K-jet Ampleness, Newton-Okounkov Bodies, and Related Criteria

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Abstract

This report investigates the relationship between k-jet ampleness of line bundles and Newton-Okounkov bodies in algebraic geometry. While k-jet ample line bundles satisfy a volume lower bound $\mathrm{vol}(\Delta(L)) \geq k^n$ and classical positivity can be characterized through geometric inclusions in Newton-Okounkov bodies, we present a critical counterexample based on Kollár's construction that disproves the conjecture that volume conditions alone determine jet ampleness. The example features "needle-like" Newton-Okounkov bodies with large volume but poor geometric shape, revealing that the relationship between these concepts is more subtle than expected—geometric shape matters more than volume, and the invariance properties under numerical equivalence differ fundamentally between the two notions, requiring sophisticated analysis beyond simple volumetric criteria. As future work, it is worth exploring the infinitesimal version of Newton–Okounkov bodies to better capture k-jet conditions and extending their construction to higher dimensions, incorporating information from multiple points and valuations.

Contents

1 Definitions and Basic Properties

Let X denote a smooth projective variety over $\mathbb{C}$, and let L be a line bundle (or Cartier divisor) on X. We will use the following standard notations throughout:

- $\mathbb{C}$: the field of complex numbers.
- X: a smooth projective variety over $\mathbb{C}$.
- L: a line bundle (or Cartier divisor) on X.
- $x \in X$: a (closed) point on X.

1.1 K-Jet Ampleness

Let $x_1, \ldots, x_r$ be r distinct reduced points on X. For each $i = 1, \ldots, r$, let $\mathfrak{m}_i$ denote the maximal ideal sheaf of the point $x_i \in X$. Note that $\mathfrak{m}_i$ is supported at x_i, and its stalk at x_i is precisely the maximal ideal (denoted $\mathfrak{m}_i \mathcal{O}_{X,x_i}$) of the local ring $\mathcal{O}_{X,x_i}$. Consider the 0-cycle $\mathcal{Z} = x_1 + \cdots + x_r$.

Definition 1.1 (k-jet ampleness)**.** *We say that a line bundle L is k-jet ample at $\mathcal{Z}$ if, for every r-tuple $(k_1, \ldots, k_r)$ of positive integers with $\sum_{i=1}^r k_i = k+1$, the natural restriction map*

$$\Gamma(L) \longrightarrow \Gamma\left(L \Big/ \bigotimes_{i=1}^{r} \mathfrak{m}_i^{k_i}\right) \cong \bigoplus_{i=1}^{r} \Gamma\left(L/\mathfrak{m}_i^{k_i}\right)$$

is surjective. Here, $L/\bigotimes_{i=1}^r \mathfrak{m}_i^{k_i}$ denotes $L \otimes \left(\mathcal{O}_X/\bigotimes_{i=1}^r \mathfrak{m}_i^{k_i}\right)$, and $\mathfrak{m}_i^{k_i}$ denotes the k_i-th power of $\mathfrak{m}_i$ in the usual sense.

We say that a line bundle L is k-jet ample on X if, for any $r \geq 1$ and any 0-cycle $\mathcal{Z} = x_1 + \cdots + x_r$, where $x_1, \ldots, x_r$ are r distinct points on X, the line bundle L is k-jet ample at $\mathcal{Z}$.

Definition 1.2 (Globally generated)**.** *A line bundle L on X is said to be globally generated (or generated by global sections) if, for every point $x \in X$, there exists a global section $s \in \Gamma(L)$ such that $s(x) \neq 0$.*

Definition 1.3 (Very ample)**.** *A line bundle L on X is said to be very ample if there exists an embedding $\varphi : X \hookrightarrow \mathbb{P}^N$ for some N, such that $L \cong \varphi^* \mathcal{O}_{\mathbb{P}^N}(1)$.*

Definition 1.4 (Ample)**.** *A line bundle L on X is said to be ample if for some integer $m > 0$, the line bundle $L^{\otimes m}$ is very ample.*

Proposition 1.5 ([Har77, Ch. II, Prop. 7.3])**.** *A line bundle L on a scheme X is* very ample *if and only if the global sections of L separate points and tangent vectors, that is:*

- ***(Separation of points):*** *For any two distinct closed points $P, Q \in X$, there exists $s \in \Gamma(X, L)$ such that $s \in \mathfrak{m}_P \mathcal{L}_P$ but $s \notin \mathfrak{m}_Q \mathcal{L}_Q$, or vice versa.*
- ***(Separation of tangent vectors):*** *For each closed point $P \in X$, the set $\{s \in \Gamma(X, L) \mid s_P \in \mathfrak{m}_P \mathcal{L}_P\}$ spans the k-vector space $\mathfrak{m}_P \mathcal{L}_P / \mathfrak{m}_P^2 \mathcal{L}_P$.*

This proposition characterizes very ample line bundles in terms of their ability to separate points and tangent vectors. The following proposition further relates jet ampleness to global generation and very ampleness.

Proposition 1.6. *Let X be a projective algebraic and L a line bundle on X.*

- *L is globally generated if and only if L is 0-jet ample.*
- *L is very ample if and only if L is 1-jet ample.*

To study the behavior of jet ampleness under tensor products, we recall the following useful lemma.

Lemma 1.7 ([BS93, Proposition 2.2])**.** *Let L_1, L_2 be line bundles on X. Assume that L_1 is a-jet ample and L_2 is b-jet ample. Then $L_1 \otimes L_2$ is $(a+b)$-jet ample.*

Corollary 1.8 ([BS93, Corollary 2.3])**.** *Let $L_1, \dots, L_k$ be very ample line bundles on X. Then $L_1 \otimes \cdots \otimes L_k$ is k-jet ample.*

Using the above results, we can now establish a general statement about the jet ampleness of sufficiently high tensor powers of an ample line bundle.

Proposition 1.9. *Let X be a smooth projective variety and L an ample line bundle on X. Then for any integer $k \geq 1$, there exists an integer N_0 such that for all $N \geq N_0$, the line bundle $L^{\otimes N}$ is k-jet ample.*

We now recall the classical results of Hartshorne concerning base point freeness and very ampleness for divisors on curves.

Proposition 1.10 ([Har77, Chapter IV, Proposition 3.1])**.** *Let D be a divisor on a curve X. Then:*

(a) The complete linear system $|D|$ has no base points if and only if for every point $P \in X$,

$$\dim |D - P| = \dim |D| - 1.$$

(b) D is very ample if and only if for every two points $P, Q \in X$ (possibly $P = Q$),

$$\dim |D - P - Q| = \dim |D| - 2.$$

Corollary 1.11 ([Har77, Chapter IV, Corollary 3.2])**.** *Let D be a divisor on a curve X of genus g.*

(a) If $\deg D \geq 2g$, then $|D|$ has no base points.

(b) If $\deg D \geq 2g+1$, then D is very ample.

Inspired by these classical results, we now present their generalization to higher order jets:

Proposition 1.12. *Let D be a divisor on a smooth projective curve X. Then D is k-jet ample if and only if for any (not necessarily distinct) points $P_1, \dots, P_r \in X$ and non-negative integers $k_1, \dots, k_r$ with $k_1 + \cdots + k_r = k+1$, we have*

$$\dim |D - k_1 P_1 - \cdots - k_r P_r| = \dim |D| - k - 1.$$

Proof. Consider the following short exact sequence of sheaves:

$$0 \longrightarrow \mathcal{O}_X(D - \sum_{i=1}^r k_i P_i) \longrightarrow \mathcal{O}_X(D) \longrightarrow \bigoplus_{i=1}^r \mathcal{O}_X(D)/\mathfrak{m}_i^{k_i} \longrightarrow 0,$$

where $\mathfrak{m}_i$ is the maximal ideal sheaf at the point P_i.

Taking global sections, we obtain the exact sequence:

$$0 \longrightarrow H^0(X, \mathcal{O}_X(D - \sum_{i=1}^r k_i P_i)) \longrightarrow H^0(X, \mathcal{O}_X(D)) \longrightarrow \bigoplus_{i=1}^r H^0(X, \mathcal{O}_X(D)/\mathfrak{m}_i^{k_i}).$$

By the definition, D is k-jet ample if and only if for any choice of points $P_1, \dots, P_r$ and non-negative integers $k_1, \dots, k_r$ with $\sum_{i=1}^r k_i = k+1$, the map

$$H^0(X, \mathcal{O}_X(D)) \to \bigoplus_{i=1}^r H^0(X, \mathcal{O}_X(D)/\mathfrak{m}_i^{k_i})$$

is surjective.

On a curve, we have $\dim H^0(X, \mathcal{O}_X(D)/\mathfrak{m}_i^{k_i}) = k_i$. Therefore,

$$\dim \left(\bigoplus_{i=1}^{r} H^0(X, \mathcal{O}_X(D)/\mathfrak{m}_i^{k_i}) \right) = \sum_{i=1}^{r} k_i = k+1.$$

The surjectivity of the map implies

$$\dim H^0(X, \mathcal{O}_X(D)) - \dim H^0(X, \mathcal{O}_X(D - \sum_{i=1}^{r} k_i P_i)) = k+1.$$

Since $\dim |D| = \dim H^0(X, \mathcal{O}_X(D)) - 1$ and similarly for the other linear system, we obtain

$$\dim |D - \sum_{i=1}^{r} k_i P_i| = \dim |D| - k - 1.$$

□

Next example will show that the very ampleness threshold for ample divisors on a surface cannot be bounded independently of the divisor.

Example 1.13 (Kollár's Example [Laz04, Example 1.5.7])**.** Let $X = E \times E$ be the product of an elliptic curve with itself, and for each integer $n \geq 2$, define the divisor

$$A_n = nF_1 + (n^2 - n + 1)F_2 - (n-1)\Delta,$$

where F_1, F_2 are the fibers of the two projections and Δ is the diagonal. Then A_n is ample.

Let $R = F_1 + F_2$, let $B \in |2R|$ be a smooth divisor, and let $f : Y \to X$ be the double cover branched along B. Set $D_n = f^* A_n$. Then D_n is ample, but the minimal integer $m(D_n)$ such that $\mathcal{O}_Y(mD_n)$ is very ample satisfies $m(D_n) > n$. Thus, $m(D_n)$ cannot be bounded independently of D_n. □

Numerical equivalence preserves ampleness but does not necessarily preserve k-jet ampleness. The first statement can be directly verified using [Laz04, Theorem 1.2.23]. For the second statement, consider the following example:

Example 1.14. Let C be a smooth curve of genus $g > 2$. Consider a divisor D of degree $\deg(D) = 2g$ on C. D is very ample if and only if, for any points $P, Q \in C$, it holds that $l(D - P - Q) = l(D) - 2$.

Now, let us examine the divisor $D' = K + P + Q$, where K is the canonical divisor and P, Q are distinct points on C. By the Riemann-Roch theorem, for a divisor D' on a smooth curve C of genus g, we have:

$$l(D') - l(K - D') = l(D') = \deg(D') - g + 1 = g + 1.$$

Next, consider $D' - P - Q = K$. By the Riemann-Roch theorem:

$$l(D' - P - Q) - l(K - K) = \deg(K) - g + 1 = g - 1.$$

Since $l(K - K) = l(0) = 1$, we find:

$$l(D' - P - Q) = g = l(D') - 1.$$

Thus, D' fails to satisfy the condition for very ampleness.

Next, consider the surface $X = C \times \mathbb{P}^1$. We analyze the divisors $D \times \mathbb{P}^1$ and $D' \times \mathbb{P}^1$. These divisors are numerically equivalent, but while $D \times \mathbb{P}^1$ is very ample, $D' \times \mathbb{P}^1$ is only ample. □

To further understand the behavior of k-jet ample line bundles, we now present a lower bound for their volume.

Definition 1.15 (Volume of a line bundle). *Let X be an irreducible projective variety of dimension n, and let L be a line bundle on X. The volume of L is defined to be the non-negative real number*

$$vol(L) = vol_X(L) = \limsup_{m\to\infty} \frac{h^0(X, L^{\otimes m})}{m^n/n!}.$$

The volume $vol(D) = vol_X(D)$ of a Cartier divisor D is defined similarly, or by passing to $\mathcal{O}_X(D)$.

Note that $\mathrm{vol}(L) > 0$ if and only if L is big. If L is nef, then it follows from asymptotic Riemann–Roch that

$$\mathrm{vol}(L) = \int_X c_1(L)^n \tag{2.9}$$

Theorem 1.16 ([BS93, Theorem 3.1]). *Let L be a k-jet ample line bundle on a smooth connected n-fold X. Then*

$$L^n \geq k^n + k^{n-1}$$

unless

$$(X, L) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)).$$

Theorem 1.17 (Lu). *Let $\pi : Y \to X$ be a finite morphism of smooth projective varieties, where:*

- *π is a degree d branched covering*
- *The branch locus is a divisor $B \subset X$*
- *$B \sim dM$ for some effective divisor M on X*
- *L is an ample line bundle on X*

If the pullback line bundle $\tilde{\pi}^*L$ is dk-jet ample on Y, then L is k-jet ample on X.

1.2 Newton-Okounkov Body

The Newton–Okounkov body is a convex geometric object associated to a big divisor or line bundle on an algebraic variety, generalizing the classical Newton polytope. It encodes asymptotic information about linear series and has deep connections to algebraic geometry and convex geometry [LM09, KK12].

Let X be an irreducible projective algebraic variety of dimension d, and let L be a big line bundle (or a big divisor) on X. Fix a complete flag of irreducible subvarieties

$$Y_\bullet : X = Y_0 \supset Y_1 \supset \cdots \supset Y_{d-1} \supset Y_d = \{pt\}$$

where $\dim Y_i = d - i$ and each Y_i is smooth at the point Y_d.

For each nonzero section $s \in H^0(X, L^{\otimes m})$ (for some $m \geq 1$), we can associate a valuation-like sequence

$$\nu(s) = (\nu_1(s), \ldots, \nu_d(s)) \in \mathbb{Z}^d$$

defined recursively as follows:

- $\nu_1(s)$ is the vanishing order of s along Y_1.
- After restricting s (modulo its vanishing along Y_1) to Y_1, $\nu_2(s)$ is the vanishing order along Y_2, and so on.

Definition 1.18 (Newton–Okounkov Body). *The Newton–Okounkov body $\Delta_{Y_\bullet}(L)$ is defined as the closed convex hull in $\mathbb{R}^d$ of all normalized valuation vectors:*

$$\Delta_{Y_\bullet}(L) = \overline{\bigcup_{m\geq 1} \left\{ \frac{1}{m}\nu(s) \mid s \in H^0(X, L^{\otimes m}) \setminus \{0\} \right\}} \subset \mathbb{R}^d$$

This convex body encodes asymptotic information about the linear series associated to L and the chosen flag $Y_\bullet$.

One of the key results, due to Lazarsfeld and Mustaţă [LM09], reveals a fundamental connection between the volume of the Newton–Okounkov body and the volume of the divisor itself:

Theorem 1.19 ([LM09, Theorem 2.3]). *Let D be a big divisor on a projective variety X of dimension d. Then*

$$\mathrm{vol}_{\mathbb{R}^d}(\Delta(D)) = \frac{1}{d!}\,\mathrm{vol}_X(D),$$

where the Okounkov body $\Delta(D)$ is constructed with respect to any choice of an admissible flag $Y_\bullet$.

These properties show that the construction of the Okounkov body is compatible with the basic operations on divisors and is intrinsically linked to their numerical classes.

Proposition 1.20 ([LM09, Proposition 4.1]). *Let D be a big divisor on X.*

1. *The Okounkov body $\Delta(D)$ depends only on the numerical equivalence class of D.*
2. *For any integer $p > 0$, one has*
$$\Delta(pD) = p \cdot \Delta(D),$$
where the expression on the right denotes the homothetic image of $\Delta(D)$ under scaling by the factor p.

We now introduce the infinitesimal case of Okounkov bodies. In the preceding sections, we mainly considered Okounkov bodies constructed with respect to a fixed global flag. To eliminate the dependence on a fixed flag, we turn to infinitesimal data, which naturally vary in families, by considering flags arising from the blow-up at a general point.

Specifically, let X be an irreducible projective variety of dimension d. Fix a smooth point $x \in X$, and choose a complete flag $V_\bullet$ of subspaces in the tangent space T_xX:

$$T_xX = V_0 \supseteq V_1 \supseteq V_2 \supseteq \cdots \supseteq V_{d-1} \supseteq \{0\}$$

Consider the blow-up

$$\mu : X' = \mathrm{Bl}_x(X) \longrightarrow X$$

of X at x, with exceptional divisor E. The projectivizations of the V_i induce a flag $F_\bullet = F(x; V_\bullet)$ on X':

$$X' \supseteq E = \mathbb{P}(T_xX) \supseteq \mathbb{P}_{\mathrm{sub}}(V_1) \supseteq \mathbb{P}_{\mathrm{sub}}(V_2) \supseteq \cdots \supseteq \mathbb{P}_{\mathrm{sub}}(V_{d-1}) = \{\mathrm{pt}\}.$$

On the other hand, let D be any divisor on X, and write $D' = \mu^*D$. Then for all m, we have

$$H^0(X, \mathcal{O}_X(mD)) = H^0(X', \mathcal{O}_{X'}(mD'))$$

Therefore, a choice of flag $Y'_\bullet$ on X' determines a valuation on sections of D, and we denote the corresponding Okounkov body by $\Delta_{Y'_\bullet}(D)$, that is,

$$\Delta_{Y'_\bullet}(D) \coloneqq \Delta_{Y'_\bullet}(D')$$

where the object on the right is constructed on X'. In particular, the flag $F_\bullet$ induced by $(x, V_\bullet)$ gives rise to a convex body $\Delta_{F_\bullet}(D) \subseteq \mathbb{R}^d$. As in Proposition 4.2, these Okounkov bodies appear as the fibers of closed convex cones

$$\Delta_{F_\bullet}(X) \subseteq \mathbb{R}^d \times N^1(X)_{\mathbb{R}}.$$

Proposition 1.21 ([LM09, Proposition 5.3]). *Let D be any big divisor on X. Then the corresponding Okounkov bodies*

$$\Delta_{F(x,V_\bullet)}(D) \subseteq \mathbb{R}^d$$

all coincide for a very general choice of $x \in X$ and the flag $V_\bullet$. The analogous statement holds for the global bodies $\Delta_{F(x,V_\bullet)}(X)$.

2 Relationship between Newton-Okounkov Bodies and Jet Ampleness

This section focuses on the connection between k-jet ampleness of line bundles and the geometry of Newton-Okounkov bodies. Our primary interest lies in understanding how these two concepts interact and inform each other in the context of algebraic geometry.

2.1 Known Results on the Relationship

In this subsection, we discuss some existing results related to the relationship between k-jet ampleness and Newton-Okounkov bodies. These results provide a foundation for understanding the interplay between these two important concepts in algebraic geometry.

Combining 1.16 and 1.19, we obtain the following result directly:

Proposition 2.1. *Let L be a k-jet ample line bundle on a smooth projective variety X of dimension n. Then the volume of the Okounkov body $\Delta(L)$ satisfies*

$$\mathrm{vol}_{\mathbb{R}^n}(\Delta(L)) \geq k^n,$$

where the Okounkov body $\Delta(L)$ is constructed with respect to any admissible flag $Y_\bullet$. Equality holds if and only if $(X, L) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k))$.

We will write

$$\Delta_\epsilon \coloneqq \{(x_1, \ldots, x_n) \in \mathbb{R}^n_+ \mid x_1 + \cdots + x_n \leq \epsilon\}$$

for the standard ϵ-simplex.

Theorem 2.2 (Theorem 2.1, [KL15b])**.** *Let D be a big $\mathbb{R}$-divisor on a smooth projective variety X of dimension n, and let $x \in X$. The following are equivalent:*

1. *$x \notin \mathbf{B}_-(D)$.*
2. *There exists an admissible flag $\mathbf{Y}_\bullet$ on X centered at x such that $0 \in \Delta_{\mathbf{Y}_\bullet}(D) \subseteq \mathbb{R}^n$.*
3. *For every admissible flag $\mathbf{Y}_\bullet$ on X centered at x, one has $0 \in \Delta_{\mathbf{Y}_\bullet}(D)$.*

Corollary 2.3 (Corollary 2.2, [KL15b])**.** *The following are equivalent for a big $\mathbb{R}$-divisor D:*

1. *D is nef.*
2. *For every point $x \in X$, there exists an admissible flag $\mathbf{Y}_\bullet$ on X centered at x such that $0 \in \Delta_{\mathbf{Y}_\bullet}(D) \subseteq \mathbb{R}^n$.*
3. *For every admissible flag $\mathbf{Y}_\bullet$, one has $0 \in \Delta_{\mathbf{Y}_\bullet}(D)$.*

Theorem 2.4 (Theorem 3.1, [KL15b])**.** *Let D be a big $\mathbb{R}$-divisor on X, and let $x \in X$ be an arbitrary (closed) point. The following are equivalent:*

1. *$x \notin \mathbf{B}_+(D)$.*
2. *There exists an admissible flag $\mathbf{Y}_\bullet$ centered at x with Y_1 ample such that $\Delta_{\epsilon_0} \subseteq \Delta_{\mathbf{Y}_\bullet}(D)$ for some $\epsilon_0 > 0$.*
3. *For every admissible flag $\mathbf{Y}_\bullet$ centered at x, there exists $\epsilon > 0$ (possibly depending on $\mathbf{Y}_\bullet$) such that $\Delta_\epsilon \subseteq \Delta_{\mathbf{Y}_\bullet}(D)$.*

Corollary 2.5 (Corollary 3.2, [KL15b])**.** *Let X be a smooth projective variety, D a big $\mathbb{R}$-divisor on X. Then the following are equivalent:*

1. *D is ample.*

2. *For every point $x \in X$, there exists an admissible flag $\mathbf{Y}_\bullet$ centered at x with Y_1 ample such that $\Delta_{\epsilon_0} \subseteq \Delta_{\mathbf{Y}_\bullet}(D)$ for some $\epsilon_0 > 0$.*

3. *For every admissible flag $\mathbf{Y}_\bullet$, there exists $\epsilon > 0$ (possibly depending on $\mathbf{Y}_\bullet$) such that $\Delta_\epsilon \subseteq \Delta_{\mathbf{Y}_\bullet}(D)$.*

The above theorems and corollaries illustrate that ampleness and nefness of a divisor D on a smooth projective variety X can be extracted from the shape of the normal Newton–Okounkov body associated with D. This highlights the role of Newton–Okounkov bodies as a bridge between geometric properties of divisors and convex geometry.

However, it is worth noting that k-jet ampleness is not consistent within numerical equivalence classes. In contrast, the normal Newton–Okounkov body remains invariant under numerical equivalence. This distinction underscores the subtle interplay between the numerical properties of divisors and their geometric interpretations.

Furthermore, the work of Küronya and Lozovanu [KL15a] provides similar insights in the context of infinitesimal Newton–Okounkov bodies. Their results show that the structure of these bodies can capture ampleness and nefness through the inclusion of specific convex polytopes, such as the inverted standard simplex.

Theorem 2.6 (Theorem 3.1, [KL15a])**.** *Let X be a smooth projective variety, D a big $\mathbb{R}$-divisor, and $x \in X$ an arbitrary point on X. Then the following are equivalent:*

1. *$x \notin B_-(D)$.*

2. *There exists an infinitesimal flag $\mathbf{Y}_\bullet$ over x such that $0 \in \widetilde{\Delta}_{\mathbf{Y}_\bullet}(D)$.*

3. *For every infinitesimal flag $\mathbf{Y}_\bullet$ over x, one has $0 \in \widetilde{\Delta}_{\mathbf{Y}_\bullet}(D)$.*

Corollary 2.7 (Corollary 3.3, [KL15a])**.** *Let X be a smooth projective variety, D a big $\mathbb{R}$-divisor on X. Then the following are equivalent:*

1. *D is nef.*

2. *For every point $x \in X$, there exists an infinitesimal flag $\mathbf{Y}_\bullet$ over x such that $0 \in \widetilde{\Delta}_{\mathbf{Y}_\bullet}(D)$.*

3. *The origin $0 \in \widetilde{\Delta}_{\mathbf{Y}_\bullet}(D)$ for every infinitesimal flag $\mathbf{Y}_\bullet$ over X.*

Definition 2.8. *(Largest inverted simplex constant) Let X be a projective variety, $x \in X$ a smooth point on X, and D a big $\mathbb{R}$-divisor with $x \notin B_-(D)$. For an infinitesimal flag $\mathbf{Y}_\bullet$ over x, write*

$$\xi_{\mathbf{Y}_\bullet}(D, x) \coloneqq \sup\{\xi > 0 \mid \Delta_\xi^{-1} \subseteq \widetilde{\Delta}_{\mathbf{Y}_\bullet}(D)\}.$$

The largest inverted simplex constant $\xi(D, x)$ of D at x is then defined as

$$\xi(D, x) \coloneqq \sup_{\mathbf{Y}_\bullet} \xi_{\mathbf{Y}_\bullet}(D, x),$$

where the supremum is taken over all infinitesimal flags $\mathbf{Y}_\bullet$ over x.

Theorem 2.9 (Theorem 4.1, [KL15a])**.** *Let X be a smooth projective variety, $x \in X$ an arbitrary (closed) point, D a big $\mathbb{R}$-divisor on X. Then the following are equivalent:*

1. *$x \notin B_+(D)$.*

2. *For every infinitesimal flag $\mathbf{Y}_\bullet$ over x, there is $\xi > 0$ such that $\Delta_\xi^{-1} \subseteq \widetilde{\Delta}_{\mathbf{Y}_\bullet}(D)$.*

3. *There exists an infinitesimal flag $\mathbf{Y}_\bullet$ over x and $\xi > 0$ such that $\Delta_\xi^{-1} \subseteq \widetilde{\Delta}_{\mathbf{Y}_\bullet}(D)$.*

Corollary 2.10 (Corollary 4.2, [KL15a]). *Let X be a smooth projective variety and D a big $\mathbb{R}$-divisor on X. Then the following are equivalent:*

1. *D is ample.*

2. *For every point $x \in X$ and every infinitesimal flag $\mathbf{Y}_\bullet$ over x, there exists a real number $\xi > 0$ for which $\Delta_\xi^{-1} \subseteq \tilde{\Delta}_{\mathbf{Y}_\bullet}(D)$.*

3. *For every point $x \in X$, there exists an infinitesimal flag $\mathbf{Y}_\bullet$ over x and a real number $\xi > 0$ such that $\Delta_\xi^{-1} \subseteq \tilde{\Delta}_{\mathbf{Y}_\bullet}(D)$.*

The connection between infinitesimal Newton–Okounkov bodies and jet separation can be established as follows:

Proposition 2.11 (Proposition 4.9, [KL15b]). *Let X be an n-dimensional smooth projective variety, D a big Cartier divisor, and x a (closed) point on X. Assume that there exists a positive real number ε and a natural number k with the property that*

$$\Delta_{n+k+\varepsilon}^{-1} \subseteq \Delta_{Y_\bullet}(\pi^*(D))$$

for every infinitesimal flag $Y_\bullet$ over x. Then $K_X + D$ separates k-jets.

2.2 An Important Counterexample (Kollár - further analysis by Lu)

In this subsection, we present an important example that demonstrates the limitations of certain conjectures regarding k-jet ample line bundles and their associated Newton–Okounkov bodies.

Based on the known results listed in the previous sections, one might conjecture the following:

1. For a line bundle L, if the volume of its Newton–Okounkov body satisfies $\mathrm{vol}(\Delta(L)) = \Omega(k^n)$, then L is k-jet ample.

2. For a k-jet ample line bundle, there exists a simplex $\Delta' \subset \Delta(L)$ whose shape can be parameterized by k.

However, we provide a counterexample to show that these conjectures are incorrect. Consider Example 1.13 and recall the notataions: let $X = E \times E$, where E is an elliptic curve, and let F_1, F_2 denote the fibers of the two projections $X \to E$. The diagonal in X is denoted by Δ. For each integer $n \geq 2$, define the divisor

$$A_n = nF_1 + (n^2 - n + 1)F_2 - (n-1)\Delta,$$

which is ample. Let $R = F_1 + F_2$, and let $B \in |2R|$ be a smooth divisor. Consider the double cover $f : Y \to X$ branched along B, and set $D_n = f^*A_n$, which is also ample.

Now, we take an admissible flag on Y: let C be the pullback of a smooth curve in the linear system $|2F_1 + 2F_2 + \Delta|$ under the map $f : Y \to X$, and let the point be a general point on the pullback curve C. Thus, we have the flag:

$$Y \supset f^*C \supset \mathrm{pt}.$$

Next, we show that f^*C is irreducible on Y. On X, consider the line bundle $M = 2F_1 + 2F_2 + \Delta$ restricted to B. Using the adjunction formula, we compute the genus of B as $g_B = 5$. Furthermore, the degree of $M|_B$ is $\deg(M|_B) = 12$, which is greater than $2g_B + 1 = 11$. Therefore, $M|_B$ is very ample. By Bertini's theorem, we can choose a section C of M that intersects B transversally. In local coordinates, it follows that the pullback f^*C is irreducible on Y.

Next, the Newton–Okounkov body of D_n can be computed using the method described in [LM09, Theorem 6.4]. For an Abelian surface, the Newton–Okounkov body is always a trapezoid. To determine the horizontal coordinate μ of the Newton–Okounkov body, we compute:

$$\mu = \max\{t : D_n - tf^*C \text{ is big}\} = T(n) = \frac{N}{16}\left(1 - \sqrt{1 - 2 \cdot \frac{16}{N^2}}\right),$$

where $N(n) = 3n^2 - 4n + 7$.

Actually, the maximum value of t can be solved by the following inequalities:

$$\begin{cases} f_*((D_n - tf^*C)^2) \geq 0, \\ f_*((D_n - tf^*C) \cdot D_n) \geq 0. \end{cases}$$

For $t \in [0, \mu]$, we need to consider the intersection number

$$(D_n - tf^*C) \cdot f^*C = 2(N - 16t),$$

as shown in the Figure 1.

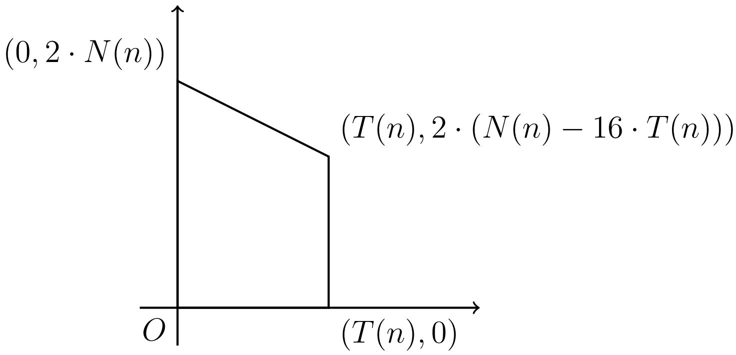

Figure 1: $\Delta(D_n)$

The trapezoid in Figure 1 can be interpreted as a "needle." As n increases, $T(n)$ approaches 0 at a rate of $\frac{1}{n^2}$, while $N(n)$ diverges at a rate of n^2.

As stated in [BS97, Corollary 2.2], nA_n is k-jet ample when $n \geq k+2$. However, nD_n is not k-jet ample. According to [BRS01, Theorem 2.1], let

$$N' = n^2 - 2n + 3, \quad N'' = \left\lceil \frac{N' + \sqrt{N'^2 - 4}}{2} \right\rceil,$$

then $(k + N'')D_n$ is k-jet ample. This implies that if we let n be a function of k, the volume lower bound of $(k + N'')D_n$ is much greater than k^2. Therefore, the first conjecture does not hold. Considering $T(n) \cdot N''(n) \to \frac{1}{3}$, as n increases. Therefore, the second conjecture is not valid.

2.3 Next Step

Although we have seen earlier that the normal Newton–Okounkov body encounters difficulties in capturing information about k-jet ampleness, it can be observed in 2.11 that the infinitesimal version can characterize the k-jet condition of a single point through its shape.

For general k-jets, it is necessary to consider the information of jets at multiple points simultaneously. On the one hand, we can follow the approach in [Tru22] to discuss the use of information from multiple points.

Furthermore, the valuation of the Newton–Okounkov body is a worthwhile aspect to investigate further, as suggested by works such as [Mou15] .

Additionally, it is also possible to explore the construction process of the Newton–Okounkov body. We may attempt to extend [LM09, Theorem 6.4] to higher dimensions based on the method in [Nak04].

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