Note on Deriving the State Transition Model for the Meteorites Project

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1 A First Attempt

From the PDF and a review of the meteorites code, we are given a motion model with terms a, b, c and the position in y is computed as:

$$y_t = c_y + b_y \Delta t + a \frac{\Delta t^2}{2} \tag{1}$$

where:
c_y is the initial position,
b_y is the initial velocity and
a is constant acceleration.
Δt is the time since the beginning of the meteorite's life.

One note here pertaining to the simulation is that since the Δt is considered from the beginning of the meteorite life to the present, the position estimate is not computed by integrating over each time step, but rather the current position is found directly from a, b, c and the Δt defined as the time from the beginning of the meteorite's life. For this reason, the values of a, b and c are truly constant.

Now turning to the Kalman Filter, in contrast to the simulation the Kalman filter iteratively propagates its estimate of the state forward over each time step so that $\Delta t = t_i - t_{i-1}$. For the Kalman filter it makes sense to choose states such that $\bar{y} = [y, \dot{y}, \ddot{y}]^T$ as this is a simple and standard state space used for a constant acceleration model in one dimension. To create the Kalman filter dynamics model, F, used to propagate the state $\bar{y}$ forward in time, we must find a consistent set of differential equations from which to construct it. Based on equation (1), we know that the acceleration will be the same throughout the entire simulation, so our state dynamics should end up with a constant acceleration model.

A reasonable approach is to assume the initial position $y = c$, initial velocity $\dot{y} = b$, and acceleration $\ddot{y} = a$. So that we have the equation representing the motion over any time interval:

$$y_t = y_{t-1} + \dot{y}_{t-1} \Delta t + \ddot{y}_{t-1} \frac{\Delta t^2}{2} \tag{2}$$

To get the remaining equations that are consistent with the first, take the derivatives of equation (2) with respect to time. The first derivative is:

$$\dot{y}_t = \dot{y}_{t-1} + \ddot{y}_{t-1} \Delta t \tag{3}$$

and the second derivative with respect to time is:

$$\ddot{y}_t = \ddot{y}_{t-1} \tag{4}$$

By inspection, we can construct each row of F from equations (2-4) and then write out the state propagation from one time step to the next as:

$$\bar{y}_t = F\bar{y}_{t-1} \tag{5}$$

which is:

$$\begin{bmatrix} y_t \\ \dot{y}_t \\ \ddot{y}_t \end{bmatrix} = \begin{bmatrix} 1 & \Delta t & \frac{\Delta t^2}{2} \\ 0 & 1 & \Delta t \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ \dot{y}_{t-1} \\ \ddot{y}_{t-1} \end{bmatrix} \tag{6}$$

2 What About the Position in x?

From the PDF document, the position in x is computed as:

$$x_t = c_x + b_x\Delta t + Sa\frac{\Delta t^2}{2} \tag{7}$$

where:

c_x is the initial position,
b_x is the initial velocity and
a is the same constant acceleration used in computing y
S is a constant scaling parameter applied to the acceleration
Δt is the time since the beginning of the meteorite's life.

Following the same approach as above but now recognizing that the motion in x is correlated to the motion in y, the iterative equation for the Kalman filter is similar to equation (2).

$$x_t = x_{t-1} + \dot{x}_{t-1}\Delta t + S\ddot{y}_{t-1}\frac{\Delta t^2}{2} \tag{8}$$

Because the value of S is provided (it is 1/3 at the time of this document), the Kalman filter does not have to estimate a separate acceleration for the x position equation. Accounting for this and following the same logic as above:

Take the first derivative with respect to time of (8) which results in:

$$\dot{x}_t = \dot{x}_{t-1} + S\ddot{y}_{t-1}\Delta t \tag{9}$$

The entire 5 state transition model including motion in y can be stated as:

$$\begin{bmatrix} x_t \\ y_t \\ \dot{x}_t \\ \dot{y}_t \\ \ddot{y}_t \end{bmatrix} = \begin{bmatrix} 1 & 0 & \Delta t & 0 & S\frac{\Delta t^2}{2} \\ 0 & 1 & 0 & \Delta t & \frac{\Delta t^2}{2} \\ 0 & 0 & 1 & 0 & S\Delta t \\ 0 & 0 & 0 & 1 & \Delta t \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ y_{t-1} \\ \dot{x}_{t-1} \\ \dot{y}_{t-1} \\ \ddot{y}_{t-1} \end{bmatrix} \tag{10}$$