Taylor Series

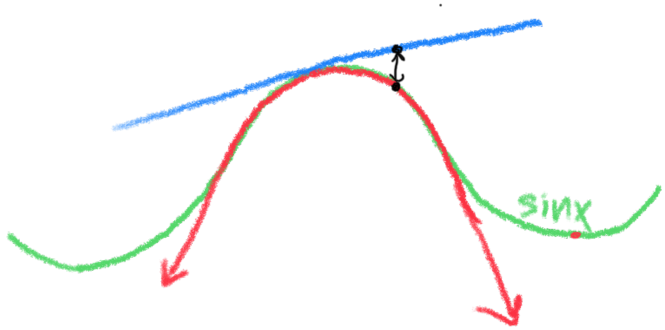
6/24/202

Lesson Objectives

1. the need for Taylor Series
2. how to come up with Taylor Series
3. how to use Taylor Series for approximation problems

1: The Need

- we can use tangent lines to approximate functions
- however, these tangent lines are not accurate enough.



1 → $y=1$. Not great.

$\sin(\frac{\pi}{2}) + \cos(\frac{\pi}{2})(x-\frac{\pi}{2}) + \frac{-\sin(\frac{\pi}{2})\cancel{x^2}}{2}$

$1 - \frac{(x-\frac{\pi}{2})^2}{2} = y$ ☺

$1 - \frac{x^2 - x\pi + \frac{\pi^2}{4}}{2} = y$

2. The Derivation

Suppose we have a function $f(x)$, and we want to find a "tangent quadratic" $g(x)$ at $x=a$.

As opposed to the line, where we had

two requirements, we now have three!

They are:

$$f(a) = g(a)$$

$$f'(a) = g'(a)$$

$$f''(a) = g''(a)$$

(think about this on your own)

A naive approach to this is

$$g(x) = f(a) + f'(a)x + \frac{1}{2}f''(a)x^2$$

This only satisfies the third criteria.

If we take away the third term, it satisfies only the second criteria. If we take away the second term as well, it only satisfies the first criteria.

In short, the higher degree terms are hindering the lower degree terms from serving their purpose.

We can solve this by setting up higher degree terms to cancel out when they are not needed. This is done by replacing

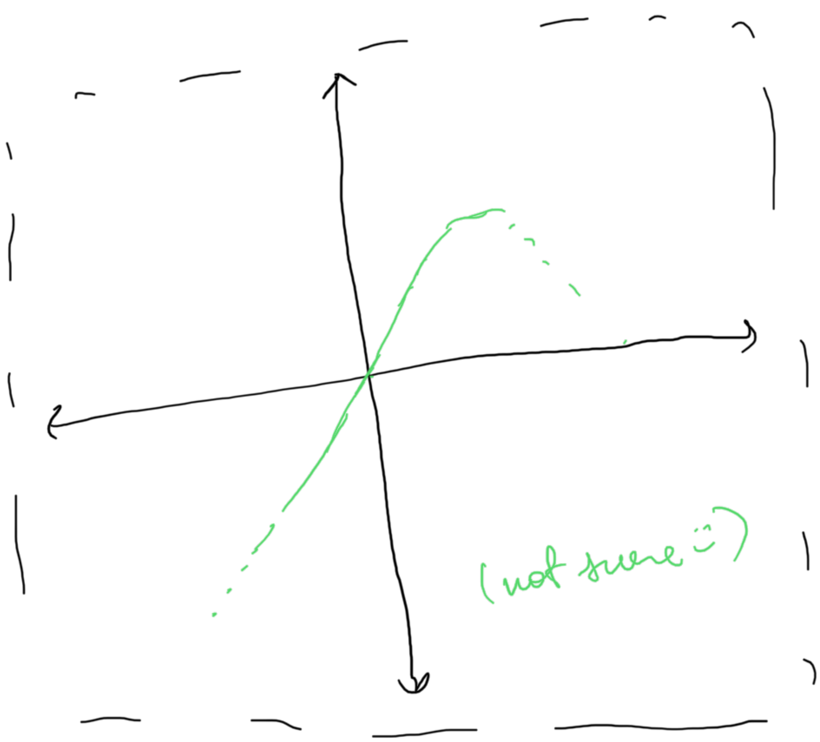
$$x \longrightarrow x - a.$$

Since these terms have the same derivative, they won't disturb the parts of our naive solution that work.

So, we have:

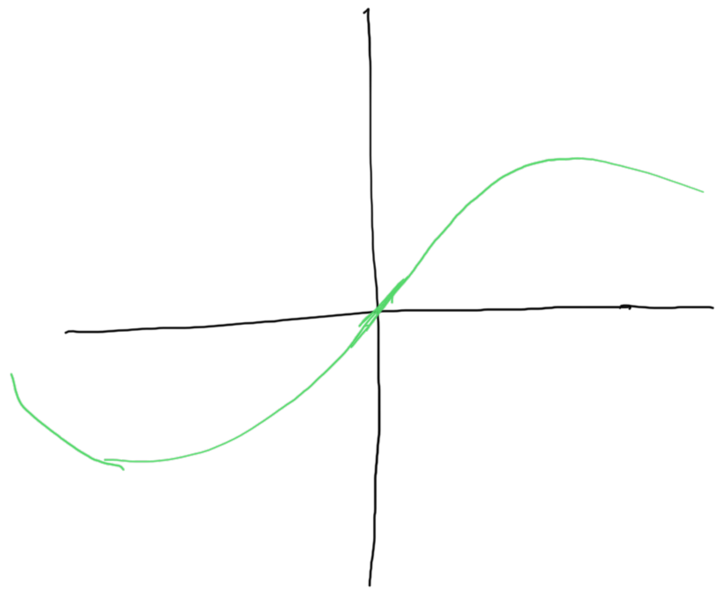
$$g(x) = f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2$$

Let's test this out:



$f(x) = \sin x \quad a = 0$

This is definitely nicer than a line. Now that we've seen greatness, let's get greedy! Upping our polynomial to the fifth degree



That's more like it. And, if we graphed an infinite degree polynomial, it would line up exactly!

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