# Linear Algebra
# Home assignment 2: Orthogonality

Solutions are by Yaroslava Lochman.

**Problem 1** (Parallel and orthogonal planes; 2pt)**.** Determine whether the given planes are:

(a) parallel:

(i) $4x - y + 2z = 5$ and $7x - 3y + 4z = 8$;

(ii) $x - 4y - 3z - 2 = 0$ and $3x - 12y - 9z - 7 = 0$.

(b) perpendicular:

(i) $3x - y + z = 0$ and $x + 2z = -1$;

(ii) $x - 2y + 3z = 4$ and $-2x + 5y + 4z = -1$.

**Solution to the problem 1.**

$$Ax + By + Cz = D \;\Rightarrow\; n = (A, B, C)^\top \text{ – the normal vector of the plane}$$

The planes are parallel $\Leftrightarrow$ the normal vectors are parallel $\Leftrightarrow$ $\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}$.
The planes are perpendicular $\Leftrightarrow$ the normal vectors are perpendicular $\Leftrightarrow$ $\langle n_1, n_2\rangle = 0$ $\Leftrightarrow$ $A_1A_2 + B_1B_2 + C_1C_2 = 0$

(a) (parallel)

(i)
$$4x - y + 2z = 5 \;\Rightarrow\; n_1 = (4, -1, 2)^\top$$
$$7x - 3y + 4z = 8 \;\Rightarrow\; n_2 = (7, -3, 4)^\top$$
$$\frac{4}{7} \neq \frac{-1}{-3} \neq \frac{2}{4}$$

Hence the planes are not parallel.

(ii)
$$x - 4y - 3z - 2 = 0 \;\Rightarrow\; n_1 = (1, -4, -3)^\top$$
$$3x - 12y - 9z - 7 = 0 \;\Rightarrow\; n_2 = (3, -12, -9)^\top$$
$$\frac{3}{1} = \frac{-12}{-4} = \frac{-9}{-3} = 3$$

Hence the planes are parallel.

(b) (perpendicular)

(i) For $3x - y + z = 0$ and $x + 2z = -1$ we have

$$\langle n_1, n_2\rangle = 3 + 0 + 2 = 5 \neq 0$$

Hence the planes are not perpendicular.

(ii) For $x-2y+3z=4$ and $-2x+5y+4z=-1$ we have

$$\langle n_1, n_2\rangle = -2-10+12=0$$

Hence the planes are perpendicular.

**Problem 2** (Orthogonal complement; 3pt)**.** (a) Let $W$ be the plane in $\mathbb{R}^3$ given by the equation $x-2y-3z=0$. Find parametric equations for $W^\perp$.

(b) Let $W$ be the line in $\mathbb{R}^3$ with parametric equations $x=2t$, $y=-5t$, $z=4t$. Find an equation for $W^\perp$.

(c) Let $W$ be the intersection of the two planes $x+y+z=0$ and $x-y+z=0$ in $\mathbb{R}^3$. Find an equation for $W^\perp$.

**Solution to the problem 2.** .

(a)

$$W = \{\begin{pmatrix} x & y & z\end{pmatrix}^\top \in \mathbb{R}^3 \,\big|\, x-2y-3z=0\} \qquad \mathbf{n} = \begin{pmatrix} 1 & -2 & -3\end{pmatrix}^\top \text{ – normal vector of the plane}$$

$$\text{So } \mathbf{x}\in W \;\Rightarrow\; \langle \mathbf{x},\mathbf{n}\rangle = 0$$

$$\mathbf{y}\in W^\perp \Rightarrow \mathbf{y}\perp W \;\Rightarrow\; \langle \mathbf{y},\mathbf{x}\rangle = 0 \;\forall \mathbf{x}\in W \;\Rightarrow\; \mathbf{y}\parallel \mathbf{n}$$

So:

$$W^\perp = \left\{ \begin{pmatrix} x & y & z\end{pmatrix}^\top \in \mathbb{R}^3 \,\middle|\, \begin{matrix} x=t \\ y=-2t \\ z=-3t \end{matrix} \quad t\in\mathbb{R} \right\} = \mathrm{ls}\{\mathbf{n}\}$$

The parametric equations for $W^\perp$ are $x=t$, $y=-2t$, $z=-3t$.

(b)

$$W = \left\{ \begin{pmatrix} x & y & z\end{pmatrix}^\top \in \mathbb{R}^3 \,\middle|\, \begin{matrix} x=2t \\ y=-5t \\ z=4t \end{matrix} \quad t\in\mathbb{R} \right\} \qquad \mathbf{v} = \begin{pmatrix} 2 \\ -5 \\ 4\end{pmatrix} \text{ – direction vector of the line}$$

$$\mathbf{y}\in W^\perp \;\Rightarrow\; \langle \mathbf{y},\mathbf{x}\rangle = 0 \;\forall \mathbf{x}\in W \;\Rightarrow\; \mathbf{y}\perp\mathbf{v} \Rightarrow W^\perp \text{ is a plane with the normal vector } \mathbf{v}$$

So:

$$W^\perp = \{\begin{pmatrix} x & y & z\end{pmatrix}^\top \in \mathbb{R}^3 \,\big|\, 2x-5y+4z=0\}$$

The equation for $W^\perp$ is $2x-5y+4z=0$.

(c)

$$W = \left\{ \begin{pmatrix} x & y & z\end{pmatrix}^\top \in \mathbb{R}^3 \,\middle|\, \begin{matrix} x+y+z=0 \\ x-y+z=0 \end{matrix} \right\}$$

$\mathbf{n}_1 = \begin{pmatrix} 1 & 1 & 1\end{pmatrix}^\top$ – normal vector of the $1^{st}$ plane.
$\mathbf{n}_2 = \begin{pmatrix} 1 & -1 & 1\end{pmatrix}^\top$ – normal vector of the $2^{nd}$ plane.

$$W = \left\{\mathbf{x}\in\mathbb{R}^3 \,\middle|\, \langle \mathbf{x},\mathbf{n}_1\rangle = 0 \;\wedge\; \langle \mathbf{x},\mathbf{n}_2\rangle = 0\right\} \text{ – is a line}$$

We have: $\mathbf{x} \perp \mathbf{n}_1$, $\mathbf{x} \perp \mathbf{n}_2$. Let $\mathbf{y} \in \text{ls}\{\mathbf{n}_1, \mathbf{n}_2\} \Rightarrow \langle \mathbf{y}, \mathbf{x} \rangle = \langle \alpha_1 \mathbf{n}_1 + \alpha_2 \mathbf{n}_2, \mathbf{x} \rangle = \alpha_1 \cdot 0 + \alpha_2 \cdot 0 = 0$ $\Rightarrow$ $\mathbf{x} \perp \text{ ls}\{\mathbf{n}_1, \mathbf{n}_2\}$. Hence:

$$W^{\perp} = \text{ls}\{\mathbf{n}_1, \mathbf{n}_2\}$$

To get an equation for the plane we need to compute its normal $\mathbf{n}$ (which is concurrently a direction vector of the line $W$). We can do it using the cross product:

$$\mathbf{n} = \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}$$

So the equation for $W^{\perp}$ is $2x - 2z = 0$.

**Problem 3** (Distance from a point; 4pt)**.** (a) Find the distance from the point $P = (1, 1, 0)$ to the line $\frac{x-1}{2} = \frac{y-2}{1} = \frac{z+1}{2}$.

(b) Let $\pi$ be a plane given by the equation $ax + by + cz + d = 0$ and $P(x_0, y_0, z_0)$ be a point outside it. Prove that the distance from $P$ to $\pi$ is given by the formula

$$\frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

Hint: if $Q$ is the point on $\pi$ realizing the distance, then $\overrightarrow{PQ}$ is collinear to $\mathbf{n} = (a, b, c)$ (why?). Take now any point $Q'$ on $\pi$ and find a projection of $\overrightarrow{PQ'}$ onto direction $\mathbf{n}$

(c) Find the distance between the point $P = (1, 0, 1)$ and the plane $2x + 2y - z = 2$.

**Solution to the problem 3.** .

(a)

$$P = (1, 1, 0) \qquad l : \frac{x-1}{2} = \frac{y-2}{1} = \frac{z+1}{2} \qquad \mathbf{v} = \begin{pmatrix} 2 & 1 & 2 \end{pmatrix}^{\top} - \text{ direction vector}$$

Let $O = (1, 2, -1)$. This point lies on the line. Then $\overrightarrow{OP} = \begin{pmatrix} 0 & -1 & 1 \end{pmatrix}^{\top}$. Let $P'$ be the point on the line realizing distance. Hence the distance can be found as:

$$\rho = \left|\overrightarrow{PP'}\right| = \sqrt{\left|\overrightarrow{OP}\right|^2 - \left|\overrightarrow{OP'}\right|^2}$$

$\left|\overrightarrow{PP'}\right|$ is the projection of $\overrightarrow{OP}$ on the line:

$$\left|\overrightarrow{OP'}\right|^2 = pr^2_{\mathbf{v}}\overrightarrow{OP} = \frac{\left\langle \overrightarrow{OP}, \mathbf{v} \right\rangle^2}{\|\mathbf{v}\|^2} = \frac{(0 - 1 + 2)^2}{2^2 + 1^2 + 2^2} = \frac{1}{9}$$

So:

$$\left|\overrightarrow{PP'}\right| = \sqrt{2 - \frac{1}{9}} = \frac{\sqrt{17}}{3}$$

(b)

$$P = (x_0, y_0, z_0) \qquad \pi: \ ax + by + cz + d = 0 \qquad \mathbf{n} = \begin{pmatrix} a & b & c \end{pmatrix}^\top \ - \ \text{normal vector}$$

Let $Q$ be the point on $\pi$ realizing the distance. Since the distance is the shortest:

$$\overrightarrow{PQ} \perp \overrightarrow{QQ'} \quad \forall Q' \in \pi \quad \text{or} \quad \overrightarrow{PQ} \perp \pi$$

($Q \in \pi$ minimizes $|PQ| \Leftrightarrow PQ \perp \pi$ – had been concluded using the Pythagorean theorem). Therefore $\overrightarrow{PQ}$ is collinear to the normal $\mathbf{n}$. So the distance $\rho = \left|\overrightarrow{PQ}\right|$ is an absolute value (since the angle between vectors can be acute or obtuse) of the projection of $\overrightarrow{PQ'}$ onto $\mathbf{n}$:

$$Q' = \begin{pmatrix} x_q & y_q & z_q \end{pmatrix}^\top \in \pi \ \Rightarrow \ ax_q + by_q + cz_q + d = 0$$

$$\overrightarrow{PQ'} = \begin{pmatrix} x_q - x_0 & y_q - y_0 & z_q - z_0 \end{pmatrix}^\top$$

$$\Rightarrow \ \rho = \left| pr_n \overrightarrow{PQ'} \right| = \frac{\left| \left\langle \overrightarrow{PQ'}, n \right\rangle \right|}{\|n\|} = \frac{\left| \left\langle \begin{pmatrix} x_q - x_0 \\ y_q - y_0 \\ z_q - z_0 \end{pmatrix}, \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\rangle \right|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|-d - ax_0 - by_0 - cz_0|}{\sqrt{a^2 + b^2 + c^2}}$$

So:

$$\rho = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

(c)

$$P = (1, 0, 1) \qquad \pi: \ 2x + 2y - z - 2 = 0$$

$$\rho = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|2 + 2 \cdot 0 - 1 + -2|}{\sqrt{2^2 + 2^2 + (-1)^2}} = \frac{1}{\sqrt{9}} = \frac{1}{3}$$

**Problem 4** (Cross product; 4pt)**.** (a) For any two vectors $\mathbf{u} = (u_1, u_2, u_3)^\top$ and $\mathbf{v} = (v_1, v_2, v_3)^\top$, their ***vector product***, or ***cross product*** $\mathbf{u} \times \mathbf{v}$ is the vector $\mathbf{w} = (w_1, w_2, w_3)^\top$ with entries

$$w_1 = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, \qquad w_2 = - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \qquad w_3 = \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}.$$

Prove that $\mathbf{w}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$ in the sense that $\mathbf{w}^\top \mathbf{u} = \mathbf{w}^\top \mathbf{v} = 0$.

Hint: these products are cofactor expansions of some $3 \times 3$ matrices

(b) Assume that $\mathbf{u}_1, \dots, \mathbf{u}_{n-1}$ are linearly independent vectors in $\mathbb{R}^n$. Find a formula analogous to that in part (a) for a vector that is orthogonal to the subspace spanned by $\mathbf{u}_1, \dots, \mathbf{u}_{n-1}$.

**Solution to the problem 4.** .

(a)

$$\mathbf{w} = \mathbf{u} \times \mathbf{v} = \begin{vmatrix} e_1 & e_2 & e_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$\mathbf{w}^\top \mathbf{u} = \begin{vmatrix} u_1 & u_2 & u_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = 0 \text{ since the } 1^{st} \text{ and } 2^{nd} \text{ rows are equal (therefore dependent)}$$

The same with $\mathbf{v}$:

$$\mathbf{w}^\top \mathbf{v} = \begin{vmatrix} v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = 0 \text{ since the } 1^{st} \text{ and } 3^{rd} \text{ rows are equal (therefore dependent)}$$

(b) Let $\mathbf{u}_i = (u_i^1, \cdots, u_i^n)^\top$. Then the answer is:

$$\mathbf{w} = \begin{vmatrix} e_1 & \cdots & e_n \\ u_1^1 & \cdots & u_1^n \\ \vdots & & \vdots \\ u_{n-1}^1 & \cdots & u_{n-1}^n \end{vmatrix}$$

One can see that $\mathbf{w}$ is orthogonal to all $\mathbf{u}_i$:

$$\mathbf{w}^\top \mathbf{u}_i = \begin{vmatrix} u_i^1 & \cdots & u_i^n \\ u_1^1 & \cdots & u_1^n \\ \vdots & & \vdots \\ u_{n-1}^1 & \cdots & u_{n-1}^n \end{vmatrix} = 0 \quad \forall i \in \overline{1, n} \quad \text{(because of the two equal rows)}$$

Therefore it is orthogonal to $\text{ls}\{\mathbf{u}_1, \ldots, \mathbf{u}_{n-1}\}$:

$$\forall \mathbf{x} \in \text{ls}\{\mathbf{u}_1, \ldots, \mathbf{u}_{n-1}\} \quad \mathbf{w}^\top \mathbf{x} = \mathbf{w}^\top \sum_{i=1}^{n-1} \alpha_i \mathbf{u}_i = \sum_{i=1}^{n-1} \alpha_i \mathbf{w}^\top \mathbf{u}_i = 0$$

**Problem 5** (Orthogonal matrices; 4pt)**.** (a) If $Q_1$ and $Q_2$ are orthogonal matrices, show that $Q_1^{-1}$ and $Q_1 Q_2$ are orthogonal as well.

(b) Prove that an orthogonal matrix that is also upper-triangular must be diagonal.

**Solution to the problem 5.** .

(a) $Q_1$ and $Q_2$ are orthogonal $\Leftrightarrow Q_1^\top = Q_1^{-1}$, $Q_2^\top = Q_2^{-1}$. So:

$$1.\ (Q_1^{-1})^\top = (Q_1^\top)^\top = Q_1 = (Q_1^{-1})^{-1}$$

Hence $Q_1^{-1}$ is orthogonal.

$$2.\ (Q_1 Q_2)^\top = Q_2^\top Q_1^\top = Q_2^{-1} Q_1^{-1} = (Q_2 Q_1)^{-1}$$

Hence $Q_1 Q_2$ is orthogonal.

(b) Let $Q = \begin{pmatrix} Q_1 & Q_2 & \cdots & Q_n \end{pmatrix}$ and $Q^\top Q = I$ :

$$\begin{pmatrix} Q_1^\top \\ Q_2^\top \\ \cdots \\ Q_n^\top \end{pmatrix} \begin{pmatrix} Q_1 & Q_2 & \cdots & Q_n \end{pmatrix} = I$$

$$\Rightarrow \begin{cases} Q_i^\top Q_i = 1 & \forall i \in \overline{1,n} \\ Q_i^\top Q_j = 0 & \forall i,j \in \overline{1,n}\ i \neq j \end{cases}$$

$$\Rightarrow Q_1^\top Q_1 = Q_{11}^2 = 1 \quad \Rightarrow Q_{11} = \pm 1 \quad \Rightarrow Q_1^\top Q_j = \pm Q_{1j} = 0\ \forall j \neq 1$$

$$\Rightarrow Q_2^\top Q_2 = Q_{12}^2 + Q_{22}^2 = 0 + Q_{22}^2 = 1 \quad \Rightarrow Q_{22} = \pm 1 \quad \Rightarrow Q_2^\top Q_j = \pm Q_{2j} = 0\ \forall j \neq 2$$

and so on. Hence $Q$ is diagonal and moreover:

$$Q = \begin{pmatrix} \pm 1 & 0 & \cdots & 0 \\ 0 & \pm 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \pm 1 \end{pmatrix}$$

**Problem 6** (Projection matrices; 3pt)**.** For the vectors $\mathbf{a}_1 = (1,0,1)$, $\mathbf{a}_2 = (0,1,2)$ and $\mathbf{b} = (-1,2,1)$

(a) find the matrix of the orthogonal projection $P_W$ onto the plane $W := \mathrm{ls}\{\mathbf{a}_1, \mathbf{a}_2\}$;

(b) find the matrix of the orthogonal projection $P_{W^\perp}$ onto the line $W^\perp$;

(c) find the components of the vector $\mathbf{b}$ with respect to the decomposition $\mathbb{R}^3 = W \oplus W^\perp$.

**Solution to the problem 6.** .

(a)

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix}$$

$$P_W = A(A^\top A)^{-1} A^\top$$

$$A^\top A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix}$$

$$(A^\top A)^{-1} = \frac{1}{6} \begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix}$$

$$P_W = \frac{1}{6} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 5 & -2 & 1 \\ -2 & 2 & 2 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 5 & -2 & 1 \\ -2 & 2 & 2 \\ 1 & 2 & 5 \end{pmatrix}$$

(b)

$$W^{\perp} = \text{ls}\{\mathbf{a}\}$$

where:

$$\mathbf{a} = \mathbf{a}_1 \times \mathbf{a}_2 = \begin{pmatrix} i & j & k \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} = -i - 2j + k = (-1, -2, 1)^{\top}$$

$$P_{W^{\top}} = \frac{\mathbf{a}\mathbf{a}^{\top}}{\mathbf{a}^{\top}\mathbf{a}} = \frac{1}{6}\begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \end{pmatrix}$$

(c)

$$\mathbf{b} = P_W\mathbf{b} + P_{W^{\top}}\mathbf{b}$$

$$P_W\mathbf{b} = \frac{1}{6}\begin{pmatrix} 5 & -2 & 1 \\ -2 & 2 & 2 \\ 1 & 2 & 5 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} -8 \\ 8 \\ 8 \end{pmatrix} = \begin{pmatrix} -4/3 \\ 4/3 \\ 4/3 \end{pmatrix}$$

$$P_{W^{\top}}\mathbf{b} = \frac{1}{6}\begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 2/3 \\ -1/3 \end{pmatrix}$$

Indeed:

$$P_W\mathbf{b} + P_{W^{\top}}\mathbf{b} = \begin{pmatrix} -4/3 \\ 4/3 \\ 4/3 \end{pmatrix} + \begin{pmatrix} 1/3 \\ 2/3 \\ -1/3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = \mathbf{b}$$

**Problem 7** (Least squares solution; 4pt)**.** Is there any value of $s$ for which $x_1 = 1$ and $x_2 = 2$ is the least squares solution of the linear system below? Explain your reasoning.

$$\begin{aligned} x_1 - \phantom{3}x_2 &= 1, \\ 2x_1 + 3x_2 &= 1, \\ 4x_1 + 5x_2 &= s. \end{aligned}$$

**Solution to the problem 7.** Let

$$A = \begin{pmatrix} 1 & -1 \\ 2 & 3 \\ 4 & 5 \end{pmatrix} = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ s \end{pmatrix}$$

Let $\mathbf{x}^* = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ be the least squares solution of the linear system $A\mathbf{x} = \mathbf{b}$. Therefore $\mathbf{e} = \mathbf{b} - A\mathbf{x}^*$ should be orthogonal to $C(A) \Leftrightarrow \langle \mathbf{e}, \mathbf{a}_i \rangle = 0 \quad i = 1, 2$

$$\mathbf{e} = \mathbf{b} - A\mathbf{x}^* = \begin{pmatrix} 1 \\ 1 \\ s \end{pmatrix} - \begin{pmatrix} 1 & -1 \\ 2 & 3 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ s \end{pmatrix} - \begin{pmatrix} -1 \\ 8 \\ 14 \end{pmatrix} = \begin{pmatrix} 2 \\ -7 \\ s - 14 \end{pmatrix}$$

$$\begin{cases} \langle \mathbf{e}, \mathbf{a}_1 \rangle = \left\langle \begin{pmatrix} 2 \\ -7 \\ s-14 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \right\rangle = 2 - 14 + 4s - 48 = 4s - 60 = 0 \quad \Rightarrow \quad s = 15 \\ \langle \mathbf{e}, \mathbf{a}_2 \rangle = \left\langle \begin{pmatrix} 2 \\ -7 \\ s-14 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix} \right\rangle = -2 - 21 + 5s - 70 = 5s - 93 = 0 \Rightarrow s = 18.6 \end{cases}$$

$$\Rightarrow s \in \varnothing$$

Hence there is no value of s for which $\mathbf{x}^*$ is the least squares solution of $A\mathbf{x} = \mathbf{b}$.

**Problem 8** (Regression; 6pt). (a) Find the least squares straight line fit to the four points $(0,1)$, $(2,0)$, $(3,1)$, and $(3,2)$.

(b) Find the quadratic polynomial that best fits the four points $(2,0)$, $(3,-10)$, $(5,-48)$, and $(6,-76)$.

(c) Find the cubic polynomial that best fits the five points $(-1,-14)$, $(0,-5)$, $(1,-4)$, $(2,1)$, and $(3,22)$.

Hint: the numbers are chosen so that $A^\top A$ can easily be inverted. If, however, this is not so, ask Python or anybody else for a help.

**Solution to the problem 8.** .

(a)

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 3 \\ 1 & 3 \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$$

The first column of $A$ corresponds to the y-intercept, and the second – to the 1-degree term.

$$A^\top A = \begin{pmatrix} 4 & 8 \\ 8 & 22 \end{pmatrix} \qquad (A^\top A)^{-1} = \frac{1}{12} \begin{pmatrix} 11 & -4 \\ -4 & 2 \end{pmatrix} \qquad (A^\top A)^{-1} A^\top = \frac{1}{12} \begin{pmatrix} 11 & 3 & -1 & -1 \\ -4 & 0 & 2 & 2 \end{pmatrix}$$

$$x = (A^\top A)^{-1} A^\top b = \frac{1}{12} \begin{pmatrix} 8 \\ 2 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 1/6 \end{pmatrix}$$

So, the quadratic polynomial $y = \frac{1}{6}x + \frac{2}{3}$ best fits the points.

(b)

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 5 & 25 \\ 1 & 6 & 36 \end{pmatrix} \quad b = \begin{pmatrix} 0 \\ -10 \\ -48 \\ -76 \end{pmatrix}$$

The first column of $A$ corresponds to the y-intercept, the second – to the 1-degree term, the third – to the 2-degree term.

$$A^\top A = \begin{pmatrix} 4 & 16 & 74 \\ 16 & 74 & 376 \\ 74 & 376 & 2018 \end{pmatrix} \qquad (A^\top A)^{-1} = \frac{1}{90} \begin{pmatrix} 1989 & -1116 & 135 \\ -1116 & 649 & -80 \\ 135 & -80 & 10 \end{pmatrix}$$

$$(A^\top A)^{-1}A^\top = \frac{1}{90}\begin{pmatrix} 297 & -144 & -216 & 153 \\ -138 & 111 & 129 & -102 \\ 15 & -15 & -15 & 15 \end{pmatrix}$$

$$x = (A^\top A)^{-1}A^\top b = \frac{1}{90}\begin{pmatrix} 180 \\ 450 \\ -270 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix}$$

So, the quadratic polynomial $y = 3x^2 + 5x + 2$ best fits the points.

(c)

$$A = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{pmatrix} \qquad b = \begin{pmatrix} -14 \\ -5 \\ -4 \\ 1 \\ 22 \end{pmatrix}$$

The first column of $A$ corresponds to the y-intercept, the second – to the 1-degree term, the third – to the 2-degree term, the fourth – to the 3-degree term.

$$A^\top A = \begin{pmatrix} 5 & 5 & 15 & 35 \\ 5 & 15 & 35 & 99 \\ 15 & 35 & 99 & 275 \\ 35 & 99 & 275 & 795 \end{pmatrix} \qquad (A^\top A)^{-1} = \frac{1}{2520}\begin{pmatrix} 1944 & -60 & -1440 & 420 \\ -60 & 1000 & -150 & -70 \\ -1440 & -150 & 1755 & -525 \\ 420 & -70 & -525 & 175 \end{pmatrix}$$

$$(A^\top A)^{-1}A^\top = \frac{1}{2520}\begin{pmatrix} 144 & 1944 & 864 & -576 & 144 \\ -1140 & -60 & 720 & 780 & -300 \\ 990 & -1440 & -360 & 1080 & -270 \\ -210 & 420 & 0 & -420 & 210 \end{pmatrix}$$

$$x = (A^\top A)^{-1}A^\top b = \frac{1}{2520}\begin{pmatrix} -12600 \\ 7560 \\ -10080 \\ 5040 \end{pmatrix} = \begin{pmatrix} -5 \\ 3 \\ -4 \\ 2 \end{pmatrix}$$

So, the cubic polynomial $y = 2x^3 - 4x^2 + 3x - 5$ best fits the points.

**Problem 9** (Least square solution; 5pt)**.** Assume $\mathbf{u}_1$ and $\mathbf{u}_2$ are two orthogonal vectors in $\mathbb{R}^n$ and set $\mathbf{a}_1 = \mathbf{u}_1$, $\mathbf{a}_2 = \mathbf{u}_1 + \varepsilon\mathbf{u}_2$ for $\varepsilon > 0$. Let also $A$ be the matrix with columns $\mathbf{a}_1$ and $\mathbf{a}_2$ and $\mathbf{b}$ a vector linearly independenet of $\mathbf{a}_1$ and $\mathbf{a}_2$. In this problem, we discuss the least square solution to the system $A\mathbf{x} = \mathbf{b}$ as $\varepsilon \to 0$.

(a) Calculate the matrix $A^\top A$, its inverse, and then $\hat{\mathbf{x}} = (A^\top A)^{-1}A^\top\mathbf{b}$ explicitly. Show that $\hat{\mathbf{x}}$ explodes as $\varepsilon \to 0$.

(b) Calculate the projection $A\hat{\mathbf{x}}$ of $\mathbf{b}$ onto $\mathrm{col}(A)$ and check that it does not depend on $\varepsilon > 0$. Explain the result.

**Solution to the problem 9.** .

(a)

$$A = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_1 + \varepsilon \mathbf{u}_2 \end{pmatrix} \qquad A^\top = \begin{pmatrix} \mathbf{a}_1^\top \\ \mathbf{a}_2^\top \end{pmatrix} = \begin{pmatrix} \mathbf{u}_1^\top \\ \mathbf{u}_1^\top + \varepsilon \mathbf{u}_2^\top \end{pmatrix}$$

$$A^\top A = \begin{pmatrix} \mathbf{a}_1^\top \\ \mathbf{a}_2^\top \end{pmatrix} \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{u}_1^\top \\ \mathbf{u}_1^\top + \varepsilon \mathbf{u}_2^\top \end{pmatrix} \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_1 + \varepsilon \mathbf{u}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{u}_1^\top \mathbf{u}_1 & \mathbf{u}_1^\top (\mathbf{u}_1 + \varepsilon \mathbf{u}_2) \\ (\mathbf{u}_1^\top + \varepsilon \mathbf{u}_2^\top)\mathbf{u}_1 & (\mathbf{u}_1^\top + \varepsilon \mathbf{u}_2^\top)(\mathbf{u}_1 + \varepsilon \mathbf{u}_2) \end{pmatrix} =$$

$$= \begin{pmatrix} \mathbf{u}_1^\top \mathbf{u}_1 & \mathbf{u}_1^\top \mathbf{u}_1 + \varepsilon \mathbf{u}_1^\top \mathbf{u}_2 \\ \mathbf{u}_1^\top \mathbf{u}_1 + \varepsilon \mathbf{u}_2^\top \mathbf{u}_1 & \mathbf{u}_1^\top \mathbf{u}_1 + \varepsilon(\mathbf{u}_2^\top \mathbf{u}_1 + \mathbf{u}_1^\top \mathbf{u}_2) + \varepsilon^2 \mathbf{u}_2^\top \mathbf{u}_2 \end{pmatrix}$$

Since $\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = 0$:

$$A^\top A = \begin{pmatrix} \|\mathbf{u}_1\|^2 & \|\mathbf{u}_1\|^2 \\ \|\mathbf{u}_1\|^2 & \|\mathbf{u}_1\|^2 + \varepsilon^2 \|\mathbf{u}_2\|^2 \end{pmatrix}$$

$$\det A^\top A = \|\mathbf{u}_1\|^2 (\|\mathbf{u}_1\|^2 + \varepsilon^2 \|\mathbf{u}_2\|^2) - \|\mathbf{u}_1\|^4 = \varepsilon^2 \|\mathbf{u}_1\|^2 \|\mathbf{u}_2\|^2$$

$$(A^\top A)^{-1} = \frac{1}{\varepsilon^2 \|\mathbf{u}_1\|^2 \|\mathbf{u}_2\|^2} \begin{pmatrix} \|\mathbf{u}_1\|^2 + \varepsilon^2 \|\mathbf{u}_2\|^2 & -\|\mathbf{u}_1\|^2 \\ -\|\mathbf{u}_1\|^2 & \|\mathbf{u}_1\|^2 \end{pmatrix} = \frac{1}{\varepsilon^2 \|\mathbf{u}_2\|^2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \frac{1}{\|\mathbf{u}_1\|^2} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$(A^\top A)^{-1} A^\top = \frac{1}{\varepsilon^2 \|\mathbf{u}_2\|^2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{u}_1^\top \\ \mathbf{u}_1^\top + \varepsilon \mathbf{u}_2^\top \end{pmatrix} + \frac{1}{\|\mathbf{u}_1\|^2} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_1^\top \\ \mathbf{u}_1^\top + \varepsilon \mathbf{u}_2^\top \end{pmatrix} =$$

$$= \frac{1}{\varepsilon^2 \|\mathbf{u}_2\|^2} \begin{pmatrix} \mathbf{u}_1^\top - \mathbf{u}_1^\top - \varepsilon \mathbf{u}_2^\top \\ -\mathbf{u}_1^\top + \mathbf{u}_1^\top + \varepsilon \mathbf{u}_2^\top \end{pmatrix} + \frac{1}{\|\mathbf{u}_1\|^2} \begin{pmatrix} \mathbf{u}_1^\top \\ 0 \end{pmatrix} = \frac{1}{\varepsilon \|\mathbf{u}_2\|^2} \begin{pmatrix} -\mathbf{u}_2^\top \\ \mathbf{u}_2^\top \end{pmatrix} + \frac{1}{\|\mathbf{u}_1\|^2} \begin{pmatrix} \mathbf{u}_1^\top \\ 0 \end{pmatrix} =$$

$$= \frac{1}{\varepsilon \|\mathbf{u}_2\|^2} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \mathbf{u}_2^\top + \frac{1}{\|\mathbf{u}_1\|^2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathbf{u}_1^\top$$

When $\mathbf{b}$ is orthogonal to $\mathbf{u}_2$, we have:

$$\hat{\mathbf{x}} = \frac{1}{\varepsilon \|\mathbf{u}_2\|^2} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \mathbf{u}_2^\top \mathbf{b} + \frac{1}{\|\mathbf{u}_1\|^2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathbf{u}_1^\top \mathbf{b} = \frac{1}{\|\mathbf{u}_1\|^2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathbf{u}_1^\top \mathbf{b}$$

that doesn't depend on $\varepsilon$. In other cases the component $\frac{1}{\varepsilon \|\mathbf{u}_2\|^2} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \mathbf{u}_2^\top \mathbf{b} \to \infty$ as $\varepsilon \to 0$ which leads to $\hat{\mathbf{x}}$ exploding as $\varepsilon \to 0$.

(b) The projection matrix:

$$P = \frac{1}{\varepsilon \|\mathbf{u}_2\|^2} \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_1 + \varepsilon \mathbf{u}_2 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \mathbf{u}_2^\top + \frac{1}{\|\mathbf{u}_1\|^2} \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_1 + \varepsilon \mathbf{u}_2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathbf{u}_1^\top =$$

$$= \frac{1}{\varepsilon \|\mathbf{u}_2\|^2} \varepsilon \mathbf{u}_2 \mathbf{u}_2^\top + \frac{1}{\|\mathbf{u}_1\|^2} \mathbf{u}_1 \mathbf{u}_1^\top = \frac{\mathbf{u}_2 \mathbf{u}_2^\top}{\|\mathbf{u}_2\|^2} + \frac{\mathbf{u}_1 \mathbf{u}_1^\top}{\|\mathbf{u}_1\|^2}$$

does not depend on $\varepsilon$ so neither does the projection $P\mathbf{b}$.

**Problem 10** (Gram–Schmidt; 3pt)**.** Use the Gram–Schmidt process to transform the basis $\mathbf{u}_1, \dots, \mathbf{u}_k$ into an orthonormal basis.

(a) $\mathbf{u}_1 = (1,3)$, $\mathbf{u}_2 = (2,-2)$;

(b) $\mathbf{u}_1 = (1,0,1)$, $\mathbf{u}_2 = (1,3,-2)$, $\mathbf{u}_3 = (0,2,1)$

**Solution to the problem 10.** .

(a) $\mathbf{u}_1 = (1,3)$, $\mathbf{u}_2 = (2,-2)$

$$\mathbf{w}_1 = \mathbf{u}_1 = \begin{pmatrix}1\\3\end{pmatrix} \qquad \mathbf{w}_2 = \mathbf{u}_2 - \frac{\mathbf{w}_1\mathbf{w}_1^\top}{\mathbf{w}_1^\top\mathbf{w}_1}\mathbf{u}_2$$

$$\frac{\mathbf{w}_1\mathbf{w}_1^\top}{\mathbf{w}_1^\top\mathbf{w}_1} = \frac{1}{10}\begin{pmatrix}1\\3\end{pmatrix}\begin{pmatrix}1 & 3\end{pmatrix} = \frac{1}{10}\begin{pmatrix}1 & 3\\3 & 9\end{pmatrix}$$

$$\mathbf{w}_2 = \begin{pmatrix}2\\-2\end{pmatrix} - \frac{1}{10}\begin{pmatrix}1 & 3\\3 & 9\end{pmatrix}\begin{pmatrix}2\\-2\end{pmatrix} = \frac{1}{10}\begin{pmatrix}20-2+6\\-20-6+18\end{pmatrix} = \frac{1}{5}\begin{pmatrix}12\\-4\end{pmatrix}$$

$$\hat{\mathbf{w}}_1 = \frac{\mathbf{w}_1}{\|\mathbf{w}\|_1} = \frac{1}{\sqrt{10}}\begin{pmatrix}1\\3\end{pmatrix}$$

$$\hat{\mathbf{w}}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}\|_2} = \frac{1}{\sqrt{10}}\begin{pmatrix}3\\-1\end{pmatrix}$$

So the answer is $\hat{\mathbf{w}}_1 = \begin{pmatrix}1/\sqrt{10}\\3/\sqrt{10}\end{pmatrix}$, $\hat{\mathbf{w}}_2 = \begin{pmatrix}3/\sqrt{10}\\-1/\sqrt{10}\end{pmatrix}$

(b) $\mathbf{u}_1 = (1,0,1)$, $\mathbf{u}_2 = (1,3,-2)$, $\mathbf{u}_3 = (0,2,1)$

$$\mathbf{w}_1 = \mathbf{u}_1 = \begin{pmatrix}1\\0\\1\end{pmatrix} \qquad \mathbf{w}_2 = \mathbf{u}_2 - \frac{\mathbf{w}_1\mathbf{w}_1^\top}{\mathbf{w}_1^\top\mathbf{w}_1}\mathbf{u}_2 \qquad \mathbf{w}_3 = \mathbf{u}_3 - \frac{\mathbf{w}_1\mathbf{w}_1^\top}{\mathbf{w}_1^\top\mathbf{w}_1}\mathbf{u}_3 - \frac{\mathbf{w}_2\mathbf{w}_2^\top}{\mathbf{w}_2^\top\mathbf{w}_2}\mathbf{u}_3$$

$$\frac{\mathbf{w}_1\mathbf{w}_1^\top}{\mathbf{w}_1^\top\mathbf{w}_1} = \frac{1}{2}\begin{pmatrix}1\\0\\1\end{pmatrix}\begin{pmatrix}1 & 0 & 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & 0 & 1\\0 & 0 & 0\\1 & 0 & 1\end{pmatrix}$$

$$\mathbf{w}_2 = \begin{pmatrix}1\\3\\-2\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1 & 0 & 1\\0 & 0 & 0\\1 & 0 & 1\end{pmatrix}\begin{pmatrix}1\\3\\-2\end{pmatrix} = \frac{1}{2}\begin{pmatrix}2-1+2\\6-0\\-4-1+2\end{pmatrix} = \frac{1}{2}\begin{pmatrix}3\\6\\-3\end{pmatrix}$$

$$\frac{\mathbf{w}_2\mathbf{w}_2^\top}{\mathbf{w}_2^\top\mathbf{w}_2} = \frac{1}{54}\begin{pmatrix}3\\6\\-3\end{pmatrix}\begin{pmatrix}3 & 6 & -3\end{pmatrix} = \frac{1}{54}\begin{pmatrix}9 & 18 & -9\\18 & 36 & -18\\-9 & -18 & 9\end{pmatrix} = \frac{1}{6}\begin{pmatrix}1 & 2 & -1\\2 & 4 & -2\\-1 & -2 & 1\end{pmatrix}$$

$$\mathbf{w}_3 = \begin{pmatrix}0\\2\\1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1 & 0 & 1\\0 & 0 & 0\\1 & 0 & 1\end{pmatrix}\begin{pmatrix}0\\2\\1\end{pmatrix} - \frac{1}{6}\begin{pmatrix}1 & 2 & -1\\2 & 4 & -2\\-1 & -2 & 1\end{pmatrix}\begin{pmatrix}0\\2\\1\end{pmatrix} = \frac{1}{6}\begin{pmatrix}0-3-4+1\\12-0-8+2\\6-3+4-1\end{pmatrix} = \begin{pmatrix}-1\\1\\1\end{pmatrix}$$

$$\hat{\mathbf{w}}_1 = \frac{\mathbf{w}_1}{\|\mathbf{w}\|_1} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix}$$

$$\hat{\mathbf{w}}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}\|_2} = \frac{1}{3\sqrt{6}}\begin{pmatrix}3\\6\\-3\end{pmatrix}$$

$$\hat{\mathbf{w}}_3 = \frac{\mathbf{w}_3}{\|\mathbf{w}\|_3} = \frac{1}{\sqrt{3}}\begin{pmatrix}-1\\1\\1\end{pmatrix}$$

So the answer is $\hat{\mathbf{w}}_1 = \begin{pmatrix}1/\sqrt{2}\\0\\1/\sqrt{2}\end{pmatrix}$, $\hat{\mathbf{w}}_2 = \begin{pmatrix}1/\sqrt{6}\\2/\sqrt{6}\\-1/\sqrt{6}\end{pmatrix}$, $\hat{\mathbf{w}}_3 = \begin{pmatrix}-1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{pmatrix}$.

**Problem 11** (QR; 5pt)**.** Find the $QR$-decomposition of the matrices below using the Gram–Schmidt algorithm:

$$(a)\ \begin{pmatrix}1 & -1\\2 & 3\end{pmatrix}; \qquad (b)\ \begin{pmatrix}1 & 2\\0 & 1\\1 & 4\end{pmatrix}; \qquad (c)\ \begin{pmatrix}1 & 0 & 2\\0 & 1 & 1\\2 & 0 & 1\end{pmatrix}$$

**Solution to the problem 11.** .

(a)

$$A = \begin{pmatrix}1 & -1\\2 & 3\end{pmatrix}$$

$$q_1 = \begin{pmatrix}1\\2\end{pmatrix} \qquad \frac{q_1 q_1^\top}{q_1^\top q_1} = \frac{1}{5}\begin{pmatrix}1 & 2\\2 & 4\end{pmatrix}$$

$$q_2 = \begin{pmatrix}-1\\3\end{pmatrix} - \frac{1}{5}\begin{pmatrix}1 & 2\\2 & 4\end{pmatrix}\begin{pmatrix}-1\\3\end{pmatrix} = \frac{1}{5}\begin{pmatrix}-5+1-6\\15+2-12\end{pmatrix} = \begin{pmatrix}-2\\1\end{pmatrix}$$

$$\hat{q}_1 = \begin{pmatrix}1/\sqrt{5}\\2/\sqrt{5}\end{pmatrix} \qquad \hat{q}_2 = \begin{pmatrix}-2/\sqrt{5}\\1/\sqrt{5}\end{pmatrix} \qquad Q = \begin{pmatrix}1/\sqrt{5} & -2/\sqrt{5}\\2/\sqrt{5} & 1/\sqrt{5}\end{pmatrix}$$

$$R = \begin{pmatrix}\hat{q}_1^\top a_1 & \hat{q}_1^\top a_2\\0 & \hat{q}_2^\top a_2\end{pmatrix} = \begin{pmatrix}5/\sqrt{5} & 5/\sqrt{5}\\0 & 5/\sqrt{5}\end{pmatrix}$$

$$\Rightarrow A = \begin{pmatrix}1 & -1\\2 & 3\end{pmatrix} = \begin{pmatrix}1/\sqrt{5} & -2/\sqrt{5}\\2/\sqrt{5} & 1/\sqrt{5}\end{pmatrix}\begin{pmatrix}\sqrt{5} & \sqrt{5}\\0 & \sqrt{5}\end{pmatrix}$$

(b)

$$A = \begin{pmatrix}1 & 2\\0 & 1\\1 & 4\end{pmatrix}$$

$$q_1 = \begin{pmatrix}1\\0\\1\end{pmatrix} \qquad \frac{q_1 q_1^\top}{q_1^\top q_1} = \frac{1}{2}\begin{pmatrix}1 & 0 & 1\\0 & 0 & 0\\1 & 0 & 1\end{pmatrix}$$

$$q_2 = \begin{pmatrix}2\\1\\4\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1&0&1\\0&0&0\\1&0&1\end{pmatrix}\begin{pmatrix}2\\1\\4\end{pmatrix} = \frac{1}{2}\begin{pmatrix}4-2-4\\2-0\\8-2-4\end{pmatrix} = \begin{pmatrix}-1\\1\\1\end{pmatrix}$$

$$\hat{q}_1 = \begin{pmatrix}1/\sqrt{2}\\0\\1/\sqrt{2}\end{pmatrix} \qquad \hat{q}_2 = \begin{pmatrix}-1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{pmatrix} \qquad Q = \begin{pmatrix}1/\sqrt{2}&-1/\sqrt{3}\\0&1/\sqrt{3}\\1/\sqrt{2}&1/\sqrt{3}\end{pmatrix}$$

$$R = \begin{pmatrix}\hat{q}_1^\top a_1 & \hat{q}_1^\top a_2\\0&\hat{q}_2^\top a_2\end{pmatrix} = \begin{pmatrix}2/\sqrt{2}&6/\sqrt{2}\\0&3/\sqrt{3}\end{pmatrix}$$

$$\Rightarrow A = \begin{pmatrix}1&2\\0&1\\1&4\end{pmatrix} = \begin{pmatrix}1/\sqrt{2}&-1/\sqrt{3}\\0&1/\sqrt{3}\\1/\sqrt{2}&1/\sqrt{3}\end{pmatrix}\begin{pmatrix}\sqrt{2}&3\sqrt{2}\\0&\sqrt{3}\end{pmatrix}$$

(c)

$$A = \begin{pmatrix}1&0&2\\0&1&1\\2&0&1\end{pmatrix}$$

$$q_1 = \begin{pmatrix}1\\0\\2\end{pmatrix} \qquad \frac{q_1q_1^\top}{q_1^\top q_1} = \frac{1}{5}\begin{pmatrix}1&0&2\\0&0&0\\2&0&4\end{pmatrix}$$

$$q_2 = \begin{pmatrix}0\\1\\0\end{pmatrix} - \frac{1}{5}\begin{pmatrix}1&0&2\\0&0&0\\2&0&4\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix} \qquad \frac{q_2q_2^\top}{q_2^\top q_2} = \begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}$$

$$q_3 = \begin{pmatrix}2\\1\\1\end{pmatrix} - -\frac{1}{5}\begin{pmatrix}1&0&2\\0&0&0\\2&0&4\end{pmatrix}\begin{pmatrix}2\\1\\1\end{pmatrix} - \begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}\begin{pmatrix}2\\1\\1\end{pmatrix} = \frac{1}{5}\begin{pmatrix}10-2-2\\5-5-0\\5-4-4\end{pmatrix} = \frac{1}{5}\begin{pmatrix}6\\0\\-3\end{pmatrix}$$

$$\hat{q}_1 = \begin{pmatrix}1/\sqrt{5}\\0\\2/\sqrt{5}\end{pmatrix} \qquad \hat{q}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix} \qquad \hat{q}_3 = \begin{pmatrix}2/\sqrt{5}\\0\\-1/\sqrt{5}\end{pmatrix} \qquad Q = \begin{pmatrix}1/\sqrt{5}&0&2/\sqrt{5}\\0&1&0\\2/\sqrt{5}&0&-1/\sqrt{5}\end{pmatrix}$$

$$R = \begin{pmatrix}\hat{q}_1^\top a_1&\hat{q}_1^\top a_2&\hat{q}_1^\top a_3\\0&\hat{q}_2^\top a_2&\hat{q}_2^\top a_3\\0&\hat{0}&\hat{q}_3^\top a_3\end{pmatrix} = \begin{pmatrix}5/\sqrt{5}&0&4/\sqrt{5}\\0&1&1\\0&0&3/\sqrt{5}\end{pmatrix}$$

$$\Rightarrow A = \begin{pmatrix}1&0&2\\0&1&1\\2&0&1\end{pmatrix} = \begin{pmatrix}1/\sqrt{5}&0&2/\sqrt{5}\\0&1&0\\2/\sqrt{5}&0&-1/\sqrt{5}\end{pmatrix}\begin{pmatrix}\sqrt{5}&0&4/\sqrt{5}\\0&1&1\\0&0&3/\sqrt{5}\end{pmatrix}$$

**Problem 12** (Householder reflection and QR; 7 pts)**.** (a) Find the unit vector $\mathbf{u} \in \mathbb{R}^2$ such that the *Householder reflection* $Q_\mathbf{u} := I - 2\mathbf{u}\mathbf{u}^\top$ maps the vector $(1,2)^\top$ onto a vector collinear to $(1,0)^\top$

(b) explain how $Q_\mathbf{u}$ helps to derive the $QR$ factorization of the matrix (a) of Problem 11.

(c) Find the $QR$-factorization of matrices in (b) and (c) of Problem 11 using the Householder reflections approach.

**Solution to the problem 12.** .

(a)

$$\mathbf{x} = \begin{pmatrix}1\\2\end{pmatrix} \quad Q_{\mathbf{u}}\mathbf{x} = \|\mathbf{x}\| \begin{pmatrix}1\\0\end{pmatrix} \quad Q_{\mathbf{u}} = I - 2\mathbf{u}\mathbf{u}^\top \quad (\|\mathbf{u}\| = 1)$$

To get $\mathbf{u}$ we need to subtract $\begin{pmatrix}1\\0\end{pmatrix}$ from $\mathbf{x}$ and normalize the vector:

$$\hat{\mathbf{u}} = \begin{pmatrix}1\\2\end{pmatrix} - \sqrt{5}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1-\sqrt{5}\\2\end{pmatrix} \qquad \mathbf{u} = \frac{\hat{\mathbf{u}}}{\|\hat{\mathbf{u}}\|} = \frac{1}{\sqrt{10-2\sqrt{5}}}\begin{pmatrix}1-\sqrt{5}\\2\end{pmatrix}$$

$$\mathbf{u}\mathbf{u}^\top = \frac{1}{10-2\sqrt{5}}\begin{pmatrix}1-\sqrt{5}\\2\end{pmatrix}\begin{pmatrix}1-\sqrt{5} & 2\end{pmatrix} = \frac{1}{10-2\sqrt{5}}\begin{pmatrix}6-2\sqrt{5} & 2-2\sqrt{5}\\2-2\sqrt{5} & 4\end{pmatrix} =$$

$$= \frac{1}{5-\sqrt{5}}\begin{pmatrix}3-\sqrt{5} & 1-\sqrt{5}\\1-\sqrt{5} & 2\end{pmatrix} = \frac{5+\sqrt{5}}{20}\begin{pmatrix}3-\sqrt{5} & 1-\sqrt{5}\\1-\sqrt{5} & 2\end{pmatrix} = \frac{1}{10}\begin{pmatrix}5-\sqrt{5} & -2\sqrt{5}\\-2\sqrt{5} & 5+\sqrt{5}\end{pmatrix}$$

$$\Rightarrow\ Q_{\mathbf{u}} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} - \frac{1}{5}\begin{pmatrix}5-\sqrt{5} & -2\sqrt{5}\\-2\sqrt{5} & 5+\sqrt{5}\end{pmatrix} = \frac{1}{5}\begin{pmatrix}\sqrt{5} & 2\sqrt{5}\\2\sqrt{5} & -\sqrt{5}\end{pmatrix} = \frac{\sqrt{5}}{5}\begin{pmatrix}1 & 2\\2 & -1\end{pmatrix}$$

And we can check:

$$Q_{\mathbf{u}}\mathbf{x} = \frac{\sqrt{5}}{5}\begin{pmatrix}1 & 2\\2 & -1\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix} = \sqrt{5}\begin{pmatrix}1\\0\end{pmatrix} \quad \| \quad \begin{pmatrix}1\\0\end{pmatrix} \qquad \|Q_{\mathbf{u}}\mathbf{x}\| = \sqrt{5} = \|\mathbf{x}\|$$

So the answer is:

$$\mathbf{u} = \frac{1}{\sqrt{10-2\sqrt{5}}}\begin{pmatrix}1-\sqrt{5}\\2\end{pmatrix}$$

(b)

$$A = \begin{pmatrix}1 & -1\\2 & 3\end{pmatrix}$$

Since $\mathbf{x} = \mathbf{a}_1$, $Q_{\mathbf{u}}$ is thus the result of the first iteration of QR factorization using the Householder reflections and:

$$Q_{\mathbf{u}}A = \begin{pmatrix}\sqrt{5} & \sqrt{5}\\0 & -\sqrt{5}\end{pmatrix}$$

give us the first row $\begin{pmatrix}\sqrt{5} & \sqrt{5}\end{pmatrix}$ of the $R$ matrix .

(c) $QR$-factorization of Problem 11 (b), (c) using the Householder reflections approach:
11. (b)

$$A = \begin{pmatrix}1 & 2\\0 & 1\\1 & 4\end{pmatrix} \qquad \mathbf{a}_1 = \begin{pmatrix}1\\0\\1\end{pmatrix}$$

$$\hat{\mathbf{u}}_1 = \mathbf{a}_1 - \|\mathbf{a}_1\|\,\mathbf{e}_1 = \begin{pmatrix}1\\0\\1\end{pmatrix} - \sqrt{2}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1-\sqrt{2}\\0\\1\end{pmatrix} \qquad \mathbf{u}_1 = \frac{1}{\sqrt{4-2\sqrt{2}}}\begin{pmatrix}1-\sqrt{2}\\0\\1\end{pmatrix}$$

$$\mathbf{u}_1\mathbf{u}_1^\top = \frac{1}{4-2\sqrt{2}}\begin{pmatrix}1-\sqrt{2}\\0\\1\end{pmatrix}\begin{pmatrix}1-\sqrt{2} & 0 & 1\end{pmatrix} = \frac{1}{4-2\sqrt{2}}\begin{pmatrix}3-2\sqrt{2} & 0 & 1-\sqrt{2}\\0 & 0 & 0\\1-\sqrt{2} & 0 & 1\end{pmatrix} =$$

$$= \frac{4+2\sqrt{2}}{8}\begin{pmatrix}3-2\sqrt{2} & 0 & 1-\sqrt{2}\\0 & 0 & 0\\1-\sqrt{2} & 0 & 1\end{pmatrix} = \frac{1}{4}\begin{pmatrix}2-\sqrt{2} & 0 & -\sqrt{2}\\0 & 0 & 0\\-\sqrt{2} & 0 & 2+\sqrt{2}\end{pmatrix}$$

$$\Rightarrow\ Q_{\mathbf{u}_1} = \frac{1}{2}\begin{pmatrix}\sqrt{2} & 0 & \sqrt{2}\\0 & 2 & 0\\\sqrt{2} & 0 & -\sqrt{2}\end{pmatrix}$$

$$Q_{\mathbf{u}_1}A = \frac{1}{2}\begin{pmatrix}\sqrt{2} & 0 & \sqrt{2}\\0 & 2 & 0\\\sqrt{2} & 0 & -\sqrt{2}\end{pmatrix}\begin{pmatrix}1 & 2\\0 & 1\\1 & 4\end{pmatrix} = \begin{pmatrix}\sqrt{2} & 3\sqrt{2}\\0 & 1\\0 & -\sqrt{2}\end{pmatrix}$$

$$\mathbf{a}_2 = \begin{pmatrix}1\\-\sqrt{2}\end{pmatrix} (\in \mathbb{R}^2)$$

$$\hat{\mathbf{u}}_2 = \mathbf{a}_2 - \|\mathbf{a}_2\|\,\mathbf{e}_1 = \begin{pmatrix}1\\-\sqrt{2}\end{pmatrix} - \sqrt{3}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1-\sqrt{3}\\-\sqrt{2}\end{pmatrix} \qquad \mathbf{u}_2 = \frac{1}{\sqrt{6-2\sqrt{3}}}\begin{pmatrix}1-\sqrt{3}\\-\sqrt{2}\end{pmatrix}$$

$$\mathbf{u}_2\mathbf{u}_2^\top = \frac{1}{6-2\sqrt{3}}\begin{pmatrix}1-\sqrt{3}\\-\sqrt{2}\end{pmatrix}\begin{pmatrix}1-\sqrt{3} & -\sqrt{2}\end{pmatrix} = \frac{1}{6-2\sqrt{3}}\begin{pmatrix}4-2\sqrt{3} & -\sqrt{2}(1-\sqrt{3})\\-\sqrt{2}(1-\sqrt{3}) & 2\end{pmatrix} =$$

$$= \frac{3+\sqrt{3}}{12}\begin{pmatrix}4-2\sqrt{3} & -\sqrt{2}(1-\sqrt{3})\\-\sqrt{2}(1-\sqrt{3}) & 2\end{pmatrix} = \frac{1}{6}\begin{pmatrix}3-\sqrt{3} & \sqrt{6}\\\sqrt{6} & 3+\sqrt{3}\end{pmatrix}$$

$$\Rightarrow\ \hat{Q}_{\mathbf{u}_2} = \frac{1}{3}\begin{pmatrix}\sqrt{3} & -\sqrt{6}\\-\sqrt{6} & -\sqrt{3}\end{pmatrix} \ \Rightarrow\ Q_{\mathbf{u}_2} = \frac{1}{3}\begin{pmatrix}3 & 0 & 0\\0 & \sqrt{3} & -\sqrt{6}\\0 & -\sqrt{6} & -\sqrt{3}\end{pmatrix}$$

$$Q_{\mathbf{u}_2}Q_{\mathbf{u}_1}A = \frac{1}{3}\begin{pmatrix}3 & 0 & 0\\0 & \sqrt{3} & -\sqrt{6}\\0 & -\sqrt{6} & -\sqrt{3}\end{pmatrix}\begin{pmatrix}\sqrt{2} & 3\sqrt{2}\\0 & 1\\0 & -\sqrt{2}\end{pmatrix} = \begin{pmatrix}\sqrt{2} & 3\sqrt{2}\\0 & \sqrt{3}\\0 & 0\end{pmatrix}$$

So the full QR factorization is:

$$R = \begin{pmatrix}\sqrt{2} & 3\sqrt{2}\\0 & \sqrt{3}\\0 & 0\end{pmatrix} \qquad Q = Q_{\mathbf{u}_1}Q_{\mathbf{u}_2} = \frac{1}{6}\begin{pmatrix}\sqrt{2} & 0 & \sqrt{2}\\0 & 2 & 0\\\sqrt{2} & 0 & -\sqrt{2}\end{pmatrix}\begin{pmatrix}3 & 0 & 0\\0 & \sqrt{3} & -\sqrt{6}\\0 & -\sqrt{6} & -\sqrt{3}\end{pmatrix} =$$

$$= \begin{pmatrix}1/\sqrt{2} & -1/\sqrt{3} & -1/\sqrt{6}\\0 & 1/\sqrt{3} & -2/\sqrt{6}\\1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6}\end{pmatrix}$$

And the reduced QR factorization is:

$$Q = \begin{pmatrix}1/\sqrt{2} & -1/\sqrt{3}\\0 & 1/\sqrt{3}\\1/\sqrt{2} & 1/\sqrt{3}\end{pmatrix} \qquad R = \begin{pmatrix}\sqrt{2} & 3\sqrt{2}\\0 & \sqrt{3}\end{pmatrix}$$

that matches the result of GS approach.

11. (c)

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix} \quad \mathbf{a}_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

$$\hat{\mathbf{u}}_1 = \mathbf{a}_1 - \|\mathbf{a}_1\| \, \mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} - \sqrt{5} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1-\sqrt{5} \\ 0 \\ 2 \end{pmatrix} \qquad \mathbf{u}_1 = \frac{1}{\sqrt{10-2\sqrt{5}}} \begin{pmatrix} 1-\sqrt{5} \\ 0 \\ 2 \end{pmatrix}$$

$$\mathbf{u}_1 \mathbf{u}_1^\top = \frac{1}{10-2\sqrt{5}} \begin{pmatrix} 1-\sqrt{5} \\ 0 \\ 2 \end{pmatrix} \begin{pmatrix} 1-\sqrt{5} & 0 & 2 \end{pmatrix} = \frac{1}{5-\sqrt{5}} \begin{pmatrix} 3-\sqrt{5} & 0 & 1-\sqrt{5} \\ 0 & 0 & 0 \\ 1-\sqrt{5} & 0 & 2 \end{pmatrix} =$$

$$= \frac{5+\sqrt{5}}{20} \begin{pmatrix} 3-\sqrt{5} & 0 & 1-\sqrt{5} \\ 0 & 0 & 0 \\ 1-\sqrt{5} & 0 & 2 \end{pmatrix} = \frac{1}{10} \begin{pmatrix} 5-\sqrt{5} & 0 & -2\sqrt{5} \\ 0 & 0 & 0 \\ -2\sqrt{5} & 0 & 5+\sqrt{5} \end{pmatrix}$$

$$\Rightarrow \; Q_{\mathbf{u}_1} = \frac{1}{5} \begin{pmatrix} \sqrt{5} & 0 & 2\sqrt{5} \\ 0 & 5 & 0 \\ 2\sqrt{5} & 0 & -\sqrt{5} \end{pmatrix}$$

$$Q_{\mathbf{u}_1} A = \frac{1}{5} \begin{pmatrix} \sqrt{5} & 0 & 2\sqrt{5} \\ 0 & 5 & 0 \\ 2\sqrt{5} & 0 & -\sqrt{5} \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \sqrt{5} & 0 & \frac{4}{5}\sqrt{5} \\ 0 & 1 & 1 \\ 0 & 0 & \frac{3}{5}\sqrt{5} \end{pmatrix}$$

$$\mathbf{a}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathbf{e}_1 (\in \mathbb{R}^2) \;\Rightarrow\; Q_{\mathbf{u}_2} = I$$

$$Q_{\mathbf{u}_2} Q_{\mathbf{u}_1} A = \begin{pmatrix} \sqrt{5} & 0 & 4/\sqrt{5} \\ 0 & 1 & 1 \\ 0 & 0 & 3/\sqrt{5} \end{pmatrix}$$

$$\Rightarrow \; R = \begin{pmatrix} \sqrt{5} & 0 & 4/\sqrt{5} \\ 0 & 1 & 1 \\ 0 & 0 & 3/\sqrt{5} \end{pmatrix} \qquad Q = Q_{\mathbf{u}_1} Q_{\mathbf{u}_2} = Q_{\mathbf{u}_1} = \begin{pmatrix} 1/\sqrt{5} & 0 & 2/\sqrt{5} \\ 0 & 1 & 0 \\ 2/\sqrt{5} & 0 & -1/\sqrt{5} \end{pmatrix}$$

that matches the result of GS approach.