# Jet ampleness and Newton-Okounkov bodies of divisors

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# Outline

1. A brief of positivity in algebraic geometry
2. Generalization of Ampleness
3. Newton-Okounkov Body

# Positivity in Algebraic Geometry

# Notation and Conventions

- We work throughout over $\mathbb{C}$.
- A scheme is a separated algebraic scheme of finite type over $\mathbb{C}$.
- A variety is a reduced and irreducible scheme.
- Let $E$ be a vector bundle on a scheme $X$, $\mathbb{P}(E)$ denotes the projective bundle of one-dimensional quotients of $E$
- The subsequent discussion will automatically invoke the concepts of divisors/bundles/linear series and intersection theory without introduction. [Ful98]

# Riemann-Roch

Here is the asymptotic form Riemann-Roch[Laz04, 1.2]:

**Theorem (Asymptotic Riemann-Roch )**

*Let $X$ be an irreducible projective variety of dimension $n$, and $D$ be a divisor on $X$. Then the Euler characteristic $\chi(X, \mathscr{O}_X(mD))$ is a polynomial of degree $\leq n$ in $m$ with*

$$\chi(X, \mathscr{O}_X(mD)) = \frac{(D^n)}{n!} m^n + O(m^{n-1}).$$

*More generally, for any coherent sheaf $\mathscr{F}$ on $X$*

$$\chi(X, \mathscr{F} \otimes \mathscr{O}_X(mD)) = rank(\mathscr{F}) \cdot \frac{(D^n)}{n!} \cdot m^n + O(m^{n-1}).$$

# Ample and Very Ample

**Motivation**[Laz04, 1.1.24]: Given a divisor $D$ on a projective variety $X$, the **positivity** of $D$ can be viewed by asking: if $D$ can be a hyperplane section under some projective embedding of $X$ or $D$ is **very ample**? (positivity on metric implies cohomological vanishing result)

## Definition

Let $X$ be a complete scheme and $L$ a line bundle on $X$.

- $L$ is **very ample** if there exists a closed embedding $X \subseteq \mathbb{P}$ of $X$ into some projective space $\mathbb{P} = \mathbb{P}^N$, such that

$$L = \mathscr{O}_X(1) = \mathscr{O}_{\mathbb{P}^N}(1)|_X.$$

- $L$ is **ample** if $L^{\otimes m}$ is very ample for some $m > 0$.

Next, we will present some results from a **cohomological** and **numerical** perspective.

# Cohomological Results

**Theorem (Cartan-Serre-Grothendieck theorem)**

*Let $L$ be a line bundle on a complete scheme $X$. TFAE*

- *$L$ is ample.*
- *$\forall$ coherent sheaf $\mathscr{F}$ on $X$, $\exists m_1 = m_1(\mathscr{F}) \in \mathbb{Z}_{>0}$ such that*

$$H^i(X, \mathscr{F} \otimes L^{\otimes m}) = 0, \forall i > 0, \forall m \geq m_1.$$

- *$\forall \mathscr{F}, \exists m_2 = m_2(\mathscr{F})$, such that $\mathscr{F} \otimes L^{\otimes m}$ is generated by global sections $\forall m \geq m_2$.*
- *$\exists m_3 \in \mathbb{Z}_{>0}, \forall m \geq m_3, L^{\otimes m_3}$ is very ample.*

# Cohomological Results

Based on the vanishing theorem, we have the following result directly.

**Proposition (Finite pullback)**

*$f : Y \to X$ is finite map of complete schemes, $L$ an ample line bundle on $X$. Then $f^*L$ is ample.*

**Corollary (Globally generated line bundles)**

*Suppose $L$ globally generated, and*

$$\varphi = \varphi_{|L|} : X \to \mathbb{P}H^0(X, L).$$

*Then $\varphi$ is finite map iff $L$ is ample, or equivalently iff*

$$deg(c_1(L) \cdot C) > 0, \forall irreducible\ curve C \subset X.$$

# Cohomological Results

The following proposition allows us to directly assume that X is integral.

**Proposition**

*Let $X$ complete, $L$ line bundle.*

- *$L$ is ample iff $L_{red}$ is ample on $X_{red}$.*
- *$L$ is ample iff the restriction of $L$ to each irreducible component of $X$ is ample.*

# Numerical Results

## Theorem (Nakai-Moishezon-Kleiman criterion)

*Let $L$ be a line bundle on a projective scheme $X$. Then $L$ is ample iff*

$$deg(c_1(L)^{dimV} \cdot V) > 0$$

*for any positive-dimensional irreducible subvariety $V \subseteq X$.*

## Corollary

*Let $D_1 \equiv_{num} D_2$, then $D_1$ ample iff $D_2$ ample.*

## Corollary

*Let $f : Y \to X$ is surjective finite map of projective schemes, and $L$ line bundle on $X$. If $f^*L$ is ample then $L$ ample.*

# Numerical Result

Based on the method of the proof of that criterion, we can also get the asymptotic results about higher cohomology.
Moreover, we can broaden the scope of $Div(X) = Div_{\mathbb{Z}}(X)$ to $Div_{\mathbb{Q}}(X)$ even $Div_{\mathbb{R}}(X)$.
The numerical criterion has nef version.

**Theorem (Kleiman criterion)**

*Let $X$ be a complete variety (or scheme). If $D$ is a nef $\mathbb{R}$-divisor on $X$. Then*

$$(D^k \cdot V) \geq 0$$

*for every irreducible subvariety $V \subseteq X$ of dimension $k$.*

This theorem can be taken as the definition of nef divisor/line bundle.

# Generalization of Ampleness

# Geometry View of Very Ampleness

In [Har77, 2.7.3], the very ampleness can be viewed as the ability to separate two points:

## Proposition

*Let $X$ be a projective variety, $L$ be a line bundle (invertible sheaf). Let $\varphi = \varphi_{|L|}$. $\varphi$ is a closed immersion iff*

- $\forall x, y \in X, \exists s \in H^0(X, L)$ *such that* $s \in \mathfrak{m}_x L_x$ *but* $s \notin \mathfrak{m}_y L_y$.
- $\forall x \in X$, *the set* $\{s : s \in \mathfrak{m}_x L_x\}$ *generates the vector space* $m_x L_x / m_x^2 L_x$.

The second point can be understood as two points coinciding with one point, becoming the infinitesimal case.
But in high dimension case, the generalization of very ampleness becomes difficult.

# K-Jet Ampleness and K-Very Ampleness

These two concepts are described in detail in [BS93] and [DRSB99].

## Definition ($K$-jet ampleness)

Let $X$ be a smooth projective variety of dimension $n$ over $\mathbb{C}$. Let $x_1, ..., x_r$ be $r$ distinct points on $X$ with their ideal sheaves $\mathfrak{m}_i$. $\mathfrak{m}_i$ is the maximal ideal of $\mathscr{O}_{X,x_i}$. Let 0-cycle $\mathscr{Z} = x_1 + ... + x_n$. We say $L$ is $k$-**jet ample at** $\mathscr{Z}$ if, for every $r$-tuple $(k_1, ..., k_r) \in \mathbb{Z}^r_{>0}$ such that $\sum k_i = k + 1$, the restriction map

$$H^0(X, L) \to H^0(X, L \otimes (\mathscr{O}_X/(\otimes \mathfrak{m}_{\mathrm{i}}{}^{k_i}))) \cong \oplus H^0(X, L \otimes (\mathscr{O}_X/\mathfrak{m}_i^{k_i}))$$

is surjective. And we say $L$ is $k$-**jet ample** if $L$ is $k$-jet at any 0-cycle on $X$ with $r \geq 1$.

# K-Jet Ampleness and K-Very Ampleness

**Definition ($k$-very ampleness)**

We say $L$ is $k$-very ampleness if the restriction map

$$H^0(X, L) \to H^0(X, L \otimes \mathscr{O}_{\mathscr{Z}})$$

is surjective for any 0-dimensional subscheme, where $length(\mathscr{O}_{\mathscr{Z}}) = k + 1$.

**Definition ($k$-jet separation)**

If we have surjective restriction map

$$H^0(X, L) \to H^0(X, L \otimes \mathscr{O}_X / \mathfrak{m}_x^{s+1}) = J_x^s(L)$$

# K-Jet Ampleness and K-Very Ampleness

- The $h^0(X, L)$ will be larger than or equal to $\binom{n+k}{n}$. Since $h^0(X, L) \geq \sum h^0(X, L_{x_i}) = \sum \binom{n+k_i}{n}$, and we can select $r = 1$ (this case will be the largest on the right hand).
- $a$-jet ample line bundle $\otimes$ $b$-jet ample line bundle will be $(a+b)$-jet ample line bundle.
- $k-$very ampleness can imply jet ampleness each other to some extend:

**Proposition**

*Assume $L$ is $k-$very ampleness, then $L$ is $\tau$-jet ample for the biggest integer $\tau$*

$$\binom{\tau+n}{n} \leq k+1.$$

*But if $L$ is $k-$jet ample, then $L$ is $k-$very ampleness.*

# Jet bundle

For the definition of jet bundle, here we follow the algebraic version [Vak98].

## Definition

Consider $X \times X$ and diagonal $\Delta$ and projections $\pi_1, \pi_2$. We let

$$(\pi_1)_*((\mathscr{O}_{X\times X}/\mathscr{I}_\Delta^{t+1}) \otimes (\pi_2)^*(L))$$

be the $t$-th jet bundle.

# Jet bundle

- Actually, we denote by $J_k(X, L)$ the $t$-th jet bundle of a line bundle $L$ on $X$ which is a vector bundle of rank $\binom{n+k}{k}$, the dimension of the space of the polynomials of degree $\leq k$ in $n$ variables.
- And there are natual maps $j_k : L \to J_t(X, L)$. When $L$ is $k$-jet ample, then $j_k$ is surjective, and $kK_X + (n+1)L$ is nef.
- More over only if $(X, L) \cong (\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(k))$, $kK_X + (n+1)L$ is numerically equivalent to zero.

# Criterion for K-Jet Ampleness on Toric Varieties

In this paper [DRSB99], a criterion for the k-jet ampleness on toric varieties is provided.

**Theorem**

*Let $L$ be a line bundle on a non-singular toric variety $X(\Delta)$, then TFAE:*

- *$L$ is $k$-jet ample.*
- *$L \cdot C \geq k$ for any $T$-invariant curve $C$.*
- *$\psi_L$ is $k$-convex.*
- *The Seshadri constant $\epsilon(L, x) \geq k$ for each $x \in X$.*

# Newton-Okounkov Body

# Historical background

Let's start with some historical background[Mer18]:

- The idea of studying the relationship between convex bodies and algebraic varieties can be traced back to the Russian school in the mid-1970s. (Bernstein, Khovanskii, Kushnirenko)
- The convex body of a polynomial $f$ is the convex hull of the multidegrees of all its monomials.
- The number of roots of a system of $n$ polynomial equations in $\mathbb{C}^n$ is equal to the mixed volume of $n$ convex bodies divided by $n!$ (Bernstein-Kushnirenko Theorem).
- Demazure introduced the correspondence between toric varieties and polytopes.
- The above correspondences only for special projective varieties. Okounkov gave a constuction of a convex body associated to an embedded variety, carring its degree as the volume. The Newton-Okounkov body is his latter systematic theory.

# Construction

Here we introduce the construction of Newton-Okounkov body[LM08].

- Let $X$ be an irreducible variety of $dim = n$ and $D$ be a divisor on $X$ (usually big).
- We fix an admissible flag:

$$Y_\bullet : Y_0 = X \supseteq Y_1 \supseteq \ldots \supseteq Y_n = \{pt\}$$

  of irreducible subvarieties of $X$, where $codim(Y_i) = i$, and each $Y_i$ is smooth at $Y_n$.
- We have high order valuation $v_{Y_\bullet}$ associated to $Y_\bullet$,

$$v_{Y_\bullet} : H^0(X, \mathscr{O}_X(D)) \to \mathbb{Z}^n$$

  satisfing
  - $v_{Y_\bullet}(s) = \infty$ iff $s = 0$;
  - Lexicographical order, and

$$v_{Y_\bullet}(s_1 + s_2) \geq min\{v_{Y_\bullet}(s_1), v_{Y_\bullet}(s_2)\}$$

    for any $s_1, s_2 \neq 0$;
  - $v_{Y_\bullet}(s \otimes t) = v_{Y_\bullet}(s) + v_{Y_\bullet}(t)$;

# Construction

- Algorithm of valuation:
  - $v_1 = ord_{Y_1}(s)$;
  - $s_1 = s|_{Y_1}$;
  - For $i = 2...n-1$:
    - $v_i = ord_{Y_i}(s_{i-1})$;
    - $v_{i+1} = s_i|_{Y_{i+1}}$;
  - Output $v_{Y_\bullet}(s) = (v_1, ..., v_n)$.
- **One-dimension leaves**: let $W \subseteq H^0(X, \mathscr{O}_X(D))$, $W_{>a} = \{s \in W : v_{Y_\bullet}(s) > a\}$ and $W_{\geq a} = \{s \in W : v_{Y_\bullet}(s) \geq a\}$, then we have

$$dim(W_{\geq a}/W_{>a}) \leq 1.$$

# Construction

- Graded semigroup:

$$\Gamma(D) = \{(v_{Y_\bullet}(s), m) : 0 \neq s \in H^0(X, \mathscr{O}_X(mD)), m \geq 0\}.$$

- Newton-Okounkov body:

$$\Delta(D) = \Delta_{Y_\bullet}(D) = \text{convex hull}(\Gamma(D)) \cap (\mathbb{R}^d \times \{1\}).$$

- Example: Let $C$ be a smooth projective curve genus $g$ with fixed point $P \in C$, we have flage $C \supseteq \{P\}$. If $c = deg(D) \geq 2g + 1$ ($D$ is very ample), $im(v) = [0, 1, ..., c - g]$. And

$$\Gamma(D) = \{(0,0)\} \cup \{(k, m) : m \geq 1, 0 \leq k \leq mc - g\}.$$

$$\Delta(D) = [0, c]$$

# Volume

The **volume** of a divisor $D$ is defined as the limit

$$vol_X(D) = \lim_{m\to\infty} \frac{h^0(X, \mathscr{O}_X(mD))}{m^d/d!}.$$

A divisor is **big** if $h^0(X, \mathscr{O}_X(mD)) \sim m^d$.

## Theorem

*Let $D$ be a big divisor on a projective $X$ of dimension $n$. Then*

$$vol_{\mathbb{R}^n}(\Delta(D)) = \frac{1}{d!} vol_X(D),$$

**Remark**:

- This theorem can be extended to the class of numerically equivalent divisors, as well as to linear systems.

# Positivity and Newton-Okounkov body

## Proposition

*Let $D$ be a big $\mathbb{R}$-divisor on a smooth projective variety $X$ of dimension $n$, let $x \in X$. TFAE*

- $x \notin \mathbb{B}_-(D)$;
- $\exists Y_\bullet$ *at* $x$, *such that* $0 \in \Delta_{Y_\bullet}(D) \subseteq \mathbb{R}^n$;
- $\forall Y_\bullet$ *at* $x$, *such that* $0 \in \Delta_{Y_\bullet}(D) \subseteq \mathbb{R}^n$.

## Corollary

*With notation as above. TFAE:*

- $D$ *is nef;*
- $\forall x \in X, \exists Y_\bullet$ *at* $x$, *such that* $0 \in \Delta_{Y_\bullet}(D) \subseteq \mathbb{R}^n$;
- $\forall x \in X, \forall Y_\bullet$ *at* $x$, *such that* $0 \in \Delta_{Y_\bullet}(D) \subseteq \mathbb{R}^n$.

# Positivity and Newton-Okounkov body

We write

$$\Delta_\epsilon = \{(x_1, ..., x_n) \in \mathbb{R}^n : x_1 + ... + x_n \leq \epsilon\}$$

for the standard $\epsilon$-simplex.

**Theorem**

*Let $D$ be a big $\mathbb{R}$-divisor on $X$, $x \in X$ be an arbitrary closed point. TFAE*

- $x \notin \mathbb{B}_+(D)$;
- $\exists Y_\bullet$ *at* $x$ *with* $Y_1$ *ample such that* $\Delta_{\epsilon_0} \subseteq \Delta_{Y_\bullet}(D)$ *for some* $\epsilon_0 > 0$;
- $\forall Y_\bullet$ *at* $x$ *such that* $\Delta_{\epsilon_0} \subseteq \Delta_{Y_\bullet}(D)$.

# Positivity and Newton-Okounkov body

## Corollary

*Let $X$ be smooth, $D$ be a big $\mathbb{R}$-divisor on $X$. TFAE:*

- *$D$ is ample;*
- *$\forall x \in X, \exists Y_\bullet$ centered at $x$ with $Y_1$ is ample such that $\Delta_{\epsilon_0} \subseteq \Delta_{Y_\bullet}(D)$ for some $\epsilon_0 > 0$.*
- *$\forall Y_\bullet, \exists \epsilon_0 > 0$ such that $\Delta_{\epsilon_0} \subseteq \Delta_{Y_\bullet}(D)$.*

**Remark**: These results are referenced in paper [KL15b]. Paper [KL15a] provides a discussion on the infinitesimal case and establishes criteria for jet separation of adjoint bundles.

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