Spread of Information With Confirmation Bias in Cyber-Social Networks

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***Abstract*—This paper provides a model to investigate information spread over cyber-social network of agents. The cyber-social network considered here comprises individuals and information sources. Each individual holds an opinion represented by a scalar that evolves over time. The information sources are stubborn, in the sense that their opinions are time-invariant. Individuals receive the opinions of information sources that are closer to their belief, confirmation bias is explicitly incorporated into the model. The proposed dynamics of cyber-social networks is adopted from DeGroot-Friedkin model, where an individual's opinion update mechanism is a convex combination of her innate opinion, her neighbors' opinions at the previous time step (obtained from the social network), and the opinions passed along by information sources from cyber layer which she follows. The characteristics of the interdependent social and cyber networks are significantly different here: the social network relies on trust and hence static, while influences from information sources to individuals are highly dynamic since they are weighted as a function of the distance between an individual state and the state of information source to account for confirmation bias. The conditions for convergence of the aforementioned dynamics to a unique equilibrium point are characterized. The estimation and exact computation of the steady-state values under non-linear and linear state-dependent weight functions are provided. Finally, the impact of polarization in the opinions of information sources on the public opinion evolution is numerically analyzed in the context of the well-known Krackhardt's advice network.**

***Index Terms*—Information spreading dynamics, confirmation bias, polarization, cyber-social networks.**

I. Introduction

Individuals form belief on various social, political and economic issues based on various factors including (i) her innate opinion, also viewed as subconscious bias, which is based on inherent personal characteristics (e.g., socio-economic conditions in which the individual grew up and/or lives in) [2]; (ii) the information received from a group of her friends, coworkers, etc. [3]; and (iii) the information received from the information sources (e.g., news agencies) and thought leaders, see e.g., [4]–[6]. Typically, a social network (friends, neighbors, coworkers) is informationally *symmetric* and *static*: when people exchange information, they influence, and also get influenced by others, while the connectivity is based on a static (information sense) network that does not depend on each individual's belief.[1] However, the interaction between the public and information sources (or thought leaders) is informationally *asymmetric*, in the sense that information sources will not update their beliefs by the information they receive from the public. In other words, the information exchange is one directional: from information sources to the public. Moreover, the influence of an information source on an individual depends on the current state of the aforementioned individual and the information source [4], [7]. This well-known behavior, namely the "confirmation bias" that is prevalent in modern societies [8], makes this network highly *dynamic*: the influence of an information source on an individual depends on the distance between the states of the individual and the information source. This entire system is an example of a cyber-social network that comprises two interacting, interdependent networks with substantially different characteristics: an informationally symmetric, static social network and an asymmetric, dynamic cyber network.

In this paper, we study the dynamics of information spread on such cyber-social networks, with a particular focus on confirmation bias. Confirmation bias refers to a type of cognitive bias that involves favoring information which confirms previously existing beliefs or biases. It is well understood that machine learning algorithms that control the information on social media news feeds automatically utilize and foster this bias without the individual's permission, or even proper understanding, see e.g., [4], and the recent information articles at popular media outlets [9], [10]. Confirmation bias has recently gained revived interest due to its role in the spread of misinformation as it creates an ideal environment for misinformation to thrive in. We consider the mathematical model presented in this paper as a tractable building block in a comprehensive exploration of mathematical underpinnings of the misinformation spread in networks, see e.g., [11]–[13] for complementary studies from our group, as well as other recent work [6], [14]–[17].

We note that there exists a large body of literature on dynamics of opinion evolution. We summarize a few popular models in the following. The DeGroot model [3] is a model of

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1. In this paper, we use "belief", "opinion" and "state" interchangeably.

TABLE I
SOCIAL NETWORK MODELS

Ref.	Dynamics	Name
[3]	$x(t+1) = Wx(t)$	DeGroot model
[2]	$x(t+1) = Wx(t) + (\mathbf{1} - W)s$	DeGroot-Friedkin model
[21]	$\begin{cases} x(t+1) = \widehat{\mathcal{W}}(x(t))\,x(t) \\ [\widehat{\mathcal{W}}(x(t))]_{ij} = \begin{cases} > 0, \lvert x_i(t) - x_j(t)\rvert < \varepsilon \\ = 0, \lvert x_i(t) - x_j(t)\rvert \geq \varepsilon \end{cases} \end{cases}$	Hegselmann-Krause model

Note: $x(t)$ is the vector of individuals' evolving opinions at time t, s is the vector of individuals' innate opinions, ε is the bound of confidence.

the conformity behavior in a social network, i.e., every individual updates her belief as an average of her neighbors. The DeGroot-Friedkin model [2] further involves innate opinions that are known to play a critical role in the expressed opinions [6], [18]–[20]. The Hegselmann-Krause model [21] addresses confirmation bias by a model where an individual completely ignores the opinions that are "too far" from hers. The specific dynamics of each of these models is described in Table I.

The recent research focus has shifted to more realistic variations of such classical models in order to capture the subtle characteristics of actual social networks. For example, the studies [22], [23] allow the individuals to have self-appraisal mechanism through updating of individuals' self-confidence level after discussion of issues. Dhamal *et al.* [6] incorporate opponent stubborn agents into DeGroot-Friedkin model [2] to study competitive information spreading in social networks. DeGroot-Friedkin model [2] has been used in several problems in social networks, including de-biasing social wisdom in online social networks [18], the competitive propagation in social networks [6], the optimal opinion conformation [19], and many more. A variant of the DeGroot-Friedkin model [2] also appears as a game-theoretic best response in a specific potential game [24]. However, the information dynamics models in the prior work do not adequately capture the characteristics of the emerging cyber-social networks that typically involve multiple interdependent, interacting networks with different properties. The following features differentiate our model from the prior work:

1) We focus on the long-term behavior, where individuals' opinions converge to possibly different values, whereas the vast majority of the existing literature studies consensus (see e.g., [3], [5], [25]), which refers to the asymptotic convergence of all opinions to the same value. It is important to note that consensus, that is a global agreement among all individuals, is virtually non-existent in practice in social networks in the presence of polarized opinions, innate opinions or confirmation bias, see e.g., political elections. Hence, we believe that our model captures real world networks more accurately.
2) Individual's opinion update mechanism is a convex combination of her innate opinion, the opinions of her neighbors at the previous time step (social network), and the opinions passed along by information sources that she follows (influence from the cyber layer).
3) The weight of influence of individuals is fixed because social influence among individuals is based on "trust", which tends to vary little over a long period of time, while the weight of influence of information sources over individuals heavily depend on the current opinions of individuals, i.e., it is state-dependent, in order to capture the impact of "confirmation bias".

The contribution of this paper is threefold, which can be summarized as follows.

- Based on the well-known DeGroot-Friedkin model [2], we propose dynamics of cyber-social networks, where the weight of influence of individuals is fixed, while the weight of influence of information sources is state-dependent (to capture confirmation bias). Sufficient conditions for asymptotic convergence to a unique equilibrium point (at a geometric rate) are characterized.
- The estimation and computation of the equilibrium point of the proposed dynamics under nonlinear and linear, state-dependent weight functions are provided.
- The effects of the distribution of information source's opinion and the distance between polarized opinions of information sources are studied in the context of the well-known Krackhardt's advice network [26].

This paper is organized as follows. In Section II, we present the notation and the network model. In Section III, we analyze the convergence of dynamics, the estimation, and computation of the unique equilibrium. We provide numerical results in Section IV, and in Section V we present conclusions and future research directions.

II. NOTATION AND NETWORK MODEL

A. Notation

We let $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ denote the set of n-dimensional real vectors and the set of $m \times n$-dimensional real matrices, respectively. Given a vector $x \in \mathbb{R}^n$ and a matrix $A \in \mathbb{R}^{n\times m}$, inequalities $x \geq 0$ and $A \geq 0$ denote element-wise inequalities. $\mathbb{N}$ represents the set of the natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We define $\mathbf{1}$ and $\mathbf{0}$, respectively, as the identity and zero matrices with proper dimensions. Moreover, we let $\mathbf{1}_n \in \mathbb{R}^n$ and $\mathbf{0}_n \in \mathbb{R}^n$ denote the vector of all ones and all zeros, respectively. The superscript '$\top$' stands for the matrix transposition. A square matrix with non-negative entries is said to be sub-stochastic (or strictly sub-stochastic) if the entries of each row of the matrix sum up to one (or less than one). The sign function is denoted by $\mathrm{sgn}(\cdot)$.

The network considered in this paper is composed of n individuals and m information sources. The interaction among the individuals is modeled by a digraph $\mathfrak{G} = (\mathbb{V}, \mathbb{E})$, where $\mathbb{V} = \{\mathrm{v}_1, \ldots, \mathrm{v}_\mathrm{n}\}$ is the set of vertices representing the individuals and $\mathbb{E} \subseteq \mathbb{V} \times \mathbb{V}$ is the set of edges of the digraph $\mathfrak{G}$ representing the influence structure. Assume that the social network has no self-loops, i.e., for any $\mathrm{v_i} \in \mathbb{V}$, $(\mathrm{v_i}, \mathrm{v_i}) \notin \mathbb{E}$. The communication from information sources to individuals is modeled by a bipartite digraph $\mathfrak{H} = (\mathbb{V} \bigcup \mathbb{K}, \mathbb{B})$, where $\mathbb{K} = \{\mathrm{u}_1, \ldots, \mathrm{u}_\mathrm{m}\}$ is the set of vertices representing the information sources and $\mathbb{B} \subset \mathbb{V} \times \mathbb{K}$ is the set of edges of the digraph.

Some important notations are highlighted as follows:

$\mathbb{V}$ the set of individuals;
$\mathbb{K}$ the set of information sources;

$\lvert\cdot\rvert$	the modulus of a real number, and the cardinality (i.e., the size) of a set;
$\lVert\cdot\rVert$	the l_1-norm of a vector $x \in \mathbb{R}^n$, i.e, $\lVert x\rVert = \sum_{i=1}^n \lvert x_i\rvert$, and the induced norm of a matrix $A \in \mathbb{R}^{n\times n}$, i.e., $\lVert A\rVert = \sup\left\{\frac{\lVert Ax\rVert}{\lVert x\rVert} : x \in \mathbb{R}^n, x \neq \mathbf{0}_n\right\}$;
$\mathbb{W}(d)$	the d^{th} element of the ordered set $\mathbb{W}$;
$\mathbf{E}_{\mathrm{U}}(\cdot)$	the expectation operator over a uniform distribution U.

B. Cyber-Social Network Model

For convenience, we refer to the stubborn agents whose opinions are time-invariant as "information sources" in the cyber layer. Other agents, who update their opinion based on their neighbors and information sources, are simply referred to as "individuals". We consider the following model which is adopted from [2]:

$$x_i(t+1) = \alpha_i(x_i(t))s_i + \sum_{j\in\mathbb{V}} w_{ij}x_j(t) + \sum_{k\in\mathbb{K}} \hat{w}_{ik}(x_i(t))y_k, \tag{1}$$

where

(i) $x_i(t) \in [0,1]$ is individual $\mathrm{v_i}$'s opinion at time t, $s_i \in [0,1]$ is her fixed innate opinion, while $y_k \in [0,1]$ is the information source $\mathrm{u_k}$'s opinion;

(ii) w_{ij} represents the weighted influence of individual $\mathrm{v_j}$ on individual $\mathrm{v_i}$,

$$w_{ij} = \begin{cases} > 0, & \text{if } (\mathrm{v_i}, \mathrm{v_j}) \in \mathbb{E} \\ = 0, & \text{otherwise,} \end{cases} \tag{2}$$

and we note that w_{ij} does not depend on time index t;

(iii) $\hat{w}_{ik}(x_i(t))$ is the weighted influence of information source $\mathrm{u_k}$ on individual $\mathrm{v_i}$ with

$$\hat{w}_{ik}(x_i(t)) \triangleq \begin{cases} g_{ik}(\lvert x_i(t) - y_k\rvert), b_{ik} = 1 \\ 0, \qquad\qquad\qquad b_{ik} = 0, \end{cases} \tag{3}$$

where $b_{ik} = 1$ if information source $\mathrm{u_k}$ has influence on individual $\mathrm{v_i}$, and $b_{ik} = 0$ otherwise, $g_{ik}(\cdot) : \mathbb{R} \to \mathbb{R}$, is a strictly decreasing function, and it satisfies $1 > g_{ik}(\lvert x_i(t) - y_k\rvert) > 0, \forall i \in \mathbb{V}, \forall k \in \mathbb{K}, \forall t \in \mathbb{N}_0$.

(iv) $\alpha_i(x_i(t))$ is referred to as the "resistance parameter" of individual $\mathrm{v_i}$, is determined in such a way that it satisfies

$$\alpha_i(x_i(t)) + \sum_{j\in\mathbb{V}} w_{ij} + \sum_{k\in\mathbb{K}} \hat{w}_{ik}(x_i(t)) = 1, \forall i \in \mathbb{V}, \forall t. \tag{4}$$

We make the following assumptions throughout this paper.

Assumption 1. The weight function $g_{ik}(\cdot) : \mathbb{R} \to \mathbb{R}$ satisfies

$$\lvert g_{ik}(z_1) - g_{ik}(z_2)\rvert \leq \mu_i \lvert z_1 - z_2\rvert, \forall i \in \mathbb{V}, \forall k \in \mathbb{K}, \tag{5}$$

for some fixed $\mu_i \in \mathbb{R}$.

We note that the resistance parameter $\alpha_i(x_i(t))$, which is state-dependent, is obtained from the relation (4). To guarantee that resistance parameters remain positive at all times, we need to impose an upper bound on μ_i, the contraction parameter associated with functions $g_{ik}(\cdot)$, $i \in \mathbb{V}, k \in \mathbb{K}$.

Assumption 2. Given $W = [w_{ij}] \in \mathbb{R}^{n\times n}$, we have

$$\lVert W\rVert + \max_{i\in\mathbb{V}}\left\{\mu_i \sum_{k\in\mathbb{K}} b_{ik}\right\} + \max_{i\in\mathbb{V}}\left\{s_i\mu_i \sum_{k\in\mathbb{K}} b_{ik}\right\} < 1. \tag{6}$$

Remark 1. Assumption 2 is stronger that what is minimally required to guarantee that the resistance parameters are positive. More precisely, to ensure the positivity of the resistance parameters, it would have been sufficient to have $\lVert W\rVert + \max_{i\in\mathbb{V}}\{\mu_i \sum_{k\in\mathbb{K}} b_{ik}\} < 1$. However, as we demonstrate later in the text, Assumption 2, in addition to rendering the resistance parameters positive, also guarantees that the dynamics (1) converges, and hence is used throughout the paper.

Remark 2 (Motivation of Weight of Influence). We assume that influence weights w_{ij}, $\forall i, j \in \mathbb{V}$, are fixed since the social influence among individuals is based on "trust," which tends to vary little over a long period of time. However, the influences of information sources over individuals depend heavily on the current opinions of individuals, due to the confirmation bias. That is why the weight of influence of information source on individual, i.e., $\hat{w}_{ik}(x_i(t))$ defined as (3), is state-dependent.

Remark 3. Equation (4) implies that the coefficients, including influence weights of individuals w_{ij}, state-dependent influence weights $\hat{w}_{ik}(x_i(t))$, and individuals' state-dependent resistance parameters $\alpha_i(x_i(t))$, are all normalized. This is a standard practice in modeling opinion evolution (see e.g., [6]) to make the dynamics invariant under translation. We also note that assuming innate opinions, initial conditions $(x_i(0), i \in \mathbb{V})$, and opinions of information sources all belong to the interval $[0,1]$, any convex combination of them also lies within that interval. This, in conjunction with (4) and (1), yields that $x_i(t)$, $\forall i \in \mathbb{V}$, will automatically lie within the interval $[0,1]$ at all future times.

Remark 4. The Lipschitz condition, i.e., Assumption 1, is a mild technical assumption widely used in control theory such as stabilization [27] and observation [28] of nonlinear systems. It is well understood that many nonlinearities can be regarded as, at least locally, Lipschitz to facilitate analysis. The following examples illustrate this point.

Example 1. Consider the function $g_{ik}(\cdot) : \mathbb{R} \to \mathbb{R}$,

$$g_{ik}(\lvert x_i - y_k\rvert) = \mu_i \ln(2 - \lvert y_k - x_i\rvert),$$

where $\mu_i \geq 0$. Without loss of generality, we let $\lvert y_k - x_i\rvert \leq \lvert y_k - x_i'\rvert$. It follows from $z \geq \ln(1+z)$ for $z > 0$, and the facts of $\lvert y_k - x_i\rvert \leq 1$ and $\lvert y_k - x_i'\rvert \leq 1$ that

$$\begin{aligned} &\left\lvert g_{ik}(\lvert x_i - y_k\rvert) - g_{ik}\left(\lvert x_i' - y_k\rvert\right)\right\rvert \\ &= \mu_i \log\left(1 + \frac{\lvert y_k - x_i'\rvert - \lvert y_k - x_i\rvert}{2 - \lvert y_k - x_i'\rvert}\right) \\ &\leq \mu_i \frac{\lvert y_k - x_i'\rvert - \lvert y_k - x_i\rvert}{2 - \lvert y_k - x_i'\rvert} \leq \mu_i \lvert x_i - x_i'\rvert. \end{aligned} \tag{7}$$

Example 2. Consider the function $g_{ik}(\cdot) : \mathbb{R} \to \mathbb{R}$,

$$g_{ik}(|x_i - y_k|) = \beta_i - \mu_i \sin(|y_k - x_i|),$$

where $\mu_i \geq 0$. Considering $z \geq \sin(z)$ for $z \geq 0$, and the well known trigonometric identity $\sin\varpi - \sin\tau = 2\cos\frac{\varpi+\tau}{2}\sin\frac{\varpi-\tau}{2}$, we have

$$\begin{aligned} &\left|g_{ik}(|x_i - y_k|) - g_{ik}\left(\left|x_i' - y_k\right|\right)\right| \\ &= \left|\mu_i\left(\sin|y_k - x_i| - \sin\left|y_k - x_i'\right|\right)\right| \\ &\leq 2\mu_i\left|\sin\frac{|y_k - x_i| - \left|y_k - x_i'\right|}{2}\right| \\ &\leq \mu_i\left||y_k - x_i| - \left|y_k - x_i'\right|\right| \leq \mu_i|x_i - x_i'|. \end{aligned} \tag{8}$$

Remark 5 (Comparison). It is well known that DeGroot model [3] has consensus-seeking dynamics. However, in many realistic social scenarios, e.g., political elections, an individual has neither a desire nor tendency for consensus or agreement. This has motivates researchers to study the disagreement beyond consensus in social networks. DeGroot-Friedkin model [2] considers the disagreement along with consensus. Perhaps, most closest prior work to our model is the Hegselmann-Krause model [21] which also studies disagreement in social networks. However, an individual in this model completely ignores the opinions that are "too far" from hers which introduces a discontinuity in the system. This non-tractable discontinuity manifests itself in the convergence analysis of this model: the equilibrium point can only be obtained by running the dynamics over the network (as opposed to estimating or computing the equilibrium, which requires substantially lower computational complexity). Moreover, the Hegselmann-Krause model does not consider the critical role of individuals' innate opinions [6], [14]–[17], [29]. The model we propose here incorporates a more tractable form of confirmation bias as well as innate opinions which are completely ignored in [21]. Our model generalizes both [3] and [2] as detailed in the following:

- In the absence of information sources, the proposed model (1) simplifies to the one in [2].
- In the absence of information sources and individuals' innate opinions, the proposed model (1) simplifies to the one in [3], with a subtle, minor difference in the choice of the influence matrices.

III. Convergence and Equilibrium Analysis

A. Convergence Analysis

In this section, we investigate the equilibrium point of the social dynamics (1) and carry out its convergence analysis. We first rewrite (1) in the following matrix form:

$$x(t+1) = \mathcal{A}(x(t))s + Wx(t) + \widehat{\mathcal{W}}(x(t))y, \tag{9}$$

where we define the following variables:

$$x(t) \triangleq [x_1(t), x_2(t), \ldots, x_n(t)]^\top \in \mathbb{R}^n, \tag{10}$$

$$s \triangleq [s_1, s_2, \ldots, s_n]^\top \in \mathbb{R}^n, \tag{11}$$

$$y \triangleq [y_1, y_2, \ldots, y_m]^\top \in \mathbb{R}^m, \tag{12}$$

$$\mathcal{A}(x(t)) \triangleq \mathrm{diag}\{\alpha_1(x_1(t)), \ldots, \alpha_n(x_n(t))\} \in \mathbb{R}^{n\times n}, \tag{13}$$

$$W \triangleq \begin{bmatrix} w_{11} & \ldots & w_{1n} \\ w_{21} & \ldots & w_{2n} \\ \vdots & \vdots & \vdots \\ w_{n1} & \ldots & w_{nn} \end{bmatrix} \in \mathbb{R}^{n\times n}, \tag{14}$$

$$\widehat{\mathcal{W}}(x(t)) \triangleq \begin{bmatrix} \hat{w}_{11}(x_1(t)) & \ldots & \hat{w}_{1m}(x_1(t)) \\ \vdots & \vdots & \vdots \\ \hat{w}_{n1}(x_n(t)) & \ldots & \hat{w}_{nm}(x_n(t)) \end{bmatrix} \in \mathbb{R}^{n\times m}. \tag{15}$$

In the following theorem, whose proof is given in Appendix A, we present the existence of a unique equilibrium point for the dynamics (1).

Theorem 1. The dynamics (1) converges to a unique equilibrium point that is independent of initial opinions.

B. Equilibrium Analysis

We use x^{ue} and x^{ie} respectively to denote the uninfluenced equilibrium point (the equilibrium point of the system in the absence of information sources) and the influenced equilibrium point (the equilibrium point in the presence of information sources).

We first focus on the analysis of the system without information sources. From (9), the dynamics of the system (in the absence of information sources) is expressed as

$$\bar{x}(t+1) = As + W\bar{x}(t), \tag{16}$$

where

$$A \triangleq \mathrm{diag}\left\{1 - \sum_{j\in\mathbb{V}} w_{1j}, \ldots, 1 - \sum_{j\in\mathbb{V}} w_{nj}\right\}. \tag{17}$$

Corollary 1. The social dynamics (16) converges to a unique equilibrium point:

$$x^{\mathrm{ue}} = (\mathbf{1} - W)^{-1}As. \tag{18}$$

Remark 6. Equation (18) appears almost identically in [18], with a subtle difference: the weight matrix W considered therein is a *strictly* sub-stochastic matrix, while W defined in (14) is a sub-stochastic matrix, under which the dynamics (16) still converges to a unique equilibrium point. Its brief proof is sketched as follows.

Proof of Corollary 1. We note that the dynamics (16) is a special case of the dynamics (9) without influences from information sources, i.e, $\hat{w}_{ik}(x_i(t)) = 0$ for $\forall i \in \mathbb{V}$, $\forall k \in \mathbb{K}$, $\forall t \in \mathbb{N}_0$. By Theorem 1, the dynamics (9) converges to a unique equilibrium point x^{ue}. Thus, at the steady state, we have $x^{\mathrm{ue}} = As + Wx^{\mathrm{ue}}$, from which, (18) follows straightforwardly. ■

We next present our result pertaining to the setting with information sources.

Corollary 2. The unique equilibrium, x^{ie}, satisfies

$$x^{\mathrm{ie}} = (\mathbf{1} - W)^{-1}\left(\mathcal{A}(x^{\mathrm{ie}})s + \widehat{\mathcal{W}}(x^{\mathrm{ie}})y\right). \tag{19}$$

Proof. It follows from (9) that at steady-state:

$$x^{\text{ie}} = \mathcal{A}(x^{\text{ie}})s + Wx^{\text{ie}} + \widehat{\mathcal{W}}(x^{\text{ie}})y. \tag{20}$$

The definition of W in (14) with the condition (6) show that W is a sub-stochastic matrix. It is well-known that $\mathbf{1} - W$ is invertible if W is a sub-stochastic matrix, thus (19) is obtained straightforwardly from (20). ∎

We next present an auxiliary relationship between x^{ue} and x^{ie}, that will be useful in the derivation of upper and lower bounds for x^{ie}. Here, we define a diagonal matrix of innate opinions:

$$S \triangleq \text{diag}\{s_1, s_2, \ldots, s_n\} \in \mathbb{R}^{n\times n}. \tag{21}$$

Next, we have the following corollary whose proof is in Appendix B.

Corollary 3. The uninfluenced equilibrium point x^{ue} in (18) and influenced equilibrium point x^{ie} in (19) satisfy

$$x^{\text{ie}} - x^{\text{ue}} = (\mathbf{1} - W)^{-1}\big(\widehat{\mathcal{W}}(x^{\text{ie}})y - S\widehat{\mathcal{W}}(x^{\text{ie}})\mathbf{1}_m\big). \tag{22}$$

1) Nonlinear Weight Functions–Estimation: We first focus on estimating the equilibrium point without requiring the full knowledge of the weight functions. Assuming the upper and lower bounds for these functions are known, we build on Corollary 3 to derive bounds for the equilibrium point.

We note that Remark 3 implies that each weight function $\hat{w}_{ik}(x_i(t))$ is bounded, i.e.,

$$0 \le \underline{\phi_i} \le \hat{w}_{ik}(x_i(t)) \le \overline{\phi_i} \le 1, \forall t \in \mathbb{N}_0, \forall i \in \mathbb{V}, \forall k \in \mathbb{K}, \tag{23}$$

and we assume $\underline{\phi_i}$ and $\overline{\phi_i}$ are known.

We then define:

$$\underline{\psi} \triangleq (\mathbf{1}-W)^{-1} \max\{\underline{\Phi}By - S\overline{\Phi}\varrho + As, \mathbf{0}_n\}, \tag{24}$$

$$\overline{\psi} \triangleq \min\Big\{(\mathbf{1} - W)^{-1} \min\{\overline{\Phi}By - S\underline{\Phi}\varrho + As, \mathbf{1}_n\}, \mathbf{1}_n\Big\}, \tag{25}$$

with

$$\underline{\Phi} \triangleq \text{diag}\Big\{\underline{\phi_1}, \underline{\phi_2}, \ldots, \underline{\phi_n}\Big\} \in \mathbb{R}^{n\times n}, \tag{26}$$

$$\overline{\Phi} \triangleq \text{diag}\{\overline{\phi_1}, \overline{\phi_2}, \ldots, \overline{\phi_n}\} \in \mathbb{R}^{n\times n}, \tag{27}$$

$$B \triangleq \begin{bmatrix} b_{11} & \ldots & b_{1m} \\ \vdots & \vdots & \vdots \\ b_{n1} & \ldots & b_{nm} \end{bmatrix} \in \mathbb{R}^{n\times m}, \tag{28}$$

$$\varrho \triangleq \left[\sum_{k\in\mathbb{K}} b_{1k}, \sum_{k\in\mathbb{K}} b_{2k}, \ldots, \sum_{k\in\mathbb{K}} b_{nk}\right]^{\top} \in \mathbb{R}^n. \tag{29}$$

We finally present upper and lower bounds of equilibrium point under general (including nonlinear and linear) weight functions in the following theorem, whose proof is given in Appendix C.

Theorem 2. The dynamics (1) converges to x^{ie} that satisfies

$$\underline{\psi} \le x^{\text{ie}} \le \overline{\psi}. \tag{30}$$

Remark 7. If (30) is satisfied with equality for the i-th entry, i.e, $\underline{\psi_i} = \overline{\psi_i}$, then the estimation of individual $\mathrm{v_i}$'s opinion at steady state is exact: $x_i^{\text{ie}} = \underline{\psi_i} = \overline{\psi_i}$. The example in Section VI-A demonstrates this point.

Remark 8. We note that $\mathbf{1} - W$ is a nonsingular M-matrix, which includes the topology information of friendship network. The properties of inverse M-matrices [30] imply that the estimation performance can be improved further by modifying the topology of friendship network through link recommendation. We leave this analysis for future work since it is beyond the scope of this paper.

2) Linear Weight Functions– Computation: We next focus on computation of the equilibrium for known, linear weight functions that are in the form of:

$$\hat{w}_{ik}(x_i^{\text{ie}}) = \big(\beta_i - \gamma_i(x_i^{\text{ie}} - y_k)\text{sgn}(x_i^{\text{ie}} - y_k)\big)b_{ik}, \tag{31}$$

$$1 > \beta_i \ge \gamma_i \ge 0. \tag{31}$$

Except special cases (see Section III-B3), it is difficult to analytically compute the equilibrium, even for the case of known, linear weight functions. However, in the following, we develop an algorithm that yields the equilibrium point. Towards this goal, we first note that $\widehat{\mathcal{W}}(x^{\text{ie}})$ in (15) can be rewritten, in terms of the linear weight functions as

$$\widehat{\mathcal{W}}(x^{\text{ie}}) = \Xi B + \Upsilon\widetilde{\mathcal{W}}(x^{\text{ie}}, y)Y - \Upsilon X\widetilde{\mathcal{W}}(x^{\text{ie}}, y), \tag{32}$$

where

$$\Xi \triangleq \text{diag}\{\beta_1, \beta_2, \ldots, \beta_n\}, \tag{33}$$

$$\Upsilon \triangleq \text{diag}\{\gamma_1, \gamma_2, \ldots, \gamma_n\}, \tag{34}$$

$$X \triangleq \text{diag}\{x_1^{\text{ie}}, x_2^{\text{ie}}, \ldots, x_n^{\text{ie}}\}, \tag{35}$$

$$Y \triangleq \text{diag}\{y_1, \ldots, y_m\}, \tag{36}$$

$$\widetilde{\mathcal{W}}(x^{\text{ie}}, y) \triangleq \begin{bmatrix} \widetilde{w}_{11}(x_1^{\text{ie}}, y_1) & \ldots & \widetilde{w}_{1m}(x_1^{\text{ie}}, y_m) \\ \vdots & \vdots & \vdots \\ \widetilde{w}_{n1}(x_n^{\text{ie}}, y_1) & \ldots & \widetilde{w}_{nm}(x_n^{\text{ie}}, y_m) \end{bmatrix}, \tag{37}$$

$$\widetilde{w}_{ik}(x_i^{\text{ie}}, y_k) \triangleq b_{ik}sgn\big(x_i^{\text{ie}} - y_k\big), i \in \mathbb{V}, k \in \mathbb{K}. \tag{38}$$

Here, the key idea in devising an algorithm is the following: although we are searching for the equilibrium point in [0,1] interval, which constitutes an infinite-dimensional optimization problem, (38) implies only a finite number of possibilities for $\widetilde{\mathcal{W}}(x^{\text{ie}}, y)$. Hence, one can go through all of these possbilities and check whether the equilibrium conditions are satisfied.

As an example, we consider the network of Fig. 1 with three individuals and two information sources. We observe from (38) and Fig. 1 that

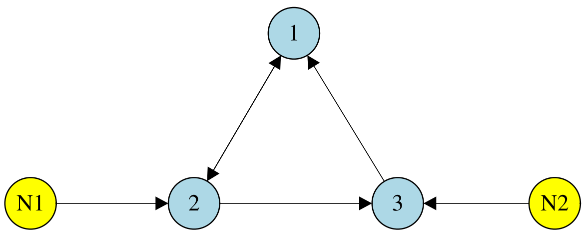

Fig. 1. A social network with three individuals and two information sources.

Algorithm 1: Computation of Equilibrium Point

Input: Ordered set $\mathbb{P}$ defined in (39) with its elements generated by (40), matrix $\Theta(d)$ defined in (43) with its entries given by (42), vector $\xi(d)$ given in (41), influence matrix in W (14), matrix A in (17), initial index $d=1$.

while $d \leq \|\mathbb{P}\|$ **do**
  Compute: $x^{\mathrm{ie}} \leftarrow (\mathbf{1}-W-\Theta(d))^{-1}(\xi(d)+As)$;
  if $b_{ik}\mathrm{sgn}(x_i^{ie}-y_k)=\widetilde{w}_{d,ik}$, for $\forall i\in\mathbb{V}$, $\forall k\in\mathbb{K}$, **then**
    Output equilibrium point: $x^{\mathrm{ie}} \leftarrow x^{\mathrm{ie}}$;
    Break;
  else
    Update index: $d \leftarrow d+1$.
  end
end

(1) each $\widetilde{w}_{ik}(x_i^{\mathrm{ie}}, y_k)$ has three possible values as: b_{ik}, 0, $-b_{ik}$;

(2) information sources N_1 and N_2 have two followers: individual 2 and individual 3.

Hence, we observe that $\widetilde{\mathcal{W}}(x^{\mathrm{ie}}, y)$ of the social network in Fig. 1 has $3^{1+1}=9$ possibilities.

Generalizing the above example, the total number of possibilities for $\widetilde{\mathcal{W}}(x^{\mathrm{ie}}, y)$ is $3^{\sum_{k\in\mathbb{K}}\mathrm{N}_k}$, where N_k is the number of followers of information source k, i.e., $\mathrm{N}_k=\sum_{i\in\mathbb{V}} b_{ik}$. To reduce the number of possibilities, we consider estimation of equilibrium point in (30) for $\widetilde{\mathcal{W}}(x^{\mathrm{ie}}, y)$, where we define an ordered set of the possibilities for $\widetilde{\mathcal{W}}(x^{\mathrm{ie}}, y)$ as follows.

$$\mathbb{P} \triangleq \left\{\widetilde{W}_1, \widetilde{W}_2, \widetilde{W}_3, \ldots, \widetilde{W}_{|\mathbb{P}|}\right\}, \tag{39}$$

and the entries of $\widetilde{W}_d \triangleq \left[\widetilde{w}_{d,ik}\right] \in \mathbb{R}^{n\times m}$, $d=1,2,\ldots,|\mathbb{P}|$, satisfy:

$$\widetilde{w}_{d,ik} = \begin{cases} -b_{ik}, & \overline{\underline{\psi}}_i < y_k \\ b_{ik}, & \underline{\underline{\psi}}_i > y_k \\ -b_{ik}, 0, & \overline{\overline{\psi}}_i = y_k \\ b_{ik}, 0, & \underline{\underline{\psi}}_i = y_k \\ -b_{ik}, b_{ik}, 0, & \text{otherwise}. \end{cases} \tag{40}$$

Considering the ordered set in (39), let us define:

$$\xi(d) \triangleq (\Xi B + \Upsilon\mathbb{P}(d)Y)y - S(\Xi B + \Upsilon\mathbb{P}(d)Y)\mathbf{1}_m, \tag{41}$$

$$\widehat{\xi}(d) \triangleq \Upsilon S\mathbb{P}(d)\mathbf{1}_m - \Upsilon\mathbb{P}(d)y, \tag{42}$$

$$\Theta(d) \triangleq \mathrm{diag}\left\{\left[\widehat{\xi}(d)\right]_1, \ldots, \left[\widehat{\xi}(d)\right]_n\right\}. \tag{43}$$

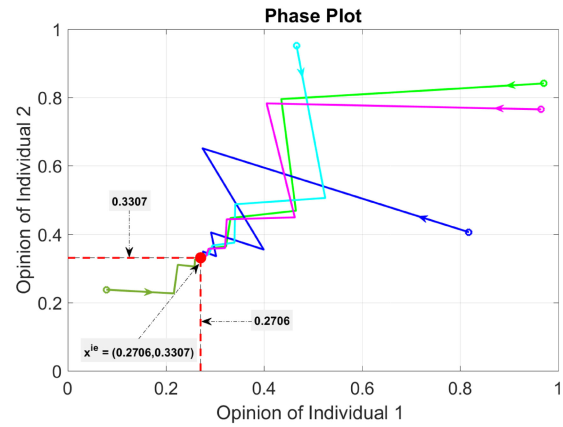

Fig. 2. Computation of equilibrium point by Algorithm 1 and phase plot of opinions under five random initial opinions (the circles are the locations of initial opinions, the arrows denote the evolving directions).

In the following, we present an algorithm that computes x^{ie} for linear weight functions.

The following theorem, whose proof is provided in Appendix D, states that the equilibrium point is achieved as an outcome of Algorithm 1.

Theorem 3. Algorithm 1 finds the unique equilibrium point (19) for the social dynamics (1), with linear state-dependent weight functions (31).

We next demonstrate the effectiveness of Algorithm 1 in the following numerical example.

Example 3. Consider a network of two individuals and one information source. Its detailed dynamics can be described as

$$\begin{aligned} x_1(t+1) &= (1-0.4-0.3(1-|x_1(t)-y_1|))s_1 \\ &\quad + 0.4x_2(t) + 0.3(1-|x_1(t)-y_1|)y_1, \\ x_2(t+1) &= (1-0.6-0.4(1-|x_2(t)-y_1|))s_2 \\ &\quad + 0.6x_1(t) + 0.4(1-|x_2(t)-y_1|)y_1. \end{aligned}$$

We set the individuals' innate opinions as $[s_1, s_2]=[0.1, 0.7]$, and the opinion of the information source $y_1=0.4$. The equilibrium point calculated by Algorithm 1 is $x^{\mathrm{ie}}=[0.2706, 0.3307]^\top$. The equilibrium point (computed via Algorithm 1) and the phase plot under five random initial opinions are shown in Fig. 2.

3) Extremal Opinion: We next focus on one particular practically important special case for which we can analytically compute the equilibrium point. We first define the following.

Definition 1 (Extremal Opinion). The opinion $\tilde{y}$ passed along by the only one information source is extremal if

$$1 \geq \tilde{y} \geq \max_{i\in\mathbb{V}}\{s_i\}, \tag{44}$$

or

$$0 \leq \tilde{y} \leq \min_{i\in\mathbb{V}}\{s_i\}. \tag{45}$$

We next define:

$$\overline{\vartheta} \triangleq (\Xi\zeta - \Upsilon\zeta\tilde{y})\tilde{y} - S(\Xi\zeta - \Upsilon\zeta\tilde{y}), \tag{46}$$

$$\underline{\vartheta} \triangleq (\Xi\zeta + \Upsilon\zeta\tilde{y})\tilde{y} - S(\Xi\zeta + \Upsilon\zeta\tilde{y}), \tag{47}$$

$$\Phi \triangleq \mathrm{diag}\big\{[\Upsilon(\zeta\tilde{y} - S\zeta)]_1, \ldots, [\Upsilon(\zeta\tilde{y} - S\zeta)]_n\big\}, \tag{48}$$

$$\zeta \triangleq [b_{11}, \ldots, b_{n1}]^\top. \tag{49}$$

The solutions of equilibrium point under (44) and (45) are presented in the following corollary, whose proof is given in Appendix E.

Corollary 4. Consider the social dynamics (1), with the linear state-dependent weight functions (31). In the case of only one information source who passes along the extreme opinion $\tilde{y}$,

- for condition (44):

$$x^{\mathrm{ie}} = (\mathbf{1} - W - \Phi)^{-1}(\overline{\vartheta} + As), \tag{50}$$

- for condition (45):

$$x^{\mathrm{ie}} = (\mathbf{1} - W + \Phi)^{-1}(\underline{\vartheta} + As). \tag{51}$$

Remark 9. It follows from (50) and (18) that

$$\begin{aligned} x^{\mathrm{ie}} - x^{\mathrm{ue}} &= (\mathbf{1} - W - \Phi)^{-1}(\overline{\vartheta} + As) - (\mathbf{1} - W)^{-1}As \\ &= \sum_{t=0}^{\infty}(W + \Phi)^t(\overline{\vartheta} + As) - \sum_{t=0}^{\infty} W^t As \\ &\geq \sum_{t=0}^{\infty} W^t(\overline{\vartheta} + As) - \sum_{t=0}^{\infty} W^t As \\ &= \sum_{t=0}^{\infty} W^t \overline{\vartheta} = (\mathbf{1} - W)^{-1}\overline{\vartheta}, \end{aligned}$$

which implies that if the weight matrix W is irreducible, the information source can drive the steady-state opinion of every individual arbitrarily away from the uninfluenced equilibrium point.

IV. Numerical Results

In this section, we first provide a numerical example to demonstrate the theoretical estimation of opinion evolution in the case of nonlinear weight functions. Then, we investigate the impact of information sources on the belief evolution.

A. Estimation Performance

We consider the cyber-social network in Fig. 3 with thirteen individuals and two information sources, where the social network consists of two disconnected communities. The innate opinions are randomly generated as $s = [0.9572, 0.4854, 0.8003, 0.1419, 0.4218, 0.9157, 0.7922, 0.9595, 0.6557, 0.0357, 0.8491, 0.9340, 0.8147]^\top$. The opinions of information sources N1 and N2 are $y = [0.1, 0.8]^\top$. The influence weights of friendship network in Fig. 3 are set as $w_{18} = w_{21} = w_{32} = w_{43} = \frac{1}{2}$, $w_{54} = w_{65} = w_{76} = w_{87} = \frac{1}{3}$, $w_{9(10)} = w_{9(13)} = \frac{1}{2}$, $w_{(10)(11)} = \frac{2}{5}$, $w_{(11)(12)} = \frac{1}{2}$, $w_{(12)(9)} = \frac{1}{10}$, $w_{(13)(9)} = \frac{7}{10}$, $w_{(13)(10)} = \frac{3}{10}$. The state-dependent weight functions of individuals v_1, v_8, v_{10} and v_{11} that are influenced by new sources N1 and N2 are set as

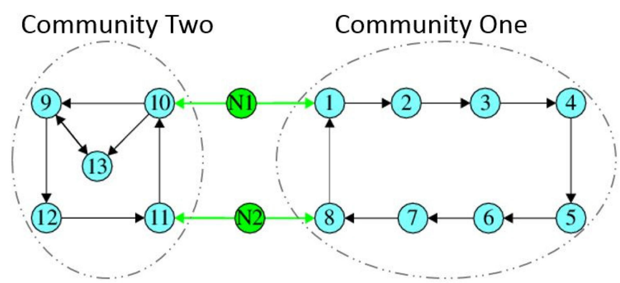

Fig. 3. Two disconnected social communities with twelve individuals and two information sources N1 and N2.

$$\hat{w}_{1(\mathrm{N}_1)}(x_1(t)) = 0.4 \ln\big(2 - |y_{\mathrm{N}_1} - x_1(t)|\big), \tag{52a}$$

$$\hat{w}_{8(\mathrm{N}_2)}(x_8(t)) = 0.4\,(1 - \sin|y_{\mathrm{N}_2} - x_8(t)|), \tag{52b}$$

$$\hat{w}_{(10)(\mathrm{N}_1)}(x_{10}(t)) = 0.4 \ln\big(2 - |y_{\mathrm{N}_1} - x_{10}(t)|\big), \tag{52c}$$

$$\hat{w}_{(11)(\mathrm{N}_2)}(x_{11}(t)) = 0.4\,(1 - \sin|y_{\mathrm{N}_2} - x_{11}(t)|), \tag{52d}$$

thus, the state-dependent weight $\widehat{\mathcal{W}}(x(t))$ defined in (15) under this setting is

$$\widehat{\mathcal{W}}(x(t)) = \left[\begin{array}{c} [\hat{w}_{1(\mathrm{N}_1)}(x_1(t)), 0] \\ \hdashline \mathbf{0}_{6\times 2} \\ \hdashline [0, \hat{w}_{8(\mathrm{N}_2)}(x_8(t))] \\ \hdashline \mathbf{0}_{1\times 2} \\ \hdashline [\hat{w}_{(10)(\mathrm{N}_1)}(x_{10}(t)), 0] \\ \hdashline [0, \hat{w}_{(11)(\mathrm{N}_2)}(x_{11}(t))] \\ \hdashline \mathbf{0}_{2\times 2} \end{array}\right].$$

The nonlinear weighted influence functions in (52) satisfy Assumptions 1 and 2 with parameters $\mu_1 = \mu_8 = \mu_{10} = \mu_{11} = 0.4$. Hence, the social network converges to a unique equilibrium, whose bounds are computed via Theorem 2 as:

$$\begin{aligned} \underline{\psi} = [&0.4961, 0.4907, 0.6455, 0.3937, 0.4124, 0.7479, 0.7774, \\ &0.5657, 0.2387, 0.2387, 0.5679, 0.8645, 0.2387]^\top, \\ \overline{\psi} = [&1.0000, 0.7856, 0.7929, 0.4674, 0.4370, 0.7561, 0.7802, \\ &1.0000, 0.5036, 0.5036, 1.0000, 0.8910, 0.5036]^\top. \end{aligned}$$

The unique equilibrium point, as well as the lower and upper estimation bounds are shown in Fig. 4. One intriguing observation from this numerical example is the following: the estimation errors of expressed opinions of information sources' immediate followers (shown in green dashed line in Fig. 4) are the largest. Hence, this numerical example suggests that we expect more accurate estimation in networks where there is a smaller number information sources (sparse cyber network) and many individuals interacting with each other (dense social network).

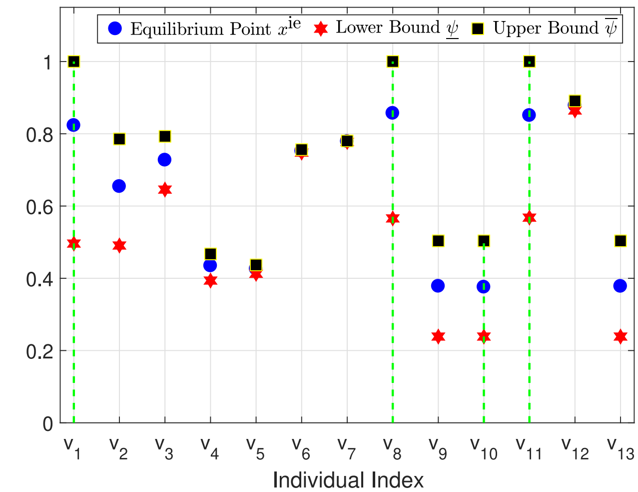

Fig. 4. Equilibrium point and its estimation bounds.

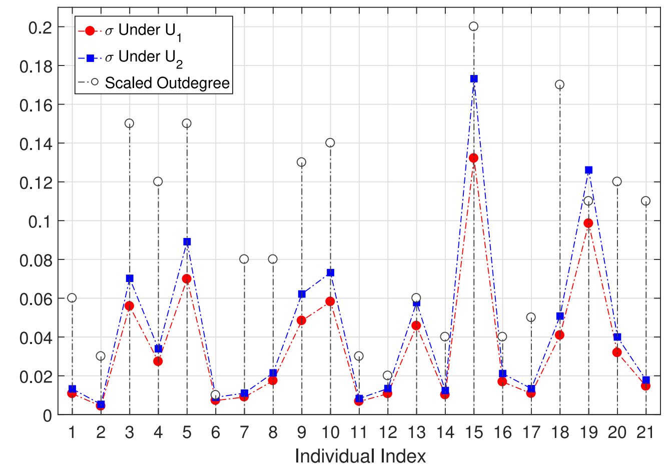

Fig. 6. Standard sample deviation under different uniform distributions, and scaled (by 1/100) outdegree distribution of 21 individuals.

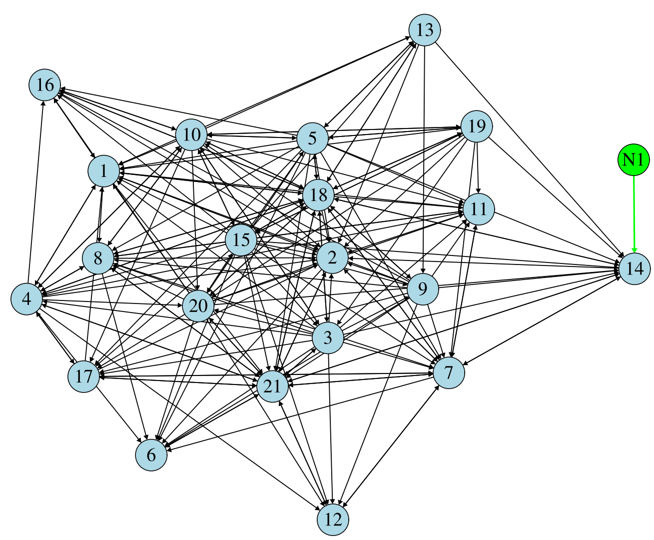

Fig. 5. Krackhardt's advice network [26] in the presence of information source N1.

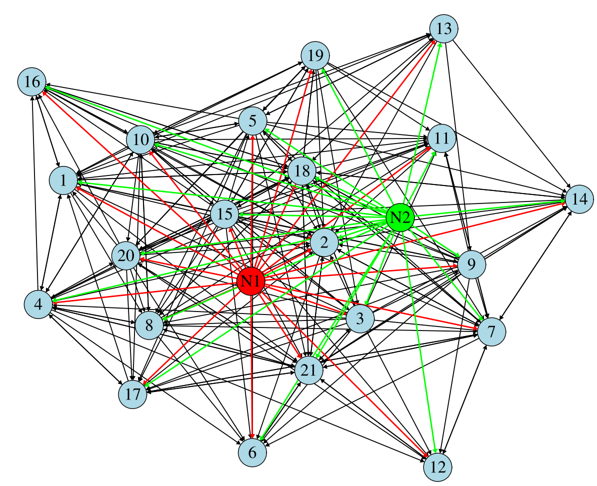

Fig. 7. Krackhardt's advice network [26] in the presence of two information sources N_1 and N_2.

B. Impact of Information Sources

We next study the impact of information sources' opinions on the evolution of overall belief over the network. Here, we use the well-known Krackhardt's advice network [26] for demonstration purposes. The communication topology of this network is shown in Fig. 5.

We plot the outdegree distribution of the 21 individuals in Fig. 6 using the source data available in [31]. Here, we consider 21 different topologies where for i-th topology, the information source N1 is connected the the i-th individual only, the remaining part (social layer) is kept unaltered.

We take the weight matrix W in (14) as follows: if individual i is connected to her neighbor j, then $w_{ij} = \frac{1}{1+\Gamma_i}$ for all the individuals j that influence individual i, where Γ_i is individual i's indegree. When an individual i is influenced by information source N1, we take her state-dependent weigh function as $\hat{w}_{i1}(x_i^{\mathrm{ie}}) = \frac{1}{1+\Gamma_i} - \frac{1}{2(1+\Gamma_i)}|x_i^{\mathrm{ie}} - y_1|$.

In the following simulations, all of the 21 individuals' innate opinions follow the uniform distribution over $[0, 1]$, i.e, $s_i \sim \mathrm{U}(0, 1)$, $\forall i \in \mathbb{V}$, and they are taken as statistically independent.

1) One Information Source Case: To isolate the impact of one information source over the options, we consider a network with one information source, denoted by N1 in Fig. 5.

The effect of the presence of information source N1 is measured by the sample deviation:

$$\sigma^2 \triangleq \frac{1}{n}\sum_{i=1}^{n} \mathbf{E}_{\mathrm{U}}\left(\left(x_i^{\mathrm{ue}} - x_i^{\mathrm{ie}}\right)^2\right). \tag{53}$$

For the opinion of information source N1, we consider two cases of uniform distributions:

$$f_{\mathrm{U}_1}(h) \triangleq \begin{cases} \frac{1}{0.8}, & h \in [0.1, 0.9] \\ 0, & \text{otherwise}, \end{cases} \tag{54}$$

$$f_{\mathrm{U}_2}(h) \triangleq \begin{cases} \frac{1}{0.4}, & h \in [0, 0.2] \cup [0.8, 1] \\ 0, & \text{otherwise}, \end{cases} \tag{55}$$

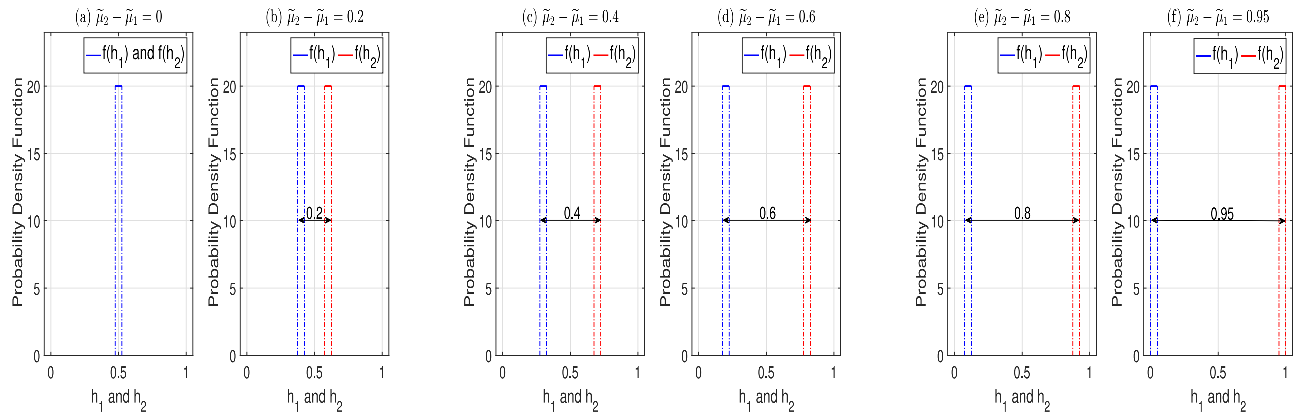

Fig. 8. Six uniform distributions of polar opinions: mean of each distribution is 0.5.

where $f_{\mathrm{U}_m}(\cdot)$ is the probability density function of U_m for $m = 1, 2$.

We note that the means of the two uniform distributions in (54) and (55), and that of the innate opinion are set to 0.5. With 10,000 random samples, the standard sample deviations σ under the two different uniform distributions are shown in Fig. 6, where individual index i denotes information source N1 influences individual v_i solely. The results in Fig. 6 numerically suggest the following intuitive observations.

- As an information source, influencing a critical individual, i.e, the individual with highest outdegree, results in largest sample deviation. This observation is particularly important for manipulative information or misinformation spread.
- The order of influences of individuals that follow the information source (measured by the sample deviation) is preserved under different distributions with the same mean, suggesting that that the shape of the distribution does not impact the previous observation significantly. Hence, the influence order is rather robust to the changes in the distribution of innate opinions and information source as long as their mean stay constant.
- Bi-model uniform distribution (U_2 in (55)) results in larger sample deviation than unimodel uniform distribution (U_1 in (54)). As we analyze in the next section, polarized opinions of information sources results in larger deviation from the average, as expected.

2) Polar Information Sources: The Krackhardt's advice network [26] that is in the presence of two information sources that pass along polar opinions is shown in Fig. 7. We use sample variance to measure the influences of different polar opinions, which is defined as follows:

$$\delta^2 \triangleq \frac{1}{n}\sum_{i=1}^{n} \mathbf{E}_{\mathrm{U}}\left(\left(x_i^{\mathrm{ie}} - \bar{x}\right)^2\right), \tag{56}$$

where $\bar{x}$ is the average of group opinions, i.e., $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i^{\mathrm{ie}}$.

The two information sources N1 and N2 consider six couples of uniform distributions, which are plotted in Fig. 8. The means of the distributions are shown as $\widetilde{\mu}_1$ and $\widetilde{\mu}_2$ for each case. As Fig. 8 depicts, we use these distributions to model polarization in the information source. We set the average of

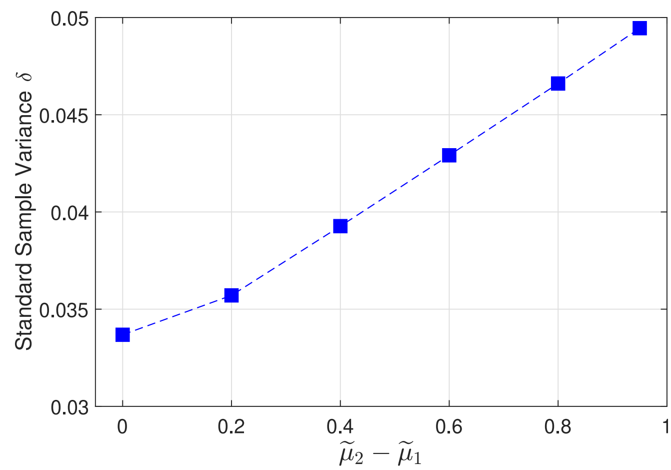

Fig. 9. Sample variance under six different cases.

the means of polar opinions ($\frac{1}{2}(\widetilde{\mu}_1 + \widetilde{\mu}_2)$) and the mean of the innate opinions to 0.5.

We plot the standard sample variances δ under six different cases, averaged over 10,000 runs, in Fig. 9. As expected, Fig. 9 demonstrates that as the distance between the means of polar opinions, $|\widetilde{\mu}_1 - \widetilde{\mu}_2|$, increases, i.e., as the information sources disperse more polarized opinions, we get an equilibrium of individuals with larger sample variance.

V. Conclusion and Discussions

In this paper, we have proposed a model for the dynamics of information spread on cyber-social networks with a particular focus on confirmation bias. We have characterized the conditions for convergence to a unique equilibrium (steady-state) point, which is independent of initial opinions. The steady-state points of the proposed social dynamics under both linear and nonlinear weight functions are studied. The estimation of equilibrium point is derived for nonlinear weight functions. An algorithm that exactly computes the equilibrium point for linear weight functions is provided. Theoretical results are verified by numerical examples. The numerical results obtained over synthetic and real networks suggest intuitive outcomes on information spreading over networks.

There are several directions for future work; some of which can be listed as follows.

- Building on the model in this paper, we will analyze (mis)-information spreading over networks for settings where different subset of information sources are controlled by different players with differing objectives, using tools game-theoretic tools (see [11]–[13] for preliminary results in this research direction).
- We will analyze the variations of the proposed model as a positive and linearized system with tolerant approximation error, see e.g., positive T-S fuzzy system, to relax the convergence condition.
- We will investigate connections between the proposed model and the DeGroot-Friedkin model [3] with stubborn agents. We are particularly interested in the relationship between the upper and lower bounds of the equilibrium point in the proposed model and the equilibrium point of the aforementioned variation of the DeGroot-Friedkin model.

Appendix A
Proof of Theorem 1

The following well-known Banach fixed-point theorem will be used to prove Theorem 1.

Theorem 4 [32]. Let $(X; d)$ be a complete metric space and $f : X \to X$ be a map such that $d(f(x); f(x')) \le c d(x; x')$ for some $0 < c < 1$ and all x and x' in X. Then, f has a unique fixed point in X. Moreover, for any $x_0 \in X$, the sequence of iterates x_0; $f(x_0)$; $f(f(x_0))$; ..., converges to the fixed point of f.

Let us consider the matrix form (9) of the dynamics (1). Choose two vectors $x \in \mathbb{R}^n$ and $x' \in \mathbb{R}^n$. Let us first define

$$f(x) \triangleq \mathcal{A}(x)s + Wx + \widehat{\mathcal{W}}(x)y.$$

Then, it follows from (9) that

$$f(x') - f(x) = (\mathcal{A}(x') - \mathcal{A}(x))s + \left(\widehat{\mathcal{W}}(x') - \widehat{\mathcal{W}}(x)\right)y + W(x' - x). \tag{57}$$

Recalling the definition of induced norm of a matrix $W \in \mathbb{R}^{n\times n}$ in Section II-A, we have

$$\|W(x' - x)\| \le \|W\|\|x' - x\|. \tag{58}$$

We conclude from (3) that the entries of the matrix $\widehat{\mathcal{W}}(x)$ can be equivalently rewritten as

$$\hat{w}_{ik}(x_i) = b_{ik} g_{ik}(|x_i - y_k|). \tag{59}$$

Under Assumption 1, and recalling that $1 \ge y_k \ge 0$, $b_{ik} = 1$ or 0, and $\mu_i \ge 0$, $\forall i \in \mathbb{V}$, $\forall k \in \mathbb{K}$, we obtain from (12), (15), (3) and (59) that

$$\begin{aligned}
&\left|\left[(\widehat{\mathcal{W}}(x') - \widehat{\mathcal{W}}(x))y\right]_i\right| \\
&= \left|\sum_{k\in\mathbb{K}} b_{ik}(g_{ik}(|x_i' - y_k|) - g_{ik}(|x_i - y_k|))y_k\right| \\
&\le \sum_{k\in\mathbb{K}} b_{ik}\left|g_{ik}(|x_i' - y_k|) - g_{ik}(|x_i - y_k|)\right|y_k.
\end{aligned} \tag{60}$$

It follows from (5) that

$$\begin{aligned}
&\left|g_{ik}\left(|x_i' - y_k|\right) - g_{ik}(|x_i - y_k|)\right| \\
&\le \mu_i\left||x_i' - y_k| - |x_i - y_k|\right| \\
&\le \mu_i\left|\left(x_i' - y_k\right) - (x_i - y_k)\right| \\
&= \mu_i\left|x_i' - x_i\right|.
\end{aligned} \tag{61}$$

Combining (60) with (61) yields

$$\begin{aligned}
\left|\left[(\widehat{\mathcal{W}}(x') - \widehat{\mathcal{W}}(x))y\right]_i\right| &\le \left|x_i' - x_i\right|\mu_i \sum_{k\in\mathbb{K}} b_{ik}y_k \\
&\le \left|x_i' - x_i\right|\mu_i \sum_{k\in\mathbb{K}} b_{ik}, \forall i \in \mathbb{V}.
\end{aligned} \tag{62}$$

From the definition of l_1-norm of a vector that is given in Section II-A, we have

$$\left\|\left(\widehat{\mathcal{W}}(x') - \widehat{\mathcal{W}}(x)\right)y\right\| = \sum_{i=1}^{n}\left|\left[\left(\widehat{\mathcal{W}}(x') - \widehat{\mathcal{W}}(x)\right)y\right]_i\right|,$$

which, in conjunction with (62), yields

$$\begin{aligned}
\left\|\left(\widehat{\mathcal{W}}(x') - \widehat{\mathcal{W}}(x)\right)y\right\| &\le \sum_{i=1}^{n}\left|x_i' - x_i\right|\mu_i \sum_{k\in\mathbb{K}} b_{ik} \\
&\le \max_{i\in\mathbb{V}}\left\{\mu_i \sum_{k\in\mathbb{K}} b_{ik}\right\}\sum_{i=1}^{n}\left|x_i' - x_i\right| \\
&= \max_{i\in\mathbb{V}}\left\{\mu_i \sum_{k\in\mathbb{K}} b_{ik}\right\}\|x' - x\|.
\end{aligned} \tag{63}$$

We note that (4) and (3) imply $\alpha_i\left(x_i'\right) - \alpha_i(x_i) = \sum_{k\in\mathbb{K}} \hat{w}_{ik}(x_i) - \sum_{k\in\mathbb{K}} \hat{w}_{ik}\left(x_i'\right)$. Under Assumption 1, it follows from (11), (13), (59), and the fact that $0 \le s_i \le 1$, $\forall i \in \mathbb{V}$, that

$$\begin{aligned}
&\left|[(\mathcal{A}(x') - \mathcal{A}(x))s]_i\right| = \left|\left(\alpha_i(x_i') - \alpha_i(x_i)\right)s_i\right| \\
&= \left|\sum_{k\in\mathbb{K}} s_i b_{ik}\left(g_{ik}(|x_i - y_k|) - g_{ik}\left(|x_i' - y_k|\right)\right)\right| \\
&\le \left|x_i' - x_i\right|s_i\mu_i \sum_{k\in\mathbb{K}} b_{ik} \le \left|x_i' - x_i\right| \max_{i\in\mathbb{V}}\left\{s_i\mu_i \sum_{k\in\mathbb{K}} b_{ik}\right\}.
\end{aligned} \tag{64}$$

It follows from (64) and the definition of l_1 norm of a vector that:

$$\begin{aligned}\|(\mathcal{A}(x')-\mathcal{A}(x))s\| &\leq \max_{i\in\mathbb{V}}\left\{s_i\mu_i\sum_{k\in\mathbb{K}}b_{ik}\right\}\sum_{i=1}^{n}|x'_i-x_i| \\ &= \max_{i\in\mathbb{V}}\left\{s_i\mu_i\sum_{k\in\mathbb{K}}b_{ik}\right\}\|x'-x\|.\end{aligned} \tag{65}$$

Combining (57) with (58), (63) and (65), we obtain

$$\begin{aligned}\|f(x')-f(x)\| \leq &\left(\|W\|+\max_{i\in\mathbb{V}}\left\{\mu_i\sum_{k\in\mathbb{K}}b_{ik}\right\}\right. \\ &\left.+\max_{i\in\mathbb{V}}\left\{s_i\mu_i\sum_{k\in\mathbb{K}}b_{ik}\right\}\right)\|x'-x\|,\end{aligned}$$

from which, we conclude that if Assumption 2 holds, the condition in Theorem 4 would be satisfied. Hence, by Theorem 4, the dynamics (1) converges to a unique equilibrium point for any initial opinion $x(0)\in\mathbb{R}^n$.

Appendix B
Proof of Corollary 3

Equations (3), (13), (15) and (4) yield

$$\begin{aligned}&\left[\mathcal{A}\left(x^{\text{ie}}\right)s+\widehat{\mathcal{W}}\left(x^{\text{ie}}\right)y\right]_i \\ &= \alpha_i\left(x_i^{\text{ie}}\right)s_i+\sum_{k\in\mathbb{K}}\hat{w}_{ik}\left(x_i^{\text{ie}}\right)y_k \\ &= \left(1-\sum_{j\in\mathbb{V}}w_{ij}-\sum_{k\in\mathbb{K}}\hat{w}_{ik}\left(x_i^{\text{ie}}\right)\right)s_i+\sum_{k\in\mathbb{K}}\hat{w}_{ik}\left(x_i^{\text{ie}}\right)y_k \\ &= \left(1-\sum_{j\in\mathbb{V}}w_{ij}\right)s_i+\sum_{k\in\mathbb{K}}\hat{w}_{ik}\left(x_i^{\text{ie}}\right)(y_k-s_i).\end{aligned} \tag{66}$$

From (17) and (11), we have

$$[As]_i = \left(1-\sum_{j\in\mathbb{V}}w_{ij}\right)s_i. \tag{67}$$

It follows from (66) and (67) that

$$\left[\mathcal{A}\left(x^{\text{ie}}\right)s+\widehat{\mathcal{W}}\left(x^{\text{ie}}\right)y\right]_i = [As]_i+\sum_{k\in\mathbb{K}}\hat{w}_{ik}\left(x_i^{\text{ie}}\right)(y_k-s_i),$$

or equivalently,

$$\mathcal{A}\left(x^{\text{ie}}\right)s+\widehat{\mathcal{W}}\left(x^{\text{ie}}\right)y = As+\widehat{\mathcal{W}}\left(x^{\text{ie}}\right)y-S\widehat{\mathcal{W}}\left(x^{\text{ie}}\right)\mathbf{1}_m, \tag{68}$$

We note that (19) subtracting (18) results in

$$x^{\text{ie}}-x^{\text{ue}} = (\mathbf{1}-W)^{-1}\left(\mathcal{A}\left(x^{\text{ie}}\right)s+\widehat{\mathcal{W}}\left(x^{\text{ie}}\right)y-As\right). \tag{69}$$

Substituting (68) into (69) yields (22).

Appendix C
Proof of Theorem 2

Noting that $\hat{w}_{ik}(x_i^{\text{ie}})\geq 0$, from (23) and (3) we have $\underline{\phi_i}\sum_{k\in\mathbb{K}}b_{ik}y_k \leq \sum_{k\in\mathbb{K}}\hat{w}_{ik}(x_i^{\text{ie}})y_k\leq\overline{\phi_i}\sum_{k\in\mathbb{K}}b_{ik}y_k$ for $\forall i\in\mathbb{V}$, which is equivalent to

$$\underline{\Phi}By\leq\widehat{\mathcal{W}}\left(x^{\text{ie}}\right)y\leq\overline{\Phi}By. \tag{70}$$

It follows from $\hat{w}_{ik}\left(x_i^{\text{ie}}\right)$ in (3) and (23) that

$$-s_i\overline{\phi_i}\sum_{k\in\mathbb{K}}b_{ik}\leq -s_i\sum_{k\in\mathbb{K}}w_{ik}\left(x_i^{\text{ie}}\right)\leq -s_i\underline{\phi_i}\sum_{k\in\mathbb{K}}b_{ik}, \forall i\in\mathbb{V}$$

which is equivalent to

$$-S\overline{\Phi}\varrho\leq -S\widehat{\mathcal{W}}\left(x^{\text{ie}}\right)\mathbf{1}_m\leq -S\underline{\Phi}\varrho. \tag{71}$$

Plugging (70) in (71), we have

$$\underline{\Phi}By-S\overline{\Phi}\varrho\leq\widehat{\mathcal{W}}\left(x^{\text{ie}}\right)y-S\widehat{\mathcal{W}}\left(x^{\text{ie}}\right)\mathbf{1}_m\leq\overline{\Phi}By-S\underline{\Phi}\varrho, \tag{72}$$

which, in conjunction with (22), results in (73). Substituting (18) into (73) yields (74).

$$\begin{aligned}(\mathbf{1}-W)^{-1}\left(\underline{\Phi}By-S\overline{\Phi}\varrho\right) &\leq x^{\text{ie}}-x^{\text{ue}} \\ &\leq (\mathbf{1}-W)^{-1}\left(\overline{\Phi}By-S\underline{\Phi}\varrho\right).\end{aligned} \tag{73}$$

$$\begin{aligned}(\mathbf{1}-W)^{-1}\left(\underline{\Phi}By-S\overline{\Phi}\varrho+As\right) &\leq x^{\text{ie}} \\ &\leq (\mathbf{1}-W)^{-1}\left(\overline{\Phi}By-S\underline{\Phi}\varrho+As\right).\end{aligned} \tag{74}$$

Remark 3 implies that $\mathbf{0}_n\leq x^{\text{ie}}\leq\mathbf{1}_n$. Then, we note from (20) that $\mathbf{0}_n\leq\mathcal{A}(x^{\text{ie}})s+Wx^{\text{ie}}+\widehat{\mathcal{W}}(x^{\text{ie}})y\leq\mathbf{1}_n$. Thus, $\mathbf{0}_n\leq\mathcal{A}(x^{\text{ie}})s+\widehat{\mathcal{W}}(x^{\text{ie}})y\leq\mathbf{1}_n$. Therefore, we conclude from (19) that

$$(\mathbf{1}-W)^{-1}\mathbf{0}_n\leq x^{\text{ie}}\leq(\mathbf{1}-W)^{-1}\mathbf{1}_n. \tag{75}$$

The left-hand side of (75), which in conjunction with that of (74), results in the lower bound of (30). The right-hand side of (75), which in conjunction with that of (74) and the fact $x^{\text{ie}}\leq\mathbf{1}_n$, results in the upper bound of (30).

Appendix D
Proof of Theorem 3

Noting that the matrices defined in (35), (21) and (34) are all diagonal, we have $S\Upsilon X=X\Upsilon S$ and $\Upsilon X=X\Upsilon$. It follows from (32) that

$$\begin{aligned}
&\widehat{\mathcal{W}}(x^{\text{ie}})y - S\widehat{\mathcal{W}}(x^{\text{ie}})\mathbf{1}_m \\
&= \Big(\varXi B + \Upsilon\widetilde{\mathcal{W}}(x^{\text{ie}}, y)Y - \Upsilon X\widetilde{\mathcal{W}}(x^{\text{ie}}, y)\Big)y \\
&\quad - S\Big(\varXi B + \Upsilon\widetilde{\mathcal{W}}(x^{\text{ie}}, y)Y - \Upsilon X\widetilde{\mathcal{W}}(x^{\text{ie}}, y)\Big)\mathbf{1}_m \\
&= \Big(\varXi B + \Upsilon\widetilde{\mathcal{W}}(x^{\text{ie}}, y)Y\Big)y - S\Big(\varXi B + \Upsilon\widetilde{\mathcal{W}}(x^{\text{ie}}, y)Y\Big)\mathbf{1}_m \\
&\quad + X\Big(\Upsilon S\widetilde{\mathcal{W}}(x^{\text{ie}}, y)\mathbf{1}_m - \Upsilon\widetilde{\mathcal{W}}(x^{\text{ie}}, y)y\Big).
\end{aligned} \tag{76}$$

Without loss of generality, we let $\widetilde{\mathcal{W}}(x^{\text{ie}}, y) = \mathbb{P}(d)$, where $\mathbb{P}$ is an ordered set of the possible choices for $\widetilde{\mathcal{W}}(x^{\text{ie}}, y)$, as defined in (39). Then, substituting (76) into (22) yields

$$x^{\text{ie}} - x^{\text{ue}} = (\mathbf{1} - W)^{-1}(X\widehat{\xi}(d) + \xi(d)), \quad 1 \leq d \leq |\mathbb{P}|. \tag{77}$$

It follows from (43), (35) and in (41) that $X\widehat{\xi}(d) = \mathbf{\Theta}(d)x^{\text{ie}}$. Then, from (77), we obtain

$$\begin{aligned}
x^{\text{ie}} - (\mathbf{1} - W)^{-1}X\widehat{\xi}(d) &= x^{\text{ie}} - (\mathbf{1} - W)^{-1}\mathbf{\Theta}(d)x^{\text{ie}} \\
&= x^{\text{ue}} + (\mathbf{1} - W)^{-1}\xi(d) \\
&= (\mathbf{1} - W)^{-1}(\xi(d) + As),
\end{aligned}$$

which is equivalent to

$$(\mathbf{1} - (\mathbf{1} - W)^{-1}\mathbf{\Theta}(d))x^{\text{ie}} = (\mathbf{1} - W)^{-1}(\xi(d) + As).$$

Therefore, we have

$$\begin{aligned}
x^{\text{ie}} &= (\mathbf{1} - (\mathbf{1} - W)^{-1}\mathbf{\Theta}(d))^{-1}(\mathbf{1} - W)^{-1}(\xi(d) + As) \\
&= ((\mathbf{1} - W)(\mathbf{1} - (\mathbf{1} - W)^{-1}\mathbf{\Theta}(d)))^{-1}(\xi(d) + As) \\
&= (\mathbf{1} - W - \mathbf{\Theta}(d))^{-1}(\xi(d) + As).
\end{aligned} \tag{78}$$

We note that the loop-stopping condition, i.e., Line 3 of Algorithm 1, is from the definition and condition of entries of $\widetilde{\mathcal{W}}(x^{\text{ie}}, y)$ in (38) and (40). Moreover, Theorem 1 states that the equilibrium point x^{ie} is unique. The computation of x^{ie} in (78) implies that once $\mathbb{P}(d) = \widetilde{\mathcal{W}}(x^{\text{ie}}, y)$ is searched, i.e., the condition in Line 3 of Algorithm 1 is satisfied, the equilibrium point x^{ie} is solved.

Appendix E
Proof of Corollary 4

Since the unique equilibrium point (19) is independent of initial opinions, it is also the equilibrium point of dynamics: $\tilde{x}(t+1) = \mathcal{A}(\tilde{x}(t))s + W\tilde{x}(t) + \widehat{\mathcal{W}}(\tilde{x}(t))\tilde{y}$ with $\tilde{x}(0) = \mathbf{0}_n$. In the situation where there is only one information source, it is straightforward to verify from the extremal opinion definition (44) and the convex combination relation (4) that $\tilde{y} \geq \max_{i\in\mathbb{V}}\{\tilde{x}_i(t)\}$, $\forall t \in \mathbb{N}_0$, which implies that

$$\tilde{y} \geq \max_{i\in\mathbb{V}}\{x_i^{\text{ie}}\}. \tag{79}$$

Under (44) and (31) $\widehat{\mathcal{W}}(x^{\text{ie}})$ can be rewritten as

$$\widehat{\mathcal{W}}(x^{\text{ie}}) = \varXi\zeta + \Upsilon X\zeta - \Upsilon\zeta\tilde{y}. \tag{80}$$

Considering (80) and following the same lines as those leading up to (76) in the proof of Theorem 3, we arrive at

$$\begin{aligned}
&\widehat{\mathcal{W}}(x^{\text{ie}})\tilde{y} - S\widehat{\mathcal{W}}(x^{\text{ie}}) \\
&= (\varXi\zeta + \Upsilon X\zeta - \Upsilon\zeta\tilde{y})\tilde{y} - S(\varXi\zeta + \Upsilon X\zeta - \Upsilon\zeta\tilde{y}) \\
&= \overline{\vartheta} + X\Upsilon(\zeta\tilde{y} - S\zeta).
\end{aligned} \tag{81}$$

Substituting (81) into (22) yields

$$x^{\text{ie}} - x^{\text{ue}} = (\mathbf{1} - W)^{-1}\big(\overline{\vartheta} + X\Upsilon(\zeta\tilde{y} - S\zeta)\big). \tag{82}$$

Noting (48) implies that $X\Upsilon(\zeta\tilde{y} - S\zeta) = \Phi x^{\text{ie}}$, and plugging (18) into (82), we have

$$\begin{aligned}
x^{\text{ie}} - (\mathbf{1} - W)^{-1}\Phi x^{\text{ie}} &= x^{\text{ie}} - (\mathbf{1} - W)^{-1}X\Upsilon(\zeta\tilde{y} - S\zeta) \\
&= x^{\text{ue}} + (\mathbf{1} - W)^{-1}\overline{\vartheta} \\
&= (\mathbf{1} - W)^{-1}\big(\overline{\vartheta} + As\big),
\end{aligned}$$

from which, we have

$$\begin{aligned}
x^{\text{ie}} &= \Big(\mathbf{1} - (\mathbf{1} - W)^{-1}\Phi\Big)^{-1}(\mathbf{1} - W)^{-1}\big(\overline{\vartheta} + As\big) \\
&= \Big((\mathbf{1} - W)\Big(\mathbf{1} - (\mathbf{1} - W)^{-1}\Phi\Big)\Big)^{-1}\big(\overline{\vartheta} + As\big) \\
&= (\mathbf{1} - W - \Phi)^{-1}\big(\overline{\vartheta} + As\big).
\end{aligned}$$

Thus, (50) is obtained. The proof of (51) follows the same steps and hence is omitted.

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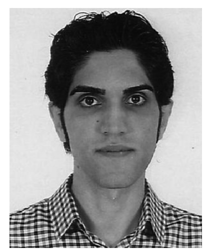

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