

## PRODUCT CONSTRUCTION OF AFFINE CODES\*

YEOW MENG CHEE<sup>†</sup>, HAN MAO KIAH<sup>†</sup>, PUNARBASU PURKAYASTHA<sup>‡</sup>,  
AND PATRICK SOLÉ<sup>§</sup>

**Abstract.** Binary matrix codes with restricted row and column weights are a desirable method of coded modulation for power line communication. In this work, we construct such matrix codes that are obtained as products of affine codes—cosets of binary linear codes. Additionally, the constructions have the property that they are systematic. Subsequently, we generalize our construction to irregular product of affine codes, where the component codes are affine codes of different rates.

**Key words.** product codes, affine codes, irregular product codes, power line communications

**AMS subject classifications.** Primary, 94B05, 94B60; Secondary, 94B20

**DOI.** 10.1137/140976996

**1. Introduction.** Product codes were introduced by Elias [10] and subsequently generalized by Forney [11] to concatenated codes. Product codes are a method of constructing larger codes from smaller codes while retaining the good rates and good decoding complexity from the smaller codes. The codewords of a product code can be written as matrices with the rows belonging to the row component code and the columns belonging to the column component code. List decoding algorithms have also been studied in this context in Barg and Zémor [4], where the min-sum algorithm was shown to be amenable to list decoding of product codes.

Product codes have been subsequently generalized to yield codes obtained from product of nonlinear codes by Amrani [2] and to multilevel product codes by Zinoviev [19]. Amrani [2] gave the construction of product codes from component nonlinear codes which are binary and systematic. The construction guarantees that all the columns of any codeword belong to the column component code; however, only the first few rows corresponding to the systematic part of the column code are guaranteed to belong to the row code. In the case where one of the component codes is linear, Amrani [2] proposed two soft-decision decoding algorithms. Irregular product codes, introduced by Alipour et al. [1], are yet another generalization of product codes where each row and column code can be a code of different rate. Irregular product codes were introduced to address the need for unequal error protection from bursty noise when some parts of the codeword are more vulnerable to burst errors than others.

In this work we study constructions of systematic nonlinear product codes which are obtained as products of affine codes—cosets of linear codes. In contrast to the work of Amrani [2], our construction guarantees that all the rows belong to the (affine) row code and all the columns belong to the (affine) column code. One primary motivation for studying such class of codes arises from a previous study on coded modulation for

---

\*Received by the editors July 10, 2014; accepted for publication (in revised form) June 3, 2015; published electronically August 26, 2015. This paper was presented in part at the IEEE International Symposium on Information Theory, 2014.

<http://www.siam.org/journals/sidma/29-3/97699.html>

<sup>†</sup>Division of Mathematical Sciences, School of Physical and Mathematical Sciences, Nanyang Technological University, Singapore (ymchee@ntu.edu.sg, hmkiah@ntu.edu.sg).

<sup>‡</sup>CGG Services Pte. Ltd., Singapore (punarbasu+paper@gmail.com). This work was done while this author was a research fellow at Nanyang Technological University, Singapore.

<sup>§</sup>Institut Mines-Télécom, Télécom ParisTech, CNRS LTCI, France (patrick.sole@telecom-paristech.fr).

power line channels by Chee, Kiah, and Purkayastha [7] that proposed a generalization of the coded modulation scheme of Vinck [18].

Chee, Kiah, and Purkayastha [7] showed that binary matrix codes with bounded column weights, in conjunction with multitone frequency shift keying, can be used to counter the harsh noise characteristics of the power line channel. Concatenated codes obtained from the concatenation of constant weight inner codes with Reed–Solomon outer codes were used to obtain families of efficiently decodable codes with good rates and good relative distances. In this work, we continue this line of investigation and introduce *binary* systematic product codes with the additional restriction that the row and column weights are bounded. The restriction on the column weight arises from the desire to be able to detect and correct impulse noise that is present in the power line channel. The restriction on the row weights allows one to detect and correct narrowband noise. It is quite evident that product codes obtained from the product of linear codes do not satisfy these restrictions. The nonlinear codes studied in this paper are constructed to satisfy these properties. The efficient decoding algorithms of product codes are directly applicable to the constructions presented in this work. As a first step to the decoding process, we subtract the coset representative that is used in the construction. The coset representative is explicitly described, as explained in the following sections.

The rest of the paper is organized as follows. In the next section we introduce the basic definitions and notation that are used throughout the rest of this paper. Section 3 discusses the general construction of  $q$ -ary systematic codes which are product of affine codes. Section 4 uses the construction in section 3 to give constructions of *binary* product codes with restricted row and column weights. This is of interest because of its application to coded modulation for power line channels. In section 5, we extend this construction to product codes which can provide unequal error protection, where different rows and columns belong to different row and column codes. This section generalizes the irregular product code construction of Alipour et al. [1], where the component codes are linear codes, to irregular product codes, where the component codes are affine codes.

**2. Notation and definitions.** Denote the set of integers  $\{1, 2, \dots, n\}$  by  $[n]$  for a positive integer  $n$ . Denote the finite field of order  $q$  by  $\mathbb{F}_q$ . A *code*  $\mathcal{C}$  of length  $n$  is a subset of  $\mathbb{F}_q^n$ , while a *linear code*  $\mathcal{C}$  of length  $n$  is a linear subspace of  $\mathbb{F}_q^n$ . The *dimension* of a linear code  $\mathcal{C}$  is given by the dimension of  $\mathcal{C}$  as a linear subspace of  $\mathbb{F}_q^n$ . Elements of  $\mathcal{C}$  are called *codewords*. Endow the space  $\mathbb{F}_q^n$  with the *Hamming distance* metric and for  $\mathbf{u} \in \mathbb{F}_q^n$ , the *Hamming weight* of  $\mathbf{u}$  is the distance of  $\mathbf{u}$  from the all-zero codeword. A code  $\mathcal{C} \subseteq \mathbb{F}_q^n$  is said to have distance  $d$  if the (Hamming) distance between any two distinct codewords of  $\mathcal{C}$  is at least  $d$ . Moreover, a linear code  $\mathcal{C}$  has distance  $d$  if the weight of all nonzero codewords in  $\mathcal{C}$  is at least  $d$ . We use the notation  $[n, k, d]$  to denote a linear code of length  $n$ , dimension  $k$ , and distance  $d$ .

Let  $m, n$  be positive integers and let  $\mathbb{F}_q^{m \times n}$  denote the set of  $m$  by  $n$  matrices over  $\mathbb{F}_q$ . The *transpose* of a matrix  $\mathbf{M}$  is denoted by  $\mathbf{M}^T$  and we regard the vector  $\mathbf{u} \in \mathbb{F}_q^n$  as a row vector, or a matrix  $\mathbf{u}$  in  $\mathbb{F}_q^{1 \times n}$ . Hence,  $\mathbf{u}^T$  denotes a column vector in  $\mathbb{F}_q^{n \times 1}$ . In addition, let  $\mathbf{0}_n$  and  $\mathbf{j}_n$  denote the all-zero and all-one vectors of length  $n$ , respectively, while  $\mathbf{I}_n$  and  $\mathbf{0}_{m \times n}$  denote the  $(n \times n)$  identity and the  $(m \times n)$  all-zero matrix, respectively. We denote the *span* of a vector  $\mathbf{u}$  by the notation  $\langle \mathbf{u} \rangle$ .

Let  $\mathcal{C}$  be a linear  $[n, k, d]$  code. After a permutation of coordinates, there exists a matrix  $\mathbf{A} \in \mathbb{F}_q^{k \times (n-k)}$  such that each codeword in  $\mathcal{C}$  can be written as  $(\mathbf{x}, \mathbf{x}\mathbf{A})$ , where  $\mathbf{x} \in \mathbb{F}_q^k$  is called the *information vector*. The matrix  $(\mathbf{I}_k | \mathbf{A})$  is said to be a *systematic encoder* of  $\mathcal{C}$ .

Let  $\mathcal{C}_1$  and  $\mathcal{C}_2$  be linear  $[n, k_1, d_1]$  and  $[n, k_2, d_2]$  codes, respectively. Suppose  $\mathcal{C}_1 \subseteq \mathcal{C}_2$  and pick  $\mathbf{u} \in \mathcal{C}_2$ . Then the set of codewords  $\mathcal{C}_1 + \mathbf{u}$  is a *coset of  $\mathcal{C}_1$  in  $\mathcal{C}_2$* . The collection of all cosets of  $\mathcal{C}_1$  in  $\mathcal{C}_2$  is denoted by  $\mathcal{C}_2/\mathcal{C}_1$ . Moreover, any coset in  $\mathcal{C}_2/\mathcal{C}_1$  is a  $(n, d_1)$  code of size  $q^{k_1}$ , and we call the coset an *affine  $[n, k_1, d_1]$  code*.

Observe that if  $(\mathbf{I}_{k_1}|\mathbf{A}_1)$  is a systematic encoder for  $\mathcal{C}_1$  and  $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2)$ , where  $\mathbf{u}_1$  is of length  $k_1$ , then  $\mathcal{C}_1 + \mathbf{u} = \mathcal{C}_1 + (\mathbf{0}_{k_1}, \mathbf{u}_2 - \mathbf{u}_1\mathbf{A}_1)$ . On the other hand, every coset in  $\mathcal{C}_2/\mathcal{C}_1$  contains at most one element of the form  $(\mathbf{0}_{k_1}, \mathbf{a})$ . Hence, for every coset  $\mathcal{C}_1 + \mathbf{u}$ , there is exactly one element of the form  $(\mathbf{0}_{k_1}, \mathbf{a})$ , and in this paper, we refer to this element as the *coset representative of  $\mathcal{C}_1 + \mathbf{u}$* . The set of all coset representatives of cosets in  $\mathcal{C}_2/\mathcal{C}_1$  is denoted  $(\mathcal{C}_2/\mathcal{C}_1)_{\text{rep}}$ .

We also consider the notion of systematicity for nonlinear codes. Let  $\mathcal{C}$  be a (matrix) code of size  $q^k$ . Then  $\mathcal{C}$  is said to be *systematic of dimension  $k$*  if there exists  $k$  coordinates such that  $\mathcal{C}$  when restricted to these  $k$  coordinates is  $\mathbb{F}_q^k$ . Observe that if  $\mathcal{C}$  is a linear  $[n, k, d]$  code, then any affine code  $\mathcal{C} + \mathbf{u}$  is systematic of dimension  $k$ .

**2.1. Matrix codes.** An  $(m \times n)$ -matrix code  $\mathcal{C}$  is a subset of  $\mathbb{F}_q^{m \times n}$ , while a *linear  $(m \times n)$ -matrix code*  $\mathcal{C}$  is a linear subspace of  $\mathbb{F}_q^{m \times n}$ , when considered as a vector space of dimension  $mn$ . Regarding each matrix in  $\mathbb{F}_q^{m \times n}$  as a vector of length  $mn$ , we have the definitions of Hamming distance, Hamming weight, and dimension. A linear  $(m \times n)$ -matrix code of dimension  $K$  and distance  $d$  is denoted by  $[m \times n, K, d]$ .

**2.2. Classical product codes.** The classical product code constructs matrix codes from two linear codes. Given a linear  $[n, k, d_C]$  code  $\mathcal{C}$  and a linear  $[m, l, d_D]$  code  $\mathcal{D}$ , let  $(\mathbf{I}_k|\mathbf{A})$  and  $(\mathbf{I}_l|\mathbf{B})$  be their respective systematic encoders. The *product code*, denoted by  $\mathcal{C} \otimes \mathcal{D}$ , is then given by the  $(m \times n)$ -matrix code (see [14, p. 568])

$$\mathcal{C} \otimes \mathcal{D} \triangleq \left\{ \left( \begin{array}{c|c} \mathbf{M} & \mathbf{MA} \\ \hline \mathbf{B}^T\mathbf{M} & \mathbf{B}^T\mathbf{MA} \end{array} \right) : \mathbf{M} \in \mathbb{F}_q^{l \times k} \right\},$$

where  $\mathbf{M}$  corresponds to the information bits. It can be shown that  $\mathcal{C} \otimes \mathcal{D}$  is a linear  $[m \times n, kl, d_D d_C]$  code. Furthermore,  $\mathcal{C} \otimes \mathcal{D}$  has the following property that depends on the *component codes  $\mathcal{C}$  and  $\mathcal{D}$* .

*Property  $(\mathcal{C}, \mathcal{D})$ .* For every  $\mathbf{N} \in \mathcal{C} \otimes \mathcal{D}$ ,

- (i) every row of  $\mathbf{N}$  belongs to  $\mathcal{C}$ , and
- (ii) every column of  $\mathbf{N}$  belongs to  $\mathcal{D}$ .

In this paper, we consider nonlinear component codes. Specifically, let  $\mathcal{C}'$  be a nonlinear code of length  $n$  and size  $q^k$  and  $\mathcal{D}'$  be a nonlinear code of length  $m$  and size  $q^l$ . We aim to construct an  $(m \times n)$ -matrix code  $\mathcal{C}' \otimes \mathcal{D}'$  of size  $q^{kl}$  such that Property  $(\mathcal{C}', \mathcal{D}')$  holds. This construction differs from the nonlinear product code construction in Amrani [2] because we guarantee that *all* the rows in every codeword belong to the row code  $\mathcal{C}'$ .

**3. Product codes from affine codes.** In this section, we provide the general construction of systematic matrix codes that are obtained as products of cosets of linear codes, i.e., as products of affine codes. Throughout this section, let  $\mathcal{C}$  and  $\mathcal{D}$  be a linear  $[n, k, d_C]$  and  $[m, l, d_D]$  codes, respectively. We consider affine codes that are obtained as cosets of the codes  $\mathcal{C}$  and  $\mathcal{D}$ , i.e., they are of the form  $\mathcal{C} + \mathbf{u}$  and  $\mathcal{D} + \mathbf{v}$ , respectively, where  $\mathbf{u}$  and  $\mathbf{v}$  are of lengths  $n$  and  $m$ , respectively. In particular, we

show that if both  $\mathcal{C}$  and  $\mathcal{D}$  contain the all-one vector, then there exists an  $(m \times n)$ -matrix code that is systematic of dimension  $kl$  with Property  $(\mathcal{C} + \mathbf{u}, \mathcal{D} + \mathbf{v})$ .

$$(3.1) \quad (\mathcal{C} + \mathbf{u}) \otimes (\mathcal{D} + \mathbf{v}) \triangleq \left\{ \left( \begin{array}{c|c} \mathbf{M} & \mathbf{MA} + \mathbf{j}_l^T \mathbf{a} \\ \hline \mathbf{B}^T \mathbf{M} + \mathbf{b}^T \mathbf{j}_k & (\mathbf{B}^T \mathbf{M} + \mathbf{b}^T \mathbf{j}_k) \mathbf{A} + \mathbf{j}_{m-l}^T \mathbf{a} \end{array} \right) : \mathbf{M} \in \mathbb{F}_q^{l \times k} \right\}.$$

$$(3.2) \quad (\mathcal{C} + \mathbf{u}) \tilde{\otimes} (\mathcal{D} + \mathbf{v}) \triangleq \left\{ \left( \begin{array}{c|c} \mathbf{M} & \mathbf{MA} + \mathbf{j}_l^T \mathbf{a} \\ \hline \mathbf{B}^T \mathbf{M} + \mathbf{b}^T \mathbf{j}_k & \mathbf{B}^T (\mathbf{MA} + \mathbf{j}_l^T \mathbf{a}) + \mathbf{b}^T \mathbf{j}_{n-k} \end{array} \right) : \mathbf{M} \in \mathbb{F}_q^{l \times k} \right\}.$$

Let  $(\mathbf{I}_k | \mathbf{A})$  and  $(\mathbf{I}_l | \mathbf{B})$  be systematic encoders for  $\mathcal{C}$  and  $\mathcal{D}$ , respectively. Recall that the set of coset representatives of cosets of  $\mathcal{C}_1$  in  $\mathcal{C}_2$  is denoted by  $(\mathcal{C}_2 / \mathcal{C}_1)_{\text{rep}}$ . Without loss of generality, pick  $\mathbf{u} = (\mathbf{0}_k, \mathbf{a}) \in (\mathbb{F}_q^n / \mathcal{C})_{\text{rep}}$  and  $\mathbf{v} = (\mathbf{0}_l, \mathbf{b}) \in (\mathbb{F}_q^m / \mathcal{D})_{\text{rep}}$ . Then a typical element in  $\mathcal{C} + \mathbf{u}$  is of the form  $(\mathbf{x}, \mathbf{x}\mathbf{A} + \mathbf{a})$ , where  $\mathbf{x}$  is the information vector of length  $k$ . Similarly, a typical element in  $\mathcal{D} + \mathbf{v}$  is of the form  $(\mathbf{x}, \mathbf{x}\mathbf{B} + \mathbf{b})$ , where  $\mathbf{x}$  is the information vector of length  $l$ .

Define  $(\mathcal{C} + \mathbf{u}) \otimes (\mathcal{D} + \mathbf{v})$  to be the  $(m \times n)$ -matrix code given by (3.1). This is obtained by the encoding the first  $k$  columns by  $\mathcal{D} + \mathbf{v}$ , followed by encoding all the rows by  $\mathcal{C} + \mathbf{u}$ . We observe that for every  $\mathbf{N} \in (\mathcal{C} + \mathbf{u}) \otimes (\mathcal{D} + \mathbf{v})$ , each row of  $\mathbf{N}$  belongs to  $\mathcal{C} + \mathbf{u}$ . However, we can guarantee only that the first  $k$  columns belong to  $\mathcal{D} + \mathbf{v}$ .

On the other hand, if we alter the definition given in (3.1) to be (3.2), where we encode the first  $l$  rows by  $\mathcal{C} + \mathbf{u}$ , followed by encoding all the columns using  $\mathcal{D} + \mathbf{v}$ , we have that every column of  $\mathbf{N}$  belongs to  $\mathcal{D} + \mathbf{v}$  for each  $\mathbf{N} \in (\mathcal{C} + \mathbf{u}) \tilde{\otimes} (\mathcal{D} + \mathbf{v})$ .

Therefore, the matrix code  $(\mathcal{C} + \mathbf{u}) \otimes (\mathcal{D} + \mathbf{v})$  meets our requirements if

$$(3.3) \quad (\mathbf{B}^T \mathbf{M} + \mathbf{b}^T \mathbf{j}_k) \mathbf{A} + \mathbf{j}_{m-l}^T \mathbf{a} = \mathbf{B}^T (\mathbf{MA} + \mathbf{j}_l^T \mathbf{a}) + \mathbf{b}^T \mathbf{j}_{n-k}, \text{ that is,} \\ \mathbf{b}^T (\mathbf{j}_k \mathbf{A} - \mathbf{j}_{n-k}) = (\mathbf{B}^T \mathbf{j}_l^T - \mathbf{j}_{m-l}^T) \mathbf{a}.$$

If (3.3) holds, then  $(\mathcal{C} + \mathbf{u}) \otimes (\mathcal{D} + \mathbf{v})$  (or equivalently,  $(\mathcal{C} + \mathbf{u}) \tilde{\otimes} (\mathcal{D} + \mathbf{v})$ ) is a coset of  $\mathcal{C} \otimes \mathcal{D}$ . That is,  $(\mathcal{C} + \mathbf{u}) \otimes (\mathcal{D} + \mathbf{v}) = (\mathcal{C} \otimes \mathcal{D}) + \mathbf{U}$ , where

$$(3.4) \quad \mathbf{U} \triangleq \left( \begin{array}{c|c} \mathbf{0}_{l \times k} & \mathbf{j}_l^T \mathbf{a} \\ \hline \mathbf{b}^T \mathbf{j}_k & \mathbf{b}^T \mathbf{j}_k \mathbf{A} + \mathbf{j}_{m-l}^T \mathbf{a} \end{array} \right).$$

**THEOREM 3.1.** *Let  $\mathcal{C}$  and  $\mathcal{D}$  be linear  $[n, k, d_{\mathcal{C}}]$  and  $[m, l, d_{\mathcal{D}}]$  codes, respectively, and  $(\mathbf{I}_k | \mathbf{A})$  and  $(\mathbf{I}_l | \mathbf{B})$  be their respective systematic encoders. Pick  $\mathbf{u} = (\mathbf{0}_k, \mathbf{a}) \in (\mathbb{F}_q^n / \mathcal{C})_{\text{rep}}$  and  $\mathbf{v} = (\mathbf{0}_l, \mathbf{b}) \in (\mathbb{F}_q^m / \mathcal{D})_{\text{rep}}$ .*

*If in addition (3.3) holds, then  $(\mathcal{C} + \mathbf{u}) \otimes (\mathcal{D} + \mathbf{v})$  defined by (3.1) is equal to  $(\mathcal{C} + \mathbf{u}) \tilde{\otimes} (\mathcal{D} + \mathbf{v})$  defined by (3.2). Moreover, the code is systematic of dimension  $kl$  and is a coset of  $\mathcal{C} \otimes \mathcal{D}$  with Property  $(\mathcal{C} + \mathbf{u}, \mathcal{D} + \mathbf{v})$ .*

We now provide a sufficient condition for (3.3) to hold. Observe that  $\mathbf{j}_n \in \mathcal{C}$  if and only if  $\mathbf{j}_k \mathbf{A} = \mathbf{j}_{n-k}$ , since  $\mathbf{j}_k (\mathbf{I}_k | \mathbf{A})$  is necessarily  $\mathbf{j}_n$ . This is because the only message vector that can give rise to the all-one vector must have all-one in the

systematic part of the codeword. Hence,  $\mathbf{j}_k \mathbf{A} - \mathbf{j}_{n-k} = \mathbf{0}_{n-k}$  and  $\mathbf{b}^T (\mathbf{j}_k \mathbf{A} - \mathbf{j}_{n-k}) = \mathbf{0}_{(m-l) \times (n-k)}$ . A similar argument holds for  $\mathbf{B}^T \mathbf{j}_l^T - \mathbf{j}_{m-l}^T$ . Hence, (3.3) holds and the coset representative  $\mathbf{U}$  given by (3.4) is

$$(3.5) \quad \mathbf{U} = \left( \begin{array}{c|c} \mathbf{0}_{l \times k} & \mathbf{j}_l^T \mathbf{a} \\ \hline \mathbf{b}^T \mathbf{j}_k & \mathbf{b}^T \mathbf{j}_{n-k} + \mathbf{j}_{m-l}^T \mathbf{a} \end{array} \right)$$

and is independent of the matrices  $\mathbf{A}$  and  $\mathbf{B}$ . The following corollary, which we refer to as Construction I, is now immediate.

**COROLLARY 3.2 (Construction I).** *Let  $\mathcal{C}$  and  $\mathcal{D}$  be linear  $[n, k, d_{\mathcal{C}}]$  and  $[m, l, d_{\mathcal{D}}]$  codes, respectively, and  $(\mathbf{I}_k | \mathbf{A})$  and  $(\mathbf{I}_l | \mathbf{B})$  be their respective systematic encoders. Pick  $\mathbf{u} = (\mathbf{0}_k, \mathbf{a}) \in (\mathbb{F}_q^n / \mathcal{C})_{\text{rep}}$  and  $\mathbf{v} = (\mathbf{0}_l, \mathbf{b}) \in (\mathbb{F}_q^m / \mathcal{D})_{\text{rep}}$ . If in addition  $\mathbf{j}_n \in \mathcal{C}$  and  $\mathbf{j}_m \in \mathcal{D}$ , then  $(\mathcal{C} + \mathbf{u}) \otimes (\mathcal{D} + \mathbf{v})$  defined by (3.1) is systematic of dimension  $kl$  and is a coset of  $\mathcal{C} \otimes \mathcal{D}$  with Property  $(\mathcal{C} + \mathbf{u}, \mathcal{D} + \mathbf{v})$ .*

Binary linear codes that contain the all-one vector are called *self-complementary codes*. Well-known examples of linear self-complementary codes include the primitive narrow-sense Bose–Chaudhuri–Hocquenghem codes, the extended Golay code, and the Reed–Muller codes [14]. Examples of  $q$ -ary linear codes that contain the all-one vector are the Reed–Solomon codes, generalized Reed–Muller codes [14], and difference matrix codes [5].

**4. Variants of Construction I.** In this section, we adopt Construction I to certain nonlinear component codes  $\mathcal{C}'$ ,  $\mathcal{D}'$  that are variants of cosets of linear codes. Several well-known families of nonlinear codes, such as Nordstrom–Robinson, Delsarte–Goethals, Kerdock, and Preparata, can be obtained as unions of cosets of linear codes (see [14, Chapter 15]). In general, it is difficult to achieve a matrix code with Property  $(\mathcal{C}', \mathcal{D}')$  of size  $q^{\log |\mathcal{C}'| \log |\mathcal{D}'|}$ . Instead, we show that it is possible to achieve a size of  $q^{\kappa \log |\mathcal{C}'| \log |\mathcal{D}'|}$  for some positive constant  $\kappa < 1$ .

A straightforward generalization of Construction I to union of cosets of linear codes can be achieved as follows. Let  $\mathcal{C}_1$  and  $\mathcal{D}_1$  be linear  $[n, k_1, d_{\mathcal{C}_1}]$  and  $[m, l_1, d_{\mathcal{D}_1}]$  such that  $\mathbf{j}_n \in \mathcal{C}_1$  and  $\mathbf{j}_m \in \mathcal{D}_1$ . Let  $\mathcal{U} \subseteq (\mathbb{F}_q^n / \mathcal{C}_1)_{\text{rep}}$  and  $\mathcal{V} \subseteq (\mathbb{F}_q^m / \mathcal{D}_1)_{\text{rep}}$ . We consider the component codes  $\mathcal{C}'$  and  $\mathcal{D}'$ , where

$$\mathcal{C}' = \bigcup_{\mathbf{u} \in \mathcal{U}} \mathcal{C}_1 + \mathbf{u}, \text{ and } \mathcal{D}' = \bigcup_{\mathbf{v} \in \mathcal{V}} \mathcal{D}_1 + \mathbf{v}.$$

Then the  $(m \times n)$ -matrix code defined by

$$(4.1) \quad \bigcup_{\mathbf{u} \in \mathcal{U}} \bigcup_{\mathbf{v} \in \mathcal{V}} (\mathcal{C}_1 + \mathbf{u}) \otimes (\mathcal{D}_1 + \mathbf{v})$$

has Property  $(\mathcal{C}', \mathcal{D}')$ . However, observe that the code has size

$$|\mathcal{U}| |\mathcal{V}| q^{k_1 l_1} = q^{k_1 l_1 + \log |\mathcal{U}| + \log |\mathcal{V}|},$$

while the sizes of  $\mathcal{C}'$  and  $\mathcal{D}'$  are  $|\mathcal{U}| q^{k_1} = q^{k_1 + \log |\mathcal{U}|}$  and  $|\mathcal{V}| q^{l_1} = q^{l_1 + \log |\mathcal{V}|}$ , respectively. Thus the size of the code obtained from (4.1) is less than  $q^{\log |\mathcal{C}'| \log |\mathcal{D}'|} = q^{(k_1 + \log |\mathcal{U}|)(l_1 + \log |\mathcal{V}|)}$ .

**4.1. Product construction of expurgated codes.** We improve the size given by (4.1) when the union of cosets of product codes has a certain structure. Specifically, we consider the instance where the cosets form an expurgated code. We describe this formally below.

In addition to the codes  $\mathcal{C}_1, \mathcal{D}_1$ , assume that  $\mathcal{C}_2$  and  $\mathcal{D}_2$  are linear  $[n, k_2, d_{\mathcal{C}_2}]$  and  $[m, l_2, d_{\mathcal{D}_2}]$  codes such that  $\mathcal{C}_1 \subset \mathcal{C}_2$  and  $\mathcal{D}_1 \subset \mathcal{D}_2$ . We consider nonlinear component codes that are obtained from expurgated codes  $\mathcal{C}_2 \setminus \mathcal{C}_1$  and  $\mathcal{D}_2 \setminus \mathcal{D}_1$ . Our objective is therefore to construct an  $(m \times n)$ -matrix code such that Property  $(\mathcal{C}_2 \setminus \mathcal{C}_1, \mathcal{D}_2 \setminus \mathcal{D}_1)$  holds.

Clearly,  $\mathcal{C}_2 \setminus \mathcal{C}_1$  and  $\mathcal{D}_2 \setminus \mathcal{D}_1$  are union of cosets of  $\mathcal{C}_1$  and  $\mathcal{D}_1$  with  $\mathcal{U} = (\mathcal{C}_2/\mathcal{C}_1)_{\text{rep}} \setminus \{\mathbf{0}_n\}$  and  $\mathcal{V} = (\mathcal{D}_2/\mathcal{D}_1)_{\text{rep}} \setminus \{\mathbf{0}_m\}$ , respectively. Then the construction described in (4.1) gives a code with size  $(q^{k_2-k_1} - 1)(q^{l_2-l_1} - 1)q^{k_1 l_1} \approx q^{k_2-k_1+l_2-l_1+k_1 l_1}$ . The distance of the product code is determined by the distance of the codes  $\mathcal{C}_2$  and  $\mathcal{D}_2$ .

On the other hand, we improve this size via the following.

*Construction IA.* Consider two intermediary codes  $\mathcal{C}_3$  and  $\mathcal{D}_3$  of dimensions  $k_2 - 1$  and  $l_2 - 1$ , respectively, such that  $\mathcal{C}_1 \subseteq \mathcal{C}_3 \subset \mathcal{C}_2$  and  $\mathcal{D}_1 \subseteq \mathcal{D}_3 \subset \mathcal{D}_2$ . Pick any  $\mathbf{u} \in (\mathcal{C}_2 \setminus \mathcal{C}_3)$  and  $\mathbf{v} \in (\mathcal{D}_2 \setminus \mathcal{D}_3)$  and observe that

$$\mathcal{C}_3 + \mathbf{u} \subset \mathcal{C}_2 \setminus \mathcal{C}_1 \text{ and } \mathcal{D}_3 + \mathbf{v} \subset \mathcal{D}_2 \setminus \mathcal{D}_1.$$

Applying Construction I to the cosets  $\mathcal{C}_3 + \mathbf{u}$  and  $\mathcal{D}_3 + \mathbf{v}$  yields a matrix code  $(\mathcal{C}_3 + \mathbf{u}) \otimes (\mathcal{D}_3 + \mathbf{v})$  with Property  $(\mathcal{C}_3 + \mathbf{u}, \mathcal{D}_3 + \mathbf{v})$  and hence Property  $(\mathcal{C}_2 \setminus \mathcal{C}_1, \mathcal{D}_2 \setminus \mathcal{D}_1)$ . Furthermore, the size of this code is  $q^{(k_2-1)(l_2-1)}$  and is significantly larger than the straightforward construction from (4.1).

**4.2. Binary matrix codes with restricted column and row weights.** In this section, we apply Construction IA to obtain matrix codes with the additional property of bounded row and column weights. The motivation for studying such matrix codes arises from the application to coded modulation for power line communication (PLC) channel. Consider a codeword  $\mathbf{N} \in \mathbb{F}_2^{m \times n}$  of a matrix code. Each row of the matrix corresponds to transmission over a particular frequency slot, while each column of the matrix corresponds to a discrete time instance. Transmission occurs at the frequency and time slots corresponding to a one in the matrix.

The different types of noises are as follows. Assuming a hard-decision threshold detector, the received signal (which may contain errors caused by noise) is demodulated to an output  $\tilde{\mathbf{N}} \in \mathbb{F}_2^{m \times n}$ . The burst and random errors that arise from the different types of noises in the PLC channel (see [3, pp. 222–223]) have the following effects on the detector output. We denote the  $(i, j)$ th entry of a matrix  $\mathbf{N}$  by  $N_{i,j}$ .

1. A narrowband noise introduces a tone at all time instances of the transmitted signals. If  $e \in [m]$  and  $e$  narrowband noise errors occur, then there is a set  $\Gamma \in \binom{[m]}{e}$  of  $e$  rows such that  $\tilde{N}_{i,j} = 1$  for  $i \in \Gamma, j \in [n]$ .
2. Impulse noise results in the entire set of tones being received at a certain time instance. If  $e \in [n]$  and  $e$  impulse noise errors occur, then there is a set  $\Pi \in \binom{[n]}{e}$  of  $e$  columns such that  $\tilde{N}_{i,j} = 1$  for  $i \in [m], j \in \Pi$ .
3. A channel fade event erases a particular tone. If  $e \in [m]$ , and  $e$  fades occur, then there is a set  $\Gamma \in \binom{[m]}{e}$  of  $e$  rows such that  $\tilde{N}_{i,j} = 0$  for all  $j \in [n]$ .
4. Background noise flips the value of the bit at a particular tone and time instance. If  $e$  background noise occurs, then there exists a set  $\Omega \in \binom{[n] \times [m]}{e}$  such that  $\tilde{N}_{i,j} = N_{i,j} + 1$  for all  $(i, j) \in \Omega$ .



We refer to [3] for an expanded description of the types of noise that are present in the power line channel.

If any row of the codeword matrix  $\mathbf{N}$  is an all-one vector, then this row is not distinguishable from an all-one row introduced by the presence of narrowband noise. Similarly, an all-one column is not distinguishable from impulse noise. Additionally, the use of multitone frequency shift keying is adopted with the understanding that the energy is concentrated on a fraction of the available frequencies (see [7]). Thus, it is desired that every row and every column of the matrix contain at least a single one, but it should not be an all-one vector. This requires the use of codes whose codewords are matrices with restricted and bounded column and row weights.

In particular, for the application to powerline communications we construct codes which are able to correct narrowband and impulse noise and random noise and also simultaneously satisfy all the following criteria (also see [7]):

- (A1) have positive rate,
- (A2) have positive relative distance,
- (A3) have efficient decoding algorithms, and
- (A4) have no restriction that the length of the code is at most the size of the alphabet.

In the following text, we use the following lemma, which was crucial in proving the so-called low symbol weight property (see [17, Proposition 1]) for  $q$ -ary affine codes.

**LEMMA 4.1.** *Let  $\mathcal{C}$  be binary linear  $[n, k, d]$  code such that  $\langle \mathbf{j}_n \rangle \subset \mathcal{C}$ . Then the codewords in  $\mathcal{C} \setminus \langle \mathbf{j}_n \rangle$  have Hamming weight bounded between  $d$  and  $n - d$ .*

*Proof.* If the Hamming weight of any vector  $\mathbf{v}$  is greater than  $n - d$ , then  $\mathbf{v} + \mathbf{j}_n$  has Hamming weight less than  $d$ . This is a contradiction.  $\square$

First, we illustrate via an example that the code obtained by straightforward expurgation does not satisfy the systematic property.

**Example 4.2.** Let  $\mathcal{C} = \mathcal{D}$  be the binary linear  $[4, 3, 2]$  code consisting of all even weight codewords. Observe that  $\mathcal{C} \setminus \langle \mathbf{j}_4 \rangle$  consists of six codewords of weight two and we are interested in constructing a  $(4 \times 4)$ -matrix code whose matrices have row weight two and column weight two.

A naive approach is to look at the  $(3 \times 3)$  information matrix and require all columns and rows to not belong to  $\{\mathbf{0}_3, \mathbf{j}_3\}$ . This approach fails as illustrated by the example codeword,

$$\left( \begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ \hline 1 & 1 & 1 & 1 \end{array} \right),$$

which contains an all-one row even though each of the component codewords in the first three rows and columns has weight exactly two.

On the other hand, consider the binary linear  $[4, 2, 2]$  code  $\mathcal{C}_3 = \{\mathbf{0}_4, \mathbf{j}_4, (1, 0, 1, 0), (0, 1, 0, 1)\}$  and let  $\mathbf{u} = (0, 0, 1, 1)$ . Then  $(\mathcal{C}_3 + \mathbf{u}) \otimes (\mathcal{C}_3 + \mathbf{u})$  yields a  $(4 \times 4)$ -matrix code whose matrices have row weight two and column weight two. Furthermore, it is systematic of dimension four.

On the other hand, it can be obtained via computer search that there are exactly 90 matrices in  $\mathcal{C} \otimes \mathcal{C}$  that have constant row weight two and constant column weight two. An exhaustive computer search shows that there do not exist five coordinates where a subset of these 90 matrices is systematic.

We proceed with the construction of matrix codes with restricted row and column weights. Let  $\mathcal{C}, \mathcal{D}$  be binary linear  $[n, k, d_{\mathcal{C}}]$  and  $[m, l, d_{\mathcal{D}}]$  codes, respectively. Suppose

$\langle \mathbf{j}_n \rangle \subset \mathcal{C}$ , and  $\langle \mathbf{j}_m \rangle \subset \mathcal{D}$ . Direct application of Construction IA yields a systematic binary  $(m \times n)$ -matrix code of dimension  $(k-1)(l-1)$  whose matrices have

- (i) row weight bounded between  $d_{\mathcal{C}}$  and  $n - d_{\mathcal{C}}$ ,
- (ii) column weight bounded between  $d_{\mathcal{D}}$  and  $m - d_{\mathcal{D}}$ .

In Example 4.2 we showed that this construction gives more desirable results and why naive methods of constructions do not work. Because of the narrowband and impulse noise present in the power line channel, we want codes with restricted column and row weights. The following proposition gives the condition under which the noises can be corrected.

**PROPOSITION 4.3.** *Let  $\mathcal{C}$ ,  $\mathcal{D}$  be binary linear  $[n, k, d_{\mathcal{C}}]$  and  $[m, l, d_{\mathcal{D}}]$  codes, respectively. Suppose  $\langle \mathbf{j}_n \rangle \subset \mathcal{C}$ , and  $\langle \mathbf{j}_m \rangle \subset \mathcal{D}$ . Then  $(\mathcal{C} \setminus \langle \mathbf{j}_n \rangle) \otimes (\mathcal{D} \setminus \langle \mathbf{j}_m \rangle)$  obtained using Construction IA yields a systematic binary  $(m \times n)$ -matrix code of dimension  $(k-1)(l-1)$  whose matrices have*

- (i) row weight bounded between  $d_{\mathcal{C}}$  and  $n - d_{\mathcal{C}}$ ,
- (ii) column weight bounded between  $d_{\mathcal{D}}$  and  $m - d_{\mathcal{D}}$ .

*Furthermore,  $(\mathcal{C} \setminus \langle \mathbf{j}_n \rangle) \otimes (\mathcal{D} \setminus \langle \mathbf{j}_m \rangle)$  is a subcode of  $\mathcal{C} \otimes \mathcal{D}$  and hence is able to correct  $e_{\text{NBD}}$  narrowband errors and  $e_{\text{IMP}}$  impulse noise errors, provided*

$$e_{\text{IMP}} < d_{\mathcal{C}}, \text{ and } e_{\text{NBD}} < d_{\mathcal{D}}.$$

*Proof.* Consider a code  $\mathcal{C}' \subset \mathcal{C}$  of dimension  $k-1$  and  $\mathcal{D}' \subset \mathcal{D}$  of dimension  $l-1$ . Using Construction IA, we consider the cosets  $\mathcal{C}' + \mathbf{u}$  and  $\mathcal{D}' + \mathbf{v}$ , where  $\mathbf{u} \in \mathcal{C} \setminus \mathcal{C}'$  and  $\mathbf{v} \in \mathcal{D} \setminus \mathcal{D}'$ . By Lemma 4.1, the weight of every vector in  $\mathcal{C}' + \mathbf{u}$  is bounded between  $d_{\mathcal{C}}$  and  $n - d_{\mathcal{C}}$ . Similarly, condition (ii) holds.

To correct  $e_{\text{NBD}}$  narrowband noise errors and  $e_{\text{IMP}}$  impulse noise errors, we use Algorithm 1, which first sets all narrowband noise and impulse noise errors to erasures, subsequently subtracts the coset leader, and then decodes the row and the column codes. In the absence of random errors, if the condition  $e_{\text{IMP}} < d_{\mathcal{C}}$  is satisfied, then the row code  $\mathcal{C}'$  can correct all the corresponding erasures. Similarly, the column code  $\mathcal{D}'$  can correct all the erasures in each column if  $e_{\text{NBD}} < d_{\mathcal{D}}$ .  $\square$

**4.2.1. Optimality of the product construction.** The affine codes obtained using the product construction Construction IA are likely not optimal for large dimensions of the matrix. Obtaining optimal codes which satisfy all the criteria (A1)–(A4) stated earlier in this subsection is still an open problem. Below, we show some examples of codes for which the construction is close to optimal.

*Example 4.4.* Consider the first order Reed–Muller code with parameters  $\mathcal{C} = [2^r, r+1, 2^{r-1}]$ . The affine code  $(\mathcal{C} \setminus \langle \mathbf{j}_n \rangle) \otimes (\mathcal{C} \setminus \langle \mathbf{j}_n \rangle)$  obtained by Construction IA is a  $(2^r \times 2^r)$ -matrix code of dimension  $r^2$  and Hamming distance  $2^{2r-2}$ . This code can correct  $e_{\text{IMP}} < 2^{r-1}$  impulse noise errors and  $e_{\text{NBD}} < 2^{r-1}$  narrowband noise errors.

Before providing the next example, we recall a “Gabidulin construction” from the thesis of the second author [13, section 5.4] for square matrix codes.

**PROPOSITION 4.5** (Kiah [13]). *Let  $d < n$  and let  $k = n - d + 1$ . Then there exists a binary  $(2n \times 2n)$  matrix code of size  $2^{nk}$  with constant column weight  $n$  that corrects  $e_{\text{IMP}}$  impulse noise errors and  $e_{\text{NBD}}$  narrowband noise errors provided that  $e_{\text{IMP}} < n$ ,  $e_{\text{NBD}} < n$ , and*

$$\left\lfloor \frac{e_{\text{IMP}}}{2} \right\rfloor + \left\lfloor \frac{e_{\text{NBD}}}{2} \right\rfloor < d.$$

The family of codes in the above proposition is obtained from Gabidulin codes [12], which are optimal rank metric codes, and can be explicitly written as follows. Let  $\mathcal{C}$



## ALGORITHM 1. DECODER FOR PRODUCT OF AFFINE CODES.

---

**Input:** detector output  $\tilde{\mathbf{N}} \in \mathbb{F}_2^{m \times n}$ , coset leader  $\mathbf{U}$   
**Output:**  $\mathbf{N}' \in \mathcal{C}' \circ \mathcal{D}'$

```

/* Consider the narrowband noise as erasures */
1 for  $i \in [m]$  do
2   if  $\tilde{\mathbf{N}}_{i,j} = 1$  for all  $j \in [n]$  then
3      $\tilde{\mathbf{N}}_{i,j} \leftarrow \varepsilon$  for all  $j \in [n]$ 
4   end
5 end

/* Consider the impulse noise as erasures */
6 for  $j \in [n]$  do
7   if  $\tilde{\mathbf{N}}_{i,j} \in \{1, \varepsilon\}$  for all  $i \in [m]$  then
8      $\tilde{\mathbf{N}}_{i,j} \leftarrow \varepsilon$  for all  $i \in [m]$ 
9   end
10 end

/* Subtract the coset leader from the nonerased coordinates */
11 for  $i \in [m], j \in [n]$  do
12   if  $\tilde{\mathbf{N}}_{i,j} \neq \varepsilon$  then
13      $\tilde{\mathbf{N}}_{i,j} \leftarrow \tilde{\mathbf{N}}_{i,j} - \mathbf{U}_{i,j}$ 
14   end
15 end
16 Decode  $\tilde{\mathbf{N}}$  to  $\mathbf{N}'$  using a product code decoder
17 return  $\mathbf{N}'$ 

```

---

denote a Gabidulin code, and let  $\mathcal{C}^*$  denote the matrix code obtained using Proposition 4.5. Then we get

$$\mathcal{C}^* = \left\{ \begin{pmatrix} \mathbf{M} & \mathbf{M} + \mathbf{J} \\ \mathbf{M} + \mathbf{J} & \mathbf{M} \end{pmatrix} : \mathbf{M} \in \mathcal{C} \right\},$$

where  $\mathbf{J}$  is the all-one matrix. We now proceed to provide an example similar to Example 4.4.

*Example 4.6.* Consider the Gabidulin code with parameters  $[n = 2^{r-1}, 1, d = 2^{r-1}]$ . Such a code has dimension  $nk = 2^{r-1}$  and can correct the same number of narrowband and impulse noise errors that the code in Example 4.4 can correct. We get the following table comparing the dimensions of the two codes obtained from Example 4.4 and Proposition 4.5, respectively.

$r$	Dimension of codes from Example 4.4, $r^2$	Dimension of codes from Proposition 4.5, $2^{r-1}$
3	9	4
4	16	8
5	25	16
6	36	32
7	49	64

Thus, beyond  $r = 6$ , the construction from the Gabidulin codes has better parameters than the product construction. However, it must be noted that in the case

of the Gabidulin construction, we have restricted the matrices in our matrix code to always be a square matrix.

For fair comparison, we have taken the affine codes to also be formed of square matrices with the same number of rows and columns as in the Gabidulin construction. The product code in general does not have this restriction. This satisfies point (A4) in the criteria that we want to satisfy. Therefore, it is of interest (and an open problem) to obtain the true upper bound for codes which are optimal and which satisfy all the four criteria (A1)–(A4).

**5. Irregular product of affine codes.** The power line channel is known to be frequency selective (see [3]), i.e., the background noise in different frequency slots is of different intensities. Thus, it is of interest to provide constructions of codes that can provide different levels of error correction over different frequencies. Such codes can be constructed as product codes where the rows of the matrix correspond to different row codes. Such codes have been studied as “generalized concatenated codes” or “multilevel concatenated codes” (see Blokh and Zyablov [6], Zinoviev [19], and Dumer [9]). The row codes, which correspond to the row encoding, in these constructions are defined over an extension field of the field of the column code. As a result, although the resulting matrix code is linear over the smaller field, the rows do not in general belong to the row code. This makes it difficult to extend the construction to product of affine codes.

Instead, we consider the case where the component codes for each row and column are different. Although the application is only for binary component codes, we provide the general theory for  $q$ -ary component codes. Such product codes were termed *irregular product codes* and were studied by Alipour et al. [1]. Specifically, they demonstrated the following proposition.

**PROPOSITION 5.1** (Alipour et al. [1]). *Let  $\mathcal{C}_i$  be a linear code of length  $n$  and dimension  $k_i$  for  $i \in [m]$  and let  $\mathcal{D}_j$  be a linear code of length  $n$  and dimension  $l_j$  for  $j \in [m]$ . Suppose that  $k_1 \leq k_2 \leq \dots \leq k_m$  and  $l_1 \leq l_2 \leq \dots \leq l_n$ . Then there exists a linear  $(m \times n)$ -matrix code of dimension  $K$ , where*

$$(5.1) \quad K \leq \sum_{j=1}^n \sum_{i=l_{j-1}+1}^{l_j} \max\{k_i - j + 1, 0\}, \text{ where } l_0 = 0,$$

and every codeword  $\mathbf{N}$  satisfies the properties that

- (i) the  $i$ th row of  $\mathbf{N}$  belongs to  $\mathcal{C}_i$  for  $i \in [m]$ , and
- (ii) the  $j$ th column of  $\mathbf{N}$  belongs to  $\mathcal{D}_j$  for  $j \in [n]$ .

Furthermore, if  $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \dots \subseteq \mathcal{C}_m$  and  $\mathcal{D}_1 \subseteq \mathcal{D}_2 \subseteq \dots \subseteq \mathcal{D}_n$ , we achieve equality in (5.1).

The encoding algorithm of irregular product codes is described in [1]. The encoding procedure encodes the rows first and then encodes the columns. The encoding assumes that the first  $k_i$  coordinates of the  $i$ th row can generate the remaining  $n - k_i$  coordinates of that row and that the first  $l_j$  coordinates of the  $j$ th column can generate the remaining  $m - l_j$  coordinates of that column. Since the generating coordinates of the code are present within the leading principal  $l_n \times k_m$  submatrix, we have the following lemma.

**LEMMA 5.2.** *The leading principal  $l_n \times k_m$  submatrix generates all the remaining coordinates of a codeword  $\mathbf{N}$  in the irregular product code.*

We apply Construction I directly to Proposition 5.1 to obtain an irregular product of affine codes.

**PROPOSITION 5.3.** *In addition to the conditions of Proposition 5.1, let  $\mathbf{j}_n \in C_i$  for  $i \in [m]$  and  $\mathbf{j}_m \in D_j$  for  $j \in [n]$ . Let  $\mathbf{u} = (\mathbf{0}_{k_m}, \mathbf{a}) \in \mathbb{F}_q^n \setminus \bigcup_{i=1}^m C_i$  and  $\mathbf{v} = (\mathbf{0}_{l_n}, \mathbf{b}) \in \mathbb{F}_q^m \setminus \bigcup_{j=1}^n D_j$ .*

*Then there exists an affine  $(m \times n)$ -matrix code of dimension  $K$  bounded by (5.1) and every codeword  $\mathbf{N}$  in the code satisfies the properties that*

- (i) *the  $i$ th row of  $\mathbf{N}$  belongs to  $C_i + \mathbf{u}$  for  $i \in [m]$ , and*
- (ii) *the  $j$ th column of  $\mathbf{N}$  belongs to  $D_j + \mathbf{v}$  for  $j \in [n]$ .*

*If  $C_1 \subseteq C_2 \subseteq \dots \subseteq C_m$  and  $D_1 \subseteq D_2 \subseteq \dots \subseteq D_n$ , we achieve equality in (5.1).*

*For  $q = 2$ , suppose there exist linear codes  $\mathcal{C}$  and  $\mathcal{D}$  such that  $\bigcup_{i=1}^m C_i \subset \mathcal{C}$  and  $\bigcup_{j=1}^n D_j \subset \mathcal{D}$ , respectively. For  $\mathbf{u} \in \mathcal{C} \setminus \bigcup_{i=1}^m C_i$  and  $\mathbf{v} \in \mathcal{D} \setminus \bigcup_{j=1}^n D_j$ , the weight of every row of any codeword is bounded between  $d_{\mathcal{C}}$  and  $n - d_{\mathcal{C}}$ , and of every column between  $d_{\mathcal{D}}$  and  $m - d_{\mathcal{D}}$ , where  $d_{\mathcal{C}}$  and  $d_{\mathcal{D}}$  are the minimum distances of  $\mathcal{C}$  and  $\mathcal{D}$ , respectively.*

*Proof.* Let  $\mathbf{N}$  be a codeword obtained by using the encoding described in [1]. We translate this codeword using the matrix

$$\mathbf{U} = \left( \begin{array}{c|c} \mathbf{0}_{k_m \times l_n} & \mathbf{j}_{l_n}^T \mathbf{a} \\ \hline \mathbf{b}^T \mathbf{j}_{k_m} & \mathbf{b}^T \mathbf{j}_{n-k_m} + \mathbf{j}_{m-l_n}^T \mathbf{a} \end{array} \right),$$

where the vector  $\mathbf{a}$  has length  $n - k_m$  and  $\mathbf{b}$  has length  $m - l_n$ . We denote the codeword  $\mathbf{N}$  by four submatrices as

$$\mathbf{N} = \begin{pmatrix} \mathbf{N}_1 & \mathbf{N}_2 \\ \mathbf{N}_3 & \mathbf{N}_4 \end{pmatrix},$$

where  $\mathbf{N}_1$  is the  $l_n \times k_m$  leading principal submatrix that generates  $\mathbf{N}_2, \mathbf{N}_3, \mathbf{N}_4$ , by Lemma 5.2. The submatrix  $\mathbf{N}_2$  is of dimension  $l_n \times (n - k_m)$ ,  $\mathbf{N}_3$  is of dimension  $(m - l_n) \times k_m$ , and  $\mathbf{N}_4$  is of dimension  $(m - l_n) \times (n - k_m)$ . Denote the corresponding matrix from the coset code as

$$\mathbf{N}' = \begin{pmatrix} \mathbf{N}_1 & \mathbf{N}'_2 \\ \mathbf{N}'_3 & \mathbf{N}'_4 \end{pmatrix},$$

where  $\mathbf{N}'_2 = \mathbf{N}_2 + \mathbf{j}_{l_n}^T \mathbf{a}$  and  $\mathbf{N}'_3 = \mathbf{N}_3 + \mathbf{b}^T \mathbf{j}_{k_m}$ .

We need to ensure that the matrix  $\mathbf{N}'_4$  obtained by encoding the rows of  $\mathbf{N}'_3$  by the row codes satisfies the condition that they belong to the row code and also satisfies that they belong to the column codes in the submatrix  $\begin{pmatrix} \mathbf{N}'_2 \\ \mathbf{N}'_4 \end{pmatrix}$ . This can be proved as follows. Let the generator matrices of  $C_i$ ,  $i = l_n + 1, \dots, m$ , be given by the matrices  $G_i = [I_{k_i} | A'_i | A_i]$ , where  $A'_i$  has dimension  $k_i \times (k_m - k_i)$  and  $A_i$  has dimension  $k_i \times (n - k_m)$ . Let  $\mathbf{b} = (b_{l_n+1}, \dots, b_m)$ . The  $i$ th row of  $\mathbf{N}'_3$  can be split into two parts, corresponding to the first two blocks of the generator matrix  $G_i$  as follows. For  $i = l_n + 1, \dots, m$  we first write the  $i$ th row of  $\mathbf{N}_3$  as  $\mathbf{N}_{3,i} = (\mathbf{n}_i, \mathbf{n}'_i)$ , where the first block has length  $k_i$  and the second block has length  $k_m - k_i$ . We obtain

$$\mathbf{N}'_{3,i} = (\mathbf{n}_i + b_i \mathbf{j}_{k_i}, \mathbf{n}'_i + b_i \mathbf{j}_{k_m - k_i}) = \mathbf{N}_{3,i} + b_i \mathbf{j}_{k_m}.$$

Encoding the first block of this row  $\mathbf{N}'_{3,i}$  with the generator matrix  $G_i$  gives the vector

$$\begin{aligned} (\mathbf{n}_i + b_i \mathbf{j}_{k_i})[I_{k_i} | A'_i | A_i] &= (\mathbf{n}_i + b_i \mathbf{j}_{k_i}, \mathbf{n}_i A'_i + b_i \mathbf{j}_{k_m - k_i}, \mathbf{n}_i A_i + b_i \mathbf{j}_{n - k_m}) \\ &= (\mathbf{n}_i, \mathbf{n}'_i, \mathbf{N}_{4,i}) + b_i \mathbf{j}_n, \end{aligned}$$

where  $\mathbf{n}_i A'_i = \mathbf{n}'_i$  is the second block of  $\mathbf{N}_{3,i}$ , and  $\mathbf{n}_i A_i = \mathbf{N}_{4,i}$  is the  $i$ th row of  $\mathbf{N}_4$ . The shift by the coset leader  $(\mathbf{0}_{k_m}, \mathbf{a})$  results in the word

$$((\mathbf{n}_i, \mathbf{n}'_i) + b_i \mathbf{j}_{k_m}, \mathbf{N}_{4,i} + b_i \mathbf{j}_{n - k_m} + \mathbf{a}) = (\mathbf{N}'_{3,i}, \mathbf{N}_{4,i} + b_i \mathbf{j}_{n - k_m} + \mathbf{a}).$$

Thus, the matrix  $\mathbf{N}'_4$  is given by the expression

$$\mathbf{N}'_4 = \mathbf{N}_4 + \mathbf{b}^T \mathbf{j}_{n - k_m} + \mathbf{j}_{m - l_n}^T \mathbf{a}.$$

A similar argument shows that the submatrix  $\begin{pmatrix} \mathbf{N}'_2 \\ \mathbf{N}'_4 \end{pmatrix}$  satisfies the corresponding column codes.

The same argument as in the proof of Proposition 4.3 shows that the row and column weights are bounded when the conditions  $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \dots \subseteq \mathcal{D}_m \subset \mathcal{C}$  and  $\mathcal{D}_1 \subseteq \mathcal{D}_2 \subseteq \dots \subseteq \mathcal{D}_m \subset \mathcal{D}$  hold.  $\square$

**6. Conclusion.** We provide new constructions of systematic nonlinear product codes that are obtained by taking product of cosets of linear codes. The constructions have the property that every row and every column belong to the row code and column code, respectively. Subsequently, we show that it is possible to construct matrix codes with restricted column and row weights. Although the primary motivation for studying such matrix codes is for coded modulation over the power line channel, the constructions can potentially be adapted to address other problems where such codes are desired, such as codes for memristor arrays and two-dimensional weight-constrained codes [16, 15].

**Acknowledgment.** The authors thank the referee for the useful comments, which helped clarify the presentation.

#### REFERENCES

- [1] M. ALIPOUR, O. ETESAMI, G. MAATOUK, AND A. SHOKROLLAHI, *Irregular product codes*, in Proceedings of the IEEE Information Theory Workshop (ITW), 2012, pp. 197–201.
- [2] O. AMRANI, *Nonlinear codes: The product construction*, IEEE Trans. Commun., 55 (2007), pp. 1845–1851.
- [3] M. ARDAKANI, G. COLAVOLPE, K. DOSTERT, H. C. FERREIRA, D. FERTONANI, T. G. SWART, A. M. TONELLO, D. UMEHARA, AND A. J. H. VINCK, *Digital transmission techniques*, in Power Line Communications: Theory and Applications for Narrowband and Broadband Communications over Power Lines, H. C. Ferreira, L. Lampe, J. Newbury, and T. G. Swart, eds., John Wiley, New York, 2010, pp. 195–310.
- [4] A. BARG AND G. ZÉMOR, *List decoding of product codes by the minsum algorithm*, in Proceedings of the IEEE International Symposium on Information Theory Proceedings (ISIT), 2011, pp. 1273–1277.
- [5] L. A. BASSALYGO, S. M. DODUNEKOV, V. A. ZINOVIEV, AND T. HELLESETH, *The Grey-Rankin bound for non-binary codes*, Probl. Inform. Trans., 42 (2006), pp. 197–203.
- [6] È. L. BLOKH AND V. V. ZYABLOV, *Coding of generalized concatenated codes*, Problemy Peredači Informacii, 10 (1974), pp. 45–50.
- [7] Y. M. CHEE, H. M. KIAH, AND P. PURKAYASTHA, *Matrix codes and multitone frequency shift keying for power line communications*, in Proceedings of the IEEE International Symposium on Information Theory, 2013, pp. 2870–2874.

- [8] Y. M. CHEE, H. M. KIAH, P. PURKAYASTHA, AND P. SOLÉ, *Product construction of affine codes*, in Proceedings of the International Symposium on Information Theory, Honolulu, HI, 2014, pp. 1441–1445.
- [9] I. DUMER, *Concatenated codes and their multilevel generalizations*, in Handbook of Coding Theory, Vol. 2, Elsevier, Amsterdam, the Netherlands, 1998, pp. 1911–1988.
- [10] P. ELIAS, *Error-free coding*, Trans. IRE Professional Group Inform. Theory, 4 (1954), pp. 29–37.
- [11] G. D. FORNEY, JR., *Concatenated Codes*, MIT Press, Cambridge, MA, 1966.
- [12] E. M. GABIDULIN AND N. I. PILIPCHUK, *Error and erasure correcting algorithms for rank codes*, Designs Codes Cryptography, 49 (2008), pp. 105–122.
- [13] H. M. KIAH, *Reliable Communications over Power Lines Through Coded Modulation Schemes*, PhD thesis, Nanyang Technological University, Singapore, 2013.
- [14] F. J. MACWILLIAMS AND N. J. A. SLOANE, *The Theory of Error-Correcting Codes*, North-Holland, Amsterdam, the Netherlands, 1977.
- [15] E. ORDENTLICH AND R. M. ROTH, *Two-dimensional weight-constrained codes through enumeration bounds*, IEEE Trans. Inform. Theory, 46 (2000), pp. 1292–1301.
- [16] E. ORDENTLICH AND R. M. ROTH, *Low complexity two-dimensional weight-constrained codes*, in Proceedings of the IEEE International Symposium on Information Theory Proceedings (ISIT), 2011, pp. 149–153.
- [17] D. J. J. VERSFELD, A. J. H. VINCK, J. N. RIDLEY, AND H. C. FERREIRA, *Constructing coset codes with optimal same-symbol weight for detecting narrowband interference in M-FSK systems*, IEEE Trans. Inform. Theory, 56 (2010), pp. 6347–6353.
- [18] A. J. H. VINCK, *Coded modulation for power line communications*, Int J. Electron. Commun., 54 (2000), pp. 45–49.
- [19] V. A. ZINOVIEV, *Generalized cascade codes*, Problemy Peredači Informacii, 12 (1976), pp. 5–15.