Math behind the project — diagonalization by FFT

Circular convolution can be written into a circulant matrix

$$k * x = Ax$$

Circulant matrix A can be written as FFT diagonalization

$$A = F^{-1} \underbrace{\operatorname{diag}(K)}_{\text{diagonal matrix whose diagonal entries are OTF values } K(u,v) = \mathcal{F}(k)} \underbrace{F}_{\text{DFT matrix}}$$

Look at our optimization objective:

$$f(x) = \frac{1}{2}\|Ax - y\|_2^2 + \frac{\lambda}{2}\|x\|_2^2$$

$$\nabla f(x) = A^T(Ax - y) + \lambda x$$

$$\Rightarrow \nabla f(x) = 0 \Rightarrow A^T(Ax - y) + \lambda \hat{x} = 0$$

$$(A^T A + \lambda I)\hat{x} = A^T y$$

$$A = F^{-1}\operatorname{diag}(K) F \Rightarrow A^T A = (F^{-1}\operatorname{diag}(\bar{K}) F)(F^{-1}\operatorname{diag}(K) F)$$

$$= F^{-1}\operatorname{diag}(|K|^2) F$$

$$\underline{(F^{-1}\operatorname{diag}(|K|^2) F + \lambda I)}\,\hat{x} = F^{-1}\operatorname{diag}(\bar{K}) F y$$

$$\underline{F^{-1}\operatorname{diag}(|K|^2 + \lambda) F}\,\hat{x} = F^{-1}\operatorname{diag}(\bar{K})\, Y$$

$$\operatorname{diag}(|K|^2 + \lambda)\,\hat{X} = \operatorname{diag}(\bar{K})\, Y \quad (X, Y \text{ are in Frequency domain})$$

$$(|K(u,v)|^2 + \lambda) \odot \hat{X} = \bar{K}(u,v) \odot Y(u,v)$$

$$\hat{X}(u,v) = \frac{\bar{K}(u,v) \odot Y(u,v)}{|K(u,v)|^2 + \lambda}$$

Proof:

$$\underline{A^T A}\, v = \mu v$$

$$(\underline{A^T A + \lambda I})\, v = (\mu + \lambda) v$$

$A^T A$ and $A^T A + \lambda I$ share the same set of ~~vector~~ eigenvectors

Eigenvalues shift by λ

If we use gradient descent:

$$\nabla f(x) = (F^{-1}\operatorname{diag}(\bar{K}) F)\,[F^{-1}\operatorname{diag}(K) F x - y]$$

$$= F^{-1}\operatorname{diag}(\bar{K})\,[\operatorname{diag}(K) X - Y].$$

$$= F^{-1}\, \bar{K}(u,v) \odot [K(u,v) \odot X(u,v) - Y(u,v)].$$