# Proofs of "Robust Operation of Distribution Systems with Uncertain Renewable Generation via Energy Sharing"

Meng Yang

January 29, 2024

## 1 Proof of Proposition 1

The KKT conditions of the second-stage sharing problem (3)-(4) (in the paper) is

$$
\begin{aligned}
2l_i\Delta p_i - \rho_i^- + \rho_i^+ + \mu_j &= 0, \forall j \in \mathcal{J}, i \in \mathcal{I}_j && \text{(A.1a)}\\
-2a\lambda + b_j - a\mu_j - Ja\eta_j &= 0, \forall j \in \mathcal{J} && \text{(A.1b)}\\
\lambda + \mu_j + \eta_j &= 0, \forall j \in \mathcal{J} && \text{(A.1c)}\\
\sum_{i\in\mathcal{I}_j}(p_i + \Delta p_i) + w_j - a\lambda + b_j &= \sum_{q\in\mathcal{Q}_j} D_q, j \in \mathcal{J} && \text{(A.1d)}\\
\sum_{j\in\mathcal{J}}(-a\lambda + b_j) &= 0 && \text{(A.1e)}\\
0 \le (\Delta p_i + r_i) &\perp \rho_i^- \ge 0, \forall j \in \mathcal{J}, i \in \mathcal{I}_j && \text{(A.1f)}\\
0 \le (-\Delta p_i + r_i) &\perp \rho_i^+ \ge 0, \forall j \in \mathcal{J}, i \in \mathcal{I}_j && \text{(A.1g)}
\end{aligned}
$$

The KKT condition of the equivalent problem (8) (in the paper) is

$$
\begin{aligned}
2l_i\Delta p_i - \frac{\sum\limits_{q\in\mathcal{Q}_j} D_q - w_j - \sum\limits_{i\in\mathcal{I}_j} p_i - \sum\limits_{i\in\mathcal{I}_j}\Delta p_i}{a(J-1)} - \delta_i^- + \delta_i^+ + \epsilon &= 0, \forall j \in \mathcal{J}, i \in \mathcal{I}_j && \text{(A.2a)}\\
\sum_{i\in\mathcal{I}}(p_i + \Delta p_i) + \sum_{j\in\mathcal{J}} w_j &= \sum_{q\in\mathcal{Q}} D_q && \text{(A.2b)}\\
0 \le (\Delta p_i + r_i) &\perp \delta_i^- \ge 0, \forall j \in \mathcal{J}, i \in \mathcal{I}_j && \text{(A.2c)}\\
0 \le (-\Delta p_i + r_i) &\perp \delta_i^+ \ge 0, \forall j \in \mathcal{J}, i \in \mathcal{I}_j && \text{(A.2d)}
\end{aligned}
$$

"$\rightarrow$": If $(\Delta p^*, b^*)$ is the NE of the sharing game (3) (in the paper), then it satisfies the KKT conditions (A.1). Sum up (A.1d) for all $j$ and substitute (A.1e) into, then (A.2b) is met. With (A.1b) $+$ $Ja$(A.1c) we have

$$(J-2)a\lambda^* + b_j^* + (J-1)a\mu_j^* = 0 \tag{A.3}$$

Together with (A.1d), we can get

$$\lambda^* + \mu_j^* + \frac{\sum\limits_{q\in\mathcal{Q}_j} D_q - w_j - \sum\limits_{i\in\mathcal{I}_j} p_i - \sum\limits_{i\in\mathcal{I}_j}\Delta p_i^*}{(J-1)a} = 0 \tag{A.4}$$

(A.1a) $-$ (A.4) gives

$$2l_i\Delta p_i^* - \frac{\sum\limits_{q\in\mathcal{Q}_j} D_q - w_j - \sum\limits_{i\in\mathcal{I}_j} p_i - \sum\limits_{i\in\mathcal{I}_j}\Delta p_i^*}{(J-1)a} - \rho_i^{*-} + \rho_i^{*+} - \lambda^* = 0 \tag{A.5}$$

Let $\delta_j^- = \rho_j^{*-}$,$\delta_j^+ = \rho_j^{*+}$ and $\epsilon = -\lambda^*$, then (A.2a) is met. Because of (A.1f) and (A.1g), (A.2c) and (A.2d) are all satisfied. As a result, $\Delta p^*$ is the optimal solution of problem (8) (in the paper).

With (A.1b) $+$ $a$(A.1c), we can get

$$\eta_j^* = \frac{-a\lambda^* + b_j^*}{(J-1)a} \tag{A.6}$$

Because of (A.1e), it is easy to obtain $\sum\limits_{j\in\mathcal{J}} \eta_j^* = 0$. Sum up (A.1a) for all $i \in \mathcal{I}_j$ gives

$$\mu_j^* = -\frac{1}{I_j}\sum_{i\in\mathcal{I}_j}(2l_i\Delta p_i^* - \rho_i^{*-} + \rho_i^{*+}) \tag{A.7}$$

Sum up (A.1c) for all $j \in \mathcal{J}$ and with $\sum_{j\in\mathcal{J}} \eta_j^* = 0$ and (A.7), we can get that

$$\lambda^* = \frac{1}{J}\sum_{j\in\mathcal{J}}\frac{1}{I_j}\sum_{i\in\mathcal{I}_j}(2l_i\Delta p_i^* - \rho_i^{*-} + \rho_i^{*+}) \tag{A.8}$$

and from (A.1d), obviously

$$b_j^* = \sum_{q\in\mathcal{Q}_j} D_q - \sum_{i\in\mathcal{I}_j}(p_i - \Delta p_i^*) - w_j + a\lambda^* \tag{A.9}$$

"$\leftarrow$" If $(p^*, b^*)$ is the optimal solution of problem (8) (in the paper) in the paper and the corresponding $b$, then it satisfies the KKT conditions (A.1) if we let

$$\begin{aligned}
\Delta p_i &= \Delta p_i^* \\
\rho_i^- &= \delta_i^{*-} \\
\rho_i^+ &= \delta_i^{*+} \\
\mu_j &= -2l_i\Delta p_i^* + \delta_i^{*-} - \delta_i^{*+} \\
\lambda &= -\epsilon^* \\
\eta_j &= -\lambda - \mu_j \\
b_j &= \sum_{q\in\mathcal{Q}_j} D_q - \sum_{i\in\mathcal{I}_j}(p_i - \Delta p_i^*) - w_j + a\lambda
\end{aligned} \tag{A.10}$$

It is worth noting that, at this time, if we sum up (A.2a) for all $i \in \mathcal{I}_j$, then

$$\frac{1}{I_j}\sum_{i\in\mathcal{I}_j}(2l_i\Delta p_i^* - \rho_i^{*-} + \rho_i^{*+}) - \frac{\sum_{q\in\mathcal{Q}_j} D_q - w_j - \sum_{i\in\mathcal{I}_j} p_i - \sum_{i\in\mathcal{I}_j}\Delta p_i^*}{a(J-1)} + \epsilon^* = 0 \tag{A.11}$$

Sum up (A.11) and together with (A.1b), we can prove that

$$\epsilon^* = -\frac{1}{J}\sum_{j\in\mathcal{J}}\frac{1}{I_j}\sum_{i\in\mathcal{I}_j}(2l_i\Delta p_i^* - \rho_i^{*-} + \rho_i^{*+}) \tag{A.12}$$

This completes the proof.

## 2 Proof of Proposition 2

If an NE of the sharing game (3) exists, then it is easy to check $\Delta p^* \in Y_{SMK}(p, r, w)$, and thus, $Y_{SMK}(p, r, w) \neq \emptyset$. If $Y_{SMK}(p, r, w) \neq \emptyset$, it means given $(p, r, w)$, there exists an $(\Delta p_i, \forall i; b_j, \forall j)$, such that (3b), (3c) and (4) are satisfied. Obviously, (8b) is met under this $\Delta p$. Sum up (3c) for all $j$ and together with (4), we can get that (8c) is satisfied. Thus, $\Delta p$ is a feasible point for the problem (8). In other words, (8) is feasible and has an optimal solution $\Delta p^*$. Let $b_j^* = \hat{b}_j, \forall j$. According to Proposition 1, $(\Delta p^*, b^*)$ is the unique NE of the sharing game (3). This completes the proof.

## 3 Proof of Proposition 3

Given the first-stage strategy $(p, r)$ and uncertain scenario $w$, under the sharing scheme, if other prosumers' bids are $b_k, k \neq j$, then by choosing

$$\Delta p_i = \Delta\tilde{p}_i, b_j = \frac{\sum_{k\neq j} b_k}{J-1}$$

with

$$\lambda = \frac{\sum_{k\neq j} b_k}{(J-1)a}$$

We have $y^j_{SMK}(p, r, w, \Delta p) = y^j_{IND}(p, r, w, \Delta\tilde{p})$, which means prosumer $j$ can achieve the same cost as under individual scheme. Because each prosumer aims at minimizing its own cost, so we always have $y^j_{SMK}(p, r, w, \Delta p^*) \leq y^j_{SMK}(p, r, w, \Delta p) = y^j_{IND}(p, r, w, \Delta\tilde{p})$. This completes the proof.

# 4 Proof of Proposition 5

Suppose that $0 < a_1 < a_2$, and $\Delta p^{1*}$ is the NE of sharing game (3) under $a = a_1$, $\Delta p^{2*}$ is the NE of sharing game (3) under $a = a_2$. According to Proposition 1, $\Delta p^{1*}$ and $\Delta p^{2*}$ are the optimal solution of problem (8) under $a = a_1$ and $a = a_2$, respectively. Due to optimality, we have

$$\begin{aligned} & f_{SMK}(\Delta p^{2*}) + \frac{\sum\limits_{j\in\mathcal{J}}(\sum\limits_{q\in\mathcal{Q}_j} D_q - w_j - \sum\limits_{i\in\mathcal{I}_j} p_i - \sum\limits_{i\in\mathcal{I}_j} \Delta p_i^{2*})^2}{2a_1(J-1)} \\ \geq\ & f_{SMK}(\Delta p^{1*}) + \frac{\sum\limits_{j\in\mathcal{J}}(\sum\limits_{q\in\mathcal{Q}_j} D_q - w_j - \sum\limits_{i\in\mathcal{I}_j} p_{it} - \sum\limits_{i\in\mathcal{I}_j} \Delta p_i^{1*})^2}{2a_1(J-1)} \end{aligned} \tag{A.13}$$

which means

$$\begin{aligned} & 2a_1(J-1)\left(f_{SMK}(\Delta p^{2*}) - f_{SMK}(\Delta p^{1*})\right) \\ \geq\ & \sum_{j\in\mathcal{J}}(\sum_{q\in\mathcal{Q}_j} D_q - w_j - \sum_{i\in\mathcal{I}_j} p_i - \sum_{i\in\mathcal{I}_j} \Delta p_i^{1*})^2 - \sum_{j\in\mathcal{J}}(\sum_{q\in\mathcal{Q}_j} D_q - w_j - \sum_{i\in\mathcal{I}_j} p_i - \sum_{i\in\mathcal{I}_j} \Delta p_i^{2*})^2 \end{aligned} \tag{A.14}$$

If we have

$$f_{SMK}(\Delta p^{1*}) < f_{SMK}(\Delta p^{2*})$$

then we have

$$\begin{aligned} & 2a_2(J-1)\left(f_{SMK}(\Delta p^{2*}) - f_{SMK}(\Delta p^{1*})\right) \\ >\ & 2a_1(J-1)\left(f_{SMK}(\Delta p^{2*}) - f_{SMK}(\Delta p^{1*})\right) \\ \geq\ & \sum_{j\in\mathcal{J}}(\sum_{q\in\mathcal{Q}_j} D_q - w_j - \sum_{i\in\mathcal{I}_j} p_i - \sum_{i\in\mathcal{I}_j} \Delta p_i^{1*})^2 - \sum_{j\in\mathcal{J}}(\sum_{q\in\mathcal{Q}_j} D_q - w_j - \sum_{i\in\mathcal{I}_j} p_i - \sum_{i\in\mathcal{I}_j} \Delta p_i^{2*})^2 \end{aligned} \tag{A.15}$$

which indicates

$$\begin{aligned} & f_{SMK}(\Delta p^{2*}) + \frac{\sum\limits_{j\in\mathcal{J}}(\sum\limits_{q\in\mathcal{Q}_j} D_q - w_j - \sum\limits_{i\in\mathcal{I}_j} p_i - \sum\limits_{i\in\mathcal{I}_j} \Delta p_i^{2*})^2}{2a_2(J-1)} \\ >\ & f_{SMK}(\Delta p^{1*}) + \frac{\sum\limits_{j\in\mathcal{J}}(\sum\limits_{q\in\mathcal{Q}_j} D_q - w_j - \sum\limits_{i\in\mathcal{I}_j} p_i - \sum\limits_{i\in\mathcal{I}_j} \Delta p_i^{1*})^2}{2a_2(J-1)} \end{aligned} \tag{A.16}$$

and is contradict to the assumption that $\Delta p^{2*}$ is the optimal solution of problem (8) (in the paper) under $a = a_2$. This completes the proof.

# 5 Proof of Proposition 7

## 5.1 For the first inequality

If the opposite holds

$$\Pi_{CTR}(\bar{p}, \bar{r}, \Delta\bar{p}(\bar{p}, \bar{r}, \bar{w})) > \Pi_{SMK}(p^*, r^*, \Delta p^*(p^*, r^*, w^*)) \tag{A.17}$$

Under the CTR scheme, fix the first-stage variable to $(p^*, r^*)$ and solve the sub "max-min" problem. Denote the obtained worst case as $\bar{w}^{'}$ and the corresponding second-stage variable as $\Delta\bar{p}^{'}(p^*, r^*, \bar{w}^{'})$.

1) If $\bar{w}^{'} = w^*$, then according to Proposition 4, we have

$$\Pi_{CTR}(p^*, r^*, \Delta\bar{p}^{'}(p^*, r^*, \bar{w}^{'})) \leq \Pi_{SMK}(p^*, r^*, \Delta p^*(p^*, r^*, w^*))$$

As a result, together with (A.17), we get

$$\Pi_{CTR}(\bar{p}, \bar{r}, \Delta\bar{p}(\bar{p}, \bar{r}, \bar{w})) > \Pi_{CTR}(p^*, r^*, \Delta\bar{p}^{'}(p^*, r^*, \bar{w}^{'}))$$

which is contradict to the assumption that $\bar{p}, \bar{r}$ is the optimal solution of robust model under CTR scheme.

2) If $\bar{w}^{'} \neq w^*$, then If

$$\Pi_{CTR}(p^*, r^*, \Delta\bar{p}^{'}(p^*, r^*, \bar{w}^{'})) \leq \Pi_{SMK}(p^*, r^*, \Delta p^*(p^*, r^*, w^*))$$

Similar contradiction as 1) can be found. Otherwise, if

$$\Pi_{CTR}(p^*, r^*, \Delta\bar{p}^{'}(p^*, r^*, \bar{w}^{'})) > \Pi_{SMK}(p^*, r^*, \Delta p^*(p^*, r^*, w^*))$$

then suppose $\Delta\bar{p}^{*'}$ is the optimal solution under scenario $\bar{w}^{'}$, because of Proposition 4 we have

$$\Pi_{SMK}(p^*, r^*, \Delta\bar{p}^{*'}(p^*, r^*, \bar{w}^{'})) \geq \Pi_{CTR}(p^*, r^*, \Delta\bar{p}^{'}(p^*, r^*, \bar{w}^{'}))$$

As a result,

$$\Pi_{SMK}(p^*, r^*, \Delta\bar{p}^{*'}(p^*, r^*, \bar{w}^{'})) > \Pi_{SMK}(p^*, r^*, \Delta p^*(p^*, r^*, w^*))$$

which is contradict to the assumption that $w^*$ is the worst case for the robust model under sharing scheme.

## 5.2 For the second inequality

If the opposite holds

$$\Pi_{SMK}(p^*, r^*, \Delta p^*(p^*, r^*, w^*)) > \Pi_{IND}(\tilde{p}, \tilde{r}, \Delta\tilde{p}(\tilde{p}, \tilde{r}, \tilde{w})) \tag{A.18}$$

Under the SMK scheme, fix the first-stage variable to $(\tilde{p}, \tilde{r})$ and solve the sub "max-min" problem. Denote the obtained worst case as $w^{*'}$ and the corresponding second-stage variable as $\Delta p^{*'}(\tilde{p}, \tilde{r}, w^{*'})$.

1) If $w^{*'} = \tilde{w}$, then according to Proposition 4, we have

$$\Pi_{SMK}(\tilde{p}, \tilde{r}, \Delta p^{*'}(\tilde{p}, \tilde{r}, w^{*'})) \leq \Pi_{IND}(\tilde{p}, \tilde{r}, \Delta\tilde{p}(\tilde{p}, \tilde{r}, \tilde{w}))$$

As a result, together with (A.18), we get

$$\Pi_{SMK}(p^*, r^*, \Delta p^*(p^*, r^*, w^*)) > \Pi_{SMK}(\tilde{p}, \tilde{r}, \Delta p^{*'}(\tilde{p}, \tilde{r}, w^{*'}))$$

which is contradict to the assumption that $p^*, r^*$ is the optimal solution of robust model under SMK scheme.

2) If $w^{*'} \neq \tilde{w}$, then If

$$\Pi_{SMK}(\tilde{p}, \tilde{r}, \Delta p^{*'}(\tilde{p}, \tilde{r}, w^{*'})) \leq \Pi_{IND}(\tilde{p}, \tilde{r}, \Delta\tilde{p}(\tilde{p}, \tilde{r}, \tilde{w}))$$

Similar contradiction as 1) can be found. Otherwise, if

$$\Pi_{SMK}(\tilde{p}, \tilde{r}, \Delta p^{*'}(\tilde{p}, \tilde{r}, w^{*'})) > \Pi_{IND}(\tilde{p}, \tilde{r}, \Delta\tilde{p}(\tilde{p}, \tilde{r}, \tilde{w}))$$

then suppose $\Delta\tilde{p}^{*'}$ is the optimal solution under scenario $w^{*'}$, because of Proposition 4 we have

$$\Pi_{IND}(\tilde{p}, \tilde{r}, \Delta\tilde{p}^{*'}(\tilde{p}, \tilde{r}, w^{*'})) \geq \Pi_{SMK}(\tilde{p}, \tilde{r}, \Delta p^{*'}(\tilde{p}, \tilde{r}, w^{*'}))$$

As a result,

$$\Pi_{IND}(\tilde{p}, \tilde{r}, \Delta\tilde{p}^{*'}(\tilde{p}, \tilde{r}, w^{*'})) > \Pi_{IND}(\tilde{p}, \tilde{r}, \Delta\tilde{p}(\tilde{p}, \tilde{r}, \tilde{w}))$$

which is contradict to the assumption that $\tilde{w}$ is the worst case for the robust model under individual scheme.

## 5.3 For the last inference

Suppose $0 < a_1 < a_2$. Under $a = a_1$, the optimal first-stage variable is $(p^*(a_1), r^*(a_1))$, the corresponding worst case is $w^*(a_1)$ and the second-stage variable is $\Delta p^*(p^*(a_1), r^*(a_1), w^*(a_1))$. Under $a = a_2$, the optimal first-stage variable is $(p^*(a_2), r^*(a_2))$, the corresponding worst case is $w^*(a_2)$ and the second-stage variable is $\Delta p^*(p^*(a_2), r^*(a_2), w^*(a_2))$.

If the opposite holds

$$\Pi_{SMK}(p^*(a_2), r^*(a_2), \Delta p^*(p^*(a_2), r^*(a_2), w^*(a_2))) > \Pi_{SMK}(p^*(a_1), r^*(a_1), \Delta p^*(p^*(a_1), r^*(a_1), w^*(a_1))) \tag{A.19}$$

Under the $a = a_2$, fix the first-stage variable to $(p^*(a_1), r^*(a_1))$ and solve the sub "max-min" problem. Denote the obtained worst case as $w_2^*(a_1)$ and the corresponding second-stage variable as $\Delta p_2^*(p^*(a_1), r^*(a_1), w_2^*(a_1))$

1) If $w_2^*(a_1) = w^*(a_1)$, then according to Proposition 4, we have

$$\Pi_{SMK}(p^*(a_1), r^*(a_1), \Delta p_2^*(p^*(a_1), r^*(a_1), w_2^*(a_1))) \leq \Pi_{SMK}(p^*(a_1), r^*(a_1), \Delta p^*(p^*(a_1), r^*(a_1), w^*(a_1)))$$

As a result, together with (A.19), we get

$$\Pi_{SMK}(p^*(a_2), r^*(a_2), \Delta p^*(p^*(a_2), r^*(a_2), w^*(a_2))) > \Pi_{SMK}(p^*(a_1), r^*(a_1), \Delta p_2^*(p^*(a_1), r^*(a_1), w_2^*(a_1)))$$

which is contradict to the assumption that $p^*(a_2), r^*(a_2)$ is the optimal solution of robust model under $a = a_2$.

2) If $w_2^*(a_1) \neq w^*(a_1)$, then If

$$\Pi_{SMK}(p^*(a_1), r^*(a_1), \Delta p_2^*(p^*(a_1), r^*(a_1), w_2^*(a_1))) \leq \Pi_{SMK}(p^*(a_1), r^*(a_1), \Delta p^*(p^*(a_1), r^*(a_1), w^*(a_1)))$$

Similar contradiction as 1) can be found. Otherwise, if

$$\Pi_{SMK}(p^*(a_1), r^*(a_1), \Delta p_2^*(p^*(a_1), r^*(a_1), w_2^*(a_1))) > \Pi_{SMK}(p^*(a_1), r^*(a_1), \Delta p^*(p^*(a_1), r^*(a_1), w^*(a_1)))$$

then suppose $\Delta p_1^*(p^*(a_1), r^*(a_1), w_2^*(a_1))$ is the optimal solution under scenario $w_2^*(a_1)$, because of Proposition 4 we have

$$\Pi_{SMK}(p^*(a_1), r^*(a_1), \Delta p_1^*(p^*(a_1), r^*(a_1), w_2^*(a_1))) \geq \Pi_{SMK}(p^*(a_1), r^*(a_1), \Delta p_2^*(p^*(a_1), r^*(a_1), w_2^*(a_1)))$$

As a result,

$$\Pi_{SMK}(p^*(a_1), r^*(a_1), \Delta p_1^*(p^*(a_1), r^*(a_1), w_2^*(a_1))) > \Pi_{SMK}(p^*(a_1), r^*(a_1), \Delta p^*(p^*(a_1), r^*(a_1), w^*(a_1)))$$

which is contradict to the assumption that $w^*(a_1)$ is the worst case for the robust model under $a = a_1$.

This completes the proof.