

Complexity of Dependences in Bounded Domains, Armstrong Codes, and Generalizations

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Abstract—The study of Armstrong codes is motivated by the problem of understanding complexities of dependences in relational database systems, where attributes have bounded domains. A (q, k, n) -Armstrong code is a q -ary code of length n with minimum Hamming distance $n - k + 1$, and for any set of $k - 1$ coordinates, there exist two codewords that agree exactly there. Let $f(q, k)$ be the maximum n for which such a code exists. In this paper, $f(q, 3) = 3q - 1$ is determined for all $q \geq 5$ with three possible exceptions. This disproves a conjecture of Sali. Furthermore, we introduce generalized Armstrong codes for branching, or (s, t) -dependencies, construct several classes of optimal Armstrong codes, and establish lower bounds for the maximum length n in this more general setting.

Index Terms—Relational database, Armstrong codes, functional dependency, extorthogonal double covers.

I. INTRODUCTION

LET A be a set of n attributes. Each attribute $x \in A$ is associated with a set Ω_x , called its *domain*. A *relation* is a finite set R of n -tuples (called *data items*), such that $R \subseteq \times_{x \in A} \Omega_x$. A relational database *table* is an $m \times n$ array where each column is indexed by an attribute and each row corresponds to a data item in R . We denote this table by $R(A)$. More specifically, if $R = \{(d_{i,x})_{x \in A} : 1 \leq i \leq m\}$, then the cell in $R(A)$ with row index i and column index x has entry $d_{i,x}$. A *relational database* is a set of tables, where different tables may be defined over different attribute sets.

For a given table $R(A)$ and $X \subseteq A$, the X -value of a data item $d = (d_x)_{x \in A}$ in $R(A)$ is the $|X|$ -tuple $d|_X = (d_x)_{x \in X}$. Let $X \subseteq A$ and $y \in A$ for a given table $R(A)$. We say that y (*functionally*) *depends* on X , written $X \rightarrow y$, if no two rows of $R(A)$ agree in X but differ in y . In other words, if the X -value of a data item is known, then its $\{y\}$ -value can be determined with certainty. A *key* for $R(A)$ is a subset $K \subseteq A$, such that $K \rightarrow b$ for all $b \in A$. A key K is called *minimal* if no subset of K is a key.

Identifying functional dependencies, especially key dependencies, is important in relational database design [3]–[5], [22]. From the schema design point of view, the question of whether a given collection Σ of functional

dependencies has an *Armstrong instance* for Σ , that is, a table that satisfies a functional dependency $X \rightarrow y$ if and only if $X \rightarrow y$ is implied by Σ , is well studied. The existence of an Armstrong instance for any given set of functional dependencies was proved by Armstrong [3] and Demetrovics [9]. Further investigations (see for example, [11]) concentrated on the minimum size of an Armstrong instance, since it is a good measure of the complexity of a set of functional dependencies, or a set of minimal keys.

Earlier work on Armstrong instances were mostly studied by assuming that the domain of each attribute is countably infinite. Recently, the study of *higher order data model* in [18] and [23] considered the question of Armstrong instances with bounded domains. Another reason for considering bounded domains is that for many attributes, their domains are well defined finite sets. For example, the age of a person can take values from the set $\{0, 1, \dots, 130\}$.

Thalheim [27] investigated the maximum number of minimal keys in the case of bounded domains and showed that restrictions on the sizes of domains make significant differences. It is natural to ask what one can say about Armstrong instances if all attributes have domains restricted to size q . Let K_n^k denote the collection of all k -subsets of an n -element attribute set A .

Definition 1: Let $q, k > 1$ be integers. Let $f(q, k)$ denote the maximum n such that there exists an Armstrong instance for K_n^k being the system of minimal keys.

The problem of determining $f(q, k)$ was introduced in [25] and investigated in [21] and [26]. The only known values of $f(q, k)$ are $f(q, 2) = \binom{q+1}{2}$, which were determined in [21].

One of the main contributions of this paper is the determination of $f(q, 3)$. We prove that $f(q, 3) = 3q - 1$ for all $q \geq 5$, except possibly for $q \in \{14, 16, 20\}$. This disproves a conjecture of Sali [24].

When functional dependencies are not known, the concept was generalized to (s, t) -dependencies to improve storage efficiency [11]–[13], [20]. In this paper, we introduce the analogous problem of determining $f(q, k)$ when extended to (s, t) -dependencies, that is the function $f_{s,t}(q, k)$ (see Section IV for detailed definition). We show that $f_{1,t}(q, 2) = \binom{q+t+1}{t+1}$, $f_{2,2}(q, 4) = 2q - 1$ and establish several lower bounds of $f_{1,t}(q, k)$ by constructive method and probabilistic method.

II. PRELIMINARIES

Throughout this paper, we view an $m \times n$ Armstrong instance with domains of size q as a q -ary code C of length n

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and size m , where the codewords are precisely the rows of the instance.

For a positive integer k , $[k]$ denotes the set of integers $\{1, 2, \dots, k\}$ and \mathbb{Z}_k denotes the ring of integers modulo k . For any subset $S \subset \mathbb{Z}_k$, let $aS \triangleq \{as : s \in S\}$ and $a + S \triangleq \{a + s : s \in S\}$.

A. Armstrong Codes

Katona et al. [21] characterized the q -ary code C corresponding to an Armstrong instance with K_n^k as the set of minimal keys, as follows:

- (i) C has minimum Hamming distance at least $n - k + 1$;
- (ii) for any set of $k-1$ coordinates there exist two codewords agreeing in exactly those coordinates.

A $(k-1)$ -set of coordinates can be considered as a “direction”, so in C the minimum distance is *attained in all directions*. Such a code C is called an *Armstrong code*, or more precisely, a (q, k, n) -Armstrong code. It is obvious that $f(q, k)$ is the maximum n such that there exists a (q, k, n) -Armstrong code. The following bounds on $f(q, k)$ are known.

Theorem 1 (Blokhuys et al. [6], Katona et al. [21], Sali and Székely [26]):

- (i) Let $q > 4$. Then $f(q, k) \geq \lceil \frac{k}{2} \log q - 1 \rceil$ for all sufficiently large k .
- (ii) $f(2, k) \geq k + 3$ for all $k \geq 7$. Further, there exists a constant $c > 1$ such that $f(2, k) \geq \lfloor ck \rfloor$ for all sufficiently large k .
- (iii) Let $q > 1$ and $k > 2$. Then

$$f(q, k) \leq q(k-1) \left(1 + \frac{q-1}{\sqrt{\frac{2(qk-q-k+2)^{k-1}}{(k-1)!} - q}} \right). \quad (1)$$

- (iv) If $q \geq 2$ and $k \geq 5$, then the bound (1) can be improved to $f(q, k) \leq q(k-1)$, except when $(k, q) \in \{(5, 2), (5, 3), (5, 4), (5, 5), (6, 2)\}$.
- (v) For fixed $q > 1$, we have

$$\frac{\sqrt{q}}{e} k < f(q, k) < (q - \log q)k$$

for all sufficiently large k .

Proposition 1 (Katona et al. [21]): For $q > 1$, $f(q, 3) \leq 3q - 1$.

B. Orthogonal Double Covers

The concept of orthogonal double covers originates in conjectures of Demetrovics et al. [10] concerning database constraints and was formalized later by Ganter et al. [15]. Let X be a finite set. A partition of X is said to *cover* $T \subseteq X$ if T is contained in some part of the partition. Let K_m denote a complete graph on m vertices. For convenience, let $a_1 K_{m_1} \cup \dots \cup a_s K_{m_s}$ denote the disjoint union of a_i copies of K_{m_i} , $1 \leq i \leq s$.

Definition 2: Let X be a set of size m . A set of partitions of X is called an *orthogonal double cover* (ODC) of K_m (with its vertices identified with elements of X) if it satisfies the following properties:

- (i) for any two partitions, there is exactly one 2-subset of X that is covered by both partitions;
- (ii) each 2-subset of X is covered by exactly two different partitions.

A construction of $(q, 3, n)$ -Armstrong codes from ODC's was introduced by Sali in [24]. View each part of a partition of X as a complete subgraph of K_m over X . Then each partition can be regarded as a disjoint union of complete subgraphs of K_m . Note that a part of size one corresponds to a complete subgraph consisting of only one isolated vertex. If an ODC consists of n partitions, each of which is isomorphic to a graph G , then we say the ODC is an *ODC by n G 's*. Suppose that G is a disjoint union of q complete subgraphs, then an ODC of K_m by n G 's gives an $m \times n$ Armstrong instance over $[q]$ as follows. For each partition of the ODC, arbitrarily order the q parts, and construct a column u of length m , with coordinates indexed by elements of X , such that for $i \in X$, $u_i = j$ if and only if i is contained in the j -th part of the partition. It is easy to check that the set of rows of this Armstrong instance is a $(q, 3, n)$ -Armstrong code.

Example 1: In [10], there is an ODC of K_7 by seven $2K_3 \cup K_1$'s over \mathbb{Z}_7 with each partition P_i , $i \in \mathbb{Z}_7$ consisting of three parts $\{i\}$, $i + \{1, 2, 4\}$ and $i + \{3, 5, 6\}$. Then a $(3, 3, 7)$ -Armstrong code is constructed as below.

3	2	2	1	2	1	1
1	3	2	2	1	2	1
1	1	3	2	2	1	2
2	1	1	3	2	2	1
1	2	1	1	3	2	2
2	1	2	1	1	3	2
2	2	1	2	1	1	3

Ganter and Gronau [14] proved that for $q \geq 5$, there exists an ODC of K_{3q-2} by $3q-2$ $(q-1)K_3 \cup K_1$'s, settling a conjecture of Demetrovics et al. [10]. This result also implies the existence of a $(q, 3, 3q-2)$ -Armstrong code. Hence, we have $f(q, 3) \geq 3q-2$. Furthermore, it is easy to show that $f(2, 3) = 4$. This led Sali [24] to make the following conjecture.

Conjecture 1 (Sali [24]): For all $q \geq 2$, $f(q, 3) = 3q - 2$.

Unfortunately, this conjecture is false. When $m \geq 2$, an ODC of K_{6m+2} by $6m+2$ $2mK_3 \cup K_2$'s has been constructed by Gronau et al. [17]. This gives a $(2m+1, 3, 6m+2)$ -Armstrong code and hence $f(2m+1, 3) \geq 6m+2$. Thus, Conjecture 1 is false for all odd $q \geq 5$. One of the primary aims of this paper is to prove that Conjecture 1 is also false for even q . In fact, we determine that $f(q, 3) = 3q - 1$ for all $q \geq 5$, with three possible exceptions.

III. $(q, 3, 3q-1)$ -ARMSTRONG CODES

We prove $f(q, 3) = 3q - 1$ by showing the existence of $(q, 3, 3q-1)$ -Armstrong codes. Our proof is constructive and uses techniques from combinatorial design theory. We briefly review some required concepts below.

A. Combinatorial Designs

A *set system* is a pair $\mathfrak{S} = (X, \mathcal{A})$, where X is a finite set of *points* and $\mathcal{A} \subseteq 2^X$. Elements of \mathcal{A} are called *blocks*.

The *order* of \mathfrak{S} is the number of points in X , and the *size* of \mathfrak{S} is the number of blocks in \mathcal{A} . Let K be a set of positive integers. A set system (X, \mathcal{A}) is K -uniform if $|A| \in K$ for all $A \in \mathcal{A}$. A *parallel class* of a set system (X, \mathcal{A}) is a set $\mathcal{P} \subseteq \mathcal{A}$ that partitions X . A *resolvable set system* is a set system whose set of blocks can be partitioned into parallel classes.

Definition 3: A triple system $\text{TS}(m, \lambda)$ is a $\{3\}$ -uniform set system (X, \mathcal{A}) of order m such that every 2-subset of X is contained in exactly λ blocks of \mathcal{A} .

Definition 4: Let (X, \mathcal{A}) be a set system and let \mathcal{G} be a partition of X into subsets, called *groups*. The triple $(X, \mathcal{G}, \mathcal{A})$ is a *group divisible design* (GDD) when every 2-subset of X not contained in a group is contained in exactly one block, and $|A \cap G| \leq 1$ for all $A \in \mathcal{A}$ and $G \in \mathcal{G}$.

We denote a GDD $(X, \mathcal{G}, \mathcal{A})$ by k -GDD if (X, \mathcal{A}) is $\{k\}$ -uniform. The *type* of a GDD $(X, \mathcal{G}, \mathcal{A})$ is the multiset $\{|G| : G \in \mathcal{G}\}$. When more convenient, the exponential notation is used to describe the type of a GDD: a GDD of type $g_1^{t_1} g_2^{t_2} \cdots g_s^{t_s}$ is a GDD where there are exactly t_i groups of size g_i , $i \in [s]$. The following results are known (see [1], [16]).

Theorem 2:

- (i) A resolvable $\text{TS}(m, 2)$ exists if and only if $m \equiv 0 \pmod{3}$ and $m \neq 6$.
- (ii) There exists a 4-GDD of type $2^u m^1$ for each $u \geq 6$, $u \equiv 0 \pmod{3}$ and $m \equiv 2 \pmod{3}$ with $2 \leq m \leq u - 1$, except for $(u, m) = (6, 5)$ and possibly except for $(u, m) \in \{(21, 17), (33, 23), (33, 29), (39, 35), (57, 44)\}$.

B. Extorthogonal Double Covers

A *suborthogonal double cover* (subODC) is a collection of partitions of $[m]$ similar to an ODC except that for any two partitions there is *at most* one 2-subset of $[m]$ covered by both partitions. SubODCs were first studied by Hartmann and Schumacher [19], who considered them as generalized ODCs under circumstances when ODCs do not exist. Here, we consider another generalization, called *extorthogonal double covers* (extODC). These are similar to ODCs, except that for any two partitions there is *at least* one 2-subset of $[m]$ covered by both partitions. We construct $(q, 3, 3q - 1)$ -Armstrong codes from a special class of extODCs of K_{3q} by qK_3 's.

Proposition 2: If there exists an extODC of K_{3q} by qK_3 's, then $f(q, 3) = 3q - 1$.

Proof: By considering 2-subsets, the number of partitions in an extODC of K_{3q} by qK_3 's is easily seen to be $2\binom{3q}{2}/3q = 3q - 1$. For each partition, arbitrarily order the q parts. Define a $3q \times (3q - 1)$ q -ary array by indexing each column by a partition and each row by a point of the extODC. For each partition, the corresponding column has the symbol i in the rows indexed by the points in the i th part. The set of rows in this array is a $(q, 3, 3q - 1)$ -Armstrong code, by the definition of an extODC. This, together with Proposition 1, implies that $f(q, 3) = 3q - 1$. \square

It is easy to see that an extODC of K_{3q} by $3q - 1$ qK_3 's is a resolvable $\text{TS}(3q, 2)$ with the additional property that every two parallel classes cover a common 2-subset.

TABLE I

BASE PARTITIONS FOR SOME SMALL extODCs

q	triples
6	$\{0, 1, 2\}\{3, 7, 12\}\{4, 15, \infty\}\{5, 8, 14\}\{6, 10, 13\}\{9, 11, 16\}$
8	$\{0, 1, 2\}\{3, 5, 8\}\{4, 12, 18\}\{6, 15, 19\}\{7, 14, \infty\}\{9, 17, 21\}$ $\{10, 13, 20\}\{11, 16, 22\}$
10	$\{0, 1, 2\}\{3, 5, 8\}\{4, 10, 20\}\{6, 23, \infty\}\{7, 11, 22\}\{9, 17, 27\}$ $\{12, 16, 25\}\{13, 21, 28\}\{14, 19, 26\}\{15, 18, 24\}$
12	$\{0, 1, 2\}\{3, 5, 8\}\{4, 7, 15\}\{6, 19, 34\}\{9, 13, 27\}\{10, 20, 25\}$ $\{11, 22, 28\}\{12, 24, 33\}\{14, 23, 30\}\{16, 26, 32\}\{17, 21, 29\}$ $\{18, 31, \infty\}$

Although $f(q, 3)$ is known for odd q , it is still interesting to know when extODCs of K_m , m odd, can exist. We have the following result for $m = 3q$, q odd.

Proposition 3: There exists an extODC of K_{3q} by qK_3 's, for all odd $q \geq 5$.

Proof: Let $u = (3q - 1)/2$. Starting from a 4-GDD $(X, \mathcal{G}, \mathcal{A})$ of type 2^u , whose existence is guaranteed by Theorem 2, we construct an extODC of K_{2u+1} over $X \cup \{\infty\}$ from \mathcal{A} . For each $x \in X$, let $\mathcal{B}_x = \{B \setminus \{x\} : x \in B \in \mathcal{A}\} \cup \{G \cup \{\infty\} : x \in G \in \mathcal{G}\}$. Then \mathcal{B}_x is a partition of $X \cup \{\infty\}$. We claim that $\{\mathcal{B}_x : x \in X\}$ is an extODC of K_{2u+1} .

Indeed, for any two partitions, say \mathcal{B}_x and \mathcal{B}_y , both of which cover $\{x, y\}$ if x, y are in the same group, and cover $B \setminus \{x, y\}$ if $x, y \in B$ are in distinct groups. For each pair $\{x, y\} \subset X \cup \{\infty\}$, if $\{x, y\} \subset G \cup \{\infty\}$ for some $G \in \mathcal{G}$, then $\{x, y\}$ is covered by two partitions \mathcal{B}_g , $g \in G$; if $x, y \in X$ are in distinct groups, then there exists exactly one block $B \in \mathcal{A}$ such that $\{x, y\} \subset B$, while $\{x, y\}$ is covered by two partitions \mathcal{B}_g , $g \in B \setminus \{x, y\}$. Hence, $\{\mathcal{B}_x : x \in X\}$ is an extODC. \square

We now construct extODCs of K_m , where $m = 3q$ is even. Define a *base partition* of order m , which is a partition P of $\mathbb{Z}_{m-1} \cup \{\infty\}$ into triples with the following two properties:

- (i) $\{\pm(a - b) : \{a, b\} \subset C \in P \text{ and } \infty \notin \{a, b\}\} = 2(\mathbb{Z}_{m-1} \setminus \{0\})$.
- (ii) $\{i : \{a, b\} + i = \{c, d\} \text{ for some } \{a, b\} \subset C, \{c, d\} \subset C' \text{ and } C, C' \in P\} \supset (\mathbb{Z}_{m-1} \setminus \{0\})$, where $\infty + i := \infty$.

Here we use angled brackets $\langle \cdot \rangle$ for multisets. For each $j \in \mathbb{Z}_{m-1}$, let $P_j = \{j + C | C \in P\}$. Then P_j , $j \in \mathbb{Z}_{m-1}$ are partitions of $\mathbb{Z}_{m-1} \cup \{\infty\}$, which form an extODC of K_m . The first property ensures that each pair occurs exactly twice, while the second ensures that any two partitions cover at least one common 2-subset.

Proposition 4: There exists an extODC of K_{3q} by qK_3 's, for $q \in \{6, 8, 10, 12\}$.

Proof: The base partitions for extODC of K_{3q} , for $q \in \{6, 8, 10, 12\}$, are given in Table I. \square

Proposition 5: There exists an extODC of K_{3q} by qK_3 's, for all even $q \geq 18$, $q \neq 20$.

Proof: Let $u = (3q - 18)/2$. There exists a 4-GDD $(X, \mathcal{G}, \mathcal{A})$ of type $2^u 17^1$ by Theorem 2. We construct an extODC of K_{3q} (on $X' = X \cup \{\infty\}$) from \mathcal{A} . Let G_0 be the long group in \mathcal{G} of size 17. By Proposition 4, there exists an extODC of K_{18} (on $G_0 \cup \{\infty\}$) by 17 $6K_3$'s over $G_0 \cup \{\infty\}$. Let the set of partitions be $\{\mathcal{C}_x : x \in G_0\}$. For each $x \in (X \setminus G_0)$, let $\mathcal{B}_x = \{B \setminus \{x\} : x \in B \in \mathcal{A}\} \cup \{G \cup \{\infty\} : x \in G \in \mathcal{G}\}$. For each $x \in G_0$, let $\mathcal{B}_x = \{B \setminus \{x\} : x \in B \in \mathcal{A}\} \cup \mathcal{C}_x$.

There are $3q - 1$ \mathcal{B}_x 's in total and each \mathcal{B}_x is a partition of X' . We claim that the set of all \mathcal{B}_x 's is an extODC.

Indeed, for any two partitions \mathcal{B}_x and \mathcal{B}_y , they both cover $\{x, y\}$ if x, y are in the same group of size 2; cover a common 2-subset if $x, y \in G_0$ since \mathcal{C}_x and \mathcal{C}_y have a common 2-subset, and both cover $B \setminus \{x, y\}$ if $x, y \in B$ are in distinct groups. For each pair $\{x, y\} \subset X'$, if $\{x, y\} \subset G \cup \{\infty\}$ for some $G \neq G_0$, then $\{x, y\}$ is covered in two partitions \mathcal{B}_g , $g \in G$. If $\{x, y\} \subset G_0 \cup \{\infty\}$, then $\{x, y\}$ is covered by both \mathcal{B}_u and \mathcal{B}_v , where $\{x, y\}$ is contained in \mathcal{C}_u and \mathcal{C}_v . If x, y are in distinct groups, then there exists exactly one block $B \in \mathcal{A}$ such that $\{x, y\} \subset B$, while $\{x, y\}$ occurs in \mathcal{B}_g , $g \in B \setminus \{x, y\}$. Hence, $\{\mathcal{B}_x : x \in X\}$ is an extODC of K_{3q} . \square

Combining Propositions 2, 3 and 5, we give the main result of this section.

Theorem 3: For all $q \geq 5$ and $q \neq 14, 16, 20$, there exists an extODC of K_{3q} by qK_3 's, and consequently $f(q, 3) = 3q - 1$.

Before closing this section, we estimate the values of $f(q, 3)$ for $q = 3$ and 4.

Proposition 6: $f(3, 3) = 7$ and $f(4, 3) \in \{10, 11\}$.

Proof: For $q = 3$, we have $f(3, 3) \leq 8$. Suppose that C is a $(3, 3, 8)$ -Armstrong code of size m . We consider the total number s of pairs of equal entries in the same coordinates of C . In any pair of codewords of C at most two coordinates can have equal entries because the minimum distance of C is six. Hence, $s \leq 2 \times \binom{m}{2}$. By the defining condition (ii) of Armstrong codes, for each pair of coordinates there is at least one pair of codewords agreeing in exactly those coordinates. Further, for different pairs of coordinates, the pairs of codewords are different, i.e., the pairs of equal entries are all different. Thus $s \geq 2 \times \binom{8}{2}$ and then $m \geq 8$. However, since C is also a ternary code of distance six, we have $m \leq 9$ [7]. Now we claim that either $m = 8$ or 9 is impossible, hence C does not exist. Consider C as an $m \times 8$ array. When $m = 8$ or 9, each column has at least 7 or 9 pairs of equal entries, respectively, which is achieved when the three symbols occur almost the same frequency. Since the total number s of such pairs in C is at most 56 if $m = 8$ or 72 if $m = 9$, there are exactly 7 or 9 such pairs in each column when $m = 8$ or 9, respectively. When $m = 8$, C is equivalent to an ODC of K_8 by eight $2K_3 \cup K_2$'s, which does not exist by [14]. When $m = 9$, C is equivalent to an extODC of K_9 by eight $3K_3$'s, which could be excluded easily by computer search. So we conclude that $f(3, 3) \leq 7$. An optimal code exists by Example 1.

For $q = 4$, we have $f(4, 3) \leq 11$. A $(4, 3, 10)$ -Armstrong code exists by the existence of an ODC of K_{10} by ten $K_4 \cup 3K_2$'s [14]. \square

IV. GENERALIZED ARMSTRONG CODES

The concept of functional dependencies was generalized by Demetrovics, Katona, and Sali [11].

Definition 5: Let $X \subseteq A$ and $y \in A$ for a given table $R(A)$. Then for positive integers $s \leq t$, we say that y (s, t) -depends on X , written $X \xrightarrow{(s,t)} y$, if there do not exist $t + 1$ data items (rows) d_1, d_2, \dots, d_{t+1} of $R(A)$ such that

- (i) $|\{d_i|_{\{x\}} : 1 \leq i \leq t + 1\}| \leq s$ for each $x \in X$, and
- (ii) $|\{d_i|_{\{y\}} : 1 \leq i \leq t + 1\}| = t + 1$.

Our usual concept of functional dependency is equivalent to the special case of $(1, 1)$ -dependency. When functional dependencies are not known, (s, t) -dependencies identified in a relational database can still be exploited for improving storage efficiency [11]–[13], [20].

Given $1 \leq s \leq t$, an (s, t) -dependent key K is a subset of the attribute set A , such that $R(A)$ satisfies (s, t) -dependencies $K \xrightarrow{(s,t)} y$ for all $y \in A$. A key K is called *minimal* if no subset of K is an (s, t) -dependent key. Here, we generalize Armstrong codes from functional dependencies into (s, t) -dependencies.

A q -ary code C is called a $(q, k, n)_{s,t}$ -Armstrong code if

- (i) for any $t + 1$ rows of C , there exist at most $k - 1$ columns such that each column has at most s distinct elements in the $t + 1$ rows, and
- (ii) for any $k - 1$ columns of C , there exist $t + 1$ rows such that each of the $k - 1$ columns has at most s distinct elements in the $t + 1$ rows. Further, there exists a column having exactly $t + 1$ distinct elements in these $t + 1$ rows.

Consider the Armstrong code defined above as an Armstrong instance with n attributes. The first property in the definition makes sure each k -subset of K_n^k is an (s, t) -dependent key, while the second property ensures that each key is minimal. It is clear that we need $q > s, t$ and $k > 1$ for a $(q, k, n)_{s,t}$ -Armstrong code to be meaningful. Note that a $(q, k, n)_{1,1}$ -Armstrong code is just a (q, k, n) -Armstrong code.

Definition 6: Let $q > t \geq s \geq 1$ and $k > 1$. Then $f_{s,t}(q, k)$ denotes the maximum n such that there exists a $(q, k, n)_{s,t}$ -Armstrong code.

As with the Armstrong codes for functional dependencies [21], we have the following restrictions on $(q, k, n)_{s,t}$ -Armstrong codes. Let ϕ be the least number of submultisets $S \subset M$ of size $t + 1$ with at most s distinct elements, where M ranges over all multisets of size m over $[q]$.

Proposition 7: Let C be a $(q, k, n)_{s,t}$ -Armstrong code and let $m = |C|$. Then $\binom{m}{t+1} \geq \binom{n}{k-1}$ and $n \cdot \phi \leq (k - 1) \binom{m}{t+1}$.

Proof: Let T be a set of $k - 1$ columns of C . By condition (ii), there exists a set R_T of $t + 1$ rows such that each column of T has at most s distinct elements in R_T . By the first defining condition (i) of a $(q, k, n)_{s,t}$ -Armstrong code, R_T is distinct for distinct T . The first inequality then follows. The second inequality holds by the definition of ϕ and the defining condition (i). \square

As in [21], the two inequalities in Proposition 7 can give two upper bounds of $f_{s,t}(q, k)$, where one is obviously increasing in m and the other could be proved to be decreasing in m . Thus there is a universal upper bound of $f_{s,t}(q, k)$ at certain m where the two upper bounds intersect. However, it is impossible to give an explicit universal upper bound in most cases. We will use this method to explore values of $f_{s,t}(q, k)$ for some special cases.

A. The Case $s = 1$ and $k = 2$

Proposition 8: When $s = 1$ and $q < m$, we have

$$\phi \geq r \binom{h+1}{t+1} + (q-r) \binom{h}{t+1} = q \binom{h}{t+1} + r \binom{h}{t},$$

where $m = qh + r$, with $0 \leq r < q$.

Proof: Similar to the proof in [21, Lemma 3.2], let m_1 and m_2 be the number of two distinct symbols in M , where M is a multiset of size m over $[q]$. The inequality follows by the fact that $\binom{m_1}{t+1} + \binom{m_2}{t+1} \geq \binom{m_1+1}{t+1} + \binom{m_2-1}{t+1}$ for all m_1 and m_2 satisfying $m_2 - m_1 \geq 2$. \square

Proposition 9: The function $g(m) = \frac{(k-1)\binom{m}{t+1}}{q\binom{h}{t+1} + r\binom{h}{t}}$ is decreasing in m , where h and r are functions of m such that $m = qh + r$ with $0 < r + 1 \leq q < m$.

Proof: We prove that $g(m) \geq g(m+1)$ in two cases. When $r + 1 < q$, $m + 1 = qh + (r + 1)$. We have to verify that

$$\frac{(k-1)\binom{m}{t+1}}{q\binom{h}{t+1} + r\binom{h}{t}} \geq \frac{(k-1)\binom{m+1}{t+1}}{q\binom{h}{t+1} + (r+1)\binom{h}{t}}.$$

After carrying out the obvious cancelations, this leads to $q - r - 1 \geq 0$ which is trivially true. When $r + 1 = q$, $m + 1 = q(h + 1)$, it is easy to check that

$$\frac{(k-1)\binom{m}{t+1}}{q\binom{h}{t+1} + r\binom{h}{t}} = \frac{(k-1)\binom{m+1}{t+1}}{q\binom{h+1}{t+1}},$$

i.e., $g(m) = g(m+1)$. \square

Proposition 10: $f_{1,t}(q, 2) = \binom{qt+1}{t+1}$.

Proof: By Proposition 7, we have $f_{1,t}(q, 2) \leq \binom{m}{t+1}$ and $f_{1,t}(q, 2) \leq \frac{\binom{m}{t+1}}{q\binom{h}{t+1} + r\binom{h}{t}}$, where $m = qh + r$, with $0 \leq r < q$. Since $\binom{m}{t+1}$ is increasing in m and $\frac{\binom{m}{t+1}}{q\binom{h}{t+1} + r\binom{h}{t}}$ is decreasing in m by Proposition 9, the upper bound $f_{1,t}(q, 2) \leq \binom{qt+1}{t+1}$ is the universal upper bound obtained by setting $m = qt + 1$ (i.e., $h = t$ and $r = 1$) where the two upper bounds intersect. The lower bound is given by construction. Construct a $(qt + 1) \times \binom{qt+1}{t+1}$ array as follows. For each column, we have exactly one subset of $t + 1$ rows with equal symbols and all other $q - 1$ symbols occurring exactly t times. We do so such that each column has a distinct subset of $t + 1$ rows with equal symbols. It is clear that this array satisfies the first property of a $(q, 2, \binom{qt+1}{t+1})_{1,t}$ -Armstrong code.

For the second property, any column of the array has $t + 1$ rows with equal symbols. Now for these $t + 1$ rows, we need a column having $t + 1$ distinct symbols in these rows. The above array does not have this property obviously. However, we can slightly rearrange the symbols occurring t times in each column to satisfy this property. We do it as follows. Let each column be indexed by the $(t + 1)$ -subset of rows which have equal symbols. Let A denote the set of all the column indices and A' be a copy of A . Define a bipartite graph with two parts A and A' , two $(t + 1)$ -subsets are adjacent if and only if they intersect at most one common symbol. This is a regular bipartite graph, thus it has a perfect matching E . Now for each edge $\{v, v'\} \in E$, where $v \in A$ and $v' \in A'$, rearrange the symbols occurring t times in the column v' , such that symbols in the rows of v are all distinct. We can do this since $|v \cap v'| \leq 1$. This rearrangement will guarantee that for each $t + 1$ rows there is a column having equal symbols and simultaneously a column having $t + 1$ distinct symbols. \square

B. The Case $s = t = 2$ and $k = 4$

Proposition 11: When $s = t = 2$ and $q < m$, we have $\phi \geq \varphi(m)$, where

$$\begin{aligned} \varphi(m) = & r\binom{h+1}{3} + (q-r)\binom{h}{3} + r\binom{h+1}{2}(m-h-1) \\ & + (q-r)\binom{h}{2}(m-h). \end{aligned}$$

Here h and r are functions of m such that $m = qh + r$, with $0 \leq r < q$.

Proof: Let m_1 and m_2 be the number of two distinct symbols in M , where M is a multiset of size m over $[q]$. The inequality follows by the fact that $\binom{m_1}{3} + \binom{m_2}{3} + \binom{m_1}{2}(m-m_1) + \binom{m_2}{2}(m-m_2) \geq \binom{m_1+1}{3} + \binom{m_2-1}{3} + \binom{m_1+1}{2}(m-m_1-1) + \binom{m_2-1}{2}(m-m_2-1)$ for all m_1 and m_2 satisfying $m_2 - m_1 \geq 2$. \square

Proposition 12: The function $k(m) = \frac{(k-1)\binom{m}{3}}{\varphi(m)}$ is decreasing in m , where $\varphi(m)$ is defined above.

Proof: As in the proof of Proposition 9, we first verify when $m = qh + r$ and $0 < r + 1 < q$ that

$$\frac{(k-1)\binom{m}{3}}{\varphi(m)} \geq \frac{(k-1)\binom{m+1}{3}}{\varphi(m+1)}.$$

Here $m + 1 = qh + (r + 1)$. Since there are a large amount of computation, we use Maple to do the cancelations. This leads to $(q - r - 1)(qh + r - 2h) \geq 0$ which is true since $q \geq r + 2 \geq 2$. If $r + 1 = q$, then $m = qh + q - 1$, $m + 1 = q(h + 1)$ and

$$\varphi(m+1) = q\binom{h+1}{3} + q\binom{h+1}{2}(m-h).$$

We can also check by Maple that

$$\frac{(k-1)\binom{m}{3}}{\varphi(m)} = \frac{(k-1)\binom{m+1}{3}}{\varphi(m+1)},$$

i.e., $k(m) = k(m+1)$. \square

When $k = 4$, we have $f_{2,2}(q, 4) \leq m$ and $f_{2,2}(q, 4) \leq k(m)$ by Propositions 7 and 11. Since $k(m)$ is decreasing by Proposition 12, we know that the universal upper bound is $f_{2,2}(q, 4) \leq m$ when $m = k(m)$. The solution is $m = 2q - 1$, which is achieved when $h = 1$ and $r = q - 1$. Hence, we have the following upper bound for $f_{2,2}(q, 4)$.

Proposition 13: $f_{2,2}(q, 4) \leq 2q - 1$.

Next, we will give a construction of an Armstrong instance of $2q - 1$ columns over q symbols for K_{2q-1}^4 being the system of minimal $(2, 2)$ -dependent keys. The construction is based on the classical near 1-factorization of complete graphs.

Let $n = 2q - 1$ and K_n be a complete graph with vertex set \mathbb{Z}_n . For each $i \in \mathbb{Z}_n$, take

$$T_i = \{\{t + i, -t + i\} : t \in [q - 1]\},$$

where the addition is in \mathbb{Z}_n . Then $\{T_i : i \in \mathbb{Z}_n\}$ is a near 1-factorization of K_n . Each T_i is a near 1-factor which misses the point i . The following fact is necessary for the construction of Armstrong codes.

Proposition 14: Let $n = 2q - 1$. For any distinct $i, j, k \in \mathbb{Z}_n$, there exist three points $x, y, z \in \mathbb{Z}_n$, such that $\{x, y\} \in T_i$, $\{y, z\} \in T_j$ and $\{z, x\} \in T_k$.

Proof: First note the fact that for each $i \in \mathbb{Z}_n$, an edge $\{x, y\}$ belongs to T_i if and only if $x + y = 2i$. Suppose $j = s + i$ for some $s \in \mathbb{Z}_n \setminus \{0\}$. Then T_i and T_j form a path from the vertex j to i as follows.

$$P = (\{s + i, -s + i\}, \{-s + i, 3s + i\}, \{3s + i, -3s + i\}, \dots, \{(2q - 3)s + i, -(2q - 3)s + i\}, \{-(2q - 3)s + i, (2q - 1)s + i\}).$$

The last vertex is i since $(2q - 1)s + i = i$. Note that the edges in P are from T_i and T_j in turn and the length of P is $2(q - 1)$. Let T be the set of edges by joining two vertices in P of distance two. We claim that $T \cap T_k \neq \emptyset$ for any $k \neq i, j$.

By the observation at the beginning, we only need to prove that there exists an edge $\{x, y\}$ in T such that $x + y = 2k$. Let $S = \{x + y : \{x, y\} \in T\}$. By the form of P , we have $S = \{4ts + 2i : t \in [q - 1]\} \cup \{-4ts + 2i : t \in [q - 2]\}$. It is easy to check that elements in S are all different, i.e., $|S| = 2q - 3$. Further, $2i \notin S$ since $s \neq 0$ and $t \neq 0$. Also, $2j \notin S$ since $4ts \neq 2s$ for all $t \in [q - 1]$ and $-4ts \neq 2s$ for all $t \in [q - 2]$. Hence, $S = \mathbb{Z}_n \setminus \{2i, 2j\}$, or for any $k \neq i, j$, $2k \in S$. This completes the proof. \square

Proposition 15: There exists a $(q, 4, 2q - 1)_{2,2}$ -Armstrong code for each $q \geq 3$.

Proof: Let $n = 2q - 1$. We construct an $n \times n$ array C over $[q]$ as follows.

The columns of C are indexed by T_i ($i \in \mathbb{Z}_n$), while the rows are indexed by the vertices of K_n , i.e., \mathbb{Z}_n . In each column, say column indexed by T_i , arbitrarily order the $q - 1$ edges of T_i , assign symbols in $[q]$ to this column, such that for each row $s \in \mathbb{Z}_n \setminus \{i\}$, symbol j is assigned if and only if s is incident to the j -th edge of T_i ; for row i , symbol q is assigned to the column.

We claim that C is a $(q, 4, 2q - 1)_{2,2}$ -Armstrong code. For each three rows x, y, z of C , we choose three columns T_i, T_j, T_k , such that $\{x, y\} \in T_i$, $\{x, z\} \in T_j$ and $\{y, z\} \in T_k$. Then these three columns have exactly two distinct symbols in rows x, y, z . For any three columns T_i, T_j, T_k , by Proposition 14, we have three rows x, y, z such that $\{x, y\} \in T_i$, $\{x, z\} \in T_j$ and $\{y, z\} \in T_k$, i.e., at most two distinct elements in these three rows. Further, since $\{T_i : i \in \mathbb{Z}_n\}$ is a near 1-factorization, no pairs from $\{x, y, z\}$ occur in any T_l with $l \neq i, j, k$. That is for any other column T_l , there are exactly three different elements in rows x, y, z . Thus we prove the claim. \square

Here is an example of applying Proposition 15 to a near 1-factorization of K_n .

Example 2: For $n = 7$, we can get a near 1-factorization of K_7 as follows:

$$\begin{aligned} T_0 &= \{\{1, 6\}, \{2, 5\}, \{3, 4\}\}, \\ T_1 &= \{\{2, 0\}, \{3, 6\}, \{4, 5\}\}, \\ T_2 &= \{\{3, 1\}, \{4, 0\}, \{5, 6\}\}, \\ T_3 &= \{\{4, 2\}, \{5, 1\}, \{6, 0\}\}, \\ T_4 &= \{\{5, 3\}, \{6, 2\}, \{0, 1\}\}, \\ T_5 &= \{\{6, 4\}, \{0, 3\}, \{1, 2\}\}, \\ T_6 &= \{\{0, 5\}, \{1, 4\}, \{2, 3\}\}. \end{aligned}$$

Then a $(4, 4, 7)_{2,2}$ -Armstrong code is constructed as below.

4	1	2	3	3	2	1
1	4	1	2	3	3	2
2	1	4	1	2	3	3
3	2	1	4	1	2	3
3	3	2	1	4	1	2
2	3	3	2	1	4	1
1	2	3	3	2	1	4

Combining Propositions 13 and 15, we determine the value of $f_{2,2}(q, 4)$.

Theorem 4: $f_{2,2}(q, 4) = 2q - 1$ for all integers $q \geq 3$.

V. LOWER BOUNDS FOR $f_{1,t}(q, k)$

Since each $(q, k, n)_{s,t}$ -Armstrong code is trivially a $(q, k, n)_{s,t}$ -Armstrong code, $1 < s \leq t$, the problem of estimating values of $f_{1,t}(q, k)$ seems more important. In this section, we focus on exploring lower bounds for $f_{s,t}(q, k)$ when $s = 1$.

A. A Construction From Reed-Solomon Codes

In this subsection, we assume that q is a prime power. Let \mathbb{F}_q be a finite field with q elements a_1, a_2, \dots, a_q . For each polynomial $f \in \mathbb{F}_q[X]$, let f_∞ denote the coefficient of X in f . A Reed-Solomon code C over \mathbb{F}_q of length $q + 1$ is constructed as follows.

$$C = \{(f(a_1), f(a_2), \dots, f(a_q), f_\infty) : f \in \mathbb{F}_q[X], \deg f < k\}.$$

Proposition 16: The code C is a $(q, k, q + 1)_{1,t}$ -Armstrong code for any $1 \leq t \leq q - 1$. Thus $f_{1,t}(q, k) \geq q + 1$ for $1 \leq t \leq q - 1$ and q a prime power.

Proof: We prove it by definition of Armstrong code. View C as a $q^k \times (q + 1)$ array with columns indexed by $(a_1, \dots, a_q, \infty)$ and rows indexed by polynomials in $\mathbb{F}_q[X]$. Since $\deg f < k$, any k coordinates determine a unique polynomial f . Hence, any two codewords agree in at most $k - 1$ positions, which means that the first condition of the definition holds. As for the second condition, choose any $k - 1$ columns, say $a_{i_1}, \dots, a_{i_{k-1}}$ and any $k - 1$ elements b_1, \dots, b_{k-1} in \mathbb{F}_q , then there are exactly q polynomials $f_l \in \mathbb{F}_q[X]$, $l \in [q]$ such that $f_l(a_{i_j}) = b_j$, $j \in [k - 1]$. Since any two codewords agree at most $k - 1$ positions, then any columns outside $\{a_{i_1}, \dots, a_{i_{k-1}}\}$ have exactly q distinct elements in the q rows f_l , $l \in [q]$. \square

B. An Existence Result Using the Probabilistic Method

In this subsection, we will give a lower bound for $f_{1,t}(q, k)$ by using a similar probabilistic method as in [26]. First, we construct a random q -ary code C of length n and size $(t + 1) \cdot \binom{n}{k-1}$ as follows.

For each subset of $k - 1$ positions $K \subset [n]$, choose a set of $t + 1$ codewords $A^K = \{A_1^K, A_2^K, \dots, A_{t+1}^K\}$ randomly such that they pairwise agree exactly at the positions in K . That is, in each position in K , a random symbol is chosen with probability $\frac{1}{q}$ and assigned to this position for all codewords in A^K . In each position out of K , $t + 1$ distinct symbols

are randomly chosen and assigned to the $t + 1$ rows. The choices are pairwise independent for distinct positions. Let $C = \cup_{K \subset [n]} A^K$. Then the choice of C makes it satisfy the second property of a $(q, k, n)_{1,t}$ -Armstrong code. Next, we will prove that C also satisfies the first property with positive probability under certain conditions.

Consider events $v(A_i^K, A_j^L)$, where $i, j \in [t + 1]$ and $K \neq L$ are $(k - 1)$ -subsets of coordinate positions, that the two codewords agree in at least k coordinates. Two such events $v(A_i^K, A_j^L)$ and $v(A_{i'}^{K'}, A_{j'}^{L'})$ are independent if $\{K, L\} \cap \{K', L'\} = \emptyset$. If for any two distinct $(k - 1)$ -subsets K, L and any pair $i, j \in [t + 1]$, event $v(A_i^K, A_j^L)$ doesn't happen, then C satisfies the first condition, i.e. C is a $(q, k, n)_{1,t}$ -Armstrong code.

Define the dependency graph $G = (V, E)$ by V being the set of events $\{v(A_i^K, A_j^L) : K \neq L, i, j \in [t + 1]\}$, and $v(A_i^K, A_j^L)$ and $v(A_{i'}^{K'}, A_{j'}^{L'})$ are connected by an edge if and only if $\{K, L\} \cap \{K', L'\} \neq \emptyset$. Thus the degree of $v(A_i^K, A_j^L)$ in the dependency graph is $2(t + 1)^2 \binom{n}{k-1} - 3(t + 1)^2 - 1$. On the other hand,

$$\text{Prob}(v(A_i^K, A_j^L)) = \sum_{l=k}^n \binom{n}{l} \left(\frac{1}{q}\right)^l \left(1 - \frac{1}{q}\right)^{n-l} = B(k, n, \frac{1}{q}).$$

By the well-known Chernoff bound,

$$B(k, n, \frac{1}{q}) \leq \left(\frac{n}{qk}\right)^k \cdot e^{k - \frac{n}{q}},$$

when $k > \frac{n}{q}$.

Now, we will apply the following famous Lovász' Local Lemma to give a lower bound for $f_{1,t}(q, k)$.

Lemma 1 [2]: Let A_1, A_2, \dots, A_n be events in an arbitrary probability space. Suppose that each event A_i is mutually independent of a set of all the other events A_j but at most d , and that $\text{Prob}(A_i) \leq p$ for all $1 \leq i \leq n$. If $ep(d + 1) \leq 1$, then $\text{Prob}(\bigwedge_{i=1}^n \overline{A_i}) > 0$.

By Lemma 1, if

$$2(t + 1)^2 \binom{n}{k-1} B(k, n, \frac{1}{q}) < \frac{1}{e}, \quad (2)$$

then

$$\text{Prob}\left(\bigcap \overline{v(A_i^K, A_j^L)}\right) > 0,$$

which means that a $(q, k, n)_{1,t}$ -Armstrong code exists with positive probability.

Assuming $n > 2k$, (2) follows from

$$2(t + 1)^2 \binom{n}{k} B(k, n, \frac{1}{q}) < \frac{1}{e}.$$

Since $\binom{n}{k} \leq \left(\frac{n \cdot e}{k}\right)^k$, it is enough to show that

$$\left(\frac{n \cdot e}{k}\right)^k \left(\frac{n}{qk}\right)^k \cdot e^{k - \frac{n}{q}} < \frac{1}{2e(t + 1)^2}.$$

Writing $n = ck$, we have

$$\frac{c^2}{q} \cdot e^{2 - \frac{c}{q}} < \sqrt[k]{\frac{1}{2e(t + 1)^2}}. \quad (3)$$

It is clear that (3) is true when $c \leq \sqrt[k]{\frac{1}{2e(t + 1)^2}} \frac{\sqrt{q}}{e}$.

Proposition 17: Assume that $t \geq 1$ and q, k are integers satisfying $q > 4e^2 \sqrt[k]{2e(t + 1)^2}$. Then a $(q, k, n)_{1,t}$ -Armstrong code exists for $n = \sqrt[k]{\frac{1}{2e(t + 1)^2}} \frac{\sqrt{q}}{e} k$, i.e.,

$$f_{1,t}(q, k) \geq \sqrt[k]{\frac{1}{2e(t + 1)^2}} \frac{\sqrt{q}}{e} k.$$

Proof: The condition $q > 4e^2 \sqrt[k]{2e(t + 1)^2}$ implies that $n > 2k$ and $qk > n$, which completes the proof by combining above analysis. \square

VI. CONCLUSION

We investigated the maximum number of minimal keys in relational database systems with attributes having bounded domains via the study of Armstrong codes. We showed that the maximum length n for which a $(q, 3, n)$ -Armstrong code can exist is $f(q, 3) = 3q - 1$ for all $q \geq 5$ with three possible exceptions, disproving a conjecture of Sali.

Our determination of $f(q, 3)$ involves introducing the new concept of extorthogonal double covers (extODC), a generalization of orthogonal double covers with property that any two partitions cover at least one common 2-subset. This new combinatorial design is interesting not only in database theory, but also in design theory. Similar to ODCs, there are several directions for the study of extODCs. For example, each partition could be extended to any spanning subgraph, or consider similar properties for hypergraphs.

Further, we generalized Armstrong codes to the case of (s, t) -dependencies. The maximum length $n = f_{s,t}(q, k)$ for which a $(q, k, n)_{s,t}$ -Armstrong code can exist seems to be quite difficult to determine. Classes of optimal Armstrong codes of this type are constructed. Several lower bounds of $f_{s,t}(q, k)$ are also established.

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