Recovering greeks from sensitivities

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ABSTRACT

We present a model-independent method for calculating delta, vega and rho based on a comparison of the sensitivities of any derivative payoff with those of its underlying observables. In doing so, we obtain generic definitions for these greeks with an intuitive geometric interpretation. By means of examples, we show how our definitions reduce to standard formulae for familiar special cases. For pure delta hedging of quanto options, we show that the standard formula for delta should be corrected to account for the convexity adjustment in the underlying forward. Finally, by applying the technique to physical and cash-settled swaptions, we illustrate that a systematic approach for calculating delta and vega ensures that contributions from the annuity are captured in a consistent manner.

1 Introduction

Delta, vega and other greeks are crucial for trading and hedging options, as they frame traders' intuition of how options behave as markets move. Valuing an option without also calculating the greeks is therefore almost useless – obtaining the greeks is an essential cost that must be paid when developing pricing tools. An option's response to changes in its underlying is particularly important because that quantity plays such a key role in the celebrated Black-Scholes analysis of the pricing problem for a European option, and the associated practice of delta-hedging. However, for some contracts with embedded optionality it is not possible to delta-hedge directly, because the underlying is not tradeable, but itself has sensitivities to the available tradeable instruments.

For example, the delta of a swaption is computed relative to the underlying swap rate, but for swaptions struck away from at-the-money, that swap is sensitive to the instruments (typically futures and other swaps) used to construct the curves used to price it. For quanto options, a pure delta-hedge requires the underlying quanto forward which often has less liquidity than the option itself. Both the quanto option and forward are sensitive to spot FX and the interest rate curve instruments in each currency.

In this work we give a generic method for calculating delta and vega from sensitivities. Based purely on sensitivities, it is both model-independent and independent of the type of option, up to identifying an underlying or set of underlyings. The intuition behind the method is that delta is recovered when the option's sensitivities are divided by the underlyings' sensitivities, and vega is formed from the remainder. We add rigor to this intuition and examine the implications.

Historically, a major drawback of any approach based on sensitivities was computational cost – calculating the sensitivity of a finanicial product to the full set of input values (typically market quotes) used to calibrate the pricing model required a

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finite difference approach, where each input in turn is adjusted by small amount and the product is recalculated each time. Recent years, however, have seen a growing number of available implementations of Algorithmic Differentiation (AD). AD can produce sensitivities at a small fraction of the cost of finite differences, by combining intermediate sensitivities calculated in closed form from their analytical formulae through the repeated application of the chain rule of differential calculus[1].

This article is not about AD, but about calculating greeks from sensitivities. The contribution of AD is to provide sensitivities at a computational cost so low that in many practical applications, our sensitivity-based method eliminates the need to consider any option- or model-specific greeks calculation in option pricers. Our method also gives a new and model-independent definition of delta, vega and rho, with an intuitive geometric interpretation, which provides the ability to calculate greeks in new situations where previously they were either ambiguous or unavailable. In addition, sensitivity-based greeks reconcile two contrasting worlds – the clean, intuitive framework of Black and Scholes[4] so fundamental to understanding the nature of derivatives, and the reality of actual tradeable instruments available for managing the market risk of a financial contract.

This paper is organized as follows. In Sec. (2) we clarify the distinction between greeks and sensitivities. In Sec. (3) we develop a generic definition of delta in terms of AD sensitivities which reduces to known option greek formulae in all special cases. In Sec. (4) we identify vega as the component of the risk vector orthogonal to delta, thereby establishing a generic definition of vega. We then form projections of these vectors suitable for calculating delta and vega as single values in Sec. (5). In Sec. (6) we address discounting risk, and show the role it plays in calculating delta and vega for path-dependent options. We then apply our technique to a series of examples in Sec. (7), including options in the Black-Scholes model, quanto options, and physically settled and cash-settled swaptions. Sec. (8) summarizes the ideas and findings covered in this article. Some remarks on Gamma in and Theta are made in the appendix, Sec. (9).

2 Greeks vs sensitivities

Before developing our sensitivity-based method, it is useful to clarify the distinction between greeks and sensitivities. The most basic greek calculation is for delta in the Black-Scholes model. For a European call option, this is given by[5]

$$\Delta_{\mathrm{BS}} \equiv \left.\frac{\partial}{\partial S_0}\right|_{r,\sigma} \left[S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2)\right] = \Phi(d_1) \tag{2}$$

where

$$d_{1,2} = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S_0}{K}\right) \pm \left(r + \frac{\sigma^2}{2}\right)T\right], \tag{3}$$

and where we have adopted the notation of Hull[6]. However, in reality we don't have a "risk free rate" r, we have discount and interest rate curves implied from instruments such as futures and swaps, and it is these instruments to which the option is sensitive through the discount factor e^{-rT}.

[1] For example, given the functions $g(z) = z^2$ and $f(x,y) = xy$, and their derivatives $\frac{\partial g}{\partial z} = 2z$, $\frac{\partial f}{\partial x} = y$ and $\frac{\partial f}{\partial y} = x$, we can calculate the sensitivities of $g(f(x,y))$ to x and y as

$$\mathrm{d}g = \frac{\partial g}{\partial z}\left(\frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y\right) = 2xy(y\,\mathrm{d}x + x\,\mathrm{d}y) \tag{1}$$

While this example is trivial, by applying the same ideas mechanically to every mathematical operation used in real pricing calculations, a full set of sensitivies can be calculated for any financial derivative product without recourse to finite differences[1], at a reduced computational cost[2]. For an accessible AD tutorial and comprehensive literature survey, we recommend Homescu[3].

For any option we can eliminate r by moving to the Black model[7] and working in forward space – instead of calculating delta as the change in present value of the option for a given change in spot underlying, we can form delta as the change in at-expiry option value for a given change in at-expiry forward value of the underlying $F_T = S_0 e^{rT}$,

$$\Delta_{\mathrm{B}} \equiv \left.\frac{\partial}{\partial F_T}\right|_{\sigma} [F_T \Phi(d_1) - K\Phi(d_2)] = \Phi(d_1) \tag{4}$$

where $d_{1,2}$ in terms of F_T are given by

$$d_{1,2} = \frac{1}{\sigma\sqrt{T}} \left[\ln\left(\frac{F_T}{K}\right) \pm \frac{\sigma^2}{2} T \right]. \tag{5}$$

Now all the details of constructing a forward curve are abstracted away in the function F_T which, when evaluated at the appropriate time, gives the forward value. We have not eliminated the sensitivity to discounting instruments, however, as long as F_T is constructed in terms of parameters derived from them such as r. In addition, in Eq. (2) we are ignoring funding spreads, dividends and other detail in the underlying's forward curve which may have been calibrated or somehow implied from option prices. Also, option contracts generally have settlement delays between expiry and delivery, so the time factor in the total variance differs from the one used in discounting. Furthermore, if the volatility input σ is obtained by interpolating market volatilities, then again we have abstracted away the market structure inside this parameter. Vega, the exposure of the option price to σ, is really a set of multiple "vegas", comprising the sensitivity to each volatility quote.

There are many ways, even for the most straightforward type of option, that the calculation of greeks diverges from the detailed reality of sensitivities, and as the complexity of the financial contract increases, this divergence grows. And yet, greeks and sensivities can be related in a straightforward manner, as we now show.

3 Generic Delta

Delta, measuring the dependence of an option on its underlying, is the most fundamental greek. A non-linear relationship between payoff and underlying is intimately linked to the notion of optionality, which makes delta an obvious starting point for reconciling the two views of market risk described in Sec. (1). We start with a full set of sensitivities and cast delta in terms of them, using a general technique described in Sec. (3.1).

3.1 Risk reprojection

Consider an arbitrary financial product with value $V(\vec{a})$ depending on a collection of N input parameters $\vec{a}$ (typically market quotes feeding calibrations). The full set of its sensitivities is given by its total derivative

$$\mathrm{d}V = \sum_{i=1}^{N} \frac{\partial V}{\partial a_i} \mathrm{d}a_i. \tag{6}$$

Given a set of M scalar-valued functions of $\vec{a}$, $\{F_j(\vec{a})\ j = 1 \ldots M\}$, we are free to add and subtract the total derivative of each to Eq. (6) weighted by an amount $\daleth_j$[2],

$$\mathrm{d}V = \sum_{i=1}^{N} \frac{\partial V}{\partial a_i} \mathrm{d}a_i - \sum_{j=1}^{M} \daleth_j \mathrm{d}F_j + \sum_{j=1}^{M} \daleth_j \mathrm{d}F_j. \tag{7}$$

[2]These weights will have an interpretation as option delta, but we shall reserve the symbols δ and Δ for use later. As the progenitor of the latin D and greek Δ, the Hebrew letter dalet ($\daleth$) is an appropriate substitute.

The ji^{th} element of the Jacobian matrix $\mathbf{J}$ relating these functions to the input parameters $\vec{a}$ is given by

$$J_{ji} = \frac{\partial F_j}{\partial a_i}. \tag{8}$$

We have

$$\mathrm{d}F_j(\vec{a}) = \sum_{i=1}^{N} J_{ji}\,\mathrm{d}a_i \tag{9}$$

and therefore

$$\begin{aligned} \mathrm{d}V &= \sum_{i=1}^{N} \left(\frac{\partial V}{\partial a_i} - \sum_{j=1}^{M} \daleth_j J_{ji} \right) \mathrm{d}a_i + \sum_{j=1}^{M} \daleth_j \mathrm{d}F_j \\ &\equiv \sum_{i=1}^{N} \nu_i\,\mathrm{d}a_i + \sum_{j=1}^{M} \daleth_j \mathrm{d}F_j \end{aligned} \tag{10}$$

which defines ν_i as

$$\nu_i = \frac{\partial V}{\partial a_i} - \sum_{j=1}^{M} \daleth_j J_{ji}. \tag{11}$$

If we can choose a set of values $\{\daleth_j\}\ j = 1\ldots M$ that eliminates each of the ν_i, we transform, or *reproject*, the sensitivities of V from the original variables $\{a_i\}\ i = 1\ldots N$ to the new variables $\{F_j\}\ j = 1\ldots M$. This is possible when $M \geq N$. When $M < N$, there is residual sensitivity to $\vec{a}$.

3.2 Geometric interpretation

The set of all possible market states is a differentiable manifold of dimension N, and the values of the parameters $\vec{a}$ form a coordinate system for that manifold. The geometry of differential manifolds[8] is a somewhat esoteric topic beyond the scope of this work. For our purposes, we can impose the restriction that the space of market states obeys the rules of Euclidean geometry. In doing so, we avoid jargon that may be less familiar to those with a background in financial mathematics, and so we make that choice immediately. The "infinitesimals" $\{\mathrm{d}a_i\}\ i = 1\ldots N$ therefore form basis vectors for the vector space $\mathbb{A}$ at $\vec{a}$ which we denote $\{\vec{e}_i\}\ i = 1\ldots N$. Given the identity metric, this basis is orthonormal,

$$\vec{e}_i \cdot \vec{e}_j = \delta_{ij}. \tag{12}$$

To interpret Eq. (10) geometrically, we define the vector

$$\mathrm{d}\vec{a} = \sum_{i=1}^{N} \mathrm{d}a_i\,\vec{e}_i, \tag{13}$$

where $\mathrm{d}a_i = \vec{e}_i \cdot \mathrm{d}\vec{a} \equiv \vec{e}_i^{\,T}\,\mathrm{d}\vec{a}$. We can then express Eq. (6) as

$$\mathrm{d}V = \vec{\nabla} V \cdot \mathrm{d}\vec{a} \equiv \vec{\nabla} V^T\,\mathrm{d}\vec{a}. \tag{14}$$

The total derivative of V is therefore a *projection* of $\vec{\nabla} V$ onto $\mathrm{d}\vec{a}$. Eq. (11) becomes

$$\vec{\nu}^T = \vec{\nabla} V^T - \vec{\daleth}^T \mathbf{J} \tag{15}$$

where the j^{th} element of the M-element vector $\vec{\daleth}$ is $\daleth_j$ and $\vec{\nu}$ is the N-element vector

$$\vec{\nu} = \sum_{i=1}^{N} \nu_i \vec{e}_i. \tag{16}$$

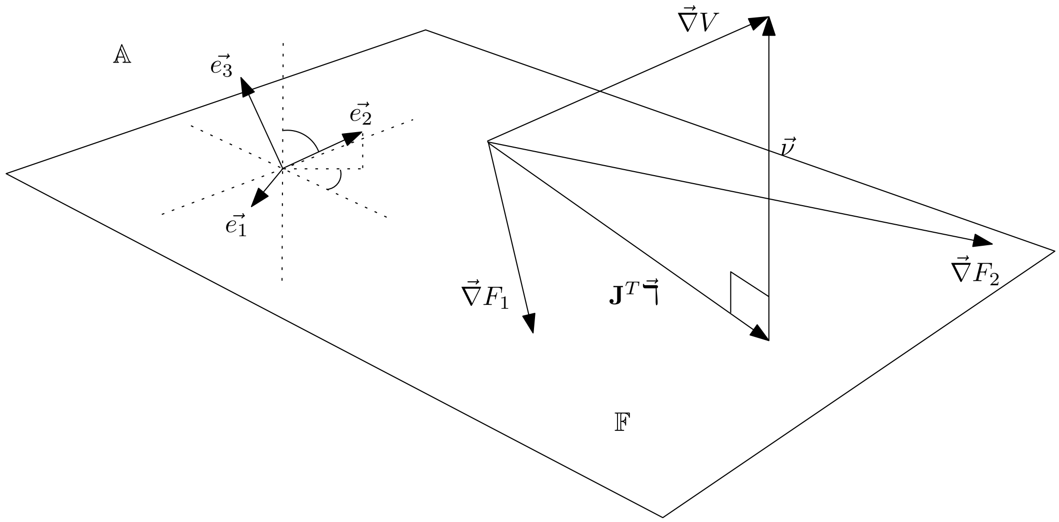


Figure 1. Illustration of the geometry of regression applied to option greeks in the case $N = 3$ and $M = 2$. The sensitivity vector $\vec{\nabla}V$ in the space $\mathbb{A}$ is resolved into two components, $\mathbf{J}^T\vec{\daleth}$ in the subspace $\mathbb{F}$ spanned by the sensitivity vectors of the underlyings $\vec{\nabla}F_1$ and $\vec{\nabla}F_2$, and a perpendicular component $\vec{\nu}$. None of these vectors is necessarily aligned with the basis $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ defined by the risk factors in the underlying sensitivity calculation.

Taking the transpose and rearranging, we have

$$\vec{\nabla}V = \mathbf{J}^T\vec{\daleth} + \vec{\nu} = \sum_{j=1}^{M} \daleth_j \vec{\nabla}F_j + \vec{\nu} \tag{17}$$

where the last step makes use of the fact that the j^{th} column of $\mathbf{J}^T$ is $\vec{\nabla}F_j$. In Eq. (17), we have decomposed the market risk of V into two components, one within the subspace $\mathbb{F}$ spanned by the vectors $\{\vec{\nabla}F_j\}$ and one in the remainder space $\overline{\mathbb{A}\cup\mathbb{F}}$. By choosing different coefficients $\{\daleth_j\}$, we can control how much the subspace $\mathbb{F}$ contributes to $\vec{\nabla}V$. The maximal contribution is obtained when $|\vec{\nu}|$ is minimized, which is achieved when $\vec{\nu}$ is orthogonal to $\mathbb{F}$,

$$\left(\mathbf{J}^T\vec{\daleth}\right) \cdot \vec{\nu} = 0. \tag{18}$$

When $M < N$, Eq. (18) is equivalent to linear regression[9], which minimizes the sum of squares of the $\{\nu_i\}$,

$$\sum_{i=1}^{N} \nu_i^2 \equiv \vec{\nu}\cdot\vec{\nu} \equiv |\vec{\nu}|^2, \tag{19}$$

giving $\vec{\daleth}$ as

$$\vec{\daleth} = \left(\mathbf{J}\mathbf{J}^T\right)^{-1}\mathbf{J}\vec{\nabla}V. \tag{20}$$

In the common case of $M = 1$ this reduces to

$$\daleth_1 = \frac{\vec{\nabla}F_1 \cdot \vec{\nabla}V}{\left|\vec{\nabla}F_1\right|^2}. \tag{21}$$

This geometry is illustrated in Fig. (1).

3.3 Option delta

Let $V(\vec{F}(\vec{a}),\vec{a})$ be the at-expiry value of a European option whose payoff depends on one or more of N input parameters $\vec{a}$, both directly and via a set observations of $M < N$ underlyings, whose expected values at each observation time (typically close to option expiry) are $\vec{F}(\vec{a})$. Then, Eq. (17) gives a decomposition of the risk vector of the option into two orthogonal components. The first component is a linear combination of the risk vectors of the option's underlyings, weighted by the M values $\{\daleth_j\}$, each of which is the delta to the corresponding underlying's at-expiry expected value. In Sec. (4) we will interpret the second component as the vega of the option.

To justify the interpretation of each $\daleth_j$ as an option delta, we proceed as follows. Changes in the option's underlyings arise through changes in the values of the input values $\vec{a}$. Denote a small change in the i^{th} element of $\vec{a}$ by δa_i, then a given "market move" is

$$\vec{\delta a} = \sum_{i=1}^{N} \delta a_i \vec{e}_i. \tag{22}$$

Given Eq. (17), the linear response in option value to a given market move is

$$\delta V = \vec{\nabla} V \cdot \vec{\delta a} = \left(\mathbf{J}^T \vec{\daleth}\right) \cdot \vec{\delta a} + \vec{v} \cdot \vec{\delta a} \equiv \delta V_\Delta + \delta V_\nu \tag{23}$$

where we have defined δV_Δ to be the contribution arising from the component of $\vec{\delta a}$ in the subspace $\mathbb{F}$ and δV_ν denotes the remainder. Any component of $\vec{\delta a}$ orthogonal to $\mathbb{F}$ makes no contribution to δV through changes in any of the underlyings. The impact of changes in the option's underlyings is contained entirely within

$$\begin{aligned}
\delta V_\Delta &= \left(\mathbf{J}^T \vec{\daleth}\right) \cdot \vec{\delta a} \\
&= \left(\sum_{j=1}^{M} \daleth_j \vec{\nabla} F_j\right) \cdot \left(\sum_{l=1}^{N} \delta a_l \vec{e}_l\right) \\
&= \left(\sum_{j=1}^{M} \daleth_j \sum_{i=1}^{N} \frac{\partial F_j}{\partial a_i} \vec{e}_i\right) \cdot \left(\sum_{l=1}^{N} \delta a_l \vec{e}_l\right) \\
&= \sum_{j=1}^{M} \daleth_j \sum_{i=1}^{N} \frac{\partial F_j}{\partial a_i} \sum_{l=1}^{N} \delta a_l \, \vec{e}_i \cdot \vec{e}_l \\
&= \sum_{j=1}^{M} \daleth_j \sum_{i=1}^{N} \frac{\partial F_j}{\partial a_i} \delta a_i \qquad (24) \\
&\equiv \sum_{j=1}^{M} \daleth_j \delta F_j \qquad (25)
\end{aligned}$$

where in Eq. (24) we made use of the orthonormality of the $\{e_i\}$ basis and Eq. (25) defines

$$\delta F_j = \sum_{i=1}^{N} \frac{\partial F_j}{\partial a_i} \delta a_i, \tag{26}$$

the impact of $\vec{\delta a}$ on the j^{th} option underlying F_j. In Eq. (25) we can see that each $\daleth_j$ is the linear response of option value to a small change in the j^{th} underlying F_j induced by an arbitrary market move, and is therefore a delta for that underlying. In Sec. (5) we will consider the problem of how to summarize the $\daleth_j$ into a single value, but before that, we focus on vega.

4 Generic Vega

For a European option in the Black model, again working with at-expiry quantities as in Eq. (4), vega is defined to be

$$\nu_B \equiv \left.\frac{\partial}{\partial \sigma}\right|_{F_T} [F_T \Phi(d_1) - K\Phi(d_2)] = F_T\, \phi(d_1)\sqrt{T} \tag{27}$$

where $\phi(x)$ denotes the standard normal density. Black vega measures the effect on the option price of the scale parameter in the distribution of F_T. For an arbitrary distribution however, particularly one generated by a stochastic volatility model, it is not always possible to identify such a parameter, and if even if we could, looking at its effect on our option price in isolation has limited use. For example in the Heston[10] model, the variance η_t follows a CIR process with mean reversion κ, long-run variance θ and volatility of volatility ξ,

$$\begin{aligned} \mathrm{d}F_t &= \sqrt{\eta_t} F_t \mathrm{d}W_t \\ \mathrm{d}\eta_t &= \kappa(\theta - \eta_t)\mathrm{d}t + \xi\sqrt{\eta_t}\mathrm{d}Z_t \end{aligned} \tag{28}$$

where $\mathrm{d}W_t\, \mathrm{d}Z_t = \rho\, \mathrm{d}t$. The width of the distribution for F_t is a function of the parameters κ, θ and ξ. The volatility of volatility, ξ, is the closest we have to a scale parameter, but if we were to define vega as the first order effect of κ it would still be of very limited use. Of primary practical use is the effect of quoted volatility on our option, which influences all three parameters through their calibration to market data, but then we have moved from the realm of clean, theoretical formulae like Eq. (27) which guide our intuition, into the realm of sensitivities, yet we are attempting to bridge the gap between these two.

Our familiar vega formula Eq. (27) arises in the approach commonly used for quoting volatility in option markets, where there are only two parameters, F_T and σ. If $V_B(F_T, \sigma)$ is the at-expiry value of an option in such a model,

$$\mathrm{d}V_B(F_T, \sigma) = \left.\frac{\partial V_B}{\partial F_T}\right|_{\sigma} \mathrm{d}F_T + \left.\frac{\partial V_B}{\partial \sigma}\right|_{F_T} \mathrm{d}\sigma \equiv \Delta_B\, \mathrm{d}F_T + \nu_B\, \mathrm{d}\sigma. \tag{29}$$

For arbitrary models, we need to generalize. One way to view vega generally is as a means of capturing the effect of non-linearity in the option's payoff. For an arbitrary payoff function $f(x)$ of some underlying observable x, the at-expiry value of the corresponding option, V, can be expressed as an expectation over the risk-neutral distribution of x,

$$V = \mathbb{E}\left[f(x)\right]. \tag{30}$$

If $f(x)$ is linear, then it and the expectation operator $\mathbb{E}\left[\right]$ commute,

$$[\mathbb{E}, f](x) \equiv \mathbb{E}\left[f(x)\right] - f(\mathbb{E}\left[x\right]) = V - f(F_T) = 0, \tag{31}$$

but non-linearity in f is required by the definition of an option, otherwise it reduces to a forward. By Jensen's inequality, for an option, the commutator $[\mathbb{E}, f](x)$ is non-zero. $f(F_T)$ is the intrinsic value of the option and the commutator is its time value in the familiar decomposition[11],

$$V = f(F_T) + [\mathbb{E}, f](x). \tag{32}$$

For a call option in the Black model, this takes the form

$$V_B(F_T, \sigma) = \max(F_T - K, 0) + \frac{\sigma F_T}{2}\int_0^T \frac{\phi(d_1)}{\sqrt{t}}\mathrm{d}t. \tag{33}$$

The first term in Eq. (32) or Eq. (33) makes no contribution to vega, only delta. The first term depends only on the first moment of the distribution $\mathbb{E}[x] = F_T$. The scale parameter can be identified with the second central moment (whether arithmetic or geometric) of the distribution, which is the only contribution to vega in the Black model. In arbitrary models, however, all higher moments make a contribution to the time value and should be included in our measure of vega.

Intuitively, in Eq. (29), we effectively expressed vega as the remainder of the total derivative of the option after removing the effect of the forward value of the underlying (the delta).

$$\text{vega} \sim \mathrm{d}V_B(F_T, \sigma) - \left.\frac{\partial V_B}{\partial F_T}\right|_{\sigma} \mathrm{d}F_T. \tag{34}$$

The key aspect of doing so is to vary the complete set of parameters on which option value depends in such a way that F_T is held constant. In the notation of Sec. (3.2), we need the component of the total derivative, $\vec{\nabla}V$, perpendicular to every vector $\vec{\nabla}F_j$. We have already identified this vector $\vec{v}$ in Eq. (15),

$$\vec{v} = \vec{\nabla}V - \mathbf{J}^T\vec{\daleth} \equiv \vec{\nabla}V - \sum_{j=1}^{M} \daleth_j \vec{\nabla}F_j \tag{35}$$

with the residual after projecting $\vec{\nabla}V$ onto the subspace $\mathbb{F}$. The single number that best represents this residual risk vector is its length $|\nu|$. This is the definition of generic vega we seek.

There is no direct analogue of the values $\{\daleth_j\}\ j = 1 \ldots M$ for vega, because there is no direct analogue of the option's underlyings $\{F_j\}\ j = 1 \ldots M$. Instead, we have a collection of N components of $\vec{v}$ on the $\{\vec{e}_i\}$ basis, $\nu_i = \vec{v} \cdot \vec{e}_i$ and, returning to Eq. (23), we can write

$$\delta V = \vec{\nabla}V \cdot \vec{\delta a} \equiv \delta V_\Delta + \delta V_\nu = \sum_{j=1}^{M} \daleth_j \delta F_j + \sum_{i=1}^{N} \nu_i \delta a_i \tag{36}$$

for an arbitrary change in market data $\vec{\delta a} = \sum_{i=1}^{N} \delta a_i \vec{e}_i$, where for the delta term we have used Eq. (25).

5 Delta and vega as scalars

In Sec. (3) and Sec. (4) we established the decomposition of the sensitivity vector $\vec{\nabla}V$ into two orthogonal components identified with delta and vega, given by Eq. (17). We then saw how the first-order change in option value resulting from an arbitrary change in market conditions decomposes into contributions through the delta and vega directions in Eq. (36).

Rather than an arbitrary market move, we may be interested in certain specific changes in the market state $\vec{a}$. For example, consider a small market move of magnitude δa in a direction parallel to $\mathbf{J}^T\vec{\daleth}$, given by the unit vector

$$\vec{e}_\Delta = \frac{\mathbf{J}^T\vec{\daleth}}{\left|\mathbf{J}^T\vec{\daleth}\right|} \equiv \frac{\sum_{j=1}^{M} \daleth_j \vec{\nabla}F_j}{\left|\sum_{j=1}^{M} \daleth_j \vec{\nabla}F_j\right|}. \tag{37}$$

The resulting first-order effect on option value contains zero contribution from vega as $\vec{e}_\Delta \cdot \vec{v} = 0$, and is given by

$$\delta V = \vec{\nabla}V \cdot (\delta a\, \vec{e}_\Delta) = \delta a\, \left|\mathbf{J}^T\vec{\daleth}\right|. \tag{38}$$

$\vec{e}_\Delta \cdot \vec{\nabla}V = \left|\mathbf{J}^T\vec{\daleth}\right| \equiv \Delta_{\max}$ is therefore *maximal* delta, corresponding to the largest effect on an option a small change in market conditions can have through the option's underlyings. Similarly, maximal vega results from a market move parallel to $\vec{v}$:

$$\delta V = \vec{\nabla}V \cdot (\delta a\, \vec{e}_\nu) = \delta a\, |\vec{v}| \tag{39}$$

where

$$\vec{e_\nu} = \frac{\vec{\nu}}{|\vec{\nu}|} \tag{40}$$

giving maximal vega as $\vec{e_\nu} \cdot \vec{\nabla} V = |\vec{\nu}| \equiv \nu_{\max}$. This allows us to express the sensitivity vector of the option neatly as

$$\vec{\nabla} V = \Delta_{\max} \vec{e_\Delta} + \nu_{\max} \vec{e_\nu}. \tag{41}$$

It is common practice to measure delta and vega by finite differences, by applying a parallel shift to the volatility surface to measure vega, and perturb each (usually the) underlying for delta. However, when option underlyings themselves contain some amount of convexity, it is not obvious a priori which subset of the quotes $\vec{a}$ are "vol quotes" and which pertain to the underlyings. They are mixed together and each quote can in general contribute to both delta and vega. Notions like "parallel vol surface shift" are arbitrary within the framework presented here and it is only by projecting the direction of a given market move onto Eq. (41) that we determine its relative contributions to delta and vega. Separating the arbitrary market move $\vec{\delta a}$ from Eq. (36) into its magnitude δa and direction $\vec{e_a}$, we have

$$\frac{\delta V}{\delta a} = \Delta_{\max} \vec{e_a} \cdot \vec{e_\Delta} + \nu_{\max} \vec{e_a} \cdot \vec{e_\nu}. \tag{42}$$

6 Discounting and rho

Discounting risk for options is commonly given the name "rho" and defined as the first order effect of changes in the risk-free rate r on the present value of an option[5]. For a European call option in the Black-Scholes model it is obtained by differentiating the present value with respect to the risk-free rate r,

$$\rho_{BS} = KTe^{-rT}\Phi(d_2). \tag{43}$$

In Sec. (3) we used the at-expiry value of the option, V, and the forward value of each underlying, F_j, thereby eliminating any discounting risk from the analysis and allowing us to focus on $\daleth_j$ and $\vec{\nu}$ as in Eq. (17). In order to examine the role of discounting risk on that analysis, we need to rework it in spot, not forward, terms. Rather than defining delta via reprojection of the risk vectors of forward value V onto forward underlying values $\{F_j\}$, we define spot delta via reprojection of the risk vectors of present option value

$$W = P_t V \tag{44}$$

onto the present value of the underlyings

$$\{G_j = P_j F_j\} \tag{45}$$

where P_t is the discount factor from option settlement t to the valuation date and P_j is the discount factor from the natural payment time of the j^{th} underlying, to the valuation date.

Note that we are not using the spot price of the underlying, because that undoes all the good work done by encapsulating funding, dividend and any other modeling details in the forwards $\{F_j\}$. By working with discounted forwards, we isolate sensitivity to discounting from sensitivity to any other detail of modeling the expected future value of underlyings. We could easily proceed with subsequent analysis of such details, but in doing so we are leaving the territory of reconciliation of detailed sensitivities with well-known option greeks – such details are already present in the sensitivities.

By reworking our reprojection analysis in present-value terms, we can relate spot delta to forward delta and calculate a generic expression for discounting risk (rho). In terms of spot delta and present-value quantities, Eq. (7) becomes

$$\mathrm{d}W = \sum_{i=1}^{N} \left(\frac{\partial W}{\partial a_i} - \sum_{j=1}^{M} \daleth_j^S \frac{\partial G_j}{\partial a_i} \right) \mathrm{d}a_i + \sum_{j=1}^{M} \daleth_j^S \mathrm{d}G_j \tag{46}$$

Applying the chain rule to Eq. (44) and Eq. (45), substituting into Eq. (46) and grouping like terms then yields

$$P_t \mathrm{d}V + V \mathrm{d}P_t = \sum_{i=1}^{N} \left(P_t \frac{\partial V}{\partial a_i} - \sum_{j=1}^{M} \daleth_j^S P_j \frac{\partial F_j}{\partial a_i} \right) \mathrm{d}a_i + \sum_{i=1}^{N} \left(V \frac{\partial P_t}{\partial a_i} - \sum_{j=1}^{M} \daleth_j^S F_j \frac{\partial P_j}{\partial a_i} \right) \mathrm{d}a_i + \sum_{j=1}^{M} \daleth_j^S \left(F_j \mathrm{d}P_j + P_j \mathrm{d}F_j \right). \tag{47}$$

Using $\sum_{i=1}^{N} \frac{\partial P_k}{\partial a_i} \mathrm{d}a_i = \mathrm{d}P_k$ for $k = j, t$ lets us simplify Eq. (47) to

$$\mathrm{d}V = \sum_{i=1}^{N} \left(\frac{\partial V}{\partial a_i} - \sum_{j=1}^{M} \frac{\daleth_j^S}{P_{jt}} \frac{\partial F_j}{\partial a_i} \right) \mathrm{d}a_i + \sum_{j=1}^{M} \frac{\daleth_j^S}{P_{jt}} \mathrm{d}F_j \tag{48}$$

where the discount factor from time t to time t_j is given by

$$P_{jt} = \frac{P_t}{P_j}. \tag{49}$$

Comparing with Eq. (20), we can see that the component of $\mathrm{d}V$ in the subspace $\mathbb{F}$ is maximized when the coefficients of $\{\mathrm{d}F_j\}$ obey

$$\daleth_j^S = P_{jt} \daleth_j \tag{50}$$

for $j = 1 \ldots M$. Each spot delta $\daleth_j^S$ is the appropriately discounted forward delta, as we would expect intuitively. For options whose underlyings are observed at expiry $P_{jt} = 1$ and $\daleth_j^S = \daleth_j$ for each of the M underlyings. In this case spot and forward delta coincide, which is no surprise given that delta is, intuitively, just the ratio of change in option value to change in underlyings, and if discounting affects both numerator and denominator in the same manner then its contribution cancels. Spot delta only differs from delta in path-dependent options, whose payoffs are functions of underlying observations made at times differing from the option expiry time, such as in American options.

Having related spot and forward delta, we can obtain a definition of discounting risk under the reprojection methodology. In the vector notation of Sec. (3.2) we have

$$\vec{\nabla} W = \sum_{j=1}^{M} \daleth_j^S \vec{\nabla} G_j + \vec{\varepsilon}, \tag{51}$$

where $\vec{\varepsilon}$ is a residual vector analogous to Eq. (17). In order to ensure that $|\vec{v}|$ is minimized, we calculate $\daleth_j$ as described in Eq. (17) and set $\daleth_j^S = P_{jt} \daleth_j$. Expanding W and G_j using the chain rule in Eq. (51) gives

$$\vec{\nabla} V = \sum_{j=1}^{M} \frac{\daleth_j^S}{P_{jt}} \vec{\nabla} F_j + \frac{1}{P_t} \left(\vec{\varepsilon} - \vec{\rho} \right) \tag{52}$$

where the discounting risk vector $\vec{\rho}$ is given by

$$\vec{\rho} = V \vec{\nabla} P_t - \sum_{j=1}^{M} F_j \daleth_j^S \vec{\nabla} P_j. \tag{53}$$

Converting from spot to forward delta with Eq. (50) and comparing with Eq. (17) allows us to identify the two components of the residual vector $\vec{\varepsilon}$,

$$\vec{\varepsilon} = \vec{\rho} + P_t \vec{v}. \tag{54}$$

In other words, the residual after removing spot delta from the sensitivities of an option's present value is composed of discounting risk rho and "spot" vega, where the latter is related to forward vega by a discount factor. This allows us to compute the discounting risk of an option of arbitrary type expiring at time t, and in an arbitrary model, as long as sensitivities are available.

6.1 Physical Options

Working in at-expiry terms and solving Eq. (17) has the advantage of efficiency for calculating delta and vega for options whose underlying observations are co-terminal with the payoff because the calculation is simplified by the absence of discounting risk. However, working in present-value terms and solving Eq. (51) allows us to compute rho in addition to delta and vega and affords a more intuitive definition of delta for path-dependent options, at the cost of the extra work of subtracting $\vec{\rho}$ from $\vec{\nabla}W$ to isolate (spot) vega.

Implicit in the analysis so far has been the assumption of cash-settlement; an option expiry t at which time a payment is made to the option's owner, and the present value of that payment, as per Eq. (44), is the option's value. This does not mean that we cannot apply the technique to physical options, but it is not trivially applicable as it is for cash settlement. For physically settled options, we need to identify a single time t to use in the discounting analysis, together with the relevant collection of observations that constitutes the underlyings.

This is often done as part of a pricing model anyway. For example, a physically settled European swaption struck at k and expiring at t is economically equivalent to a payment of $\max(s_t - k, 0)$ times the value of the underlying annuity, where s_t is the value of the swap rate[12]. Even though the contract references the swap itself, s_t identifies a single time to use in the discounting analysis. As another example, while an American option may be exercised at any time until its maturity t_m, some quasi closed form pricing models value such an option relative the equivalent European option expiring at $t = t_m$ by approximating the extra value held in the right to exercise early.[13] In doing so, they effectively express the physical American option as a cash-settled equivalent with expiry t.

It is difficult to determine algorithmically the cash-settled equivalent of an arbitrary financial derivative contract allowing for any collection of (perhaps nested) choice rights afforded to the holder. However, for any payment obligation (whether convexity is present or not) and for any physical option for which a cash-settled equivalent is available, option greeks may be calculated using the reprojection approach described in this work.

Indeed, Eq. (51) can be applied to a portfolio of such derivatives as a means of detecting optionality (or any other form of convexity) as long as both the sensitivities of the portfolio and those of each observable referenced in the payoffs of the derivatives, at each observation time, are available. Then, after removing the contribution of delta and rho using Eq. (50) and Eq. (53), only vega remains. If $|\vec{v}| = 0$, then there is no optionality – it is a statically hedgeable delta-one portfolio which can be verified explicitly through the delta calculation as a consistency check. If on the other hand there is non-zero vega, then there is optionality present which requires dynamic hedging to replicate.

7 Applications

In this section we apply the ideas presented in Sec. (3) and Sec. (4) to a series of examples. In Sec. (7.1) we show how our generic definitions of delta and vega reduce to the well-known formulae Eq. (4), Eq. (27), and Eq. (43) in the Black model. In Sec. (7.2) we derive corrections to standard formulae for a quanto option's greeks that account for the convexity adjustment of the underlying forward, and in Sec. (7.3) we demonstrate in the context of swaptions how the systematic nature of our approach to calculating option greeks captures the contribution from the annuity.

7.1 The Black Scholes model

The simplest illustrative example of the ideas described in this article is the derivation of Eq. (4) from Eq. (2). This has the advantage of being readily tractable without recourse to reprojection, giving a verification of the method in a familiar setting. The vector $\vec{a}$ of input parameters is given by $\vec{a}^T = (S_0, r, \sigma)$, so $N = 3$. In terms of these inputs, the at-expiry value of the option is

$$V(\vec{a}) = \alpha S_0 \Phi(d_1) - K\Phi(d_2) \tag{55}$$

where $\alpha = e^{rT}$, $\Phi(x)$ denotes the cumulative standard normal distribution, and

$$d_{1,2} = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S_0}{K}\right) \pm \left(r + \frac{\sigma^2}{2}\right)T\right]. \tag{56}$$

Therefore the gradient vector $\vec{\nabla}V$ is given by

$$\vec{\nabla}V = \left(\frac{\partial V}{\partial S_0}, \frac{\partial V}{\partial r}, \frac{\partial V}{\partial \sigma}\right)^T = \alpha\left(\Phi(d_1), tS_0\Phi(d_1), \sqrt{t}S_0\phi(d_1)\right)^T \tag{57}$$

where $\phi(x)$ denotes the standard normal density. We have a single underlying $F_T = S_0\alpha$, so $M = 1$, and

$$\vec{\nabla}F_T = \left(\frac{\partial F_T}{\partial S_0}, \frac{\partial F_T}{\partial r}, \frac{\partial F_T}{\partial \sigma}\right)^T = \alpha\,(1, tS_0, 0)^T. \tag{58}$$

Given the single underlying, there is a single weight $\daleth_1$ and the residual vector $\vec{v}$ in Eq. (15) takes the form

$$\vec{v} = \vec{\nabla}V - \daleth_1\vec{\nabla}F_T = \alpha\left(\Phi(d_1) - \daleth_1, tS_0\Phi(d_1) - \daleth_1 tS_0, \sqrt{t}S_0\phi(d_1)\right)^T \tag{59}$$

and its square length is given by

$$v \cdot v = \alpha^2\left((\Phi(d_1) - \daleth_1)^2 + (tS_0\Phi(d_1) - \daleth_1 tS_0)^2 + \left(\sqrt{t}S_0\phi(d_1)\right)^2\right). \tag{60}$$

This is minimized when

$$\daleth_1 = \Phi(d_1) = \Delta_B = \Delta_{BS}, \tag{61}$$

as in Eq. (4), and the resulting residual (vega) vector is

$$\vec{v} = \left(0, 0, \sqrt{t}F_T\phi(d_1)\right) \tag{62}$$

as in Eq. (27). Clearly, $|\vec{v}| = \nu_B$, and direct computation of $\daleth_1$ via Eq. (21) gives

$$\daleth_1 = \frac{\vec{\nabla}V \cdot \vec{\nabla}F_T}{\left|\vec{\nabla}F_T\right|^2} = \frac{\alpha^2\left(\Phi(d_1) + t^2S_0^2\Phi(d_1)\right)}{\alpha^2\left(1 + t^2S_0^2\right)} = \Phi(d_1) = \Delta_B = \Delta_{BS}, \tag{63}$$

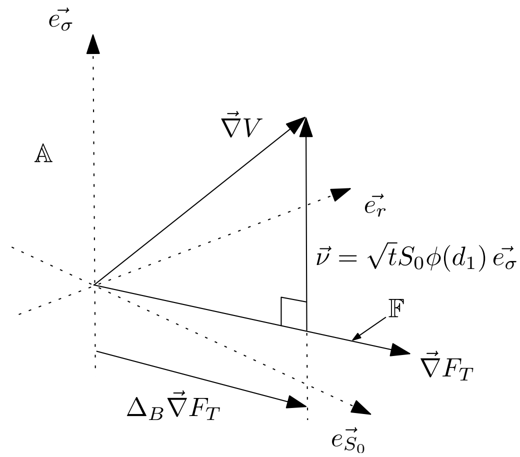


Figure 2. The geometry of Black-Scholes delta and vega. Here, the dimensionality N of the sensitivity space $\mathbb{A}$ is 3, and that of the space of underlyings $\mathbb{F}$ is 1. The forward F_T has no sensitivity to σ and therefore $\vec{\nabla} F_T$ lies entirely within the S_0-r plane and $\vec{\nu}$ is parallel to $\vec{e_\sigma}$.

again agreeing with Eq. (4). The geometry is shown in Fig. (2).

Rho and spot vega are easily calculated as well. In the Black Scholes model the discounting risk vector of Eq. (53) reduces to

$$\vec{\rho} = \left(V - F_T \Delta^S\right) \mathrm{d}P_T = -K\Phi(d_2)\, \mathrm{d}(e^{-rt}). \tag{64}$$

The discount rate component (coefficient of dr) recovers Eq. (43)

$$\rho_{BS} = KTe^{-rT}\Phi(d_2). \tag{65}$$

As shown in Eq. (54), spot vega is obtained from forward vega by discounting from expiry. Applying this discounting term to the vega calculated above produces the standard Black Scholes result

$$\nu_{BS} = \sqrt{t} e^{-rT} F_T \phi(d_1) = \sqrt{t} S_0 \phi(d_1). \tag{66}$$

This simple example verifies that the textbook formulae for Δ_{BS}, ν_{BS}, and ρ_{BS} can all be recovered with the ideas presented in this paper.

7.2 Quanto options

The Black-Scholes example described in Sec. (7.1) is somewhat trivial because vega was already orthogonal to delta; $\vec{\nabla} F$ lay entirely within the S_0-r plane, so it was really a 2-d problem despite having $N = 3$. In contrast, a quanto option's forward is convexity-adjusted and so depends on volatility,[14] which mixes the dimensions of the input space $\mathbb{A}$. If σ_X is the volatility of the FX rate and ρ is the correlation between the Brownian motion driving a lognormal process for the FX rate and that for the underlying, the at-expiry expected value of the underlying in the Black model is given by

$$F'_T = F_T e^{-\rho\sigma_X\sigma t} = S_0 e^{-(r+\rho\sigma_X\sigma)t}. \tag{67}$$

7.2.1 Quanto delta

Reprojection as described in Sec. (3) gives a delta Δ which is the partial derivative of (at-expiry) option value V_Q with respect to changes in the convexity-adjusted forward F'_T at constant vega, which is initially unknown. The magnitude and direction of vega is another output of the method. This definition of delta is the natural generalization of Eq. (4) to convexity-adjusted settings such as a quanto option, and is not the same as simply taking

$$\Delta'_B = \Phi(d'_1) \equiv \Phi\left(\frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{F'_T}{K}\right) + \frac{\sigma^2}{2}T\right]\right) \tag{68}$$

as delta, in which we have simply replaced F_T with F'_T in Eq. (4). Eq. (68) is intuitively appealing because to value a quanto option in the Black model, replacing the forward with Eq. (67) is correct – both F_T and F'_T have the same Black volatility. However, the reprojection-based delta Δ_Q corresponds to an infinitesimal move in a direction in the space $\mathbb{A}$ that results in maximal change of F'_T, whereas the naive delta in Eq. (68) corresponds to a direction corresponding to constant σ, and because F'_T depends on σ through the convexity adjustment in Eq. (67), by keeping σ fixed a contribution to the change in F'_T is missed. The difference is typically small, but the arbitrage-free replicating portfolio argument that leads to the Black formula in the first place is based on F'_T, not F_T. Furthermore, neither of these values are of primary interest for hedging – the sensitivities $\vec{\nabla}V_Q$ represent the most salient information for practical risk management – but if we are attempting to reconcile the information in $\vec{\nabla}V_Q$ with the intuition about option behaviour developed in a lognormal setting, we should apply a systematic method rather than ad hoc approaches like that leading to Eq. (68).

For tractability, recognizing that ρ and σ_X play the same role, we combine them into a single variable $\beta = \rho\sigma_X$, and work with F_T given that Sec. (7.1) has already dealt with the effect of separate S_0 and r. We therefore work in a space $\mathbb{A}$ of dimensionality $N = 3$ where $\vec{a} = (F_t, \beta, \sigma)^T$ and with the single underlying

$$F'_T = F_T e^{-\beta\sigma T}, \tag{69}$$

so the dimensionality M of the subspace $\mathbb{F}$ is 1. The total derivative of V_Q is given by

$$\mathrm{d}V_Q = \Delta'_B \mathrm{d}F'_T + \nu'_B \mathrm{d}\sigma \tag{70}$$

where $\nu'_B = \sqrt{T} F'_T \phi(d'_1)$, and the total derivative of the convexity-adjusted forward is

$$\mathrm{d}F'_T = F'_T\left[\frac{\mathrm{d}F_T}{F_T} - T\left(\beta\mathrm{d}\sigma + \sigma\mathrm{d}\beta\right)\right]. \tag{71}$$

Adopting the notation from Sec. (3.2), the gradient vector of V_Q in $\mathbb{A}$ is

$$\vec{\nabla}V_Q = \Delta'_B \vec{\nabla}F'_T + \nu'_B \vec{e_\sigma} \tag{72}$$

and that of the underlying is

$$\vec{\nabla}F'_T = F'_T\left[\frac{\vec{e_{F_T}}}{F_T} - T\left(\beta\vec{e_\sigma} + \sigma\vec{e_\beta}\right)\right]. \tag{73}$$

Therefore, by Eq. (21), we can calculate the delta of a quanto option in the Black model as

$$\Delta_Q = \frac{\vec{\nabla}V_Q \cdot \vec{\nabla}F'_T}{\left|\vec{\nabla}F'_T\right|^2} = \Delta'_B + \nu'_B\frac{\vec{e_\sigma}\cdot\vec{\nabla}F'_T}{\left|\vec{\nabla}F'_T\right|^2} = \Delta'_B + \nu'_B\frac{\vec{e_\sigma}\cdot\vec{e_{F'_T}}}{\left|\vec{\nabla}F'_T\right|} \tag{74}$$

where the unit vector $\vec{e_{F'_T}} = \frac{\vec{\nabla} F'_T}{|\vec{\nabla} F'_T|}$. We can see that Δ_Q contains a contribution from ν'_B that depends on the angle between $\vec{e_\sigma}$ and $\vec{\nabla} F'_T$, which grows from zero with β. In other words, when convexity is present in the underlying, delta to the convexity-adjusted forward contains a contribution from the naive vega ν'_B.

Evaluating an explicit form for Δ_Q is straightforward given Eq. (73), although in order to study both the effect of correlation ρ and FX volatility σ_X explicitly, in addition to that of spot S_0 and the interest rate r, we elect to work with the orthonormal basis $\{\vec{e_{S_0}}, \vec{e_r}, \vec{e_\rho}, \vec{e_{\sigma_X}}, \vec{e_\sigma}\}$ in which

$$\begin{aligned} \vec{e_{F_T}} &= \frac{\partial F_T}{\partial S_0}\vec{e_{S_0}} + \frac{\partial F_T}{\partial r}\vec{e_r} = F_T\left(\frac{1}{S_0}\vec{e_{S_0}} - T\vec{e_r}\right) \\ \vec{e_\beta} &= \frac{\partial \beta}{\partial \rho}\vec{e_\rho} + \frac{\partial \beta}{\partial \sigma_X}\vec{e_{\sigma_X}} = \sigma_X \vec{e_\rho} + \rho \vec{e_{\sigma_X}} \end{aligned} \tag{75}$$

while $\vec{e_\sigma}$ remains unchanged. Defining $L_T = -\ln\frac{F'_T}{F_T} = \beta\sigma T$, and introducing ξ, from Eq. (73) we have

$$\vec{e_\sigma} \cdot \vec{e_{F'_T}} = -\beta T F'_T \tag{76}$$

and $\left|\vec{\nabla} F'_T\right|^2 = {F'_T}^2 \xi^2$ where

$$\xi^2 = \frac{1}{S_0^2} + T^2 + L_T^2\left(\frac{1}{\rho^2} + \frac{1}{\sigma_X^2} + \frac{1}{\sigma^2}\right) \tag{77}$$

which when substituted into Eq. (74) yields

$$\Delta_Q = \Delta'_B - \nu'_B\,\frac{\frac{F'_T L_T}{\sigma}}{{F'_T}^2\xi^2} = \Delta'_B - \frac{\phi(d'_1)\beta T^{3/2}}{\xi^2}. \tag{78}$$

Fig. (3) shows the deviation of Δ_Q from Δ'_B as a function of correlation over a range of FX volatilities.

7.2.2 Quanto vega

In terms of our quanto example, Eq. (35) reads

$$\vec{\nu_Q} = \vec{\nabla} V_Q - \Delta_Q \vec{\nabla} F'_T. \tag{79}$$

Eliminating $\vec{\nabla} V_Q$ using Eq. (72) gives the vega vector

$$\vec{\nu_Q} = \left(\Delta'_B - \Delta_Q\right)\vec{\nabla} F'_T + \nu'_B \vec{e_\sigma} = \nu'_B\left(\vec{e_\sigma} - \vec{e_\sigma}\cdot\vec{e_{F'_T}}\ \vec{e_{F'_T}}\right) \tag{80}$$

whose length, using Eq. (76) is

$$|\vec{\nu_Q}| = \nu'_B\sqrt{1 - \left(\frac{\beta T}{\xi}\right)^2}. \tag{81}$$

The orthogonality of $\vec{\nu_Q}$ and $\vec{\nabla} F'_T$ is particularly clear given Eq. (80), where

$$\vec{\nu_Q}\cdot\vec{\nabla} F'_T = \nu'_B\,\xi\,\vec{e_\sigma}\cdot\vec{e_{F'_T}}\left(1 - \vec{e_{F'_T}}\cdot\vec{e_{F'_T}}\right) = 0. \tag{82}$$

Fig. (4) shows the deviation of $|\vec{\nu_Q}|$ from ν'_B as a function of correlation over a range of FX volatilities, in two stock volatility regimes, and Fig. (5) shows the correlation that results in maximal vega as a function of σ and σ_X.

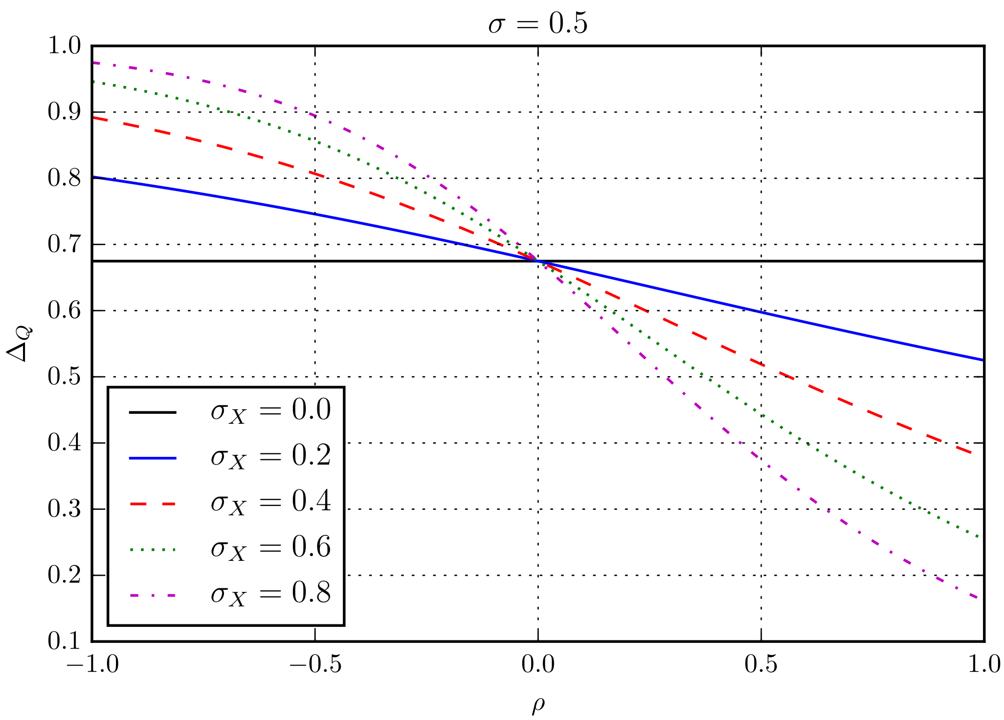
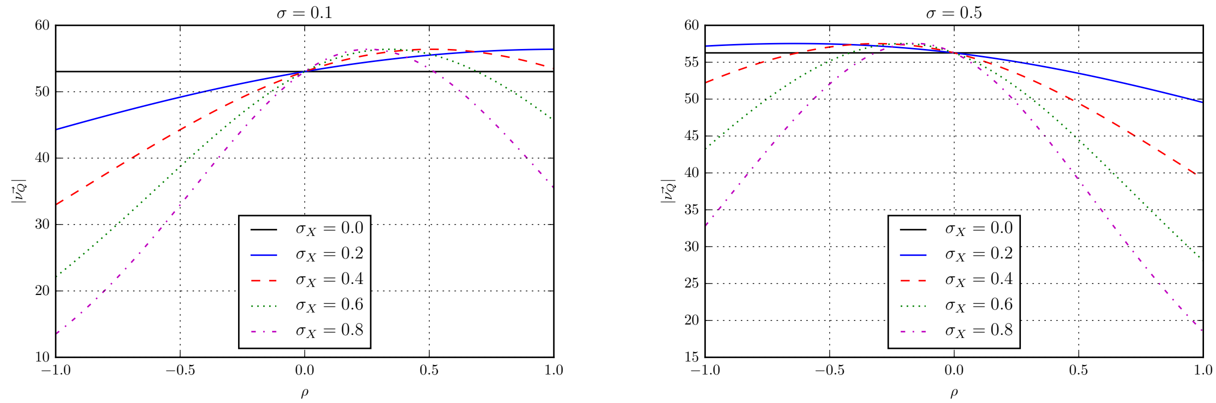


Figure 3. The delta of a quanto call option in the Black model, Δ_Q from Eq. (78), as a function of correlation ρ, for a selection of values of FX volatility σ_X. The spot stock price $S_0 = 100$, strike $K = 103$, interest rate $r = 5\%$, expiry $T = 2$ years and stock volatility $\sigma = 50\%$. When $\sigma_X = 0$, the quanto delta reduces to Eq. (68), ie $\Delta_Q = \Delta'_B$, shown by the horizontal solid black line. All of the lines intersect with that line at $\rho = 0$. In general, however, $\Delta_Q \neq \Delta'_B$.



(a) **(b)**

Figure 4. The vega of a quanto call option in the Black model, as the magnitude of the vector $\vec{\nu}_Q$ in Eq. (81), as a function of correlation ρ, for a selection of values of FX volatility σ_X. The spot stock price $S_0 = 100$, strike $K = 103$, interest rate $r = 5\%$, expiry $T = 2$ years and σ takes the two values 10% in **(a)** and 50% in **(b)**. When $\sigma_X = 0$, vega takes the value ν_B' shown by the solid black horizontal line, and vega is independent of $\sigma_X = 0$ for $\rho = 0$, as expected. Interestingly, in most of the parameter space shown here, there is a correlation ρ_* for which $|\vec{\nu}_Q|$ is maximal.

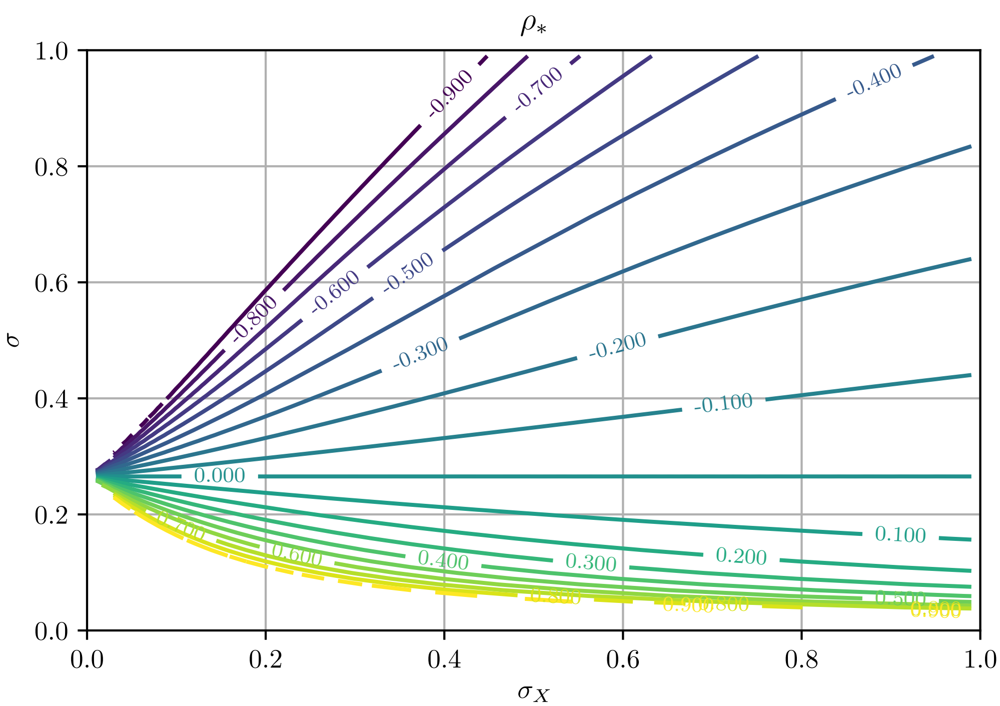


Figure 5. The correlation between Brownian motions driving lognormal stock and FX rate processes that results in maximal vega defined by Eq. (81) as a function of stock volatility σ and FX rate volatility σ_X. The spot stock price $S_0 = 100$, strike $K = 103$, expiry $T = 2$ years and interest rate $r = 5\%$.

7.3 Swaptions

The contrast between physically and cash-settled swaptions provides another interesting example. As in the Black-Scholes model described in Sec. (7.1), vega is orthogonal to delta. However, as we will show below, the contribution to delta from the annuity is captured in a systematic way using the ideas described in this article.

The present value of a physically-settled swaption (PSS) can be written[6]

$$V_{PSS} = P(0,T)\mathbb{E}^T\left[A(T)\left(S_T - K\right)^+\right], \tag{83}$$

where $P(t,T)$ is the time t value of a zero-coupon bond maturing at T, $A(T) = \sum_{i=1}^n \alpha_i P(T,t_i)$ is the annuity, α_i is the accrual of the i^{th} payment in the underlying swap, T is the swaption expiry, n is the number of fixed swap payments, S_T is the time T forward swap rate at expiry, and K is the strike. As discussed in Sec. (6.1), we have modeled the physical swaption with its cash-settled equivalent – a single cash flow at time T whose present value is the same as the swaption. By assuming the forward swap rate follows the Black model, the swaption value can be calculated as

$$V_{PSS} = A(0)\left[\Phi(d_1)S_0 - \Phi(d_2)K\right], \tag{84}$$

where $d_{1,2}$ are as defined in Eq. (56), but with $r = 0$.

A cash-settled swaption (CSS) has the same payoff, but the annuity $A(T)$ is replaced by the cash-settled annuity $C(S_T)$. The present value of a cash-settled swaption is

$$V_{CSS} = P(0,T)\mathbb{E}^T\left[C(S_T)\left(S_T - K\right)^+\right], \tag{85}$$

where

$$C(S_T) = \sum_{i=1}^n \frac{n}{\left(1+\alpha S_T\right)^i} \tag{86}$$

and the accrual α is calculated as the total length of the swap divided by n. For simplicity, we will use the “market” formula for cash-settled swaptions,[15] which assumes Eq. (84) holds when the physically-settled annuity $A(0)$ is replaced by the cash-settled annuity $C(S_0)$. However, the ideas presented below still apply when a terminal swap rate model is used.

In common practice, the delta of a swaption is often obtained by differentiating Eq. (84) with respect to the swap rate S_0 at constant volatility

$$\Delta' = A\Delta_B + \frac{V}{A}\frac{\partial A}{\partial S_0}, \tag{87}$$

where $A = A(0)$ or $A = C(S_0)$ for a physically or cash-settled swaption, respectively. This is the Black delta scaled by the annuity, plus a correction arising from the dependence of the annuity on the swap rate. For a physically-settled swaption, the correction term vanishes when the annuity is assumed to be independent of the swap rate. Therefore $\Delta'_{PSS} = A(0)\Delta_B$. In contrast, the cash-settled annuity does have an explicit dependence on the swap rate, which implies that $\Delta'_{CSS} < C(S_0)\Delta_B$ because $C(S_0)$ is a positive-valued monotonically decreasing function of S_0.

For comparison, we will now compute delta using the generic approach presented in this paper. As in the above examples, we will work in a space $\mathbb{A}$ of dimensionality N=3, where $\vec{a} = (r,z,\sigma)^T$, where r is a parameter describing the risk-free rate, and

z is a parameter describing the spread of Libor over this risk-free rate. The single underlying is S_0, so the dimensionality M of the subspace $\mathbb{F}$ is again 1. In order to apply the reprojection technique of Sec. (3), we compute the usual gradient vectors

$$\vec{\nabla}V = \left(\frac{\partial V}{\partial r}, \frac{\partial V}{\partial z}, \frac{\partial V}{\partial \sigma}\right)^T = A\left(\Phi(d_1)\frac{\partial S_0}{\partial r} + \frac{V}{A}\frac{\partial A}{\partial r}, \Phi(d_1)\frac{\partial S_0}{\partial z} + \frac{V}{A}\frac{\partial A}{\partial z}, A\sqrt{t}S_0\phi(d_1)\right)^T \tag{88}$$

and

$$\vec{\nabla}S_0 = \left(\frac{\partial S_0}{\partial r}, \frac{\partial S_0}{\partial z}, \frac{\partial S_0}{\partial \sigma}\right)^T = \left(\frac{\partial S_0}{\partial r}, \frac{\partial s}{\partial z}, 0\right)^T, \tag{89}$$

where we have assumed that $A = A(r,z)$ and $S_0 = S_0(r,z)$. Substituting these into Eq. (21), we obtain

$$\Delta = A\Delta_B + \frac{V}{A}\frac{\frac{\partial A}{\partial r}\frac{\partial S_0}{\partial r} + \frac{\partial A}{\partial z}\frac{\partial S_0}{\partial z}}{\left(\frac{\partial S_0}{\partial r}\right)^2 + \left(\frac{\partial S_0}{\partial z}\right)^2}. \tag{90}$$

The residual (vega) vector is given by the annuity scaled Black-Scholes result

$$\vec{v} = \left(0, 0, A\sqrt{t}S_0\phi(d_1)\right), \tag{91}$$

which is expected given the orthogonality of delta and vega. Applying the chain rule to the cash-settled swaption annuity gives

$$\frac{\partial C(S_0)}{\partial x} = \frac{\partial C(S_0)}{\partial S_0}\frac{\partial S_0}{\partial x} \tag{92}$$

for $x = r, z$. Substituting these expressions into Eq. (90) one can show $\Delta_{CSS} = \Delta'_{CSS}$. However, in the case of a physically-settled swaption, the reprojection approach captures a correction term due to the annuity

$$\Delta_{PSS} - \Delta'_{PSS} = \frac{V}{A}\frac{\frac{\partial A}{\partial r}\frac{\partial S_0}{\partial r}}{\left(\frac{\partial S_0}{\partial r}\right)^2 + \left(\frac{\partial S_0}{\partial z}\right)^2}. \tag{93}$$

While this amount is typically small, it is not negligible, particularly for in-the-money physically-settled swaptions.

Which physically-settled swaption delta is correct? Of course both deltas are correct according to their respective definitions, so a more meaningful question is which delta is more useful? Δ'_{PSS} is equivalent to calculating the change in swaption value relative to the change in the underlying swap rate *at constant annuity*. In contrast, Δ_{PSS} is the change in swaption value relative to the change in the underlying swap rate *at constant vega*. Alternatively, Δ_{PSS} can be thought of as the change in option value that can be captured by the sensitivities of the underlying swap rate. In realistic market conditions, it is highly unlikely that changes in the swap rate would be unaccompanied by changes in the risk-free rate, yet this is exactly what Δ'_{PSS} implies. By defining delta in terms of reprojection, we guarantee that any variation of the swaption price that can be attributed to the underlying swap is captured in delta. This gives further support for the use of a systematic method rather than ad hoc approaches like those leading to Eq. (68) and Eq. (87).

In order to analyze the deltas numerically, we make a few simplifying assumptions about the model for the swap rate, but emphasize that the approach described in this article can be applied to any modeling choice. We define the zero coupon bond price and Libor accruing curves as

$$P_D(0,t) = (1+\alpha r)^{-t}, \quad P_L(0,t) = (1+\alpha(r+z))^{-t}, \tag{94}$$

that is, both curves are flat in simple rate terms and the Libor rate is specified as a spread z over the risk-free rate r. Assuming that the fixed and floating schedules of the underlying swap have the same coupon length α and identical payment times, the swap rate is just $r+z$. The physically and cash-settled swaption deltas of Eq. (90) are then

$$\Delta_{PSS} = A(0)\Phi(d_1) + \frac{V}{2A(0)}\frac{\partial A(0)}{\partial r} \tag{95}$$

and

$$\Delta_{CSS} = C(S_0)\Phi(d_1) + \frac{V}{2C(S_0)}\left(\frac{\partial C(S_0)}{\partial r} + \frac{\partial C(S_0)}{\partial z}\right). \tag{96}$$

Fig. (6) shows the various swaption deltas as a function of strike under these modeling assumptions.

8 Conclusion

In this work we have reconciled two contrasting approaches to calculating the sensitivities, or greeks, required for hedging options. Textbook formulae form the basis of our intuitive understanding of how options relate to the markets in which they and their underlying assets are traded, but such formulae lack the detail present in real models and are only available for a subset of models and implementations thereof for a given type of option. Such detail is available in the full set of sensitivities of the model to its inputs including market quotes. While historically it has been too expensive to compute a comprehensive set of sensitivities for most models, with Algorithmic Differentiation they are now available often at a computational cost comparable to valuation itself.

We have presented a method of moving from the typically high-dimensional space of all sensitivities of a real option to the key greeks that characterize option behaviour, based on reprojecting the sensitivities onto those of the option's underlyings. In doing so, a new geometric interpretation of delta and vega is obtained. The method is model-independent and works for any type of payoff where it is possible to identify a collection of observations playing the role of "underlyings". The only requirement is the full set of sensitivities for both option and underlying observations. Historically this requirement would have been onerous, but with the advent of systems with Algorithmic Differentiation as a central feature, it is now a reality.

Computing delta and vega by reprojection gives results that reduce to standard formulae for the relevant special cases. However, when the option and its underlyings share dependence on the same variables, some important differences emerge between reprojection and traditional approaches to calculating greeks. We demonstrated this in the case of a quanto option in the Black model, where both delta and vega receive a convexity correction, and in the case of swaptions where the role played by the annuity can be analyzed consistently, resulting in a correction to delta in the physical settled case.

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Appendix

While the focus of our article is on the fundamental first-order greeks delta, vega and rho, many other greeks exist through higher-order derivatives of option price with respect to various parameters, and by considering the effect of the passage of time on the option. In this appendix, we make some remarks about two of these, Gamma and Theta.

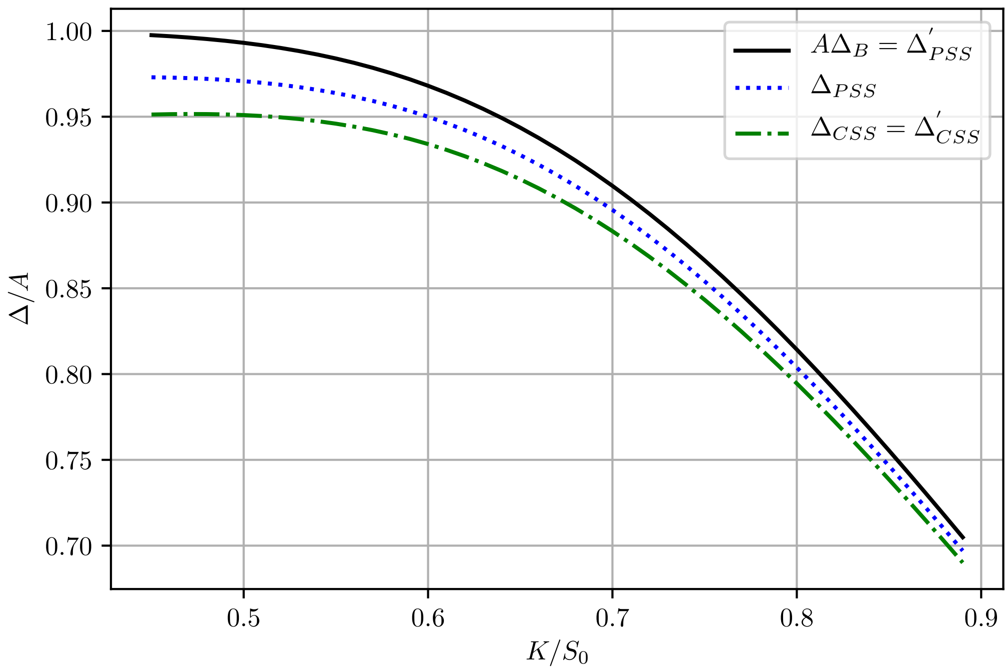

Figure 6. Delta for physically and cash-settled swaptions as a function of moneyness K/S_0. The interest rate is $r = 5\%$, the spread is $z = 1\%$, the expiry is $T = 2$ years, the swap length is also 2 year, and the swap rate volatility is $\sigma = 30\%$. Δ'_x denotes the delta that is obtained by differentiating the swaption value with respect to the underlying the swap rate as in Eq. (87) where x represents cash (CSS) and physical (PSS) swaptions, whereas Δ_x (with no prime) is the reprojection-based value. For physically-settled swaptions $\Delta'_{PSS} = A\Delta_B$, which is equivalent to calculating the change in swaption value while holding the annuity (and volatility) constant. For cash-settled swaptions the annuity depends on the swap rate explicitly, which leads to an additional term in Eq. (87). This also results in $\Delta/A < 1$, even for deep in-the-money swaptions, because the option is no longer equivalent to a long position in the underlying swap rate in that limit. This is due to the payoff formula including a contribution that is a non-linear function of the swap rate. The general approach for calculating Δ presented in this paper captures this annuity correction for both physically and cash-settled swaptions.

Gamma

Risk reprojection, as described in Sec. (3.1) can be generalized to higher order terms in a straightforward manner. To second order, the change in value of an arbitrary financial product can be expressed as

$$\begin{aligned}\delta V(\vec{a}) &= \sum_{i=1}^{N}\frac{\partial V}{\partial a_i}\delta a_i+\frac{1}{2}\sum_{i,l=1}^{N}\frac{\partial^2 V}{\partial a_i\partial a_l}\delta a_i\delta a_l \\ &= \sum_{i=1}^{N}\left(\frac{\partial V}{\partial a_i}-\sum_{j=1}^{M}J_{ji}\urcorner_j\right)\delta a_i+\frac{1}{2}\sum_{i,l}^{N}\left(\frac{\partial^2 V}{\partial a_i\partial a_l}-\sum_{j,k=1}^{M}\Gamma_{jk}J_{ji}J_{kl}\right)\delta a_i\delta a_l \qquad (97)\\ &\quad+\sum_{j=1}^{M}\urcorner_j\delta F_j+\frac{1}{2}\sum_{j,k=1}^{M}\Gamma_{jk}\delta F_j\delta F_k+O(\delta F^3) \qquad (98)\end{aligned}$$

where in addition to the weights $\{\urcorner_j\}$ from Sec. (3.1), we have introduced an $M\times M$ matrix of weights Γ_{jk}. By minimizing the contribution of Eq. (97) to $\delta V(\vec{a})$ over both the $\{\urcorner_j\}$ and $\{\Gamma_{jk}\}$, we calculate both delta and gamma in terms of the first and second order derivatives of $V(\vec{a})$ with respect to $\vec{a}$.

However, this is of no practical use unless both the first and second order derivatives of $V(\vec{a})$ with respect to $\vec{a}$ are available. While modern AD implementations provide complete first order derivatives, and higher order derivatives can be calculated in principle[16], second order derivatives are impractical to provide comprehensively, because the cost scales as N^2. For this reason, we do not pursue a reprojection-based approach here. Instead, for higher order sensitivities in general, we recommend finite difference applied to AD as the most productive strategy. The efficiency of AD means that the cost of bumping $\vec{\nabla}V(\vec{a})$ is usually of the same order as bumping $V(\vec{a})$.

The method of finite differences can be applied at any stage of the process. For example, given a bump defined by the small vector $\vec{\delta a}$ in Eq. (22), a forward difference of the sensitivities $\vec{\nabla}V(\vec{a})$ is given by

$$\frac{\vec{\nabla}V(\vec{a}+\vec{\delta a})-\vec{\nabla}V(\vec{a})}{\left|\vec{\delta a}\right|}. \qquad (99)$$

Backward and centered differences are defined similarly. If the bump is aligned with the i^{th} input, $\vec{\delta a}=\delta a\,\vec{e}_i$, we obtain an approximation of one column of the Hessian matrix $H_{il}=\frac{\partial^2 V}{\partial a_i\partial a_l}\,l=1\ldots N$. Alternatively, we can apply the same bump to Eq. (20) for each element of the vector $\vec{\urcorner}$. The j^{th} element of the corresponding gamma vector under forward difference is then given by

$$\Gamma_j=\frac{1}{\delta a}\left(\urcorner_j(\vec{a}+\delta a\,\vec{e}_\Delta)-\urcorner_j(\vec{a})\right). \qquad (100)$$

In addition, we can evaluate a finite difference of maximal delta defined in Sec. (5).

Time decay and theta

The value of an option changes with time even if all other factors are held constant, and the name given to this time decay is "theta", Θ. For a European call option in the Black-Scholes model[5],

$$\Theta=-\frac{S_0\phi(d_1)\sigma}{2T}-rKe^{-rT}\Phi(d_2). \qquad (101)$$

As with rho, the time decay of real options differs significantly from the stylized form of Eq. (101). For example, the passage of time influences the option not only directly but through forward value F_T, but forward contracts specify a payment date, not a payment time, and so we are limited to a resolution of business days when measuring the contribution of the forward. However, many conventions for converting from dates to times include weekends and other non-business days, giving rise to jumps in theta. One approach to mitigate this could be coarse-graining the finite difference calculation for theta then scaling to give a daily measure, but theta changes the most when very close to expiry, where such a coarse-graining would not work.

In contrast to rho, the time decay of options is not a measure of market risk, so there is no possibility of reconciling formulae such as Eq. (101) with a set of sensitivities $\vec{\nabla}V(\vec{a})$. Fortunately, there is no pressing need, as theta is best regarded as a subset of profit-and-loss analysis. If a connection to textbook formulae is required, then we advocate calculating an equivalent theta – compute the equivalent theta in a model of choice. For example, given an at-expiry option price V and an at-expiry expected value of the underlying F in an arbitrary model, a volatility may be implied from the Black model as long as F is positive. We can then evaluate Eq. (101). For negative values of F, we must choose an appropriate model that admits such values, but the same approach applies.

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