n)}{ln(5)}}{\frac{1}{3n^{\frac{2}{3}}}}$$

$$L := \lim_{n\to\infty} \frac{3n^{\frac{2}{3}} * ln(n)}{ln(5)}$$

$$L := \lim_{n \to \infty} \frac{3n^{\frac{2}{3}}}{ln(5)} * \lim_{n \to \infty} \frac{ln(x)}{ln(5)}$$
$$L := \infty$$

so that $\sqrt[3]{n} \leq O(log_5(n^2))$

Assuming transivity the order of growth is stated below:

$$\sqrt[3]{n}, log_{2.5}(n), log_3(n)), log_5(n), log_5(n^2), (log_3(n))^3$$

where

$$log_{2.5}(n) = \Theta(log_3(n)) = \Theta(log_5(n)) = \Theta(log_5(n^2))$$

(2d) Construct specific functions $f(n)$ and $g(n)$ such that $f(n) = \Theta(g(n))$ but $2^{f(n)} \neq \Theta(2^{g(n)})$. Formally show that $2^{f(n)} \neq \Theta(2^{g(n)})$ here.
Solution: Using limit comparison, let $f(n) = n$ and $g(n) = 2n$

$$L := \lim_{n \to \infty} \frac{n}{2n}$$
$$L := \lim_{n \to \infty} \frac{1}{2} = \frac{1}{2}$$

because $L := \frac{1}{2}$ and $0 < \frac{1}{2} < \infty$, $f(n) = \Theta(g(n))$

Now let $f(n) = 2^n$ and $g(n) = 2^{2n}$

$$L := \lim_{n \to \infty} \frac{2^n}{2^{2n}}$$
$$L := \lim_{n \to \infty} 2^{n-2n}$$
$$L := \lim_{n \to \infty} 2^{-n}$$
$$L := \lim_{n \to \infty} \frac{1}{2^n} = 0$$

therefore $f(n) \leq O(g(n)) and f(n) \neq \Theta(g(n))$
so substituting we can see that $f(n) = n$ and $g(n) = 2n$ where

$$n = \Theta(2n)$$

but

$$2^n \neq \Theta(n^{2n})$$

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(2e) *Logarithms in exponents. [**Hint:** Review the logarithm change of base formula, as well as the rules of logarithms.]*

$$n^{\log_4(n)} \qquad n^{\log_5(n)} \qquad n^{1/\log_3(n)} \qquad n \qquad 1$$

Solution:
Using the limit comparison test, let $f(n) = 1$ and $g(n) = n$

$$L := \lim_{n \to \infty} \frac{1}{n} = 0$$

showing that $1 \leq O(n)$

(2f) *Exponentials. [**Hint:** Recall the Ratio and Root Tests from Michael's Calculus Notes.]*

$$n! \qquad 3^n \qquad 3^{5n} \qquad 3^{n\log_4(n)} \qquad 3^{n+13}$$

Solution:

Using the root test

$$L := \lim_{n \to \infty} \mid a_n \mid^{\frac{1}{n}}$$

Let

$$a_n = \frac{3^n}{3^{5n}}$$

$$L := \lim_{n \to \infty} (\frac{3^n}{3^{5n}})^{\frac{1}{n}}$$

$$L := \lim_{n \to \infty} \frac{3}{3^5}$$

$$L :\approx 0.0123$$

Because $0 < L < \infty$, $3^n \leq \Theta(3^{5n})$

Now performing a ratio test

$$L := \lim_{n \to \infty} \mid \frac{a_{n+1}}{a_n} \mid$$

Let $a_n = \frac{3^n}{n!}$

$$L := \lim_{n \to \infty} \frac{\frac{3^{n+1}}{(n+1)n!}}{\frac{3^n}{n!}}$$

Simplifying

$$L := \lim_{n \to \infty} \frac{3^{n+1} * n!}{(n+1) * n! * 3^n}$$

Simplifying further

$$L := \lim_{n \to \infty} \frac{3}{n+1} = 0$$

Therefore $3^n \leq O(n!)$

Performing a ratio test again,letting $a_n = \frac{3^{n+13}}{3^n}$

$$L := \lim_{n \to \infty} \frac{\frac{3^{n+14}}{3^{n+1}}}{\frac{3^{n+13}}{3^n}}$$

Simplifying we obtain:

$$L := \lim_{n \to \infty} \frac{3^{2*n+14}}{3^{2n+14}} = 1$$

Because $0 < L < \infty$, $3^{n+13} = \Theta(3^n)$

Lastly, completing another root test will provide the growth rate order.
Let $a_n = \frac{3^n}{3^{nlog_4(n)}}$

$$L := \lim_{n \to \infty} (\frac{3^n}{3^{nlog_4(n)}})^{\frac{1}{n}}$$

$$L := \lim_{n \to \infty} \frac{3}{3^{log_4(n)}} = 0$$

Therefore, $3^n \leq O(3^{n*log_4(n)})$

Assuming transitivity, $3^n, 3^{n+13}, 3^{5n}, 3^{nlog_4(n)}, n!$
where $3^n = \Theta(3^{n+13}) = \Theta(3^{5n}) = \Theta(3^{nlog_4(n)})$