K-jet Ampleness and Newton-Okounkov Bodies

A Counterexample Analysis

Yi Lu

July 5, 2025

Outline

Basic Setup

- Let X be an irreducible projective variety of dimension d
- Let L be a big line bundle on X
- Fix a complete flag of irreducible subvarieties:

$$X = Y_0 \supset Y_1 \supset \cdots \supset Y_{d-1} \supset Y_d = \{pt\}$$

where $\dim Y_i = d - i$ and each Y_i is smooth at point Y_d

Valuation Map - Definition

For any divisor D on X, define the valuation function:

$$\nu = \nu_{Y_\bullet} = \nu_{Y_\bullet, D} : H^0(X, \mathcal{O}_X(D)) \to \mathbb{Z}^d \cup \{\infty\}$$

For each nonzero section $s \in H^0(X, \mathcal{O}_X(D))$:

$$\nu(s) = (\nu_1(s), \ldots, \nu_d(s))$$

Recursive construction:

- $\nu_1(s) =$ vanishing order of s along Y_1
- After restricting s to Y_1, $\nu_2(s) =$ vanishing order along Y_2
- Continue recursively until reaching the point Y_d

Valuation Properties

The valuation $\nu_{Y_\bullet}$ satisfies three key properties:

Valuation-like Properties

(i) $\nu_{Y_\bullet}(s) = \infty$ if and only if $s = 0$
(ii) Ordering $\mathbb{Z}^d$ lexicographically:

$$\nu_{Y_\bullet}(s_1 + s_2) \geq \min\{\nu_{Y_\bullet}(s_1), \nu_{Y_\bullet}(s_2)\}$$

for any non-zero sections $s_1, s_2 \in H^0(X, \mathcal{O}_X(D))$
(iii) Given sections $s \in H^0(X, \mathcal{O}_X(D))$ and $t \in H^0(X, \mathcal{O}_X(E))$:

$$\nu_{Y_\bullet, D+E}(s \otimes t) = \nu_{Y_\bullet, D}(s) + \nu_{Y_\bullet, E}(t)$$

Newton-Okounkov Body Definition

Definition

The Newton-Okounkov body $\Delta_{Y_\bullet}(L)$ is the closed convex hull:

$$\Delta_{Y_\bullet}(L) = \overline{\bigcup_{m\geq 1} \left\{ \frac{1}{m}\nu(s) \mid s \in H^0(X, L^{\otimes m}) \setminus \{0\} \right\}} \subset \mathbb{R}^d$$

Key Properties:

- Encodes asymptotic information about linear series
- Volume formula: $\mathrm{vol}_{\mathbb{R}^d}(\Delta(D)) = \frac{1}{d!}\mathrm{vol}_X(D)$
- Depends only on numerical equivalence class
- The valuation properties ensure convexity and well-definedness

Why Valuation Properties Matter

Property (ii) - Subadditivity:

- Ensures the Newton-Okounkov body is convex
- Critical for asymptotic analysis of linear series
- Connects to tropical geometry

Property (iii) - Multiplicativity:

- Makes the construction functorial
- Allows tensor product computations
- Essential for relating different line bundles

Lexicographic ordering:

- Natural total order on $\mathbb{Z}^d$
- Respects the flag structure $Y_0 \supset Y_1 \supset \cdots \supset Y_d$
- Enables well-defined minimum in property (ii)

Infinitesimal Newton-Okounkov Bodies

Construction:

- Fix smooth point $x \in X$ and flag $V_\bullet$ in tangent space $T_x X$
- Consider blow-up $\mu : X' = \mathrm{Bl}_x(X) \to X$
- Induced flag on X':

$$X' \supseteq E = \mathbb{P}(T_x X) \supseteq \mathbb{P}(V_1) \supseteq \cdots \supseteq \{pt\}$$

- The valuation properties transfer to the blown-up setting
- For very general choice of $(x, V_\bullet)$, the bodies coincide

Kollár's Construction - Base Example

Setup: $X = E \times E$ (product of elliptic curves)
For each $n \geq 2$, define:

$$A_n = nF_1 + (n^2 - n + 1)F_2 - (n-1)\Delta$$

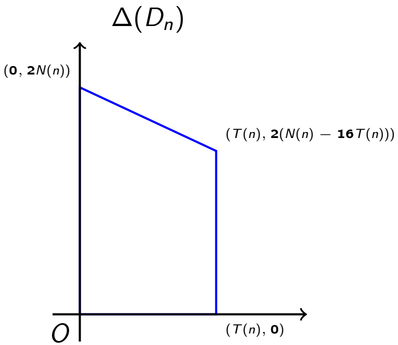
where:

- F_1, F_2 are fibers of the two projections
- Δ is the diagonal
- A_n is ample

Let $R = F_1 + F_2$, $B \in |2R|$ smooth, $f : Y \to X$ double cover branched along B
Set $D_n = f^* A_n$ (also ample)

The "Needle" Newton-Okounkov Body

Flag choice: $Y \supset f^*C \supset \mathrm{pt}$
where C is pullback of smooth curve in $|2F_1 + 2F_2 + \Delta|$



Key insight: As $n \to \infty$: $T(n) \sim \frac{1}{n^2}$, $N(n) \sim n^2$

Critical Calculations

Volume computation:

$$N(n) = 3n^2 - 4n + 7$$

$$T(n) = \frac{N(n)}{16}\left(1 - \sqrt{1 - \frac{32}{N(n)^2}}\right)$$

Jet ampleness results:

- nA_n is k-jet ample when $n \geq k + 2$
- But nD_n is **not** k-jet ample!
- Need $(k + N'')D_n$ where $N'' \sim n^2$

The "needle" effect:

- D_n has fixed volume: $\mathrm{vol}(\Delta(D_n)) = 4$
- But $(k + N'')D_n$ has large volume: $\mathrm{vol}(\Delta((k + N'')D_n)) \sim (k + n^2)^2$
- Poor geometric shape: width $T(n) \sim \frac{1}{n^2}$
- Valuation detects this pathological behavior

Inverse of Di Rocco-Szemberg Theorem

Let t be the restriction of τ to Y and $\tilde{\pi} : Y \to X$ the restriction of p. The projection formula gives the decomposition:

$$H^0(Y, \tilde{\pi}^* L) = \bigoplus_{q=0}^{d-1} t^q H^0(X, L - qM)$$

Theorem (Lu)

If $\tilde{\pi}^* L$ is dk-jet ample, then L is k-jet ample.

Failed Conjectures

Natural conjectures that our example disproves:

Conjecture 1 (Volume-based)

If $\mathrm{vol}(\Delta(L)) = \Omega(k^n)$, then L is k-jet ample.

Conjecture 2 (Shape-based)

For k-jet ample L, there exists simplex $\Delta' \subset \Delta(L)$ whose shape is parameterized by k.

Our counterexample: D_n requires coefficient $\sim n^2$ for k-jet ampleness!
The valuation properties alone cannot prevent this pathological behavior.

The Fundamental Issue

Why volume fails:

- Newton-Okounkov bodies can be "needle-like"
- High volume concentrated in wrong directions
- Valuation captures vanishing orders but not geometric distribution

Numerical equivalence vs. jet ampleness:

- Newton-Okounkov bodies: invariant under numerical equivalence
- k-jet ampleness: **not** invariant under numerical equivalence
- This fundamental mismatch explains the failure
- Valuation properties preserve numerical equivalence

Infinitesimal Bodies - A Partial Solution

Better approach: Infinitesimal Newton-Okounkov bodies

Key Result

If for infinitesimal flag $Y_\bullet$ over point x:

$$\Delta^{-1}_{n+k+\varepsilon} \subseteq \Delta_{Y_\bullet}(\pi^*(D))$$

then $K_X + D$ separates k-jets at x.

Advantages:

- Can capture single-point k-jet conditions
- More sensitive to local geometry
- Better suited for jet separation analysis
- Valuation properties still hold in infinitesimal setting

Limitations:

- Only handles single points (but general k-jet ampleness requires multiple points simultaneously
- Adjoint bundle appears

Future Directions

Promising research directions:

1. **Multi-point information:** Extend infinitesimal construction to handle multiple points simultaneously
2. **Refined valuations:** Investigate valuations that capture geometric shape, not just vanishing orders
3. **Higher dimensions:** Extend computational methods beyond abelian surfaces
4. **Shape-sensitive measures:** Develop invariants that combine valuation data with geometric constraints

Key insight: Need to incorporate **geometric shape** information beyond standard valuation properties!

Conclusion

Main contributions:

- **Counterexample:** Disproved volume-based characterization of k-jet ampleness
- **Valuation analysis:** Showed that standard valuation properties cannot prevent pathological behavior
- **Geometric insight:** Shape matters more than volume for jet conditions
- **Theoretical gap:** Identified fundamental mismatch between numerical invariance and jet ampleness

Broader impact:

- Reveals limitations of classical valuation theory for jet ampleness
- Suggests need for geometric refinements of Newton-Okounkov bodies

Thank you for your attention!