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TIME-DEPENDENT RESULTS IN STORAGE THEORY

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CONTENTS

Introduction

The probability theory of storage systems formulated by P. A. P. Moran in 1954 has now developed into an active branch of applied probability. An excellent account of the theory, describing results obtained up to 1958 is contained in Moran's (1959) monograph. Considerable progress has since been

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made in several directions—the study of the time-dependent behaviour of stochastic processes underlying Moran's original model, modifications of this model, as well as the formulation and solution of new models. The aim of this paper is to give an expository account of these developments; a comprehensive treatment will be found in the author's forthcoming book [Prabhu (1964)].

Briefly, Moran's storage model can be described as follows. The amount of water which flows into a dam (the input) will vary from time to time, and will thus have a probability distribution. Apart from a possible overflow, which may occur if the dam is of finite capacity, this water is stored, and released according to a specified rule. The stored water is used for generation of hydro-electric power, and the released water (the output) for irrigation purposes. The central characteristic of the system is the storage function, giving the amount of water stored in the dam at various points of time.

From the above description, it is seen that the dam model is a particular case of the general storage (or inventory) model, in which both input and output are random variables. Such models arise in economics and in business administration, and have been extensively studied during the last few years. A valuable review of these models has been given by Gani (1957), and a systematic study has been made by Arrow, Karlin and Scarf (1958).

Associated with a storage model there is a cost structure of the system, in terms of costs involved in operating the system, deficit costs, and revenues from the amount of material sold; an important problem which arises in the analysis of the system is the so called optimization problem, where it is desired to find an optimum ordering (or release) policy, such that the cost function is a minimum. References to earlier work done on optimization problems in water storage systems will be found in Arrow, Karlin and Scarf, (1958). An elementary treatment of such a problem associated with Moran's model was given by Prabhu (1960 a), while Jarvis (1963) has applied a modification of Moran's model to the Ord River Project in Western Australia. Bather (1962, 1963) has considered optimal regulation policies for storage models which are somewhat more general than Moran's. Avi-Itzhak and Ben-Tuvia (1963) have applied storage theory to the problem of utilizing water by surface reservoirs in Israel.

In the deterministic case where both input and output are known quantities, the storage function can be uniquely determined in advance. The more interesting case, however, is the one in which at least one of these is a random variable and the storage function is therefore a stochastic process. This process depends on the given release policy and the nature of supply and demand, but does not involve the cost structure of the model. An investigation of this process is obviously essential for a detailed analysis of the system. In particular, the stationary distribution of the storage function, which describes the 'long run' behaviour of the system, when it has settled down to statistical equilibrium, is of some interest.

On the other hand, for a deeper understanding of the model, it is necessary to study the time-dependent behaviour of the process.

The present review will not be concerned with optimization problems, nor, except for occasional references, and a single particular case, with stationary solutions; an account of the latter will be found in Moran's (1959) monograph. Here we shall deal mostly with time-dependent solutions for Moran's original model and for various other models which have been recently developed.

This paper is divided into three parts. Part I (Sections 1–3) is concerned with Moran's original discrete time model for a dam. A description of this model is given in Section 1 together with an extension of it due to Lloyd (1963 a), and the time-dependent behaviour of the storage function is discussed in Section 2. For the finite dam (dam with finite capacity) with discrete inputs, Weesakul (1961 a) has obtained a formal solution involving determinants, which have been explicitly evaluated in a particular case. Elegant results can be obtained for the infinite dam by applying the fluctuation theory of random variables. In the special case where unit amounts of water are released (when available) from the infinite dam, explicit results due to Yeo [(1960), (1961 a)] exist. In Section 3 we consider the problem of emptiness in the dam, which was first formulated in continuous time by Kendall (1957); the discrete time version of this problem has been studied by Prabhu (1958 a), Ghosal [(1960 a), (1960 b)], Weesakul (1961 b) and others.

Part II (Sections 4–11) deals with an extension to continuous time of Moran's basic model. The initial attempts in this direction made use of limiting methods [Moran (1956), Downton (1957)], and were sometimes based on heuristic arguments [see discussion in Kendall (1957)]. These methods, however, are cumbersome and obscure the essential features of the basic stochastic processes. Systematic attempts to formulate a continuous time storage theory were made by Kendall (1957), Gani and Prabhu [(1958), (1959 a), (1959 b), (1963)], Gani and Pyke (1960 a), and others. The two important but by no means easy tasks in this connection are (a) the specification of the input process, and (b) the formulation of the storage function in continuous time; these are described respectively in Sections 4 and 5. The most significant result here is the one due to Gani and Pyke (1960 a) which expresses the storage function as the supremum of an infinitely divisible sochastic process. From this, using a result due to Baxter and Donsker (1957), the double transform of the distribution function of the storage function can be obtained (Section 6).

Section 7 is concerned with the wet period in a dam—the time the dam with a given initial content will take to dry up. Elegant results have been obtained by Kendall (1957) for the wet period in an infinite dam. For the finite dam Phatarfod (1963) has shown how useful results can be obtained by applying the extension of Wald's fundamental identity of sequential analysis due to Dvoretzky, Kiefer and Wolfowitz (1953).

In Section 8 we consider a dam subject to simple Poisson inputs, for which the stationary solution (in the finite case) had been obtained by Gani (1955), and an equivalent result (for an apparently different problem) by Moran (1955). Later, time-dependent solutions were obtained by Gani and Prabhu (1959 b) for the infinite dam.

Analogies between storage and queueing models have been pointed out by several authors [see, in particular, Gani and Prabhu (1957), Prabhu (1960 b) and Langbein (1958), (1961)]. In discrete time, it was observed that the analogy rested on the mathematical formalisms of the models rather than their physical details. However, in continuous time there exists an exact analogy, as was pointed out by Prabhu (1960 b) with reference to a single server queue with Poisson arrivals. In Section 9 we give Prabhu's analysis of the waiting time in this queueing system using methods of storage theory. These methods were later extended by Gani (1962 a) to a dam model with non-homogeneous Poisson inputs; these results are described in Section 10. We remark here that storage methods can also be applied to the ruin problem of collective risk theory [see Prabhu (1961)].

In Section 11 we consider inputs of continuous type. The transition distribution function of the storage function was explicitly obtained in this case by Gani and Prabhu (1963) by solving the integro-differential equation of the process. The limiting behaviour of the process was also studied.

Finally, in Part III (Sections 12–15) we consider some generalizations of Moran's model. In Section 12 we describe a model due to Gani and Pyke (1960 a) in which the output consists of a random as well as a deterministic component. Section 13 deals with a dam with ordered inputs. Time-dependent solutions for such a dam with discrete inputs were obtained by Yeo (1961 b), while Gani [(1961), (1962 b)] has studied the case of the dam in continuous time fed by two Poisson type inputs. The problem of emptiness for this second model is found to be equivalent to that of a system of two dams in parallel. In Section 14 we give an account of storage models with random linear inputs and outputs recently studied by Miller (1963). Lastly in Section 15, we evaluate the present position of the field, mention some unsolved problems and give further possible directions for research. In the list of references we have included papers not cited in the text, with a view to making it a reasonably up to date bibliography on storage theory.

I. The Dam Process in Discrete Time

1. Moran's model for the dam and an extension to it

In the basic storage model considered by Moran (1954), the content Z_n of a dam of finite capacity k, is defined at discrete times $n = 0, 1, 2, \dots$ by the recurrence relations

$$(1.1) \qquad Z_{n+1} = \min\{k, Z_n + X_n\} - \min\{m, Z_n + X_n\} \qquad (0 < m < k).$$

Here (a) X_n denotes the amount of water which has flowed into the dam during the time interval $(n, n+1)$ (say, the nth year), and it is assumed that $X_0, X_1, X_2, \dots$

are mutually independent and identically distributed random variables; (b) the term $\min\{k, Z_n + X_n\}$ represents a possible overflow at time $n+1$, which occurs if and only if $Z_n + X_n > k$, the content of the dam being k after the overflow; and (c) the term $\min\{m, Z_n + X_n\}$ indicates a release policy of "meeting the demand if physically possible", according to which, at time $n+1$, an amount m of water is released, unless the dam contains less than m, in which case the entire available amount is released.

The assumption of serial independence of inputs has been made mainly to simplify the model, but is difficult to justify in practice. Thus, for instance, if the inputs depend on rainfall, they may be subject to long term trends and periodic patterns. The validity of an independence hypothesis was examined by Bhat and Gani (1959) for the annual discharge of water of 17 Australian rivers; they found evidence of serial correlation, slight in some cases, strong in others, but nevertheless state that the assumption of independence is not entirely unrealistic, and provides a useful first approximation in the formulation of the model. We later consider a model of Lloyd (1963 a) which takes account of the possible dependence of consecutive inputs.

The demand for water is deterministic in the sense that a fixed amount m of water is required every time, and is released if available. More realistic release rules have been suggested by Holdaway [see Moran (1955), Ghosal (1959), (1960 a)]; however, the stochastic processes arising from such a model are more complicated than in the present case.

From the assumptions (a), (b) and (c), it follows that the sequence $\{Z_n, n = 0, 1, \ldots\}$ of random variables defined by (1.1) forms a time-homogeneous Markov chain. In the case where the inputs X_n have a discrete probability distribution, with

$$\Pr\{X_n = j\} = g_j \quad (j = 0, 1, \ldots), \tag{1.2}$$

this chain has a finite number, $k-m$, of states, and the matrix $P = (P_{ij})$ of its transition probabilities

$$P_{ij} = \Pr\{Z_{n+1} = j \mid Z_0 = i\} \quad (0 \leqq 1,\ j \leqq k - m) \tag{1.3}$$

is given by

$$P = \begin{array}{c|cccccccc}
\nearrow & 0 & 1 & 2 & \cdot & \cdot & \cdot & k-m-1 & k-m \\
\hline
0 & G_m & g_{m+1} & g_{m+2} & \cdot & \cdot & \cdot & g_{k-1} & h_k \\
1 & G_{m-1} & g_m & g_{m-1} & \cdot & \cdot & \cdot & g_{k-2} & h_{k-1} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
m & G_0 & g_1 & g_2 & \cdot & \cdot & \cdot & g_{k-m-1} & h_{k-m} \\
m+1 & 0 & g_0 & g_1 & \cdot & \cdot & \cdot & g_{k-m-2} & h_{k-m-1} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
k-m & 0 & 0 & 0 & \cdot & \cdot & \cdot & g_{m-1} & h_m,
\end{array} \tag{1.4}$$

where $G_i = \sum_0^i g_j$, $h_i = \sum_i^\infty g_i$ $(i \geqq 0)$ and it is assumed that $m < \frac{1}{2}k$. If the X_n have a continuous distribution with

$$G(x) = \Pr\{X_n \leqq x\} \quad (0 \leqq x < \infty) \tag{1.5}$$

then the Markov chain $\{Z_n\}$ has a continuous infinity of states belonging to the closed interval $[0, k-m)$; its transition distribution function (d.f.) is given by

$$P(x; y) = \Pr\{Z_{n+1} \leqq y \mid Z_n = x\}$$

$$= \begin{cases} G(x-x) & \text{if } 0 \leqq x \leqq m,\ y = 0 \\ G(m+y-x) & \text{if } \max(0, x-m) \leqq y < k-m \\ 1 - G(k-x) & \text{if } 0 \leqq x \leqq k-m,\ y = k-m, \end{cases} \tag{1.6}$$

$P(x; y)$ being zero for all other combinations of x and y.

Earlier work on this subject was almost entirely devoted to the study of the limiting behaviour of Z_n as $n \to \infty$ (when the process settles down to a statistical equilibrium). An account of various stationary solutions thus obtained may be found in Moran's (1959) monograph; we also quote the papers by Prabhu (1958 a) and Ghosal (1962). Here we remark that the existence of the limiting random variable $Z^* = \lim_{n\to\infty} Z_n$ can be established by a reasoning similar to the one used by Loynes (1962) for more general models.

We conclude this section by reviewing briefly some recent results due to Lloyd (1963 a) for dams with serially correlated inputs. He considers a model in which the inputs X_n are bounded, and form a Markov chain. Thus $X_n \leqq l$ with probability one for some $l > 0$, and the transition probabilities of the Markov chain $\{X_n\}$ are given by

$$g_{ij} = \Pr\{X_{n+1} = j \mid X_n = i\} \qquad (i, j = 0, 1, \ldots, l). \tag{1.7}$$

Further, the inputs are assumed to have a stationary distribution $\{\pi_j\}$, where the π_j satisfy the equations

$$\pi_j = \sum_{i=0}^{l} \pi_i g_{ij} \, (j = 0, 1, \ldots, l), \qquad \pi_0 + \pi_1 + \ldots + \pi_l = 1. \tag{1.8}$$

It is clear that here (as in Moran's model) Z_{n+1} depends on Z_n, and moreover, that X_{n+1} depends on X_n. The process is therefore determined by the variables (Z_n, X_n) and is thus a bivariate Markov chain; let its transition probabilities be denoted by

$$P(j, j' \mid i, i') = \Pr\{Z_{n+1} = j, X_{n+1} = j' \mid Z_n = i, X_n = i'\}. \tag{1.9}$$

It is found that

$$P(j, j' \mid i, i') = g_{i'j'} \Pr\{Z_{n+1} = j \mid Z_n = i, X_n = i'\}, \tag{1.10}$$

further specification of this probability depending on the release policy. If, as in Moran's model, m units of water are released whenever available, then it follows from (1.1) that $\Pr\{Z_{n+1} = j \mid Z_n = i,\ X_n = i'\} = \delta_{jw}$, where $\delta_{jw} = 1$ if $j = w$ and $= 0$ if $j \neq w$, and where

$$w = \begin{cases} 0 & \text{if } \ i + i' \leqq m \\ i + i' - m & \text{if } \ m < i + i' < k \\ k - m & \text{if } \ i + i' \geqq k. \end{cases} \tag{1.11}$$

The unconditional probabilities

$$u_n(j,j') = \Pr\{Z_n = j \ X_n = j'\} \tag{1.12}$$

satisfy the recurrence relations

$$u_{n+1}(j,j') = \sum_{i,i'} u_n(i,i')P(j,j' \mid i,i'). \tag{1.13}$$

It is intuitively evident that the limiting distribution $\{u(j,j')\}$ exists and from (1.13) it will be found that this satisfies the equations

$$u(j,j') = \sum_{i,i'} u(i,i')\, P(j,j' \mid i,i') \qquad (j,j' = 0,1,\dots,l), \tag{1.14}$$

together with $\sum u(j,j') = 1$. Lloyd obtains an explicit solution to (1.14) in the particular case where the inputs take the values $m - 1, m, m + 1$ with the probabilities

$$\Pr\{X_n = m, m-1, m+1\} = (q, r, p) \qquad (q + r + p = 1), \tag{1.15}$$

and the matrix of transition probabilities (1.7) is given by

$$\begin{array}{c|ccc} \nearrow & m-1 & m & m+1 \\ \hline m-1 & 1-2b & b & b \\ m & a & 1-2a & a \\ m+1 & c & c & 1-2c \end{array} \tag{1.16}$$

$$(0 < \min(a,b,c),\ \max(a,b,c) < \tfrac{1}{2}).$$

The probability distribution (1.15) is stationary if we choose a, b, c so that $a = \theta/r$, $b = \theta/q$, $c = \theta/p$, for some $\theta > 0$; the significance of this parameter θ lies in the fact that the serial correlation coefficient (with lag one) of the inputs is given by

$$\rho(X_n, X_{n-1}) = 1 - 6\theta/\{p + q - (p - q)^2\}. \tag{1.17}$$

After some straightforward calculations, the joint distribution of (Z_n, X_n) in statistical equilibrium is obtained; from this it is found that the dam content Z^* has the quasi-geometric distribution given by

$$\begin{aligned}
\Pr\{Z^* = 0\} &= d(2a+b)/3ab,\\
\Pr\{Z^* = 1\} &= d[4a+b+c-3(ab+bc+ca)]/a(2-3b),\\
\Pr\{Z^* = j\} &= g^{j-1}\Pr\{Z^* = 1\} \quad (j = 2,3,\dots,k-m-1)\\
\Pr\{Z^* = k-m\} &= d(2a+c)g^{k-m-1}/3ac,
\end{aligned}$$

where $g = (2-3c)/(2-3b)$, and d is the normalizing constant.

2. Time-dependent solutions for the discrete dam

2.1. *The finite dam.* For a detailed analysis of the model formulated in Section 1, it is obviously necessary to investigate the transient behaviour of the Markov chain $\{Z_n\}$. Weesakul (1961 a) has studied the case of the finite dam with discrete inputs and release $m = 1$. Let

$$P_{ij}^{(n)} = \Pr\{Z_n = j \mid Z_0 = i\} \qquad (i,j = 0,1,\dots,k-1,\ n \geqq 1) \tag{2.1}$$

be the transition probabilities of the Markov chain $\{Z_n\}$; also let $P_{ij}^{(0)} = \delta_{ij}$, where $\delta_{ij} = 1$ or 0 according as $i = j$ or $i \neq j$. It is known that the $P_{ij}^{(n)}$ are the elements of the matrix P^n , where P is the transition probability matrix (1.4) Writing $(P_{ij}^{(n)}) = P{\cdot}P{\cdot}^{-2}\,P$ we see that

$$P_{ij}^{(n)} = Q_i P^{n-2} R_j \quad (n \geqq 2), \tag{2.2}$$

where Q_i is the ith row vector and R_j the jth column vector of P. Now let us define for some suitable θ the transform

$$G_{ij}(\theta) = \sum_{n=2}^{\infty} P_{ij}^{(n)}\theta^n; \tag{2.3}$$

then from (2.2) we obtain

$$G_{ij}(\theta) = \theta^2 Q_i (I - \theta P)^{-1} R_j, \tag{2.4}$$

where $I = (\delta_{ij})$ is the identity matrix of order k, and the existence of the inverse matrix $(I-\theta P)^{-1}$ is ensured if we take, for instance, θ such that $\max_j |\theta g_j| < 1$. Consider, in particular, the case where the input has the geometric distribution

$$g_j = \Pr\{X_n = j\} = ab^j \quad (0 < a < 1,\ b = 1-a,\ j = 0,1,\dots); \tag{2.5}$$

the matrix P is here given by

(2.6)
$$P = \begin{array}{c|cccccc}
\nearrow & 0 & 1 & 2 & \cdots & k-2 & k-1 \\
\hline
0 & a+ab & ab^2 & ab^3 & \cdots & ab^{k-1} & b^k \\
1 & a & ab & ab^2 & \cdots & ab^{k-2} & b^{k-1} \\
2 & 0 & a & ab & \cdots & ab^{k-3} & b^{k-2} \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
k-1 & 0 & 0 & 0 & \cdots & a & b
\end{array}$$

The result (2.4) can be written in this case as

(2.7
$$G_{ij}(\theta) = \frac{\theta^2 \sum_{\nu=0}^{j+1} ab^{j-\nu+1} A_\nu}{|I - \theta p|},$$

where A_ν is the determinant of the matrix $(I - \theta P)$ with the νth row replaced by Q_i. Weesakul evaluates $G_{ij}(\theta)$ as from (2.7); for $i = 1$ it is found that

(2.8)
$$G_{1j}(\theta) = \frac{(a\theta)^2 b^{j+1}}{1-\theta} \sum_{\nu=1}^{j+2} U_\nu(\theta)$$

where

(2.9)
$$U_1(\theta) = \frac{\lambda_1^{k-1} - \lambda_2^{k-1} - b\theta(\lambda_1^{k-2} - \lambda_2^{k-2})}{\lambda_1^{k+1} - \lambda_2^{k+1}}$$

$$U_\nu(\theta) = b^2(1 - a\theta)\frac{\lambda_1^{k-\nu} - \lambda_2^{k-\nu} - b\theta(\lambda_1^{k-\nu-1} - \lambda_2^{k-\nu-1})}{\lambda_1^{k+1} - \lambda_2^{k+1}}$$
$$(1 < \nu \leqq j+2),$$

(2.10)
$$\lambda_1 = \frac{1 + \sqrt{1 - 4ab\theta}}{2}, \qquad \lambda_2 = \frac{1 - \sqrt{1 - 4ab\theta}}{2}.$$

He also determines the explicit expressions for $P_{ij}^{(n)}$; these, however, are rather complicated and will not be quoted here.

2.2. *The infinite dam: a special case.* For the case where the capacity $k \to \infty$, but the release is still $m = 1$, Yeo has obtained the transition probabilities of $\{Z_n\}$, firstly (1960) for the special case where the input distribution is geometric, and later (1961 a) for the general case where the inputs are of discrete additive type. The transition probability matrix in the latter case is given by

$$(2.11) \qquad P \equiv \begin{array}{c|ccccccc} \nearrow & 0 & 1 & 2 & . & . & . \\ \hline 0 & k_0+k_1 & k_2 & k_3 & . & . & . \\ 1 & k_0 & k_1 & k_2 & . & . & . \\ 2 & 0 & k_0 & k_1 & . & . & . \\ 3 & 0 & 0 & k_0 & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{array}$$

where $k_j = \Pr\{X_n = j\}$ $(j = 0,1,\ldots)$. Let $k_j^{(n)} = \Pr\{X_0 + X_1 + \ldots + X_{n-1} = j\}$, $k_j^{(1)} = k_j$, and $K_j^{(n)} = k_0^{(n)} + k_1^{(n)} + \ldots + k_j^{(n)}$ $(j = 0,1,\ldots,n = 1,2,\ldots)$; since the inputs are assumed to be additive, the probability distribution $\{k_j^{(n)}\}$ is the n-fold convolution of $\{k_j\}$ with itself. From the well known relations $P_{ij}^{(n+1)} = \sum_\nu P_{i\nu}^{(n)} P_{\nu j}$ we obtain for $n \geqq 0$,

$$(2.12) \qquad \begin{aligned} P_{i0}^{(n+1)} &= (k_0 + k_1)P_{i0}^{(n)} + k_0 P_{i1}^{(n)} \\ P_{ij}^{(n+1)} &= \sum_{\nu=0}^{j+1} k_{j-\nu+1} P_{i\nu}^{(n)} \quad (j \geqq 1). \end{aligned}$$

Let $K(z) = \sum_0^\infty k_j z^n$ $(|z| < 1)$ be the probability generating function (p.g.f.) of X_n and

$$(2.13) \qquad \phi_n(z) = \sum_{j=0}^{\infty} P_{ij}^{(n)} z^j \qquad (|z| < 1, n = 0,1,\ldots)$$

the p.g.f. of Z_n; $\phi_0(z) = z^i$. Multiplying the equations (2.12) by $1, z, z^2, \ldots$ successively and adding, we obtain the difference equation

$$(2.14) \qquad \phi_{n+1}(z) - \frac{K(z)}{z}\phi_n(z) = -k_0 \frac{1-z}{z} P_{i0}^{(n)}.$$

Now, let

$$(2.15) \qquad \Phi(z,\theta) = \sum_{n=0}^{\infty} \phi_n(z)\theta^n \quad (|\theta| < 1)$$

be the transform of $\phi_n(z)$ with respect to n; then (2.14) yields the result

$$(2.16) \qquad \Phi(z,\theta) = \left\{1 - \frac{\theta K(z)}{z}\right\}^{-1} \left\{z^i - k_0 \frac{\theta}{z}(1-z)\Phi(0,\theta)\right\},$$

where $\Phi(0,\theta) = \sum_{n=0}^{\infty} P_{i0}^{(n)}\theta^n$ is the transform of the probability $P_{i0}^{(n)}$ that the dam is empty at time n. Expanding the right hand side of (2.16) in powers of $K(z)/z$, and noting that $[K(z)]^n$ is the p.g.f. of $\{k_j^{(n)}\}$, we obtain

(2.17) $$\Pr\{Z_n \leqq j \mid Z_0 = i\} = K^{(n)}_{n+j+i} - k_0 \sum_{m=2}^{n-i} P^{(n-m)}_{i0} k^{(m-1)}_{m+j}.$$

It will be observed that this last expression contains the probability $P^{(n)}_{i0}$; by using a combinatorial method (see Section 8) due to Gani and Prabhu (1959 b), Yeo obtains this as

(2.18) $$P^{(n)}_{i0} = \Pr\{Z_n = 0 \mid Z_0 = i\} = (k_0)^{-1} \sum_{v=i+1}^{n+1} \frac{v}{n+1} k^{(n+1)}_{n+1-v}.$$

The results (2.17) and (2.18) can also be obtained by the combinatorial methods developed by Prabhu and Bhat (1963); see also Takács (1964).

2.3. *The infinite dam: the general case.* In the general case where $m \geqq 1$, and the inputs may be discrete or continuous, elegant results can be obtained for the infinite dam. Let us note that the recurrence relation (1.1) now reduces to

$$Z_{n+1} = \max(0, Z_n + X_n - m);$$

solving this successively for $n = 0, 1, 2, \dots$ we obtain

(2.19) $$Z_n = \max\{S_n - S_r \, (0 \leqq r \leqq n), Z_0 + S_n\},$$

where $S_n = Y_0 + Y_1 + \dots + Y_{n-1}$ $(n \geqq 1)$, $S_0 = 0$, and $Y_n = X_n - m$, so that Y_n is the net input (input minus the amount demanded for consumption) during the interval $(n, n+1)$. Now since the random variables Y_n are independently and identically distributed, $S_n - S_r$ has the same distribution as S_{n-r}, and therefore the distribution of Z_n is the same as that of the random variable

(2.20) $$\max\{S_r \, (0 \leqq r \leqq n), Z_0 + S_n\}.$$

It may be noted, however, that this last result holds under the weaker assumption that $Y_1, Y_2, \dots, Y_n$ are reversible, i.e., $(Y_1, Y_2, \dots, Y_n)$ and $(Y_n, Y_{n-1}, \dots, Y_1)$ have the same joint distribution [Gani and Pyke (1960 a)]. Consider the special case where $Z_0 = 0$; (2.20) is then the maximal term among the partial sums $(S_0, S_1, \dots, S_n)$. The study of such a maximal term and other related quantities has been the subject of some highly stimulating research during the last few years [see Spitzer (1956), which also contains further references on the subject]. We find that

(2.21) $$\sum_0^\infty t^n E(e^{-\theta Z_n}) = \exp\left\{\sum_1^\infty \frac{t^n}{n} G_n(nm) + \sum_1^\infty \frac{(te^{\theta m})^n}{n} \int_{nm}^\infty e^{-\theta x} dG_n(x)\right.$$
$$(|t| < 1, \operatorname{Re}(\theta) > 0),$$

where $G_n(x) = \ = \Pr\{S_n + nm \leqq x\}$ is the n-fold convolution of $G(x) = G_1(x)$ with itself $(n \geqq 1)$. Further, let $B \equiv \sum_1^\infty n^{-1} \Pr\{S_n > 0\}$. Then as $n \to \infty$, $Z_n \to \infty$ or

$Z_n \to Z^* < \infty$, with probability one, according as the series B diverges or converges; in the case $B<\infty$, we have

$$(2.22) \qquad E(e^{-\theta Z^*}) = \exp\left\{ -\sum_1^\infty \frac{1}{n} \int_{0-}^{\infty} (1-e^{-\theta x})\, dG_n(x+nm) \right\} (\mathrm{Re}(\theta) > 0).$$

In the present situation it is reasonable to assume that the inputs X_n have a finite mean, and ignore the trivial case where $\mathrm{P}\{X_n = 0\} = 1$; then it will be found that

$$(2.23) \qquad B = \infty \text{ if } E(X_n) \geqq m; \quad B < \infty \text{ if } E(X_n) < m.$$

The results (2.21) $-$(2.23) are due to Spitzer (1956). As an application, let us consider the case where the inputs have the gamma distribution

$$(2.24) \qquad dG(x) = \Pr\{x < X_n < x+dx\} = \frac{\lambda^p}{(p-1)!} e^{-\lambda x} x^{p-1} dx \quad (0 < x < \infty),$$

where $\lambda > 0$ and p is a positive integer. Following Kemperman [(1961), pp.71–76], we can simplify the right hand side expressions in (2.21) and (2.22); thus

$$(2.25) \qquad \sum_0^\infty t^n E(e^{-\theta Z_n}) = \frac{1}{1-t}\left(\prod_{r=1}^{p} \frac{1+\theta/\lambda}{1+\theta/\theta_r}\right)$$

where $\theta_1, \theta_2, \dots, \theta_p$ are the roots of the equation $(\lambda - \theta)^p = \lambda^p t e^{-\theta m}$ such that $\mathrm{Re}(\theta_r) > 0$. Further, if $p < \lambda m$, then

$$(2.26) \qquad E(e^{-\theta Z^*}) = \prod_{r=1}^{p} \left(\frac{1+\theta/\lambda}{1+\theta/\theta_r^0}\right)$$

where $\theta_1^0, \theta_2^0, \dots, \theta_p^0$ are the roots of the equation $(\lambda - \theta)^p = \lambda^p e^{-\theta m}$, with $\mathrm{Re}(\theta_r^0) > 0$. From (2.26) we find that the limiting d.f. can be expressed as a weighted sum of negative exponential terms [cf. Lindley (1952)].

The results (2.21) and (2.22) assume simpler forms when the X_n have a discrete distribution. Again, following Kemperman [(1961), pp. 71-76], we find that

$$(2.27) \qquad \sum_0^\infty t^n E(\theta^{Z_n}) = \frac{1}{\theta^m - tK(\theta)} \prod_{r=1}^{m} \left(\frac{\theta - \xi}{1-\xi_r}\right) \qquad (|t| < 1, |\theta| \leqq 1),$$

where $K(\theta) = E(\theta^{X_n})$ is the probability generating function (p.g.f.) of X_n, and $\xi_1, \xi_2, \dots, \xi_m$ are the roots of the functional equation $\xi^m = tK(\xi)$ such that $|\xi_r| < 1$; moreover, if $E(X_n) = m_1 < m$, then the limiting distribution of Z_n has the p.g.f. given by

$$(2.28) \qquad U(\theta) = E(\theta^{Z^*}) = \frac{(m-m_1)(1-\theta)}{K(\theta) - \theta^m} \prod_{r=1}^{m-1} \left(\frac{\theta - \zeta_r}{1-\zeta_r}\right)$$

where $\zeta_1, \zeta_2, \dots, \zeta_{m-1}$ are the roots of the equation $\zeta^m = K(\zeta)$ within the unit circle.

3. The problem of emptiness

3.1. *Emptiness in a finite dam.* Suppose that at time $t=0$ the dam contains an amount $Z_0 > 0$ of water, and let T be the first subsequent time at which it becomes empty; T is called the 'wet period' in the dam, and the problem of finding its distribution is of some practical importance in storage theory. This problem was first formulated by Kendall (1957) for a continuous time (see Section 7). In discrete time the random variable T is defined by

$$T = \min\{n \mid Z_n = 0\}\,; \tag{3.1}$$

clearly, T is the first passage time of the Markov chain $\{Z_n\}$ from a given initial state to state 0. Weesakul (1961 b) has investigated the case where the inputs are discrete and the release $m=1$. Let $T=T_i$ be the wet period initiated by a dam content $Z_0 = i$ $(1 \leqq i \leqq k-1)$; the probability distribution of T_i is given by

$$\begin{aligned} f_{i0}^{(n)} &= \Pr\{T_i = n\} \\ &= \Pr\{Z_r > 0\ (r=1,2,\dots,n-1);\ Z_n = 0 \mid Z_0 = i\}\ \ (n \geqq 1). \end{aligned} \tag{3.2}$$

The relations $f_{i0}^{(n)} = \sum_{v=1}^{k-1} P_{iv} f_{v0}^{(n-1)} (n \geqq 2), f_{i0}^{(1)} = P_{i0}$ give

$$f_{i0}^{(1)} = g_0 \delta_{i1}, f_{i0}^{(n)} = \sum_{v=1}^{k-2} g_{v-i+1} f_{v0}^{(n-1)} + h_{k-i} f_{k-1,0}^{(n-1)}\ \ (n \geqq 2). \tag{3.3}$$

Now let $\phi^{(n)}$ be the column vector with the elements $(f_{10}^{(n)}, f_{20}^{(n)}, \dots, f_{k-1,0}^{(n)})$ $(n \geqq 1)$, and Q the matrix

$$Q = \begin{bmatrix} g_1 & g_2 & \cdot & \cdot & \cdot & g_{k-2} & h_{k-1} \\ g_0 & g_1 & \cdot & \cdot & \cdot & g_{k-3} & h_{k-2} \\ 0 & g_0 & \cdot & \cdot & \cdot & g_{k-4} & h_{k-3} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & g_0 & h_1 \end{bmatrix}. \tag{3.4}$$

Then the equations (3.3) can be written as $\phi^{(n)} = Q\phi^{(n-1)}$ $(n \geqq 2)$; solving this successively for $n = 2,3,\dots,$ we obtain $\phi^{(n)} \cdot Q^{n-1}\phi^{(1)} = Q \cdot Q^{n-2} \cdot \phi^{(1)}$. Hence we find that

$$f_{i0}^{(n)} = \gamma_i Q^{n-2} \phi^{(1)} \qquad (n \geqq 2) \tag{3.5}$$

where γ_i is the ith row vector of Q. If we now define the p.g.f. of T_i as

$$F_i(\theta) = \sum_{n=1}^{\infty} f_{i0}^{(n)} \theta^n$$

for some suitable θ, then (3.5) yields the result

$$F_i(z) = \theta g_0 \delta_{i1} + \theta^2 \gamma_i (I - \theta Q)^{-1} \phi^{(1)} \tag{3.7}$$

where the existence of the inverse matrix $(I-\theta Q)^{-1}$ is ensured if we take, for instance, z so that $\max |zg_j| < 1$. Note that the vector $\phi^{(1)} = (g_0, 0, 0, \ldots, 0)'$, and therefore the result (3.7) can be simplified as

$$F_i(z) = \theta g_0 \delta_{i1} + \theta^2 g_0 \frac{|E|}{|I - zQ|}, \tag{3.8}$$

where E is the matrix $(I-\theta Q)$ with the first row replaced by γ_i. It is known from the theory of finite Markov chains that $\Pr\{T_i < \infty\} = 1$, so that the dam with an initial content i will eventually dry up. This can also be verified by putting $\theta = 1$ in (3.8); further, the mean duration of a wet period can also be obtained from (3.8). As an application, let us consider the input having a geometric distribution (2.5). In this case Weesakul evaluates the two determinants on the right hand side of (3.8) and obtains

$$F_i(\theta) = (a\theta)^i \, \frac{(\lambda_1^{k-i} - \lambda_2^{k-i}) - b\theta(\lambda_1^{k-i-1} - \lambda_2^{k-i-1})}{(\lambda_1^k - \lambda_2^k - b\theta(\lambda_1^{k-1} - \lambda_2^{k-1})}, \tag{3.9}$$

λ_1, λ_2 being given by (2.10). Explicit expressions for the probabilities $f_{i0}^{(n)}$ can also be obtained from (3.9).

During a wet period as defined by (3.1), it is possible that the dam overflows. If an overflow is not allowed, then the appropriate definition of the wet period is

$$T_i = \min\{n \,|\, Z_n + X_n \leqq 1 \text{ or } Z_n + X_n > k\} \tag{3.10}$$

since emptiness occurs only when $Z_n + X_n \leqq 1$ and overflow when $Z_n + X_n > k$. For the probability distribution of T_i we have

$$\begin{aligned} &\Pr\{T_i = n, Z_n = 0\} \\ &= \Pr\{1 < Z_r + X_r \leqq k (r = 0, 1, \ldots, n-2); Z_{n-1} + X_{n-1} \leqq 1 \,|\, Z_0 = i\} (n \geqq 1). \end{aligned} \tag{3.11}$$

The p.g.f. of the wet period in this case is given by a formula of the type (3.7), but where now the last column in Q is to be replaced by $(g_{k-1}, g_{k-2}, \ldots, g_1)$. This result is also due to Weesakul, who applies it to the geometric distribution (2.5) and obtains an expression similar to (3.9). In the general case, from the p.g.f. of the wet period one can obtain the probability

$$V_i = \Pr\{T_i < \infty, Z_{T_i} = o\} \qquad (1 \leqq i \leqq k-1) \tag{3.12}$$

that the dam ever dries up before overflowing; it is seen (and intuitively obvious) that emptiness is no longer a certain event. Note that the dam process can be considered as a random walk between two impenetrable barriers at 0 and $k-1$, and V_i is the probability of absorption at 0 (before reaching the barrier at $k-1$); a systematic method for evaluating the V_i explicitly was given by Prabhu (1958 a). Thus in the case of the geometric input (2.5) he finds that

$$V_i = \begin{cases} \dfrac{1-\rho^{k+1-i}}{1-\rho^{k+1}} & \text{if } \rho \neq 1 \\[2ex] 1 - \dfrac{i}{k+1} & \text{if } \rho = 1 \end{cases} \tag{3.13}$$

where $\rho = ba^{-1}$ is the mean output. In the general case lower and upper bounds for V_i can be obtained.

Let us now consider the case where the inputs are continuous and have the d.f. (1.5). Let the initial content of the dam be $Z_0 = u > 0$, and consider the problem of emptiness before an overflow. The probability $V(u) = \Pr\{T < \infty, Z_T = 0\}$ that the dam dries up before overflowing is seen to satisfy the integral equation

$$V(u) = G(m-u) + \int_{0+}^{k-m} d_x G(x-u+m)\, V(x), \tag{3.14}$$

subject to the boundary conditions $V(0) = 1$ and $V(u) = 0$ for $u > k-m$ [Ghosal (1960 a), (1960 b)]. For the input with the negative exponential distribution

$$dG(x) = \mu e^{-ux}\, dx \qquad (0 < x < \infty), \tag{3.15}$$

Ghosal obtains

$$V(u) = \begin{cases} 1 - ce^{\mu u} & (0 < u \leqq m) \\[1ex] 1 - c \sum\limits_{q=0}^{n} e^{-\mu(qm-u)} \dfrac{\mu^q}{q!} (qm-u)^q & \\ \qquad (nm < u \leqq nm+m,\ n = 1, 2, \ldots, \left[\dfrac{k}{m}\right] - 2) \end{cases} \tag{3.16}$$

where

$$c = \left\{ \sum_{q=0}^{[k/m]} e^{-\mu(qm-k)} \frac{\mu^q}{q!} (qm-k)^q \right\}^{-1}, \tag{3.17}$$

and $[x]$ is the largest integer contained in x (it will be observed that $V(u)$ as obtained here has two points of discontinuity at $u = 0$ and $u = k-m$).

3.2. *Emptiness in an infinite dam.* In the last section it was found that while formal solutions are available for the problem of emptiness in a finite dam, these do not always yield pleasant expressions for the required probabilities. In the case of an infinite dam, however, elegant solutions are once again available. Here the probability of an overflow does not arise, and the wet period is therefore defined by (3.1); in view of (2.19), this can be written as

$$T = \min\{n \mid Z_0 + S_n \leqq 0\}, \tag{3.18}$$

where the S_n are the partial sums of the sequence of random variables $Y_n = X_n - m$ $(n = 0, 1, \ldots)$. First passage times like (3.18) have been extensively studied by Kemperman (1961); applying his results (pp. 83–86) we find that the p.g.f. of $T \equiv T(u)$ (where $Z_0 = u > 0$) is given by

$$E\{z^{T(u)}\} = 1 - e^{-\sum_1^\infty z^n/nG_n(nm)} \{1 + V(-u, z)\} \quad (|z| < 1), \tag{3.19}$$

where the function $V(y, z)$ is defined by

$$V(y,z) = \sum_1^\infty z^n \Pr\{S_1 \leqq 0, S_2 \leqq 0, \ldots, S_{n-1} \leqq 0, y \leqq S_n \leqq 0\} \tag{3.20}$$

its Laplace-Stieltjes transform being

$$\int_{-\infty}^{0+} e^{\theta y}(-d_y)\, V(y,z) = \exp\left\{ -\sum_1^\infty \frac{(ze^{-\theta m})^n}{n!} \int_{-\infty}^{nm} e^{\theta x} dG_n(x) \right\} \quad (|z| < 1, \mathrm{Re}(\theta) > 0). \tag{3.21}$$

In (3.19) and (3.21) the notation is the same as in Section 2.3. Further, if $A = \sum_1^\infty (1/n) \Pr\{S_n \leqq 0\}$, then the probability that the dam ever dries up is given by

$$\Pr\{T(u) < \infty\} = \begin{cases} 1 & \text{if } A = \infty \\ 1 - e^{-A}[1 + V(-u, 1)] & \text{if } A < \infty. \end{cases} \tag{3.22}$$

If we assume that the inputs have a finite mean (say m_1), then as in (2.3) we find that $A < \infty$ if $m_1 > m$ and $A = \infty$ if $m_1 \leqq m$. Thus the wet period is of finite duration if and only if the mean input does not exceed the amount released.

Explicit results are available in the case where the inputs are discrete and the release $m = 1$. Let $Z_0 = i > 0$ and $T = T_i$; then in the notations of Section 2.2, we have

$$g(i, n) = \Pr\{T_i = n\} = \begin{cases} 0 & \text{if } n < i \\ \dfrac{i}{n} k_{n-i}^{(n)} & \text{if } n \geqq i \end{cases} \tag{3.23}$$

for the probability distribution of T_i, and

$$\Pr\{T_i < \infty\} = \begin{cases} 1 & \text{if } m_1 \leqq 1 \\ \xi^i & \text{if } m_1 > 1 \end{cases} \tag{3.24}$$

for that probability that the dam eventually dries up, ζ being the largest positive root of the equation $\zeta = K(\zeta)$. The results (3.23) and (3.24) are analogous to those established by Kendall (1957) in continuous time [equations (7.5) and (7.7)]; (3.23) can be proved by induction, starting with $g(i,1) = k_0\delta_{i1}$, and using the relation

$$g(i,n) = \sum_{\nu>0} k_{\nu-i+1}\, g(\nu, n-1) \qquad (n \geqq 2). \tag{3.25}$$

Gani (1958) formulated the problem of emptiness in this dam as an occupancy problem and obtained the results for two special cases by the method of truncated polynomials. We also mention the two recent papers on the proof of (3.23) by Lloyd (1963 b) using inductive methods, and Mott (1963) relying on a combinatorial lemma. Lastly we note that Yeo's result (2.18) can be written as

$$P_{i0}^{(n)} = (k_0)^{-1} \sum_{\nu=i+1}^{n+1} g(\nu, n+1). \tag{3.26}$$

II. The Dam Process in Continuous Time

4. The input process

In setting up a model for the dam in continuous time, our first task would obviously be a reasonably realistic specification of the input into the dam. In the discrete time model of Section 1, if we denote by X_n the total input during an interval $(0, n)$ $(n \geqq 1)$, then it was assumed that $\{X_n\}$ is a sequence of partial sums of mutually independent and identically distributed random variables. The natural generalization of such an input in continuous time would therefore be a process $X(t)$ $(t \geqq 0)$ with stationary and independent increments (time-homogeneous additive process). Such a process is known to be infinitely divisible, i.e the input during an interval $(0, t]$ can be expressed as the sum of inputs during the n subintervals $(0, t/n]$, $(t/n, 2t/n], \ldots, ((n-1)t/n, t]$ for every positive integer n. This again is in agreement with our concept of inputs into a dam. Clearly, $X(t) \geqq 0$, and we shall assume that the mean input is finite; for the purpose of analyzing the process we shall further assume that $X(t)$ is separable and centered. It is known that such a process has a characteristic function (c.f.) of the form

$$E\{e^{i\theta Z(t)}\} = e^{t\phi(\theta)}, \tag{4.1}$$

where

$$\phi(\theta) = \int_0^\infty (e^{i\theta x} - 1)\, dL(x), \tag{4.2}$$

$L(x)$ being a function which is continuous to the right, non-decreasing and is such that $L(\infty) = 0$. Further, almost all sample functions of $X(t)$ have right and left hand limits at every t, and they increase only by jumps, the number of jumps in $(0, t]$ of magnitude $\geqq x \geqq 0$ being a Poisson process with mean $-tL(x)$ [Lévy (1948), Doob (1953)].

As examples of such an input, let us consider the following.

(i) *The compound Poisson input.* Let $B(x)$ be a d.f. with $B(x) = 0$ for $x \leqq 0$, and $B(\infty) = 1$, and let $\psi(\theta)$ be the corresponding c.f. Let $0 < \lambda < \infty$, and put $L(x) = \lambda B(x) - \lambda$ $(x \geqq 0)$; then $L(x)$ is seen to satisfy the conditions stated above, (in particular, note that $L(0) = -\lambda > -\infty$). We have $\phi(\theta) = \lambda\psi(\theta) - \lambda$ and the c.f. (4.1) is therefore given by

$$E\{e^{i\theta X(t)}\} = e^{-\lambda t[1 - \psi(\theta)]}. \tag{4.3}$$

The process $X(t)$ thus has a compound Poisson distribution. Here $X(t)$ increases by jumps whose magnitudes have a d.f. $B(x)$ $(x \geqq 0)$, there being finitely many jumps in every finite interval since $L(0) > -\infty$. A special case of this input with $B(x) = 0$ for $x < h$, and $= 1$ for $x \geqq h$ was considered by Moran (1955) and Gani (1955) (see Section 8).

(ii) *The gamma input.* Let $0 < \rho < \infty$, and put

$$L(x) = -\int_x^\infty e^{-u/\rho} \frac{du}{u} \qquad (x > 0); \tag{4.4}$$

$L(x)$ is of the required type, and $L(0) = -\infty$. We have

$$\phi(\theta) = \int_0^\infty (e^{i\theta x} - 1) e^{-x/\rho} \frac{dx}{x} = -\log(1 - \rho i\theta), \tag{4.5}$$

so that the c.f. of $X(t)$ is given by

$$E\{e^{i\theta X(t)}\} = (1 - \rho i\theta)^{-t}. \tag{4.6}$$

Thus $X(t)$ in this case has a gamma distribution; the process increases by jumps such that

$$\Pr\{x < dX(t) < x + dx\} = \frac{e^{-x/\rho}}{x} dx\, dt + o(dt) \quad (x > 0) \tag{4.7}$$

where $dX = X(t + dt) - X(t)$, but since $L(0) = -\infty$, there may be an infinity of such jumps in a finite interval. This process was considered by Moran (1956).

5. Formulation of the dam process in continuous time

Suppose that the input into a dam has been specified by (4.1) and (4.2). For simplicity, let us assume that the dam has infinite capacity. Let the release be continuous and occur at a unit rate except when the dam is empty. If $Z(t)$ denotes the content of the dam at time t, it is clear that $Z(t)$ satisfies the relation

$$Z(t + dt) = Z(t) + dX(t) - (1 - r)dt, \tag{5.1}$$

where rdt $(0 \leqq r \leqq 1)$ is that part of the interval $(t, t + dt)$ during which the dam is empty. More correctly, we may write

(5.2) $$Z(t) = Z(0) + X(t) - t + \int_0^t \zeta(\tau)d\tau,$$

where $\zeta(t)$ is a random variable such that $\zeta(t) = 1$ if $Z(t) = 0$, and $= 0$ if $Z(t) > 0$, so that the integral in (5.2) gives the length of the dry period in the dam during $(0, t)$. However, as Kingman (1963) has pointed out, the specification (5.2) is not valid for arbitrarily defined inputs. Consider, for instance, the deterministic input $X(t) = \frac{1}{2}t$; here (5.2) gives $Z'(t) = -\frac{1}{2} + \zeta(t)$ for almost all t. At a point t such that $\zeta(t) = 1$, we have $Z(t) = 0$, $Z'(t) = -\frac{1}{2}$, which contradicts the fact that $Z(t) \geqq 0$. Thus $\zeta(t) = 0$ for almost all t, and $Z(t) = Z(0) - \frac{1}{2}t$, which is again a contradiction. Thus (5.2) breaks down in this case. Kingman therefore suggests the modified formulation

(5.3) $$Z(t) = Z(0) + Y(t) + \int_0^t \zeta(\tau)dY_-(\tau),$$

where $Y(t) = X(t) - t$ is the net input during $(0, t]$, and $Y_-(t)$ is the total negative variation of $Y(t)$ in $(0, t]$. This specification is equivalent to (5.2) if and only if $X'(t) = 0$ for all t, which condition is satisfied in our case since $X(t)$ increases only by jumps.

Let us recall that in the discrete time model the storage function Z_n was given by (2.19) in terms of the partial sums of the random variables Y_n $(n \geqq 0)$. It is therefore natural to ask whether the analogous formula

(5.4) $$Z(t) = [\sup_{0 \leqq \tau \leqq t} \{Y(t) - Y(\tau -)\}, \quad Z(0) + Y(t)]$$

holds for the continuous time storage process $Z(t)$. For the case where $Z(0) = 0$ and $X(t)$ has a compound Poisson distribution, this was proved by Reich (1958) for the waiting time in the queueing system $M/G/1$. Gani and Pyke (1960 a) extended this result to the case where $Z(0) \geqq 0$, and the net input $Y(t) = X(t) - t$ is such that jumps of positive or negative magnitudes occur in the process $X(t)$ in such a way that $L(0) > -\infty$. (This, however, is different from the processes under discussion in the present section, and will be considered in Section 12). These results do not hold when $L(0) = -\infty$ as, for instance, in the case of the gamma input; however, Gani and Pyke (1960 a) have proved that the process defined by (5.4) is the limit with probability one of a sequence of discrete time dam processes defined as follows. Let the time unit be 2^{-n}, and let us suppose that the net input $U_k^{(n)}$ during the interval $\{k2^{-n}, (k+1)2^{-n}\}$ is given by

(5.5) $$U_k^{(n)} = Y\{(k+1)2^{-n}\} - Y(k2^{-n}),$$

where $Y(t)$ is a separable centered infinitely divisible process with $Y(0) = 0$; the random variables $U_k^{(n)}(k \geqq 0)$ are therefore independent and identically distributed for each n. Let us write $Y_k^{(n)} = Y(k2^{-n})$, then as in (2.19), the storage function $Z_k^{(n)}$ is given by

$$Z_k^{(n)} = \max\left\{\max_{0\leq r\leq k}(Y_k^{(n)} - Y_{r-}^{(n)}),\ Z_0^{(n)} + Y_k^{(n)}\right\}. \tag{5.6}$$

Let us now define a stochastic process $Z^{(n)}(t)$ in continuous time $t \geqq 0$, and depending on a parameter n, as follows:

$$Z^{(n)}(t) = Z_k^{(n)} \text{ for } k2^{-n} \leqq t < (k+1)2^{-n} \quad (k \geqq 0). \tag{5.7}$$

In view of (5.6) we can write for $t \geqq 0$,

$$Z^{(n)}(t) = \max\left\{\max_{0\leq r\leq [t2^n]}(Y_{[t2^n]}^{(n)} - Y_{r-}^{(n)}),\ Z_0^{(n)} + Y_{[t2^n]}^{(n)}\right\}. \tag{5.8}$$

Let $Z_0^{(n)} = Z(0)$ (fixed); then as $n \to \infty$, $Z^{(n)}(t) \to Z(t)$ with probability one, where $Z(t)$ is defined by (5.4). Recently, without recourse to discrete time analogues, Kingman (1963) has proved that if $Y(t)$ is a right continuous function of bounded variation in every finite subinterval of $t \geqq 0$, which has no downward jumps and satisfies $Y(0) = 0$, then the unique non-negative measurable solution of (5.3) is given by (5.4).

6. The distribution of storage

The dam process $Z(t)$ defined by (5.4) is clearly a time-homogeneous Markov process of mixed type. In the special case where $Z(0) = 0$, an expression for its double transform, analogous to the discrete time result (2.21), can be found as follows. Since the process $Y(t)$ has stationary and independent increments, $Y(t) - Y(\tau -)$ has the same distribution as $Y(t-\tau)$, and therefore the distribution of $Z(t)$ is the same as that of

$$\max\left[\sup_{0\leqq\tau\leqq t}\{Y(\tau)\},\ Z(0) + Y(t)\right], \tag{6.1}$$

which reduces to

$$\sup_{0\leqq\tau\leqq t}\{Y(\tau)\} \tag{6.2}$$

if $Z(0) = 0$. Now, for any separable centered infinitely divisible process $Y(t)$, Baxter and Donsker (1957) have shown that

$$\begin{aligned} &s\int_{t=0}^{\infty}\int_{u=0-}^{\infty} e^{-st-\theta u} d_u \Pr\left\{\sup_{0\leqq\tau\leqq t} Y(\tau) < u\right\} \\ &\quad = \exp\int_{w=s}^{\infty}\int_{t=0}^{\infty} e^{-wt}[\psi(\theta,t) - 1]\,dt\,dw \quad (\theta > 0,\ s > 0), \end{aligned} \tag{6.3}$$

where

$$\psi(\theta,t) = 1 + \int_0^{\infty}(e^{-\theta y} - 1)\, d_y \Pr\{Y(t) < y\}. \tag{6.4}$$

Applying these results to our process $Z(t)$, we obtain

$$\text{(6.5)}\qquad \begin{aligned} &s\int_{t=0}^{\infty}\int_{u=0-}^{\infty} e^{-st-\theta u} d_u \Pr\{Z(t)<u\} \\ &= \exp\int_{w=s}^{\infty}\int_{t=0}^{\infty} e^{-wt}\int_{x=0}^{\infty}(e^{-\theta x}-1)\,d_x K(t+x,t)\ (\theta>0,\ s>0), \end{aligned}$$

where $K(x,t) = \Pr\{Y(t)+t \leqq x\}$ is the d.f. of the input $X(t)$. Although the explicit evaluation of the integral on the right hand side of (6.5) is difficult, the distribution problem of $Z(t)$ can be considered to have been formally solved at least in the case where $Z(0) = 0$.

7. The wet period in a dam

We now consider the problem of emptiness in a dam, the discrete time version of which was discussed in Section 3. Suppose that the input $X(t)$ into the dam during an interval $(0,t]$ is a time-homogeneous additive process such that its probability density function (p.d.f.) is given by

$$\text{(7.1)}\qquad \Pr\{x < X(t) < x+dx\} = k(x,t)dx \qquad (x \geqq 0, t > 0)$$

and its Laplace transform, by

$$\text{(7.2)}\qquad E\{e^{-\theta X(t)}\} = \int_0^\infty e^{-\theta x}k(x,t)dx = e^{-t\xi(\theta)} \quad (\mathrm{Re}(\theta)>0),$$

where $\xi(\theta)$ is a function specified as in Section 4. Let the mean input be finite (say ρt); then $\xi(\theta) = \rho\theta + o(\theta)$ as $\theta \to 0+$. Also, let the release occur continuously at a unit rate except when the dam is empty. Let us first consider the case of the infinite dam; for the wet period initiated by a content $Z(0) = u > 0$ in this dam we have the definition

$$\text{(7.3)}\qquad T(u) = \inf\{t \mid u + X(t) - t \leqq 0\}$$

analogous to (3.18). Clearly, $T(u) \geqq u$, and we have

$$\text{(7.4)}\qquad T(u) = u + T\{X(u)\}.$$

Further it is obvious that the random variable $T(u)$ is itself additive in the parameter u, so that its Laplace transform must be of the form $E\{e^{-\theta T(u)}\} = e^{-u\eta(\theta)}$; from (7.4) it is found that the function $\eta(\theta)$ is the unique solution of the functional equation $\eta(\theta) = \theta + \xi\{\eta(\theta)\}$ subject to the condition $\eta(\infty) = \infty$. Moreover, the probability that the wet period is of finite duration is given by

$$\text{(7.5)}\qquad \Pr\{T(u) < \infty\} = \begin{cases} 1 & \text{if } \rho \leqq 1 \\ e^{-u\zeta} & \text{if } \rho > 1, \end{cases}$$

where ζ is the largest positive root of the equation $\zeta = \xi(\zeta)$. Lastly, let us assume that $T(u)$ has a p.d.f., say $g(u,t)$; from (7.4) we find that $g(u,t)$ satisfies the integral equation

$$g(u,t) = \int_0^\infty k(x,u)\, g(x,t-u)\, dx. \tag{7.6}$$

It is now observed that the function defined by

$$g(u,t) = \begin{cases} 0 & \text{if } t \leqq u \\ \dfrac{u}{t} k(t-u,t) & \text{if } t > u \end{cases} \tag{7.7}$$

satisfies (7.6); assuming that (7.6) has a unique solution, we conclude that the p.d.f. of the wet period is given by (7.7). The results (7.4) – (7.7) are due to Kendall (1957) who also formulated the problem. In the special case of the gamma input [Moran (1956)] we have

$$k(x,t) = e^{-x/\rho} \frac{x^{t-1}}{\rho^t \Gamma(t)} \quad (x \geqq 0, t > 0) \tag{7.8}$$

for the p.d.f. of $X(t)$, so that

$$g(u,t) = \frac{u e^{-(t-u)/\rho} (t-u)^{t-1}}{\rho^t \Gamma(t+1)} \qquad (t \geqq u > 0). \tag{7.9}$$

In the case of the finite dam, the problem of emptiness is much more difficult. However, consider the random variable

$$T(u) = \inf \{ t \mid Z(t) = 0 \text{ or } Z(t) = k \} \tag{7.10}$$

where $Z(0) = u > 0$; $T(u)$ is the time at which the dam either dries up or overflows for the first time, starting with an initial content u. We can write (7.10) as

$$\begin{aligned} T(u) &= \inf \{ t \mid u + X(t) - t \leqq 0 \text{ or } u + X(t) - t \geqq k \} \\ &= \inf \{ t \mid Y(t) \leqq -u \text{ or } Y(t) \geqq k - u \}, \end{aligned} \tag{7.11}$$

where $Y(t) = X(t) - t$ is the net input into the dam; $T(u)$ is thus the first passage time of the process $Y(t)$ from the origin to a point outside the open interval $(-u, k-u)$. Now let $E\{e^{\theta Y(t)}\} = e^{t\phi(\theta)}$ be the moment generating function (m.g.f.) of $Y(t)$; then using the extension of Wald's fundamental identity of sequential analysis to a continuous parameter due to Dvoretzky, Kiefer and Wolfowitz (1953) we have that

$$E\{e^{\theta_0 Y(T+)}\} = 1, \tag{7.12}$$

where θ_0 is the nonzero real solution of the equation $\phi(\theta) = 0$, and $T \equiv T(u)$. If the barriers $-u$, $k-u$ are exactly reached, then the identity (7.12) gives $P_u e^{-u\theta_0} + (1 - P_u) e^{(k-u)\theta_0} = 1$, or

$$P_u = \frac{e^{u\theta_0} - e^{k\theta_0}}{1 - e^{k\theta_0}} \tag{7.13}$$

for the probability P_u of absorption at $-u$ (i.e. P_u is the probability that the dam dries up before overflowing). If the barriers are not exactly reached, then the result (7.13) holds approximately. Thus in the case of the gamma input with the p.d.f. (7.8) we find that P_u is given approximately by (7.13), where θ_0 is the nonzero real root of the equation $e^{-\theta} = 1 - \rho\theta$. If ρ is close to unity, then $\theta_0 \simeq 2(1-\rho)$, and

$$P_u \sim \frac{e^{2u(1-\rho)} - e^{2k(1-\rho)}}{1 - e^{2k(1-\rho)}}. \tag{7.14}$$

The results (7.13) and (7.14) are due to Phatarfod (1963).

8. A dam with simple Poisson inputs

Consider a dam of finite capacity k, which is fed by inputs of Poisson type such that during an interval $(t, t + dt)$, an amount $dX(t)$ of either h ($< k$) units or no units flows into the dam with probabilities $\lambda dt + o(dt)$ and $1 - \lambda dt + o(dt)$ respectively. The input $X(t)$ during $(0, t]$ has thus the d.f.

$$K(x, t) = \Pr\{X(t) \leqq x\} = \sum_{r=0}^{[x/h]} e^{-\lambda t} \frac{(\lambda t)^r}{r!} \tag{8.1}$$

and the c.f.

$$E\{e^{i\theta X(t)}\} = e^{-\lambda t(1 - e^{i\theta h})}. \tag{8.2}$$

It is clear from (8.2) that $X(t)$ is a time-homogeneous additive process. The content $Z(t)$ of the dam at time $t \geqq 0$ is defined by the equation

$$Z(t + dt) = \min\{k, Z(t) + dX(t)\} - (1 - r)\, dt, \tag{8.3}$$

where rdt $(0 \leqq r \leqq 1)$ is defined as in (5.1), and $\min\{k, Z(t) + dX(t)\}$ indicates that there will be an overflow whenever $Z(t) + dX(t) > k$, leaving only the amount k in the dam. This model was considered by Gani (1955), who obtained the stationary distribution of $Z(t)$. To study the time-dependent behaviour of the process $Z(t)$, let us denote its transition d.f. by

$$F(z_0; z, t) = \Pr\{Z(t) \leqq z \mid Z(0) = z_0\} \quad (z \geqq 0, t \geqq 0); \tag{8.4}$$

for convenience we shall sometimes write this d.f. as $F(z, t)$. We have $F(z, t) = 0$ for $z < 0$ and $F(z, t) = 1$ for $z \geqq k$; it is also known [Doob (1953), pp. 269–270] that $F(z, t)$ is differentiable with respect to t for all $z \geqq 0$. By considering $F(z, t)$ over two consecutive intervals $(0, t)$, $(t, t + dt)$ we find that

$$\begin{aligned} &F(z, t + dt) = F(z + dt, t)(1 - \lambda dt) + F(z - h + dt, t)\lambda dt + o(dt) \ (0 \leqq z < k), \\ &F(k, t + dt) - F(k - dt, t + dt) = \lambda dt\{F(k, t) - F(k - h, t)\} + o(dt). \end{aligned} \tag{8.5}$$

From (8.5), by letting $dt \to 0$ we obtain the differential equation

$$(8.6) \qquad \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial z}\right) F(z,t) = -\lambda F(z,t) + \lambda F(z-h,t) \qquad (0 \leqq z \leqq k),$$

[Gani and Prabhu (1959 a)]. It is seen from (8.6) that $F(z,t)$ is continuous for all $0 < z \leqq k$ but has a discontinuity at $z = 0$, the concentration $F(0,t)$ being the probability of emptiness of the dam at time t. Further, $F(z,t)$ has a derivative $\partial F/\partial z$ continuous everywhere except at $z = 0$, $z = h$ and $z = k$. At $z = 0$ the right derivative exists, and at $z = h$ both the left and right derivatives exist (but are distinct); at these points $\partial F/\partial z$ in (8.6) denotes the right derivative. At $z = k$ the left derivative exists and $\partial F/\partial z$ denotes this derivative in (8.6).

It is not known how the differential equation (8.6) can be solved for $k < \infty$. Consider, however, the infinite dam, where $k = \infty$. Here (8.6) holds for all $0 \leqq z < \infty$; this also arises as a special case of Takács' (1955) integro-differential equation for the queue with Poisson arrivals. A formal solution of (8.6) can be deduced from the one obtained by Takács for his equation (by using Laplace-Stieltjes transforms) (see Section 9). However, Gani and Prabhu (1959 b) obtained an explicit solution as follows. They perform a change of variables from (z,t) to (u,t), where $u = t + z$. Then if $F(Z,t) \to G(u,t)$ it is seen that $G(u,t)$ is itself a d.f., having properties similar to those of $F(z,t)$, and that the equation (8.6) reduces to

$$(8.7) \qquad \frac{\partial G}{\partial u} = -\lambda G(u,t) + \lambda G(u-h,t) \qquad (t \leqq u < \infty).$$

This new equation is simpler than (8.6) and can be solved step by step in the ranges $[0,h)$, $[h,2h), \ldots$. From the general solution for $G(u,t)$ so obtained, the expression for $F(z,t)$ is found to be

$$(8.8) \qquad F(z,t) = \sum_{r=0}^{[z/h]} F(0, z+t-rh) e^{-\lambda(rh-z)} \frac{\lambda^r}{r!} (rh-z)^r,$$

the probability $F(0,t)$ having been determined directly by combinatorial methods as follows. Firstly, let us observe that the dam with an initial content $u(>0)$ will become empty only at times $u, u+h, u+2h, \ldots$, so that the wet period $T(u)$ in this case is a discrete random variable with values $u + nh$ $(n = 0, 1, \ldots)$. From Kendall's result (7.7) (which applied only to the case of inputs having a continuous distribution) it appears possible that the distribution of $T(u)$ is given by

$$(8.9) \qquad \begin{aligned} g(u, u+nh) &= \Pr\{T(u) = u + nh\} \\ &= e^{-\lambda(u+nh)} \frac{\lambda^n}{n!} u(u+nh)^{n-1} \qquad (n \geqq 0). \end{aligned}$$

By taking transforms of (8.9) Gani and Prabhu verified that (8.9) does in fact give the required distribution. Gani (1958) established this result by the method of truncated polynomials. Now let $X_0, X_1, X_2, \ldots$ be the inputs into the dam during the intervals $(0,u], (u,u+h], (u+h,u+2h], \ldots$; then the probability $g(u,u+nh)$ can be reformulated as follows:

$$(8.10) \qquad G(u,u+nh) = \Pr\{X_0 + X_1 + \ldots + X_{j-1} \geqq j \ (j=1,2,\ldots,n-1), \\ X_0 + X_1 + \ldots + X_{n-1} = n, \ X_n = 0\}.$$

Note that $X_0, X_1, X_2, \ldots$ are mutually independent Poisson random variables, X_0 having mean λu, and $(n \geqq 1)$ having mean λh. Next, consider the probability $F(0,t)$ of emptiness (not necessarily fot the first time). Let $[t - z_0/h] = n$, and let $F_r(0,t)$ be the conditional probability of emptiness at time t, on the assumption that r $(\leqq n)$ inputs have flowed into the dam during $(0,t]$. We can then write

$$(8.11) \qquad \begin{aligned} F_r(0,t) &= \Pr\{Z(t) = 0, \ X(t) = rh \,|\, Z(0) = z_0\} \\ &= \Pr\{X_0 + X_1 + \ldots + X_{j-1} \geqq j \ (j=0,1,\ldots,r-1), \\ &\qquad X_0 + X_1 + \ldots + X_{r-1} = r, \ X_r = 0\} \end{aligned}$$

where $X_0, X_1, X_2, \ldots, X_r$ are the inputs during $(0, z_0 + nh - rh], (t-rh, t-rh+h]$, $(t-rh+h, t-rh+2h], \ldots, (t-h,t]$, these being mutually independent Poisson variables. However, from (8.10) we see that this last probability is the same as the first emptiness probability $g(u,u+rh)$ with u replaced by $t - rh$; thus

$$(8.12) \qquad F_r(0,t) = e^{-\lambda t} \frac{\lambda^r}{r!}(t-rh)t^{r-1} \qquad (0 \leqq r \leqq n).$$

It follows that

$$(8.13) \qquad F(0,t) = \sum_{r=0}^{n} e^{\lambda - t} \frac{\lambda^r}{r!} t^{r-1}(t-rh),$$

and the solution (8.7) is thus complete.

9. The queue $M/G/1$ as a storage system

Consider a queueing system in which customers arrive in a Poisson process with intensity $\lambda(t)$ and are served at a single counter on a "first come, first served" basis, the service time having a d.f. $B(x)$. Let $W(t)$ be the virtual waiting time in this system, i.e., the time an intending customer at time t would have to wait for his service to commence. It is seen that $W(t)$ is a (non-homogeneous) Markov process; let its transition d.f. be denoted by

$$(9.1) \qquad F(z_0, \tau; z, t) = \Pr\{W(t) \leqq z \,|\, W(\tau) = z_0\}.$$

We shall denote this d.f. briefly as $F(z,t)$. By considering $F(z,t)$ over the intervals (τ, t), $(t, t+dt)$ Takács (1955) obtains the relation

$$F(z,t+dt)=[1-\lambda(t)dt]F(z+dt,t) \tag{9.2}$$
$$+\ \lambda(t)dt\int_0^x d_yF(y,t)\,B(x-y)+o(dt),$$

from which, by proceeding to the limit as $dt\to 0$, he obtains the integro-differential equation

$$\frac{\partial F}{\partial t}-\frac{\partial F}{\partial z}=-\lambda(t)\ F(z,t)+\lambda(t)\int_0^x d_yF(y,t)\ B(x-y). \tag{9.3}$$

The properties of $F(z,t)$ are similar to those of $F(z,t)$ in Section 8; thus, in particular $F(z,t)$ has a discontinuity at $z=0$, $F(0,t)$ being the probability that the server is idle at time t. For $\mathrm{Re}(\theta)>0$, let us define the Laplace-Stieltjes transforms

$$\Phi(\theta,t)=\int_{0-}^{\infty}e^{-\theta z}d_zF(z,t),\qquad \psi(\theta)=\int_0^{\infty}e^{-\theta x}dB(x) \tag{9.4}$$

of $F(z,t)$ and $B(x)$ respectively. Then taking transforms of both sides of (9.3) we obtain the differential equation

$$\frac{\partial}{\partial t}\ \Phi(\theta,t)=\Phi(\theta,t)\{\theta-\lambda(t)+\lambda(t)\psi(\theta)\}-\theta F(0,t) \tag{9.5}$$

for $\Phi(\theta,t)$; Takács proves that (9.5) has the unique solution given by

$$\Phi(\theta,t)=e^{\theta(t-z_0)-\Lambda(t)[1-\psi(\theta)]} \tag{9.6}$$
$$-\ \theta\int_0^t e^{\theta(t-u)-[\Lambda(t)-\Lambda(u)][1-\psi(\theta)]}F(0,u)\,du,$$

where

$$\Lambda(t)=\int_0^t \lambda(u)\,du. \tag{9.7}$$

As observed by Prabhu (1960 b), when $\lambda(t)=\lambda$ (constant), so that the underlying processes are time-homogeneous, the queueing system described above is analogous to an infinite dam model. In fact, if N is the number of customers who arrive during the interval $(0,t]$, then their total service time has the d.f. $B_N(x)$, and, since N is a random variable having the Poisson distribution with mean λt, it follows that the total service time $X(t)$ of customers arriving during $(0,t]$ has the compound Poisson distribution with

$$K(x,t)\ =\Pr\{X(t)\leqq x\}\ =\ \sum_{n=0}^{\infty}\ e^{-\lambda t}\,\frac{(\lambda t)^n}{n!}\ B_n(x), \tag{9.8}$$

where $B_n(x)$ is the n-fold convolution of $B(x)$ with itself ($n\geqq 1$), and $B_0(x)=0$ if $x<0$ and $=1$ if $x\geqq 0$. This total service time $X(t)$ can be considered as the 'service potential' (or load), which is steadily exhausted by the server at unit rate per unit time except when it is zero. Viewed in this manner, the waiting time $W(t)$ in this queue $M/G/1$ reduces to a special case of the dam model under our investigation, where the input $X(t)$ has the d.f. (9.8). Clearly, $X(t)$ is a time-homogeneous additive process, whose Laplace-Stieltjes transform is found to be

$$\int_0^\infty e^{-\theta x} d_x K(x,t) = e^{-\lambda t[1-\psi(\theta)]} \qquad (\mathrm{Re}(\theta) > 0). \tag{9.9}$$

Gani's dam model considered in Section 8 arises as a special case when $B(x) = 0$ if $x < h$, and $= 1$ if $x \geqq h$. In the general case where $\lambda(t) = \lambda$ but $B(x)$ is an arbitrary d.f., we can write $F(z_0, \tau; z, t) \equiv F(z_0; z, t-\tau)$; the equation (9.6) can then be simplified as

$$\theta^{-1}\Phi(\theta,t) = \theta^{-1} e^{\theta(t-z_0)-\lambda t[1-\psi(\theta)]} - \int_0^t e^{\theta\tau - \lambda\tau[1-\psi(\theta)]} F(0, t-\tau) d\tau. \tag{9.10}$$

Identifying the transforms on both sides of (9.10), Prabhu obtains the explicit expression

$$F(z,t) = K(t+z-z_0, t) - \int_0^t F(0, t-\tau)\, dK(\tau+z, \tau) \tag{9.10}$$

for the transition d.f. of the storage function $W(t)$, where

$$dK(t+z,t) = d_x K(x,t)\big|_{x=t+z} = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!}\, dB_n(t+z). \tag{9.11}$$

However, the right hand side of (9.10) contains the unknown probability $F(0,t)$ that the server is idle (dam is empty) at time t. Prabhu obtains this by first considering the random variable $T(u) = \inf\{t \mid X(t) - t \leqq 0\}$ which is the busy period initiated by a waiting time $W(0) = u$ (wet period in the dam whose initial content was u). From Kendall's result (7.7) it seems likely that the distribution of $T(u)$ is given by

$$dG(u,t) = \Pr\{t < T(u) < t + dt\} = \frac{u}{t}\, dK(t-u, t) \tag{9.12}$$

in the notation (9.11); Prabhu (1960c) proved that (9.12) gives the distribution of $T(u)$. He introduces a new variable N, which is the number of customers served during a busy period, excluding those initially present (number of inputs during a wet period in the dam), and considers the joint distribution $dG_n(u,t)$ of $T(u)$ and N. It is found that $dG_0(u,t) = e^{-\lambda t} dB_0(t-x)$ and that $dG_n(u,t)$ $(n \geqq 1)$ satisfies the relation

$$dG_n(u,t) = \int_{\tau=0}^{t} \int_{y=0}^{t-u} \lambda e^{-\lambda\tau} d\tau\, dG_{n-1}(u-\tau+y, t-\tau)\, dB(y). \tag{9.13}$$

Solving (9.13) successively for $n = 1,2,\dots$, Prabhu arrives at the general solution

$$dG_n(u,t) = e^{-\lambda t}\frac{\lambda^n}{n!}\,t^{n-1}u\,dB_n(t-u) \qquad (n = 0,1,\dots), \tag{9.14}$$

and by adding (9.14) over $n = 0,1,2,\dots$, at the result

$$dG(u,t) = \sum_0^\infty e^{-\lambda t}\frac{\lambda^n t^{n-1}}{n!}\,u\,dB_n(t-u) \tag{9.15}$$

which is seen to be identical with (9.12). Using this the probability $F(0,t)$ is then obtained; correcting a minor error in the result in its original form, it can be written as

$$F(0,t)\,dt = \begin{cases} 0 & \text{if } t < z_0 \\ e^{-\lambda t}dt + \displaystyle\int_{u=z_0}^{t} dG(u,t)\,du & \text{if } t \geqq z_0. \end{cases} \tag{9.16}$$

More directly, (9.16) can be obtained from a relation of the type (9.13). Since $F(0,t)$ is now known, the result (9.10) is complete, and gives the transition d.f. of the storage function $W(t)$.

We remark that the Laplace-Stieltjes transform of $T(u)$ is easily found to be $e^{-u\eta(\theta)}$, where $\eta(\theta)$ satisfies the functional equation $\eta = \theta + \lambda - \lambda\psi(\eta)$, and, moreover, that $\Pr\{T(u) < \infty\} = 1$ or $e^{-u\zeta}$ according as $\rho \equiv -\lambda\psi'(0) \leqq 1$ or > 1 ζ being the least positive root of the equation $\zeta = \lambda - \lambda\psi(\zeta)$. Here ρ is the relative traffic intensity of the queueing system (mean input into the dam per unit time). These results may be compared with Kendall's results (Section 7) for the wet period in the case of continuous inputs. Lastly, we have

$$\int_0^\infty e^{-\theta t}F(0,t)\,dt = e^{-z_0\eta(\theta)}/\eta(\theta) \tag{9.17}$$

a result due to Beneš (1957). Recently, Takács (1964) has used combinatorial methods to derive (9.10) and (9.16).

10. A dam process with non-homogeneous Poisson inputs

It is clear that the analogy described in the last section between the queueing system $M/G/1$ and the dam model remains valid even if the input process $X(t)$ is non-homogeneous. In this case $X(t)$ has the d.f.

$$\begin{aligned} K(\tau; x, t) &= \Pr\{X(t) \leqq x \mid X(\tau) = 0\} \\ &= \sum_{n=0}^\infty e^{-[\Lambda(t)-\Lambda(\tau)]}\frac{1}{n!}[\Lambda(t) - \Lambda(\tau)]^n B_n(x). \end{aligned} \tag{10.1}$$

This case has been studied by Reich [(1958), (1959)], but the following approach due to Gani (1962 a) is much simpler. As in the homogeneous case, the probability of emptiness has first to be determined. Let

(10.2) $$dG(u;\tau,\tau+t) = \Pr\{\tau+t < T(u) < \tau+t+dt\}$$

denote the distribution of first emptiness times; using arguments similar to Kendall's (1957) (see Section 7) we obtain for $t \geqq u$,

(10.3) $$dG(u;\tau,\tau+t) = \int_{0-}^{t-u} d_v K(v;\tau,\tau+u)\, dG(v;\tau+u,\tau+t),$$

while $dG(u;\tau,\tau+t) = 0$ for $t < u$. In the case where all the inputs are of unit size, so that

(10.4) $$B(x) = 0 \text{ for } x<1, \text{ and} = 1 \text{ for } x \geqq 1,$$

the distribution of $T(u)$ takes the form

(10.5) $$dG(u;\tau,\tau+t) = \begin{cases} g(u;\tau,\tau+u+n) & \text{for } t = u+n \ (n = 0,1,\ldots) \\ 0 & \text{otherwise,} \end{cases}$$

and (10.3) becomes

(10.6) $$g(u;\tau,\tau+u+n) = \sum_{j=0}^{n} e^{-[\Lambda(\tau+u)-\Lambda(\tau)]} \frac{1}{j!} [\Lambda(\tau+u)-\Lambda(\tau)]^j g(j;\tau+u,\tau+u+n)$$

where $g(0;\tau,\tau) = 1$ for $\tau \geqq 0$. Explicit values for $g(u;\tau,\tau+u+n)$ can be obtained recursively from (10.6); thus

(10.7) $$\begin{aligned} g(u;0,u) &= e^{-\Lambda(u)} \\ g(u;0,u+1) &= e^{-\Lambda(u+1)}\Lambda(u) \\ g(u;0,u+2) &= e^{-\Lambda(u+2)} \frac{1}{2!} \{2\Lambda(u)\Lambda(u+1) - \Lambda^2(u)\} \end{aligned}$$

and so on. These can also be systematically obtained by the method of truncated polynomials [Gani (1958)]. In the case where the input has a continuous distribution,

(10.8) $$dG(u;\tau,\tau+t) = \begin{cases} e^{-[\Lambda(\tau+u)-\Lambda(\tau)]} & \text{for } t = u \\ g(u;\tau,\tau+t)\, dt & \text{for } t > u \end{cases}$$

with a discrete concentration at $t = u$ and a continuous probability distribution for $t > u$; the equation (10.3) can be then written in terms of $g(u;\tau,\tau+t)$.

The probability of emptiness $F(z_0,\tau;0,\tau+t)$ is given by

(10.9) $$F(z_0,\tau;0,\tau+t) = \sum_{j=0}^{[t-z_0]} g(t-j;\tau,\tau+t)$$

when inputs are discrete, and

$$(10.10)\qquad F(z_0,\tau;0,\tau+t)=e^{-[\Lambda(\tau+t)-\Lambda(\tau)]}+\int_v^t g(v;\tau,\tau+t)\,dv$$

when they are continuous. Gani proves (10.10) by a geometrical argument, the proof of (10.9) being similar.

The transform (9.6) can be inverted precisely as in the homogeneous case, and yields the integral representation

$$(10.11)\qquad F(z_0,0;z,t)=K(t+z-z_0,t)-\int_0^t F(z_0,0;0,t-\tau)\,dK(t-\tau;\tau+z,t).$$

In the case of discrete inputs of unit size, Gani obtains the result in a slightly different form

$$(10.12)\qquad F(z_0,0;z,t)=\sum_{r=0}^{[z]}\frac{[\Lambda(t)-\Lambda(z+t-r)]^r}{r!}\,e^{-[\Lambda(t)-\Lambda(z+t-r)]}\times F(z_0,0;0,z+t-r).$$

by a direct solution of the integro-differential equation (9.3) using the method used in Section 8 for the homogeneous case.

11. The case of continuous inputs

In the models considered in Sections 8–10, the d.f. of the dam content $Z(t)$ was obtained by solving the forward Kolmogorov equation of the process, which took the form of a differential or an integro-differential equation. It would have been noted, however, that since the input process $X(t)$ in these cases had a compound Poisson distribution, it was easy to obtain these equations. The question naturally arises whether this method of investigation could be extended to the general case where the input is specified as in Section 5. Gani and Prabhu (1963) studied the case where the input $X(t)$ is time-homogeneous, non-negative, infinitely divisible process with a continuous d.f. $K(x,t)$, having finite mean and variance. The Laplace-Stieltjes transform of $X(t)$ is given by

$$(11.1)\qquad E\{e^{-\theta X(t)}\}=\int_0^\infty e^{-\theta x}d_xK(x,t)=e^{-t\xi(\theta)}\quad(\mathrm{Re}(\theta)>0),$$

where

$$(11.2)\qquad \xi(\theta)=\int_0^\infty(1-e^{-\theta u})\,\lambda(u)\,du,$$

where $\lambda(u)\geqq 0$ is finite for $u>0$, but $\lambda(u)\to\infty$ as $u\to 0$. The properties of this process are similar to those described in Section 4. Thus, in particular, the probability of a jump $dX(t)$ of magnitude $u<dX(t)<u+du$ in a time interval $(t,t+dt)$ is given by

$$(11.3)\qquad dK(u,dt)=\lambda(u)\,dt+o(dt).$$

Further, we have for $t > 0$,

$$\Pr\{X(t) = 0\} = K(0,t) = e^{-t\int_0^\infty \lambda(u)du} \tag{11.4}$$

[Doob (1953)], and since this is zero, $\lambda(u) \to \infty$ as $u \to 0$. The mean and variance of the process per unit time are respectively

$$\rho = \int_0^\infty u\lambda(u)du < \infty, \quad \sigma^2 = \int_0^\infty u^2\lambda(u)\,du < \infty. \tag{11.5}$$

For $t = 0$ it follows from (11.1) that $K(x,0) = 1$, and from continuity properties of the transform, we have that $K(x,t) \to 1$ as $t \to 0$ $(x > 0)$. Lastly, since $K(x,t)$ is assumed to be continuous, we have

$$\frac{d}{dx}K(x,t) = k(x,t) \qquad (0 \leqq x < \infty,\ t > 0). \tag{11.6}$$

It can be easily verified that Moran's (1956) gamma type input is a process of this type, with the p.d.f. given by (7.8).

Let us now consider an infinite dam which is fed by an input specified by the equations (9.1)-(9.6), and whose content $Z(t)$ at time $t \geqq 0$ is defined by the relation (5.1). To derive the forward Kolmogorov equation of $Z(t)$, Gani and Prabhu consider a discrete time model in which (a) the units of time and input are both equal to Δ, (b) the inputs $X(\Delta)$ over the time intervals $(n\Delta, n\Delta + \Delta)$ are independent and identically distributed random variables with

$$\begin{aligned} \Pr\{X(\Delta) = 0\} &= e^{-\lambda\Delta} \\ \Pr\{X(\Delta) = i\Delta\} &= (1 - e^{-\lambda\Delta})\,\frac{\lambda_i}{\lambda} \quad (i \geqq 1), \end{aligned} \tag{11.7}$$

where $\lambda_i \equiv \lambda_i(\Delta)$, $\lambda \equiv \lambda(\Delta) = \sum_1^\infty \lambda_i$, $0 < \rho(\Delta) \equiv E\{X(\Delta)\} < \infty$, and $0 < \sigma^2(\Delta) \equiv E\{X(\Delta)\}^2 - \rho^2(\Delta) < \infty$; and (c) the release is Δ at the end of each time interval, except when the dam is empty. The content $Z(n\Delta)$ at time $n\Delta$ in this dam is then given by

$$Z(n\Delta) = \max\{Y(n\Delta) - Y(m\Delta) \quad (0 \leqq m \leqq n),\ z_0 + Y(n\Delta)\}, \tag{11.8}$$

where $z_0 = Z(0)$ and $Y(n\Delta) = X(n\Delta) - n\Delta$, the net input during $(0,n\Delta)$ [cf. equation (2.19)]. Let $F(j\Delta,n\Delta) = \Pr\{Z(n\Delta) \leqq j\Delta\}$ be the d.f.of $Z(n\Delta)$; then we have

$$\begin{aligned} F(j\Delta - \Delta, n\Delta + \Delta) &= F(j\Delta, n\Delta)e^{-\lambda\Delta} + \sum_{i=1}^\infty F(j\Delta - i\Delta, n\Delta)(1 - e^{-\lambda\Delta})\frac{\lambda_i}{\lambda}(j > 0), \\ F(0, n\Delta + \Delta) &= F(\Delta, n\Delta)e^{-\lambda\Delta}. \end{aligned} \tag{11.9}$$

Now let us assume that as $\Delta \to 0$, and $i\Delta \to u$,

$$\frac{\lambda_i}{\lambda} \to \lambda(u),\ \lambda(\Delta) \to \infty \quad \text{but } \lambda\Delta \to 0 \tag{11.10}$$

and

$$\frac{\rho(\Delta)}{\Delta} \to \rho\ (0 < \rho < \infty),\quad \frac{\sigma^2(\Delta)}{\Delta} \to \sigma^2\ (0 < \sigma^2 < \infty). \tag{11.11}$$

From these assumptions it is seen that as $\Delta \to 0$ the input $X(n\Delta)$ in the discrete model tends to the continuous process $X(t)$ defined by (11.1)–(11.6). The storage function $Z(n\Delta)$ defined by (11.8) will tend with probability one to the continuous time storage function $Z(t)$ defined by (5.4). Further, the difference equation (11.9) for $Z(n\Delta)$ tends to the integro-differential equation

$$\frac{\partial}{\partial t}F(z,t) - \frac{\partial}{\partial z}F(z,t) = \int_0^\infty \{F(z-u,t) - F(z,t)\}\lambda(u)du \quad (0 \leqq z < \infty) \tag{11.12}$$

for the transition d.f. $F(z,t) \equiv F(z_0;z,t)$ of $Z(t)$, defined by (8.4). Here $F(z,t)$ has properties similar to those in the case of Poisson inputs described in Section 8 [see remarks following (8.6)].

The equation (11.12) yields the solution

$$F(z,t) = K(t+z-z_0,t) - \int_0^{t-z_0} F(0,t-\tau)\, k(\tau+z,\tau)d\tau, \quad (0 \leqq z < \infty), \tag{11.13}$$

where it is understood that the integral vanishes if $t - z_0 < 0$ and $K(x,t) = 0$ for $x \leqq 0$. From (11.13) it is seen that the probability $F(0,t)$ of emptiness of the dam at time t satisfies the integral equation

$$F(0,t) + \int_0^{t-z} F(0,t-\tau)\, k(\tau,\tau)d\tau = K(t-z_0,t). \tag{11.14}$$

The solution of (11.14) is found to be

$$F(0,t) = \begin{cases} 0 & \text{if } t < z_0 \\ \displaystyle\int_{z_0}^{t} g(u,t)\, du & \text{if } t \geqq z_0, \end{cases} \tag{11.15}$$

where $g(u,t)$ is the p.d.f. of the wet period $T(u)$ and is given by (7.7); the result (11.15) may be compared with the corresponding result (3.26) in the discrete time case. Since $F(0,t)$ is given by (11.15) the solution (11.13) of the integro-differential equation (9.12) is now complete.

From (11.15) it is found that the Laplace transform of $F(0,t)$ is given by

$$F^*(s) = \int_0^\infty e^{-\theta t}F(0,t)\, dt = e^{-z_0\eta(s)}/\eta(s) \tag{11.16}$$

where $\eta(s)$ is given as in Section 7, (cf. 9.17), and more generally from (9.12) that the double transform

(11.17)
$$\psi(\theta, z) = \int_{t=0}^{\infty} \int_{z=0-}^{\infty} e^{-st-\theta z} d_z F(z,t)\, dt$$
$$= \frac{e^{-\theta z_0} - \theta e^{-z_0 \eta(s)}/\eta(s)}{s - \theta + \xi(\theta)} \qquad (\mathrm{Re}[s - \theta + \zeta(\theta)] > 0).$$

The results (11.16) and (11.17) enable us to investigate the limiting behaviour of $Z(t)$ as $t \to \infty$. For, it may be easily shown that $\lim_{t\to\infty} F(z_0; z, t)$ exists independently of $z_0 \geqq 0$; let us denote this limit by $\Phi(z)$ for $z \geqq 0$. From (11.16) it follows that

(11.18)
$$\Phi(0) = \lim_{t\to\infty} F(z_0;0,t) = \lim_{s\to 0+} sF^*(s)$$
$$= \begin{cases} 0 & \text{if } \rho \geqq 1, \\ 1-\rho & \text{if } \rho < 1, \end{cases}$$

and further, from (11.17) we find that the Laplace-Stieltjes transform of $\Phi(z)$ is given by

(11.19)
$$\psi(\theta) = \int_{0-}^{\infty} e^{-\theta z} d\Phi(z) = \lim_{s\to 0+} s\Psi(\theta,s)$$
$$= \begin{cases} 0 & \text{if } \rho \geqq 1 \\ \dfrac{\theta(1-\rho)}{\theta - \xi(\theta)} & \text{if } \rho < 1. \end{cases}$$

The formula (11.19) was obtained by Downton, Lindley and Smith by heuristic arguments; Daniels inverted this transform to obtain the stationary d.f. $\Phi(z)$ in the form

(11.20)
$$\Phi(z) = 1 - (1-\rho) \int_0^{\infty} k(t+z, t)\, dt \qquad (\rho < 1)$$

[see discussion in Kendall (1957)]. Gani and Prabhu (1963) obtained (11.20) directly by letting $t \to \infty$ in (11.13). Finally we remark that Beneš (1960) and Reich (1961) have obtained results similar to those described in this section but their approach and methods are quite different from those used by Gani and Prabhu.

III. Some Further Storage Models

12. Gani and Pyke's model

As a generalization of the dam model described in sections 4–6, Gani and Pyke (1960a) have considered a storage model in which the net input $Y(t)$ is a separable infinitely divisible process, which can be represented as the difference of two infinitely divisible non-negative processes. Such a process is centered; if, further, we assume that it has a finite mean, then the c.f. of $Y(t)$ is of the form

$$E\{e^{i\theta Y(t)}\} = e^{t\phi(\theta)}, \tag{12.1}$$

where

$$\phi(\theta) = i\alpha\theta + \left(\int_{-\infty}^{0-} + \int_{0}^{\infty}\right) (e^{i\theta x} - 1)\, dL(x), \tag{12.2}$$

where α is real and L is a right continuous function which is non-decreasing on both $(-\infty, 0)$ and $[0, \infty)$ and satisfies $L(\pm\infty) = 0$. We have

$$E\{Y(1)\} = a + \int_{-\infty}^{\infty} x\, dL(x). \tag{12.3}$$

The term $i\alpha\theta$ in (12.2) indicates that $Y(t)$ contains a deterministic term αt. Let $\alpha \leqq 0$; then the net input is of the form $Y(t) = X_1(t) - X_2(t) + \alpha t$, $(X_1(t) \geqq 0, X_2(t) \geqq 0)$, where $X_1(t)$ is the actual input, and the random component $X_2(t)$ and the deterministic component $(-\alpha t)$ together constitute the output. From the general theory of additive processes it follows that almost all sample functions of $Y(t) - \alpha t$ have right and left hand limits at every t, and that they increase or decrease only by jumps, the number of jumps in $(0, t]$ of magnitude $\geqq x \geqq 0$ $(\geqq x < 0)$ being a random variable having a Poisson distribution with mean $-tL(x)$ $[tL(x)]$. If now we assume that $L(0) > -\infty$, then in any finite interval $(0, t]$, only a finite number of jumps can occur, and moreover,

$$\begin{aligned} &\Pr\{Y(t) \text{ is non-increasing in } (0, t]\} \\ &\qquad = \Pr\{\text{no. of jumps of positive magnitude in } (0, t]\} = e^{-\lambda t} \end{aligned} \tag{12.4}$$

where $\lambda = -L(0) < \infty$. Gani and Pyke then define a stochastic process $Z(t)$ $(t \geqq 0)$ to represent the content of a dam over time as follows. Set $Z(0) = z_0 \geqq 0$. Define the random variables

$$\begin{aligned} \tau_1 &= \inf\{t \geqq 0 \mid z_0 + Y(t) \leqq 0\} \\ T_1 &= \sup\{t \geqq \tau_1 \mid Y(t) \text{ is non-increasing in } (\tau_1, t)\}; \end{aligned} \tag{12.5}$$

since $Y(t)$ is separable, τ_1, and T_1 are both random variables (not necessarily finite). Set

$$Z(t) = \begin{cases} z_0 + Y(t) & \text{if } 0 \leqq t < \tau_1 \\ 0 & \text{if } \tau_1 \leqq t < T_1, \\ Y(T_1+) - Y(T_1-) & \text{if } t = T_1. \end{cases} \tag{12.6}$$

Now define recursively for $k > 1$, the random variables

$$\tau_k = \inf\{t \geqq T_{k-1} \mid Z(T_{k-1}) + Y(t) \leqq 0\} \tag{12.7}$$
$$T_k = \sup\{t \geqq \tau_k \mid Y(t) \text{ is non-increasing in } (\tau_k, t)\}$$

and set

$$Z(t) = \begin{cases} Y(t) - Y(T_{k-1}) & \text{if } T_{k-1} < t < \tau_k \\ 0 & \text{if } \tau_k \leqq t < T_k \\ Y(T_k+) - Y(T_k-) & \text{if } t = T_k. \end{cases} \tag{12.8}$$

The interpretation of this process $Z(t)$ as a dam process is as follows. The initial content of the dam is z_0, and if the net input during $(0,t]$ is $Y(t)$, then the content at time t will equal $z_0 + Y(t)$ until this quantity first becomes non-positive, namely at time τ_1. When this happens, the content is set equal to zero and remains there until the net input process has a positive jump, namely at time T_1; the magnitude of this jump is then the new 'initial' content and the procedure then repeats itself.

It is clear that $\tau_1, \tau_2 - T_1, \tau_3 - T_2, \ldots$ are the successive wet periods in the dam, and $T_k - \tau_k$ $(k \geqq 1)$ the successive dry periods; since the process $Y(t)$ is separable, these are well defined random variables and therefore the process $Z(t)$ defined in terms of these random variables is also well defined. Gani and Pyke proved that the above constructive definition of $Z(t)$ is equivalent to the more direct one given by (5.4). They also considered the distribution of times the dam remains in empty and non-empty states.

13. Dams with ordered inputs

13.1. *The first emptiness problem in a dam with ordered inputs.* It occurs frequently in practice that the input into a dam varies seasonally, so that its distribution during the annual wet season is different from the one during the dry season. Gani [(1961), (1962 b)] and Yeo (1961 b) have considered models of a dam with such an ordered input. In Gani's (1961) model a dam of infinite capacity is fed by ordered inputs α_1, α_2 $(\alpha_1 > 0, \alpha_2 > 0)$ such that the distribution of the input $X(t)$ during $(0,t]$ is given by

$$\Pr\left\{X(t) = \left[\frac{j+1}{2}\right]\alpha_1 + \left[\frac{j}{2}\right]\alpha_2\right\} = f(j,t) \quad (j = 0, 1, \ldots). \tag{13.1}$$

Here t may be discrete or continuous, but it is assumed that the input process $X(t)$ is additive, so that its p.g.f. is of the form

$$\psi(\theta, t) = \sum_{j=0}^{\infty} \theta^j f(j,t) = [\psi(\theta, 1)]^t \quad (0 \leqq \theta < 1). \tag{13.2}$$

In continuous time $t \geqq 0$, the only distribution $f(j,t)$ which corresponds to a non-negative integer-valued stochastic process $X(t)$ is the Poisson, but in the discrete case where $t = 0, 1, 2, \ldots$, other distributions exist. The following formulae apply to both discrete as well as continuous times. The dam with these ordered inputs is subject to a continuous release at constant unit rate except when it is empty.

Let T_1 and T_2 be the durations of the wet periods initiated by a dam content z, the first inputs being α_1 and α_2 respectively. It is clear that the possible values of T_1 and T_2 are of the form

$$(13.3)\qquad \begin{aligned} T_1 &= z + \left[\frac{n+1}{2}\right]\alpha_1 + \left[\frac{n}{2}\right]\alpha_2 \\ T_2 &= z + \left[\frac{n}{2}\right]\alpha_1 + \left[\frac{n+1}{2}\right]\alpha_2 \qquad (n = 0, 1, \ldots); \end{aligned}$$

let $g_1(z,t)$ and $g_2(z,t)$ be the corresponding probabilities. We have

$$(13.4)\qquad g_i(z,t) = f(0,z) \text{ if } t = z \ (i = 1,2),$$

$$(13.5)\qquad \begin{aligned} g_1(z,t) &= \sum_{k=1}^{[n/2]} f(2k,z)\, g_1(k\alpha_1 + k\alpha_2, t-z) \\ &+ \sum_{k=1}^{[(n+1)/2]} f(2k-1,z)\, g_2(k\alpha_1 + k\alpha_2 - \alpha_2, t-z) \ (t > z), \end{aligned}$$

and a similar relation for $g_2(z,t)$ obtained from (13.5) by interchanging α_1 and α_2, and, g_1 and g_2. Now for $0 \leqq \theta \leqq 1$ let $\phi_1(\theta,z) \equiv \phi(\theta,\alpha_1,\alpha_2,z)$ be the p.g.f. of T_1 and $\phi_2(\theta,z) = \phi(\theta,\alpha_2,\alpha_1,z)$ that of T_2. Also, let Φ be the column vector

$$(13.6)\qquad \Phi = \{\phi_1(\theta,z_1), \phi_2(\theta,z_2), \phi_3(\theta,z_3), \phi_4(\theta,z_4), \cdots\}'$$

where

$$(13.7)\qquad \begin{aligned} z_{2j} &= \left[\frac{j+1}{2}\right]\alpha_1 + \left[\frac{2j+1}{4}\right]\alpha_2, \\ z_{2j-1} &= \left[\frac{2j+1}{4}\right]\alpha_1 + \left[\frac{j+1}{2}\right]\alpha_2 \qquad (j = 1, 2, \ldots). \end{aligned}$$

Then from the relation (13.5) Gani obtains the formula

$$(13.8)\qquad \Phi = A + \sum_{n=1}^{\infty} B^n A \quad B^0 = I,$$

where A is the infinite column vector with the elements $\theta^{z_r} f(0,z_r)$ $(r = 1,2,\ldots)$, the elements of the infinite matrix B are functions of z_r and $f(j,z_r)$, and the infinite series on the right hand side of (13.8) is convergent for $0 \leqq \theta \leqq 1$. Thus (13.8) gives a formal solution for the problem of first emptiness in the dam with ordered inputs specified by (13.1) and (13.2). It may be verified that when $\alpha_1 = \alpha_2$, the expressions for $\phi_i(\theta, z)$ $(i = 1,2)$ reduce to the known form of the p.g.f. of the wet period (cf. Sections 3 and 8).

In the case of a Poisson input with $f(j,t) = e^{-\lambda t}(\lambda t)^j / j!$ $(j = 0, 1, \dots)$ Gani evaluates the probabilities $g_i(z,t)$ $(i = 1,2)$ directly from the equations (13.5). Thus

$$
\begin{aligned}
& g_1(z, z+\alpha_1) &&= e^{-\lambda(z+\alpha_1)}\lambda z \\
(13.9) \quad & g_1(z, z+\alpha_1+\alpha_2) &&= e^{-\lambda(z+\alpha_1+\alpha_2)}\frac{\lambda^2}{2!}\, z(z+2\alpha_1) \\
& g_1(z, z+2\alpha_1+\alpha_2) &&= e^{-\lambda(z+2\alpha_1+\alpha_2)}\frac{\lambda^2}{3!}\, z\{z^2+3z(\alpha_1+\alpha_2)+3(\alpha_1^2+2\alpha_1\alpha_2)\} \\
& \cdots
\end{aligned}
$$

and similarly for $g_2(z,t)$. In the more general case Gani characterizes first emptiness as an occupancy problem, and for Poisson inputs, obtains the probabilities (13.9) by a rapid computational procedure, using the method of truncated polynomials. Upper and lower bounds for these probabilities have been derived by Gani and Pyke (1960 b).

As a generalization of the model described in this section, Gani and Pyke (1960 b) consider the dam fed by certain inputs $\alpha_1, \alpha_2, \dots, \alpha_p$ not all equal, which follows one another cyclically in that order. The emptiness probabilities $g_i(z,t)$ $(i = 1, 2, \dots, p)$ here satisfy recurrence relations similar to (13.5), which are rather unwieldy; simple inequalities are derived for them, which are found to be more useful in practice.

Finally we remark that the emptiness problem with two ordered inputs is formally equivalent to that of a system of two dams in parallel, both subject to a steady release at constant unit rate, and fed by a discrete additive input process such that inputs are always directed to the dam with lesser content [Gani (1961)].

13.2. *The time-dependent solution for a dam with ordered Poisson inputs.* We now describe a model for a dam fed by two ordered Poisson inputs, considered by Gani (1962 b). Here the dam content $Z(t)$ at time $t \geqq 0$ is defined by the equation.

$$(13.10) \qquad Z(t+dt) = Z(t) + dX(t) - (1-r)\,dt$$

where rdt $(0 \leqq r \leqq 1)$ is the time during $(t, t+dt)$ that the dam is empty, and the input $X(t)$ during a time interval $(0,t]$ is given by

$$(13.11) \qquad X(t) = \left[\frac{N(t)+1}{2}\right]\alpha_1 + \left[\frac{N(t)}{2}\right]\alpha_2 \quad (\alpha_1 \neq \alpha_2, \alpha_1 > 0, \alpha_2 > 0),$$

$N(t)$ being a Poisson process with mean λt. The interpretation of (13.11) is that inputs of constant sizes α_1, α_2 arrive consecutively in that order, in a Poisson process with mean λt. The process $Z(t)$ defined by (13.10) is not Markovian; $\{Z(t), N(t)\}$, however, jointly define a time-homogeneous Markov process. Let

$$\text{(13.12)}\qquad \begin{aligned} F_1(z_0; z,t) &= \Pr\{Z(t) \leqq z;\, N(t) \text{ odd} \mid Z(0) = z_0\} \\ F_2(z_0; z,t) &= \Pr\{Z(t) \leqq z;\, N(t) \text{ even} \mid Z(0) = z_0\}. \end{aligned}$$

The transition d.f. of $Z(t)$ is then given by

$$\text{(13.13)}\qquad \begin{aligned} F(z_0; z,t) &= \Pr\{Z(t) \leqq z \mid Z(0) = z_0\} \\ &= F_1(z_0; z,t) + F_2(z_0; z,t). \end{aligned}$$

Each of the functions F_1 and F_2 has properties similar to those of the transition d.f. of the dam with a single Poisson input described in Section 8; as indicated there, we may derive for these d.f.'s two differential equations

$$\text{(13.14)}\qquad \begin{aligned} \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial z}\right) F_1(z_0; z,t) &= -\lambda F_1(z_0; z,t) + \lambda F_2(z_0; z-\alpha_1, t) \\ \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial z}\right) F_2(z_0; z,t) &= -\lambda F_2(z_0; z,t) + \lambda F_1(z_0; z-\alpha_2, t)(z \geqq 0, t \geqq 0), \end{aligned}$$

the solutions of which are found to be

$$\text{(13.15)}\qquad \begin{aligned} &F_1(z_0; z,t) \\ &= \sum_{k=0}^{[(n-1)/2]} F_1(0, t+z-k\alpha_1-k\alpha_2) f(2k, z-k\alpha_1-k\alpha_2) \\ &+ \sum_{k=1}^{[n/2]} F_2(0, t+z-k\alpha_1-k\alpha_2+\alpha_2) f(2k-1, z-k\alpha_1-k\alpha_2+\alpha_2) \\ &\left(\left[\frac{n}{2}\right]\alpha_1 + \left[\frac{n-1}{2}\right]\alpha_2 \leqq z < \left[\frac{n+1}{2}\right]\alpha_1 + \left[\frac{n}{2}\right]\alpha_2 \quad n = 1,2,\ldots\right), \end{aligned}$$

and similarly for $F_2(z_0; z,t)$, where $f(j,z) = e^{-\lambda z}(\lambda z)^j/j!$ $(j = 0, 1, \ldots)$. The result (13.15) contains the probabilities $F_i(z_0; 0,t)$ of emptiness of the dam at time t; to determine these, consider the first emptiness probabilities $g_i(z,t)$ which has been already obtained [equations (13.9)]. Now let us define the two probabilities $g_{11}(z,t)$ and $g_{12}(z,t)$ for $t = z + [(n+1)/2]\alpha_1 + [n/2]\alpha_2$ as follows:

$$\text{(13.16)}\qquad g_{11}(z,t) = \begin{cases} 0 & \text{if } n \text{ is even} \\ g_1(z,t) & \text{if } n \text{ is odd} \end{cases}$$

$$\text{(13.17)}\qquad g_{12}(z,t)] = \begin{cases} g_1(z,t) & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Clearly, $g_1(z,t) = g_{11}(z,t) + g_{12}(z,t)$. Using combinatorial arguments Gani proves that

$$(13.18) \qquad F_1(z;0,t) = \sum_{n=0}^{\infty} g_{11}\left(t - \frac{n+1}{2}\alpha_1 - \left[\frac{n}{2}\right]\alpha_2, t\right)$$

and similarly for $F_2(z;0,t)$. The solution (13.15) is thus complete.

Yeo (1961 b) considers a discrete model with two ordered inputs which are independent and additive. Proceeding as in the case of a single input (see Section 2) he obtains the transition probabilities of the dam content.

14. A model with random linear inputs and outputs

This model was first introduced by Gaver and Miller (1962), and originally arose in the study of some diffusion processes. Thus, in particular, Goldstein (1951) had considered a diffusion model for a large number of non-interacting particles moving along a straight line with a uniform velocity v. Starting from an origin, half of these particles move to the right and the other half to the left, and thereafter, at the end of each time interval τ (fixed) each particle either continues to move in the same direction with probability p or moves in the opposite direction with probability q $(p + q = 1)$. Let $\gamma(n, v)$ be the fraction of the number of particles at a distance vd (where $d = v\tau$) from the origin after a time $n\tau$; the moment generating function of $\gamma(n,v)$ is obtained by the usual arguments. From this, by putting $2d = dy$, $ndy = 2D$, $nq/p = B$, and letting $n \to \infty$, Goldstein obtains a c.f., which, on inversion, gives the d.f. of the distance of a particle from the origin, in the continuous time diffusion model which is the limit of the discrete model. The difference equation for $\gamma(n,v)$ gives rise to the differential equation of the continuous time model. [These limiting techniques may be compared with those of Moran (1956).]

Gaver and Miller's model, however, is designed to describe storage processes; here there is a barrier at the origin, which changes the structure of the process. The model itself is described as follows. Let $Z(t)$ be the storage level at time $t \geqq 0$ and assume that $Z(0) = 0$; $Z(t)$ increases and decreases alternately over random intervals of time. If $Z(t) = 0$ for some $t > 0$, it remains at zero until the end of the current period of decrease. The rates of increase and decrease are respectively $+1$ and -1. Let $v_1, v_2, v_3, \ldots$ denote the successive intervals of increase, and $u_1, u_2, u_3, \ldots$ the successive intervals of decrease; it is assumed that $\{v_i\}$ and $\{u_i\}$ are independent random variables. The v_i are identically distributed with the d.f. $B(x)$, and the u_i are identically distributed with the d.f. $A(x)$ $(0 \leqq x < \infty)$. For convenience let us assume that the initial period is an interval of decrease, say u_0. Then from the above description we find that

$$(14.1) \qquad Z(t) = 0 \text{ for } 0 \leqq t \leqq u_0,$$

$$(14.2) \qquad Z(t+dt) = \begin{cases} Z(t) + dt & \text{for } t_n \leqq t < t + dt \leqq t_n + v_{n+1} \\ \max(0, Z(t) - dt) & \text{for } t_n + v_{n+1} \leqq t \leqq t_{n+1} \; (n = 0,1,2,\ldots), \end{cases}$$

where $t_0 = u_0$ and $t_n = u_0 + \sum_1^n (v_i + u_i)$ $(n \geqq 1)$.

In the general case where both d.f.'s $A(x)$ and $B(x)$ are arbitrary, the system will be denoted by $B/A/\infty$, where the symbol ∞ indicates that there is no upper barrier on the height of the process. This symbol is to be replaced by K if there is an upper reflecting barrier at height K similar to the lower reflecting barrier at 0.

Except in the special case where both $A(x)$ and $B(x)$ are negative exponential (M) d.f.'s, the process $Z(t)$ is non-Markovian. However, the systems $M/A/\infty$ and $B/M/\infty$, where one of these distributions is a negative exponential, while the other is arbitrary, can be analyzed by the use of supplementary variables, as was shown by Miller (1963). As an illustration we describe below Miller's results for the system $B/M/\infty$.

Let $A(x) = 1 - e^{-\lambda x} (\lambda > 0)$, and $dB(x) = b(x)\,dx \quad (x > 0)$. For $t \geqq 0$ let us define the random variable $Y(t)$ by

$$Y(t) = \begin{cases} 0 & \text{if } 0 \leqq t \leqq t_0 \\ t - t_n & \text{if } t_n \leqq t < t_{n+1} \ (n = 0, 1, \dots); \end{cases} \tag{14.3}$$

$Y(t)$ is the amount of time which has elapsed since the beginning of the last preceding interval of increase. In terms of the renewal process $\{t_n\}$, $Y(t)$ is the age of the article in use at time t, and from renewal theory it will be found that it has a frequency density function. The state of the system will be defined by three variables, namely, $Z(t)$, $Y(t)$ and whether $Z(t)$ is increasing or decreasing; this definition makes the process Markovian. Let

$$\begin{aligned} F_+(x,y,t) &= \Pr\{Z(t) \leqq x, y \leqq Y(t) \leqq y + dy, Z(t) \text{ increasing}\} \\ & \qquad\qquad\qquad\qquad\qquad\qquad (x > 0,\ y \geqq 0) \\ F_-(x,t) &= \Pr\{Z(t) \leqq x, Z(t) \text{ decreasing}\} \qquad (x \geqq 0) \end{aligned} \tag{14.4}$$

The function $F_-(x,t)$ has a discontinuity at $x = 0$, $F_-(0,t)$ being the discrete probability of the store being empty at time t. Considering $Z(t)$ over the infinitesimal time interval $[t, t + dt]$, we find that

$$\begin{aligned} F_+(x,y,t+dt) &= \{1 - f^*(y - dt\}F_+(x - dt, y - dt, t) + o(dt) \\ & \qquad\qquad\qquad\qquad\qquad\qquad (x > 0, y > 0) \end{aligned} \tag{14.5}$$

and

$$\begin{aligned} F_-(x,t+dt) &= (1 - \lambda dt) F_-(x + dt, t) \\ & \quad + \int_0^\infty d_y F_+(x + dt, y, t) f^*(y) dt + o(dt) \qquad (x \geqq 0), \end{aligned} \tag{14.6}$$

where $f^*(y) = b(y)/[1 - B(y)]$. Let us assume that $F_+(x,y,t)$ and $F_-(x,t)$ have continuous partial derivatives with respect to $x, y, t > 0$; then expanding both sides of (14.5) and (14.6) in terms of these partial derivatives and letting $dt \to 0$, we obtain the forward Kolmogorov equations of the process:

$$(14.7)\qquad \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial t}\right) F_+(x,y,t) = -f^*(y)\, F_+(x,y,t),\ 0<y<\min(x,t),$$

$$(14.8)\qquad \frac{\partial}{\partial t}F_-(x,t) - \frac{\partial}{\partial x}F_-(x,t) = -\lambda F_-(x,t) + \int_0^\infty F_+(x,y,t)\, f^*(y)\, dy \qquad (x,t>0).$$

The boundary conditions are

$$(14.9)\qquad \begin{aligned} F_+(x,y,0) &= 0 \quad \text{for } x \geqq 0,\ y \geqq 0, \\ F_-(x,0) &= 1 \quad \text{for } x \geqq 0 \end{aligned}$$

and

$$(14.10)\qquad F_+(x,0,t) = \lambda F_-(x,t) \text{ for } x>0.$$

The solution to the linear, first order partial differential equation (14.7) subject to the boundary condition (14.10) is given by

$$(14.11)\qquad F_+(x,y,t) = \lambda[1-B(y)]\, F_-(x-y,\, t-y).$$

Substituting (14.11) into (14.8) we obtain the integro-differential equation

$$(14.12)\qquad \frac{\partial}{\partial t} F_-(x,t) - \frac{\partial}{\partial x} F_-(x,t) = -\lambda F_-(x,t) + \lambda \int_0^\infty F_-(x-y,t-y)\, b(y) dy,$$

which is similar to Takács' equation (9.3) for the waiting time in the queueing system $M/G/1$, the only difference being in the integrand. Let

$$(14.13)\qquad \psi(\theta) = \int_0^\infty e^{-\theta x}\, dB(x)$$

$$(14.14)\qquad \Phi_-(\theta,s) = \int_{t=0}^\infty \int_{x=0-}^\infty e^{-\theta x - st} d_x F_-(x,t)\, dt$$

for suitable values of θ's. Then proceeding as in the system $M/G/1$ we find that

$$(14.15)\qquad \Phi_-(\theta,s) = \frac{1-\theta F_-^*(0,s)}{s-\theta+\lambda-\lambda\psi(\theta+s)},$$

where

$$(14.16)\qquad F_-^*(0,s) = \int_0^\infty e^{-st} F_-(0,t)\, dt.$$

To evaluate the unknown transform $F_-^*(0,s)$ occurring in (14.15) we consider the distribution of non-empty periods in the system. A non-empty period T commences at a point at which the storage level leaves zero and lasts until the first subsequent return of the level to zero; thus, for instance, (14.17) $$T = \inf\{t > t_0 \mid Z(t) = 0\}.$$

Clearly, $T = \sum_1^{N-1}(v_i + u_i) + v_N + u_N'$, where

$$N = \min\{n > 0 \mid Z(t_n) = 0\}$$

(14.18) $$= \min\{n > 0 \mid (v_1 + v_2 + \ldots + v_n) - (u_1 + u_2 + \ldots + u_n) \leqq 0\}$$

and u_N' is such that $(v_1 + v_2 + \ldots + v_N) - (u_1 + u_2 + \ldots + u_{N-1} + u_N') = 0$. Thus $T = 2T_1$, where $T_1 = v_1 + v_2 + \ldots + v_N$. However, it is seen that T_1 is the busy period in the queueing system $M/G/1$, and therefore it follows from the remarks in the last paragraph of Section 9 that the Laplace-Stieltjes transform $\Gamma(\theta) = E(e^{-\theta N})$ satisfies the functional equation

(14.19) $$\Gamma(\theta) = \psi(2\theta + \lambda - \lambda\Gamma(\theta))$$

with $\Gamma(\infty) = 0$. From Takács' (1955) result we can then find that the transform $F_-^*(0,s)$ is given by

(14.20) $$F_-^*(0,s) = \{s + \lambda - \lambda\Gamma(s)\}^{-1}$$

so that, if (14.19) can be solved explicitly for $\Gamma(\theta)$, the expression (14.15) is also explicit. The transform of $F_+(x,y,t)$, namely

(14.21) $$\Phi_+(\theta,\omega,s) = \int_{t=0}^{\infty}\int_{x=0}^{\infty}\int_{y=0}^{\infty} e^{-\theta x - \omega y - st} d_x F_+(x,y,t)\, dt\, dy$$

can be obtained from (14.11) and (14.15); thus

(14.22) $$\Phi_+(\theta,\omega,s) = \lambda\Phi_-(\theta,s)\frac{1-\psi(\theta+\omega+s)}{\theta+\omega+s}.$$

The limiting distribution of $Z(t)$ as $t \to \infty$ can be investigated by using the results (14.14), (14.20) and (14.22).

15. Some general comments: unsolved problems and directions for research

We conclude this survey by drawing attention to some unsolved problems and indicating possible directions for research.

The various extensions of Moran's original model, discussed in Parts II and III, represent major advances in storage theory; although most of these extensions deal with dams of infinite capacity, the results obtained can be considered as useful approximations for the finite capacity case. However, these models are subject to the same criticism as Moran's model, in that they are all based on the assumption of mutual independence of inputs. Some indication of the kind of results which might emerge if this assumption is removed, is provided by Lloyd's (1963a) work on dams with correlated inputs; his results hold in statistical equilibrium, but time-dependent solutions, when they become available, will present a more complete picture. More light will be thrown on the situation when a theory dealing with correlated inputs is developed in continuous time, for it is there that the independence assumption is least likely to hold.

A second unsatisfactory feature of the theory in its present position is the rather oversimplified release rule, according to which a constant amount of water is released, if available (the only exception to this is in Gani and Pyke's (1960a) model). In the general inventory model the output at any time t is a function of the demand and the amount in stock at time t. For a dam, it is realistic to take the outflow to be proportional to the square root of the storage height, this height depending on the section of the dam, and the theory should therefore take account of this. Another interesting but yet unsolved problem is that of dams in series.

Our final comment concerns techniques used in storage theory. It is interesting to note that the stochastic processes which have arisen in storage theory so far, are all Markovian, and that the tools used in their analysis involve forward Kolmogorov equations, combinatorial arguments and some limiting methods. As in other branches of probability, combinatorial methods are likely to prove increasingly useful in dealing with storage problems. A technique which is familiar in the theory of Markov processes, but one which has not so far been used in storage theory, is one using operators. An operator-theoretic treatment of the integral equation (5.2) is likely to yield the integro-differential equation (11.12) directly, and it seems also possible that such a treatment of this integral equation, suitably modified to take account of dependence of inputs, would yield the required solutions. It is to be anticipated that complicated stochastic processes, not necessarily Markovian, would arise from more refined storage models, and that the tools necessary to deal with them would have to be more powerful than those used so far.

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