Introduction to Measurements and Error Analysis

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1 True Value, Error and Measurement

Numerical statements can be commonly used in our daily life. For example, the man is 26 years old, or the price of the cell phone is 1300 yuan. In fact, these statements contain values and units. The numerical statements convey some information which is good enough for communications. But did you notice in the "uncertainty" in the normal converstations? I.e. the person who claimed 26 years old is not exactly 26 years old but may be 26 years x month y days z hours a minutes ... old. and the cell phone may be slightly cheaper than 1300 yuan in some stores. Such "uncertainty" is ignorable in the common conversation, but in scientific research, mechanical engineering, material fabrication and etc, the "uncertainty" is critical. Now let's perform a scientific measurement. For example, what is the length of the pencil in Figure 1?

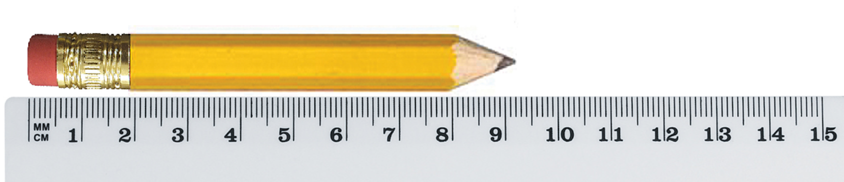

Figure 1: Measurement: length of a pencil.

Obviously, the length of the pencil is between 9cm and 10cm. But is the length 9.1cm, 9.2cm or 9.3cm? Given the precision of the ruler, we can estimate the length of the pencil as "around 9.2cm". But 9.2cm is not the "true value" of the length. In the daily conversation about the length of the pencil, reporting 9.2cm is enough. In contrast, the uncertainty should be presented in a scientific report. As a result, a measurement should be expressed as:

$$Measurement = value(best\, estimate) \pm uncertainty \quad unit \tag{1}$$

Note that the "±" operator is for notation and there is no need to add or subtract. The length is between 9.1cm and 9.3cm. So, to be precise, we write the length as 9.2±0.1 cm as the format provided in Equation 1. The previous example shows how scientists deal with not knowing the true value: by providing the best estimate with a range of uncertainty. In scientific research,

"No measurement made is ever exact!"

The difference between our measurement and the true value is called "experimental error" or simply "error". The error is NOT mistakes or blunders.

$$Error = Measurement - True\, value \tag{2}$$

Given an object, it is impossible to get the "exact" length (true value) we need. Instead, we must deal with the errors. As the strategy introduced previously, we perform multiple measurements and analyze the instrument. Though each measurement differs from the true value by an error, we can summarize all the measurements and errors as the best estimate and uncertainty to approach the true value.

As explained, each measurement is involved by several sources of error. In general, the errors can be classified into two categories.

1. Systematic (determinate) error can also be called statistical bias which always occurs in the same way. The systematic error is originated from instruments, theory and method, environment and the observer. E.g. the instruments we used can never be the perfect ones; the theoretical formulas ignore the air resistance; the temperature does not match the required temperature; the observer who read the meter has some personal habit and etc.

2. Random (indeterminate) error, also called noise, varies in each measurement and cannot be controlled. It occurs randomly when we push our measurement to the limit of the instrument. E.g. in the pencil measurement example, the limit of the ruler is mm. So the random error shows when we try to read the accurate length.

Having the knowledge of the two types of the errors, let's look at how to approach to the best estimate and the uncertainty mathematically.

2 Best Estimate and Uncertainty

When performing multiple measurement, the result can be improved. By taking the advantage of average of several measurement we get our best estimate.

$$l_{best} = \bar{l} = \frac{1}{N}(l_1 + l_2 + \cdots + l_N) \tag{3}$$

The total uncertainty depends on the two classes of the errors. We call the uncertainty from the random error "Type A evaluation of uncertainty" or u_A while the uncertainty from the systematic error "Type B evaluation of uncertainty" or u_B. Then the total uncertainty is calculated using Equation 4.

$$u = \sqrt{u_A^2 + u_B^2} \tag{4}$$

The Type A evaluation of uncertainty can be resolved statistically. Suppose we have multiple measurement for an object and the measurement is only affected by the random error, then result follows a Normal distribution (Gaussian function): there will be more measurement results close to the true value than the ones differ a lot. Assume that we use the same ruler to measure the pencil for 100 times, the result is provided in Table 1. Note that the last digit of each measurement is estimated.

9.13	9.18	9.25	9.19	9.25	9.21	9.15	9.16	9.20	9.19
9.19	9.27	9.21	9.18	9.16	9.25	9.20	9.15	9.26	9.23
9.21	9.24	9.22	9.17	9.19	9.22	9.18	9.22	9.16	9.23
9.28	9.22	9.23	9.16	9.21	9.18	9.24	9.20	9.15	9.27
9.17	9.18	9.26	9.14	9.13	9.25	9.24	9.19	9.12	9.23
9.17	9.21	9.25	9.17	9.22	9.21	9.23	9.26	9.17	9.25
9.16	9.16	9.25	9.21	9.22	9.20	9.11	9.26	9.25	9.21
9.13	9.22	9.18	9.21	9.23	9.22	9.20	9.22	9.21	9.24
9.17	9.23	9.14	9.21	9.22	9.18	9.22	9.18	9.17	9.19
9.19	9.15	9.21	9.19	9.21	9.16	9.22	9.19	9.20	9.16

Table 1: Pencil measurement: 50 times (unit: cm)

By applying the histogram analysis, we can get a general idea of the measurement as shown in Figure 2 (left). As we improve the measurement by performing more and more measurements, the histogram will eventually shift to a normal distribution as shown in Figure 2 (right).

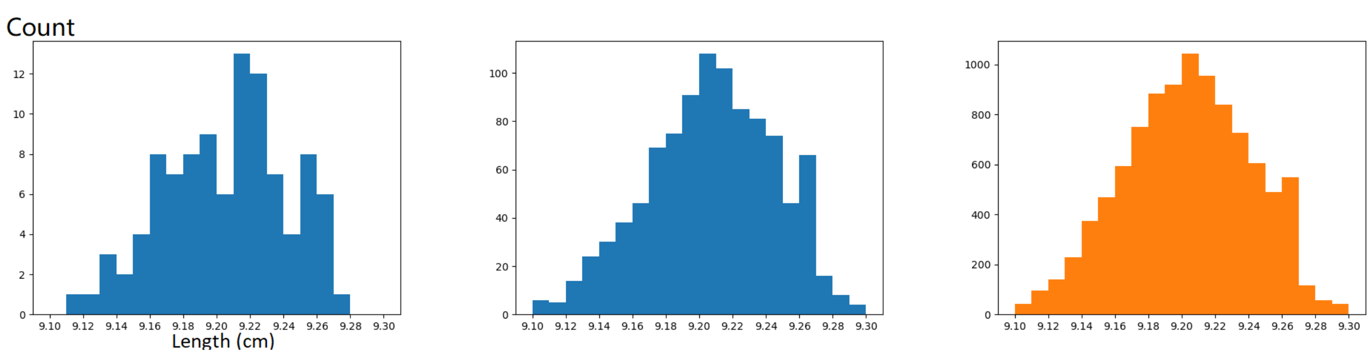

Figure 2: Histogram of 100 measurement (left), 1k measurement (middle) and 10k measurement (right).

The statistical distribution behind the random error is the normal distribution. Therefore, we can evaluate the fluctuation of the errors by the standard deviation which is calculated as

$$\sigma_l = \sqrt{\frac{1}{N-1}}\sqrt{\left[\left(l_1-\bar{l}\right)^2+\left(l_2-\bar{l}\right)^2+\cdots+\left(l_N-\bar{l}\right)^2\right]} \tag{5}$$

The Type A evaluation of uncertainty is then evaluated as

$$u_A = \frac{t(N)}{\sqrt{N}}S = t(N)\sqrt{\frac{1}{N(N-1)}}\sqrt{\left[\left(l_1-\bar{l}\right)^2+\left(l_2-\bar{l}\right)^2+\cdots+\left(l_N-\bar{l}\right)^2\right]} \tag{6}$$

Please memorize Equation 6 for it is widely used in many experiments! N is the time of measurement, the larger the more accurate. $t(N)$ is a factor which is mathematically obtained from normal distribution as shown in Table 2. u_A is proportional to the standard deviation of the length distribution.

N	3	4	5	6	7	8	9	10	11
$t_{0.683}$	1.32	1.20	1.14	1.11	1.09	1.08	1.07	1.06	1.05
$t_{0.95}$	4.30	3.18	2.78	2.57	2.45	2.36	2.31	2.26	2.23
$t_{0.99}$	9.92	5.84	4.60	4.03	3.71	3.50	3.36	3.25	3.17

Table 2: t factor at different confidence interval with different measurement times.

We will use $t_{0.95}$ row through all the experiments covered by this lab manual. For example, if one measures some part for 5 times, the factor taken should be 2.78.

Equation 6 provides us a statistical to approach for u_A. But for u_B, it varies in different cases which depends on the instruments and how we use the instrument. We can intuitively understand that it is related to the instrument producer, working condition and the Least Count, the smallest division marked on the instrument (For example, the Least Count for the ruler in Figure 1 is 0.1cm). Therefore, the instruments such as ruler, scale, thermal meter and etc show their own uncertainties during the usage. Such phenomena do NOT mean the poor quality, but because of the variation in the mass production. Image that you use an lathe to fabricate 100 scales, there are always some small differences between every 2 scales. All the instrument related uncertainty is summarized as u_B which can be evaluated using Equation 7.

$$u_B = \frac{\Delta}{C} \tag{7}$$

The constant factor C is a statistical factor which depends on the fabrication of the instrument which represents the distribution of fabrication error among all the instruments. $\Delta$is determined by the error due to the Least Count and the particular instrument. In different experiments, the two values will be posted on the instruction. In general, when the two types of uncertainty are determined, the total uncertainty is calculated using Equation 4.

3 Significant Figures and Significance Arithmetic Rules

Given an object and a meter, one can get an estimate instead of the true value. The record of your measurement should contain one extra figure than the Least Count. For example, the pencil in Figure 1 should be record as 9.24cm (you may have different estimate such as 9.25cm, 9.28cm, 9.21cm, etc). The last figure 4 is the estimate which is one order smaller than the Least Count. The significant figures for the length of the pencil is 9.24 including the definite figures, 9.2 and the extra figure, 4.

In our lab, all of the measurements MUST have the extra figure of estimate if possible. For example, one student measured the diameter of a cap nut using the centimeter for 10 times. The results are recorded as shown in Table 3.

Cap nut diameter (unit: mm)				
14.298	14.256	14.290	14.262	14.234
14.263	14.242	14.272	14.278	14.216

Table 3: 10 record of the measurement result for a cap nut using centimeter.

If the length of the object falls exactly on the tick in some cases, then the extra figure should be 0.

We then apply some calculation to the measurement results. After each calculation, the extra figure on the final result should match the extra figure on the data. Here is an example of taking the averaged diameter in Table 3. A bar is used to denote the extra figure in the equation.

$$d_{avg} = \frac{1}{10}\left(14.29\bar{8}+\cdots+14.21\bar{6}\right) = 14.26\bar{1}\bar{1} \rightarrow 14.26\bar{1}\,mm \tag{8}$$

The addition and division of the measurement results are just normal operations, but the final result contains as much extra figures as the measurement. In each diameter measurement, there is only one extra figure. Therefore, the averaged diameter contains one extra figure by rounding off the last figure. So you CANNOT keep more extra figure than the measurement. In addition, we use the round-to-even rule: "round half to even".

The Type A uncertainty can be calculated using Equation 6. Note that the uncertainty is "aligned" to the averaged diameter by figures.

$$u_A = 2.26\sqrt{\frac{1}{10 \times 9}}\sqrt{(14.29\bar{8} - 14.26\bar{1}\bar{1})^2 + \cdots + (14.21\bar{6} - 14.26\bar{1}\bar{1})^2} = 0.0180876\cdots \to 0.018\,mm \tag{9}$$

The measurement of the cap nut yields $14.261 \pm 0.018\,mm$.

In addition, there can be algebra operation applied to two numbers. The result shall match the larger number in additions or subtractions, match the fewer significant figures in multiplication or division. For example:

$$32.\bar{1} + 3.27\bar{6} = 35.\bar{3}\bar{7}\bar{6} \to 35.\bar{4} \tag{10}$$

$$26.\bar{6}\bar{5} - 3.92\bar{6} = 22.\bar{7}\bar{2}\bar{4} \to 22.\bar{7}\bar{2} \tag{11}$$

$$5.34\bar{8} \times 20.\bar{5} = 10\bar{9}.\bar{6}\bar{3}\bar{4}\bar{0} \to 11\bar{0} \tag{12}$$

$$3764.\bar{3} \div 21.\bar{7} = 17\bar{3}.\bar{4}\bar{7}\bar{0} \to 17\bar{3} \tag{13}$$

Follow the algebra operation rules, we can monitor the results by keeping the precision of the measurements in the measurement results. In contrast, if we ignore the operation laws, we may end up with a "more accurate result" but some of the figures are artifact which are beyond the scope of the instruments.

For the other operations including trigonometric, exponential, logarithmic evaluations, the Condition Number should be used to determine the significant figures. The one variable Condition Number can be calculated as

$$Condition\,Number = \left|\frac{xf'(x)}{f(x)}\right|. \tag{14}$$

From the definition of the Condition Number, one can see that it measures the change of a function which is also depends on the measured value. In general, if the order of magnitude of the condition number is N_o, then we should round the result with N_o figures fewer. For example, we measure an angle which is 35.58°. And we compute the Condition Number and get 0.868 which has order of magnitude 0. The result contains as many figures as the angle.

$$\sin(35.5\bar{8}^\circ) = 0.5818391\cdots \to 0.581\bar{8}\,. \tag{15}$$

Same rule apply to exponential operations. When we compute $e^{5.6142}$, the Condition Number is 5.6142 which has the order of magnitude 1. Therefore, the result should be 1 figure fewer than 5.6142.

$$e^{5.614\bar{2}} = 274.293\cdots \to 274.\bar{2} \tag{16}$$

The Formula for the Condition Number for commonly used function is provided in Table 4.

Function	Condition Number
e^x	$\lvert x\rvert$
$\ln(x)$	$\frac{1}{\lvert\ln(x)\rvert}$
$\sin(x)$	$\lvert x\cot(x)\rvert$
$\cos(x)$	$\lvert x\tan(x)\rvert$
$\arcsin(x)$	$\left\lvert\frac{x}{\sqrt{1-x^2}\arcsin(x)}\right\rvert$
$\arccos(x)$	$\left\lvert\frac{x}{\sqrt{1-x^2}\arccos(x)}\right\rvert$
$\arctan(x)$	$\left\lvert\frac{x}{(1+x^2)\arctan(x)}\right\rvert$

Table 4: Condition Number of the mostly used functions.

To conclude, the calculation with significant figures is different from the operation we learn, please pay special attention to the round issues.

	description	example
Significant figures	definite figures + 1 estimate figures	9.24cm
rounding	round half to even	$3.266 \to 3.26$ $3.276 \to 3.28$
+ or -	match the larger number	$32.\bar{1} + 3.27\bar{6} = 35.\bar{3}\bar{7}\bar{6} \to 35.\bar{4}$ $26.\bar{6}\bar{5} - 3.92\bar{6} = 22.\bar{7}\bar{2}\bar{4} \to 22.\bar{7}\bar{2}$
$\times$ or $\div$	match the fewer significant figures	$5.34\bar{8} \times 20.\bar{5} = 10\bar{9}.\bar{6}\bar{3}\bar{4}\bar{0} \to 11\bar{0}$ $3764.\bar{3} \div 21.\bar{7} = 17\bar{3}.\bar{4}\bar{7}\bar{0} \to 17\bar{3}$
uncertainty	match the averaged value in decimal	$14.26\bar{1} \pm 0.018\, cm$
Other operation	Check Condition Number	

Table 5: Summary for the significant figures and significance arithmetic rules.

4 Propagation of Uncertainty

In practice, many quantities cannot be measured directly but derived from other measured quantities. The uncertainty of each measurement combined to the total uncertainty through the derivation. The combination of uncertainties depends on the formula used to calculate the final quantity. If you familiar with calculus, recall that the total differential of a function $w = f(x)$ is

$$\Delta w = \frac{\partial f}{\partial x} \Delta x \,. \tag{17}$$

The deviation of w can be related to the deviations of x. Suppose we somehow measured x with uncertainty u_x. The u_x can be mathematically understood as the deviation, Δx. Then the uncertainty of w, u_w is

$$u_w = \sqrt{(\Delta w)^2} = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 u_x^2} \tag{18}$$

Similarly, when there are more than 1 measurements needed in some experiments and the function is $w = f(x, y, z, \cdots)$, the uncertainty of w is accumulated as

$$u_w = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 u_x^2 + \left(\frac{\partial f}{\partial y}\right)^2 u_y^2 + \left(\frac{\partial f}{\partial z}\right)^2 u_z^2 + \cdots} \,. \tag{19}$$

If the function f is given, we can take the derivatives and obtain the result listed in Table 6. To apply, simply match the formula for a particular lab to one in the table for the detailed form of uncertainty.

Formula expression	Total uncertainty
$w = x + y$ or $w = x - y$	$u_w = \sqrt{u_x^2 + u_y^2}$
$w = xy$	$\frac{u_w}{w} = \sqrt{\left(\frac{u_x}{x}\right)^2 + \left(\frac{u_y}{y}\right)^2}$
$w = \frac{x}{y}$	$\frac{u_w}{w} = \sqrt{\left(\frac{u_x}{x}\right)^2 + \left(\frac{u_y}{y}\right)^2}$
$w = \frac{x^k y^m}{z^n}$	$\frac{u_w}{w} = \sqrt{k^2 \left(\frac{u_x}{x}\right)^2 + m^2 \left(\frac{u_y}{y}\right)^2 + n^2 \left(\frac{u_z}{z}\right)^2}$
$w = kx$	$u_w = k u_x \, OR \, \frac{u_w}{w} = \frac{u_x}{x}$
$w = \sqrt[k]{x}$	$\frac{u_w}{w} = \frac{1}{k} \frac{u_x}{x}$
$w = \sin x$	$u_w = \cos x u_x$
$w = \ln x$	$u_w = \frac{u_x}{x}$

Table 6: Propagation of uncertainty for common functions.

5 Data Interpretation: Ordinary Least Square Fitting

In the previous introduction, we learned how to overcome the error to approach the true value at some level. Sometimes, the relationship of two quantities, especially for the linear relationship. In the formal studies, we have seen linear formulas such as s=vt, F=ma, U=IR and many more. But when the error presents in each measurement, how do we monitor the linear relation? In this section, let us discuss the Ordinary Least Square (OLS) method and its applications.

For demonstration purpose, the data collected in an experiment is shown in Table 7.

t/c°	10.0	20.0	30.0	40.0	50.0	60.0
$\gamma/(10^{-3}N\cdot m^{-1})$	74.22	72.75	71.78	69.56	67.91	66.18

Table 7: Tension factor measurement at different temperature.

By plotting the raw data, we see the 6 data points are very close to a straight line and the small deviations should be the errors due to the uncertainties. Though it is impossible to fit the straight line to all the points, there is always a "compromise" such that none of the points stays far from the straight line. The idea of the linear fitting is to find 2 parameters k and b such that $y = k\,x + b$ lies close to all points. To find the slope and intercept, we define a "cost function" as

$$J(k, b) = (k\,t_1 + b - \gamma_1)^2 + \cdots + (k\,t_6 + b - \gamma_6)^2\ . \tag{20}$$

In Equation 20, $k\,t_1 + b - \gamma_1$, also named residual, represents the difference of the value from the linear model, $k\,t_i + b$ and the measurement, γ_i. Thus the squares of the differences sum to the total cost. For convenient, we define two matrices X and Y,

$$X = \begin{bmatrix} 1 & t_1 \\ 1 & t_2 \\ 1 & t_3 \\ 1 & t_4 \\ 1 & t_5 \\ 1 & t_6 \end{bmatrix},\ Y = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \\ \gamma_5 \\ \gamma_6 \end{bmatrix}. \tag{21}$$

Since the cost function is proportional to the total residual which reflect the deviation of all data points to the straight line, we then can minimize the cost function to find the best "compromise" which corresponds to the smallest deviation. Thus, the parameter k and b can be solved.

$$\begin{bmatrix} b \\ k \end{bmatrix} = \frac{X^T\cdot Y}{X^T\cdot X} \tag{22}$$

The "Least" in the name refers to the minimization process and the "Square" refers to the definition of the cost function. With the OLS method, we can obtain b and k easily using Equation 22. It is also very easy to extend the application from few observations to a large number of observations. The most important application is in data fitting. The raw data and fitted data is plotted in Figure 3.

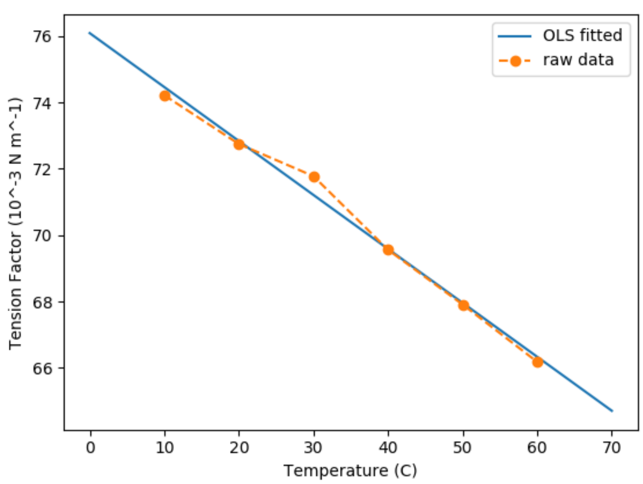

Figure 3: OLS method: raw data and fitted line.

In fact, we can expand Equation 22 if the number of data points is not large (typically 4~10 data points). If there are only 4 points, the slope and intercept can be expressed as

$$b = \frac{\left(x_1^2 + x_2^2 + x_3^2 + x_4^2\right)(y_1 + y_2 + y_3 + y_4) - (x_1 + x_2 + x_3 + x_4)(x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4)}{4\left(x_1^2 + x_2^2 + x_3^2 + x_4^2\right) - (x_1 + x_2 + x_3 + x_4)^2} \tag{23}$$

$$k = \frac{4\left(x_1 y_1 + x_2 y_2 + x_3 y_3 + x_4 y_4\right) - \left(x_1 + x_2 + x_3 + x_4\right)\left(y_1 + y_2 + y_3 + y_4\right)}{4\left(x_1^2 + x_2^2 + x_3^2 + x_4^2\right) - \left(x_1 + x_2 + x_3 + x_4\right)^2} \tag{24}$$

For simplicity, we define:

$$\begin{gathered} xx = x_1^2 + x_2^2 + x_3^2 + x_4^2, xy = x_1 y_1 + x_2 y_2 + x_3 y_3 + x_4 y_4, \\ x_{sum} = x_1 + x_2 + x_3 + x_4, y_{sum} = y_1 + y_2 + y_3 + y_4 \end{gathered} \tag{25}$$

Then the slope and the intercept are as simple as

$$b = \frac{xx * y_{sum} - xy * x_{sum}}{4 * xx - x_{sum}^2},\ k = \frac{4 * xy - x_{sum} * y_{sum}}{4 * xx - x_{sum}^2} \tag{26}$$

The two equations can be extended to apply to larger data sets.

In summary, the OLS method can be applied to find the linear correspondence in measurements to overcome the uncertainty. As shown in Figure 3, OLS helps us to find the relationship that the Tension Factor reduces linearly with the temperature raise. By checking the slope, -0.1627, we know the factor decrease by about 0.1627 per temperature raise by degree. Using OLS we can evaluate how one quantity change with another.