

2. THE SIMPLEX METHOD

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THE SIMPLEX METHOD

Standard form.

Assuming that the right-hand-side vector is nonnegative, $\mathbf{b} \geq \mathbf{0}$,

$$\begin{aligned} \max(\min) z &= \mathbf{c}^T \mathbf{x} \\ \text{subject to} \\ \mathbf{Ax} &= \mathbf{b} \\ \mathbf{x} &\geq \mathbf{0} \end{aligned}$$

1. Model manipulation

To transform any linear model into the standard form:

1. The objective function.

$$\min z = \sum_{j=1}^n c_j x_j \iff \max (-z) = \sum_{j=1}^n -c_j x_j.$$

2. Constraints.

$$(a) \sum_{j=1}^n a_{ij} x_j \geq b_i \iff \sum_{j=1}^n -a_{ij} x_j \leq -b_i.$$

$$(b) \sum_{j=1}^n a_{ij} x_j \leq b_i \iff \sum_{j=1}^n a_{ij} x_j + y = b_i, \quad y \geq 0.$$

$$\sum_{j=1}^n a_{ij} x_j \geq b_i \iff \sum_{j=1}^n a_{ij} x_j - y = b_i, \quad y \geq 0.$$

$$(c) \sum_{j=1}^n a_{ij} x_j = b_i \iff \sum_{j=1}^n a_{ij} x_j \leq b_i, \quad \sum_{j=1}^n a_{ij} x_j \geq b_i.$$

3. Variables.

- If $x_j \leq 0$, then $x_j = -x'_j$, where $x'_j \geq 0$.
- If x_j is unrestricted in sign, then

$$x_j = x'_j - x''_j, \quad \text{where} \quad x'_j, x''_j \geq 0.$$

2. Solving linear models

- \mathbf{x} is a **solution** if it satisfies $\mathbf{Ax} = \mathbf{b}$.
- A solution \mathbf{x} is **feasible** if $\mathbf{x} \geq \mathbf{0}$.
- Let \mathbf{B} be a basic matrix constituted by extracting m columns out of \mathbf{A} . \mathbf{x}_B is a **basic solution** if

$$\mathbf{Bx}_B = \mathbf{b}.$$

All nonbasic variables are zero, $\mathbf{x}_N = \mathbf{0}$. If $\mathbf{x}_B \geq \mathbf{0}$ then it is a **basic feasible solution**.

- A basic feasible solution is **degenerate** if at least one component of \mathbf{x}_B is zero.
- The set of all feasible solutions of a linear problem is the **feasible region** or the **convex set of feasible solutions**, and it is noted by F .
- \mathbf{x}^* denotes the **optimal solution**, and $z^* = \mathbf{c}^T \mathbf{x}^*$ the **optimal objective value**.
- A linear model is **unbounded** if there exists no finite optimal value for the objective function,

$$z^* \rightarrow +\infty \quad \text{or} \quad z^* \rightarrow -\infty.$$

- A model has **multiple optimal solutions** if it has more than one optimal solution.

3. Extreme points and basic feasible solutions

Theorem 1 Consider a linear model in standard form:

$$\begin{aligned} \max \quad & z = \mathbf{c}^T \mathbf{x} \\ \text{subject to} \quad & \\ & \mathbf{Ax} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

\mathbf{x} is a *basic feasible solution* if and only if \mathbf{x} is an *extreme point* of F .

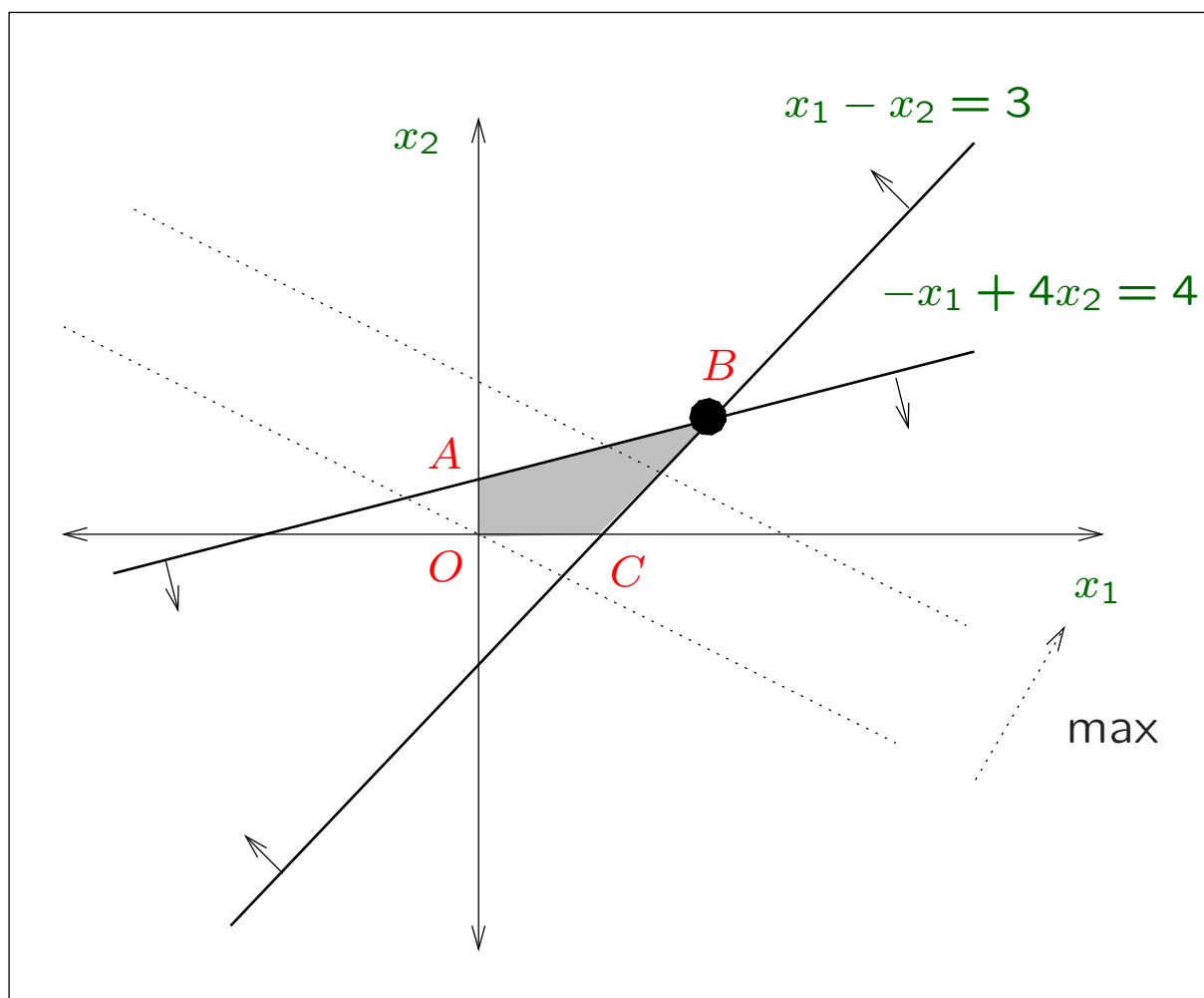
Theorem 2 Consider a linear model in standard form:

$$\begin{aligned} \max \quad & z = \mathbf{c}^T \mathbf{x} \\ \text{subject to} \quad & \\ & \mathbf{Ax} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

The *optimal value* of the objective function is obtained at an *extreme point* of the feasible region F .

Example. Consider the following linear model:

$$\begin{aligned} \max \quad & z = x_1 + 2x_2 \\ \text{subject to} \quad & -x_1 + 4x_2 \leq 4 \\ & x_1 - x_2 \leq 3 \\ & x_1, x_2 \geq 0 \end{aligned}$$



$$O = (0, 0), \quad A = (0, 1), \quad B = \left(\frac{16}{3}, \frac{7}{3}\right), \quad C = (3, 0).$$

The linear model in standard form:

$$\begin{aligned} \max \quad & z = x_1 + 2x_2 + 0x_3 + 0x_4 \\ \text{subject to} \quad & \end{aligned}$$

$$-x_1 + 4x_2 + x_3 = 4$$

$$x_1 - x_2 + x_4 = 3$$

$$x_1, x_2, x_3, x_4 \geq 0$$

The algebraic calculation of two basic solutions:

- Basis $\mathbf{B} = (\mathbf{a}_1 \ \mathbf{a}_2)$.

$$\mathbf{x}_B = \begin{pmatrix} -1 & 4 \\ 1 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{16}{3} \\ \frac{7}{3} \end{pmatrix}$$

The basic solution is **feasible**, and it corresponds to the **extreme point** B .

- Basis $\mathbf{B} = (\mathbf{a}_1 \ \mathbf{a}_4)$.

$$\mathbf{x}_B = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} -4 \\ 7 \end{pmatrix}$$

This basic solution is **not feasible**, because there are negative components. Thus, it does not correspond to any extreme point in the graphical representation.

4. The simplex method

4.1 Definitions and notation

Assume there are m linearly independent rows and n columns in matrix \mathbf{A} , $n > m$.

Let \mathbf{B} be a basis formed by m linearly independent columns of \mathbf{A} (assume they are the first m columns).

$$\begin{aligned} \max \quad & z = (\mathbf{c}_B^T \mid \mathbf{c}_N^T) \begin{pmatrix} \mathbf{x}_B \\ - \\ \mathbf{x}_N \end{pmatrix} \\ \text{subject to} \quad & (\mathbf{B} \mid \mathbf{N}) \begin{pmatrix} \mathbf{x}_B \\ - \\ \mathbf{x}_N \end{pmatrix} = \mathbf{b} \\ & \mathbf{x}_B, \mathbf{x}_N \geq \mathbf{0} \end{aligned}$$

The linear model can be written as:

$$\begin{aligned} \max \quad & z = \mathbf{c}_B^T \mathbf{x}_B + \mathbf{c}_N^T \mathbf{x}_N \\ \text{subject to} \quad & \mathbf{B}\mathbf{x}_B + \mathbf{N}\mathbf{x}_N = \mathbf{b} \\ & \mathbf{x}_B, \mathbf{x}_N \geq \mathbf{0} \end{aligned}$$

- **Basic solution.** By setting $\mathbf{x}_N = 0$, the system of linear equations is:

$$\mathbf{B}\mathbf{x}_B = \mathbf{b} \rightarrow \mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} \text{ where } \mathbf{x}_B = \begin{pmatrix} x_{B1} \\ x_{B2} \\ \vdots \\ x_{Bm} \end{pmatrix}.$$

- **Objective value.** Let $\mathbf{c}_B^T = (c_{B1}, c_{B2}, \dots, c_{Bm})$,

$$z = \mathbf{c}_B^T \mathbf{x}_B = (c_{B1}, c_{B2}, \dots, c_{Bm}) \begin{pmatrix} x_{B1} \\ x_{B2} \\ \vdots \\ x_{Bm} \end{pmatrix} = \sum_{i=1}^m c_{Bi} x_{Bi}.$$

- **Coordinate vector.** $\mathbf{a}_j = \mathbf{B}\mathbf{y}_j \rightarrow \mathbf{y}_j = \mathbf{B}^{-1}\mathbf{a}_j$.

$$\mathbf{a}_j = y_{1j}\mathbf{a}_1 + y_{2j}\mathbf{a}_2 + \dots + y_{mj}\mathbf{a}_m = \sum_{i=1}^m y_{ij}\mathbf{a}_i.$$

- **Reduced cost coefficients $z_j - c_j$.** The scalar z_j .

$$z_j = \mathbf{c}_B^T \mathbf{y}_j = c_{B1}y_{1j} + c_{B2}y_{2j} + \dots + c_{Bm}y_{mj} = \sum_{i=1}^m c_{Bi}y_{ij}.$$

4.2 Improvement of a basic feasible solution

Theorem 3 Consider a linear model in maximization standard form. Let \mathbf{B} be a basis formed by columns of \mathbf{A} , and let $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$ be the basic feasible solution relative to \mathbf{B} , and $z = \mathbf{c}_B^T \mathbf{x}_B$ the corresponding objective value.

If there exists a nonbasic vector \mathbf{a}_j in matrix \mathbf{A} such that $z_j - c_j < 0$ and for such vector \mathbf{a}_j there exists a positive coordinate y_{ij} , $i = 1, \dots, m$, then there exists another basic feasible solution $\hat{\mathbf{x}}_B$, where

$$\hat{z} = \hat{\mathbf{c}}_B^T \hat{\mathbf{x}}_B \geq z = \mathbf{c}_B^T \mathbf{x}_B.$$

4.3 Selection of entering and leaving vectors

- Entering vector rule. Select \mathbf{a}_k to enter the basis, where

$$z_k - c_k = \min_j \{z_j - c_j / z_j - c_j < 0\}$$

The rule guarantees that $\hat{z} \geq z$.

- Leaving vector rule. Select \mathbf{a}_r to leave the basis, where

$$\frac{x_{Br}}{y_{rk}} = \min_i \left\{ \frac{x_{Bi}}{y_{ik}} / y_{ik} > 0 \right\}$$

The rule guarantees that $\hat{\mathbf{x}}_B \geq \mathbf{0}$.

4.4 Rules to compute $\hat{\mathbf{x}}_B$ and \hat{z}

- Calculation of the new basic feasible solution:

$$\hat{\mathbf{x}}_B = \begin{cases} x_{Bi} - x_{Br} \frac{y_{ik}}{y_{rk}} & i \neq r \\ \frac{x_{Br}}{y_{rk}} & i = r \end{cases}$$

- Calculation of the new objective value:

$$\hat{z} = z - \frac{x_{Br}}{y_{rk}} (z_k - c_k)$$

Theorem 4 (Optimality condition). *Consider a linear model in standard form.*

$$\begin{aligned} \max \quad & z = \mathbf{c}^T \mathbf{x} \\ \text{subject to} \quad & \\ & \mathbf{Ax} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

Let \mathbf{B} be a basis formed by columns of \mathbf{A} , and let $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$ be its corresponding basic feasible solution, and $z = \mathbf{c}_B^T \mathbf{x}_B$ its objective value.

If $z_j - c_j$ is greater than or equal to zero for every vector \mathbf{a}_j of matrix \mathbf{A} , then \mathbf{x}_B is an optimal basic feasible solution to the model.

4.5 Unbounded solution

Theorem 5 Consider a linear model in maximization standard form. Let \mathbf{B} be a basis formed by columns of \mathbf{A} , and let $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$ be its corresponding basic feasible solution, and $z = \mathbf{c}_B^T \mathbf{x}_B$ its objective value.

If there exists a nonbasic vector \mathbf{a}_k in matrix \mathbf{A} such that $z_k - c_k < 0$, and for such vector \mathbf{a}_k all coordinates y_{ik} are less than or equal to zero, $i = 1, \dots, m$, then the solution to the model is unbounded.

4.6 Multiple optimal solutions

Theorem 6 Consider a linear model in maximization standard form. Let \mathbf{B} be a basis formed by columns of \mathbf{A} , and let $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$ be its corresponding basic feasible solution, and $z = \mathbf{c}_B^T \mathbf{x}_B$ its objective value.

If $z_j - c_j \geq 0$ holds for every vector \mathbf{a}_j of matrix \mathbf{A} , then the solution \mathbf{x}_B is optimal. Moreover, if there exists a nonbasic vector \mathbf{a}_k such that $z_k - c_k = 0$, and at least one coordinate $y_{ik} > 0$, $i = 1, \dots, m$, then there exist multiple optimal solutions.

Theorem 7 Consider a linear model in maximization standard form. Let $\mathbf{x}_1, \dots, \mathbf{x}_p$ be optimal basic feasible solutions of the model. Then, every generalized convex linear combination of them is an optimal feasible solution of the model.

Theorem 8 Let us consider a linear model in maximization standard form. Let \mathbf{B} be a basis formed by columns of \mathbf{A} , and let $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$ be its corresponding basic feasible solution, and $z = \mathbf{c}_B^T \mathbf{x}_B$ its objective value. If $z_j - c_j \geq 0$ holds for every vector \mathbf{a}_j of matrix \mathbf{A} , then the solution \mathbf{x}_B is optimal.

If there exists a nonbasic vector \mathbf{a}_k in matrix \mathbf{A} such that $z_k - c_k = 0$, and for such vector \mathbf{a}_k all coordinates y_{ik} are less than or equal to zero, $i = 1, \dots, m$, then there are multiple optimal solutions with unbounded variables.

4.7 The initial basic feasible solution

Relative to the canonical basis. Two cases:

An initial canonical basis formed by slack variables

$$\begin{array}{ll} \max z = \mathbf{c}^T \mathbf{x} & \max z = \mathbf{c}^T \mathbf{x} + \mathbf{0}^T \mathbf{y} \\ \text{subject to} & \rightarrow \text{subject to} \\ \mathbf{Ax} \leq \mathbf{b} & \mathbf{Ax} + \mathbf{Iy} = \mathbf{b} \\ \mathbf{x} \geq \mathbf{0} & \mathbf{x}, \mathbf{y} \geq \mathbf{0} \end{array}$$

- Calculation of the basic solution. It is feasible.

$$\mathbf{x}_B = \mathbf{B}^{-1} \mathbf{b} = \mathbf{Ib} = \mathbf{b} \geq \mathbf{0}.$$

- Calculation of the objective value. As all vectors in the initial canonical basis \mathbf{B} correspond to slack variables, $\mathbf{c}_B^T = \mathbf{0}$ holds.

$$z = \mathbf{c}_B^T \mathbf{x}_B = \mathbf{0}^T \mathbf{x}_B = 0.$$

- Coordinate vectors.

$$\mathbf{y}_j = \mathbf{B}^{-1} \mathbf{a}_j = \mathbf{Ia}_j = \mathbf{a}_j.$$

- Calculation of the reduced cost coefficients $z_j - c_j$. As all vectors in the initial canonical basis \mathbf{B} correspond to slack variables, $\mathbf{c}_B^T = \mathbf{0}$ holds.

$$z_j - c_j = \mathbf{c}_B^T \mathbf{y}_j - c_j = 0 - c_j = -c_j.$$

Artificial variables in the initial basis

If once the model is in standard form matrix \mathbf{A} has **no identity submatrix**, then we introduce **artificial variables** to find a starting canonical basis and its corresponding basic feasible solution. Example:

$$\begin{aligned} \max \quad z &= 3x_1 + x_2 \\ \text{subject to} \quad & \\ & x_1 + x_2 \leq 3 \\ & x_1 + 2x_2 \geq 2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Add the slack variable x_3 , subtract x_4 and add an artificial variable, $w_1 \geq 0$ to get a canonical basis.

$$\begin{aligned} x_1 + x_2 + x_3 &= 3 \\ x_1 + 2x_2 - x_4 + w_1 &= 2 \end{aligned}$$

The basic feasible solution relative to the canonical basis $\mathbf{B} = (\mathbf{a}_3 \ \mathbf{a}_{w1})$:

$$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \geq \mathbf{0}.$$

However, \mathbf{x}_B is not a solution to the initial linear model, because the artificial variable $w_1 = 2 > 0$; constraint $x_1 + 2x_2 - x_4 = 2$ does not hold.

4.8 The simplex tableau

In the process of computing the optimal solution to a linear model in standard form, all the calculations that correspond to each of the bases are gathered in a tableau: **the simplex tableau**.

		Original variables			Auxiliary variables				
		x_1	\dots	x_n	x_{n+1}	\dots	x_j	\dots	
		$z_1 - c_1$	\dots	$z_n - c_n$	$z_{n+1} - c_{n+1}$	\dots	$z_j - c_j$	\dots	z
c_{B1}	\mathbf{a}_{B1}	y_{11}	\dots	y_{1n}	$y_{1,n+1}$	\dots	$y_{1,j}$	\dots	x_{B1}
\vdots	\vdots		\vdots			\vdots			\vdots
c_{Bi}	\mathbf{a}_{Bi}	y_{i1}	\dots	y_{in}	$y_{i,n+1}$	\dots	$y_{i,j}$	\dots	x_{Bi}
\vdots	\vdots		\vdots			\vdots			\vdots
c_{Bm}	\mathbf{a}_{Bm}	y_{m1}	\dots	y_{mn}	$y_{m,n+1}$	\dots	$y_{m,j}$	\dots	x_{Bm}

If the initial canonical basis \mathbf{B} is formed by just slack variables, the simplex tableau can be written like this:

		x_1	x_2	\dots	x_n	x_{n+1}	x_{n+2}	\dots	x_{n+m}
		$\mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{A} - \mathbf{c}^T$				$\mathbf{c}_B^T \mathbf{B}^{-1}$			$\mathbf{c}_B^T \mathbf{x}_B$
\mathbf{c}_B	\mathbf{B}	$\mathbf{B}^{-1} \mathbf{A}$				\mathbf{B}^{-1}			\mathbf{x}_B

5. The Big-M method

By introducing artificial variables to a linear model, we change the problem. In order to return to the original problem, we must force artificial variables to zero.

The Big-M method consists in **penalizing** the artificial variables in the objective function

Example:

$$\max z = -5x_1 + 6x_2 + 7x_3 + 0x_4 + 0x_5 - Mw_1 - Mw_2$$

subject to

$$2x_1 + 10x_2 - 6x_3 - x_4 + w_1 = 30$$

$$\frac{5}{2}x_1 - 3x_2 + 5x_3 + x_5 = 10$$

$$2x_1 + 2x_2 + 2x_3 + w_2 = 5$$

$$x_1, x_2, x_3, x_4, x_5, w_1, w_2 \geq 0$$

Add w_1 , w_2 to obtain the initial $B = I$, and **penalize** them in the objective function. The simplex tableau:

		x_1	x_2	x_3	x_4	x_5	w_1	w_2	
		$-4M + 5$	$-12M - 6$	$4M - 7$	M	0	0	0	$-35M$
$-M$	a_{w1}	2	10	-6	-1	0	1	0	30
0	a_5	$\frac{5}{2}$	-3	5	0	1	0	0	10
$-M$	a_{w2}	2	2	2	0	0	0	1	5

6. The simplex algorithm

Write the model in maximization standard form.

* **Step 1.** Construct the initial simplex tableau.

* **Step 2.**

- If there exists $z_j - c_j < 0$, then the solution may be improved. Go to Step 4.
- If $z_j - c_j \geq 0$ for every vector \mathbf{a}_j of matrix \mathbf{A} , then go to Step 3.

* **Step 3.**

- If there exists an artificial variable with a positive value then the problem is **infeasible**. Stop.
- If there is no artificial variable in the basis, then the solution \mathbf{x}_B in the tableau is **optimal**.
 - * $z_j - c_j \geq 0$ holds for every vector \mathbf{a}_j of matrix \mathbf{A} . If $z_j - c_j > 0$ for every nonbasic vector \mathbf{a}_j , then \mathbf{x}_B is the **unique optimal solution**. Stop.
 - * $z_j - c_j \geq 0$ holds for every vector \mathbf{a}_j . If there exists a nonbasic vector \mathbf{a}_k such that $z_k - c_k = 0$, and at least one of its coordinates y_{ik} is greater than zero, $i = 1, \dots, m$, then **another basic feasible solution** can be computed. The problem has **multiple optimal solutions**. Go to Step 5.
 - * $z_j - c_j \geq 0$ holds for every vector \mathbf{a}_j . If there exists a nonbasic vector \mathbf{a}_k such that $z_k - c_k = 0$, and if $y_{ik} \leq 0$, $i = 1, \dots, m$ holds for such vector \mathbf{a}_k , then the problem has **multiple optimal solutions**, but they are not basic solutions. Stop.

* **Step 4.**

- If there exists an a_j such that $z_j - c_j < 0$, and $y_j \leq 0$, then the solution is **unbounded**. Stop.
- If there exists an a_j such that $z_j - c_j < 0$, and there exists at least a coordinate greater than zero in vector y_j , then go to Step 5.

* **Step 5.** Select an entering vector a_k and a leaving vector a_r , according to the following rules:

- Vector a_k is selected to enter the basis, such that:

$$z_k - c_k = \min_j \{ z_j - c_j \mid z_j - c_j \leq 0 \}.$$

The k th column is called the **pivot column**.

- Vector a_r is selected to leave the basis, such that:

$$\frac{x_{Br}}{y_{rk}} = \min_i \left\{ \frac{x_{Bi}}{y_{ik}} \mid y_{ik} > 0 \right\}.$$

The r th row is called the **pivot row**.

The coordinate y_{rk} is called the **pivot element**.

* **Step 6.** Update the tableau.

- The new pivot row is computed by dividing the current pivot row by the pivot element y_{rk} .
- To update all other rows i : the new row is equal to the current row – the row multiplier \times the current pivot row. The multiplier for row i : $m_i = \frac{y_{ik}}{y_{rk}}$, $i = 1, \dots, m$, $i \neq r$.

Go to Step 2.

8. The two-phase method

Analog to the Big-M method. The aim of the two methods is to reach an optimal solution where all the artificial variables are zero.

The two-phase method minimizes the sum of the artificial variables in a first phase.

- * **Phase 1.** The constraints of the linear model are considered, but a new objective function is defined: **to minimize the sum of the artificial variables**. The simplex algorithm is applied. Two cases may arise:
 - If the optimal objective function value is greater than zero, then the original linear model is **infeasible**.
 - Otherwise, the original linear model is **feasible**. Go to Phase 2.
- * **Phase 2.** **The original objective function is optimized**. Take the optimal tableau obtained in Phase 1 as starting point. The artificial variables are discarded. The reduced cost coefficients $z_j - c_j$ must be updated. Apply the simplex algorithm.

9. The revised simplex method

Given a linear model in standard form, \mathbf{B} and \mathbf{x}_B ,

$$\begin{aligned} \max \quad z &= \mathbf{c}^T \mathbf{x} \\ \text{subject to} \\ \mathbf{Ax} &= \mathbf{b} \\ \mathbf{x} &\geq \mathbf{0} \end{aligned}$$

Vector \mathbf{a}_k enters the basis, $z_j - c_j = \mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{a}_j - c_j$.

$$z_k - c_k = \min_j \{ z_j - c_j \mid z_j - c_j \leq 0 \}.$$

By using the coordinates $\mathbf{y}_k = \mathbf{B}^{-1} \mathbf{a}_k$ and \mathbf{x}_B , vector \mathbf{a}_r is selected to leave the basis.

$$\frac{x_{Br}}{y_{rk}} = \min_i \left\{ \frac{x_{Bi}}{y_{ik}} \mid y_{ik} > 0 \right\}.$$

We need to know the inverse basis \mathbf{B}^{-1} to make all the necessary calculations. All the rest are known parameters, present in the linear model.

We just need a reduced tableau.

		x_{n+1}	x_{n+2}	\dots	x_{n+m}
		$\mathbf{c}_B^T \mathbf{B}^{-1}$			
		$\mathbf{c}_B^T \mathbf{x}_B$			
\mathbf{c}_B	\mathbf{B}	\mathbf{B}^{-1}			
		\mathbf{x}_B			

10. Some remarks

1. Round-off errors. Computers can carry only a limited number of decimal places of accuracy, and so, computations involving real numbers are approximate, and the errors tend to accumulate. If the error becomes significant, the optimal basic feasible solution obtained may not satisfy the constraints, or if it does, it may not be the optimal solution.

2. Artificial variables in the optimal basis. In some cases, once the simplex algorithm has reached the optimal tableau, and even though we may force the artificial variables to zero, one or more of these variables may still be basic. This fact can show two different situations: either there are redundant constraints in the model, or the solution is degenerate.

3. Cycling. If the solution is degenerate, it may cause the algorithm to cycle and iterate indefinitely without ever reaching termination but simply switching from one basis to another. This may happen if ties at the leaving vector rule are not correctly broken.

4. Computational complexity of the simplex algorithm. In most of the cases, the algorithm is observed to take $\frac{3m}{2}$ iterations, being A an $m \times n$ matrix. This means that the convergence of the algorithm is more sensitive to the number of constraints than to the number of variables of the model.