OPTION VALUATION UNDER THE EFFECT OF TIME DILATION

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Abstract: 110 years ago, Einstein published his Theory of Special Relativity. In this thought experiment, we introduce Einstein's idea of time dilation to option valuation. We analyse the effect of time dilation on the time decay of an European long call option.

Keywords: BLACK SCHOLES, EINSTEIN, TIME DILATION

1 INTRODUCTION

In the following thought experiment, we look at the time decay of a financial option in two systems moving relatively to one another. In his Theory of Special Relativity, Einstein concluded that in a moving system, time passes more slowly than in a stationary system considered relatively. This phenomenon is called time dilation or, put more simply: Moving clocks tick more slowly. In our study, we continue a current thought experiment considering the time in a spaceship flying approximately at the speed of light relative to the time on earth. We transfer this thought experiment to the world of derivatives and assume the following scenario: An investor concludes a financial option on earth. At the same instant, his twin brother concludes an option bearing the same conditions on board a spaceship flying at approximately the speed of light. Here, the earth and the spaceship constitute the reference system.

Section 2 provides a short overview of Einstein's Theory of Relativity and the idea of time dilation. In Section 3, we present the Black and Scholes formula for option valuation. Section 4 introduces the idea of time dilation to option valuation. Section 5 concludes.

2 BASES OF EINSTEINS'S THEORY

The Theory of Relativity was developed by Albert Einstein and it describes the motion of bodies at approximately the speed of light. Einstein's theory is divided into the (i) Theory of Special Relativity (STR) (Einstein, 1905) and the (ii) Theory of General Relativity (GTR) (Einstein, 1914). The STR studies the motion of physical variables in time and space and was extended by the GTR, which treating the curvature created by mass in the space-time continuum.

The STR is based on two essential principle, which are (i) the speed of light is constant in all space-time models and (ii) there is no such thing as absolute space or absolute time. Time and space are relative to one another. The time dilation can be derived from these two principles. It says that both time and space depend on the observer's motion state. If the observer is in the state of a uniform motion, a watch moving relatively to it ticks more slowly.

The relative time difference between the observer and the object in motion, can be calculated as follows

$$\hat{t} = t\sqrt{1 - \left(\frac{\upsilon}{c}\right)^2}, \tag{1}$$

where $\hat{t}$ is the moment in the system in motion, t is the moment in the stationary system, c is the vacuum speed of light (299.792.458 m/s) and $\upsilon = \frac{\lambda}{c}$ is a factor of the speed of light, indicating the speed of the system in motion relative to the stationary system. Note however, $0 < \lambda < 1$ as a system in motion never reaches the factor zero or one. For further information on the derivation of Equation 1 we refer the interested reader to Einstein (1905).

From the perspective of the system in motion, the course of time relative to the stationary system decelerates by

$$\sqrt{1 - \left(\frac{\upsilon}{c}\right)^2}. \tag{2}$$

The moment t in the stationary system can be determined by simply rearranging Equation 1

$$t = \frac{\hat{t}}{\sqrt{1 - \left(\frac{\upsilon}{c}\right)^2}}. \tag{3}$$

For the further explanations, we define $\hat{t}$ as the moment in a spaceship moving relatively to the earth and t as the moment on earth.

In our thought experiment we only deal with the time dilation as described in the STR and with its effect on the time decay of an option.

3 PRINCIPLE OF OPTION PRICE CALCULATION

In general, the price of an option can be explained as the sum of its (i) time value and (ii) intrinsic value. The time value represents the probability that the option will be in-the-money by the end of its term. The intrinsic value is the difference between the option strike price K and the underlying price S

$$\begin{aligned} \textit{Intrinsic Value Call} &= max(S - K; 0) \\ \textit{Intrinsic Value Put} &= max(K - S; 0). \end{aligned}$$

As the functions show, the intrinsic value cannot be negative. Moreover, without time value the price of an out-of-the money option would be zero. Options with longer time to maturity have a higher time value than options with shorter time to maturity. This is because with longer maturity the probability increases that the option's intrinsic value is greater than zero before expiry.

The intrinsic value of an option is easy to calculate. However, to determine the time value of an option more sophisticated methods are needed. The financial literature proposes several methods to calculate the price of an option (Van Hulle (1988); Boyle (1977); Schwartz (1977)). A very common approach to determine the price of a European long call option at time t was introduced by Black and Scholes (1973)

$$C(S, t) = SN(d_1) - Ke^{r(t-t^*)}N(d_2) \tag{4}$$

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$$d_1 = \frac{ln(\frac{S}{K}) + \left(r + \frac{1}{2}\sigma^2\right)(t^* - t)}{\sigma\sqrt{(t^* - t)}}$$
$$d_2 = \frac{ln(\frac{S}{K}) + \left(r - \frac{1}{2}\sigma^2\right)(t^* - t)}{\sigma\sqrt{(t^* - t)}},$$

where $N(d)$ is the cumulative normal density function, r the risk free interest rate and t^* the maturity date of the option. Based on the Black and Scholes formula, the option's sensitivity to a change in time is

$$\theta = \frac{\delta C(S,t)}{\delta t}. \tag{5}$$

Hence, if all other parameters are constant the option value changes over time by the value of θ.

4 TIME DILATION AND BLACK SCHOLES

In this section, we introduce the idea of time dilation to the Black Scholes model. The derived formulas are applied to or option thought experiment introduced in Section 1. For the valuation of the two European long call options we assume an underlying price S of 50, a strike price K of 50, a risk free interest rate r of 2%, no dividend payments, volatility σ of 30%, the maturity of the earth option to be three years and λ to be 0.5.

For each point in time t the remaining time to maturity of the earth option m_t^E is simply given by

$$m_t^E = t_E^* - t \tag{6}$$

where t_E^* is the maturity date of the earth option. To calculate the maturity date t_S^* of the space option at earth time, we introduce t_E^* to Formula 3

$$t_S^* = \frac{t_E^*}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}. \tag{7}$$

Hence, the time to maturity of the space option m_t^S at earth time t is given by

$$m_t^S = t_E^* - \hat{t}. \tag{8}$$

Compared to m_t^E, the time to maturity of the space option decelerates by Formula 2. The maturity of the options is indicated in years. Correspondingly, we calculate $365 \times t_E^*$ or $365 \times t_S^*$ theta values for the earth respectively for the space option, to represent the time decay of the respective option during its maturity.

Figure 1 shows the effect of the speed of the spaceship on the maturity of the space option at earth time. As discernible, the maturity of the space option at earth time increases exponentially, up to a factor of the speed of light. The more the spaceship approaches the speed of light, the more the time in the spaceship decelerates relatively to the time on earth. For instance if the maturity of the earth option is three years and the spaceship moves with the factor 0.5 of the speed of light, the space option's maturity at earth time is 3.46 years. At a factor of 0.75 it is already 4.54 years.

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FIGURE 1

The effect of λ on the maturity of the space option at earth time.

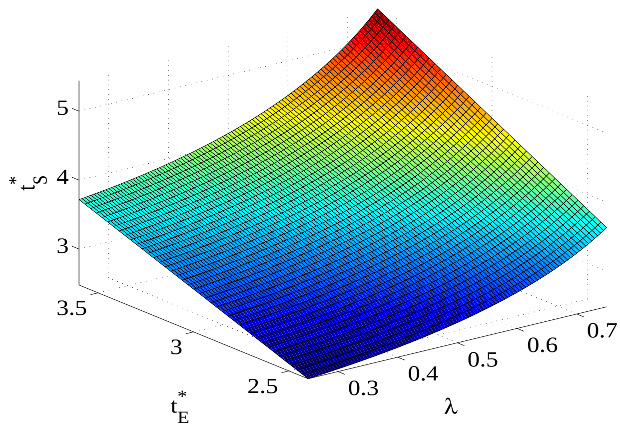
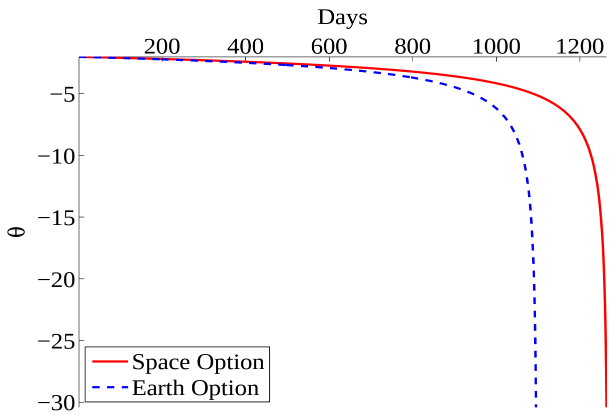


FIGURE 2

θ value for call option.



If λ is constant the maturity of the earth option and the maturity of the space option at earth time behave linearly.

At an increasing speed of the spaceship, the time decay of the option decelerates by the factor of Formula 2. Figure 2 shows the theta value of the earth and the space option.

The theta value of the earth option reaches its minimum at maturity, which is after three year or $365 \times 3 = 1095$ days. The space option, however, experiences the highest time decay 169 days after the expiration of the earth option, which is at day 1264.

5 CONCLUSION

Both options experience their own time decay; The space option expires significantly later than the earth option. The more the speed of the spaceship accelerates, the longer the space options maturity at earth time. Hence, the time decay of the space option decelerates with increasing speed of the spaceship. In the above mentioned thought experiment, the time value elasticity of the space option relative to the earth option is obvious. Hence, it can be concluded that the time dilation described by Albert Einstein can also be applied to financial options. Finally, the time value evolution clearly shows a shift of the time decay in conformity with the patterns described by Einstein.

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