Outline

- Logarithms
- Floors & Ceilings
- Sequences & Series
- Limits
- Derivatives
- Permutations
- Combinations

Logarithms

$$y = b^x$$

$$log_b(y) = log_b(b^x)$$

$$\boldsymbol{log_b(y) = x}$$

Logarithms (continued)

$$log_a(xy) = log_a(x) + log_a(y)$$

$$log_a\frac{x}{y} = log_a(x) - log_a(y)$$

Logarithms (continued)

$$log_a x^y = y log_a(x)$$

$$log_a(x) = \frac{log_b(x)}{log_b(a)}$$

$$x^{log_b(y)} = y^{log_b(x)}$$

Logarithms (continued)

$$lg(x) = log_2(x)$$

$$ln(x) = log_e(x)$$

$$log^k(n) = (log(n))^k$$

$$lglg(n) = lg(lg(n))$$

Floors & Ceilings

- Floors

  $\lfloor x \rfloor$ is the largest integer not greater than x.

  For $x \in \mathbb{R}, x - 1 < \lfloor x \rfloor \leq x$.

- Ceilings

  $\lceil x \rceil$ is the smallest integer not less than x.

  For $x \in \mathbb{R}, x \leq \lceil x \rceil < x + 1$.

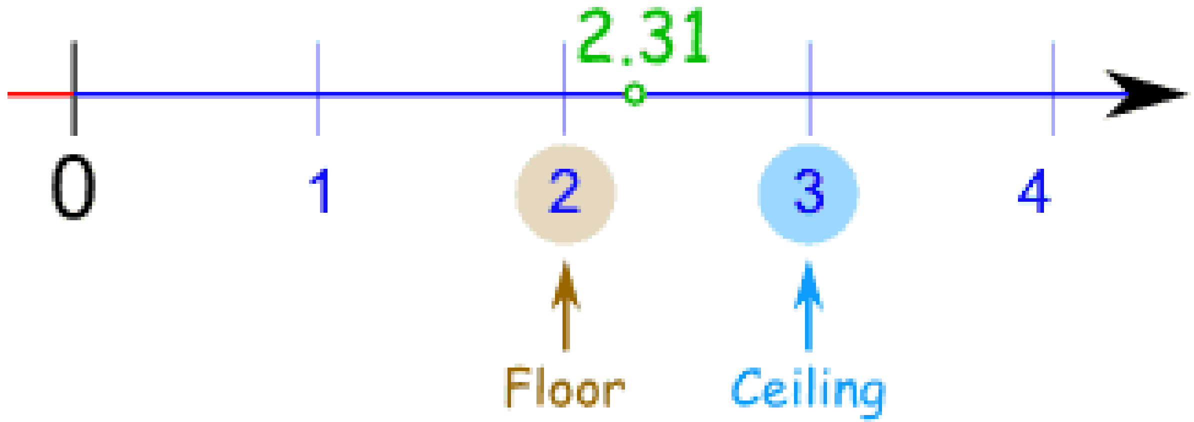
Floors & Ceilings – Example 1

- What is the floor and ceiling of 2.31?

Source: http://www.mathsisfun.com/sets/function-floor-ceiling.html

Floors & Ceilings – Example 1

- What is the floor and ceiling of 2.31?



- $\lfloor 2.31 \rfloor = 2$
- $\lceil 2.31 \rceil = 3$

Source: http://www.mathsisfun.com/sets/function-floor-ceiling.html

Floors & Ceilings – Example 2

- What is the floor and ceiling of 5?

Source: http://www.mathsisfun.com/sets/function-floor-ceiling.html

Floors & Ceilings – Example 2

- What is the floor and ceiling of 5?

The Floor of 5 is **5**
The Ceiling of 5 is **5**

Source: http://www.mathsisfun.com/sets/function-floor-ceiling.html

Floors & Ceilings – Example 3

x	**Floor**	**Ceiling**
-1.1	-2	-1
0	0	0
1.01	1	2
2.9	2	3
3	3	3

Source: http://www.mathsisfun.com/sets/function-floor-ceiling.html

Floors & Ceilings - Properties

$$x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1$$

$$\forall n \in \mathbb{Z}, \lceil \tfrac{n}{2} \rceil + \lfloor \tfrac{n}{2} \rfloor = n$$

$$\forall n \in \mathbb{R}^{+} \cup 0, \text{ and } \forall a, b \in \mathbb{Z}^{+}, \lceil \frac{\lceil \frac{n}{a} \rceil}{b} \rceil = \lceil \frac{n}{ab} \rceil$$

$$\forall a, b \in \mathbb{Z}^{+}, \lceil \tfrac{a}{b} \rceil \leq \tfrac{a+(b-1)}{b}$$

Arithmetic Sequences & Series

- The Arithmetic Sequence is a sequence of numbers such that the difference between successive terms in the sequence is constant.
- The first n values of the arithmetic sequence are:
  - $a, a+d, a+2d, a+3d, \ \dots \ , a + (n-1)d.$
  - a – initial value
  - d – difference

- Example: 1, 4, 7, 10, 13, 16, 19, … (difference of 3).

Arithmetic Sequences & Series

- The Arithmetic Series is the sum of the terms in the Arithmetic Sequence.

$$\sum_{i=0}^{n-1}(a+id) = \frac{(2a+(n-1)d)n}{2}$$

- Let $a_1 = a$ and $a_n = a + (n-1)d$

$$\sum_{i=0}^{n-1}(a+id) = \frac{(a_1+a_n)n}{2}$$

Geometric Sequences & Series

- The Geometric Sequence is a sequence of numbers where each successive term is found by multiplying the previous term by a fixed, non-zero, common ratio.
- The first n values of the geometric sequence are:
  - $a,\ ar,\ ar^2,\ ar^3,\ \ldots, ar^{n-1}$
  - a – initial value
  - $r \neq 0$ – fixed multiplier
- Example: 1, 2, 4, 8, 16, 32, … (common ratio of 2).

Geometric Sequences & Series

- The Geometric Series is the sum of the terms in the Geometric Sequence.

$$\sum_{i=0}^{n-1}(ar^i) = \frac{a(1-r^n)}{1-r}$$

- When $-1 < r < 1$, the sum of the in infinite geometric progression converges to:

$$\sum_{i=0}^{\infty}(ar^i) = \frac{a}{1-r}$$

Harmonic Series

- The first n values are:

$$1, {}^{1}/_{2}, {}^{1}/_{3}, \dots, {}^{1}/_{n}$$

- The sum of these values can be represented with:

$$H_n = \sum_{i=0}^{n} \frac{1}{i}$$

- The harmonic series does not converge, but satisfies the following property:

$$\ln(n + 1) < H_n \leq 1 + \ln(n)$$

Limits - Definition

A limit is a way of determining trends for values that may or may not exist.

The definition of a limit follows:

$$lim_{x\rightarrow c}f(x) = l$$

$$\Longleftrightarrow$$

$$\forall \varepsilon > 0, \exists \delta > 0$$

$$\text{such that if } 0 < |x - c| < \delta,$$

$$\text{then} |f(x) - l| < \varepsilon$$

Limits - Rules

$$lim_{x \to c} b = b$$

$$lim_{x \to c} x = c$$

$$lim_{x \to c} x^n = c^n$$

Limits – Rules (continued)

Constants can be pulled out of limits

$$lim_{x\rightarrow c}(a)(f(x)) = (a)(l) \text{ when } a \in \mathbb{R} \text{ and } lim_{x\rightarrow c}f(x) = l$$

Limits – Rules (continued)

The limit of a sum is the sum of the limits

$$lim_{x \to c}\{f(x) + g(x)\} = lim_{x \to c} f(x) + lim_{x \to c} g(x)$$

Limits – Rules (continued)

The limit of a product is the product of the limits

$$lim_{x \to c}\{f(x) \times g(x)\} = lim_{x \to c}f(x) \times lim_{x \to c}g(x)$$

Limits – Rules (continued)

The limit of a quotient is the quotient of the limits, when divisor is not 0.

$$lim_{x\to c}\{f(x)\div g(x)\} = lim_{x\to c}f(x)\div lim_{x\to c}g(x)$$

$$lim_{x\to c}g(x) \neq 0$$

Derivatives - Definition

Derivatives are a measure of how a function changes with respect to its input.

For a real-valued function of a single real variable, the derivative at a point is the slope of the tangent line to the graph of the function at that point.

Derivatives - Rules

- When $p(x) = x^n$ and $n \neq 0$, $p^{'}(x) = (n)(x^{n-1})$.
- $\{(f(x))(g(x))\}^{'} = (f(x))(g^{'}(x)) + (f^{'}(x))(g(x))$
- $(\frac{f}{g})^{'}(x) = \frac{(g(x))(f^{'}(x)) - (f(x))(g^{'}(x))}{(g(x))^2}$
- $\frac{\partial}{\partial x}(f(g(x))) = f^{'}(g(x))g^{'}(x)$
- $\frac{\partial}{\partial x} ln(x) = \frac{1}{x}$

Derivatives - Rules

- $\frac{\partial}{\partial x}(e^x) = e^x$
- $\frac{\partial}{\partial x}(e^{f(x)}) = (e^{f(x)})(f^{'}(x))$
- $\frac{\partial}{\partial x}(p^x) = (p^x)(ln(p))$
- $\frac{\partial}{\partial x}(p^{g(x)}) = (p^{g(x)})(g^{'}(x))(ln(p))$
- $\frac{\partial}{\partial x}(log_p(x)) = \frac{1}{(x)(ln(p))}$
- $\frac{\partial}{\partial x}(log_p(g(x))) = \frac{g^{'}(x)}{(g(x))(ln(p))}$

L'Hopital's Rule

Assume $f(x)$ and $g(x)$ are both differentiable, with derivatives $f'(x)$ and $g'(x)$ respectively. Further, assume that $c \in \mathbb{R}$.

If $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{0}{0}$ or $\lim\limits_{x \to c} \frac{f(x)}{g(x)} = \frac{\pm\infty}{\pm\infty}$

and $\lim\limits_{x \to c} \frac{f\prime(x)}{g\prime(x)}$ exists,

then $\lim\limits_{x \to c} \frac{f(x)}{g(x)} = \lim\limits_{x \to c} \frac{f\prime(x)}{g\prime(x)}$

Permutations

- A K-Permutation is an ordered subsequence of k distinct elements of a set S.

- The number of k-permutations of a set S,

  with $|S| = n$ is:

  $$n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!}$$

Permutations - Example

- When $S = \{a, b, c\}$,

  the 2-permutations are $\{ab, ac, ba, bc, ca, cb\}$.

- The number of 2-permutations of S $(k = 2)$,

  with $|S| = n = 3$ is:

  $n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!}$

Permutations - Example

- When $S = \{a, b, c\}$,

  the 2-permutations are $\{ab, ac, ba, bc, ca, cb\}$.

- The number of 2-permutations of S,

  with $|S| = 3$ is:

$$3(3-1) = \frac{3!}{(3-2)!} = \frac{6}{1} = 6$$

Combinations

- A K-Combination is an un-ordered subsequence of k distinct elements of a set S.

- The number of k-combinations of a set S,

  with $|S| = n$ is:

$$\frac{n!}{(n-k)!k!}$$

Combinations - Example

- When $S = \{a, b, c\}$,

  the 2-combinations are $\{ab, ac, bc\}$.

- The number of 2-combinations of a set S $(k = 2)$,

  with $|S| = n = 3$ is:

$$\frac{n!}{(n-k)!k!}$$

Combinations - Example

- When $S = \{a, b, c\}$,

  the 2-combinations are $\{ab, ac, bc\}$.

- The number of 2-combinations of a set S $(k = 2)$,

  with $|S| = n = 3$ is:

$$\frac{3!}{(3-2)!\,(2!)} = \frac{6}{2} = 3$$

Combinations – Binomial Coefficients

- We use the notation $\binom{n}{k}$ (read: n choose k) to denote the number of k-combinations.

Combinations – Binomial Coefficients

- Some Binomial Coecient properties:
  - $\binom{n}{k} = \binom{n}{n-k}$
  - $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$
  - $\binom{n}{k} \geq \left(\frac{n}{k}\right)^k$
  - $\binom{n}{k} \leq \frac{(n^k)}{k!}$

Combinations – Binomial Coefficients

Binomial Coefficients can be used in binomial expansion. Binomial expansion is given by:

$$(x+a)^n = \sum_{k=0}^{n} \binom{n}{k} x^k a^{n-k}$$

In particular, when x = a = 1, we have:

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$