A GRAPHICAL NOTE ON EUROPEAN PUT THETAS

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INTRODUCTION

Theta measures the *time decay* of an option, meaning it measures the sensitivity of an option's price to a change in the time remaining until expiration. For European call options it is well known that theta is positive, indicating that the call's price decreases due to the passage of time, ceteris paribus.[1] However, thetas for European put options can be either positive or negative, meaning that the put's price can decrease or increase due to the passage of time alone. Furthermore, this can happen for quite reasonable values of the stock-to-exercise price ratio, the stock volatility, and the riskless interest rate.

It is important for several reasons for option traders and researchers to fully understand what happens to an option's price as time passes

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[1]Sometimes theta is referred to as the *negative* of the partial derivative; see, for example, Cox and Rubinstein (1985, pp. 304–305).

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and its expiration date draws nearer. First, only one of the parameters that affect the price of an option will change in a known manner, and that is the time to expiration. Second, while it is quite likely that the values of other relevant parameters will change, the magnitude of the change can be small or large. However, the magnitude of the change in time to expiration will be large over the life of the option. Third, it is important to know a strategy's *position theta* (that is, the average theta of its components), as it will have a bearing on how the position's value changes when the prices of the underlying assets change and time passes.[2] For example, the position theta of a delta-neutral strategy (meaning the average delta is zero) will indicate how the position's value will change over time when the prices of the underlying assets change by sufficiently small amounts.[3] In such a situation, it would be advantageous to have a negative position theta. Finally, theta-neutral strategies will result in position values that will not change for small changes in time to expiration. Such strategies might prove useful under certain circumstances, either alone or in conjunction with another strategy. For example, a theta-neutral strategy can be coupled with another strategy to minimize the time decay of a position.[4]

The purpose of this article is to graphically analyze the comparative statistics of European puts with respect to their time to expiration.[5] That is, the partial derivative of a put's price with respect to its time to expiration is explored graphically to more fully understand when it is positive and negative, and just how it changes due to the passage of time.[6] In doing so, a fuller understanding of put thetas is provided.

THE BLACK–SCHOLES PUT PRICE

The familiar Black–Scholes (1973) put pricing formula is:

$$P = Xe^{-rT}N(-d_2) - SN(-d_1) \tag{1}$$

with

$$d_1 = [\ln(S/X) + (r + 0.5\sigma^2)T]/\sigma\sqrt{T} \tag{2a}$$

$$d_2 = d_1 - \sigma\sqrt{T} \tag{2b}$$

[2] See Cox and Rubinstein (1985, pp. 305–306).
[3] See Cox and Rubinstein (1985, pp. 307–317.)
[4] See Stoll and Whaley (1993, pp. 307–309).
[5] This is done by using the *Mathematica* system (developed by Wolfram Research); see Miller (1993).
[6] While the partial derivatives for calls have been analyzed in an intuitive manner by Chance (1994), he refers only indirectly to put thetas. All references to puts and calls should be interpreted as referring to *European* puts and calls.

where X is the exercise price, r is the annualized riskless rate, S and σ are the underlying stock's price and annualized volatility, $N(\cdot)$ denotes the standard normal cumulative probability distribution, and T is the time to expiration.[7] It is commonly known that P is a convex function of S that asymptotically approaches the following lower boundary:

$$P = Xe^{-rT} - S \tag{3}$$

for relatively low values of S and zero for relatively high values of S.[8]

Opposing Forces and Put Thetas

As described by Jarrow and Rudd (1983, p. 214), the relative importance of two opposing forces determines the sign of the put theta derived from eq. (1). One force is represented by the risklessly discounted present value of the exercise price, Xe^{-rT}. As time passes, this present value increases, thereby increasing the put's price. But as time passes, it is less likely that the stock price's volatility will lead either an out-of-the-money put option into the money or an in-the-money put option more deeply into the money, thereby decreasing the put's price. Thus, the two forces (the *present value force* and the *volatility force*) work in opposite directions, so that their relative strengths determine the change in the put's price due to the passage of time.[9]

Three-Dimensional Analysis of Put Prices

Figure 1 illustrates in three dimensions the opposing forces at work on a put option having $X = \$100$ where $\sigma = 30\%$ and $r = 4\%$.[10] Looking from left to right along the S axis illustrates that P is convex and downward sloping in S for any T. The aforementioned ambiguity of

[7]The Black–Scholes model for ordinary puts given in eq. (1) can be altered to allow for dividends and the analysis of puts on futures. For example, Merton's (1973) model allows for dividends by simply substituting Se^{-yT} for S in eq. (1) where y is the continuous dividend yield on the stock. In turn, Black's (1976) model for puts on futures is derived by substituting $Fe^{(y-r)T}$ for S in Merton's model or, equivalently, Fe^{-rT} for S in the Black–Scholes model [see, for example, Hull (1993, pp. 247–248 and 258–265)], resulting in:

$$P = e^{-rT}[XN(-d_2) - FN(-d_1)]$$

where $d_1 = [\ln(F/X) + 0.5\sigma^2 T]/\sigma\sqrt{T}$ and $d_2 = d_1 - \sigma\sqrt{T}$.

[8]The boundary condition for puts on futures corresponding to eq. (3) is: $P = (X - F)e^{-rT}$. For more on boundary conditions, see Puttonen (1993).

[9]These two forces are also present in puts on futures.

[10]All the figures shown in this article are representative of the figures obtained when different (but reasonable) values are used for the parameters.

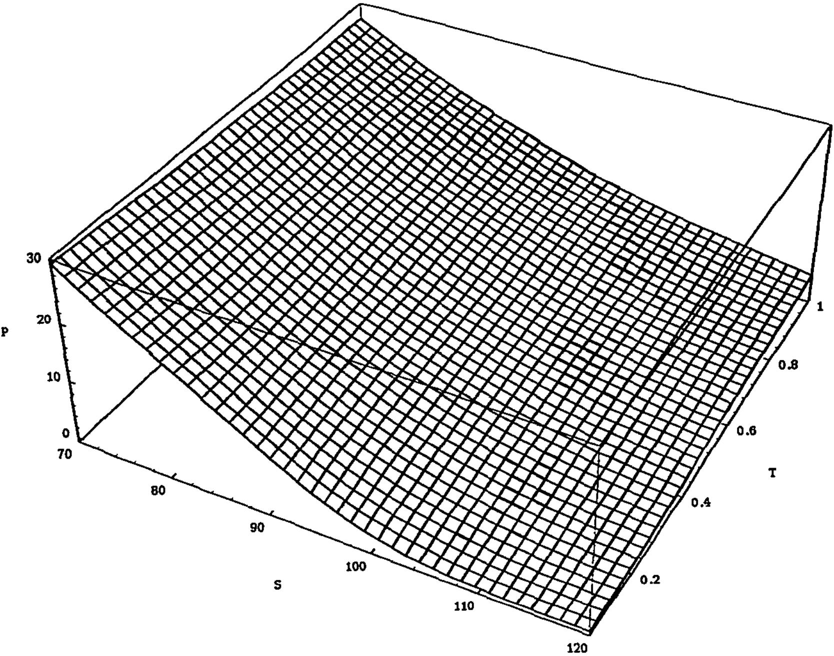

FIGURE 1
Put price as a function of stock price S and time to expiration T (assuming $X = \$100$, $r = 4\%$, $\sigma = 30\%$).

sign in the change in P due to the passage of time can also be observed by looking from right to left along the T axis.[11] Doing so clearly shows that when the put option is well out of the money (e.g., $S = \$120$), P decreases as time passes, since the volatility force dominates the present value force. Because S has little chance of dropping below X, time-induced increases in the present value of X exert little force on P. Conversely, when the put is deep in the money (e.g., $S = \$70$), the present value force exceeds the volatility force, resulting in a positive change in P due to the passage of time.

[11]Some textbooks contain two-dimensional diagrams that represent vertical slices taken from Figure 1. For example, Cox and Rubinstein (1985, p. 220) indicate the level of P associated with various levels of T for a given S. However, none of the books examined showed a put that was deep enough in the money so that P actually increased as T decreased for a given S, which is where theta is negative.

This phenomenon may be illustrated also by examining the put theta (Θ) itself:

$$\Theta = rXe^{-rT}[N(d_2) - 1] + S\sigma n(d_1)/2\sqrt{T} \tag{4}$$

where $n(\cdot)$ is the standard normal probability density.[12] The first term on the right-hand side, $rXe^{-rT}[N(d_2) - 1]$, can be thought of as representing the present value force described earlier. It is always negative, since it is the product of a positive number, rXe^{-rT}, and a negative number, $[N(d_2) - 1]$. The second term on the right-hand side, $S\sigma n(d_1)/2\sqrt{T}$, can be thought of as representing the volatility force, and is always positive since all of its components are positive. So, if the present value force dominates, then $|rXe^{-rT}[N(d_2) - 1]| > S\sigma n(d_1)/2\sqrt{T}$ and hence $\Theta < 0$, resulting in the put increasing in price as time passes and the expiration date approaches. Conversely, if the volatility force dominates, then $|rXe^{-rT}[N(d_2) - 1]| < S\sigma n(d_1)/2\sqrt{T}$ and hence $\Theta > 0$, resulting in the put decreasing in price as time passes and the expiration date approaches.

Figure 2a presents a view of Θ as a function of S and T.[13] Looking along the T axis confirms that $\Theta < 0$ for all T when S is roughly less than \$90 (this can be confirmed in Fig. 2b, which expands the area between \$75 and \$90), so that P increases as T decreases. However, this does not remain true for any level of S. For sufficiently large S (e.g., S = \$120), it shows that Θ first increases and then decreases but remains positive as T decreases. Note that Θ is most sensitive to changes in S and T when the stock is selling for a price near X, provided the put is near expiration.[14]

[12]See Chance (1994, p. 45).

[13]The corresponding equation for puts on futures is: $\Theta = e^{-rT}\{[Xn(-d_2)][d_1/2T] - [Fn(-d_1)][d_2/2T]\} - rP$ where F denotes the current futures price. Three-dimensional graphs for P and Θ of puts on futures (see footnote 7) are qualitatively similar to the graphs of ordinary options given in Figures 1–4. Wolf (1984, pp. 496–497, 503) discusses thetas for puts on futures, mentioning that they can be negative and giving an example where they are positive. Similarly, Shastri and Tandon (1986, pp. 603, 605) illustrate a case involving positive thetas for puts on futures.

[14]Some textbooks contain two-dimensional figures that represent vertical slices taken from Figure 2. For example, Cox and Rubinstein (1985, p. 231) indicate the level of Θ associated with various levels of T for a given S, as well as the level of Θ associated with various levels of S for a given T.

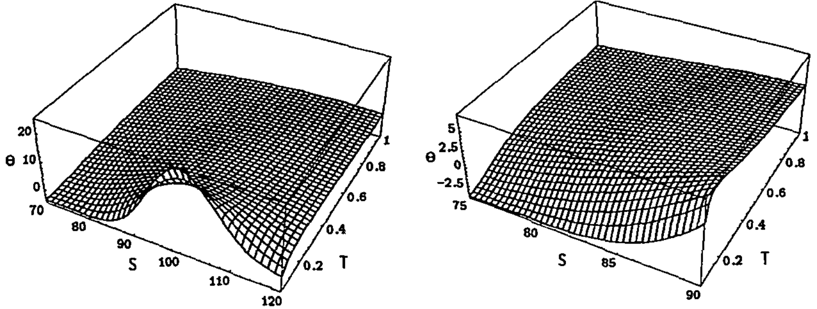

FIGURE 2
Put theta as a function of stock price S and time to expiration T (assuming $X = \$100$, $r = 4\%$, $\sigma = 30\%$). (a) Stock prices between \$70 and \$120. (b) Stock prices between \$75 and \$90.

PUT PRICES, INTRINSIC VALUES, AND TIME TO EXPIRATION

Jarrow and Rudd (1983, p. 219) describe and illustrate the movement that occurs in the (P, S) plane as time passes.[15] They note that this movement is more complex than the movement associated with calls for two related reasons.

First, prior to expiration, the analogous call price function $C(S)$ always lies entirely above the intrinsic value function defined by the kinked line:

$$IV_c(S) = \max(0, S - X) \tag{5}$$

That is, $C(S) \geq IV_c(S) \forall S$. However, the graph of the put price function $P(S)$ intersects its intrinsic value function, lying below it [but bounded by eq. (3)] for stock prices that are substantially in the money and above it for stock prices that are either at or out of the money. That is, prior to expiration and given the following intrinsic value function for puts:

$$IV_p(S) = \max(0, X - S) \tag{6}$$

it is known that $IV_p(S) > P(S) \geq Xe^{-rT} - S$ if $S < X$ and $IV_p(S) < P(S)$ if $S \geq X$.[16]

Second, $C(S)$ moves downward toward $IV_c(S)$ as $T \rightarrow 0$, until the two functions coincide at $T = 0$. Hence, the movement of the call price

[5]Jarrow and Rudd are cited here because it was the only textbook found that attempts to describe the movement of the put price function $P(S)$ as time passes.
[6]Actually $IV_p(S) < P(S)$ not only for $S \geq X$, but also for stock prices that are slightly less than X, as is illustrated in Figure 3.

function is unambiguous in that, for any stock price, the value of a call will decrease with the passage of time.[17] In contrast, since $P(S)$ intersects $IV_p(S)$, it cannot merely move in one direction as time passes to collapse to $IV_p(S)$ at $T = 0$. Instead, $P(S)$ undergoes a clockwise pivoting motion [meaning $P(S)$ increases for low S and decreases for high S] as time passes until it coincides with $IV_p(S)$ at $T = 0$. However, the pivot point S' is not fixed for all T, as will be shown shortly.

Figure 3 shows the price functions for two puts having expiration dates T equal to $T1$ and $T2$ where $T1 < T2$, along with the associated intrinsic value function (note how these puts are priced below their intrinsic value when they are deep in the money). These two put price functions intersect each other at a point denoted I_{12}. Jarrow and Rudd (1983, Figure 15-1, p. 219) erred in not having their put price functions intersecting even though any two put price functions *must* intersect. This is because (1) when $S = 0$, $P_{T1} > P_{T2}$, since, in this situation, $P = Xe^{-rT}$ and $Xe^{-rT1} > Xe^{-rT2}$; and (2) when S is arbitrarily large, $P_{T1} < P_{T2}$. This can be seen by examining eq. (1) and Figure 1. It follows by continuity that the two price functions must intersect.[18]

As the time to expiration is shortened from $T2$, the put price function $P(S)$ pivots clockwise, getting closer and closer to $IV_p(S)$ as $T \rightarrow 0$. The pivot point (S', P') is the point where $P(S)$ at time T

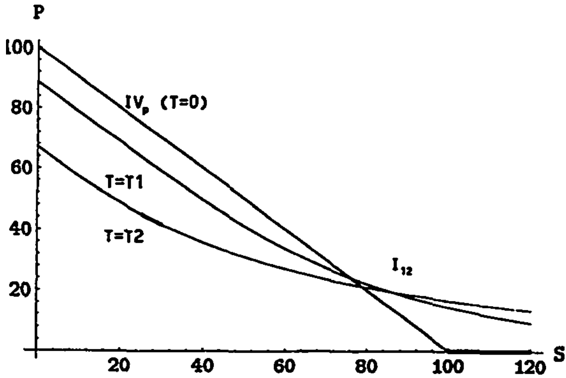

FIGURE 3
Put price as a function of stock price S with $T1 < T2$ (assuming $X = \$100$, $r = 4\%$, $\sigma = 30\%$).

[7]See Hull (1993, p. 160) for a graphical illustration.
[8]They will intersect at only one point because $\partial P/\partial S < 0$ and $\partial^2 P/\partial S^2 > 0$.

intersects $P(S)$ at time $T - \Delta T$.[19] However, the pivoting that takes place is complicated because the point at which the pivoting takes place moves southeast (that is, down and to the right) as $T \to 0$. More specifically, $\Theta = 0$ at the stock price S' associated with the pivot point for any given T. Thus, if the actual stock price is less than S', the theta will be negative and the passage of time will lead to an increase in the put's price. Conversely, if the actual stock price is greater than S', the theta will be positive and the passage of time will lead to a decrease in the put's price. That is, for a given T, $\Theta < 0$ if $S < S'$ and $\Theta > 0$ if $S > S'$. This is illustrated in Figure 4, which shows the stock price S' where $\Theta = 0$ for any given T. Notice how the function is negatively sloped and convex, indicating that, as time passes, the value of S' gets larger. Consequently, the corresponding put price P' gets smaller and, thus, the pivot point (S', P') moves southeast as time passes.[20]

SUMMARY

While it is well known that European put prices can increase or decrease with the passage of time, analytical exploration of the effect of time to expiration on these prices is cumbersome due to the complexity of the

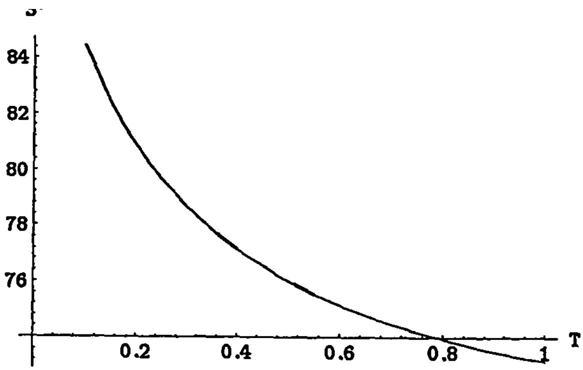

FIGURE 4
Stock prices where theta is zero for various times to expiration T (assuming $X = \$100$, $r = 4\%$, $\sigma = 30\%$)

[19]The values of $T1$ and $T2$ giving rise to the intersection point I_{12} in Figure 3 correspond to three years and ten years, respectively. While they should differ by an infinitesimally small amount ΔT, a large ΔT (seven years) is used to make P_{T1} and P_{T2} have visually different locations.

[20]This can be seen by taking any two values of T and S' from Figure 4 and then inserting them into eq. (1) to determine the corresponding values of P'. The smaller value of T will have a larger value of S' and smaller value of P', thereby showing the direction of the shift in the pivot point.

partial derivative, known as theta. Such an exploration is done here with graphs. In doing so, it is shown how the put price function converges to its intrinsic value function in a complex clockwise pivoting motion where the pivot point, rather than being fixed, moves as the time to expiration gets shorter. A description of just how this motion is linked to the put's theta is provided. More specifically, the pivot point corresponds to the stock price where the put's theta is zero. If the stock price is greater than this, the put will decrease in price with the passage of time, but if the stock price is less, the put will increase in price. This analysis should help traders better understand how the passage of time influences the value of any actual or contemplated position that involves put options.

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