# Inflation and Dark Energy Reconciled

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**Abstract**

This report will discuss the cosmological properties of modified non-metric gravity, in particular how it can potentially be relevant for early and late epochs which would be precious for describing both inflation and dark energy within the same theory. This will be done by studying the dynamical system of cosmological solutions with specific forms of $f(Q)$ where $Q$ is the non-metric scalar.

# Contents

# 1 Geometrical considerations

## 1.1 Standard General Relativity

The theory of General Relativity (GR) describes gravity through a geometrical perspective which arises from the equivalence principle stated by Einstein in the beginning of the 20th century. Unlike Newton's law of gravitation the gravitational interaction is not described with forces anymore but instead relies on the notion of curvature to explain the relationship between matter and gravity. This relationship is given by the following tensorial equations, the Einstein field equations (EFE):

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{1}$$

with $R_{\mu\nu}$ the Ricci tensor, $R := g^{\alpha\beta} R_{\alpha\beta}$ the Ricci scalar, both linked to the curvature of spacetime, $g_{\mu\nu}$ the metric tensor of spacetime and $T_{\mu\nu}$ the energy-momentum tensor linked to the matter presence. Note that we are using units where the speed of light $c = 1$ and from now on where the universal gravitational constant $G = \frac{1}{8\pi}$.

Adopting the Lagrangian formalism the EFE can be retrieved from the least action principle ($\delta\mathcal{S} = 0$) by variation with respect to the metric $g_{\mu\nu}$ applied to the functional:

$$\mathcal{S}_{GR}[g] := \int \mathrm{d}^4x \sqrt{-g} \left(\frac{R}{2} + \mathcal{L}_m\right) \tag{2}$$

This is the Einstein-Hilbert action with a minimally coupled matter term represented by the Lagrangian density $\mathcal{L}_m$ and it is equivalent to the EFE. In fact, this formulation will be more useful for deriving equations of motion and we will thus continue to use it in the rest of the report.

## 1.2 Alternative geometrical interpretations

What is important to know is that the geometrical interpretation of gravity is not restricted to curvature. Indeed there are two additional geometrical properties of spacetime which can be used to formulate the effects of gravity: non-metricity or torsion. This means that gravity can also be fully ascribed either to non-metricity or to torsion instead of curvature. Mathematically these two properties can respectively be represented by the tensors:

$$\begin{aligned} Q_{\alpha\mu\nu} &:= \nabla_\alpha g_{\mu\nu} \\ T^\alpha{}_{\mu\nu} &:= 2\Gamma^\alpha{}_{[\mu\nu]} \end{aligned} \tag{3}$$

with $\Gamma^\alpha{}_{\mu\nu}$ being the general connection of the manifold, $Q_{\alpha\mu\nu}$ and $T^\alpha{}_{\mu\nu}$ are called the non-metricity and torsion tensors and they vanish in the standard formulation of GR. Indeed the latter states that the spacetime is metric and torsionless.

As for the Riemann curvature tensor $R^\alpha{}_{\beta\mu\nu}$, which can be geometrically understood through the angle difference yielded by a parallel transport of a vector on a curved manifold, their meaning can be visualised in the figure 1. Intuitively this means that instead of curving the space, these two different formulations of GR state that gravity respectively either twists or stretches it. Scalar quantities can be extracted from the non-metricity and torsion tensors in order to formulate these theories using an action as

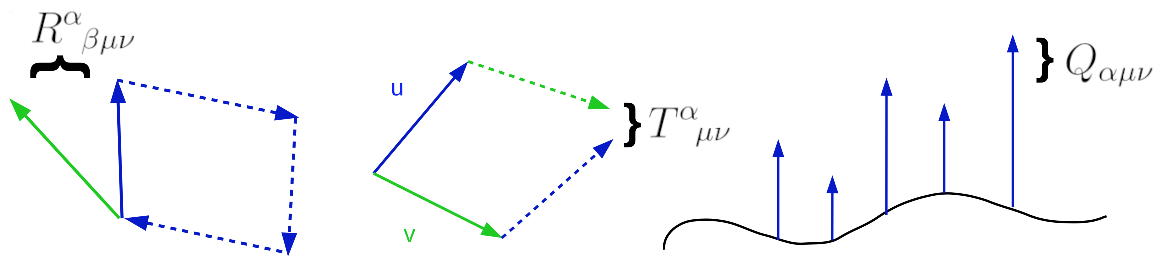



Figure 1: Visualisation of the geometrical meaning of the curvature, torsion and non-metricity tensors respectively [1]. When spacetime admits torsion parallelograms do not close and if it is non-metric the norm of vector fields is not conserved when transported along a curve.

| GR | TEGR | STEGR |
|---|---|---|
| curvature | torsion | non-metricity |
| $R \neq 0$ | $T \neq 0$ | $Q \neq 0$ |
| $Q = 0 = T$ | $Q = 0 = R$ | $T = 0 = R$ |

Table 1: Spacetime properties of the three equivalent geometrical descriptions of gravity.

in 2. They are respectively $Q$ and $T$ yielded from independent second order contractions of the aforementioned tensors. Due to the (anti-)symmetry of these tensors with respect to certain indices the contractions are constrained to specific forms but are not unique:

$$\begin{aligned} Q &:= a_1 Q_{\alpha\beta\gamma} Q^{\alpha\beta\gamma} + a_2 Q_{\alpha\beta\gamma} Q^{\beta\alpha\gamma} + a_3 Q_{\alpha\lambda}{}^{\lambda} Q^{\alpha}{}_{\lambda}{}^{\lambda} + a_4 Q^{\lambda}{}_{\lambda\alpha} Q^{\lambda}{}_{\lambda}{}^{\alpha} + a_5 Q_{\alpha\lambda}{}^{\lambda} Q^{\lambda}{}_{\lambda}{}^{\alpha} \\ T &:= b_1 T_{\alpha\mu\nu} T^{\alpha\mu\nu} + b_2 T_{\alpha\mu\nu} T^{\mu\alpha\nu} + b_3 T^{\beta}{}_{\alpha\beta} T^{\beta\alpha}{}_{\beta} \end{aligned} \tag{4}$$

It can be formally shown [2] that these three formulations (based only on curvature, torsion or non-metricity) are perfectly equivalent to each other if the $\{a_i\}$ and $\{b_i\}$ are properly defined in 4 (this suitable choice for them is $a_1 = \frac{1}{4} = -a_3,\ a_2 = -\frac{1}{2} = -a_5,\ a_4 = 0$ and $b_1 = -\frac{1}{4},\ b_2 = -\frac{1}{2},\ b_3 = 1$). This equivalence corresponds to what is called the "geometrical trinity of gravity". The formulation based on torsion but a flat and metric spacetime is then the Teleparallel Equivalent of General Relativity (TEGR) where the teleparallel denomination simply comes from the fact that the spacetime is not curved hence the parallel transport of a vector will not alter its orientation. The one based on a flat, torsionless and non-metric spacetime is the Symmetric Teleparallel Equivalent of General Relativity (STEGR) and its name is due to the $\mu \leftrightarrow \nu$ symmetry of the non-metricity tensor $Q_{\alpha\mu\nu}$. For clarification the properties of the three alternative formulations are summarised in the table 1.

## 1.3 Coincident General Relativity

In spite of this essential triangular equivalence STEGR bears an outstanding property. Due to the flatness condition, the connection can be parameterised by an element $\Lambda^{\alpha}{}_{\mu} \in GL(4, \mathbb{R})$:

$$R^{\alpha}{}_{\beta\mu\nu} = 0 \qquad \Rightarrow \qquad \Gamma^{\alpha}{}_{\mu\nu} = (\Lambda^{-1})^{\alpha}{}_{\lambda} \partial_{\mu} \Lambda^{\lambda}{}_{\nu} \tag{5}$$

Moreover [3], the torsionless condition gives an additional constraint which leads to the parametrisation through a set of diffeomorphisms $\xi^\alpha$ as follows:

$$T^\alpha{}_{\mu\nu} = 0 \qquad \Rightarrow \qquad \partial_{[\mu}\Lambda^\lambda{}_{\nu]} = 0 \qquad \Rightarrow \qquad \Lambda^\alpha{}_\mu = \partial_\mu \xi^\alpha \tag{6}$$

The outstanding property is the combination of 5 coupled with 6 which yields:

$$\Gamma^\alpha{}_{\mu\nu} = (\frac{\partial \xi^\lambda}{\partial x^\alpha})^{-1}\partial_\mu\partial_\nu\xi^\lambda \tag{7}$$

This is an important consequence since with the coordinate transformation $\xi^\alpha = x^\alpha$ (where the origin of spacetime coincides with the origin of the tangent space parameterised by $\xi^\alpha$) the connection vanishes and there are no inertial effects. Indeed in this gauge the covariant derivative is reduced to the normal partial derivative which tremendously simplifies computations. This gauge is commonly denoted by the "coincident gauge" and thus the non-metric formulation in this gauge is called Coincident General Relativity (CGR) [4]. In CGR the non-metricity scalar reduces to:

$$Q = g^{\mu\nu}(\left\{{}^{\alpha}_{\beta\mu}\right\}\left\{{}^{\beta}_{\nu\alpha}\right\} - \left\{{}^{\alpha}_{\beta\alpha}\right\}\left\{{}^{\beta}_{\mu\nu}\right\}) \tag{8}$$

where the $\left\{{}^{\alpha}_{\mu\nu}\right\}$ are the Christoffel symbols of the Levi-Civita connection which we are familiar with from standard GR:

$$\left\{{}^{\alpha}_{\mu\nu}\right\} := \frac{1}{2}g^{\alpha\lambda}(2\partial_{(\mu}g_{\nu)\lambda} - \partial_\lambda g_{\mu\nu}) \tag{9}$$

As we can see these include only first derivatives of the metric which means that this will also be the case in the action. This reveals itself to be powerful since there is no need for an additional boundary term anymore when considering bounded spacetime manifolds (Gibbons-Hawking-York term [5]) in order to have a properly defined action principle. The latter leads to simplifications for instance in the fields of energetics, thermodynamics and quantum theory.

So considering the above discussion the CGR paradigm seems better suited for the geometrical treatment of gravity.

## 1.4 Modified non-metric gravity

Despite GR being successful in extensively describing gravity it still fails to encompass several cosmological observations which will be discussed in section 2. This motivates the consideration of general non-linear extensions of the theory by promoting $Q$ to $f(Q)$ in the action. This new framework is part of what is called "modified gravity". It is important to note that the triangular equivalence mentioned in section 1.2 is broken by this consideration meaning that $f(Q)$, $f(T)$ and $f(R)$ theories express totally different dynamical behaviours. The fact that the properties of $f(R)$ and $f(T)$ have been quite deeply studied in the literature and the conclusion of section 1.3 points to giving a closer look to $f(Q)$ theory, modified non-metric gravity. Since $f(Q) = Q$ is equivalent to the Einstein-Hilbert action, it makes sense to consider its corrections that is to study functions of the form $f(Q) = Q + ...$ with the correction terms giving rise to the modification of gravity. The desired action that will be studied in this report is thus finally:

$$\mathcal{S}[g] = \int \mathrm{d}^4x\sqrt{-g}(\frac{f(Q)}{2} + \mathcal{L}_m) \tag{10}$$

# 2 Cosmological background evolution

## 2.1 Modified Friedmann equations

In order to study the cosmological properties of $f(Q)$ we proceed in the usual way by considering the flat ($k = 0$) Friedmann–Lemaître–Robertson–Walker (FLRW) metric describing an expanding (or contracting) Euclidean, homogeneous and isotropic universe. The assumption of flatness is supported by several persuasive arguments, both observational and theoretical [6]. Consequently the line element used throughout the report is:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j \tag{11}$$

with $a(t)$ being the scale factor. We can define the expansion rate as $H := \frac{\dot{a}}{a}$, this is the Hubble parameter. Now from 8 we can have an explicit value for the non-metricity scalar: $Q = 6H^2$.

The Friedmann equations describing the cosmological background evolution are obtained from the least action principle on the action in 10 with the metric from 11. They are modified by the $Q \mapsto f(Q)$ promotion in the following way:

$$\begin{cases} 3H^2 = \rho \\ \dot{H} = -\frac{1}{2}(\rho + p) \end{cases} \qquad \longmapsto \qquad \begin{cases} 6f'H^2 - \frac{1}{2}f = \rho \\ (12f''H^2 + f')\dot{H} = -\frac{1}{2}(\rho + p) \end{cases} \tag{12}$$

with $f'$ and $f''$ denoting the first and second derivatives of $f(Q)$ with respect to $Q$, $\rho$ and $p$ respectively being the energy density and pressure of the matter described by the matter Lagrangian $\mathcal{L}_m$. Indeed by definition of the energy-momentum tensor:

$$T_{\mu\nu} := -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}} \tag{13}$$

By the aforementioned cosmological principle of homogeneity and isotropy this tensor must be of the form $T^{\mu}_{\ \nu} = \mathrm{diag}(-\rho, p, p, p)$ describing a perfect fluid which then yields the equations in 12. From the conservation of this tensor we can have the continuity equation:

$$\nabla_\nu T^{\mu\nu} = 0 \qquad \Leftrightarrow \qquad \dot{\rho} = 3H(\rho + p) \tag{14}$$

Nevertheless note that the continuity equation can be also retrieved from the Friedmann equations. The time dependence of the parameters $H$, $\rho$ and $p$ is omitted for clarity but should not be forgotten since the two Friedmann equations rule their time evolution. However since there are only two independent equations for three parameters we introduce the equation of state relating $\rho$ and $p$ as follows:

$$p = w\rho \tag{15}$$

with $w$ called the equation of state parameter. For relativistic matter (radiation) we have that $p = \frac{\rho}{3}$ meaning that $w = \frac{1}{3}$ and for non-relativistic matter (dust) $p = 0$ giving $w = 0$.

## 2.2 Expansion and contraction

Studying the evolution of $H$ amounts to studying the evolution of the universe geometry since it is directly related to the scale factor $a$ which rules the spatial part of the line element in 11. In particular just from its sign we can deduce if the universe is contracting or expanding:

$$H \begin{cases} > 0 & \Rightarrow \text{expansion} \\ = 0 & \Rightarrow \text{fixed Minkowski spacetime} \\ < 0 & \Rightarrow \text{contraction} \end{cases} \tag{16}$$

Then logically from the time derivative of the Hubble parameter $\dot{H}$ we can also know if this expansion or contraction is happening in an accelerated, constant or decelerated manner:

$$\dot{H} + H^2 = \frac{\ddot{a}}{a} \begin{cases} > 0 & \Rightarrow \text{accelerated} \\ = 0 & \Rightarrow \text{constant} \\ < 0 & \Rightarrow \text{decelerated} \end{cases} \tag{17}$$

Translating these different scenarios in terms of the matter presence can be done by looking at the modified Friedmann equations and recasting them in a similar shape of the standard Friedmann equations in 12:

$$\begin{aligned} 3H^2 &= \rho_{eff} \\ \dot{H} &= -\frac{1}{2}(\rho_{eff} + p_{eff}) \end{aligned} \tag{18}$$

with $\rho_{eff}$ and $p_{eff}$ being the effective pressure and energy density of the total fluid containing the modifications yielded by $f(Q)$. They are defined as follows:

$$\rho_{eff} := \frac{\rho + \frac{f}{2}}{2f'} \qquad\qquad p_{eff} := \frac{p + 3H^2(f' + 8\dot{H}f'') - \frac{f}{2}}{f'} \tag{19}$$

Now from the equation of state of these effective quantities $w_{eff} = \frac{p_{eff}}{\rho_{eff}}$ we can derive equivalent conditions for 17 which thus gives a relation between the dominant constituent of the universe and its evolution:

$$w_{eff} \begin{cases} > -\frac{1}{3} & \Rightarrow \text{accelerated} \\ = -\frac{1}{3} & \Rightarrow \text{constant} \\ < -\frac{1}{3} & \Rightarrow \text{decelerated} \end{cases} \tag{20}$$

Current observations (namely of the cosmic microwave background anisotropies [7] and of the luminosity distance of type Ia supernovae [8]) point at an expanding universe which went through different epochs of accelerated and decelerated expansion. In the following subsections we will review the current most successful theoretical explanations that comply with these observations. These are essentially the descriptions that we are aiming to unify through the additional degrees of freedom gained by the $Q \mapsto f(Q)$ promotion.

## 2.3 Radiation and matter domination

The dynamics of the early universe after the conjectured Big Bang were ruled by relativistic particles, mainly photons and neutrinos, for about 47'000 years [9]. They are usually both referred as radiation and its energy density was dominating those of the other constituents. Since the equation of state parameter of radiation is $w = \frac{1}{3}$ the expansion of the universe was decelerating.

After radiation domination ended, the energy density of non-relativistic matter (representing constituents such as dust) was the prevailing form of energy in the universe. Due to its vanishing pressure this matter-dominated universe was also expanding in a decelerating manner. This era lasted until the universe was around 9.8 billion years old [10].

## 2.4 Dark energy

Having left the phase of matter domination, we currently live in an epoch where the universe is experiencing accelerated expansion and this requires $w_{eff} < -\frac{1}{3}$ as discussed in section 2.2. Such an equation of state does not correspond to familiar matter which motivates the consideration of a new dominant component of the universe: "dark energy".

The simplest possibility for dark energy is to consider a cosmological constant $\Lambda$ which is introduced in the action seen in 10. This constant can be interpreted as the intrinsic or fundamental energy density of space and when it is cast into the form of an energy-momentum tensor we see that $p_\Lambda = -\rho_\Lambda$ which corresponds to $w_\Lambda = -1$ agreeing with the condition in 20. An example of such a theory is the popular $\Lambda CDM$ model also called the standard model of Big Bang cosmology since it is the simplest theory providing a satisfying account of several properties of the universe such as the current accelerating expansion, the cosmic microwave background (CMB) or the large scale structure (LSS) in the distribution of galaxies.

Another possibility is to make the constant dynamical by turning the energy density of $\Lambda$ into the potential energy of a scalar field, often referred to as "quintessence". It can be shown [11] that such form of dark energy exhibits an equation of state parameter also agreeing with 20.

## 2.5 Cosmic inflation

One of the greatest cosmological mysteries that we are faced with is how could the universe be so homogeneous on such large scales, the so called horizon problem. This motivates the scrutiny of the initial conditions of the universe and the lookout for an explanation combining rapid expansion and causal connection between the ingredients, yielding the observed smoothness.

The most accepted theory explaining the initial perturbations of the early universe with then grew into the structures that we observe nowadays, solving the horizon problem, is cosmic inflation. In this paradigm the universe went through a very brief (less than $10^{-32}$ seconds) phase of accelerated expansion which expanded the universe by a factor of around $10^{26}$ [6]. This phase precedes the radiation-dominated era described in section 2.3.

This exponential expansion cannot be driven by ordinary matter as seen in section 2.2 but it can also not be driven by the dark energy discussed in section 2.4. Indeed the scales are very different since the energy density required for inflation is at least 60 orders of magnitude larger than the dark energy density which is measured nowadays [6]. Hence, another new constituent in the form of a dynamical scalar field was conjectured to drive this inflationary phase: the "inflaton". If this field denoted by $\phi$ is minimally coupled to gravity its presence would modify the GR action as such:

$$\mathcal{S} = \int \mathrm{d}^4x\sqrt{-g}(\frac{Q}{2} + \mathcal{L}_m - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi))) \tag{21}$$

with the two additional terms being respectively the kinetic and potential terms of the field $\phi$. The energy-momentum tensor for this field yields the following energy density and pressure (noting that we only keep the homogeneous part of the field):

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) \qquad\qquad p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{22}$$

The most accepted scenario for cosmic inflation is the one assuming that the scalar field $\phi$ is slowly rolling toward its ground state. This slow roll is induced by an almost flat potential $V(\phi)$ dominating the kinetic energy of $\phi$. Inflation ends once the ground state, the minimum of the potential, is reached by the field. Since the dominating potential energy is nonzero then by 22 the pressure is negative. More specifically, from $V(\phi) \gg \dot{\phi}$, the equation of state parameter for the inflaton becomes $w_\phi \approx -1$ and we recover an accelerated expansion as explained in section 2.2. This scenario bears the name of "slow roll inflation".

An important fact is that the inflaton field can emerge from modified gravity. Indeed the extra degrees of freedom yielded by the $Q \mapsto f(Q)$ promotion can correspond to a scalar field. Usually in order to explicit those degrees of freedom in the form of a scalar-tensor theory we perform a conformal transformation which is a local angle-preserving rescaling of the metric. This allows to transform dynamics from the Jordan to the Einstein frame where the Jordan frame is the one used to describe the action in 10 and the Einstein frame the action in 21. However it is not guaranteed that the scalar field will be minimally coupled to gravity in every regime [12].

One example of a successful modified gravity theory explaining inflation is the Starobinksy model [13]. In this model the function $f(R)$ is of the form $f(R) = R + R^2/6M^2$ where $M$ is some constant. Performing the aforementioned conformal transformation this function corresponds in the Einstein frame to a potential of the form $V(\phi) = M^2(1 - e^{-\sqrt{2/3}\phi})^2$ allowing a slow roll of the scalar field.

# 3 Choice of $f(Q)$

In order to study in more detail the above discussion we need at one point to choose an Ansatz for the explicit form of $f(Q)$. To restrict the endless possibilities for this function we can summon both theoretical and observational considerations. This section presents some examples.

## 3.1 Theoretical considerations

Recalling section 1.4 it is reasonable to assume that $f(Q)$ would be based on a correction on top of simply having $Q$. Mathematically speaking this motivates writing the function as a power series including all natural powers of the non-metric scalar. Or analogously to the Starobinsky model we could for instance think of just keeping the quadratic correction by setting $f(Q) = Q + \alpha Q^2$ with $\alpha$ some constant.

The evolution of cosmological perturbations is also an important consideration in the case of discriminating potential forms of the function. Indeed since all structure in the universe arises from the perturbations applied to the homogeneous and isotropic background it is a crucial feature that needs to be compatible with the chosen function. More specifically it should be checked that there are no coupling problems in the scalar, vector or tensor sector of the perturbations. This procedure is elaborated in great detail in [14].

## 3.2 Observational considerations

The observational data recovered from measurements is precious when it comes to constrain theoretical models. A great number of experiments have been set up in order to extract as much information from phenomena such as gravitational waves, the CMB, gravitational lensing or baryon acoustic oscillations (BAO). Quantities can be extracted from these measurements which are also theoretically calculable allowing us to test the models and their parameters.

In particular when taking under scrutiny the compatibility of $f(Q)$ and inflation by applying the conformal transformation discussed in section 2.5, we can compare the properties of the induced slow roll potential with quantities which are measurable today. For instance in [12] they compared these theoretical implications of two different forms of $f(Q)$ with observational data extracted from the 2018 Planck mission [15], namely the ratio of tensor to scalar power spectra and the scalar spectral index both linked to the primordial fluctuations power spectrum.

# 4 Dynamical analysis

## 4.1 Autonomous system

We know that the equations ruling the evolution of spacetime are highly non-linear and do not admit a general solution. However it is possible to study the global behaviour of different cosmological scenarios by considering the evolution of spacetime as a complex dynamical system [16]. This allows us to make use of our familiar mathematical understanding of differential equations to treat dynamical systems.

The modified Friedmann equations seen in 18 can be cast into a one-dimensional autonomous ordinary differentiable equation (ODE) for the Hubble parameter $H$. Indeed from the second Friedmann equation we have a relation for the time derivative $\dot{H}$ which when combined with the first equation yields:

$$\dot{H} = -\frac{3}{2}(1 + w_{eff})H^2 =: F(H, w_{eff}) \tag{23}$$

with $w_{eff}$ defined in section 2.2 containing the modifications from $f(Q)$. We can also write 23 in terms of the standard equation of state parameter $w$:

$$\dot{H} = -\frac{1}{2}(1+w)\frac{6f'H^2 - \frac{f}{2}}{12H^2 f'' + f'} =: \tilde{F}(H, w) \tag{24}$$

Note that 23 and 24 are completely equivalent and just represent two ways of expressing the Friedmann equations as an ODE since in both cases $\dot{H}$ only depends on $H$ in a way ruled by $F$ or $\tilde{F}$ respectively (they are not denoted by the same function because of their different arguments). The evolution of the Hubble parameter also naturally depends on the universe content described by the equation of state parameter (but this will be either a fixed quantity or it will depend on $H$).

## 4.2 Fixed points and stability

Studying the dynamical evolution of the Hubble parameter amounts to analysing $F$ or $\tilde{F}$ without needing to know the exact analytical solution of 23 or 24 (from now on the discussion will only focus on 23 for pure readability reasons since they both are equivalent).

The first property that we are interested in are fixed points. These occur when $H$ remains constant and thus $\dot{H} = 0$. From the ODE we deduce that the fixed points correspond to the roots of $F$. We denote them by $H^*$ and thus $F(H^*) = 0$. Physically speaking they describe a de Sitter solution when nonzero since the Hubble parameter $H = H^*$ is constant and if a trajectory starts from a fixed point they will by definition remain with this same Hubble parameter.

A second important property is the stability of these fixed points. Indeed we can ask ourselves what would happen if there is a small deviation from $H^*$. This is ruled by the first derivative of $F$ with respect to $H$ and yields three different types of fixed points:

- $H^*$ stable $\Leftrightarrow F'(H^*) > 0:$ the deviation from the fixed point decays exponentially such that the trajectory is attracted to it. $H^* \neq 0$ is called a stable de Sitter solution.

- $H^*$ semi-stable $\Leftrightarrow F'(H^*) = 0:$ since the derivative changes sign at this point it is unstable on one side and stable on the other.

- $H^*$ unstable $\Leftrightarrow F'(H^*) < 0:$ the deviation from the fixed point grows exponentially such that the trajectory is repelled by it. $H^* \neq 0$ is called an unstable de Sitter solution.

From equation 24 we observe that fixed points occur when $H^2 = \frac{f(Q)}{12f'(Q)}$. Recalling that in CGR the non-metric scalar takes the value $Q = 6H^2$ we understand that the number of fixed points and their properties directly depend on the shape of $f(Q)$.

## 4.3 Phase portrait

The above dynamical properties can be visualised through the phase space diagram of $H$ corresponding to drawing $\dot{H}$ versus $H$. The ODE in the form of 23 will be

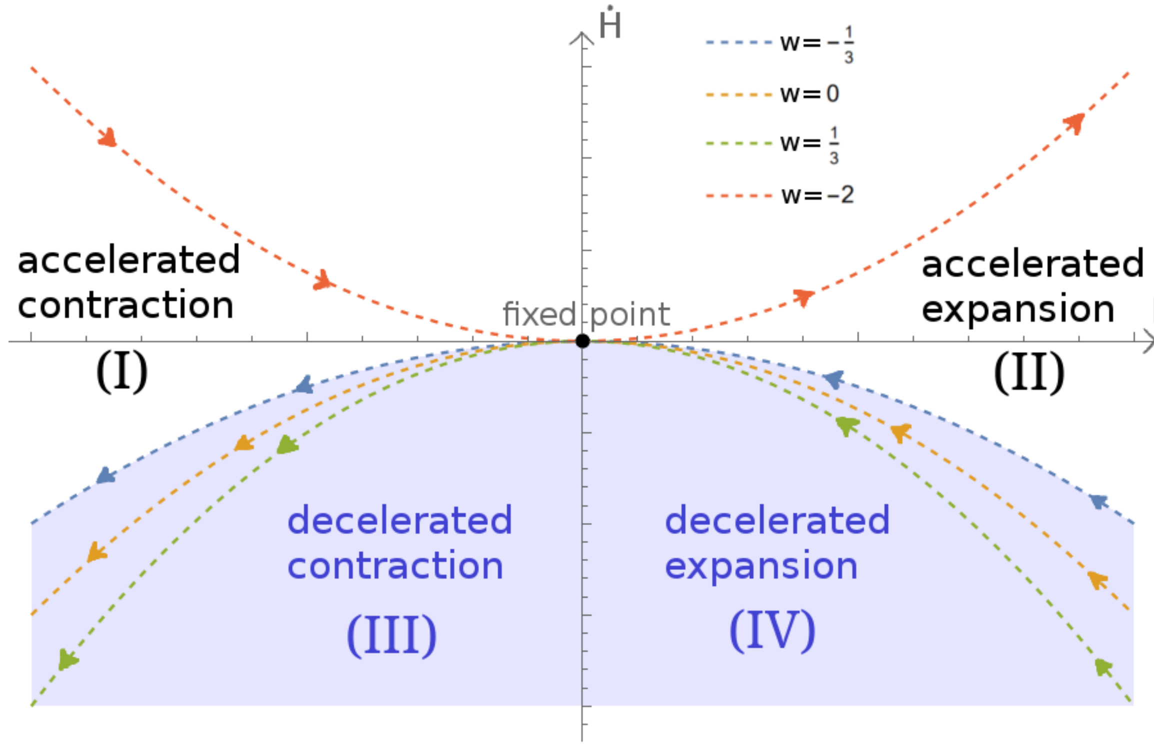



Figure 2: Main patches of the phase portrait for a universe with fixed effective equation of state $w_{eff} = w \in \mathbb{R}$ (c.f. 23). Different equation of state parameters are given and the fixed point is denoted by a black dot (units are omitted since the plot is scale-invariant).

more convenient to understand the different sections of this phase portrait. Indeed when assuming that $w_{eff}$ is constant and thus independent of $H$ we can easily plot the cosmological trajectories for a universe with fixed effective equation of state.

Figure 2 shows the phase portrait for such a universe for different equation of state parameters. The relations derived in 2.2 allow us to split the phase portrait into sections corresponding to distinct cosmological scenarios.

First recalling 16 we can split the portrait in two: the left part corresponding to $H < 0$ and thus a contraction of space (zone I and III) and the right one where $H > 0$ (zone II and IV) i.e. where space is expanding. Secondly from 20 we can again divide the portrait in two zones where the universe is either contracting/expanding in an accelerated manner (zone I and II) or decelerated manner (zone III and IV). Finally we can also mathematically deduce the orientation of the flow depending on the patch where it is taking place: for the upper patch where $\dot{H} > 0$ the flow will go from left to right since $H$ grows larger. Analogously for the lower patch where $\dot{H} < 0$ the flow goes from right to left corresponding to a decrease of $H$.

In this diagram there is only one fixed point located at $(H, \dot{H}) = (0, 0)$ which is simply a Minkowskian universe as seen in 16. This trivial setting is due to the fact that we did not let $w_{eff}$ depend on $H$ yet.

To visualise the cosmological trajectory for any arbitrary fixed effective equation of state (i.e. $\forall w_{eff} \in \mathbb{R}$) we can plot the phase portrait as a vector field which would indicate the general trend of trajectories (each one is defined by an equation of state

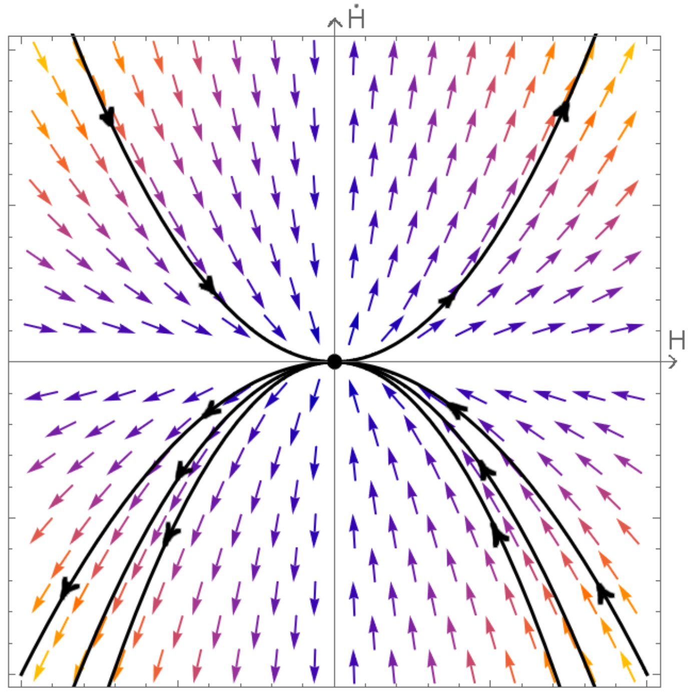



Figure 3: Phase portrait for all fixed effective equations of state (c.f. 23). The colour of the arrows represent their magnitude, brighter being greater. The four trajectories from figure 2 are reproduced to show how they fit in the general trend (units are omitted since the plot is scale-invariant).

parameter or initial conditions for the Hubble parameter). This is done in figure 3. Moreover such a plot favors an even simpler visualisation of the stability of a fixed point, in this case we clearly observe that it is semi-stable. Note that this phase portrait is a generalisation of figure 2 and the trajectories plotted on the latter are reproduced.

Visually a stable fixed point is then where a flow stops and an unstable fixed point corresponds to the start of a flow. They respectively induce a convergence and divergence of the trajectories in the phase space.

## 4.4 Sought cosmological trajectory

Now that the important principles of the dynamical analysis have been introduced we are ready to define the trajectory we are looking for. The main goal of this report is to seek a unification of the theory of cosmic inflation with the actual dark energy dominated universe. As seen earlier we can characterise these two epochs through the background behaviour namely its expansion.

Concerning the initial conditions of the trajectory, recalling section 2.5, the majority of models predict that the universe at that very early epoch was best described by a de Sitter universe. This means that the trajectory we are looking for should bear a constant $H > 0$ at its beginning. It could be achieved by a deviation from an unstable de Sitter fixed point.

Moreover there is observational evidence that in late times when dark energy is the dominant constituent, see section 2.4, the universe is evolving like a de Sitter universe:

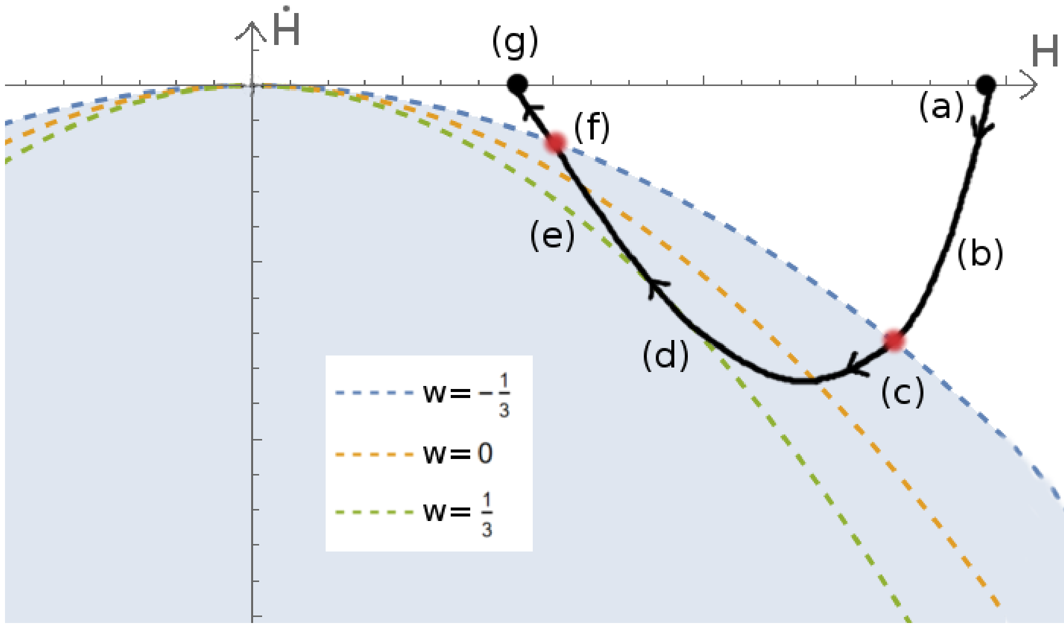



Figure 4: Example of a cosmological trajectory compatible with the desired features. The black dots represent fixed points and red dots represent a transition from decelerated to accelerated (or accelerated to decelerated) expansion. The different parts of the trajectory are labelled by the letters listed in section 4.4.

we say that the universe is as asymptotically de Sitter. The dynamical implication in phase space is that the sought trajectory should end in a stable de Sitter fixed point.

Altogether, considering the epochs between inflation and dark energy domination (see section 2.3), we need to find a trajectory incorporating the following features:

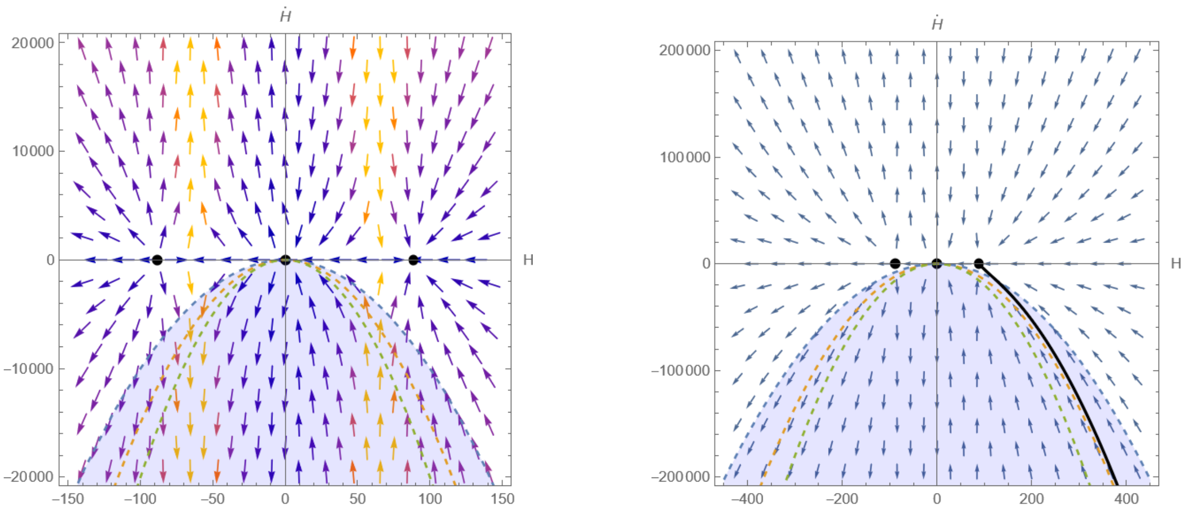
- **(a)** de Sitter universe with positive $H$
- **(b)** Accelerated expansion of space corresponding to the inflationary phase
- **(c)** Graceful exit from inflation consisting of a transition to a decelerated expansion
- **(d)** Radiation dominated universe going through decelerated expansion
- **(e)** Matter dominated universe still going through decelerated expansion
- **(f)** Transition to a dark energy dominated universe expanding in an accelerated way
- **(g)** Asymptotic convergence towards a stable de Sitter universe with positive $H$

Figure 4 summarises all of the above features in the now familiar phase portrait focusing on zones II and IV since we are looking at an expanding universe.

### 4.5 Dynamical effective equation of state

We are now ready to study cosmological trajectories for general equation of state parameters $w_{eff} = w_{eff}(H)$ corresponding to trajectories affected by $f(Q)$. Recalling section 3 we can test the two Ansatz functions suggested by [12] and see if they exhibit the desired features described in the last section.

The first function strongly resembles the square correction described in section 3.1. It is based on the so called "$Q^n$ model": $f(Q) = Q + \alpha Q^n$ with $\alpha = -4.5 \cdot 10^{-6}$ and $n = 2.04$. The phase portrait for this function is shown in the figure 5a. As we can see it has one stable and one unstable de Sitter fixed points in addition to the usual Minkowski fixed point. This gives rise to trajectories which can transition from decelerated to accelerated expansion and then converge to a de Sitter universe. Figure 5b shows an example of such a trajectory which thus verifies conditions (e), (f) and (g) from section 4.4.




(a) General phase portrait

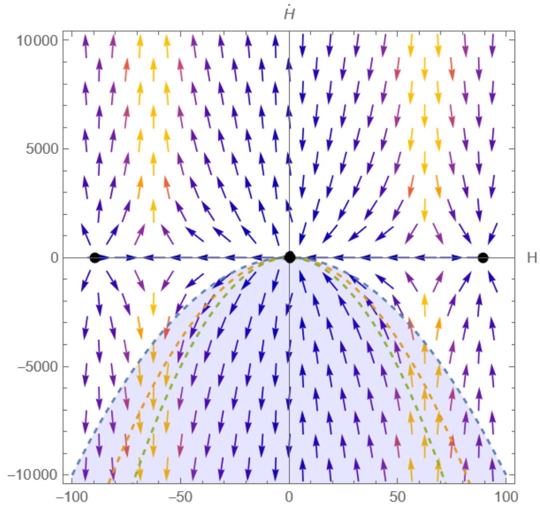
(b) Selected trajectory

Figure 5: $f(Q) = Q + \alpha Q^n$ with $\alpha = -4.5 \cdot 10^{-6}$ and $n = 2.04$

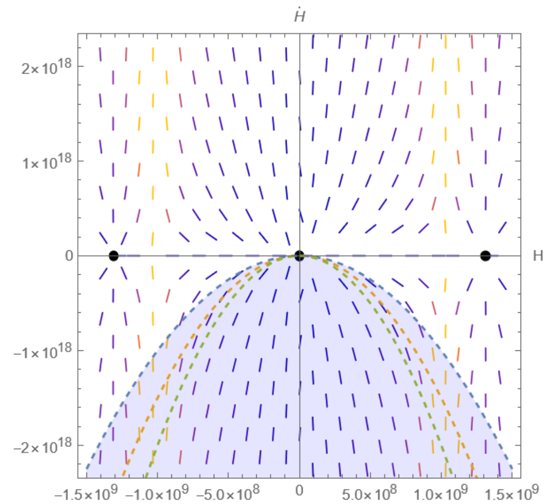
The second function consists of the quadratic model corrected by a logarithmic term: $f(Q) = Q + \alpha Q^2 + \beta Q^2 \ln(Q)$ with $\alpha = 10^{-8}$ and $\beta = -0.025\alpha$. The figure 6a displays the phase portrait for this function and figure 6b shows the same diagram but with a larger scale (the 3 fixed points from figure 6a are too close to be resolved). We see that it is similar to the plot of the above perturbed $Q^n$ model except that it has 2 additional fixed points of the same behaviour which are due to the logarithmic correction. Nevertheless it does not bring new possibilities for interesting trajectories compared to the square correction.

We can also study functions derived from a general logarithmic correction to the $Q^n$ model written as follows: $f(Q) = Q + \alpha Q^n + \beta Q^m \ln(Q)$. One interesting example is the one with $\alpha = 1, \beta = -1, n = 2$ and $m = \frac{3}{2}$. Its phase portrait is depicted in figure 7a and it seems that there are trajectories which could include a graceful exit from inflation and then a transition to radiation and matter domination. Indeed in figure 7b is an example of a trajectory exactly achieving this. We see that it incorporates features (b), (c), (d) and (e) which is non-negligible.

The most compelling case is probably the general logarithmic correction with $\alpha = \frac{1}{100}, \beta = 1, n = \frac{5}{2}$ and $m = \frac{3}{2}$. The phase portrait for this instance, displayed in figure 8a, looks like it bears a richer structure than the above plots in the trends of trajectories. On figure 8b is plotted a trajectory which begins with an accelerated expansion then a

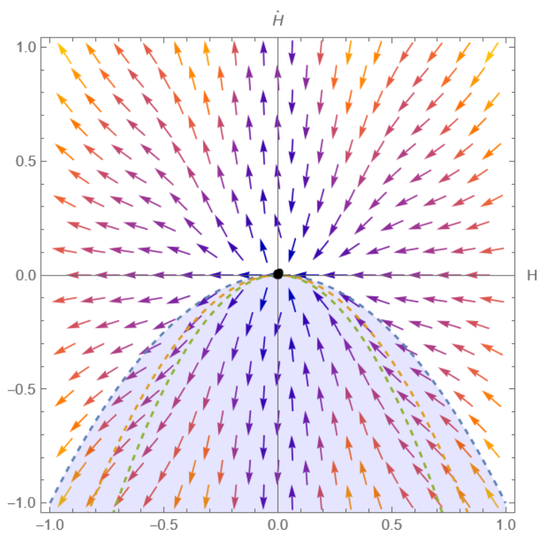



(a) General phase portrait

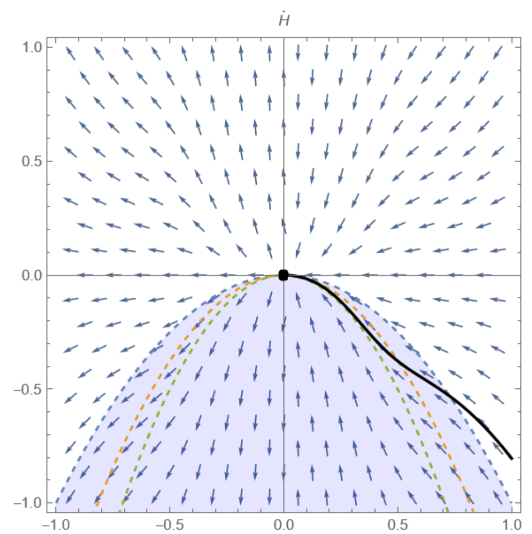



(b) Zoomed out general phase portrait

Figure 6: $f(Q) = Q + \alpha Q^2 + \beta Q^2 \ln(Q)$ with $\alpha = 10^{-8}$ and $\beta = -0.025\alpha$

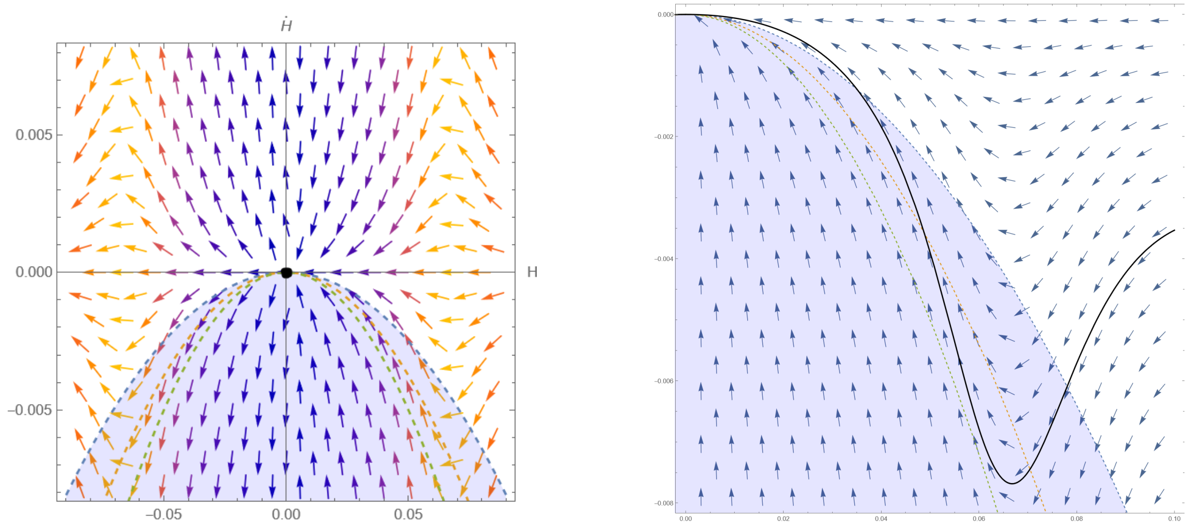



(a) General phase portrait




(b) Selected trajectory

Figure 7: $f(Q) = Q + \alpha Q^n + \beta Q^m \ln(Q)$ with $\alpha = 1, \beta = -1, n = 2$ and $m = \frac{3}{2}$




(a) General phase portrait

(b) Selected trajectory

Figure 8: $f(Q) = Q + \alpha Q^n + \beta Q^m \ln(Q)$ with $\alpha = \frac{1}{100}, \beta = 1, n = \frac{5}{2}$ and $m = \frac{3}{2}$

graceful exit from inflation, goes through the radiation- followed by matter-domination and finally transitions to accelerated expansion. That is a trajectory satisfying (b), (c), (d), (e) and also (f) which is the greatest number of validated features compared to all of the above functions.

We thus saw how different forms of $f(Q)$ influence the configuration of the phase portrait and what features they each possess. However we still are missing the function which aligns all the aspects enumerated in section 4.4. If we manage to combine all of the analytical behaviours seen above in one function it could potentially achieve the sought trajectory.

## 4.6 Universe content

If we desire to study how the constituents of the universe evolve with its geometry during the trajectory we can decide to change the way we draw the phase diagram. What could make sense in that case is to plot the energy density $\rho$ against $H$ (having information about $\rho$, the pressure $p$ is deducible from the equation of state in 15). Indeed we can consider that plotting $H$ and $\dot{H}$ is redundant since both variables give an indication about the space geometry.

The system that we would study is described by two coupled autonomous ODEs ruling the evolution of the parameters $H$ and $\rho$. The first one originates from the Friedmann equations in 12 and the second one is simply the continuity equation in 14. In both cases we make use of the equation of state to get rid of $p$.

Conventionally instead of working with the energy density $\rho$ we introduce what is called the density parameter $\Omega$, defined as being the ratio between $\rho$ and the critical energy density $\rho_c$. The latter is the density required to achieve an expanding universe. From the standard Friedmann equation we see that this is $\rho_c = 3H^2$ (remember that we work in natural units where $c = 1$ and $G = \frac{1}{8\pi}$). So ultimately we have that $\Omega = \frac{\rho}{3H^2}$ meaning that if $\Omega > 1 (< 1)$ then the universe expands (contracts).

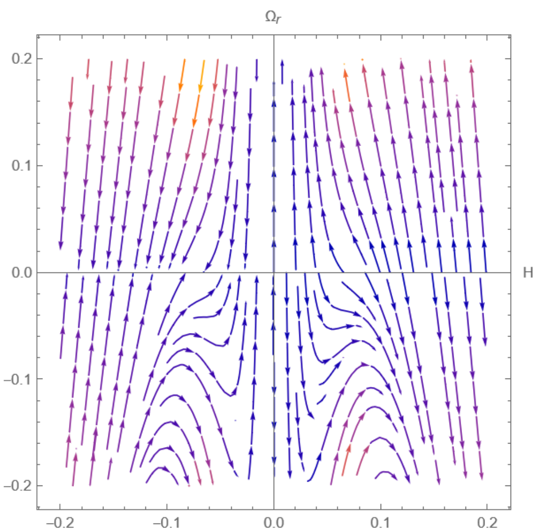
Now this equivalent parameter can be defined for all different possible constituents.

For instance in a universe where only radiation and matter can be present ($\rho = \rho_r + \rho_m$) we have that the total density parameter is naturally $\Omega = \Omega_m + \Omega_r$ where $\Omega_m = \frac{\rho_m}{3H^2}$ and $\Omega_r = \frac{\rho_r}{3H^2}$. This allows us to study the universe at early epochs (relevant for inflation) by assuming that $\Omega = \Omega_r$ or late epochs (relevant for dark energy) by assuming that $\Omega = \Omega_m$.

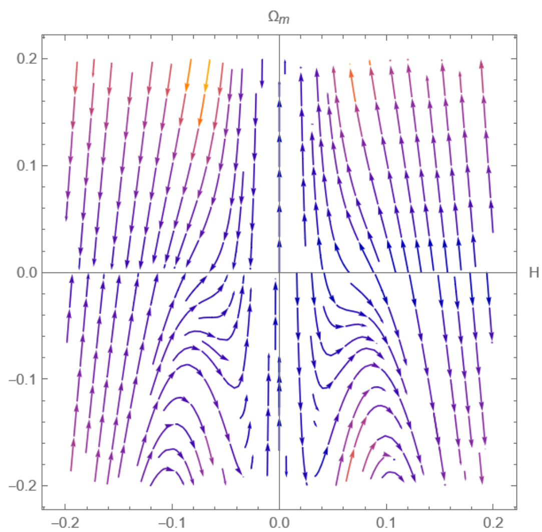
The system of equations that we want to study is finally (the $\dot{H}$ term in the second ODE can be replaced by $F_1(H, \Omega)$ keeping the system autonomous):

$$\begin{aligned} \dot{H} &= -\frac{1}{2}(1+w)\frac{3H^2\Omega}{12H^2 f'' + f'} =: F_1(H, \Omega) \\ \dot{\Omega} &= (3(1+w)H - 2\frac{\dot{H}}{H})\Omega =: F_2(H, \Omega) \end{aligned} \tag{25}$$

Figures 9a and 9b show this phase portrait for the last function from section 4.5 ($f(Q) = Q + \alpha Q^n + \beta Q^m \ln(Q)$ with $\alpha = \frac{1}{100}, \beta = 1, n = \frac{5}{2}$ and $m = \frac{3}{2}$) within a radiation universe ($\Omega = \Omega_r$) and a matter universe ($\Omega = \Omega_m$). Both plots look relatively similar since the dominating constituent only changes the ruling ODEs through the equation of state parameter which in 25 simply acts as a scaling factor. We see that the plot show some trajectories having a relatively constant Hubble parameter at their starting point. Recalling section 4.4 this is a feature that we are looking for. Altogether, such plots should be studied in greater detail for other functions which incorporate more features than this one.




(a) Radiation universe




(b) Matter universe

Figure 9: Phase portraits for the system described in 25 with $f(Q)$ being a special case of the general logarithmic correction.

# 5 Conclusion and future perspectives

In summary, we have reviewed in this report how GR can be stated in an equivalent manner through non-metricity instead of curvature. We went through interesting properties of this formulation and in particular how the specific choice of a gauge (CGR) can remarkably simplify computations and offer precious theoretical advantages. We then saw how modified non-metric gravity in this coincident gauge alters the familiar Friedmann equations ruling the background evolution and what is the relation of the latter with the dominating constituent in the universe. After reassessing the main cosmological epochs complying with current observations we paved our way through the popular theories incorporating these observations and understood what wished to be unified by $f(Q)$ theories.

As a way to look for this unification we introduced notions from dynamical analysis which amounts to treat the evolution of spacetime as a complex system. This allowed us to define more rigorously what features were desired in order to include both early epoch and late epoch behaviours in one continuous trajectory. We saw multiple examples of trajectories bearing some wanted features of this unification. The sought trajectory combining all of them is thus still to be found but there were rather interesting cases that could hint at the direction in which we should further explore.

Here is a non-exhaustive list of possible future concepts which would be relevant to investigate as a follow-up to the above discussion:

**Choice of function:** We can naturally study any other form of $f(Q)$ as long as it meets the theoretical and observational considerations developed in section 3. Nevertheless by taking them under further scrutiny we could maybe challenge those considerations which will open the door for completely different forms of $f(Q)$ exhibiting the desired features.

**Dynamical connection:** Instead of working in the CGR paradigm where the connection vanishes it could be interesting to look upon dynamical choices (corresponding to $\Gamma_{\mathbb{Q}}^{(I)}$ and $\Gamma_{\mathbb{Q}}^{(II)}$ in [3]) which possess additional degrees of freedom (carried by $\mathcal{N}$ and $C_2$). The latter could be chosen as perhaps bearing the ability to induce an accelerated expansion of space at early and late epochs. Note that this new choice of connection would also alter the metric field equations and consequently the Friedmann equations, meaning that 12 will not hold anymore.

**Conformal transformation:** The action in 10 could be examined as a scalar-tensor theory after transforming the dynamics to the Einstein frame. In [12] this is done but then emerges a coupling between gravity and a scalar field which is non-minimal. In the mentioned paper, this coupling is neglected for particular regimes but when left intact it can be studied in more detail to see if it holds relevant expansion properties. The said action in the Einstein frame is finally of the form:

$$\mathcal{S} = \int \mathrm{d}^4x\sqrt{-g}(\frac{Q}{2} - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) - U(\phi)g^{\alpha\beta}Q^\mu{}_{\alpha\beta}\partial_\mu\phi + \mathcal{L}_m)$$

where $U(\phi) = -\frac{7}{8\sqrt{5}}(\exp(-\frac{\phi}{\sqrt{5}}) - 1)$ and $V(\phi)$ is defined in the paper. The non-minimal coupling thus arises from the mixed term involving the non-metricity tensor and the scalar field.

## Acknowledgments

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