ZD1

2024年4月30日 16:00

$X = \mathbb{P}^n$, $[x_0 : \cdots : x_n]$ $\qquad [1:0:\cdots:0]$

Flag: $X \supseteq V(x_1) \supseteq V(x_1, x_2) \supseteq \cdots \supseteq V(x_1, \cdots, x_n)$.

$D = \mathcal{O}(1)$

$\nu_1(x_0^{a_0} x_1^{a_1} \cdots x_n^{a_n}) = (a_1, \cdots, a_n)$ $\qquad$ all exponents in $\mathbb{N}_k$. $\sum_0^n a_i = k$.

W_1: we consider all polynomials as monomial subalgebra with degree 1

Since local equations of each subvarieties in Y is x_i $\quad k[x_0, \cdots, x_n]$

we only to concern the case that

$x_0^{a_0} \cdots x_n^{a_n}$; $\sum a_i = 1$. $\qquad i = 0 \cdots n$ $\quad \sum a$

$P(D) \cap (\mathbb{Z}^n_{\geq 0} \times \{1\}) = \{(e_i, 1)\} \cup \{0\}$.

Similarly, W_2 $\quad x_0^{a_0} \cdots x_n^{a_n}$; $\sum a_i = 2$.

$\Rightarrow P(D) \cap (\mathbb{Z}^n_{\geq 0} \times \{2\}) = 2 \cdot (P(D) \cap (\mathbb{Z}^n_{\geq 0} \times \{1\}))$.

$\vdots$

W_n, $\quad x_0^{a_0} \cdots x_n^{a_n}$; $\sum a_i = n$, $\quad a_i \geq 0$, $a_i \leq n$. $\longrightarrow (a_1, \cdots, a_n)$

$\Rightarrow P(D) \cap (\mathbb{Z}^n_{\geq 0} \times \{n\}) = n \cdot (P(D) \cap (\mathbb{Z}^n_{\geq 0} \times \{1\}))$.

Khovanskii base is after simple computation.

$x_0, x_2, \cdots, x_n$.

$\downarrow \quad \downarrow \quad \downarrow$

$e_1 \quad e_2 \quad \cdots \quad e_n$, $\rightarrow$ standard base

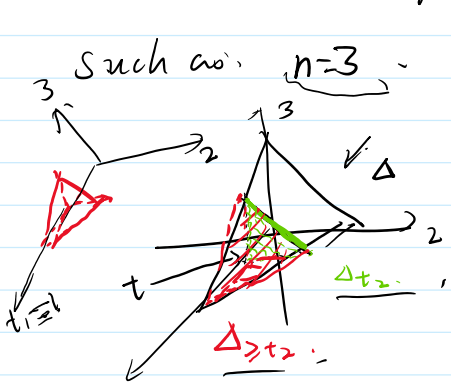
$\lambda_i \geq 0$

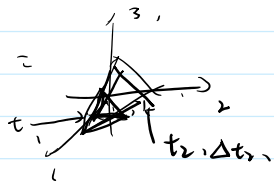
$\Delta(X, D) = \{\sum \lambda_i e_i : \sum \lambda_i = 1\}$.

$E = Y_1 = V(x_1)$

$\Delta(D)_{\nu_1 = t} = \{\sum_{i \geq 1} \lambda_i e_i : \lambda_1 = t, \sum \lambda_i = 1\}$

such as: $n = 3$. $\qquad \Delta(D) =$





e_1, e_2, e_3

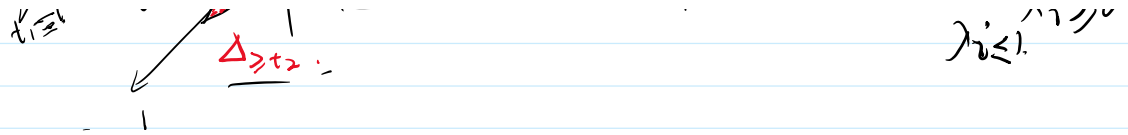
$\sum \lambda_i e_i$

$\sum \lambda_i = 1$.

$\lambda_i \geq 0$

$\lambda_i \leq 1$.

} Simplex



Zariski Decomposition ← Fujita.

$\kappa(X, D)$. X normal complete, alg surf over k.

D - div

Let $N(X, D) = \{ m > 0 : H^0(X, \mathcal{O}_X(mD)) \neq 0 \}$

$N(X, D) \neq \emptyset$ $\forall m, n \in \underbrace{N(X, D)}_{\text{(semigroup, +)}} \Rightarrow mn \in N(X, D)$

$= \emptyset$ D-div $-\infty = \kappa(X, D)$ Kepper

$\neq \emptyset$, graded k-alg Kodira

$R(X, D) = \bigoplus_{i \geq 0}^{\infty} H^0(X, \mathcal{O}_X(iD))$. domain.

$\operatorname{trdeg}_k (R(X, D)) - 1 = \kappa(X, D)$.

Thm. $D \in \operatorname{Div}(X)$. $\kappa(X, D) \geq 0$.

(i). $\kappa = \max_{n \in N(X, D)} \dim(\varphi_{|mD|}(X))$.

$\varphi_{|mD|} : X \longrightarrow \mathbb{P}^{N(n)} = \mathbb{P}(H^0(X, \mathcal{O}_X(nD))^\vee)$

(ii). $\exists!$ $\kappa \geq 0$. $n_0, a, b > 0$. when $m \gg 0$.

$am^\kappa \leq \dim_k H^0(X, \mathcal{O}_X(mn_0 D)) \leq bm^n$.

$\kappa(X, D) \leq \dim(X) = 2$. Rmk: (for all dim. (i) & (ii)).

Def. $D \in \operatorname{Div}(X)$.

(i). D p eff if $\forall H$ amp. $(D \cdot H) \geq 0$.

(ii) D n ef if $\forall C$ integral curve. $(D \cdot C) \geq 0$.

Lemma Let D be a $\mathbb{Q}$-div. on X. D pef iff $(D.P) \geq 0$. $\forall$ nef P on X.

pf: D integral by multiplying suitable integer.

$\Leftarrow$: obvious

$\Rightarrow$ D pef. P nef. H amp on X.

$\forall n \geq 0$. $H + nP$ is ample.

$(D.(H+nP)) \geq 0$. $\Rightarrow$ $D.P \geq 0$. $\square$

Lemma. D on X, D not pef.

$\Rightarrow \forall Z \in \mathrm{Div}(X)$, $H^0(X, \mathcal{O}_X(nD+Z)) = 0$. $\forall n \gg 0$.

pf: D not pef, $\exists$ amp H.

$(D.H) < 0$.

$\Rightarrow (nD+Z).H < 0$. for $n \gg 0$.

&

$|nD+Z| = \emptyset$. $\square$.

Lemma D nef. $\mathbb{R}$-div on X. Then $\kappa(X,D) = 2$. iff

$(D^2) > 0$.

pf: Assume D integral. $D^2 > 0$.

$\forall m > 0$.

$h^0(mD) + h^2(mD) \geq \frac{1}{2} m^2 (D^2) - \frac{1}{2} m (D.K) + \chi(\mathcal{O}_X)$.

Serre duality $\longrightarrow$ $h^0(K - mD) = h^2(mD)$

If $-D$ pef:

$((-D).D) \geq 0$. (D nef) contradicted to $D^2 > 0$

$\Downarrow$

$-D$ not pef $\Rightarrow$ $h^0(K - mD) = 0$. $\forall$ $m \gg 0$.

D not pef $\Rightarrow$ $h^0(K-mD)=0$. $\forall$ $m \gg 0$.

Then,

$$h^0(mD) \geq \frac{1}{2} m^2 (D^2) - \frac{1}{2} m \cdot _____ \quad \forall m \gg 0.$$

$\Rightarrow$ $\kappa(X, D) = 2$.

Conversely, $\kappa(X, D) = 2$.

D nef, $(D^2) \geq 0$

Assume $D^2 = 0$. $\quad h^0(mD) \geq cm^2 \quad m \gg 0$.

$|mD| = |M| + Z$

fixed part

Then.

$$0 = (mD)^2 = M^2 + (M \cdot Z) + m(D \cdot Z)$$

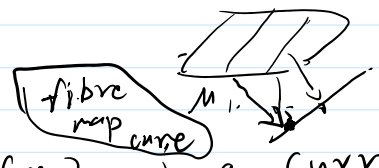
$$= (mD)(M+Z)$$

$$= mD \cdot M + m(D \cdot Z) = (M+Z) M + m(D \cdot Z).$$

$|M|$ has no fixed ..., M nef, $(M^2) \geq 0$.

$(M \cdot Z) \geq 0$

$D \cdot Z \geq 0$.



ruled?

$C_1 \times C_2$.

elliptic surface.

$M^2 = 0 \Rightarrow \varphi_{|mD|}(X)$ is a curve.

contradicting the $\kappa(X, D) = 2$. $\times$

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Lemma. H nef. $\mathbb{Q}$-div $\quad E$ eff $\mathbb{Q}$-div

$((H+E) \cdot C) \geq 0 \quad \forall C \Rightarrow H+E$ nef.

component of E.

$E = \sum m_i C_i \quad \forall i$. $\quad \forall C \in$ integral curve on X.

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$E = \sum n_i C_i$. $\forall i$. $\forall C \in$ integral curve on X.

$((H+E) \cdot C_i) \geq 0$. $\Big\} \rightsquigarrow (H+E) \cdot C \begin{cases} 0 \\ \geq 0 \end{cases}$

$H \cdot C_i \geq 0$. $\boxed{H \cdot C \geq 0}$ $(E \cdot C \overset{?}{\geq} 0)$

$n_i \geq 0$.

$(\sum n_i C_i \cdot C) \geq 0$.

irr curve

Fulton intersection $\boxed{C_i \cdot C_j \geq 0 ?}$ $C^2 \begin{cases} \not\geq 0 \\ \geq 0 \end{cases}$

$E^2 = -1$.

Lemma. $C_1, \dots, C_q$ integral curves X

$\|(C_i \cdot C_j)\|_{i,j} = \begin{bmatrix} C_1 \cdot C_1 & \cdots & C_1 \cdot C_q \\ \vdots & & \vdots \\ C_q \cdot C_1 & \cdots & C_q \cdot C_q \end{bmatrix}$.

neg-define.

$D = \sum_{i=1}^{q} a_i C_i$. s.t. $(D \cdot C_i) \leq 0$, $\forall i = 1, \dots, q$.

pos, neg $\Rightarrow D \geq 0$

pf: $D = A - B$, $A, B \geq 0$. no common components.

$0 \geq (D \cdot B) = (A \cdot B) - (B^2)$.

$(A \cdot B) \geq 0$. whence $B^2 \geq 0$.

$B^2 \leq 0$ (neg def) $\Big] \Rightarrow B^2 = 0$.

whence $B = 0$. $\square$

Lemma. $C_1, \dots, C_q$. int curves X, $\|C_i \cdot C_j\|$. neg-def.

$D = \sum a_i C_i$ $\mathbb{Q}$-div. D'. pef $\mathbb{Q}$-div.

$((D' - D) \cdot C_i) \leq 0$. $\forall i = 1, \dots, q$.

$\Rightarrow$. $D' - D$ pef. (pf) proof

pf.

H nef $\mathbb{Q}$-div. $\therefore$ $\exists Y = \sum \lambda_i C_i$

H nef $\mathbb{Q}$-div, $\exists Y = \sum \lambda_i C_i$

$(Y \cdot C_i) = -(H \cdot C_i)$, $\forall i = 1, \cdots, q$.

Since $(Y \cdot C_i) = -(H \cdot C_i) \leq 0$,

$Y \geq 0$.

$((D' - D) \cdot Y) \leq 0$.

$((H + Y) \cdot C_i) = 0 \Rightarrow H + Y$ nef.

D' psef $\Rightarrow ((H+Y) \cdot D') \geq 0$.

$$((D' - D) \cdot H) = (D' \cdot H) - (D \cdot H) = (D' \cdot H) + (D \cdot Y)$$
$$\geq -(D' \cdot Y) + (D \cdot Y) = -((D' - D) \cdot Y) \geq 0.$$

$\square$

Lemma. $C_1, \cdots, C_q$, irr curves X,

$I(C_1, \cdots, C_q)$ neg-semidef.

Assume $I(C_1, \cdots, C_r)$ neg-def, $(r < q)$.

$\Rightarrow \forall j > r$, $\exists$ eff $\mathbb{Q}$-div $D_j = \sum_{i=1}^{r} a_{ji} C_i$ s.t.

$$((D_j + C_j)) \cdot C_i = 0 \quad \forall i = 1, \cdots, q$$

Pf: $1 \leq p \leq q$. denote $V(C_1, \cdots, C_p)$, $\mathbb{Q}$-vec space by $C_1, \cdots, C_p$

$V_1 = V(C_1, \cdots, C_r)$ $V_2 = \{D \in V \mid (D \cdot D') = 0, \forall D' \in V\}$.

$$V = V_1 \oplus V_2.$$

$\forall j > r$, $\exists D_j \in V_1$, $E_j \in V_2$. s.t.

$$C_j = -D_j + E_j.$$

Since $E_j = C_j + D_j \in V_2$.

$((D_j + C_j) \cdot C_i) = 0$, $\forall i = 1, \cdots, q$.

$(D_j \cdot C_i) = -(C_j \cdot C_i) \leq 0$. $\forall i = 1, \cdots, r$.

$D_j \geq 0$. $\forall j > r$. $\square$

Lemma, $C_1, \dots, C_q$ be irr curve. X.

D pseff $\mathbb{Q}$-div. s.t. $(D \cdot C_r) \leq 0$, $\forall r = 1, \dots, q$.

Assume, $\exists r$. $1 \leq r < q$. $(D \cdot C_j) < 0$. $\forall j = r+1, \dots, q$.

If $I(C_1, \dots, C_r)$ neg det. $\Rightarrow$ $I(C_1, \dots, C_q)$ also neg-det

pf:

Assume $I(C_1, \dots, C_q)$ not neg-det.

$\exists$ $\mathbb{Q}$-div $Z \in V(C_1, \dots, C_q)$, $(Z^2) \geq 0$.

$\in \mathrm{Div}(X)$.

<u>Z. eff</u>. $Z = A - B$. $A \geq 0$, $B \geq 0$. no common.

$0 \leq (Z^2) = (A^2) + (B^2) - 2(A \cdot B)$.

$(A \cdot B) \geq 0$. $A^2 \geq 0$. or $B^2 \geq 0$.

We can use A or B instead of Z. ✓

Assume $Z > 0$. By R.R.

$$h^0(nZ) + \underline{h^0(K - mD)} \geq \frac{n^2}{2} Z^2 - \frac{n}{2}(Z \cdot K) + \chi(\mathcal{O}_X)$$

$H \cdot Z > 0$. $\forall$ amp H on X. whence. $(H \cdot (-Z)) < 0$.

<u>$-Z$ cannot be pseff</u>.

$h^0(K - nZ) = 0$, $n \gg 0$.

$\kappa(X, Z) = 2$.

$\varphi_{|nZ|}(X)$. is surface. if $n \gg 0$.

Replace $\underline{Z}$, $\varphi_{|Z|}(X)$. is a surface

F is fixed part of $|Z|$.

F is fixed part of $|Z|$.

$$|Z| = |Z'| + F.$$

$Z' \geq 0$, $Z' \in V(C_1, \cdots, C_q)$.

$\kappa(X, Z') = \kappa(X, Z) = 2$.

? Z' nef.

$(Z')^2 > 0$.

Since $Z' \in V(C_1, \cdots, C_q)$ $Z' \geq 0$.

$(D \cdot Z') \leq 0$.

pseff nef.

$\Rightarrow$, $(D \cdot Z') = 0$.

$\Rightarrow D \cdot Z' \geq 0$.

$Z' \in V(C_1, \cdots, C_r)$.

Contradiction $I(C_1, \cdots, C_r)$ is neg-def

$\Rightarrow$, $I(C_1, \cdots, C_q)$ neg semidef.

Assume. not neg-def. $\forall j = r+1, \cdots, q$.

$\exists$ eff $\mathbb{Q}$-div $D_j \in V(C_1, \cdots, C_r)$.

s.t. $((D_j + C_j) \cdot C_i) = 0$, $\forall i = 1, \cdots, q$.

$D_j + C_j$ nef. $(C_i) = 0$, $E = D_j + C_j$.

By hypothesis. $(D, (D_j + C_j)) \leq D \cdot C_j < 0$.

contradict D pseff

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Cor D pseff X, $\exists$ curve C s.t. s.t. $(D C) < 0$.

pf: Neron-Severi Thm. $NS(X)$ finite rank.

pf: Neron-Severi Thm. $NS(X)$ finite rank.

$C_1, \cdots, C_q$. irr cur $(D \cdot C_i) < 0$.

$I(C_1, \cdots, C_q)$. neg-def.

$\square$.

Thm. (Zariski-Fujita)

D pseff $\mathbb{Q}$-div. $D \overset{\text{unique}}{=\!=} \underset{\text{nef}}{P} + \underset{\text{eff} \geq 0}{N}$.

If $N > 0$ $(P \cdot C_i) = 0$, $\forall i = 1, \cdots, q$.

$I(C_1, \cdots, C_q)$ neg def.

(C_i). irreducible component of $\operatorname{supp}(N)$

pf: (Existence)

D nef. $P = D$, $N = 0$.

Otherwise, $C_1, \cdots, C_{q_1}$ int cur on X. s.t. $(D \cdot C_i) < 0$.

$I(C_1, \cdots, C_{q_1})$. neg-def.

$\exists!$ $\mathbb{Q}$-div $N_1 \in V(C_1, \cdots, C_{q_1})$ s.t. $(N_1 \cdot C_i) = (D \cdot C_i)$

Since $(D \cdot C_i) < 0$, $\Rightarrow$ $N_1 \geq 0$.

$D_1 := D - N_1$ is nef. $P = D_1$ $N = N_1$.

Otherwise $C_{q_1+1}, \cdots, C_{q_2}$. $(D_1 \cdot C_j) < 0$

$I(C_1, \cdots, C_{q_2})$. neg-def.

$\exists!$ $N_2 \in V(C_1, \cdots, C_{q_2})$ $(N_2 \cdot C_j) = (D_1 \cdot C_j)$

$N_2 \geq 0$. $D_2 = D_1 - N_2$.

$\operatorname{rk}(NS(X)) < +\infty$.

Lemma. $C_1, \dots, C_q$. int cur on X. $I(C_1, \dots, C_q)$ neg def

D' eff $\mathbb{Q}$-div on X. $\underset{\mathbb{Q}\text{-div}}{D} \in V(C_1, \dots, C_q)$.

Sol: $(D'-D) \cdot C_i \leq 0$. $\Rightarrow D'-D \geq 0$.

pf: $D' = Z + D_1'$. $Z \in V(C_1, \dots, C_q)$. $C_i \notin \mathrm{Supp}(D_1')$

$\underline{(D_1' \cdot C_i) \geq 0}$. $\Rightarrow (Z \cdot C_i) = (D' \cdot C_i) - (D_1' \cdot C_i) \leq$

$(D' \cdot C_i) \leq (D \cdot C_i)$

$\Rightarrow Z - D \geq 0$. $\Rightarrow D' - D \geq 0$. $\square$

$D = P + N = P' + N'$. $\mathrm{Supp}(N) = C_1 \cup \dots \cup C_q$.

$(P \cdot C_i) \leq 0$. $(\underset{\text{nef}}{P'} \cdot C_i) \geq 0$. $(N \cdot C_i) = (D \cdot C_i) = (P' \cdot C_i) + (N' \cdot C_i)$

$\geq (N' \cdot C_i)$.

$\Rightarrow \left.\begin{array}{l} N' \geq N \\ N \geq N' \end{array}\right\} \Rightarrow N' = N$. $\square$

Finish proof. example, Kh $\to$ OK... hdg det pers.

3.14.2024