Linear Algebra
Homework 1: Prerequisites

Solutions are by Yaroslava Lochman.

Problem 1 (System of linear equations; 3pt)**.** Determine all the values of k for which the matrix below is the augmented matrix of a consistent linear system.

$$(a)\quad \left(\begin{array}{cc|c} 1 & k & 4 \\ 3 & 6 & 8 \end{array}\right) \qquad (b)\quad \left(\begin{array}{cc|c} 1 & 4 & -2 \\ 3 & k & -6 \end{array}\right) \qquad (c)\quad \left(\begin{array}{cc|c} -4 & 12 & k \\ 2 & -6 & -3 \end{array}\right)$$

Solution to the problem 1. Let $(A|b)$ denote the augmented matrix of linear system. The linear system is consistent when there exists a solution which is equivalent to $\operatorname{rank}(A \mid b) = \operatorname{rank}(A)$. To calculate rank we'll apply elementary transformations to matrices below.

(a)

$$\left(\begin{array}{cc|c} 1 & k & 4 \\ 3 & 6 & 8 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & k & 4 \\ 0 & 6-3k & -4 \end{array}\right)$$

$$\Rightarrow \begin{cases} \operatorname{rank}(A \mid b) = 2 \ \forall k \in \mathbb{R} \\ \operatorname{rank}(A) = \left[\begin{array}{l} 2,\ k \neq 2 \\ 1,\ k = 2 \end{array}\right. \end{cases} \Rightarrow\ k \neq 2$$

(b)

$$\left(\begin{array}{cc|c} 1 & 4 & -2 \\ 3 & k & -6 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 4 & -2 \\ 0 & k-12 & 0 \end{array}\right)$$

$$\Rightarrow \begin{cases} \operatorname{rank}(A \mid b) = \left[\begin{array}{l} 2,\ k \neq 12 \\ 1,\ k = 12 \end{array}\right. \\ \operatorname{rank}(A) = \left[\begin{array}{l} 2,\ k \neq 12 \\ 1,\ k = 12 \end{array}\right. \end{cases} \Rightarrow\ k \in \mathbb{R}$$

(c)

$$\left(\begin{array}{cc|c} -4 & 12 & k \\ 2 & -6 & -3 \end{array}\right) \sim \left(\begin{array}{cc|c} 2 & -6 & -3 \\ 0 & 0 & k-6 \end{array}\right)$$

$$\Rightarrow \begin{cases} \operatorname{rank}(A \mid b) = \left[\begin{array}{l} 2,\ k \neq 6 \\ 1,\ k = 6 \end{array}\right. \\ \operatorname{rank}(A) = 1 \ \forall k \in \mathbb{R} \end{cases} \Rightarrow\ k = 6$$

Problem 2 (System of linear equations; 4pt)**.** Let

$$\left(\begin{array}{ccc|c} a & 0 & b & 2 \\ a & a & 4 & 4 \\ 0 & a & 2 & b \end{array}\right)$$

be the augmented matrix for a linear system. Find for what values of a and b the system has

(a) a unique solution;
(b) a one-parameter solution set;
(c) a two-parameter solution set;
(d) no solution.

Solution to the problem 2. Analogically let $(A|b)$ denote the augmented matrix of linear system. We'll apply elementary transformations:

$$\left(\begin{array}{ccc|c} a & 0 & b & 2 \\ a & a & 4 & 4 \\ 0 & a & 2 & b \end{array}\right) \sim \left(\begin{array}{ccc|c} a & 0 & b & 2 \\ 0 & a & 2 & b \\ 0 & a & 4-b & 2 \end{array}\right) \sim \left(\begin{array}{ccc|c} a & 0 & b & 2 \\ 0 & a & 2 & b \\ 0 & 0 & 2-b & 2-b \end{array}\right)$$

(a) For a unique solution we need:

$$\text{rank}(A \mid b) = rank(A) = 3 \Leftrightarrow \begin{cases} b \neq 2 \\ a \neq 0 \end{cases}$$

(b) For a one-parameter solution set we need:

$$\text{rank}(A \mid b) = rank(A) = 2 \Leftrightarrow \begin{cases} b = 2 \\ a \neq 0 \end{cases}$$

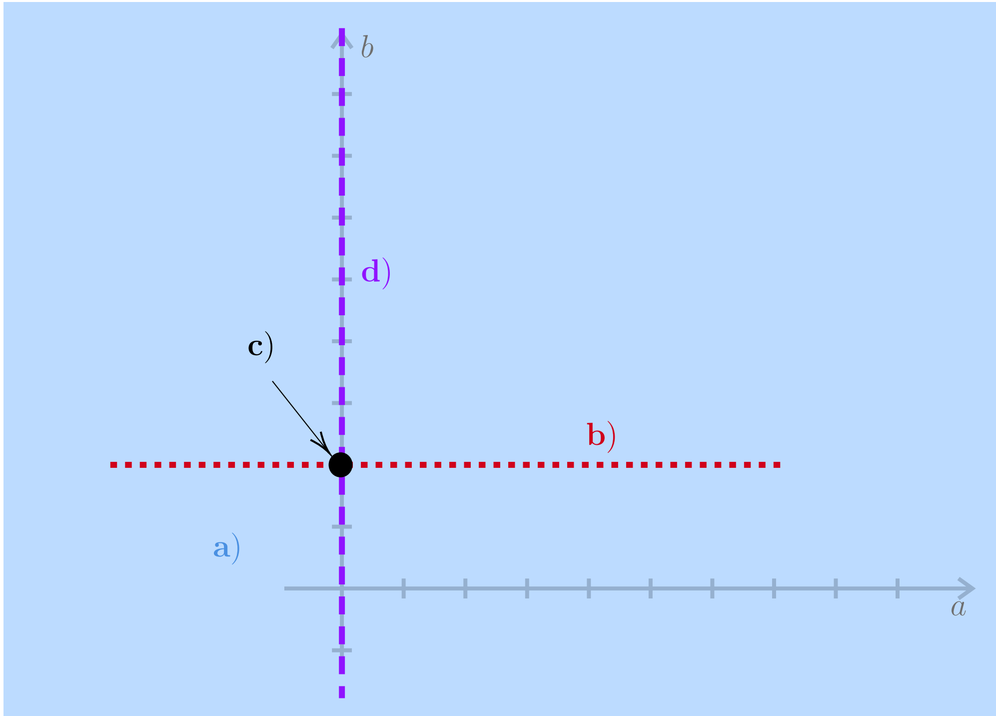
(c) For a two-parameter soltion set we need:

$$\text{rank}(A \mid b) = rank(A) = 1 \Leftrightarrow \begin{cases} b = 2 \\ a = 0 \end{cases}$$

(d) No solution is equivalent to:

$$\text{rank}(A \mid b) \neq rank(A) \Leftrightarrow \begin{cases} b \neq 2 \\ a = 0 \end{cases}$$

The parameter space might look like this:



Problem 3 (System of linear equations; 6pt)**.** Write a system of linear equations consisting of m equations in n unknowns with

(a) no solutions; (b) exactly one solution; (c) infinitely many solutions

for (i) $m = n = 3$; (ii) $m = 3$ and $n = 2$; (iii) $m = 2$, $n = 3$.

Solution to the problem 3. .

(i)

$$(a)\begin{cases} x_1 + 9x_2 + x_3 = 1 \\ 9x_1 + 81x_2 + 9x_3 = 101 \\ 8x_1 - 21x_2 + 14x_3 = 20 \end{cases} \quad (b)\begin{cases} 5x_1 + 3x_2 + 2x_3 = 1 \\ x_1 + 4x_2 + 3x_3 = 1 \\ 2x_1 + 1x_2 + 1x_3 = 1 \end{cases} \quad (c)\begin{cases} 3x_1 - x_2 - 2x_3 = 5 \\ 5x_1 + 12x_2 - 6x_3 = 23 \\ 9x_1 - 3x_2 - 6x_3 = 15 \end{cases}$$

(ii)

$$(a)\begin{cases} 2x_1 + x_2 = 12 \\ x_1 + x_2 = 3 \\ 3x_1 + 3x_2 = 1 \end{cases} \quad (b)\begin{cases} 2x_1 + x_2 = 12 \\ x_1 + x_2 = 3 \\ 3x_1 + 3x_2 = 9 \end{cases} \quad (c)\begin{cases} x_1 - x_2 = 2 \\ -x_1 + x_2 = -2 \\ 5x_1 - 5x_2 = 10 \end{cases}$$

(iii)

$$(a)\begin{cases} 15x_1 + 5x_2 + 10x_3 = 25 \\ 3x_1 + x_2 + 2x_3 = 10 \end{cases} \quad (b)\ \begin{array}{c}\text{there is no such system,} \\ \text{not enough equations} \\ \text{for 3 unknowns.} \\ \text{(should be} \geq 3\text{)}\end{array} \quad (c)\begin{cases} x_1 - x_2 + x_3 = 5 \\ 13x_1 + 4x_2 - 8x_3 = -9 \end{cases}$$

Problem 4 (System of linear equations; 4pt)**.** The following are coefficient matrices of linear systems. For each system, what can you say about the number of solutions to the corresponding system **(i)** in the homogeneous case (when $b_1 = \cdots = b_m = 0$) and **(ii)** for a generic RHS?

$$\text{(a)} \begin{pmatrix} 1 & 4 \\ 2 & 1 \end{pmatrix}, \qquad \text{(b)} \begin{pmatrix} 1 & 4 & 3 \\ 2 & 1 & 0 \end{pmatrix}, \qquad \text{(c)} \begin{pmatrix} 2 & 1 \\ 1 & 4 \\ 0 & 3 \end{pmatrix}, \qquad \text{(d)} \begin{pmatrix} 1 & 4 & 3 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

Solution to the problem 4. Let $(A \mid b)$ denote the augmented matrix of linear system.

(a)

$$\begin{pmatrix} 1 & 4 \\ 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 4 \\ 0 & -7 \end{pmatrix}$$

(i) $\operatorname{rank}(A) = 2 \;\Rightarrow$ exactly one solution.

(ii) $\operatorname{rank}(A \mid b) = \operatorname{rank}(A) = 2 \;\Rightarrow$ exactly one solution.

(b)

$$\begin{pmatrix} 1 & 4 & 3 \\ 2 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 4 & 3 \\ 0 & -7 & -6 \end{pmatrix}$$

(i) $\operatorname{rank}(A) = 2 < 3 \;\Rightarrow$ the infinite number of solutions.

(ii) $\operatorname{rank}(A \mid b) = \operatorname{rank}(A) = 2 < 3 \;\Rightarrow$ the infinite number of solutions.

(c)

$$\begin{pmatrix} 2 & 1 \\ 1 & 4 \\ 0 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 4 \\ 0 & -7 \\ 0 & 3 \end{pmatrix}$$

(i) $\operatorname{rank}(A) = 2 \;\Rightarrow$ exactly one solution.

(ii) $\operatorname{rank}(A) = 2$; $\operatorname{rank}(A \mid b)$ may be 2 or 3, if 2 $\Rightarrow$ exactly one solution, otherwise there is no solution.

(d)

$$\begin{pmatrix} 1 & 4 & 3 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 4 & 3 \\ 0 & -7 & -6 \\ 0 & -3 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 4 & 3 \\ 0 & -7 & -6 \\ 0 & 0 & 4/7 \end{pmatrix}$$

(i) $\operatorname{rank}(A) = 3 \;\Rightarrow$ exactly one solution.

(ii) $\operatorname{rank}(A \mid b) = \operatorname{rank}(A) = 3 \;\Rightarrow$ exactly one solution.

Problem 5 (System of linear equations; linear dependence; 3pt)**.** Prove that any $n+1$ vectors in $\mathbb{R}^n$ are linearly dependent.

Hint: regard a linear combination of these vectors resulting in a zero vector as a homogeneous linear system and show that it possesses a non-trivial solution

Solution to the problem 5. Let $\{a_1 \dots a_{n+1}\}$ be the set of $n+1$ vectors in $\mathbb{R}^n$. If $\{a_1 \dots a_n\}$ is a linearly dependent set then $\{a_1 \dots a_{n+1}\}$ is also a linearly dependent set. Now consider $\{a_1 \dots a_n\}$ is a linearly independent set. We need to prove that there exist $x_1 \dots x_{n+1}$, not equal to 0 simultaneously, such that

$$\sum_{1}^{n+1} x_i a_i = 0$$

If we denote

$$A = \begin{pmatrix} a_1 & \cdots & a_{n+1} \end{pmatrix} = \begin{pmatrix} a_1^1 & \cdots & a_n^1 & a_{n+1}^1 \\ \vdots & & \vdots & \vdots \\ a_1^n & \cdots & a_n^n & a_{n+1}^n \end{pmatrix} \qquad x = \begin{pmatrix} x_1 & \cdots & x_{n+1} \end{pmatrix}^\top$$

then we need to find a non-trivial solution of $Ax = 0$. Since $\{a_1 \dots a_n\}$ is a linearly independent set $\Rightarrow \operatorname{rank}(A) = n$ (suppose that $a_1^1 \neq 0$):

$$\Rightarrow \begin{pmatrix} a_1^1 & \cdots & a_n^1 \\ \vdots & & \vdots \\ a_1^n & \cdots & a_n^n \end{pmatrix} \sim \begin{pmatrix} a_1^1 & a_2^1 & \cdots & a_n^1 \\ 0 & \hat{a}_2^2 & \cdots & \hat{a}_n^2 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \hat{a}_n^n \end{pmatrix} \quad \text{and} \quad \hat{a}_k^k \neq 0 \quad \forall k = \overline{1, n}$$

$$\Rightarrow A = \begin{pmatrix} a_1^1 & \cdots & a_n^1 & a_{n+1}^1 \\ \vdots & & \vdots & \vdots \\ a_1^n & \cdots & a_n^n & a_{n+1}^n \end{pmatrix} \sim \begin{pmatrix} a_1^1 & a_2^1 & \cdots & a_n^1 & a_{n+1}^1 \\ 0 & \hat{a}_2^2 & \cdots & \hat{a}_n^2 & \hat{a}_{n+1}^2 \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \hat{a}_n^n & \hat{a}_{n+1}^n \end{pmatrix}$$

$$\hat{a}_n^n x_n + \hat{a}_{n+1}^n x_{n+1} = 0$$

If $\hat{a}_{n+1}^n = 0$ then $x_n = 0$ and $x_{n+1} \in \mathbb{R} \Rightarrow$ with $x_{n+1} \neq 0$ the non-trivial solution is found. Otherwise $x_{n+1} = -\frac{\hat{a}_n^n}{\hat{a}_{n+1}^n} x_n$. We can substitute x_{n+1} by this and get $n \times n-1$ system now. And so on analogically we can reach zero coefficient or remaining x_1 and x_2 and see that we may choose one of these values so we can get a non-trivial solution.

Problem 6 (Gauss elimination; determinants; 2pt)**.** Determine all the values of k for which the column vectors below are linearly dependent:

$$\text{(a)} \quad \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}, \begin{pmatrix} 6 \\ k \\ 1 \end{pmatrix}; \qquad \text{(b)} \quad \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -4 \\ -2 \end{pmatrix}, \begin{pmatrix} k \\ 3 \\ -3 \end{pmatrix}$$

Solution to the problem 6. Let A denote a matrix composed of given column vectors. Vectors are linearly dependent $\Leftrightarrow$ $\det A = 0$ $\Leftrightarrow$ $\operatorname{rank}(A) < 3$.

(a)

$$\begin{pmatrix} 1 & 2 & 6 \\ 2 & -3 & k \\ -1 & 5 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 6 \\ 0 & -7 & k-24 \\ 0 & 7 & 7 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 6 \\ 0 & 7 & 7 \\ 0 & 0 & k-17 \end{pmatrix} \Rightarrow k = 17$$

(b)

$$\begin{pmatrix} -1 & 2 & k \\ 2 & -4 & 3 \\ 1 & -2 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & -3 \\ 0 & 0 & 9 \\ 0 & 0 & k-3 \end{pmatrix} \Rightarrow k \in \mathbb{R}$$

Problem 7 (Matrix algebra; 4pt)**.** Let $\mathbf{0}_n$ and I_n denote respectively the zero and identity matrices of size n (say, $n = 10$).

(a) Is there an $n \times n$ matrix A such that $A \neq \mathbf{0}_n$ and $A^2 = \mathbf{0}_n$? Justify your answer.

(b) Is there an $n \times n$ matrix A such that $A \neq \mathbf{0}_n, I_n$ and $A^2 = A$? Justify your answer.

(c) Is there an $n \times n$ matrix A such that $A \neq I_n$ and $A^2 = I_n$? Justify your answer.

(d) Are there $n \times n$ matrices A and B such that $A \neq \mathbf{0}_n$, $B \neq \mathbf{0}_n$, $AB \neq \mathbf{0}_n$ but $BA = \mathbf{0}_n$?

Hint: analyse the case $n = 2$ to guess the answer and then try to see the pattern in higher dimensions

Solution to the problem 7.

(a) Yes, e.g.:
If $n \mod 2 = 0$ then:

$$A = \begin{pmatrix} 1 & -1 & \cdots & 1 & -1 \\ 1 & -1 & \cdots & 1 & -1 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & -1 & \cdots & 1 & -1 \\ 1 & -1 & \cdots & 1 & -1 \end{pmatrix}$$

(for $n = 2$ we have $A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$ so $A^2 = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1-1 & 1-1 \\ -1+1 & -1+1 \end{pmatrix} = \mathbf{0}_n$ etc.).

If $n \mod 2 \neq 0$ then:

$$A = \begin{pmatrix} 1 & -1 & \cdots & 1 & -1 & 0 \\ 1 & -1 & \cdots & 1 & -1 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 1 & -1 & \cdots & 1 & -1 & 0 \\ 1 & -1 & \cdots & 1 & -1 & 0 \end{pmatrix}$$

(for $n = 3$: $A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix}$ so $A^2 = \begin{pmatrix} 1-1+0 & 1-1+0 & 1-1+0 \\ -1+1+0 & -1+1+0 & -1+1+0 \\ 0 & 0 & 0 \end{pmatrix} = \mathbf{0}_n$ etc.).

(b) Yes, e.g. let B be an $n \times n$ matrix such that $B^\top B$ is not singular. Then $B(B^\top B)^{-1}B^\top$ is idempotent:

$$B(B^\top B)^{-1}(B^\top B)(B^\top B)^{-1}B^\top = B(B^\top B)^{-1}B^\top$$

Since we can find such B (e.g. $B = \begin{pmatrix} 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \\ 0 & \cdots & 0 & 5 \end{pmatrix}$) then $A = B(B^\top B)^{-1}B^\top \neq I_n, \neq \mathbf{0}_n$.

(c) Yes, e.g.:

$$A = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & \iddots & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{pmatrix}$$

So $A^2 = I_n$

(d) Yes, e.g.:

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}$$

and if $n \mod 2 = 0$ then:

$$B = \begin{pmatrix} 1 & -1 & \cdots & 1 & -1 \\ 1 & -1 & \cdots & 1 & -1 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & -1 & \cdots & 1 & -1 \\ 1 & -1 & \cdots & 1 & -1 \end{pmatrix}$$

otherwise:

$$B = \begin{pmatrix} 1 & -1 & \cdots & 1 & -1 & 0 \\ 1 & -1 & \cdots & 1 & -1 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 1 & -1 & \cdots & 1 & -1 & 0 \\ 1 & -1 & \cdots & 1 & -1 & 0 \end{pmatrix}$$

One can see that $BA = \mathbf{0}_n$ and that $AB \neq \mathbf{0}_n$ (the element $(AB)_{11}$ is already equal to $n \neq 0$)

Problem 8 (Determinants and cross-products; 4pt)**.** A parallelepiped has edges from $(0;0;0)$ to $(2;1;1)$, $(1;2;1)$, and $(1;1;2)$. Find its volume and also find the area of each parallelogram face.
Hint: a cross-, or vector-product in $\mathbb{R}^3$ is handy here. Also, recall the geometric meaning of determinant

Solution to the problem 8.
Volume:

$$V = |a \cdot (b \times c)| = \left| \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \right| = \left| \det \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \right| = |8+1+1-2-2-2| = 4$$

Area of parallelograms based on ab, bc and ac:

$$S_{ab} = \left| \det \begin{pmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} \right| = \left| \det \begin{pmatrix} i & j & k \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix} \right| = \left| \det \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} i - \det \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} j + \det \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} k \right| =$$

$$= \left| (-1, -1, 3)^\top \right| = \sqrt{1+1+9} = \sqrt{11}$$

$$S_{bc} = \left|\det\begin{pmatrix} i & j & k \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}\right| = \left|\det\begin{pmatrix} i & j & k \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}\right| = \left|\det\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} i - \det\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} j + \det\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} k\right| =$$

$$= \left|(3, -1, -1)^\top\right| = \sqrt{11}$$

$$S_{ac} = \left|\det\begin{pmatrix} i & j & k \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}\right| = \left|\det\begin{pmatrix} i & j & k \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}\right| = \left|\det\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} i - \det\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} j + \det\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} k\right| =$$

$$= \left|(1, -3, 1)^\top\right| = \sqrt{11}$$

Problem 9 (Determinants and matrix algebra; 2pt)**.** Assume that 3×3 matrices A, B and C are as follows

$$A = \begin{pmatrix} \text{row 1} \\ \text{row 2} \\ \text{row 3} \end{pmatrix} \qquad B = \begin{pmatrix} \text{row 1} + \text{row 2} \\ \text{row 2} + \text{row 3} \\ \text{row 3} + \text{row 1} \end{pmatrix} \qquad C = \begin{pmatrix} \text{row 1} - \text{row 2} \\ \text{row 2} - \text{row 3} \\ \text{row 3} - \text{row 1} \end{pmatrix}$$

Given that $\det(A) = 5$, find $\det(B)$ and $\det(C)$.

Hint: use the elementary row operations to produce B from A; an alternative (and more elegant) way is to find a matrix B' such that $B = B'A$; the same for C

Solution to the problem 9.

$$B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \text{row 1} \\ \text{row 2} \\ \text{row 3} \end{pmatrix} = B'A \Rightarrow B' = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

Hence

$$\det B = \det(B'A) = \det B' \det A = (1 + 1 + 0) \det A = 10$$

$$C = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \text{row 1} \\ \text{row 2} \\ \text{row 3} \end{pmatrix} = C'A \Rightarrow C' = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}$$

Hence

$$\det C = \det(C'A) = \det C' \det A = (1 - 1 + 0) \det A = 0$$

Problem 10 (Determinants; eigenvalues and their properties; 3pt)**.** Using any of the methods, find all λ for which the matrix below is singular:

$$A - \lambda I = \begin{pmatrix} a-\lambda & b & c & d \\ a & b-\lambda & c & d \\ a & b & c-\lambda & d \\ a & b & c & d-\lambda \end{pmatrix}$$

Hint: one approach is to calculate the determinant and find its roots. An alternative approach is to identify the λ's looked for as eigenvalues of A. Note A is of rank 1; what conclusions on eigenvalues can you derive?

Solution to the problem 10. One can see that:

$$A = \begin{pmatrix} a & b & c & d \\ a & b & c & d \\ a & b & c & d \\ a & b & c & d \end{pmatrix}$$

$A - \lambda I$ is singular $\Leftrightarrow det(A - \lambda I) = 0 \Leftrightarrow Av = \lambda v$ has a non-zero solution w.r.t v.

$$\Rightarrow av_1 + bv_2 + cv_3 + dv_4 = \lambda v_1 = \lambda v_2 = \lambda v_3 = \lambda v_4 \neq 0$$

$$\Rightarrow \begin{cases} v_1 = v_2 = v_3 = v_4 \neq 0 \\ (a+b+c+d)v_1 = \lambda v_1 \end{cases}$$

$$\Rightarrow \lambda = a+b+c+d$$

Problem 11 (Rank of a matrix; 4pt)**.** (a) Assume that A and B are matrices such that AB is well defined. By comparing the column spaces of A and AB, show that $\text{rank}(AB) \leq \text{rank}(A)$. Transpose to conclude that also $\text{rank}(AB) \leq \text{rank}(B)$.

(b) Assume that A and B are non-square matrices such that both AB and BA exist. Show that at least one of AB or BA is singular.
Hint: in (b), show that at least one of AB and BA is not of full rank

Solution to the problem 11. (a) Let A be a $n \times m$ matrix and B – a $m \times k$ matrix, so AB is $n \times k$ matrix. We can rewrite matrices in the form of column-vectors:

$$A = \begin{pmatrix} a_1 & \dots & a_m \end{pmatrix} \quad B = \begin{pmatrix} b_1 & \dots & b_k \end{pmatrix} \;\Rightarrow\; AB = \begin{pmatrix} Ab_1 & \dots & Ab_k \end{pmatrix}$$

$\text{rank}(AB) = \dim Im(AB) = \dim span\{Ab_1 \dots Ab_k\} \leq^* \dim span\{a_1 \dots a_m\} = \dim Im(A) = rank(A)$

Below is shown that * is satisfied:

$$\forall y \in Im(AB) = span\{Ab_1 \dots Ab_k\} \; \exists x \in \mathbb{R}^k : \; y = ABx = A(Bx) \;\; (\exists Bx = z \in \mathbb{R}^m)$$

$$\Rightarrow y \in Im(A) = span\{a_1 \dots a_m\}$$

Hence

$$Im(AB) \subset Im(A) \ \Rightarrow \ \dim Im(AB) \leq \dim Im(A)$$

$$\text{rank}(AB) = \text{rank}(AB)^\top = \text{rank}(B^\top A^\top) \leq \text{rank}(B^\top) = \text{rank}(B)$$

So

$$\text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\}$$

(b) Both existing AB and BA means that if A is a $n \times m$ matrix then B is a $m \times n$ matrix. Let $n < m$. BA is a $m \times m$ matrix. And

$$\text{rank}(BA) \leq \min\{\text{rank}(B), \text{rank}(A)\} \leq n < m \ \Rightarrow \ \det(BA) = 0$$

Hence BA is singular.
And vice versa, if $m < n$ then AB is singular.

Problem 12 (Trace of a matrix; 2pt)**.** Are there $n \times n$ matrices A and B such that $AB - BA = I_n$?

Solution to the problem 12. No, since $(AB - BA = I_n) \Rightarrow (\text{tr}(AB - BA) = \text{tr}(I_n))$ one can show that the necessary condition isn't satisfied:
$\text{tr}(AB - BA) = \text{tr}(AB) - \text{tr}(BA) = \text{tr}(AB) - \text{tr}(AB) = 0$, but: $\text{tr}(I_n) = n \neq 0$

Problem 13 (Bases; 3pt)**.** For what numbers c are the following sets of vectors bases for $\mathbb{R}^3$?

(a) $(c, 1, 1)^\top$, $(1, -1, 2)^\top$, $(3, 4, -1)^\top$;

(b) $(c, 1, 1)^\top$, $(1, -1, 2)^\top$, $(-2, 2, -4)^\top$;

(c) $(c, 1, 1)^\top$, $(1, 1, 0)^\top$, $(0, 1, 2)^\top$, $(3, 0, -1)^\top$;

(d) $(c, 1, 1)^\top$, $(1, 0, 1)^\top$

Solution to the problem 13. .
Basis of $\mathbb{R}^3$ is a set of 3 linearly independent vectors.

(a) The set should be linearly independent so we have a linear system:

$$\begin{pmatrix} c & 1 & 3 \\ 1 & -1 & 4 \\ 1 & 2 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & -5 \\ 0 & 1-2c & 3+c \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & -5 \\ 0 & 0 & 14-7c \end{pmatrix}$$

$$\Rightarrow \ 14 - 7c \neq 0 \ \Rightarrow \ c \neq 2$$

(b) One can notice that $(1, -1, 2)^\top$ and $(-2, 2, -4)^\top$ are linearly dependent therefore the vectors in the whole set are linearly dependent so it already can't be a basis. Hence there is no proper number: $c \in \varnothing$.

(c) The number of vectors in the set is $4 \neq 3$, so it can't be a basis for any value of c. Hence $c \in \varnothing$

(d) Same, the number of vectors in the set is $2 \neq 3$, so it can't be a basis for any value of c. Hence $c \in \varnothing$

Problem 14 (Bases; transition matrices; 4pt)**.** Consider the bases $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $B' = \{\mathbf{v}'_1, \mathbf{v}'_2, \mathbf{v}'_3\}$ for $\mathbb{R}^3$, where

$$\mathbf{v}_1 = \begin{pmatrix} -3 \\ 0 \\ -3 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 6 \\ -1 \end{pmatrix}$$
$$\mathbf{v}'_1 = \begin{pmatrix} -6 \\ -6 \\ 0 \end{pmatrix}, \quad \mathbf{v}'_2 = \begin{pmatrix} -2 \\ -6 \\ 4 \end{pmatrix}, \quad \mathbf{v}'_3 = \begin{pmatrix} -2 \\ -3 \\ 7 \end{pmatrix}$$

(a) Find the transition matrix $P_{B \to B'}$ from B to B'.

(b) Compute the coordinate vector $(\mathbf{u})_B$ for $\mathbf{u} = (-5, 8, -5)^\top$.

(c) Use the transition matrix $P_{B \to B'}$ to compute the coordinate vector $(\mathbf{u})_{B'}$.

(d) Check your work by computing $(\mathbf{u})_{B'}$ directly.

Solution to the problem 14. Given the bases of B and B' we can construct the transition matrices from each to $R = \{e_1, e_2, e_3\}$.

$$P_{B \to R} = \begin{pmatrix} -3 & -3 & 1 \\ 0 & 2 & 6 \\ -3 & -1 & -1 \end{pmatrix} \qquad P_{B' \to R} = \begin{pmatrix} -6 & -2 & -2 \\ -6 & -6 & -3 \\ 0 & 4 & 7 \end{pmatrix}$$

(a) $P_{B \to B'} = P_{R \to B'} P_{B \to R} = P^{-1}_{B' \to R} P_{B \to R}$. We have to find the $P^{-1}_{B' \to R}$:

$$\det P_{B' \to R} = 252 + 48 - 72 - 84 = 144$$

$$P^{-1}_{B' \to R} = \frac{1}{144} \begin{pmatrix} -30 & 42 & -24 \\ 6 & -42 & 24 \\ -6 & -6 & 24 \end{pmatrix}^\top = \frac{1}{24} \begin{pmatrix} -5 & 1 & -1 \\ 7 & -7 & -1 \\ -4 & 4 & 4 \end{pmatrix}$$

Now we can compute $P_{B \to B'}$:

$$P_{B \to B'} = \frac{1}{24} \begin{pmatrix} -5 & 1 & -1 \\ 7 & -7 & -1 \\ -4 & 4 & 4 \end{pmatrix} \begin{pmatrix} -3 & -3 & 1 \\ 0 & 2 & 6 \\ -3 & -1 & -1 \end{pmatrix} = \frac{1}{12} \begin{pmatrix} 9 & 9 & 1 \\ -9 & -17 & -17 \\ 0 & 8 & 8 \end{pmatrix}$$

(b) $(\mathbf{u})_B = P_{R \to B} \mathbf{u} = P^{-1}_{B \to R} \mathbf{u}$. We have to find the $P^{-1}_{B \to R}$:

$$\det P_{B \to R} = 6 + 54 + 6 - 18 = 48$$

$$P_{B\to R}^{-1} = \frac{1}{48}\begin{pmatrix} 4 & -18 & 6 \\ -4 & 6 & 6 \\ -20 & 18 & -6 \end{pmatrix}^\top = \frac{1}{24}\begin{pmatrix} 2 & -2 & -10 \\ -9 & 3 & 9 \\ 3 & 3 & -3 \end{pmatrix}$$

Hence

$$(\mathbf{u})_B = \frac{1}{24}\begin{pmatrix} 2 & -2 & -10 \\ -9 & 3 & 9 \\ 3 & 3 & -3 \end{pmatrix}\begin{pmatrix} -5 \\ 8 \\ -5 \end{pmatrix} = \frac{1}{24}\begin{pmatrix} 24 \\ 24 \\ 24 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

(c)

$$(\mathbf{u})_{B'} = P_{B\to B'}(\mathbf{u})_B = \frac{1}{12}\begin{pmatrix} 9 & 9 & 1 \\ -9 & -17 & -17 \\ 0 & 8 & 8 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{12}\begin{pmatrix} 19 \\ -43 \\ 16 \end{pmatrix}$$

(d)

$$(\mathbf{u})_{B'} = P_{R\to B'}\mathbf{u} = P_{B'\to R}^{-1}\mathbf{u} = \frac{1}{24}\begin{pmatrix} -5 & 1 & -1 \\ 7 & -7 & -1 \\ -4 & 4 & 4 \end{pmatrix}\begin{pmatrix} -5 \\ 8 \\ -5 \end{pmatrix} = \frac{1}{24}\begin{pmatrix} 38 \\ -86 \\ 32 \end{pmatrix} = \frac{1}{12}\begin{pmatrix} 19 \\ -43 \\ 16 \end{pmatrix}$$

which matches the previous one.

Problem 15 (Linear transformations; 2pt)**.** (a) For what real values of parameters a and b there is a linear transformation of the space $\mathbb{R}^3$ sending the vectors $\mathbf{u}_1 = (1,0,0)^\top, \mathbf{u}_2 = (1,a,0)^\top$, and $\mathbf{u}_3 = (0,1,b)^\top$ into the vectors $\mathbf{v}_1 = (0,0,1)^\top, \mathbf{v}_2 = (0,b,1)^\top$, and $\mathbf{v}_3 = (a,1,0)^\top$ respectively?

(b) For what a and b such a transformation is unique?

(c) For what a and b there exists an orthogonal transformation with this property?

Hint: yes, that's the problem from your entrance exam. However, nobody solved it correctly, and now you have a second chance!

Solution to the problem 15. Let $u = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $v = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Since we are dealing with $\mathbb{R}^3$ we need $\mathbf{u}_i \neq \mathbf{u}_j \quad \forall i \neq j$ and $\mathbf{v}_i \neq \mathbf{v}_j \quad \forall i \neq j$ to have a unique transformation. Otherwise if some vectors match (which means that we have less than 3 unique vectors in u or v) e.g. $\mathbf{u}_1 = \mathbf{u}_2$ then there exists a linear transformation only if the corresponding vectors $\mathbf{v}_1 = \mathbf{v}_2$ also match and then the transformation is not unique. Otherwise we have $A\mathbf{u}_1 = \mathbf{v}_1$ and $A\mathbf{u}_2 = \mathbf{v}_2$ so $A\mathbf{u}_1 = A\mathbf{u}_2 = \mathbf{v}_2 \neq \mathbf{v}_1$ which is impossible.

(a) In addition to finding such a, b that there is a unique transformation (which is described below), we have to find such a, b that $card(u) = card(v) < 3$ and matching vectors in u correspond to the same matching vectors in v. One can see that $a = 0$ leads to $\mathbf{u}_1 = \mathbf{u}_2$, hence $\mathbf{v}_1 = \mathbf{v}_2$ that forces $b = 0$. If $a = 0 \wedge b \neq 0$ then (as described above) a contradiction arises, the same with $a \neq 0 \wedge b = 0$. So the answer is $(a,b) \in \{(0,0)\} \cup \{(a,b) \in \mathbb{R}^2 \mid a \neq 0 \wedge b \neq 0\}$. Indeed, we have

$$\mathbf{u}_1 = \mathbf{u}_2 = (1,0,0)^\top \quad \mathbf{u}_3 = (0,1,0)^\top \quad \mathbf{v}_1 = \mathbf{v}_2 = (0,0,1)^\top \quad \mathbf{v}_3 = (0,1,0)^\top$$

$$A\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix} \Rightarrow column_1(A) = \begin{pmatrix}0\\0\\1\end{pmatrix}$$
$$A\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix} \Rightarrow column_2(A) = \begin{pmatrix}0\\1\\0\end{pmatrix}$$

Hence

$$A = \begin{pmatrix}0 & 0 & a_1\\0 & 1 & a_2\\1 & 0 & a_3\end{pmatrix} \quad a_i \in \mathbb{R} \quad i = \overline{1,3}$$

(b) So, for the unique transformation we need to have 3 unique vectors in u and 3 unique vectors in v which is equivalent to $a \neq 0 \wedge b \neq 0$. To make sure we'll find this linear transformation A. The transition scheme:

$$\begin{tikzcd}
\mathbb{R}^3_e \arrow[r, "A_e"] & \mathbb{R}^3_e \\
\mathbb{R}^3_u \arrow[u, bend left, "P_{u\to e}"] \arrow[ur, "A_{ve}"] \arrow[r, "A_u"] & \mathbb{R}^3_u \arrow[u, bend right, "P_{u\to e}"']
\end{tikzcd}$$

$$P_{u\to e} = \begin{pmatrix}1 & 1 & 0\\0 & a & 1\\0 & 0 & b\end{pmatrix} \qquad A_{ve} = \begin{pmatrix}0 & 0 & a\\0 & b & 1\\1 & 1 & 0\end{pmatrix}$$

We need to find $A = A_e$:

$$A_e = A_{ve}P_{e\to u} = A_{ve}P^{-1}_{u\to e}$$

$$\det P_{u\to e} = ab \qquad P^{-1}_{u\to e} = \frac{1}{ab}\begin{pmatrix}ab & -b & 1\\0 & b & -1\\0 & 0 & a\end{pmatrix}$$

$$\Rightarrow A_e = \frac{1}{ab}\begin{pmatrix}0 & 0 & a\\0 & b & 1\\1 & 1 & 0\end{pmatrix}\begin{pmatrix}ab & -b & 1\\0 & b & -1\\0 & 0 & a\end{pmatrix} = \frac{1}{ab}\begin{pmatrix}0 & 0 & a^2\\0 & b^2 & a-b\\ab & 0 & 0\end{pmatrix} = \begin{pmatrix}0 & 0 & \frac{a}{b}\\0 & \frac{b}{a} & \frac{a-b}{ab}\\1 & 0 & 0\end{pmatrix}$$

One can see that indeed $\forall i = \overline{1,3} \quad A_e\mathbf{u}_i = \mathbf{v}_i$. So the answer is $a \neq 0 \wedge b \neq 0$.

(c) A is orthogonal when $A^\top = A^{-1}$. First let's check for a unique linear transformation when $a \neq 0 \wedge b \neq 0$:

$$A^\top = \begin{pmatrix}0 & 0 & 1\\0 & \frac{b}{a} & 0\\\frac{a}{b} & \frac{a-b}{ab} & 0\end{pmatrix}$$

$$\det A = -1 \qquad A^{-1} = -\begin{pmatrix} 0 & -\frac{b-a}{ab} & -\frac{b}{a} \\ 0 & -\frac{a}{b} & 0 \\ -1 & 0 & 0 \end{pmatrix}^\top = \begin{pmatrix} 0 & 0 & 1 \\ \frac{b-a}{ab} & \frac{a}{b} & 0 \\ \frac{b}{a} & 0 & 0 \end{pmatrix}$$

Hence $a = b \Rightarrow A^\top = A^{-1}$.
Second, let's check for a non-unique linear transformation when $a = b = 0$:

$$A^\top = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ a_1 & a_2 & a_3 \end{pmatrix}$$

$$\det A = -a_1 \qquad A^{-1} = \frac{1}{-a_1}\begin{pmatrix} a_3 & a_2 & -1 \\ 0 & -a_1 & 0 \\ -a_1 & 0 & 0 \end{pmatrix}^\top = \begin{pmatrix} -\frac{a_3}{a_1} & 0 & 1 \\ -\frac{a_2}{a_1} & 1 & 0 \\ \frac{1}{a_1} & 0 & 0 \end{pmatrix}^\top$$

$$\Rightarrow \begin{cases} a_3 = 0 \\ a_2 = 0 \\ a_1 = \frac{1}{a_1} \end{cases} \Leftrightarrow \begin{cases} a_1 = \pm 1 \\ a_3 = 0 \\ a_2 = 0 \end{cases}$$

Hence for $a = b = 0$ there exists such orthogonal transformation.
So the answer is $(a, b) \in \{(a, a) \mid a \in \mathbb{R}\}$.