Proofs of "Robust Operation of Distribution System with Uncertain Renewable Generation via Energy Sharing"

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1 Proof of Proposition 1

The KKT conditions of the second-stage sharing problem (3) is

$$\begin{aligned}
2l_i\Delta p_{it} - \rho_{it}^- + \rho_{it}^+ + \mu_{jt} &= 0, \forall j \in \mathcal{J}, i \in \mathcal{I}_j, \forall t && \text{(1a)}\\
-2a\lambda_t + b_{jt} - a\mu_{jt} - Ia\eta_{jt} &= 0, \forall j \in \mathcal{J}, \forall t && \text{(1b)}\\
\lambda_t + \mu_{jt} + \eta_{jt} &= 0, \forall j \in \mathcal{J}, \forall t && \text{(1c)}\\
\sum_{i\in\mathcal{I}_j}(p_{it} + \Delta p_{it}) + w_{jt} - a\lambda_t + b_{jt} &= \sum_{q\in\mathcal{Q}_j} D_{qt}, j \in \mathcal{J}, \forall t && \text{(1d)}\\
\sum_{j\in\mathcal{J}}(-a\lambda_t + b_{jt}) &= 0, \forall t && \text{(1e)}\\
0 \le (\Delta p_{it} + r_{it}) &\perp \rho_{it}^- \ge 0, \forall j \in \mathcal{J}, i \in \mathcal{I}_j, \forall t && \text{(1f)}\\
0 \le (-\Delta p_{it} + r_{it}) &\perp \rho_{it}^+ \ge 0, \forall j \in \mathcal{J}, i \in \mathcal{I}_j, \forall t && \text{(1g)}
\end{aligned}$$

The KKT condition of the equivalent problem (7) is

$$\begin{aligned}
2l_i\Delta p_{it} - \frac{\sum\limits_{q\in\mathcal{Q}_j} D_{qt} - w_{jt} - \sum\limits_{i\in\mathcal{I}_j} p_{it} - \sum\limits_{i\in\mathcal{I}_j}\Delta p_{it}}{a(I-1)} - \delta_{it}^- + \delta_{it}^+ + \epsilon_t &= 0, \forall j \in \mathcal{J}, i \in \mathcal{I}_j, \forall t && \text{(2a)}\\
\sum_{i\in\mathcal{I}}(p_{it} + \Delta p_{it}) + \sum_{j\in\mathcal{J}} w_{jt} &= \sum_{q\in\mathcal{Q}} D_{qt}, \forall t && \text{(2b)}\\
0 \le (\Delta p_{it} + r_{it}) &\perp \delta_{it}^- \ge 0, \forall j \in \mathcal{J}, i \in \mathcal{I}_j, \forall t && \text{(2c)}\\
0 \le (-\Delta p_{it} + r_{it}) &\perp \delta_{it}^+ \ge 0, \forall j \in \mathcal{J}, i \in \mathcal{I}_j, \forall t && \text{(2d)}
\end{aligned}$$

"$\rightarrow$": If $(\Delta p^*, b^*)$ is the NE of the sharing game (3), then it satisfies the KKT conditions (1). Sum up (1d) for all j and substitute (1e) into, then (2b) is met. With (1b) + Ia(1c) we have

$$(I-2)a\lambda_t + b_{jt} + (I-1)a\mu_{jt} = 0 \tag{3}$$

Together with (1d), we can get

$$\lambda_t + \mu_{jt} + \frac{\sum\limits_{q\in\mathcal{Q}_j} D_{qt} - w_{jt} - \sum\limits_{i\in\mathcal{I}_j} p_{it} - \sum\limits_{i\in\mathcal{I}_j}\Delta p_{it}}{(I-1)a} = 0 \tag{4}$$

(1a) − (5) gives

$$2l_i\Delta p_{it} - \frac{\sum\limits_{q\in\mathcal{Q}_j} D_{qt} - w_{jt} - \sum\limits_{i\in\mathcal{I}_j} p_{it} - \sum\limits_{i\in\mathcal{I}_j}\Delta p_{it}}{(I-1)a} - \rho_{it}^- + \rho_{it}^+ - \lambda_t = 0 \tag{5}$$

Let $\delta_{jt}^- = \rho_{jt}^-$,$\delta_{jt}^+ = \rho_{jt}^+$ and $\epsilon_t = -\lambda_t$, then (2a) is met. Because of (1f) and (1g), (2c) and (2d) are all satisfied. As a result, Δp^* is the optimal solution of problem (7).

With (1b) + a(1c), we can get

$$\eta_{jt} = \frac{-a\lambda_t + b_{jt}}{(I-1)a} \tag{6}$$

Because of (1e), it is easy to obtain $\sum\limits_{j\in\mathcal{J}} \eta_{jt} = 0$. Sum up (1a) for all $i \in \mathcal{I}_j$ gives

$$\mu_{jt} = -\frac{1}{I_j}\sum_{i\in\mathcal{I}_j}(2l_i\Delta p_{it} - \rho_{it}^- + \rho_{it}^+) \tag{7}$$

Sum up (1c) for all $j \in \mathcal{J}$ and with $\sum\limits_{j\in\mathcal{J}} \eta_{jt} = 0$ and (7), we can get that

$$\lambda_t = \frac{1}{J}\sum_{j\in\mathcal{J}}\frac{1}{I_j}\sum_{i\in\mathcal{I}_j}(2l_i\Delta p_{it} - \rho_{it}^- + \rho_{it}^+) \tag{8}$$

and from (1d), obviously

$$b_{jt} = \sum_{q\in\mathcal{Q}_j} D_{qt} - \sum_{i\in\mathcal{I}_j}(p_{it} - \Delta p_{it}) - w_{jt} + a\lambda_t \tag{9}$$

"$\leftarrow$" If (p^*, b^*) is the optimal solution of problem (7) in the paper and the corresponding b, then it satisfies the KKT conditions (1) if we let

$$\begin{aligned}
\Delta p_{it} &= \Delta p_{it}^* \\
\rho_{it}^- &= \delta_{it}^{*-} \\
\rho_{it}^+ &= \delta_{it}^{*+} \\
\mu_{jt} &= -2l_i\Delta p_{it}^* + \delta_{it}^{*-} - \delta_{it}^{*+} \\
\lambda_t &= -\epsilon_t^* \\
\eta_{jt} &= -\lambda_t - \mu_{jt} \\
b_{jt} &= \sum_{q\in\mathcal{Q}_j} D_{qt} - \sum_{i\in\mathcal{I}_j}(p_{it} - \Delta p_{it}^*) - w_{jt} + a\lambda_t
\end{aligned} \tag{10}$$

It is worth noting that, at this time, if we sum up (2a) for all $i \in \mathcal{I}_j$, then

$$\frac{1}{I_j}\sum_{i\in\mathcal{I}_j}(2l_i\Delta p_{it} - \rho_{it}^- + \rho_{it}^+) - \frac{\sum\limits_{q\in\mathcal{Q}_j} D_{qt} - w_{jt} - \sum\limits_{i\in\mathcal{I}_j} p_{it} - \sum\limits_{i\in\mathcal{I}_j}\Delta p_{it}}{a(I-1)} + \epsilon_t = 0 \tag{11}$$

Sum up (11) and together with (1b), we can prove that

$$\epsilon_t = -\frac{1}{J}\sum_{j\in\mathcal{J}}\frac{1}{I_j}\sum_{i\in\mathcal{I}_j}(2l_i\Delta p_{it} - \rho_{it}^- + \rho_{it}^+) \tag{12}$$

This completes the proof.

2 Proof of Proposition 2

Given the first-stage strategy (p, r) and uncertain scenario w, under the sharing scheme, if other prosumers' bids are $b_{kt}, k \neq j$, then by choosing

$$\Delta p_{it} = \Delta\tilde{p}_{it}, b_{jt} = \frac{\sum_{k\neq j} b_{kt}}{I-1}$$

with

$$\lambda_t = \frac{\sum_{k\neq j} b_{kt}}{(I-1)a}$$

We have $y^j_{SMK}(p, r, w, \Delta p) = y^j_{IND}(p, r, w, \Delta\tilde{p})$, which means prosumer j can achieve the same cost as under individual scheme. Because each prosumer aims at minimization its own cost, so we always have $y^j_{SMK}(p, r, w, \Delta p^*) \leq y^j_{SMK}(p, r, w, \Delta p) = y^j_{IND}(p, r, w, \Delta\tilde{p})$. This completes the proof.

3 Proof of Proposition 4

Suppose that $0 < a_1 < a_2$, and Δp^{1*} is the NE of sharing game (3) under $a = a_1$, Δp^{2*} is the NE of sharing game (3) under $a = a_2$. According to Proposition 1, Δp^{1*} and Δp^{2*} are the optimal solution of problem (7) under $a = a_1$ and $a = a_2$, respectively. Due to optimality, we have

$$\begin{aligned}
& f_{SMK}(\Delta p^{2*}) + \frac{\sum\limits_{j\in\mathcal{J}}(\sum\limits_{q\in\mathcal{Q}_j} D_{qt} - w_{jt} - \sum\limits_{i\in\mathcal{I}_j} p_{it} - \sum\limits_{i\in\mathcal{I}_j}\Delta p_{it}^{2*})^2}{2a_1(I-1)} \\
\geq\ & f_{SMK}(\Delta p^{1*}) + \frac{\sum\limits_{j\in\mathcal{J}}(\sum\limits_{q\in\mathcal{Q}_j} D_{qt} - w_{jt} - \sum\limits_{i\in\mathcal{I}_j} p_{it} - \sum\limits_{i\in\mathcal{I}_j}\Delta p_{it}^{1*})^2}{2a_1(I-1)}
\end{aligned} \tag{13}$$

which means

$$\begin{aligned}
& 2a_1(I-1)\left(f_{SMK}(\Delta p^{2*})-f_{SMK}(\Delta p^{1*})\right) \\
\geq\ & \sum_{j\in\mathcal{J}}(\sum_{q\in\mathcal{Q}_j} D_{qt}-w_{jt}-\sum_{i\in\mathcal{I}_j} p_{it}-\sum_{i\in\mathcal{I}_j}\Delta p_{it}^{1*})^2-\sum_{j\in\mathcal{J}}(\sum_{q\in\mathcal{Q}_j} D_{qt}-w_{jt}-\sum_{i\in\mathcal{I}_j} p_{it}-\sum_{i\in\mathcal{I}_j}\Delta p_{it}^{2*})^2
\end{aligned} \tag{14}$$

If we have

$$f_{SMK}(\Delta p^{1*})<f_{SMK}(\Delta p^{2*})$$

then we have

$$\begin{aligned}
& 2a_2(I-1)\left(f_{SMK}(\Delta p^{2*})-f_{SMK}(\Delta p^{1*})\right) \\
>\ & 2a_1(I-1)\left(f_{SMK}(\Delta p^{2*})-f_{SMK}(\Delta p^{1*})\right) \\
\geq\ & \sum_{j\in\mathcal{J}}(\sum_{q\in\mathcal{Q}_j} D_{qt}-w_{jt}-\sum_{i\in\mathcal{I}_j} p_{it}-\sum_{i\in\mathcal{I}_j}\Delta p_{it}^{1*})^2-\sum_{j\in\mathcal{J}}(\sum_{q\in\mathcal{Q}_j} D_{qt}-w_{jt}-\sum_{i\in\mathcal{I}_j} p_{it}-\sum_{i\in\mathcal{I}_j}\Delta p_{it}^{2*})^2
\end{aligned} \tag{15}$$

which indicates

$$\begin{aligned}
& f_{SMK}(\Delta p^{2*})+\frac{\sum_{j\in\mathcal{J}}(\sum_{q\in\mathcal{Q}_j} D_{qt}-w_{jt}-\sum_{i\in\mathcal{I}_j} p_{it}-\sum_{i\in\mathcal{I}_j}\Delta p_{it}^{2*})^2}{2a_2(I-1)} \\
>\ & f_{SMK}(\Delta p^{1*})+\frac{\sum_{j\in\mathcal{J}}(\sum_{q\in\mathcal{Q}_j} D_{qt}-w_{jt}-\sum_{i\in\mathcal{I}_j} p_{it}-\sum_{i\in\mathcal{I}_j}\Delta p_{it}^{1*})^2}{2a_2(I-1)}
\end{aligned} \tag{16}$$

and is contradict to the assumption that Δp^{2*} is the optimal solution of problem (7) under $a=a_2$. This completes the proof.

4 Proof of Proposition 6

4.1 For the first inequality

If the opposite holds

$$\Pi_{AGG}(\bar{p},\bar{r},\Delta\bar{p}(\bar{p},\bar{r},\bar{w}))>\Pi_{SMK}(p^*,r^*,\Delta p^*(p^*,r^*,w^*)) \tag{17}$$

Under the AGG scheme, fix the first-stage variable to (p^*,r^*) and solve the sub "max-min" problem. Denote the obtained worst case as $\bar{w}^{'}$ and the corresponding second-stage variable as $\Delta\bar{p}^{'}(p^*,r^*,\bar{w}^{'})$.

1) If $\bar{w}^{'}=w^*$, then according to Proposition 3, we have

$$\Pi_{AGG}(p^*,r^*,\Delta\bar{p}^{'}(p^*,r^*,\bar{w}^{'}))\leq\Pi_{SMK}(p^*,r^*,\Delta p^*(p^*,r^*,w^*))$$

As a result, together with (17), we get

$$\Pi_{AGG}(\bar{p},\bar{r},\Delta\bar{p}(\bar{p},\bar{r},\bar{w}))>\Pi_{AGG}(p^*,r^*,\Delta\bar{p}^{'}(p^*,r^*,\bar{w}^{'}))$$

which is contradict to the assumption that $\bar{p},\bar{r}$ is the optimal solution of robust model under AGG scheme.

2) If $\bar{w}^{'}\neq w^*$, then If

$$\Pi_{AGG}(p^*,r^*,\Delta\bar{p}^{'}(p^*,r^*,\bar{w}^{'}))\leq\Pi_{SMK}(p^*,r^*,\Delta p^*(p^*,r^*,w^*))$$

Similar contradiction as 1) can be found. Otherwise, if

$$\Pi_{AGG}(p^*,r^*,\Delta\bar{p}^{'}(p^*,r^*,\bar{w}^{'}))>\Pi_{SMK}(p^*,r^*,\Delta p^*(p^*,r^*,w^*))$$

then because of Proposition 3, suppose $\Delta\bar{p}^{*'}$ is the optimal solution under scenario $\bar{w}^{'}$, we have

$$\Pi_{SMK}(p^*,r^*,\Delta\bar{p}^{*'}(p^*,r^*,\bar{w}^{'}))\geq\Pi_{AGG}(p^*,r^*,\Delta\bar{p}^{'}(p^*,r^*,\bar{w}^{'}))$$

As a result,

$$\Pi_{SMK}(p^*,r^*,\Delta\bar{p}^{*'}(p^*,r^*,\bar{w}^{'}))>\Pi_{SMK}(p^*,r^*,\Delta p^*(p^*,r^*,w^*))$$

which is contradict to the assumption that w^* is the worst case for the robust model under sharing scheme.

4.2 For the second inequality

If the opposite holds

$$\Pi_{SMK}(p^*, r^*, \Delta p^*(p^*, r^*, w^*)) > \Pi_{IND}(\tilde{p}, \tilde{r}, \Delta\tilde{p}(\tilde{p}, \tilde{r}, \tilde{w})) \tag{18}$$

Under the SMK scheme, fix the first-stage variable to $(\tilde{p}, \tilde{r})$ and solve the sub "max-min" problem. Denote the obtained worst case as $w^{*'}$ and the corresponding second-stage variable as $\Delta p^{*'}(\tilde{p}, \tilde{r}, w^{*'})$.

1) If $w^{*'} = \tilde{w}$, then according to Proposition 3, we have

$$\Pi_{SMK}(\tilde{p}, \tilde{r}, \Delta p^{*'}(\tilde{p}, \tilde{r}, w^{*'})) \leq \Pi_{IND}(\tilde{p}, \tilde{r}, \Delta\tilde{p}(\tilde{p}, \tilde{r}, \tilde{w}))$$

As a result, together with (18), we get

$$\Pi_{SMK}(p^*, r^*, \Delta p^*(p^*, r^*, w^*)) > \Pi_{SMK}(\tilde{p}, \tilde{r}, \Delta p^{*'}(\tilde{p}, \tilde{r}, w^{*'}))$$

which is contradict to the assumption that p^*, r^* is the optimal solution of robust model under SMK scheme.

2) If $w^{*'} \neq \tilde{w}$, then If

$$\Pi_{SMK}(\tilde{p}, \tilde{r}, \Delta p^{*'}(\tilde{p}, \tilde{r}, w^{*'})) \leq \Pi_{IND}(\tilde{p}, \tilde{r}, \Delta\tilde{p}(\tilde{p}, \tilde{r}, \tilde{w}))$$

Similar contradiction as 1) can be found. Otherwise, if

$$\Pi_{SMK}(\tilde{p}, \tilde{r}, \Delta p^{*'}(\tilde{p}, \tilde{r}, w^{*'})) > \Pi_{IND}(\tilde{p}, \tilde{r}, \Delta\tilde{p}(\tilde{p}, \tilde{r}, \tilde{w}))$$

then because of Proposition 3, suppose $\Delta\tilde{p}^{*'}$ is the optimal solution under scenario $w^{*'}$, we have

$$\Pi_{IND}(\tilde{p}, \tilde{r}, \Delta\tilde{p}^{*'}(\tilde{p}, \tilde{r}, w^{*'})) \geq \Pi_{SMK}(\tilde{p}, \tilde{r}, \Delta p^{*'}(\tilde{p}, \tilde{r}, w^{*'}))$$

As a result,

$$\Pi_{IND}(\tilde{p}, \tilde{r}, \Delta\tilde{p}^{*'}(\tilde{p}, \tilde{r}, w^{*'})) > \Pi_{IND}(\tilde{p}, \tilde{r}, \Delta\tilde{p}(\tilde{p}, \tilde{r}, \tilde{w}))$$

which is contradict to the assumption that $\tilde{w}$ is the worst case for the robust model under individual scheme.

4.3 For the last inference

Suppose $0 < a_1 < a_2$. Under $a = a_1$, the optimal first-stage variable is $(p^*(a_1), r^*(a_1))$, the corresponding worst case is $w^*(a_1)$ and the second-stage variable is $\Delta p^*(p^*(a_1), r^*(a_1), w^*(a_1))$. Under $a = a_2$, the optimal first-stage variable is $(p^*(a_2), r^*(a_2))$, the corresponding worst case is $w^*(a_2)$ and the second-stage variable is $\Delta p^*(p^*(a_2), r^*(a_2), w^*(a_2))$.

If the opposite holds

$$\Pi_{SMK}(p^*(a_2), r^*(a_2), \Delta p^*(p^*(a_2), r^*(a_2), w^*(a_2))) > \Pi_{SMK}(p^*(a_1), r^*(a_1), \Delta p^*(p^*(a_1), r^*(a_1), w^*(a_1))) \tag{19}$$

Under the $a = a_2$, fix the first-stage variable to $(p^*(a_1), r^*(a_1))$ and solve the sub "max-min" problem. Denote the obtained worst case as $w_2^*(a_1)$ and the corresponding second-stage variable as $\Delta p_2^*(p^*(a_1), r^*(a_1), w_2^*(a_1))$

1) If $w_2^*(a_1) = w^*(a_1)$, then according to Proposition 4, we have

$$\Pi_{SMK}(p^*(a_1), r^*(a_1), \Delta p_2^*(p^*(a_1), r^*(a_1), w_2^*(a_1))) \leq \Pi_{SMK}(p^*(a_1), r^*(a_1), \Delta p^*(p^*(a_1), r^*(a_1), w^*(a_1)))$$

As a result, together with (19), we get

$$\Pi_{SMK}(p^*(a_2), r^*(a_2), \Delta p^*(p^*(a_2), r^*(a_2), w^*(a_2))) > \Pi_{SMK}(p^*(a_1), r^*(a_1), \Delta p_2^*(p^*(a_1), r^*(a_1), w_2^*(a_1)))$$

which is contradict to the assumption that $p^*(a_2), r^*(a_2)$ is the optimal solution of robust model under $a = a_2$.

2) If $w_2^*(a_1) \neq w^*(a_1)$, then If

$$\Pi_{SMK}(p^*(a_1), r^*(a_1), \Delta p_2^*(p^*(a_1), r^*(a_1), w_2^*(a_1))) \leq \Pi_{SMK}(p^*(a_1), r^*(a_1), \Delta p^*(p^*(a_1), r^*(a_1), w^*(a_1)))$$

Similar contradiction as 1) can be found. Otherwise, if

$$\Pi_{SMK}(p^*(a_1), r^*(a_1), \Delta p_2^*(p^*(a_1), r^*(a_1), w_2^*(a_1))) > \Pi_{SMK}(p^*(a_1), r^*(a_1), \Delta p^*(p^*(a_1), r^*(a_1), w^*(a_1)))$$

then because of Proposition 4, suppose $\Delta p_1^*(p^*(a_1), r^*(a_1), w_2^*(a_1))$ is the optimal solution under scenario $w_2^*(a_1)$, we have

$$\Pi_{SMK}(p^*(a_1), r^*(a_1), \Delta p_1^*(p^*(a_1), r^*(a_1), w_2^*(a_1))) \geq \Pi_{SMK}(p^*(a_1), r^*(a_1), \Delta p_2^*(p^*(a_1), r^*(a_1), w_2^*(a_1)))$$

As a result,

$$\Pi_{SMK}(p^*(a_1), r^*(a_1), \Delta p_1^*(p^*(a_1), r^*(a_1), w_2^*(a_1))) > \Pi_{SMK}(p^*(a_1), r^*(a_1), \Delta p^*(p^*(a_1), r^*(a_1), w^*(a_1)))$$

which is contradict to the assumption that $w^*(a_1)$ is the worst case for the robust model under $a = a_1$.

This completes the proof.