

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	sum
4.5	4.5	3	4	3.5	2	2	1	4	2	5	3.5	4	4	1	4	39

R.H.

Linear Algebra: Homework 3 @ CSDS UCU

Autumn 2018

Well done,
Yaroslavo!

Linear Algebra

best combination

Homework 3: Eigenvalues and eigenvectors

Solutions are by Yaroslava Lochman.

39 out of 40
grade: 19.5 out of 20

4.5/5

Problem 1 (Oblique projectors; 5pt). Assume that \mathbb{R}^n is represented as the *direct* (but not necessarily *orthogonal*) sum $M_1 \dot{+} M_2$ of two its subspaces M_1 and M_2 . In particular, every $\mathbf{x} \in \mathbb{R}^n$ can be represented in a unique way as $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$ with some $\mathbf{x}_j \in M_j$, and the mapping $P_j : \mathbf{x} \mapsto \mathbf{x}_j$ is called the (*oblique*) *projector onto M_j parallel to M_{3-j}* . It is easy to show that P_j satisfy the following properties: $P_1 + P_2 = I_n$, $P_j^2 = P_j$ and $P_1 P_2 = P_2 P_1 = 0$.

- Show that any matrix P satisfying the relation $P^2 = P$ is a projector onto some subspace L parallel to M , and identify these L and M .
- Show that the projector P is an orthogonal projector if and only if the matrix P is symmetric.
- Assume that two transformations P_1 and P_2 of \mathbb{R}^n satisfy the following conditions: $P_1 + P_2 = I_n$ and $P_1 P_2 = 0$. Prove that P_1 and P_2 are projectors and that $P_2 P_1 = 0$.

Solution to the problem 1. .

- $P^2 = P$. Let $P : \mathbb{R}^n \rightarrow \mathbb{R}^m$. Since P should be able to be applied to $Px \Rightarrow m = n$.

$$\forall z \in \mathbb{R}^n : z = Pz + (z - Pz) \quad \begin{cases} Pz \in \text{Im}P \\ P(z - Pz) = Pz - P^2z = 0 \Rightarrow z - Pz \in \text{Ker}P \end{cases}$$

So $\mathbb{R}^n = \text{Im}P + \text{Ker}P$. Let's show that $\text{Im}P \cap \text{Ker}P = \{0\}$ and hence $\mathbb{R}^n = \text{Im}P \dot{+} \text{Ker}P$.

$$\forall y \in \text{Im}P \cap \text{Ker}P : \begin{cases} Py = 0 \\ \exists x \in \mathbb{R}^n : y = Px \Rightarrow Py = P^2x = Px = y \end{cases} \Rightarrow y = Py = 0$$

Let's show now that $\forall z \in \mathbb{R} \exists! x \in \text{Ker}P, \exists! y \in \text{Im}P : z = x + y$. Let it be false, meaning $\exists x_1, x_2 \in \text{Ker}P, y_1, y_2 \in \text{Im}P$ s.t. :

$$\begin{cases} z = x_1 + y_1 \\ z = x_2 + y_2 \end{cases} \Rightarrow x_1 - x_2 (\in \text{Ker}P) = y_1 - y_2 (\in \text{Im}P)$$

$$\Rightarrow \begin{cases} x_1 - x_2 \in \text{Im}P \cap \text{Ker}P = \{0\} \\ y_1 - y_2 \in \text{Im}P \cap \text{Ker}P = \{0\} \end{cases} \Rightarrow \begin{cases} x_1 = x_2 \\ y_1 = y_2 \end{cases}$$

So we have:

$$\mathbb{R}^n = \text{Im}P \dot{+} \text{Ker}P = (\text{in other words}) = \mathcal{C}(P) \dot{+} \mathcal{N}(P)$$

- Let P now be also a symmetric matrix: $P = P^\top$. Let's show that $\mathcal{C}(P) \perp \mathcal{N}(P)$. From the second fundamental theorem of L:

$$\mathcal{C}(P) \perp \mathcal{N}(P^\top)$$

$$\text{Since } P = P^\top \Rightarrow \mathcal{N}(P) = \mathcal{N}(P)^\top \Rightarrow \mathcal{C}(P) \perp \mathcal{N}(P)$$

$$\Rightarrow \mathbb{R}^n = \mathcal{C}(P) \oplus \mathcal{N}(P)$$

this is sufficiency
and what about
necessity

(c) $P_1 : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $P_2 : \mathbb{R}^n \rightarrow \mathbb{R}^n$.

$$\begin{cases} P_1 + P_2 = I_n \\ P_1 P_2 = 0_n \end{cases} \Rightarrow \begin{cases} P_1 = P_1(P_1 + P_2) = P_1^2 + P_1 P_2 = P_1^2 \Rightarrow P_1 \text{ is a projector.} \\ P_2 = (P_1 + P_2)P_2 = P_1 P_2 + P_2^2 = P_2^2 \Rightarrow P_2 \text{ is a projector.} \end{cases}$$

$$P_1 = (P_1 + P_2)P_1 = P_1^2 + P_2 P_1 = P_1 + P_2 P_1 \Rightarrow P_2 P_1 = 0_n$$



4.5/5 Problem 2 (Projectors; 5pt). (a) Find a matrix of oblique projector in \mathbb{R}^3 onto the subspace $U = \text{ls}\{(1, 0, 1)^\top\}$ parallel to the subspace $W = \text{ls}\{(1, 1, 0)^\top, (0, 1, 1)^\top\}$.

(b) Find a projection matrix of \mathbb{R}^3 onto the subspace $U = \text{ls}\{(1, 2, 1)^\top, (1, 0, -1)^\top\}$ parallel to the subspace $W = \text{ls}\{(1, 0, 1)^\top\}$.

(c) Is it possible to fill in the missing entries in the matrix

$$A = \begin{pmatrix} 1 & * & 0 \\ 0 & \frac{1}{2} & * \\ * & * & * \end{pmatrix}$$

to get a matrix of an orthogonal projection in \mathbb{R}^3 ? If so, find the subspace U of \mathbb{R}^3 such that A is an orthogonal projection onto U .

Hint: Do you see why only one of (a) or (b) needs to be worked out in detail?

Solution to the problem 2. Let's find a general form of the matrix of the projector.

Let $u_1 \dots u_k$ be linearly independent in \mathbb{R}^n , $U = \text{ls}\{u_1 \dots u_k\}$, $\mathbb{U} = (u_1 \dots u_k)$,
 $w_1 \dots w_{n-k}$ - linearly independent in \mathbb{R}^n , $W = \text{ls}\{w_1 \dots w_{n-k}\}$, $\mathbb{W} = (w_1 \dots w_{n-k})$ s.t.:

$$\mathbb{R}^n = U \dot{+} W$$

Let's find a matrix of oblique projector in \mathbb{R}^n onto U parallel to W .

$$\forall b \in \mathbb{R}^n : p = Pb \in U \Rightarrow \exists x : p = \mathbb{U}x$$

$$(b - \mathbb{U}x) \text{ should be parallel to } W \Rightarrow (b - \mathbb{U}x) \perp W^\perp$$

Let V be W^\perp , $\dim(V) = n - \dim(W) = n - n + k = k$, so:

$$V = \text{ls}\{v_1 \dots v_k\}, \mathbb{V} = (v_1 \dots v_k), \mathbb{V}^\top \mathbb{W} = \mathbf{0}_{k \times (n-k)}$$

$$\Rightarrow \mathbb{V}^\top (b - \mathbb{U}x) = \mathbf{0}_k \Rightarrow \mathbb{V}^\top \mathbb{U}x = \mathbb{V}^\top b$$

$\mathbb{V}^\top \mathbb{U}$ is $k \times k$ non-singular since $\mathcal{N}(\mathbb{V}^\top \mathbb{U}) = \{0\}$:

$$\mathbb{V}^\top \mathbb{U}x = 0 \Rightarrow \mathbb{U}x \in W \text{ but } W \cap U = 0$$

So there will be no linear combination of columns of \mathbb{U} that lies in $W \Rightarrow x = 0$

$$\Rightarrow x = (\mathbb{V}^\top \mathbb{U})^{-1} \mathbb{V}^\top b \Rightarrow p = \mathbb{U}x = \mathbb{U}(\mathbb{V}^\top \mathbb{U})^{-1} \mathbb{V}^\top b = Pb \Rightarrow P = \mathbb{U}(\mathbb{V}^\top \mathbb{U})^{-1} \mathbb{V}^\top$$

(a) $U = \text{ls}\{(1, 0, 1)^\top\}$, $W = \text{ls}\{(1, 1, 0)^\top, (0, 1, 1)^\top\}$. Let's first check that $\mathbb{R}^n = U \dot{+} W$:

$$\det \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} = 1 \Rightarrow \text{the system of columns is linearly independent.}$$


$$\Rightarrow \mathbb{R}^n = \text{ls} \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\} = U \dot{+} W$$

So

$$\mathbb{U} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \mathbb{W} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \mathbb{V} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \text{ since:}$$

$$V = W^\perp = \text{ls} \left\{ \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} \right\} = \text{ls} \{(1, -1, 1)^\top\}$$

$$\mathbb{V}^\top \mathbb{U} = (1 \quad -1 \quad 1) \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 2 \Rightarrow P = \mathbb{U}(\mathbb{V}^\top \mathbb{U})^{-1} \mathbb{V}^\top = \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} (1 \quad -1 \quad 1) = \frac{1}{2} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 1 & -1 & 1 \end{pmatrix}$$

Answer: $P = \frac{1}{2} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 1 & -1 & 1 \end{pmatrix}$ 

(b) $U = \text{ls}\{(1, 2, 1)^\top, (1, 0, -1)^\top\}$, $W = \text{ls}\{(1, 0, 1)^\top\}$. Let's first check that $\mathbb{R}^n = U \dot{+} W$:

$$\det \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 0 \\ 1 & -1 & 1 \end{pmatrix} = -4 \Rightarrow \text{the system of columns is linearly independent.}$$

$$\Rightarrow \mathbb{R}^n = \text{ls} \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\} = U \dot{+} W$$

So

$$\mathbb{U} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 1 & -1 \end{pmatrix} \quad \mathbb{W} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \mathbb{V} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \text{ since:}$$

$$v_1 \perp w_1 \Rightarrow v_1 \text{ can be } (0, 1, 0)^\top$$

$$v_2 \in \{w_1, v_1\}^\perp \Rightarrow v_2 \text{ can be } \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = (-1, 0, 1)^\top \Rightarrow V = \text{ls} \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$\mathbb{V}^\top \mathbb{U} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \quad (\mathbb{V}^\top \mathbb{U})^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\Rightarrow P = \mathbb{U}(\mathbb{V}^\top \mathbb{U})^{-1} \mathbb{V}^\top = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ -1 & 1 & 1 \end{pmatrix}$$

In fact if $P_{(a)}$ is a projector from task (a), and $P_{(b)}$ – from task (b), then $P_{(b)} = I - P_{(a)}$, since $U_{(a)} = W_{(b)}$. Let's check:

$$I - P_{(a)} = \frac{1}{2} \left(\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} - \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 1 & -1 & 1 \end{pmatrix} \right) = \frac{1}{2} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ -1 & 1 & 1 \end{pmatrix} = P_{(b)}$$

Answer: $P = \frac{1}{2} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ -1 & 1 & 1 \end{pmatrix}$ ✓

(c)

$$A = \begin{pmatrix} 1 & x_2 & 0 \\ 0 & \frac{1}{2} & x_4 \\ x_1 & x_3 & x_5 \end{pmatrix} = |A \text{ should be symmetric}| = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & x_3 \\ 0 & x_3 & x_5 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} + x_3^2 & \frac{1}{2}x_3 + x_3x_5 \\ 0 & \frac{1}{2}x_3 + x_3x_5 & x_3^2 + x_5^2 \end{pmatrix} = \begin{pmatrix} 1 & x_2 & 0 \\ 0 & \frac{1}{2} & x_4 \\ x_1 & x_3 & x_5 \end{pmatrix} = A$$

$$\begin{cases} \frac{1}{4} + x_3^2 = \frac{1}{2} \\ x_3(x_5 - \frac{1}{2}) = 0 \\ x_5^2 - x_5 + x_3^2 = 0 \end{cases} \Rightarrow \begin{cases} x_5 = \frac{1}{2} \\ x_3 = \pm \frac{1}{2} \end{cases} \Rightarrow A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

It is enough to take only one of these two vectors

$$U = \mathcal{C}(A) = \text{ls} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \right\}$$

3/3 Problem 3 (Eigenvalues; 3 pt). Find all the eigenvalues of the following matrices by **inspection**:

(a) $\begin{pmatrix} 1/2 & 1 \\ 1/2 & 0 \end{pmatrix}$, (b) $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, (c) $\begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}$, (d) $\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$, (e) $\begin{pmatrix} 0 & 0 & 2 \\ 0 & 5 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.

Hint: Look for constant row/column sums, diagonal entries with zeros in the corresponding row or column otherwise, use eigenvalue sum/product rules, try subtracting λI for “tempting” candidates for λ etc

Solution to the problem 3. .

(a) $A = \begin{pmatrix} 1/2 & 1 \\ 1/2 & 0 \end{pmatrix}$ $A - I = \begin{pmatrix} -1/2 & 1 \\ 1/2 & -1 \end{pmatrix}$ $\det(A - I) = 0 \Rightarrow \lambda_1 = 1$

$$\lambda_2 = \text{tr } A - \lambda_1 = 1/2 - 1 = -1/2$$

Answer: $\lambda_1 = 1, \lambda_2 = -1/2$ ✓

(b) $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ $\det A = 0 \Rightarrow \lambda_1 = 0$

$$\lambda_2 = \operatorname{tr} A - \lambda_1 = 2$$

Answer: $\lambda_1 = 0, \lambda_2 = 2$ ✓

(c) $A = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}$ $A - I = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}$ $\det(A - I) = 0 \Rightarrow \lambda_1 = 1$

$$\lambda_2 = \operatorname{tr} A - \lambda_1 = 5 - 1 = 4$$

Answer: $\lambda_1 = 1, \lambda_2 = 4$ ✓

(d) $A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$ $A + I = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix}$ $\det(A + I) = 0 \Rightarrow \lambda_1 = -1$

$$\operatorname{rank}(A + I) = 1 \Rightarrow \dim N(A + I) = 3 - 1 = 2 \Rightarrow \lambda_2 = \lambda_1 = -1$$

$$\Rightarrow \lambda_3 = \operatorname{tr} A + 1 + 1 = 3$$

Answer: $\lambda_1 = \lambda_2 = -1, \lambda_3 = 3$ ✓

(e) $A = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 5 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ $A - 5I = \begin{pmatrix} -5 & 0 & 2 \\ 0 & 0 & 0 \\ 1 & 0 & -4 \end{pmatrix}$ $\det(A - 5I) = 0 \Rightarrow \lambda_1 = 5$ ✓

$$\begin{cases} 5 + \lambda_2 + \lambda_3 = \operatorname{tr} A = 6 \\ 5\lambda_2 \cdot \lambda_3 = \det A = -5 \end{cases} \Rightarrow \begin{cases} \lambda_2 + \lambda_3 = 1 \\ \lambda_3 = -\frac{1}{\lambda_2} \end{cases} \Rightarrow \left| \lambda - \frac{1}{\lambda} = 1, \quad \lambda^2 - \lambda - 1 = 0 \right| \Rightarrow \lambda_{2,3} = \frac{1 \pm \sqrt{5}}{2} \quad \times$$

Answer: $\lambda_1 = 5, \lambda_2 = \frac{1+\sqrt{5}}{2}, \lambda_3 = \frac{1-\sqrt{5}}{2}$

4/4

Problem 4 (Eigenvalues and eigenvectors; 4 pt). For the matrix A in each part below, find the eigenvalues and eigenvectors of A, A^2, A^{100}, A^{-1} and e^{tA} :

(a) $\begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix},$ (b) $\begin{pmatrix} 4 & 0 & -1 \\ 0 & -1 & 4 \\ 0 & 2 & 1 \end{pmatrix}$

Solution to the problem 4. Let A be $k \times k$ matrix. Let λ_i and \mathbf{v}_i be the eigenvalue and corresponding eigenvector of A ; $\lambda_i^{(n)}$ and $\mathbf{v}_i^{(n)}$ – the eigenvalue and corresponding eigenvector of A^n ; $\hat{\lambda}_i$ and $\hat{\mathbf{v}}_i$ – the eigenvalue and corresponding eigenvector of e^{tA} , $i = \overline{1, k}$.

In case when $A = PDP^{-1}$ where $P = (\mathbf{v}_1 \ \dots \ \mathbf{v}_n)$ and $D = \operatorname{diag}\{\lambda_1, \dots, \lambda_n\}$ we have:

$$A^n = P D^n P^{-1} \Rightarrow \begin{cases} \lambda_i^{(n)} = \lambda_i^n \\ \mathbf{v}_i^{(n)} = \mathbf{v}_i \end{cases} \quad i = \overline{1, k}$$

And since $e^x = \sum_{n=1}^{\infty} \frac{x^n}{n!}$:

$$e^{tA} = \sum_{n=1}^{\infty} \frac{t^n A^n}{n!} = \sum_{n=1}^{\infty} \frac{t^n P D^n P^{-1}}{n!} = P \sum_{n=1}^{\infty} \frac{t^n D^n}{n!} P^{-1} = P e^{tD} P^{-1} \Rightarrow \begin{cases} \hat{\lambda}_i = e^{t\lambda_i} \\ \hat{\mathbf{v}}_i = \mathbf{v}_i \end{cases} \quad i = \overline{1, k}$$

$$(a) \quad A = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} \quad A - I = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} \quad \det(A - I) = 0 \Rightarrow \lambda_1 = 1$$

$$(A - \lambda_1 I)\mathbf{v}_1 = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} \mathbf{v}_1 = 0 \Rightarrow \mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\lambda_2 = \text{tr } A - \lambda_1 = 5 - 1 = 4 \quad (A - \lambda_2 I)\mathbf{v}_2 = \begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix} \mathbf{v}_2 = 0 \Rightarrow \mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$P = (\mathbf{v}_1 \quad \mathbf{v}_2) = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} \quad | \text{ self-check: } \det P = 3 \neq 0 \quad | \quad D = \text{diag}\{\lambda_1, \lambda_2\}$$

So $A = PDP^{-1}$, and we can easily calculate eigenvalues and eigenvectors (using the conclusions above):

Answer: $\lambda_1 = 1 \quad \lambda_2 = 4$

$$\lambda_1^{(2)} = 1 \quad \lambda_2^{(2)} = 4^2 = 16$$

$$\lambda_1^{(100)} = 1 \quad \lambda_2^{(100)} = 4^{100}$$

$$\lambda_1^{(-1)} = 1 \quad \lambda_2^{(-1)} = 4^{-1} = 1/4 \quad \checkmark$$

$$\hat{\lambda}_1 = e^t \quad \hat{\lambda}_2 = e^{4t}$$

$$\{\mathbf{v}_1, \mathbf{v}_2\} = \left\{ \mathbf{v}_1^{(2)}, \mathbf{v}_2^{(2)} \right\} = \left\{ \mathbf{v}_1^{(100)}, \mathbf{v}_2^{(100)} \right\} = \left\{ \mathbf{v}_1^{(-1)}, \mathbf{v}_2^{(-1)} \right\} = \{\hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2\} = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\} \checkmark$$

$$(b) \quad A = \begin{pmatrix} 4 & 0 & -1 \\ 0 & -1 & 4 \\ 0 & 2 & 1 \end{pmatrix} \quad A - 4I = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -5 & 4 \\ 0 & 2 & -3 \end{pmatrix} \quad \det(A - 4I) = 0 \Rightarrow \lambda_1 = 4$$

$$(A - 4I)\mathbf{v}_1 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -5 & 4 \\ 0 & 2 & -3 \end{pmatrix} \mathbf{v}_1 = 0 \Rightarrow \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \checkmark$$

$$A + 3I = \begin{pmatrix} 7 & 0 & -1 \\ 0 & 2 & 4 \\ 0 & 2 & 4 \end{pmatrix} \quad \det(A + 3I) = 0 \Rightarrow \lambda_2 = -3 \quad \checkmark$$

$$(A + 3I)\mathbf{v}_2 = \begin{pmatrix} 7 & 0 & -1 \\ 0 & 2 & 4 \\ 0 & 2 & 4 \end{pmatrix} \mathbf{v}_2 = 0 \Rightarrow \mathbf{v}_2 = \begin{pmatrix} 1 \\ -14 \\ 7 \end{pmatrix} \quad \checkmark$$

$$\lambda_3 = \text{tr } A - 4 + 3 = 3 \quad (A - 3I)\mathbf{v}_3 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -4 & 4 \\ 0 & 2 & -2 \end{pmatrix} \mathbf{v}_3 = 0 \Rightarrow \mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \checkmark$$

$$P = (\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -14 & 1 \\ 0 & 7 & 1 \end{pmatrix} \quad | \text{ self-check: } \det P = -21 \neq 0 \quad | \quad D = \text{diag}\{\lambda_1, \lambda_2, \lambda_3\}$$

So $A = PDP^{-1}$, and we can easily calculate eigenvalues and eigenvectors (using the conclusions above):

Answer: $\lambda_1 = 4 \quad \lambda_2 = -3 \quad \lambda_3 = 3$

$$\lambda_1^{(2)} = 16 \quad \lambda_2^{(2)} = 9 \quad \lambda_3^{(2)} = 9$$

$$\lambda_1^{(100)} = 4^{100} \quad \lambda_2^{(100)} = 3^{100} \quad \lambda_3^{(100)} = 3^{100}$$

$$\lambda_1^{(-1)} = 1/4 \quad \lambda_2^{(-1)} = -1/4 \quad \lambda_3^{(100)} = 1/3$$

$$\hat{\lambda} = e^{4t} \quad \hat{\lambda} = e^{-4t} \quad \hat{\lambda} = e^{3t}$$



$$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \{v_1^{(2)}, v_2^{(2)}, v_3^{(2)}\} = \{v_1^{(100)}, v_2^{(100)}, v_3^{(100)}\} =$$

$$= \{v_1^{(-1)}, v_2^{(-1)}, v_3^{(-1)}\} = \{\hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2, \hat{\mathbf{v}}_3\} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -14 \\ 7 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

3.5/4

Problem 5 (Difference equation; 4 pt). A *generalized Fibonacci* sequence g_n is defined through the relations $g_0 = 0$, $g_1 = 1$, $g_2 = 2$ and $g_{n+3} = 3g_{n+2} - g_{n+1} - g_n$ for $n \in \mathbb{Z}_+$. Find the formula for the n^{th} generalized Fibonacci number g_n . To this end,

- represent the difference equation in the form $\mathbf{x}_{n+1} = A\mathbf{x}_n$ for a suitable 3×3 matrix A , find its eigenvalues and eigenvectors and then diagonalize A ;
- find a general solution \mathbf{x}_n and then the one satisfying the initial condition $\mathbf{x}_0 = (0, 1, 2)^\top$.

Solution to the problem 5. .


(a)

$$\mathbf{x}_n = \begin{pmatrix} g_n \\ g_{n+1} \\ g_{n+2} \end{pmatrix} \Rightarrow \mathbf{x}_0 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \quad \mathbf{x}_{n+1} = \begin{pmatrix} g_{n+1} \\ g_{n+2} \\ g_{n+3} \end{pmatrix} = \begin{pmatrix} g_{n+1} \\ g_{n+2} \\ 3g_{n+2} - g_{n+1} - g_n \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & 3 \end{pmatrix} \mathbf{x}_n$$

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & 3 \end{pmatrix} \quad x_{n+1} = Ax_n \quad \checkmark$$

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -1 & -1 & 3 - \lambda \end{vmatrix} = \lambda^2(3 - \lambda) - 1 - \lambda = -\lambda^3 + 3\lambda^2 - \lambda - 1 = 0$$

$$\lambda^3 - 3\lambda^2 + \lambda + 1 = 0 \quad (\lambda - 1)(\lambda^2 - 2\lambda - 1) = 0 \quad \Rightarrow \lambda_1 = 1, \lambda_{2,3} = 1 \pm \sqrt{2} \quad \checkmark$$

$$(A - \lambda_1)\mathbf{v} = 0 \quad \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ -1 & -1 & 2 \end{pmatrix} \mathbf{v}_1 = 0 \quad \Rightarrow \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$



$$(A - \lambda_2)\mathbf{v} = 0 \quad \begin{pmatrix} -1 - \sqrt{2} & 1 & 0 \\ 0 & -1 - \sqrt{2} & 1 \\ -1 & -1 & 2 - \sqrt{2} \end{pmatrix} \mathbf{v}_2 = 0 \quad \Rightarrow \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 + \sqrt{2} \\ 3 + 2\sqrt{2} \end{pmatrix}$$

$$(A - \lambda_3)\mathbf{v} = 0 \quad \begin{pmatrix} -1 + \sqrt{2} & 1 & 0 \\ 0 & -1 + \sqrt{2} & 1 \\ -1 & -1 & 2 + \sqrt{2} \end{pmatrix} \mathbf{v}_3 = 0 \quad \Rightarrow \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 - \sqrt{2} \\ 3 - 2\sqrt{2} \end{pmatrix}$$


$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 + \sqrt{2} & 0 \\ 0 & 0 & 1 - \sqrt{2} \end{pmatrix} \quad P = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 + \sqrt{2} & 1 - \sqrt{2} \\ 1 & 3 + 2\sqrt{2} & 3 - 2\sqrt{2} \end{pmatrix}$$

$$\det P = (1 + \sqrt{2})(3 - 2\sqrt{2}) + 1 - \sqrt{2} + 3 + 2\sqrt{2} - 1 - \sqrt{2} - (1 - \sqrt{2})(3 + 2\sqrt{2}) - 3 + 2\sqrt{2} =$$

$$= 3 - 2\sqrt{2} + 3\sqrt{2} - 4 - 3 - 2\sqrt{2} + 3\sqrt{2} + 4 + 2\sqrt{2} = 4\sqrt{2}$$

$$P^{-1} = \frac{1}{4\sqrt{2}} \begin{pmatrix} 2\sqrt{2} & 4\sqrt{2} & -2\sqrt{2} \\ -2 + \sqrt{2} & 2 - 2\sqrt{2} & \sqrt{2} \\ 2 + \sqrt{2} & -2 - 2\sqrt{2} & \sqrt{2} \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 2 & 4 & -2 \\ 1 - \sqrt{2} & -2 + \sqrt{2} & 1 \\ 1 + \sqrt{2} & -2 - \sqrt{2} & 1 \end{pmatrix}$$


So:

$$A = PDP^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 + \sqrt{2} & 1 - \sqrt{2} \\ 1 & 3 + 2\sqrt{2} & 3 - 2\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 + \sqrt{2} & 0 \\ 0 & 0 & 1 - \sqrt{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 1 & -\frac{1}{2} \\ \frac{1-\sqrt{2}}{4} & \frac{-2+\sqrt{2}}{4} & \frac{1}{4} \\ \frac{1+\sqrt{2}}{4} & \frac{-2-\sqrt{2}}{4} & \frac{1}{4} \end{pmatrix}$$


(b)

$$\mathbf{x}_n = PD^n P^{-1} x_0 = PD^n c_0 = (\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3) \begin{pmatrix} \lambda_1^n & 0 & 0 \\ 0 & \lambda_2^n & 0 \\ 0 & 0 & \lambda_3^n \end{pmatrix} c_0 = c_0^{(1)} \lambda_1^n \mathbf{v}_1 + c_0^{(2)} \lambda_2^n \mathbf{v}_2 + c_0^{(3)} \lambda_3^n \mathbf{v}_3 =$$

$$= c_0^{(1)} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_0^{(2)} (1 + \sqrt{2})^n \begin{pmatrix} 1 \\ 1 + \sqrt{2} \\ 3 + 2\sqrt{2} \end{pmatrix} + c_0^{(3)} (1 - \sqrt{2})^n \begin{pmatrix} 1 \\ 1 - \sqrt{2} \\ 3 - 2\sqrt{2} \end{pmatrix}$$

$$c_0 = P^{-1} x_0 = \frac{1}{4} \begin{pmatrix} 2 & 4 & -2 \\ 1 - \sqrt{2} & -2 + \sqrt{2} & 1 \\ 1 + \sqrt{2} & -2 - \sqrt{2} & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \frac{\sqrt{2}}{4} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$\Rightarrow \mathbf{x}_n = PD^n c_0 = \frac{\sqrt{2}}{4}(\lambda_2^n \mathbf{v}_2 + \lambda_3^n \mathbf{v}_3) = \frac{\sqrt{2}}{4} \left((1 + \sqrt{2})^n \begin{pmatrix} 1 \\ 1 + \sqrt{2} \\ 3 + 2\sqrt{2} \end{pmatrix} - (1 - \sqrt{2})^n \begin{pmatrix} 1 \\ 1 - \sqrt{2} \\ 3 - 2\sqrt{2} \end{pmatrix} \right)$$

$$\text{Also : } \mathbf{x}_n = \frac{\sqrt{2}}{4} \lambda_2^n \left(\mathbf{v}_2 + \left(\frac{\lambda_3}{\lambda_2} \right)^n \mathbf{v}_3 \right) \underset{n \rightarrow \infty}{\sim} \frac{\sqrt{2}}{4} \lambda_2^n \mathbf{v}_2 \quad \text{since } \lambda_2 > \lambda_3$$

without formula for g_n

2/4

Problem 6 (Jordan form; 4 pt). For each of the following matrices A , find P so that $P^{-1}AP$ is in the Jordan form (ie, either diagonal or a Jordan block), and write this Jordan form:

$$(a) \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}, \quad (b) \begin{pmatrix} 3 & 2 \\ 1 & -1 \end{pmatrix}, \quad (c) \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}, \quad (d) \begin{pmatrix} -5 & 2 \\ -\frac{1}{2} & -3 \end{pmatrix}.$$

Hint: You do not have to calculate $P^{-1}AP$; you can just write it out!

Solution to the problem 6.

(a)

$$A = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} \quad \lambda_1 = 1 \quad \lambda_2 = 5 - 1 = 4 \quad J = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \quad \checkmark$$

What P did you find?

(b)

$$A = \begin{pmatrix} 3 & 2 \\ 1 & -1 \end{pmatrix} \quad \begin{cases} \lambda_1 + \lambda_2 = 2 \\ \lambda_1 \cdot \lambda_2 = -5 \end{cases} \quad \begin{cases} \lambda_2 = 2 - \lambda_1 \\ \lambda_1^2 - 2\lambda_1 - 5 = 0 \end{cases}$$

$$\Rightarrow \lambda_{1,2} = 1 \pm \sqrt{6} \quad J = \begin{pmatrix} 1 + \sqrt{6} & 0 \\ 0 & 1 - \sqrt{6} \end{pmatrix} \quad \checkmark$$

(c)

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix} \quad \lambda_1 = 4 \quad \lambda_2 = 3 - 4 = -1 \quad J = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix} \quad \checkmark$$

(d)

$$A = \begin{pmatrix} -5 & 2 \\ -\frac{1}{2} & -3 \end{pmatrix} \quad \begin{cases} \lambda_1 + \lambda_2 = -8 \\ \lambda_1 \cdot \lambda_2 = 16 \end{cases} \quad \lambda_1 = \lambda_2 = -4 \quad J = \begin{pmatrix} -4 & 1 \\ 0 & -4 \end{pmatrix} \quad \checkmark$$

2/2

Problem 7 (Eigenvalues and eigenvectors; 2pt). What are the eigenvalues and eigenvectors of the linear transformation A defined on \mathbb{C}^3 via $A(x_1, x_2, x_3) = (x_3, x_1, x_2)$?

Solution to the problem 7. .

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_3 \\ x_1 \\ x_2 \end{pmatrix} \quad A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\det(A - \lambda) = \begin{vmatrix} -\lambda & 0 & 1 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{vmatrix} = -\lambda^3 + 1 = 0$$

$$\lambda^3 - 1 = (\lambda - 1)(\lambda^2 + \lambda + 1) \Rightarrow \lambda_1 = 1 \quad \lambda_{2,3} = \frac{-1 \pm i\sqrt{3}}{2} \quad \checkmark$$

$$\begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \mathbf{v} = 0 \Rightarrow \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1-i\sqrt{3}}{2} & 0 & 1 \\ 1 & \frac{1-i\sqrt{3}}{2} & 0 \\ 0 & 1 & \frac{1-i\sqrt{3}}{2} \end{pmatrix} \mathbf{v} = 0 \Rightarrow \mathbf{v}_2 = \begin{pmatrix} 1 \\ -\frac{2}{1-i\sqrt{3}} \\ \frac{-1+i\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{-1-i\sqrt{3}}{2} \\ \frac{-1+i\sqrt{3}}{2} \end{pmatrix} \quad \checkmark$$

$$\begin{pmatrix} \frac{1+i\sqrt{3}}{2} & 0 & 1 \\ 1 & \frac{1+i\sqrt{3}}{2} & 0 \\ 0 & 1 & \frac{1+i\sqrt{3}}{2} \end{pmatrix} \mathbf{v} = 0 \Rightarrow \mathbf{v}_3 = \begin{pmatrix} 1 \\ -\frac{2}{1+i\sqrt{3}} \\ \frac{-1-i\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{-1+i\sqrt{3}}{2} \\ \frac{-1-i\sqrt{3}}{2} \end{pmatrix}$$

Answer: $\lambda_1 = 1, \lambda_2 = \frac{-1+i\sqrt{3}}{2}, \lambda_3 = \frac{-1-i\sqrt{3}}{2} \quad \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ \frac{-1-i\sqrt{3}}{2} \\ \frac{-1+i\sqrt{3}}{2} \end{pmatrix}, \begin{pmatrix} 1 \\ \frac{-1+i\sqrt{3}}{2} \\ \frac{-1-i\sqrt{3}}{2} \end{pmatrix} \right\}$

1/2

Problem 8 (Eigenvalues and eigenvectors; 2pt). Let \mathbf{a} be a non-zero vector in \mathbb{R}^3 and A a linear transformation of \mathbb{R}^3 given by $A\mathbf{x} = \mathbf{x} \times \mathbf{a}$, where “ \times ” denotes the cross-product (i.e., vector product). Find the eigenvalues and eigenvectors of A .

Hint: do not use coordinates; use the geometric meaning of the cross product instead

Solution to the problem 8. Let's show that $\mathcal{N}(A) = \text{ls}\{\mathbf{a}\}$, $\mathcal{C}(A) = \text{ls}\{\mathbf{a}\}^\perp$

$$A\mathbf{x} = 0 \Leftrightarrow \mathbf{x} \times \mathbf{a} = 0 \Leftrightarrow \mathbf{x} \parallel \mathbf{a} \Rightarrow \mathcal{N}(A) = \text{ls}\{\mathbf{a}\}$$

$$\left\{ \begin{array}{l} \forall \mathbf{y} \in \mathcal{C}(A) \exists \mathbf{x} : \mathbf{x} \times \mathbf{a} = \mathbf{y} \Rightarrow \mathbf{y} \perp \mathbf{a} \Rightarrow \mathbf{y} \in \text{ls}\{\mathbf{a}\}^\perp \Rightarrow \mathcal{C}(A) \subset \text{ls}\{\mathbf{a}\}^\perp \\ \dim \mathcal{C}(A) = 3 - \dim \mathcal{N}(A) = 3 - 1 = 2 \end{array} \right. \Rightarrow \mathcal{C}(A) = \text{ls}\{\mathbf{a}\}^\perp$$

$$\text{rank } A = \dim \mathcal{C}(A) = 2 \Rightarrow \det A = 0 \Rightarrow \lambda = 0 \quad \mathbf{v} \in \mathcal{N}(A) \Rightarrow \mathbf{v} = \mathbf{a}$$

Indeed if we look at the equation $\mathbf{v} \times \mathbf{a} = \lambda \mathbf{v}$ we see that the result of the cross-product with \mathbf{v} (which should be orthogonal to \mathbf{v} in case \mathbf{v} is not parallel to \mathbf{a} , otherwise zero) is parallel to \mathbf{v} , so the only possible solution is $\lambda = 0$ and $\mathbf{v} = \mathbf{a}$.

Answer: For this linear transformation $\lambda = 0$ is the only real one eigenvalue with corresponding eigenvector $\mathbf{v} = \mathbf{a}$.

but A is 3x3 matrix. So there are 3 eigenvalues

4/4

Problem 9 (Eigenvalues and eigenvectors; 4pt). (a) Assume that \mathbf{u} and \mathbf{v} are two non-zero (column) vectors of \mathbb{R}^n . Find eigenvalues and eigenvectors of the matrix $A := \mathbf{u}\mathbf{v}^\top$.

(b) Assume that a and b are two non-equal numbers. Find eigenvalues and eigenvectors of the $n \times n$ matrix

$$\begin{pmatrix} a & b & b & \dots & b \\ b & a & b & \dots & b \\ b & b & a & \dots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \dots & a \end{pmatrix}$$

Hint: think geometrically! What does the matrix A do? In (b), what happens if $a = b$?

Solution to the problem 9. .

(a)

$$A = \mathbf{u}\mathbf{v}^\top$$

$$\forall \mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{u}\mathbf{v}^\top \mathbf{x} = (\mathbf{v}^\top \mathbf{x})\mathbf{u} \in \text{ls}\{\mathbf{u}\} \Rightarrow \mathcal{C}(A) = \text{ls}\{\mathbf{u}\}$$

$$A\mathbf{x} = 0 \Leftrightarrow (\mathbf{v}^\top \mathbf{x})\mathbf{u} = 0 \Leftrightarrow \mathbf{v}^\top \mathbf{x} = 0 \Leftrightarrow \mathbf{x} \in \text{ls}\{\mathbf{v}\}^\perp \Rightarrow \mathcal{N}(A) = \text{ls}\{\mathbf{v}\}^\perp$$

$$\dim \mathcal{N}(A) = n - 1$$

$$\text{rank } A = \dim \mathcal{C}(A) = 1 \Rightarrow \det A = 0 \text{ and } \lambda_1 = \dots = \lambda_{n-1} = 0 \quad \checkmark$$

$$\{\mathbf{v}_1 \dots \mathbf{v}_{n-1}\} \text{ should be the basis of } \text{ls}\{\mathbf{v}\}^\perp \text{ since } A\mathbf{x} = 0 \forall \mathbf{x} \in \text{ls}\{\mathbf{v}\}^\perp$$

$$\lambda_n = \text{tr}(\mathbf{u}\mathbf{v}^\top) - 0 = \mathbf{v}^\top \mathbf{u} \quad A\mathbf{v}_n = (\mathbf{v}^\top \mathbf{v}_n)\mathbf{u} = (\mathbf{v}^\top \mathbf{u})\mathbf{v}_n \Rightarrow \mathbf{v}_n = \mathbf{u} \quad \checkmark$$

$$\textbf{Answer: } \lambda_1 = \dots = \lambda_{n-1} = 0, \quad \{\mathbf{v}_1 \dots \mathbf{v}_{n-1}\} = \text{basis } \mathbf{v}^\perp \quad \lambda_n = \mathbf{v}^\top \mathbf{u}, \quad \mathbf{v}_n = \mathbf{u} \quad \checkmark$$

Also one can notice that $A = \mathbf{u}\mathbf{v}^\top = \mathbf{v}^\top \mathbf{u} \frac{\mathbf{u}\mathbf{v}^\top}{\mathbf{v}^\top \mathbf{u}} = \mathbf{v}^\top \mathbf{u} P$, where $P = \frac{\mathbf{u}\mathbf{v}^\top}{\mathbf{v}^\top \mathbf{u}}$ is an oblique projector onto \mathbf{u} parallel to \mathbf{v}^\perp . So A also projects every \mathbf{x} onto \mathbf{u} and the multiplies it by $\mathbf{v}^\top \mathbf{u}$

(b) Let $\mathbf{1} = (1 \dots 1)^\top$.

$$A = \begin{pmatrix} a & b & b & \dots & b \\ b & a & b & \dots & b \\ b & b & a & \dots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \dots & a \end{pmatrix} = b \begin{pmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{pmatrix} + (a - b)I = b \mathbf{1}\mathbf{1}^\top + (a - b)I$$

If $a = b$: $A = b \mathbf{1}\mathbf{1}^\top = nb \frac{\mathbf{1}\mathbf{1}^\top}{n} = nbP_1$ and we have a case similar to (a): $\lambda_1 = \dots = \lambda_{n-1} = 0$, $\{\mathbf{v}_1 \dots \mathbf{v}_{n-1}\} = \text{basis } \mathbf{1}^\perp$, $\lambda_n = nb$, $\mathbf{v}_n = \mathbf{1}$. Moreover P_1 is orthogonal projector onto $\mathbf{1}$. Now $A = (a - b)I + nbP_1$

$$A\mathbf{x} = (a - b)\mathbf{x} + nbP_1\mathbf{x} = (a - b)\mathbf{x} + nbk\mathbf{1}$$

$$A\mathbf{v} = \lambda\mathbf{v} \quad (a - b)\mathbf{v} + nbP_1\mathbf{v} = \lambda\mathbf{v}$$

$$\Rightarrow \begin{cases} P_1 \mathbf{v} = 0 \\ \mathbf{v} \parallel \mathbf{1} \end{cases} \Rightarrow \begin{cases} \mathbf{v} \perp \mathbf{1} \\ (a-b)\mathbf{v} = \lambda \mathbf{v} \\ \mathbf{v} \parallel \mathbf{1} \\ (a-b)\mathbf{v} + nb\mathbf{v} = \lambda \mathbf{v} \end{cases} \Rightarrow \begin{cases} \mathbf{v}_1, \dots, \mathbf{v}_{n-1} - \text{basis of } \mathbf{1}^\perp \\ \lambda_1 = \dots = \lambda_{n-1} = a-b \\ \mathbf{v}_n = \mathbf{1} \\ \lambda_n = a + (n-1)b \end{cases}$$

Answer: $\lambda_1 = \dots = \lambda_{n-1} = a-b$, $\{\mathbf{v}_1 \dots \mathbf{v}_{n-1}\} = \text{basis } \mathbf{1}^\perp$ $\lambda_n = a + (n-1)b$, $\mathbf{v}_n = \mathbf{1}$ ✓

2/4

- Problem 10** (Commuting matrices; 4pt). (a) Assume that $n \times n$ matrices A and B commute, i.e., $AB = BA$, and that \mathbf{v} is an eigenvector of A corresponding to an eigenvalue λ . Is it true that \mathbf{v} is also an eigenvector for B ? Justify your answer (ie., prove the claim or disprove it by example; in the latter case suggest extra conditions under which \mathbf{v} is an eigenvector of B)
- (b) Assume that A commutes with B and B commutes with C . Is it true that A and C must commute?

Solution to the problem 10. .

(a)

$$AB = BA \quad A\mathbf{v} = \lambda\mathbf{v}$$

If $B\mathbf{v} = 0 \Rightarrow \mathbf{v}$ is an eigenvector of B corresponding to 0 eigenvalue.

If not, then:

$$AB\mathbf{v} = BA\mathbf{v} = B\lambda\mathbf{v} = \lambda(B\mathbf{v}) = A(B\mathbf{v})$$

and $B\mathbf{v}$ is an eigenvector of A corresponding to λ . If λ is **distinct** then the corresponding eigenspace has $\dim = 1 \Rightarrow B\mathbf{v} = \mu\mathbf{v} \Rightarrow \mathbf{v}$ is an eigenvector of B (with eigenvalue μ).

So if A is diagonalisable then eigenvectors for A are also eigenvectors for B .

5/5

Problem 11 (Symmetric matrices; 5pt). Find a matrix P that orthogonally diagonalizes A , and determine $P^{-1}AP$:

$$(a) \quad \begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix}; \quad (b) \quad \begin{pmatrix} -2 & 0 & -36 \\ 0 & 3 & 0 \\ -36 & 0 & -23 \end{pmatrix}; \quad (c) \quad \begin{pmatrix} 1 & -4 & 2 \\ -4 & 1 & -2 \\ 2 & -2 & -2 \end{pmatrix}$$

Solution to the problem 11. .

(a)

$$A = \begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix} \quad \lambda_1 = 1 \quad \lambda_2 = 7 - 1 = 6$$

$$\begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} \mathbf{v} = 0 \Rightarrow \mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \hat{\mathbf{v}}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} -1 & -2 \\ -2 & -4 \end{pmatrix} \mathbf{v} = 0 \Rightarrow \mathbf{v}_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad \hat{\mathbf{v}}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$P = (\hat{\mathbf{v}}_1 \quad \hat{\mathbf{v}}_2)$$

$$\text{Answer: } P = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \quad P^{-1}AP = \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}$$

$$\text{or } P^{-1}AP = P^T AP = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$$

(b)

$$A = \begin{pmatrix} -2 & 0 & -36 \\ 0 & 3 & 0 \\ -36 & 0 & -23 \end{pmatrix} \quad \lambda_1 = 3 \quad \begin{cases} \lambda_2 + \lambda_3 = -25 \\ \lambda_2 \lambda_3 = -1250 \end{cases} \quad \lambda_2 = 25 \quad \lambda_3 = -50$$

$$\begin{pmatrix} -5 & 0 & -36 \\ 0 & 0 & 0 \\ -36 & 0 & -26 \end{pmatrix} \mathbf{v} = 0 \quad \mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \hat{\mathbf{v}}_1$$

$$\begin{pmatrix} -27 & 0 & -36 \\ 0 & -22 & 0 \\ -36 & 0 & -48 \end{pmatrix} \mathbf{v} = 0 \quad \mathbf{v}_2 = \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix} \quad \hat{\mathbf{v}}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{1}{5} \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix}$$

$$\begin{pmatrix} 48 & 0 & -36 \\ 0 & 53 & 0 \\ -36 & 0 & 27 \end{pmatrix} \mathbf{v} = 0 \quad \mathbf{v}_3 = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} \quad \hat{\mathbf{v}}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \frac{1}{5} \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$$

$$P = (\hat{\mathbf{v}}_1 \quad \hat{\mathbf{v}}_2 \quad \hat{\mathbf{v}}_3)$$

$$\text{Answer: } P = \frac{1}{5} \begin{pmatrix} 0 & 4 & 3 \\ 5 & 0 & 0 \\ 0 & -3 & 4 \end{pmatrix} \quad P^{-1}AP = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & -50 \end{pmatrix}$$

$$\text{or } P^{-1}AP = P^T AP = \frac{1}{5} \begin{pmatrix} 0 & 5 & 0 \\ 4 & 0 & -3 \\ 3 & 0 & 4 \end{pmatrix} \begin{pmatrix} -2 & 0 & -36 \\ 0 & 3 & 0 \\ -36 & 0 & -23 \end{pmatrix} \frac{1}{5} \begin{pmatrix} 0 & 4 & 3 \\ 5 & 0 & 0 \\ 0 & -3 & 4 \end{pmatrix}$$

(c)

$$A = \begin{pmatrix} 1 & -4 & 2 \\ -4 & 1 & -2 \\ 2 & -2 & -2 \end{pmatrix} \quad A + 3I = \begin{pmatrix} 4 & -4 & 2 \\ -4 & 4 & -2 \\ 2 & -2 & 1 \end{pmatrix}$$

$$\text{rank}(A + 3I) = 1 \Rightarrow \lambda_1 = \lambda_2 = -3 \quad \lambda_3 = 0 + 3 + 3 = 6$$


$$\begin{pmatrix} 4 & -4 & 2 \\ -4 & 4 & -2 \\ 2 & -2 & 1 \end{pmatrix} \mathbf{v} = 0 \Rightarrow 2v_1 - 2v_2 + v_3 = 0 \Rightarrow v_3 = 2v_2 - 2v_1 \Rightarrow \mathbf{v} = v_1 \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + v_2 \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \Rightarrow \text{GS} \Rightarrow \hat{\mathbf{v}}_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \quad \hat{\mathbf{v}}_2 = \frac{1}{3\sqrt{5}} \begin{pmatrix} 4 \\ 5 \\ 2 \end{pmatrix}$$

$$\left(\mathbf{v}_1^T \mathbf{v}_1 = 5 \quad \mathbf{v}_1^T \mathbf{v}_2 = -4 \quad \hat{\mathbf{v}}_2 = \frac{\mathbf{v}_2 + \frac{4}{5}\mathbf{v}_1}{\|\cdot\|} = \frac{1}{3\sqrt{5}} \begin{pmatrix} 4 \\ 5 \\ 2 \end{pmatrix} \right)$$

$$\begin{pmatrix} -5 & -4 & 2 \\ -4 & -5 & -2 \\ 2 & -2 & -8 \end{pmatrix} \mathbf{v} = 0 \quad \begin{cases} -5v_1 - 4v_2 + 2v_3 = 0 \\ -4v_1 - 5v_2 - 2v_3 = 0 \\ v_1 = v_2 + 4v_3 \end{cases} \Rightarrow -9v_2 - 18v_3 = 0$$

$$\Rightarrow \begin{cases} v_2 = -2v_3 \\ v_1 = 2v_3 \end{cases} \quad \mathbf{v}_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \quad \hat{\mathbf{v}}_3 = \frac{1}{3} \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \quad P = (\hat{\mathbf{v}}_1 \quad \hat{\mathbf{v}}_2 \quad \hat{\mathbf{v}}_3)$$

Answer: $P = \frac{1}{15} \begin{pmatrix} 3\sqrt{5} & 4\sqrt{5} & 10 \\ 0 & 5\sqrt{5} & -10 \\ -6\sqrt{5} & 2\sqrt{5} & 5 \end{pmatrix} \quad P^{-1}AP = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 6 \end{pmatrix}$ 

$$\text{or } P^{-1}AP = \frac{1}{15} \begin{pmatrix} 3\sqrt{5} & 0 & -6\sqrt{5} \\ 4\sqrt{5} & 5\sqrt{5} & 2\sqrt{5} \\ 10 & -10 & 5 \end{pmatrix} \begin{pmatrix} 1 & -4 & 2 \\ -4 & 1 & -2 \\ 2 & -2 & -2 \end{pmatrix} \frac{1}{15} \begin{pmatrix} 3\sqrt{5} & 4\sqrt{5} & 10 \\ 0 & 5\sqrt{5} & -10 \\ -6\sqrt{5} & 2\sqrt{5} & 5 \end{pmatrix}$$

3.5/4

Problem 12 (Symmetric matrices; 4pt). (a) Find all values of a and b for which there exists a 3×3 symmetric matrix with eigenvalues $\lambda_1 = -1$, $\lambda_2 = 3$, $\lambda_3 = 7$ and the corresponding eigenvectors

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} a \\ b \\ 0 \end{pmatrix}.$$

(b) Reconsider the problem if $\lambda_3 = 3$.


(c) Find all symmetric matrices A satisfying the conditions of parts (a) and (b).

Solution to the problem 12. .

(a) For the symmetric matrix the eigenvectors corresponding to distinct eigenvalues are necessary orthogonal:

$$\langle \mathbf{v}_1, \mathbf{v}_3 \rangle = 0 + b + 0 = 0 \Rightarrow b = 0 \quad \langle \mathbf{v}_2, \mathbf{v}_3 \rangle = a + b + 0 = 0 \Rightarrow a = 0$$

Since eigenvector should be nonzero there is no values of a, b satisfying the condition.

Answer: $(a, b) \in \emptyset$ 

(b) $\lambda_2 = \lambda_3 = 3$ is a multiple eigenvalue. For the symmetric matrix we can find such P s.t. P is orthogonal and $P^T AP = \text{diag}\{-1, 3, 3\}$. So \mathbf{v}_3 only needs to be orthogonal to \mathbf{v}_1 (since they correspond to different eigenvalues).

$$\langle \mathbf{v}_1, \mathbf{v}_3 \rangle = 0 + b + 0 = 0 \Rightarrow b = 0 \Rightarrow \mathbf{v}_3 = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}.$$

So now we see that $\mathbf{v}_3 \in \text{ls}\{e_1\}$ and $a \neq 0$. Let's check and find A :

$$\hat{\mathbf{v}}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \quad \hat{\mathbf{v}}_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\mathbf{v}_2^\top \mathbf{v}_2 = 3 \quad \mathbf{v}_2^\top \mathbf{v}_3 = a \quad \hat{\mathbf{v}}_3 = \frac{\mathbf{v}_3 - \frac{a}{3}\mathbf{v}_2}{\|\cdot\|} = \frac{a}{3} \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} / \left\| \frac{a}{3} \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \right\| = \frac{\operatorname{sgn} a}{\sqrt{6}} \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$$

We're not interested in the orientations of vectors, so let $\hat{\mathbf{v}}_3 = \frac{1}{\sqrt{6}} (2 \ -1 \ 1)^\top$

$$P = (\hat{\mathbf{v}}_1 \ \hat{\mathbf{v}}_2 \ \hat{\mathbf{v}}_3) = \frac{1}{\sqrt{6}} \begin{pmatrix} 0 & \sqrt{2} & 2 \\ \sqrt{3} & \sqrt{2} & -1 \\ -\sqrt{3} & \sqrt{2} & -1 \end{pmatrix}$$

$$\begin{aligned} PDP^{-1} &= PDP^\top = \frac{1}{6} \begin{pmatrix} 0 & \sqrt{2} & 2 \\ \sqrt{3} & \sqrt{2} & -1 \\ -\sqrt{3} & \sqrt{2} & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 0 & \sqrt{3} & -\sqrt{3} \\ \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 2 & -1 & 1 \end{pmatrix} = \\ &= \frac{1}{6} \begin{pmatrix} 0 & 3\sqrt{2} & 6 \\ -\sqrt{3} & 3\sqrt{2} & -3 \\ \sqrt{3} & 3\sqrt{2} & 3 \end{pmatrix} \begin{pmatrix} 0 & \sqrt{3} & -\sqrt{3} \\ \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 2 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 2 \\ 0 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix} \quad \leftarrow \text{miscellaneous} \end{aligned}$$

Answer: $a \neq 0, b = 0$

(c) **Answer:** (a): $A \in \emptyset$ (b): $A = \begin{pmatrix} 3 & 0 & 2 \\ 0 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}$

4/4

Problem 13 (Symmetric matrices; 4pt). A 3×3 symmetric matrix A has eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = \lambda$ and the corresponding eigenvectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} a \\ b \\ 0 \end{pmatrix}.$$

- (a) Find all possible values of a , b , and λ .
 (b) Find the corresponding matrices A .

Solution to the problem 13. .

- (a) 1. Let $\lambda \neq 1$ and $\lambda \neq 2$. For the symmetric matrix the eigenvectors corresponding to distinct eigenvalues are necessary orthogonal:

$$\langle \mathbf{v}_1, \mathbf{v}_3 \rangle = a + 0 = 0 \Rightarrow a = 0 \quad \langle \mathbf{v}_2, \mathbf{v}_3 \rangle = a + b + 0 = 0 \Rightarrow b = 0$$

Since eigenvector should be nonzero, $(a, b) \in \emptyset$. 

2. Let $\lambda = 1$. $\lambda_1 = \lambda_3 = 1$ is a multiple eigenvalue. So \mathbf{v}_3 should be orthogonal to \mathbf{v}_2 (since they correspond to different eigenvalues).

$$\langle \mathbf{v}_2, \mathbf{v}_3 \rangle = a + b = 0 \Rightarrow b = -a \Rightarrow \mathbf{v}_3 = a \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

Since eigenvector should be nonzero, $a \neq 0$.

3. Let $\lambda = 2$. $\lambda_2 = \lambda_3 = 2$ is a multiple eigenvalue. So \mathbf{v}_3 should be orthogonal to \mathbf{v}_1 .

$$\langle \mathbf{v}_1, \mathbf{v}_3 \rangle = a + 0 = 0 \Rightarrow a = 0 \Rightarrow \mathbf{v}_3 = b \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$



Since eigenvector should be nonzero, $b \neq 0$.

Answer: $(\lambda, a, b) \in \{(1, k, -k) \mid k \neq 0\} \cup \{(2, 0, k) \mid k \neq 0\}$

(b)

$$\hat{\mathbf{v}}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \hat{\mathbf{v}}_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

For $\lambda = 1$:

$$\mathbf{v}_1^\top \mathbf{v}_1 = 2 \quad \mathbf{v}_1^\top \mathbf{v}_3 = a \quad \hat{\mathbf{v}}_3 = \frac{\mathbf{v}_3 - \frac{a}{2} \mathbf{v}_1}{\|\cdot\|} = \frac{a}{2} \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} / \|\cdot\|$$

Let's set (similarly to Problem 12): $\hat{\mathbf{v}}_3 = \frac{1}{\sqrt{6}} (1 \ -2 \ -1)^\top$

$$P = (\hat{\mathbf{v}}_1 \ \hat{\mathbf{v}}_2 \ \hat{\mathbf{v}}_3) = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{3} & \sqrt{2} & 1 \\ 0 & \sqrt{2} & -2 \\ \sqrt{3} & -\sqrt{2} & -1 \end{pmatrix}$$



$$\begin{aligned} PDP^{-1} &= PDP^\top = \frac{1}{6} \begin{pmatrix} \sqrt{3} & \sqrt{2} & 1 \\ 0 & \sqrt{2} & -2 \\ \sqrt{3} & -\sqrt{2} & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 & \sqrt{3} \\ \sqrt{2} & \sqrt{2} & -\sqrt{2} \\ 1 & -2 & -1 \end{pmatrix} = \\ &= \frac{1}{6} \begin{pmatrix} \sqrt{3} & 2\sqrt{2} & 1 \\ 0 & 2\sqrt{2} & -2 \\ \sqrt{3} & -2\sqrt{2} & -1 \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 & \sqrt{3} \\ \sqrt{2} & \sqrt{2} & -\sqrt{2} \\ 1 & -2 & -1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 4 & 1 & -1 \\ 1 & 4 & -1 \\ -1 & -1 & 4 \end{pmatrix} \end{aligned}$$

Answer: $A = \frac{1}{3} \begin{pmatrix} 4 & 1 & -1 \\ 1 & 4 & -1 \\ -1 & -1 & 4 \end{pmatrix}$



For $\lambda = 2$:

$$\mathbf{v}_2^\top \mathbf{v}_2 = 3 \quad \mathbf{v}_2^\top \mathbf{v}_3 = b \quad \hat{\mathbf{v}}_3 = \frac{\mathbf{v}_3 - \frac{b}{3} \mathbf{v}_2}{\|\cdot\|} = \frac{b}{3} \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} / \|\cdot\|$$

Let's set (similarly to Problem 12): $\hat{\mathbf{v}}_3 = \frac{1}{\sqrt{6}} (-1 \ 2 \ 1)^\top$

$$P = (\hat{\mathbf{v}}_1 \ \hat{\mathbf{v}}_2 \ \hat{\mathbf{v}}_3) = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{3} & \sqrt{2} & -1 \\ 0 & \sqrt{2} & 2 \\ \sqrt{3} & -\sqrt{2} & 1 \end{pmatrix}$$



$$PDP^{-1} = PDP^\top = \frac{1}{6} \begin{pmatrix} \sqrt{3} & \sqrt{2} & -1 \\ 0 & \sqrt{2} & 2 \\ \sqrt{3} & -\sqrt{2} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 & \sqrt{3} \\ \sqrt{2} & \sqrt{2} & -\sqrt{2} \\ -1 & 2 & 1 \end{pmatrix} =$$

$$\frac{1}{6} \begin{pmatrix} \sqrt{3} & 2\sqrt{2} & -2 \\ 0 & 2\sqrt{2} & 4 \\ \sqrt{3} & -2\sqrt{2} & 2 \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 & \sqrt{3} \\ \sqrt{2} & \sqrt{2} & -\sqrt{2} \\ -1 & 2 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 3 & 0 & -1 \\ 0 & 4 & 0 \\ -1 & 0 & 3 \end{pmatrix}$$

Answer: $A = \begin{pmatrix} 3 & 0 & -1 \\ 0 & 4 & 0 \\ -1 & 0 & 3 \end{pmatrix}$

4/4

Problem 14 (Quadratic forms; 4 pt). Find an orthogonal change of variables that eliminates the cross product terms in the quadratic form Q , and express Q in terms of the new variables:

(a) $Q(x_1, x_2) = 2x_1^2 + 2x_2^2 - 2x_1x_2$;

(b) $Q(x_1, x_2, x_3) = 3x_1^2 + 4x_2^2 + 5x_3^2 + 4x_1x_2 - 4x_2x_3$

Solution to the problem 14. .

(a)

$$Q(*\mathbf{x}) = 2x_1^2 + 2x_2^2 - 2x_1x_2 = \mathbf{x}^\top A \mathbf{x} \quad A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\lambda_1 = 1 \quad \lambda_2 = 4 - 1 = 3$$

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \mathbf{v} = 0 \quad \mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \mathbf{v} = 0 \quad \mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad P^\top = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$Q(*\mathbf{x}) = \mathbf{x}^\top A \mathbf{x} = \mathbf{x}^\top P D P^\top \mathbf{x} = |y = P^\top \mathbf{x}| = \mathbf{y}^\top D \mathbf{y} = y_1^2 + 3y_2^2 = Q(*\mathbf{y}) \quad \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Answer: The orthogonal change of variables: $P^\top = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $Q(*\mathbf{y}) = y_1^2 + 3y_2^2$ ✓

(b)

$$Q(*\mathbf{x}) = 3x_1^2 + 4x_2^2 + 5x_3^2 + 4x_1x_2 - 4x_2x_3 = \mathbf{x}^\top A \mathbf{x} \quad A = \begin{pmatrix} 3 & 2 & 0 \\ 2 & 4 & -2 \\ 0 & -2 & 5 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\det(A - \lambda I) = -\lambda^3 + \text{tr } A \lambda^2 - \lambda \sum M_2 + \det A = -\lambda^3 + 12\lambda^2 - 39\lambda + 28$$

$$\lambda^3 - 12\lambda^2 + 39\lambda - 28 = (\lambda - 1)(\lambda^2 - 11\lambda + 28) = (\lambda - 1)(\lambda - 4)(\lambda - 7)$$

$$\Rightarrow \lambda_1 = 1 \quad \lambda_2 = 4 \quad \lambda_3 = 7$$

$$\begin{pmatrix} 2 & 2 & 0 \\ 2 & 3 & -2 \\ 0 & -2 & 4 \end{pmatrix} \mathbf{v} = 0 \quad \mathbf{v}_1 = \frac{1}{3} \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 2 & 0 \\ 2 & 0 & -2 \\ 0 & -2 & 1 \end{pmatrix} \mathbf{v} = 0 \quad \mathbf{v}_2 = \frac{1}{3} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} -4 & 2 & 0 \\ 2 & -3 & -2 \\ 0 & -2 & -2 \end{pmatrix} \mathbf{v} = 0 \quad \mathbf{v}_3 = \frac{1}{3} \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$$

$$P = \frac{1}{3} \begin{pmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ -1 & 2 & -2 \end{pmatrix} \quad P^\top = \frac{1}{3} \begin{pmatrix} 2 & -2 & -1 \\ 2 & 1 & 2 \\ 1 & 2 & -2 \end{pmatrix}$$

$$Q(*\mathbf{x}) = \mathbf{x}^\top A \mathbf{x} = \mathbf{x}^\top P D P^\top \mathbf{x} = |y = P^\top \mathbf{x}| = \mathbf{y}^\top D \mathbf{y} = y_1^2 + 4y_2^2 + 7y_3^2 = Q(*\mathbf{y}) \quad \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

Answer: The orthogonal transformation: $P^\top = \frac{1}{3} \begin{pmatrix} 2 & -2 & -1 \\ 2 & 1 & 2 \\ 1 & 2 & -2 \end{pmatrix}$, $Q(*\mathbf{y}) = y_1^2 + 4y_2^2 + 7y_3^2$ ✓

1/2 Problem 15 (Quadratic forms; 2 pt). Find all values of k for which the quadratic form

$$5x_1^2 + x_2^2 + kx_3^2 + 4x_1x_2 - 2x_1x_3 - 2x_2x_3$$

is positive definite.

Solution to the problem 15. .

$$Q(*\mathbf{x}) = 5x_1^2 + x_2^2 + kx_3^2 + 4x_1x_2 - 2x_1x_3 - 2x_2x_3 = \mathbf{x}^\top A \mathbf{x} \quad A = \begin{pmatrix} 5 & 2 & -1 \\ 2 & 1 & -1 \\ -1 & -1 & k \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$p_A(\lambda) = -\lambda^3 + \text{tr } A \lambda^2 - \lambda \sum M_2 + \det A = -\lambda^3 + (6+k)\lambda^2 - (6k-1)\lambda + k - 2 =$$

$$= -(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$$

Q is positive definite $\Leftrightarrow A$ is positive definite $\Leftrightarrow \lambda_i > 0 \ \forall i \in \overline{1,3}$

Let $k = 2$, then:

$$p_A(0) = k - 2 = 0 \Rightarrow \lambda_1 = 0 \quad \lambda^2 - 8\lambda + 11 = 0 \quad \lambda_{1,2} = 4 \pm \sqrt{5} > 0$$

So all the eigenvalues are non-negative. If we take $k > 2$ then the roots will be positive.

Answer: $k > 2$

how did you obtain this conclusion?

4/4 Problem 16 (Quadratic forms; 4 pt). Consider the matrix A and the vector \mathbf{v}_1 , where

$$A = \begin{pmatrix} 3 & -2 & 1 \\ -2 & 6 & -2 \\ 1 & -2 & 3 \end{pmatrix}, \quad \mathbf{v}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

- (a) Show that \mathbf{v}_1 is an eigenvector of A , and find its corresponding eigenvalue. Then find the other two eigenvalues and eigenvectors and orthogonally diagonalize A .
- (b) Is the quadratic form $Q(\mathbf{x}) = \mathbf{x}^\top A \mathbf{x}$ positive definite, negative definite or indefinite? Find the principal axes of Q and the corresponding transition matrix.

Solution to the problem 16. .

(a)

$$A = \begin{pmatrix} 3 & -2 & 1 \\ -2 & 6 & -2 \\ 1 & -2 & 3 \end{pmatrix} \quad \mathbf{v}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Let's check that $\exists \lambda : (A - \lambda I)\mathbf{v}_1 = 0$

$$\begin{pmatrix} 3-\lambda & -2 & 1 \\ -2 & 6-\lambda & -2 \\ 1 & -2 & 3-\lambda \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = 0 \Leftrightarrow \begin{cases} \lambda - 3 + 1 = 0 \\ 2 - 2 = 0 \\ -1 + 3 - \lambda = 0 \end{cases} \Leftrightarrow \lambda_1 = 2$$

$$\text{rank}(A - 2I) = \text{rank} \begin{pmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{pmatrix} = 1 \Rightarrow \lambda_2 = \lambda_1 = 2 \quad \lambda_3 = 12 - 2 - 2 = 8$$

$$\begin{pmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{pmatrix} \mathbf{v}_2 = 0 \quad v_1 = 2v_2 - v_3 \Rightarrow \mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + v_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$


$$\Rightarrow \mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \Rightarrow GS :$$

$$\mathbf{v}_1^\top \mathbf{v}_1 = 2 \quad \hat{\mathbf{v}}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \quad \mathbf{v}_1^\top \mathbf{v}_2 = -2 \quad \hat{\mathbf{v}}_2 = \frac{\mathbf{v}_2 + \mathbf{v}_1}{\|\mathbf{v}_2 + \mathbf{v}_1\|} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$(A - 8I)\mathbf{v} = 0 \Rightarrow \begin{cases} -5v_1 - 2v_2 + v_3 = 0 \\ -2v_1 - 2v_2 - 2v_3 = 0 \\ v_1 = 2v_2 + 5v_3 \end{cases} \quad \begin{cases} v_2 = -2v_3 \\ v_1 = 2v_2 + 5v_3 = v_3 \end{cases} \quad \mathbf{v}_3 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$P = \frac{1}{\sqrt{6}} \begin{pmatrix} -\sqrt{3} & \sqrt{2} & 1 \\ 0 & \sqrt{2} & -2 \\ \sqrt{3} & \sqrt{2} & 1 \end{pmatrix} \quad P^{-1} = P^\top = \frac{1}{\sqrt{6}} \begin{pmatrix} -\sqrt{3} & 0 & \sqrt{3} \\ \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 1 & -2 & 1 \end{pmatrix}$$

$$A = PDP^\top = \frac{1}{\sqrt{6}} \begin{pmatrix} -\sqrt{3} & \sqrt{2} & 1 \\ 0 & \sqrt{2} & -2 \\ \sqrt{3} & \sqrt{2} & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{pmatrix} \frac{1}{\sqrt{6}} \begin{pmatrix} -\sqrt{3} & 0 & \sqrt{3} \\ \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 1 & -2 & 1 \end{pmatrix}$$

- (b) $Q(\mathbf{x})$ is positive definite since all eigenvalues of A are positive. Principal axes of Q correspond to the eigenvectors of A : $\frac{1}{\sqrt{6}} \left\{ \begin{pmatrix} -\sqrt{3} \\ 0 \\ \sqrt{3} \end{pmatrix}, \begin{pmatrix} \sqrt{2} \\ \sqrt{2} \\ \sqrt{2} \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\}$ 

And the transition matrix is P^\top since it transforms a vector coordinates from the canonical basis to the eigenvectors basis:

$$\frac{1}{\sqrt{6}} \begin{pmatrix} -\sqrt{3} & 0 & \sqrt{3} \\ \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 1 & -2 & 1 \end{pmatrix} \quad \img alt="green checkmark" data-bbox="665 260 705 290"/>$$