Linear Algebra
Homework 3: Eigenvalues and eigenvectors

Solutions are by Yaroslava Lochman.

Problem 1 (Oblique projectors; 5pt)**.** Assume that $\mathbb{R}^n$ is represented as the *direct* (but not necessarily *orthogonal*) sum $M_1 \dotplus M_2$ of two its subspaces M_1 and M_2. In particular, every $\mathbf{x} \in \mathbb{R}^n$ can be represented in a unique way as $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$ with some $\mathbf{x}_j \in M_j$, and the mapping $P_j : \mathbf{x} \mapsto \mathbf{x}_j$ is called the *(oblique) projector onto* M_j *parallel to* M_{3-j}. It is easy to show that P_j satisfy the following properties: $P_1 + P_2 = I_n$, $P_j^2 = P_j$ and $P_1P_2 = P_2P_1 = 0$.

(a) Show that any matrix P satisfying the relation $P^2 = P$ is a projector onto some subspace L parallel to M, and identify these L and M.

(b) Show that the projector P is an orthogonal projector if and only if the matrix P is symmetric.

(c) Assume that two transformations P_1 and P_2 of $\mathbb{R}^n$ satisfy the following conditions: $P_1+P_2 = I_n$ and $P_1P_2 = 0$. Prove that P_1 and P_2 are projectors and that $P_2P_1 = 0$.

Solution to the problem 1. .

(a) $P^2 = P$. Let $P : \mathbb{R}^n \to \mathbb{R}^m$. Since P should be able to be applied to $Px \;\Rightarrow\; m = n$.

$$\forall z \in \mathbb{R}^n : \; z = Pz \; + \; (z - Pz) \qquad \begin{cases} Pz \in ImP \\ P(z - Pz) = Pz - P^2z = 0 \;\Rightarrow\; z - Pz \in KerP \end{cases}$$

So $\mathbb{R}^n = ImP + KerP$. Let's show that $ImP \cap KerP = \{0\}$ and hence $\mathbb{R}^n = ImP \dotplus KerP$.

$$\forall y \in ImP \cap KerP : \begin{cases} Py = 0 \\ \exists x \in \mathbb{R}^n : y = Px \;\Rightarrow\; Py = P^2x = Px = y \end{cases} \Rightarrow\; y = Py = 0$$

Let's show now that $\forall z \in \mathbb{R}\ \exists! x \in KerP,\ \exists! y \in ImP : z = x + y$. Let it be false, meaning $\exists x_1, x_2 \in KerP,\ y_1, y_2 \in ImP$ s.t. :

$$\begin{cases} z = x_1 + y_1 \\ z = x_2 + y_2 \end{cases} \Rightarrow\; x_1 - x_2 (\in KerP) = y_1 - y_2 (\in ImP)$$

$$\Rightarrow\; \begin{cases} x_1 - x_2 \in ImP \cap KerP = \{0\} \\ y_1 - y_2 \in ImP \cap KerP = \{0\} \end{cases} \Rightarrow\; \begin{cases} x_1 = x_2 \\ y_1 = y_2 \end{cases}$$

So we have:

$$\mathbb{R}^n = ImP \dotplus KerP = \text{ (in other words) } = \mathcal{C}(P) \dotplus \mathcal{N}(P)$$

(b) Let P now be also a symmetric matrix: $P = P^\top$. Let's show that $\mathcal{C}(P) \perp \mathcal{N}(P)$. From the second fundamental theorem of L:

$$\mathcal{C}(P) \perp \mathcal{N}(P^\top)$$

Since $P = P^\top \;\Rightarrow\; \mathcal{N}(P) = \mathcal{N}(P)^\top \;\Rightarrow\; \mathcal{C}(P) \perp \mathcal{N}(P)$

$$\Rightarrow\; \mathbb{R}^n = \mathcal{C}(P) \oplus \mathcal{N}(P)$$

(c) $P_1 : \mathbb{R}^n \to \mathbb{R}^n,\ \ P_2 : \mathbb{R}^n \to \mathbb{R}^n.$

$$\begin{cases} P_1 + P_2 = I_n \\ P_1 P_2 = 0_n \end{cases} \Rightarrow \begin{cases} P_1 = P_1(P_1+P_2) = P_1^2 + P_1P_2 = P_1^2 \ \Rightarrow\ P_1 \text{ is a projector.} \\ P_2 = (P_1+P_2)P_2 = P_1P_2 + P_2^2 = P_2^2 \ \Rightarrow\ P_2 \text{ is a projector.} \end{cases}$$

$$P_1 = (P_1+P_2)P_1 = P_1^2 + P_2P_1 = P_1 + P_2P_1 \Rightarrow P_2P_1 = 0_n$$

Problem 2 (Projectors; 5pt)**.** (a) Find a matrix of oblique projector in $\mathbb{R}^3$ onto the subspace $U = \operatorname{ls}\{(1,0,1)^\top\}$ parallel to the subspace $W = \operatorname{ls}\{(1,1,0)^\top, (0,1,1)^\top\}$.

(b) Find a projection matrix of $\mathbb{R}^3$ onto the subspace $U = \operatorname{ls}\{(1,2,1)^\top, (1,0,-1)^\top\}$ parallel to the subspace $W = \operatorname{ls}\{(1,0,1)^\top\}$.

(c) Is it possible to fill in the missing entries in the matrix

$$A = \begin{pmatrix} 1 & * & 0 \\ 0 & \frac{1}{2} & * \\ * & * & * \end{pmatrix}$$

to get a matrix of an orthogonal projection in $\mathbb{R}^3$? If so, find the subspace U of $\mathbb{R}^3$ such that A is an orthogonal projection onto U.

Hint: Do you see why only one of (a) or (b) needs to be worked out in detail?

Solution to the problem 2. Let's find a general form of the matrix of the projector.
Let $u_1 \dots u_k$ be linearly independent in $\mathbb{R}^n$, $U = \operatorname{ls}\{u_1 \dots u_k\}$, $\mathbb{U} = \begin{pmatrix} u_1 & \dots & u_k \end{pmatrix}$,
$w_1 \dots w_{n-k}$ – linearly independent in $\mathbb{R}^n$, $W = \operatorname{ls}\{w_1 \dots w_{n-k}\}$, $\mathbb{W} = \begin{pmatrix} w_1 & \dots & w_{n-k} \end{pmatrix}$ s.t.:

$$\mathbb{R}^n = U \dotplus W$$

Let's find a matrix of oblique projector in $\mathbb{R}^n$ onto U parallel to W.

$$\forall b \in \mathbb{R}^n : p = Pb \in U \ \Rightarrow\ \exists x : p = \mathbb{U}x$$

$$(b - \mathbb{U}x) \text{ should be parallel to } W \ \Rightarrow\ (b - \mathbb{U}x) \perp W^\perp$$

Let V be $W^\perp$, $\dim(V) = n - \dim(W) = n - n + k = k$, so:
$V = \operatorname{ls}\{v_1 \dots v_k\}$, $\mathbb{V} = \begin{pmatrix} v_1 & \dots & v_k \end{pmatrix}$, $\mathbb{V}^\top \mathbb{W} = \mathbf{0_{k\times(n-k)}}$

$$\Rightarrow\ \mathbb{V}^\top(b - \mathbb{U}x) = \mathbf{0_k} \ \Rightarrow\ \mathbb{V}^\top\mathbb{U}x = \mathbb{V}^\top b$$

$\mathbb{V}^\top\mathbb{U}$ is $k \times k$ non-singular since $\mathcal{N}(\mathbb{V}^\top\mathbb{U}) = \{0\}$:

$$\mathbb{V}^\top\mathbb{U}x = 0 \ \Rightarrow\ \mathbb{U}x \in W \quad \text{but} \quad W \cap U = 0$$

So there will be no linear combination of columns of $\mathbb{U}$ that lies in W $\ \Rightarrow\ x = 0$

$$\Rightarrow\ x = (\mathbb{V}^\top\mathbb{U})^{-1}\mathbb{V}^\top b \ \Rightarrow\ p = \mathbb{U}x = \mathbb{U}(\mathbb{V}^\top\mathbb{U})^{-1}\mathbb{V}^\top b = Pb \ \Rightarrow\ P = \mathbb{U}(\mathbb{V}^\top\mathbb{U})^{-1}\mathbb{V}^\top$$

(a) $U = \operatorname{ls}\{(1,0,1)^\top\}$, $W = \operatorname{ls}\{(1,1,0)^\top, (0,1,1)^\top\}$. Let's first check that $\mathbb{R}^n = U \dotplus W$:

$$\det \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} = 1 \;\Rightarrow\; \text{the system of columns is linearly independent.}$$

$$\Rightarrow\; \mathbb{R}^n = \operatorname{ls}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\} = U \dotplus W$$

So

$$\mathbb{U} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \mathbb{W} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \mathbb{V} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \text{ since:}$$

$$V = W^\perp = \operatorname{ls}\left\{ \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} \right\} = \operatorname{ls}\left\{ (1,-1,1)^\top \right\}$$

$$\mathbb{V}^\top \mathbb{U} = \begin{pmatrix} 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 2 \;\Rightarrow\; P = \mathbb{U}(\mathbb{V}^\top \mathbb{U})^{-1} \mathbb{V}^\top = \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 1 & -1 & 1 \end{pmatrix}$$

Answer: $P = \frac{1}{2} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 1 & -1 & 1 \end{pmatrix}$

(b) $U = \operatorname{ls}\{(1,2,1)^\top, (1,0,-1)^\top\}$, $W = \operatorname{ls}\{(1,0,1)^\top\}$. Let's first check that $\mathbb{R}^n = U \dotplus W$:

$$\det \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 0 \\ 1 & -1 & 1 \end{pmatrix} = -4 \;\Rightarrow\; \text{the system of columns is linearly independent.}$$

$$\Rightarrow\; \mathbb{R}^n = \operatorname{ls}\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\} = U \dotplus W$$

So

$$\mathbb{U} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 1 & -1 \end{pmatrix} \quad \mathbb{W} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \mathbb{V} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \text{ since:}$$

$$v_1 \perp w_1 \;\Rightarrow\; v_1 \text{ can be } (0,1,0)^\top$$

$$v_2 \in \{w_1, v_1\}^\perp \;\Rightarrow\; v_2 \text{ can be } \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = (-1,0,1)^\top \;\Rightarrow\; V = \operatorname{ls}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$\mathbb{V}^\top \mathbb{U} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \qquad (\mathbb{V}^\top \mathbb{U})^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\Rightarrow\ P = \mathbb{U}(\mathbb{V}^\top\mathbb{U})^{-1}\mathbb{V}^\top = \frac{1}{2}\begin{pmatrix}1 & 1\\ 2 & 0\\ 1 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & 1 & -1\\ 0 & 2 & 0\\ -1 & 1 & 1\end{pmatrix}$$

In fact if $P_{(a)}$ is a projector from task (a), and $P_{(b)}$ – from task (b), then $P_{(b)} = I - P_{(a)}$, since $U_{(a)} = W_{(b)}$. Let's check:

$$I - P_{(a)} = \frac{1}{2}\left(\begin{pmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{pmatrix} - \begin{pmatrix}1 & -1 & 1\\ 0 & 0 & 0\\ 1 & -1 & 1\end{pmatrix}\right) = \frac{1}{2}\begin{pmatrix}1 & 1 & -1\\ 0 & 2 & 0\\ -1 & 1 & 1\end{pmatrix} = P_{(b)}$$

Answer: $P = \frac{1}{2}\begin{pmatrix}1 & 1 & -1\\ 0 & 2 & 0\\ -1 & 1 & 1\end{pmatrix}$

(c)

$$A = \begin{pmatrix}1 & x_2 & 0\\ 0 & \frac{1}{2} & x_4\\ x_1 & x_3 & x_5\end{pmatrix} = |\text{A should be symmetric}| = \begin{pmatrix}1 & 0 & 0\\ 0 & \frac{1}{2} & x_3\\ 0 & x_3 & x_5\end{pmatrix}$$

$$A^2 = \begin{pmatrix}1 & 0 & 0\\ 0 & \frac{1}{4} + x_3^2 & \frac{1}{2}x_3 + x_3x_5\\ 0 & \frac{1}{2}x_3 + x_3x_5 & x_3^2 + x_5^2\end{pmatrix} = \begin{pmatrix}1 & x_2 & 0\\ 0 & \frac{1}{2} & x_4\\ x_1 & x_3 & x_5\end{pmatrix} = A$$

$$\begin{cases}\frac{1}{4} + x_3^2 = \frac{1}{2}\\ x_3(x_5 - \frac{1}{2}) = 0\\ x_5^2 - x_5 + x_3^2 = 0\end{cases} \Rightarrow \begin{cases}x_5 = \frac{1}{2}\\ x_3 = \frac{1}{2}\end{cases} \Rightarrow A = \begin{pmatrix}1 & 0 & 0\\ 0 & \frac{1}{2} & \frac{1}{2}\\ 0 & \frac{1}{2} & \frac{1}{2}\end{pmatrix}$$

$$U = \mathcal{C}(A) = \text{ls}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\ \frac{1}{2}\\ \frac{1}{2}\end{pmatrix}, \begin{pmatrix}0\\ \frac{1}{2}\\ \frac{1}{2}\end{pmatrix}\right\}$$

Problem 3 (Eigenvalues; 3 pt)**.** Find all the eigenvalues of the following matrices by **inspection**:

(a) $\begin{pmatrix}1/2 & 1\\ 1/2 & 0\end{pmatrix}$, (b) $\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}$, (c) $\begin{pmatrix}3 & 2\\ 1 & 2\end{pmatrix}$, (d) $\begin{pmatrix}0 & 1 & 2\\ 1 & 0 & 2\\ 1 & 1 & 1\end{pmatrix}$, (e) $\begin{pmatrix}0 & 0 & 2\\ 0 & 5 & 0\\ 1 & 0 & 1\end{pmatrix}$.

Hint: Look for constant row/column sums, diagonal entries with zeros in the corresponding row or column otherwise, use eigenvalue sum/product rules, try subtracting λI for "tempting" candidates for λ etc

Solution to the problem 3. .

(a) $A = \begin{pmatrix}1/2 & 1\\ 1/2 & 0\end{pmatrix}$ $\quad A - I = \begin{pmatrix}-1/2 & 1\\ 1/2 & -1\end{pmatrix}$ $\quad \det(A - I) = 0 \Rightarrow \lambda_1 = 1$

$$\lambda_2 = \operatorname{tr} A - \lambda_1 = 1/2 - 1 = -1/2$$

Answer: $\lambda_1 = 1,\ \lambda_2 = -1/2$

(b) $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad \det A = 0 \;\Rightarrow\; \lambda_1 = 0$

$$\lambda_2 = \operatorname{tr} A - \lambda_1 = 2$$

Answer: $\lambda_1 = 0,\; \lambda_2 = 2$

(c) $A = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} \qquad A - I = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} \quad \det(A - I) = 0 \;\Rightarrow\; \lambda_1 = 1$

$$\lambda_2 = \operatorname{tr} A - \lambda_1 = 5 - 1 = 4$$

Answer: $\lambda_1 = 1,\; \lambda_2 = 4$

(d) $A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix} \qquad A + I = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix} \quad \det(A + I) = 0 \;\Rightarrow\; \lambda_1 = -1$

$$\operatorname{rank}(A + I) = 1 \;\Rightarrow\; \dim N(A + I) = 3 - 1 = 2 \;\Rightarrow\; \lambda_2 = \lambda_1 = -1$$

$$\Rightarrow \lambda_3 = \operatorname{tr} A + 1 + 1 = 3$$

Answer: $\lambda_1 = \lambda_2 = -1,\; \lambda_3 = 3$

(e) $A = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 5 & 0 \\ 1 & 0 & 1 \end{pmatrix} \qquad A - 5I = \begin{pmatrix} -5 & 0 & 2 \\ 0 & 0 & 0 \\ 1 & 0 & -4 \end{pmatrix} \quad \det(A - 5I) = 0 \;\Rightarrow\; \lambda_1 = 5$

$$\begin{cases} 5 + \lambda_2 + \lambda_3 = \operatorname{tr} A = 6 \\ 5\lambda_2 \cdot \lambda_3 = \det A = -5 \end{cases} \Rightarrow \begin{cases} \lambda_2 + \lambda_3 = 1 \\ \lambda_3 = -\frac{1}{\lambda_2} \end{cases} \Rightarrow \left| \lambda - \frac{1}{\lambda} = 1, \quad \lambda^2 - \lambda - 1 = 0 \right| \Rightarrow \lambda_{2,3} = \frac{1 \pm \sqrt{5}}{2}$$

Answer: $\lambda_1 = 5, \quad \lambda_2 = \frac{1+\sqrt{5}}{2}, \quad \lambda_3 = \frac{1-\sqrt{5}}{2}$

Problem 4 (Eigenvalues and eigenvectors; 4 pt)**.** For the matrix A in each part below, find the eigenvalues and eigenvectors of A, A^2, A^{100}, A^{-1} and e^{tA}:

$$\text{(a)} \quad \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}, \qquad \text{(b)} \quad \begin{pmatrix} 4 & 0 & -1 \\ 0 & -1 & 4 \\ 0 & 2 & 1 \end{pmatrix}$$

Solution to the problem 4. Let A be $k \times k$ matrix. Let λ_i and $\mathbf{v}_i$ be the eigenvalue and corresponding eigenvector of A; $\quad \lambda_i^{(n)}$ and $v_i^{(n)}$ – the eigenvalue and corresponding eigenvector of A^n; $\quad \hat{\lambda}_i$ and $\hat{\mathbf{v}}_i$ – the eigenvalue and corresponding eigenvector of e^{tA}, $\quad i = \overline{1, k}$.
In case when $A = PDP^{-1}$ where $P = \begin{pmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_n \end{pmatrix}$ and $D = diag\{\lambda_1, \dots, \lambda_n\}$ we have:

$$A^n = PD\cancel{P^{-1}}\cancel{P}D\cancel{P^{-1}} \cdots \cancel{P}DP^{-1} = PD^nP^{-1} \;\Rightarrow\; \begin{cases} \lambda_i^{(n)} = \lambda_i^n \\ \mathbf{v}_i^{(n)} = \mathbf{v}_i \end{cases} \quad i = \overline{1, k}$$

And since $e^x = \sum_{n=1}^{\infty} \frac{x^n}{n!}$:

$$e^{tA} = \sum_{n=1}^{\infty} \frac{t^n A^n}{n!} = \sum_{n=1}^{\infty} \frac{t^n PD^nP^{-1}}{n!} = P \sum_{n=1}^{\infty} \frac{t^n D^n}{n!} P^{-1} = Pe^{tD}P^{-1} \;\Rightarrow\; \begin{cases} \hat{\lambda}_i = e^{t\lambda_i} \\ \hat{\mathbf{v}}_i = \mathbf{v}_i \end{cases} \quad i = \overline{1, k}$$

(a) $A = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} \qquad A - I = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} \quad \det(A - I) = 0 \ \Rightarrow\ \lambda_1 = 1$

$$(A - \lambda_1 I)\mathbf{v}_1 = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} \mathbf{v}_1 = 0 \ \Rightarrow\ \mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\lambda_2 = \operatorname{tr} A - \lambda_1 = 5 - 1 = 4 \qquad (A - \lambda_2 I)\mathbf{v}_2 = \begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix} \mathbf{v}_2 = 0 \ \Rightarrow\ \mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$P = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} \qquad |\ \text{self-check: } \det P = 3 \neq 0\ | \qquad D = diag\{\lambda_1, \lambda_2\}$$

So $A = PDP^{-1}$, and we can easily calculate eigenvalues and eigenvectors (using the conclusions above):

Answer: $\lambda_1 = 1 \quad \lambda_2 = 4$

$\lambda_1^{(2)} = 1 \quad \lambda_2^{(2)} = 4^2 = 16$

$\lambda_1^{(100)} = 1 \quad \lambda_2^{(100)} = 4^{100}$

$\lambda_1^{(-1)} = 1 \quad \lambda_2^{(-1)} = 4^{-1} = 1/4$

$\hat{\lambda}_1 = e^t \quad \hat{\lambda}_2 = e^{4t}$

$$\{\mathbf{v}_1, \mathbf{v}_2\} = \left\{\mathbf{v}_1^{(2)}, \mathbf{v}_2^{(2)}\right\} = \left\{\mathbf{v}_1^{(100)}, \mathbf{v}_2^{(100)}\right\} = \left\{\mathbf{v}_1^{(-1)}, \mathbf{v}_2^{(-1)}\right\} = \{\hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2\} = \left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right\}$$

(b) $A = \begin{pmatrix} 4 & 0 & -1 \\ 0 & -1 & 4 \\ 0 & 2 & 1 \end{pmatrix} \qquad A - 4I = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -5 & 4 \\ 0 & 2 & -3 \end{pmatrix} \quad \det(A - 4I) = 0 \ \Rightarrow\ \lambda_1 = 4$

$$(A - 4I)\mathbf{v}_1 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -5 & 4 \\ 0 & 2 & -3 \end{pmatrix} \mathbf{v}_1 = 0 \ \Rightarrow\ \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$A + 3I = \begin{pmatrix} 7 & 0 & -1 \\ 0 & 2 & 4 \\ 0 & 2 & 4 \end{pmatrix} \quad \det(A + 3I) = 0 \ \Rightarrow\ \lambda_2 = -3$$

$$(A + 3I)\mathbf{v}_2 = \begin{pmatrix} 7 & 0 & -1 \\ 0 & 2 & 4 \\ 0 & 2 & 4 \end{pmatrix} \mathbf{v}_2 = 0 \ \Rightarrow\ \mathbf{v}_2 = \begin{pmatrix} 1 \\ -14 \\ 7 \end{pmatrix}$$

$$\lambda_3 = \operatorname{tr} A - 4 + 3 = 3 \qquad (A - 3I)\mathbf{v}_3 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -4 & 4 \\ 0 & 2 & -2 \end{pmatrix} \mathbf{v}_3 = 0 \ \Rightarrow\ \mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$P = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -14 & 1 \\ 0 & 7 & 1 \end{pmatrix} \qquad |\ \text{self-check: } \det P = -21 \neq 0\ | \qquad D = diag\{\lambda_1, \lambda_2, \lambda_3\}$$

So $A = PDP^{-1}$, and we can easily calculate eigenvalues and eigenvectors (using the conclusions above):

Answer: $\lambda_1 = 4 \quad \lambda_2 = -3 \quad \lambda_3 = 3$

$$\lambda_1^{(2)} = 16 \quad \lambda_2^{(2)} = 9 \quad \lambda_3^{(2)} = 9$$

$$\lambda_1^{(100)} = 4^{100} \quad \lambda_2^{(100)} = 3^{100} \quad \lambda_3^{(100)} = 3^{100}$$

$$\lambda_1^{(-1)} = 1/4 \quad \lambda_2^{(-1)} = -1/4 \quad \lambda_3^{(100)} = 1/3$$

$$\hat{\lambda} = e^{4t} \quad \hat{\lambda} = e^{-4t} \quad \hat{\lambda} = e^{3t}$$

$$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \left\{v_1^{(2)}, v_2^{(2)}, v_3^{(2)}\right\} = \left\{v_1^{(100)}, v_2^{(100)}, v_3^{(100)}\right\} =$$

$$= \left\{v_1^{(-1)}, v_2^{(-1)}, v_3^{(-1)}\right\} = \{\hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2, \hat{\mathbf{v}}_3\} = \left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\-14\\7\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix} \right\}$$

Problem 5 (Difference equation; 4 pt)**.** A *generalized Fibonacci* sequence g_n is defined through the relations $g_0 = 0$, $g_1 = 1$, $g_2 = 2$ and $g_{n+3} = 3g_{n+2} - g_{n+1} - g_n$ for $n \in \mathbb{Z}_+$. Find the formula for the n^{th} generalized Fibonacci number g_n. To this end,

(a) represent the difference equation in the form $\mathbf{x}_{n+1} = A\mathbf{x}_n$ for a suitable 3×3 matrix A, find its eigenvalues and eigenvectors and then diagonalize A;

(b) find a general solution $\mathbf{x}_n$ and then the one satisfying the initial condition $\mathbf{x}_0 = (0, 1, 2)^\top$.

Solution to the problem 5. .

(a)

$$\mathbf{x}_n = \begin{pmatrix} g_n \\ g_{n+1} \\ g_{n+2} \end{pmatrix} \Rightarrow \mathbf{x}_0 = \begin{pmatrix}0\\1\\2\end{pmatrix} \quad \mathbf{x}_{n+1} \begin{pmatrix} g_{n+1} \\ g_{n+2} \\ g_{n+3} \end{pmatrix} = \begin{pmatrix} g_{n+1} \\ g_{n+2} \\ 3g_{n+2} - g_{n+1} - g_n \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & 3 \end{pmatrix} \mathbf{x}_n$$

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & 3 \end{pmatrix} \qquad x_{n+1} = Ax_n$$

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -1 & -1 & 3-\lambda \end{vmatrix} = \lambda^2(3-\lambda) - 1 - \lambda = -\lambda^3 + 3\lambda^2 - \lambda - 1 = 0$$

$$\lambda^3 - 3\lambda^2 + \lambda + 1 = 0 \qquad (\lambda - 1)(\lambda^2 - 2\lambda - 1) = 0 \qquad \Rightarrow \lambda_1 = 1,\ \lambda_{2,3} = 1 \pm \sqrt{2}$$

$$(A-\lambda_1)\mathbf{v}=0 \qquad \begin{pmatrix}-1 & 1 & 0\\ 0 & -1 & 1\\ -1 & -1 & 2\end{pmatrix}\mathbf{v}_1=0 \qquad \Rightarrow \ \mathbf{v}_1=\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$(A-\lambda_2)\mathbf{v}=0 \qquad \begin{pmatrix}-1-\sqrt{2} & 1 & 0\\ 0 & -1-\sqrt{2} & 1\\ -1 & -1 & 2-\sqrt{2}\end{pmatrix}\mathbf{v}_2=0 \ \Rightarrow \ \mathbf{v}_2=\begin{pmatrix}1\\1+\sqrt{2}\\3+2\sqrt{2}\end{pmatrix}$$

$$(A-\lambda_3)\mathbf{v}=0 \qquad \begin{pmatrix}-1+\sqrt{2} & 1 & 0\\ 0 & -1+\sqrt{2} & 1\\ -1 & -1 & 2+\sqrt{2}\end{pmatrix}\mathbf{v}_3=0 \ \Rightarrow \ \mathbf{v}_2=\begin{pmatrix}1\\1-\sqrt{2}\\3-2\sqrt{2}\end{pmatrix}$$

$$D=\begin{pmatrix}1 & 0 & 0\\ 0 & 1+\sqrt{2} & 0\\ 0 & 0 & 1-\sqrt{2}\end{pmatrix} \quad P=\begin{pmatrix}1 & 1 & 1\\ 1 & 1+\sqrt{2} & 1-\sqrt{2}\\ 1 & 3+2\sqrt{2} & 3-2\sqrt{2}\end{pmatrix}$$

$$\det P=(1+\sqrt{2})(3-2\sqrt{2})+1-\sqrt{2}+3+2\sqrt{2}-1-\sqrt{2}-(1-\sqrt{2})(3+2\sqrt{2})-3+2\sqrt{2}=$$
$$=3-2\sqrt{2}+3\sqrt{2}-4-3-2\sqrt{2}+3\sqrt{2}+4+2\sqrt{2}=4\sqrt{2}$$

$$P^{-1}=\frac{1}{4\sqrt{2}}\begin{pmatrix}2\sqrt{2} & 4\sqrt{2} & -2\sqrt{2}\\ -2+\sqrt{2} & 2-2\sqrt{2} & \sqrt{2}\\ 2+\sqrt{2} & -2-2\sqrt{2} & \sqrt{2}\end{pmatrix}=\frac{1}{4}\begin{pmatrix}2 & 4 & -2\\ 1-\sqrt{2} & -2+\sqrt{2} & 1\\ 1+\sqrt{2} & -2-\sqrt{2} & 1\end{pmatrix}$$

So:

$$A=PDP^{-1}=\begin{pmatrix}1 & 1 & 1\\ 1 & 1+\sqrt{2} & 1-\sqrt{2}\\ 1 & 3+2\sqrt{2} & 3-2\sqrt{2}\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1+\sqrt{2} & 0\\ 0 & 0 & 1-\sqrt{2}\end{pmatrix}\begin{pmatrix}\frac{1}{2} & 1 & -\frac{1}{2}\\ \frac{1-\sqrt{2}}{4} & \frac{-2+\sqrt{2}}{4} & \frac{1}{4}\\ \frac{1+\sqrt{2}}{4} & \frac{-2-\sqrt{2}}{4} & \frac{1}{4}\end{pmatrix}$$

(b)

$$\mathbf{x}_n=PD^nP^{-1}x_0=PD^nc_0=\begin{pmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3\end{pmatrix}\begin{pmatrix}\lambda_1^n & 0 & 0\\ 0 & \lambda_2^n & 0\\ 0 & 0 & \lambda_3^n\end{pmatrix}c_0=c_0^{(1)}\lambda_1^n\mathbf{v}_1+c_0^{(2)}\lambda_2^n\mathbf{v}_2+c_0^{(3)}\lambda_3^n\mathbf{v}_3=$$

$$=c_0^{(1)}\begin{pmatrix}1\\1\\1\end{pmatrix}+c_0^{(2)}(1+\sqrt{2})^n\begin{pmatrix}1\\1+\sqrt{2}\\3+2\sqrt{2}\end{pmatrix}+c_0^{(3)}(1-\sqrt{2})^n\begin{pmatrix}1\\1-\sqrt{2}\\3-2\sqrt{2}\end{pmatrix}$$

$$c_0=P^{-1}x_0=\frac{1}{4}\begin{pmatrix}2 & 4 & -2\\ 1-\sqrt{2} & -2+\sqrt{2} & 1\\ 1+\sqrt{2} & -2-\sqrt{2} & 1\end{pmatrix}\begin{pmatrix}0\\1\\2\end{pmatrix}=\frac{\sqrt{2}}{4}\begin{pmatrix}0\\1\\1\end{pmatrix}$$

$$\Rightarrow \mathbf{x}_n = PD^n c_0 = \frac{\sqrt{2}}{4}(\lambda_2^n \mathbf{v}_2 + \lambda_3^n \mathbf{v}_3) = \frac{\sqrt{2}}{4}\left((1+\sqrt{2})^n \begin{pmatrix} 1 \\ 1+\sqrt{2} \\ 3+2\sqrt{2} \end{pmatrix} + (1-\sqrt{2})^n \begin{pmatrix} 1 \\ 1-\sqrt{2} \\ 3-2\sqrt{2} \end{pmatrix}\right)$$

$$\text{Also : } \mathbf{x}_n = \frac{\sqrt{2}}{4}\lambda_2^n \left(\mathbf{v}_2 + \left(\frac{\lambda_3}{\lambda_2}\right)^n \mathbf{v}_3\right) \underset{n\to\infty}{\sim} \frac{\sqrt{2}}{4}\lambda_2^n \mathbf{v}_2 \quad \text{since } \lambda_2 > \lambda_3$$

Problem 6 (Jordan form; 4 pt)**.** For each of the following matrices A, find P so that $P^{-1}AP$ is in the Jordan form (ie, either diagonal or a Jordan block), and write this Jordan form:

$$\text{(a)} \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}, \qquad \text{(b)} \begin{pmatrix} 3 & 2 \\ 1 & -1 \end{pmatrix}, \qquad \text{(c)} \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}, \qquad \text{(d)} \begin{pmatrix} -5 & 2 \\ -\frac{1}{2} & -3 \end{pmatrix}.$$

Hint: You do not have to calculate $P^{-1}AP$; you can just write it out!

Solution to the problem 6. .

(a)

$$A = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} \qquad \lambda_1 = 1 \quad \lambda_2 = 5 - 1 = 4 \qquad J = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$$

(b)

$$A = \begin{pmatrix} 3 & 2 \\ 1 & -1 \end{pmatrix} \qquad \begin{cases} \lambda_1 + \lambda_2 = 2 \\ \lambda_1 \cdot \lambda_2 = -5 \end{cases} \qquad \begin{cases} \lambda_2 = 2 - \lambda_1 \\ \lambda_1^2 - 2\lambda_1 - 5 = 0 \end{cases}$$

$$\Rightarrow \lambda_{1,2} = 1 \pm \sqrt{6} \qquad J = \begin{pmatrix} 1+\sqrt{6} & 0 \\ 0 & 1-\sqrt{6} \end{pmatrix}$$

(c)

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix} \qquad \lambda_1 = 4 \quad \lambda_2 = 3 - 4 = -1 \qquad J = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}$$

(d)

$$A = \begin{pmatrix} -5 & 2 \\ -\frac{1}{2} & -3 \end{pmatrix} \qquad \begin{cases} \lambda_1 + \lambda_2 = -8 \\ \lambda_1 \cdot \lambda_2 = 16 \end{cases} \qquad \lambda_1 = \lambda_2 = -4 \qquad J = \begin{pmatrix} -4 & 1 \\ 0 & -4 \end{pmatrix}$$

Problem 7 (Eigenvalues and eigenvectors; 2pt)**.** What are the eigenvalues and eigenvectors of the linear transformation A defined on $\mathbb{C}^3$ via $A(x_1, x_2, x_3) = (x_3, x_1, x_2)$?

Solution to the problem 7. .

$$\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}x_3\\x_1\\x_2\end{pmatrix}\qquad A=\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$$

$$\det(A-\lambda)=\begin{vmatrix}-\lambda&0&1\\1&-\lambda&0\\0&1&-\lambda\end{vmatrix}=-\lambda^3+1=0$$

$$\lambda^3-1=(\lambda-1)(\lambda^2+\lambda+1)\qquad\Rightarrow\ \lambda_1=1\quad\lambda_{2,3}=\frac{-1\pm i\sqrt{3}}{2}$$

$$\begin{pmatrix}-1&0&1\\1&-1&0\\0&1&-1\end{pmatrix}\mathbf{v}=0\ \Rightarrow\ \mathbf{v}_1=\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\begin{pmatrix}\frac{1-i\sqrt{3}}{2}&0&1\\1&\frac{1-i\sqrt{3}}{2}&0\\0&1&\frac{1-i\sqrt{3}}{2}\end{pmatrix}\mathbf{v}=0\ \Rightarrow\ \mathbf{v}_2=\begin{pmatrix}1\\-\frac{2}{1-i\sqrt{3}}\\\frac{-1+i\sqrt{3}}{2}\end{pmatrix}=\begin{pmatrix}1\\\frac{-1-i\sqrt{3}}{2}\\\frac{-1+i\sqrt{3}}{2}\end{pmatrix}$$

$$\begin{pmatrix}\frac{1+i\sqrt{3}}{2}&0&1\\1&\frac{1+i\sqrt{3}}{2}&0\\0&1&\frac{1+i\sqrt{3}}{2}\end{pmatrix}\mathbf{v}=0\ \Rightarrow\ \mathbf{v}_3=\begin{pmatrix}1\\-\frac{2}{1+i\sqrt{3}}\\\frac{-1-i\sqrt{3}}{2}\end{pmatrix}=\begin{pmatrix}1\\\frac{-1+i\sqrt{3}}{2}\\\frac{-1-i\sqrt{3}}{2}\end{pmatrix}$$

Answer: $\lambda_1=1,\ \lambda_2=\frac{-1+i\sqrt{3}}{2},\ \lambda_3=\frac{-1-i\sqrt{3}}{2}\quad\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}=\left\{\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}1\\\frac{-1-i\sqrt{3}}{2}\\\frac{-1+i\sqrt{3}}{2}\end{pmatrix},\begin{pmatrix}1\\\frac{-1+i\sqrt{3}}{2}\\\frac{-1-i\sqrt{3}}{2}\end{pmatrix}\right\}$

Problem 8 (Eigenvalues and eigenvectors; 2pt)**.** Let $\mathbf{a}$ be a non-zero vector in $\mathbb{R}^3$ and A a linear transformation of $\mathbb{R}^3$ given by $A\mathbf{x}=\mathbf{x}\times a$, where "$\times$" denotes the cross-product (i.e., vector product). Find the eigenvalues and eigenvectors of A.

Hint: do not use coordinates; use the geometric meaning of the cross product instead

Solution to the problem 8. Let's show that $\mathcal{N}(A)=\mathrm{ls}\{a\}$, $\mathcal{C}(A)=\mathrm{ls}\{a\}^\perp$

$$A\mathbf{x}=0\ \Leftrightarrow\ \mathbf{x}\times a=0\ \Leftrightarrow\ \mathbf{x}\parallel a\ \Rightarrow\ \mathcal{N}(A)=\mathrm{ls}\{a\}$$

$$\begin{cases}\forall\mathbf{y}\in\mathcal{C}(A)\ \exists\mathbf{x}:\mathbf{x}\times a=\mathbf{y}\ \Rightarrow\ \mathbf{y}\perp a\ \Rightarrow\ \mathbf{y}\in ls\{a\}^\perp\ \Rightarrow\ \mathcal{C}(A)\subset ls\{a\}^\perp\\ \dim\mathcal{C}(A)=3-\dim\mathcal{N}(A)=3-1=2\end{cases}\Rightarrow\ \mathcal{C}(A)=ls\{a\}^\perp$$

$$\operatorname{rank}A=\dim\mathcal{C}(A)=2\ \Rightarrow\ \det A=0\ \Rightarrow\ \lambda=0\quad\mathbf{v}\in\mathcal{N}(A)\Rightarrow\mathbf{v}=a$$

Indeed if we look at the equation $\mathbf{v}\times a=\lambda\mathbf{v}$ we see that the result of the cross-product with $\mathbf{v}$ (which should be orthogonal to $\mathbf{v}$ in case $\mathbf{v}$ is not parallel to a, otherwise zero) is parallel to $\mathbf{v}$, so the only possible solution is $\lambda=0$ and $\mathbf{v}=a$.

Answer: For this linear transformation $\lambda=0$ is the only real one eigenvalue with corresponding eigenvector $\mathbf{v}=a$.

Problem 9 (Eigenvalues and eigenvectors; 4pt)**.** (a) Assume that $\mathbf{u}$ and $\mathbf{v}$ are two non-zero (column) vectors of $\mathbb{R}^n$. Find eigenvalues and eigenvectors of the matrix $A := \mathbf{u}\mathbf{v}^\top$.

(b) Assume that a and b are two non-equal numbers. Find eigenvalues and eigenvectors of the $n \times n$ matrix

$$\begin{pmatrix} a & b & b & \dots & b \\ b & a & b & \dots & b \\ b & b & a & \dots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \dots & a \end{pmatrix}$$

Hint: think geometrically! What does the matrix A do? In (b), what happens if $a = b$?

Solution to the problem 9. .

(a)

$$A = \mathbf{u}\mathbf{v}^\top$$

$$\forall \mathbf{x} \in \mathbb{R}^n: \; A\mathbf{x} = \mathbf{u}\mathbf{v}^\top\mathbf{x} = (\mathbf{v}^\top\mathbf{x})\mathbf{u} \in \mathrm{ls}\{\mathbf{u}\} \;\Rightarrow\; \mathcal{C}(A) = \mathrm{ls}\{\mathbf{u}\}$$

$$A\mathbf{x} = 0 \;\Leftrightarrow\; (\mathbf{v}^\top\mathbf{x})\mathbf{u} = 0 \;\Leftrightarrow\; \mathbf{v}^\top\mathbf{x} = 0 \;\Leftrightarrow\; \mathbf{x} \in \mathrm{ls}\{\mathbf{v}\}^\perp \;\Rightarrow\; \mathcal{N}(A) = \mathrm{ls}\{\mathbf{v}\}^\perp$$

$$\dim \mathcal{N}(A) = n - 1$$

$$\operatorname{rank} A = \dim \mathcal{C}(A) = 1 \;\Rightarrow\; \det A = 0 \text{ and } \lambda_1 = \dots = \lambda_{n-1} = 0$$

$$\{\mathbf{v}_1 \dots \mathbf{v}_{n-1}\} \text{ should be the basis of } \mathrm{ls}\{\mathbf{v}\}^\perp \text{ since } A\mathbf{x} = 0 \;\forall \mathbf{x} \in \mathrm{ls}\{\mathbf{v}\}^\perp$$

$$\lambda_n = \operatorname{tr}(\mathbf{u}\mathbf{v}^\top) - 0 = \mathbf{v}^\top\mathbf{u} \qquad A\mathbf{v}_n = (\mathbf{v}^\top\mathbf{v}_n)\mathbf{u} = (\mathbf{v}^\top\mathbf{u})\mathbf{v}_n \;\Rightarrow\; \mathbf{v}_n = \mathbf{u}$$

Answer: $\lambda_1 = \dots = \lambda_{n-1} = 0, \quad \{\mathbf{v}_1 \dots \mathbf{v}_{n-1}\} = \text{basis } \mathbf{v}^\perp \qquad \lambda_n = \mathbf{v}^\top\mathbf{u}, \quad \mathbf{v}_n = \mathbf{u}$

Also one can notice that $A = \mathbf{u}\mathbf{v}^\top = \mathbf{v}^\top\mathbf{u}\frac{\mathbf{u}\mathbf{v}^\top}{\mathbf{v}^\top\mathbf{u}} = \mathbf{v}^\top\mathbf{u}\; P$, where $P = \frac{\mathbf{u}\mathbf{v}^\top}{\mathbf{v}^\top\mathbf{u}}$ is an oblique projector onto $\mathbf{u}$ parallel to $\mathbf{v}^\perp$. So A also projects every $\mathbf{x}$ onto $\mathbf{u}$ and the multiplies it by $\mathbf{v}^\top\mathbf{u}$

(b) Let $\mathbf{1} = (1 \dots 1)^\top$.

$$A = \begin{pmatrix} a & b & b & \dots & b \\ b & a & b & \dots & b \\ b & b & a & \dots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \dots & a \end{pmatrix} = b \begin{pmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{pmatrix} + (a-b)I = b\;\mathbf{1}\mathbf{1}^\top + (a-b)I$$

If $a = b$: $A = b\;\mathbf{1}\mathbf{1}^\top = nb\;\frac{\mathbf{1}\mathbf{1}^\top}{n} = nbP_{\mathbf{1}}$ and we have a case similar to (a): $\lambda_1 = \dots = \lambda_{n-1} = 0$, $\{\mathbf{v}_1 \dots \mathbf{v}_{n-1}\} = \text{basis } \mathbf{1}^\perp$ $\lambda_n = nb$, $\mathbf{v}_n = \mathbf{1}$. Moreover $P_{\mathbf{1}}$ is orthogonal projector onto $\mathbf{1}$. Now $A = (a-b)I + nbP_{\mathbf{1}}$

$$A\mathbf{x} = (a-b)\mathbf{x} + nbP_{\mathbf{1}}\mathbf{x} = (a-b)\mathbf{x} + nbk\mathbf{1}$$

$$A\mathbf{v} = \lambda\mathbf{v} \qquad (a-b)\mathbf{v} + nbP_{\mathbf{1}}\mathbf{v} = \lambda\mathbf{v}$$

$$\Rightarrow \begin{cases} P_{\mathbf{1}}\mathbf{v} = 0 \\ \mathbf{v} \parallel \mathbf{1} \end{cases} \Rightarrow \left[\begin{array}{l} \begin{cases} \mathbf{v} \perp \mathbf{1} \\ (a-b)\mathbf{v} = \lambda\mathbf{v} \end{cases} \\ \begin{cases} \mathbf{v} \parallel \mathbf{1} \\ (a-b)\mathbf{v} + nb\mathbf{v} = \lambda\mathbf{v} \end{cases} \end{array}\right. \Rightarrow \begin{cases} \mathbf{v}_1, \cdots, \mathbf{v}_{n-1} - \text{basis of } \mathbf{1}^{\perp} \\ \lambda_1 = \cdots = \lambda_{n-1} = a - b \\ \mathbf{v}_n = \mathbf{1} \\ \lambda_n = a + (n-1)b \end{cases}$$

Answer: $\lambda_1 = \cdots = \lambda_{n-1} = a-b, \quad \{\mathbf{v}_1 \ldots \mathbf{v}_{n-1}\} = \text{basis } \mathbf{1}^{\perp} \qquad \lambda_n = a+(n-1)b, \quad \mathbf{v}_n = \mathbf{1}$

Problem 10 (Commuting matrices; 4pt)**.** (a) Assume that $n \times n$ matrices A and B commute, i.e., $AB = BA$, and that $\mathbf{v}$ is an eigenvector of A corresponding to an eigenvalue λ. Is it true that $\mathbf{v}$ is also an eigenvector for B? Justify your answer (ie., prove the claim or disprove it by example; in the latter case suggest extra conditions under which $\mathbf{v}$ is an eigenvector of B)

(b) Assume that A commutes with B and B commutes with C. Is it true that A and C must commute?

Solution to the problem 10. .

(a)

$$AB = BA \quad A\mathbf{v} = \lambda\mathbf{v}$$

If $B\mathbf{v} = 0 \;\Rightarrow\; \mathbf{v}$ is an eigenvector of B corresponding to 0 eigenvalue.
If not, then:

$$AB\mathbf{v} = BA\mathbf{v} = B\lambda\mathbf{v} = \lambda(B\mathbf{v}) = A(B\mathbf{v})$$

and $B\mathbf{v}$ is an eigenvector of A corresponding to λ. If λ **is distinct** then the corresponding eigenspace has dim $= 1 \;\Rightarrow\; B\mathbf{v} = \mu\mathbf{v} \;\Rightarrow\; \mathbf{v}$ is an eigenvector of B (with eigenvalue μ).

So if A is diagonalisable then eigenvecors for A are also eigenvectors for B.

Problem 11 (Symmetric matrices; 5pt)**.** Find a matrix P that orthogonally diagonalizes A, and determine $P^{-1}AP$:

$$(a) \quad \begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix}; \qquad (b) \quad \begin{pmatrix} -2 & 0 & -36 \\ 0 & 3 & 0 \\ -36 & 0 & -23 \end{pmatrix}; \qquad (c) \quad \begin{pmatrix} 1 & -4 & 2 \\ -4 & 1 & -2 \\ 2 & -2 & -2 \end{pmatrix}$$

Solution to the problem 11. .

(a)

$$A = \begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix} \qquad \lambda_1 = 1 \;\; \lambda_2 = 7 - 1 = 6$$

$$\begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}\mathbf{v} = 0 \;\Rightarrow\; \mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \hat{\mathbf{v}}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} -1 & -2 \\ -2 & -4 \end{pmatrix}\mathbf{v} = 0 \;\Rightarrow\; \mathbf{v}_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \qquad \hat{\mathbf{v}}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$P = \begin{pmatrix} \hat{\mathbf{v}}_1 & \hat{\mathbf{v}}_2 \end{pmatrix}$$

Answer: $P = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \qquad P^{-1}AP = \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}$

$$\text{or } P^{-1}AP = P^\top AP = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix}\frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$$

(b)

$$A = \begin{pmatrix} -2 & 0 & -36 \\ 0 & 3 & 0 \\ -36 & 0 & -23 \end{pmatrix} \qquad \lambda_1 = 3 \quad \begin{cases} \lambda_2 + \lambda_3 = -25 \\ \lambda_2\lambda_3 = -1250 \end{cases} \quad \lambda_2 = 25 \quad \lambda_3 = -50$$

$$\begin{pmatrix} -5 & 0 & -36 \\ 0 & 0 & 0 \\ -36 & 0 & -26 \end{pmatrix}\mathbf{v} = 0 \quad \mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \hat{\mathbf{v}}_1$$

$$\begin{pmatrix} -27 & 0 & -36 \\ 0 & -22 & 0 \\ -36 & 0 & -48 \end{pmatrix}\mathbf{v} = 0 \quad \mathbf{v}_2 = \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix} \quad \hat{\mathbf{v}}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{1}{5}\begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix}$$

$$\begin{pmatrix} 48 & 0 & -36 \\ 0 & 53 & 0 \\ -36 & 0 & 27 \end{pmatrix}\mathbf{v} = 0 \quad \mathbf{v}_3 = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} \quad \hat{\mathbf{v}}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \frac{1}{5}\begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$$

$$P = \begin{pmatrix} \hat{\mathbf{v}}_1 & \hat{\mathbf{v}}_2 & \hat{\mathbf{v}}_3 \end{pmatrix}$$

Answer: $P = \frac{1}{5}\begin{pmatrix} 0 & 4 & 3 \\ 5 & 0 & 0 \\ 0 & -3 & 4 \end{pmatrix} \qquad P^{-1}AP = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & -50 \end{pmatrix}$

$$\text{or } P^{-1}AP = P^\top AP = \frac{1}{5}\begin{pmatrix} 0 & 5 & 0 \\ 4 & 0 & -3 \\ 3 & 0 & 4 \end{pmatrix}\begin{pmatrix} -2 & 0 & -36 \\ 0 & 3 & 0 \\ -36 & 0 & -23 \end{pmatrix}\frac{1}{5}\begin{pmatrix} 0 & 4 & 3 \\ 5 & 0 & 0 \\ 0 & -3 & 4 \end{pmatrix}$$

(c)

$$A = \begin{pmatrix} 1 & -4 & 2 \\ -4 & 1 & -2 \\ 2 & -2 & -2 \end{pmatrix} \qquad A + 3I = \begin{pmatrix} 4 & -4 & 2 \\ -4 & 4 & -2 \\ 2 & -2 & 1 \end{pmatrix}$$

$$\operatorname{rank}(A+3I) = 1 \ \Rightarrow \lambda_1 = \lambda_2 = -3 \quad \lambda_3 = 0 + 3 + 3 = 6$$

$$\begin{pmatrix} 4 & -4 & 2 \\ -4 & 4 & -2 \\ 2 & -2 & 1 \end{pmatrix}\mathbf{v} = 0 \ \Rightarrow \ 2v_1 - 2v_2 + v_3 = 0 \ \Rightarrow \ v_3 = 2v_2 - 2v_1 \ \Rightarrow \ \mathbf{v} = v_1\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + v_2\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \ \Rightarrow \ \text{GS} \ \Rightarrow \ \hat{\mathbf{v}}_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \quad \hat{\mathbf{v}}_2 = \frac{1}{3\sqrt{5}}\begin{pmatrix} 4 \\ 5 \\ 2 \end{pmatrix}$$

$$\left(\mathbf{v}_1^\top\mathbf{v}_1 = 5 \quad \mathbf{v}_1^\top\mathbf{v}_2 = -4 \quad \hat{\mathbf{v}}_2 = \frac{\mathbf{v}_2 + \frac{4}{5}\mathbf{v}_1}{\|\cdot\|} = \frac{1}{3\sqrt{5}}\begin{pmatrix} 4 \\ 5 \\ 2 \end{pmatrix} \right)$$

$$\begin{pmatrix} -5 & -4 & 2 \\ -4 & -5 & -2 \\ 2 & -2 & -8 \end{pmatrix} \mathbf{v} = 0 \qquad \begin{cases} -5v_1 - 4v_2 + 2v_3 = 0 \\ -4v_1 - 5v_2 - 2v_3 = 0 \\ v_1 = v_2 + 4v_3 \end{cases} \Rightarrow \; -9v_2 - 18v_3 = 0$$

$$\Rightarrow \begin{cases} v_2 = -2v_3 \\ v_1 = 2v_3 \end{cases} \qquad \mathbf{v}_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \qquad \hat{\mathbf{v}}_3 = \frac{1}{3}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \qquad P = \begin{pmatrix} \hat{\mathbf{v}}_1 & \hat{\mathbf{v}}_2 & \hat{\mathbf{v}}_3 \end{pmatrix}$$

Answer: $P = \frac{1}{15}\begin{pmatrix} 3\sqrt{5} & 4\sqrt{5} & 10 \\ 0 & 5\sqrt{5} & -10 \\ -6\sqrt{5} & 2\sqrt{5} & 5 \end{pmatrix} \qquad P^{-1}AP = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 6 \end{pmatrix}$

$$\text{or } P^{-1}AP = \frac{1}{15}\begin{pmatrix} 3\sqrt{5} & 0 & -6\sqrt{5} \\ 4\sqrt{5} & 5\sqrt{5} & 2\sqrt{5} \\ 10 & -10 & 5 \end{pmatrix} \begin{pmatrix} 1 & -4 & 2 \\ -4 & 1 & -2 \\ 2 & -2 & -2 \end{pmatrix} \frac{1}{15}\begin{pmatrix} 3\sqrt{5} & 4\sqrt{5} & 10 \\ 0 & 5\sqrt{5} & -10 \\ -6\sqrt{5} & 2\sqrt{5} & 5 \end{pmatrix}$$

Problem 12 (Symmetric matrices; 4pt)**.** (a) Find all values of a and b for which there exists a 3×3 symmetric matrix with eigenvalues $\lambda_1 = -1$, $\lambda_2 = 3$, $\lambda_3 = 7$ and the corresponding eigenvectors

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \mathbf{v}_3 = \begin{pmatrix} a \\ b \\ 0 \end{pmatrix}.$$

(b) Reconsider the problem if $\lambda_3 = 3$.

(c) Find all symmetric matrices A satisfying the conditions of parts (a) and (b).

Solution to the problem 12. .

(a) For the symmetric matrix the eigenvectors corresponding to distinct eigenvalues are necessary orthogonal:

$$\langle \mathbf{v}_1, \mathbf{v}_3 \rangle = 0 + b + 0 = 0 \;\Rightarrow\; b = 0 \qquad \langle \mathbf{v}_2, \mathbf{v}_3 \rangle = a + b + 0 = 0 \;\Rightarrow\; a = 0$$

Since eigenvector should be nonzero there is no values of a, b satisfying the condition.
Answer: $(a, b) \in \varnothing$

(b) $\lambda_2 = \lambda_3 = 3$ is a multiple eigenvalue. For the symmetric matrix we can find such P s.t. P is orthogonal and $P^\top AP = diag\{-1, 3, 3\}$. So $\mathbf{v}_3$ only needs to be orthogonal to $\mathbf{v}_1$ (since they correspond to different eigenvalues).

$$\langle \mathbf{v}_1, \mathbf{v}_3 \rangle = 0 + b + 0 = 0 \;\Rightarrow\; b = 0 \;\Rightarrow\; \mathbf{v}_3 = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}.$$

So now we see that $\mathbf{v}_3 \in \text{ls}\{e_1\}$ and $a \neq 0$. Let's check and find A:

$$\hat{\mathbf{v}}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \qquad \hat{\mathbf{v}}_2 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\mathbf{v}_2^\top \mathbf{v}_2 = 3 \quad \mathbf{v}_2^\top \mathbf{v}_3 = a \quad \hat{\mathbf{v}}_3 = \frac{\mathbf{v}_3 - \frac{a}{3}\mathbf{v}_2}{\|\cdot\|} = \frac{a}{3}\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} / \left\| \frac{a}{3}\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \right\| = \frac{\operatorname{sgn} a}{\sqrt{6}}\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

We're not interested in the orientations of vectors, so let $\hat{\mathbf{v}}_3 = \frac{1}{\sqrt{6}}\begin{pmatrix} 2 & -1 & 1 \end{pmatrix}^\top$

$$P = \begin{pmatrix} \hat{\mathbf{v}}_1 & \hat{\mathbf{v}}_2 & \hat{\mathbf{v}}_3 \end{pmatrix} = \frac{1}{\sqrt{6}}\begin{pmatrix} 0 & \sqrt{2} & 2 \\ \sqrt{3} & \sqrt{2} & -1 \\ -\sqrt{3} & \sqrt{2} & 1 \end{pmatrix}$$

$$PDP^{-1} = PDP^\top = \frac{1}{6}\begin{pmatrix} 0 & \sqrt{2} & 2 \\ \sqrt{3} & \sqrt{2} & -1 \\ -\sqrt{3} & \sqrt{2} & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 0 & \sqrt{3} & -\sqrt{3} \\ \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 2 & -1 & 1 \end{pmatrix} =$$

$$= \frac{1}{6}\begin{pmatrix} 0 & 3\sqrt{2} & 6 \\ -\sqrt{3} & 3\sqrt{2} & -3 \\ \sqrt{3} & 3\sqrt{2} & 3 \end{pmatrix}\begin{pmatrix} 0 & \sqrt{3} & -\sqrt{3} \\ \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 2 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 2 \\ 0 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}$$

Answer: $a \neq 0$, $b = 0$

(c) **Answer:** (a): $A \in \varnothing$ (b): $A = \begin{pmatrix} 3 & 0 & 2 \\ 0 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}$

Problem 13 (Symmetric matrices; 4pt)**.** A 3×3 symmetric matrix A has eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = \lambda$ and the corresponding eigenvectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \qquad \mathbf{v}_3 = \begin{pmatrix} a \\ b \\ 0 \end{pmatrix}.$$

(a) Find all possible values of a, b, and λ.

(b) Find the corresponding matrices A.

Solution to the problem 13. .

(a) 1. Let $\lambda \neq 1$ and $\lambda \neq 2$. For the symmetric matrix the eigenvectors corresponding to distinct eigenvalues are necessary orthogonal:

$$\langle \mathbf{v}_1, \mathbf{v}_3 \rangle = a + 0 = 0 \;\Rightarrow\; a = 0 \qquad \langle \mathbf{v}_2, \mathbf{v}_3 \rangle = a + b + 0 = 0 \;\Rightarrow\; b = 0$$

Since eigenvector should be nonzero, $(a, b) \in \varnothing$.
2. Let $\lambda = 1$. $\lambda_1 = \lambda_3 = 1$ is a multiple eigenvalue. So $\mathbf{v}_3$ should be orthogonal to $\mathbf{v}_2$ (since they correspond to different eigenvalues).

$$\langle \mathbf{v}_2, \mathbf{v}_3 \rangle = a + b = 0 \;\Rightarrow\; b = -a \;\Rightarrow\; \mathbf{v}_3 = a\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

Since eigenvector should be nonzero, $a \neq 0$.
3. Let $\lambda = 2$. $\lambda_2 = \lambda_3 = 2$ is a multiple eigenvalue. So $\mathbf{v}_3$ should be orthogonal to $\mathbf{v}_1$.

$$\langle \mathbf{v}_1, \mathbf{v}_3 \rangle = a + 0 = 0 \;\Rightarrow\; a = 0 \;\Rightarrow\; \mathbf{v}_3 = b \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Since eigenvector should be nonzero, $b \neq 0$.
Answer: $(\lambda, a, b) \in \{(1, k, -k) \mid k \neq 0\} \cup \{(2, 0, k) \mid k \neq 0\}$

(b)

$$\hat{\mathbf{v}}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad \hat{\mathbf{v}}_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

For $\lambda = 1$:

$$\mathbf{v}_1^\top \mathbf{v}_1 = 2 \quad \mathbf{v}_1^\top \mathbf{v}_3 = a \quad \hat{\mathbf{v}}_3 = \frac{\mathbf{v}_3 - \frac{a}{2}\mathbf{v}_1}{\|\cdot\|} = \frac{a}{2} \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} / \|\cdot\|$$

Let's set (similarly to Problem 12): $\hat{\mathbf{v}}_3 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & -2 & -1 \end{pmatrix}^\top$

$$P = \begin{pmatrix} \hat{\mathbf{v}}_1 & \hat{\mathbf{v}}_2 & \hat{\mathbf{v}}_3 \end{pmatrix} = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{3} & \sqrt{2} & 1 \\ 0 & \sqrt{2} & -2 \\ \sqrt{3} & -\sqrt{2} & -1 \end{pmatrix}$$

$$PDP^{-1} = PDP^\top = \frac{1}{6} \begin{pmatrix} \sqrt{3} & \sqrt{2} & 1 \\ 0 & \sqrt{2} & -2 \\ \sqrt{3} & -\sqrt{2} & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 & \sqrt{3} \\ \sqrt{2} & \sqrt{2} & -\sqrt{2} \\ 1 & -2 & -1 \end{pmatrix} =$$

$$\frac{1}{6} \begin{pmatrix} \sqrt{3} & 2\sqrt{2} & 1 \\ 0 & 2\sqrt{2} & -2 \\ \sqrt{3} & -2\sqrt{2} & -1 \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 & \sqrt{3} \\ \sqrt{2} & \sqrt{2} & -\sqrt{2} \\ 1 & -2 & -1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 4 & 1 & -1 \\ 1 & 4 & -1 \\ -1 & -1 & 4 \end{pmatrix}$$

Answer: $A = \frac{1}{3} \begin{pmatrix} 4 & 1 & -1 \\ 1 & 4 & -1 \\ -1 & -1 & 4 \end{pmatrix}$

For $\lambda = 2$:

$$\mathbf{v}_2^\top \mathbf{v}_2 = 3 \quad \mathbf{v}_2^\top \mathbf{v}_3 = b \quad \hat{\mathbf{v}}_3 = \frac{\mathbf{v}_3 - \frac{b}{3}\mathbf{v}_2}{\|\cdot\|} = \frac{b}{3} \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \|\cdot\|$$

Let's set (similarly to Problem 12): $\hat{\mathbf{v}}_3 = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 & 2 & 1 \end{pmatrix}^\top$

$$P = \begin{pmatrix} \hat{\mathbf{v}}_1 & \hat{\mathbf{v}}_2 & \hat{\mathbf{v}}_3 \end{pmatrix} = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{3} & \sqrt{2} & -1 \\ 0 & \sqrt{2} & 2 \\ \sqrt{3} & -\sqrt{2} & 1 \end{pmatrix}$$

$$PDP^{-1} = PDP^\top = \frac{1}{6} \begin{pmatrix} \sqrt{3} & \sqrt{2} & -1 \\ 0 & \sqrt{2} & 2 \\ \sqrt{3} & -\sqrt{2} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 & \sqrt{3} \\ \sqrt{2} & \sqrt{2} & -\sqrt{2} \\ -1 & 2 & 1 \end{pmatrix} =$$

$$\frac{1}{6}\begin{pmatrix}\sqrt{3} & 2\sqrt{2} & -2\\ 0 & 2\sqrt{2} & 4\\ \sqrt{3} & -2\sqrt{2} & 2\end{pmatrix}\begin{pmatrix}\sqrt{3} & 0 & \sqrt{3}\\ \sqrt{2} & \sqrt{2} & -\sqrt{2}\\ -1 & 2 & 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}3 & 0 & -1\\ 0 & 4 & 0\\ -1 & 0 & 3\end{pmatrix}$$

Answer: $A = \frac{1}{3}\begin{pmatrix}3 & 0 & -1\\ 0 & 4 & 0\\ -1 & 0 & 3\end{pmatrix}$

Problem 14 (Quadratic forms; 4 pt)**.** Find an orthogonal change of variables that eliminates the cross product terms in the quadratic form Q, and express Q in terms of the new variables:

(a) $Q(x_1, x_2) = 2x_1^2 + 2x_2^2 - 2x_1x_2$;

(b) $Q(x_1, x_2, x_3) = 3x_1^2 + 4x_2^2 + 5x_3^2 + 4x_1x_2 - 4x_2x_3$

Solution to the problem 14. .

(a)

$$Q(*\mathbf{x}) = 2x_1^2 + 2x_2^2 - 2x_1x_2 = \mathbf{x}^\top A\mathbf{x} \qquad A = \begin{pmatrix}2 & -1\\ -1 & 2\end{pmatrix} \qquad \mathbf{x} = \begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

$$\lambda_1 = 1 \quad \lambda_2 = 4 - 1 = 3$$

$$\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\mathbf{v} = 0 \quad \mathbf{v}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}$$

$$\begin{pmatrix}-1 & -1\\ -1 & -1\end{pmatrix}\mathbf{v} = 0 \quad \mathbf{v}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix}$$

$$P = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} \qquad P^\top = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$

$$Q(*\mathbf{x}) = \mathbf{x}^\top A\underline{x} = \mathbf{x}^\top PDP^\top\mathbf{x} = \left|y = P^\top\mathbf{x}\right| = \mathbf{y}^\top D\mathbf{y} = y_1^2 + 3y_2^2 = Q(*\mathbf{y}) \qquad \mathbf{y} = \begin{pmatrix}y_1\\ y_2\end{pmatrix}$$

Answer: The orthogonal change of variables: $P^\top = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$, $\quad Q(*\mathbf{y}) = y_1^2 + 3y_2^2$

(b)

$$Q(*\mathbf{x}) = 3x_1^2 + 4x_2^2 + 5x_3^2 + 4x_1x_2 - 4x_2x_3 = \mathbf{x}^\top A\mathbf{x} \qquad A = \begin{pmatrix}3 & 2 & 0\\ 2 & 4 & -2\\ 0 & -2 & 5\end{pmatrix} \quad \mathbf{x} = \begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}$$

$$\det(A - \lambda I) = -\lambda^3 + \operatorname{tr} A\lambda^2 - \lambda\sum M_2 + \det A = -\lambda^3 + 12\lambda^2 - 39\lambda + 28$$

$$\lambda^3 - 12\lambda^2 + 39\lambda - 28 = (\lambda - 1)(\lambda^2 - 11\lambda + 28) = (\lambda - 1)(\lambda - 4)(\lambda - 7)$$

$$\Rightarrow \lambda_1 = 1 \quad \lambda_2 = 4 \quad \lambda_3 = 7$$

$$\begin{pmatrix}2 & 2 & 0\\ 2 & 3 & -2\\ 0 & -2 & 4\end{pmatrix}\mathbf{v}=0 \quad \mathbf{v}_1=\frac{1}{3}\begin{pmatrix}2\\ -2\\ -1\end{pmatrix}$$

$$\begin{pmatrix}-1 & 2 & 0\\ 2 & 0 & -2\\ 0 & -2 & 1\end{pmatrix}\mathbf{v}=0 \quad \mathbf{v}_2=\frac{1}{3}\begin{pmatrix}2\\ 1\\ 2\end{pmatrix}$$

$$\begin{pmatrix}-4 & 2 & 0\\ 2 & -3 & -2\\ 0 & -2 & -2\end{pmatrix}\mathbf{v}=0 \quad \mathbf{v}_3=\frac{1}{3}\begin{pmatrix}1\\ 2\\ -2\end{pmatrix}$$

$$P=\frac{1}{3}\begin{pmatrix}2 & 2 & 1\\ -2 & 1 & 2\\ -1 & 2 & -2\end{pmatrix} \qquad P^\top=\frac{1}{3}\begin{pmatrix}2 & -2 & -1\\ 2 & 1 & 2\\ 1 & 2 & -2\end{pmatrix}$$

$$Q(*\mathbf{x})=\mathbf{x}^\top A\underline{\mathrm{x}}=\mathbf{x}^\top PDP^\top\mathbf{x}=\left|y=P^\top\mathbf{x}\right|=\mathbf{y}^\top D\mathbf{y}=y_1^2+4y_2^2+7y_3^2=Q(*\mathbf{y}) \qquad \mathbf{y}=\begin{pmatrix}y_1\\ y_2\\ y_3\end{pmatrix}$$

Answer: The orthogonal transformation: $P^\top=\frac{1}{3}\begin{pmatrix}2 & -2 & -1\\ 2 & 1 & 2\\ 1 & 2 & -2\end{pmatrix}$, $\quad Q(*\mathbf{y})=y_1^2+4y_2^2+7y_3$

Problem 15 (Quadratic forms; 2 pt)**.** Find all values of k for which the quadratic form

$$5x_1^2+x_2^2+kx_3^2+4x_1x_2-2x_1x_3-2x_2x_3$$

is positive definite.

Solution to the problem 15. .

$$Q(*\mathbf{x})=5x_1^2+x_2^2+kx_3^2+4x_1x_2-2x_1x_3-2x_2x_3=\mathbf{x}^\top A\mathbf{x} \qquad A=\begin{pmatrix}5 & 2 & -1\\ 2 & 1 & -1\\ -1 & -1 & k\end{pmatrix} \quad \mathbf{x}=\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}$$

$$p_A(\lambda)=-\lambda^3+\operatorname{tr}A\lambda^2-\lambda\sum M_2+\det A=-\lambda^3+(6+k)\lambda^2-(6k-1)\lambda+k-2=$$
$$=-(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-v_3)$$

Q is positive definite $\Leftrightarrow$ A is positive definite $\Leftrightarrow \lambda_i>0\ \forall i\in\overline{1,3}$
Let k = 2, then:

$$p_A(0)=k-2=0 \;\Rightarrow\; \lambda_1=0 \qquad \lambda^2-8\lambda+11=0 \qquad \lambda_{1,2}=4\pm\sqrt{5}>0$$

So all the eigenvalues are non-negative. If we take $k>2$ then the roots will be positive.
Answer: $k>2$

Problem 16 (Quadratic forms; 4 pt)**.** Consider the matrix A and the vector $\mathbf{v}_1$, where

$$A = \begin{pmatrix} 3 & -2 & 1 \\ -2 & 6 & -2 \\ 1 & -2 & 3 \end{pmatrix}, \qquad \mathbf{v}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

(a) Show that $\mathbf{v}_1$ is an eigenvector of A, and find its corresponding eigenvalue. Then find the other two eigenvalues and eigenvectors and orthogonally diagonalize A.

(b) Is the quadratic form $Q(\mathbf{x}) = \mathbf{x}^\top A\mathbf{x}$ positive definite, negative definite or indefinite? Find the principal axes of Q and the corresponding transition matrix.

Solution to the problem 16. .

(a)

$$A = \begin{pmatrix} 3 & -2 & 1 \\ -2 & 6 & -2 \\ 1 & -2 & 3 \end{pmatrix} \quad \mathbf{v}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Let's check that $\exists\lambda : \ (A - \lambda I)\mathbf{v}_1 = 0$

$$\begin{pmatrix} 3-\lambda & -2 & 1 \\ -2 & 6-\lambda & -2 \\ 1 & -2 & 3-\lambda \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = 0 \Leftrightarrow \begin{cases} \lambda - 3 + 1 = 0 \\ 2 - 2 = 0 \\ -1 + 3 - \lambda = 0 \end{cases} \Leftrightarrow \lambda_1 = 2$$

$$\operatorname{rank}(A - 2I) = \operatorname{rank}\begin{pmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{pmatrix} = 1 \Rightarrow \lambda_2 = \lambda_1 = 2 \qquad \lambda_3 = 12 - 2 - 2 = 8$$

$$\begin{pmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{pmatrix} \mathbf{v}_2 = 0 \qquad v_1 = 2v_2 - v_3 \mathbf{v} = v_2 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + v_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow \mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \quad \Rightarrow GS:$$

$$\mathbf{v}_1^\top \mathbf{v}_1 = 2 \qquad \hat{\mathbf{v}}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \qquad \mathbf{v}_1^\top \mathbf{v}_2 = -2 \qquad \hat{\mathbf{v}}_2 = \frac{\mathbf{v}_2 + \mathbf{v}_1}{\|\mathbf{v}_2 + \mathbf{v}_1\|} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$(A - 8I)\mathbf{v} = 0 \Rightarrow \begin{cases} -5v_1 - 2v_2 + v_3 = 0 \\ -2v_1 - 2v_2 - 2v_3 = 0 \\ v_1 = 2v_2 + 5v_3 \end{cases} \qquad \begin{cases} v_2 = -2v_3 \\ v_1 = 2v_2 + 5v_3 = v_3 \end{cases} \qquad \mathbf{v}_3 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$P = \frac{1}{\sqrt{6}} \begin{pmatrix} -\sqrt{3} & \sqrt{2} & 1 \\ 0 & \sqrt{2} & -2 \\ \sqrt{3} & \sqrt{2} & 1 \end{pmatrix} \qquad P^{-1} = P^\top = \frac{1}{\sqrt{6}} \begin{pmatrix} -\sqrt{3} & 0 & \sqrt{3} \\ \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 1 & -2 & 1 \end{pmatrix}$$

$$A = PDP^\top = \frac{1}{\sqrt{6}} \begin{pmatrix} -\sqrt{3} & \sqrt{2} & 1 \\ 0 & \sqrt{2} & -2 \\ \sqrt{3} & \sqrt{2} & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{pmatrix} \frac{1}{\sqrt{6}} \begin{pmatrix} -\sqrt{3} & 0 & \sqrt{3} \\ \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 1 & -2 & 1 \end{pmatrix}$$

(b) $Q(\mathbf{x})$ is positive definite since all eigenvalues of A are positive. Principal axes of Q correspond to the eigenvectors of A: $\frac{1}{\sqrt{6}}\left\{\begin{pmatrix}-\sqrt{3}\\0\\\sqrt{3}\end{pmatrix}, \begin{pmatrix}\sqrt{2}\\\sqrt{2}\\\sqrt{2}\end{pmatrix}, \begin{pmatrix}1\\-2\\1\end{pmatrix}\right\}$

And the transition matrix is $P^\top$ since it transforms a vector coordinates from the canonical basis to the eigenvectors basis:

$$\frac{1}{\sqrt{6}}\begin{pmatrix}-\sqrt{3} & 0 & \sqrt{3}\\ \sqrt{2} & \sqrt{2} & \sqrt{2}\\ 1 & -2 & 1\end{pmatrix}$$