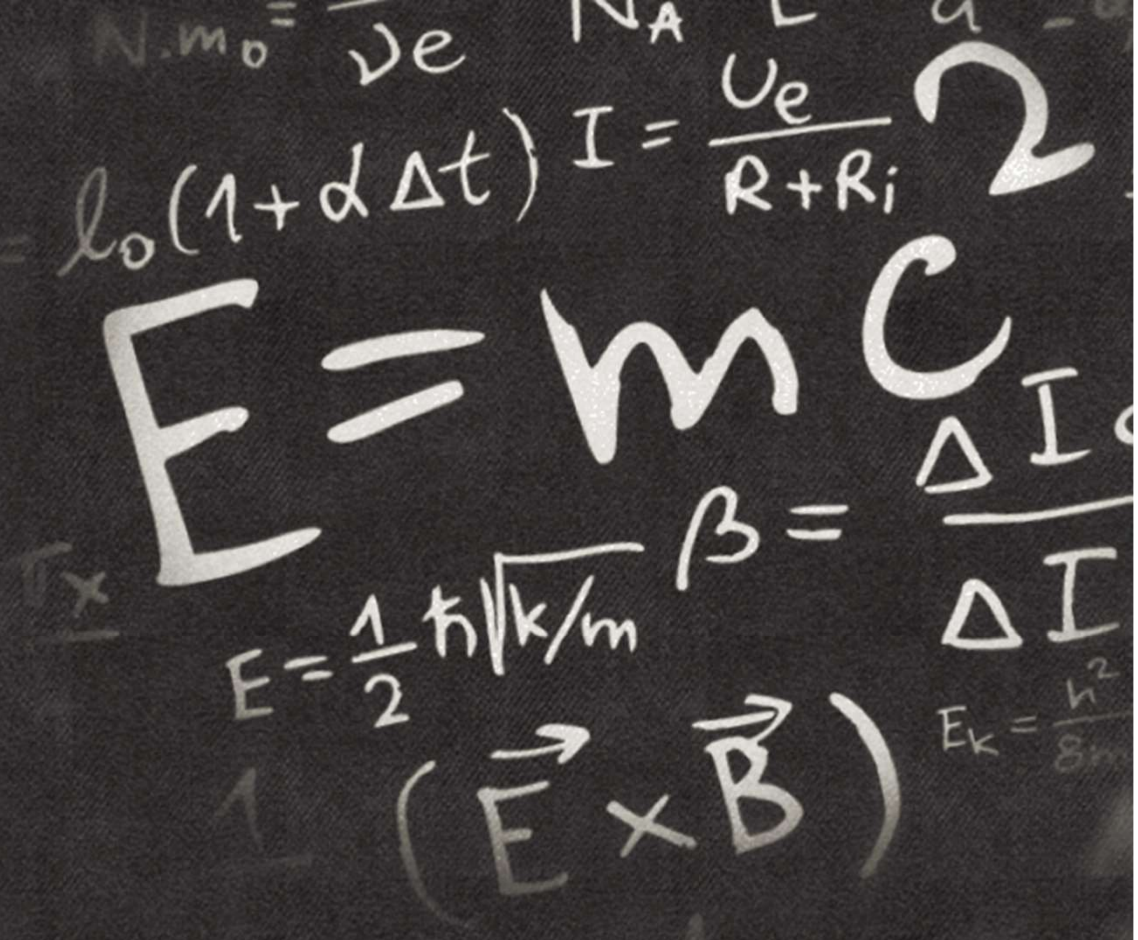
Error Analysis

Uncertainties and Error Propagation



PHY 206 Physics Laboratory III

2022/04

Presented by
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Topics to Learn

We sperate the video into 2 Parts

- Part I : Express number properly
- Part II: Systematic and random errors

Part I : How to Express a number properly

Learning Outcomes

- To round a number correctly
- To identify significant figures
- To express a number with scientific notation
- To apply correct rounding rules in arithmetic operations.

Round a number

4.25 cm?

4.257861453 cm?

Rules of Rounding Numbers

- **Rule 1:** If the remainder is less than 5, drop the last digit. Rounding to one decimal place: 5.34 → 5.3

- **Rule 2:** If the remainder is greater than 5, increase the final digit by 1. Rounding to one decimal place: 5.79 → 5.8

Rule 3: If the remainder is exactly 5 then round the last digit to the closest even number. This is to prevent rounding bias. Rounding to one decimal place: 3.55 → 3.6, also 3.65 → 3.6

Pause the video and Try the following exercises.

1) Round 6.5199 to one decimal place
2) Round 25.1521 to two decimal places
3) Round 13.9595 to three decimal places
4) Round 0.26925 to four decimal places

Answers:

1) 6.6
2) 25.05
3) 109.920
4) 0.5678

Instruction: Round to N decimal places → Look at (N+1)th decimal place → Follow rules

- Less than 5, drop the last digit (Nth decimal place unchanged)
- More than 5, Nth decimal place increase by 1)
- Equals 5, Nth decimal place goes to nearest even number.

Scientific Notation

321000 m	$= 3.21 \times 10^5$ m
0.000156 m	$= 1.56 \times 10^{-4}$ m

- A way of expressing numbers that are too big or too small to be conveniently written in decimal form.

$$\boxed{m \times 10^n}$$

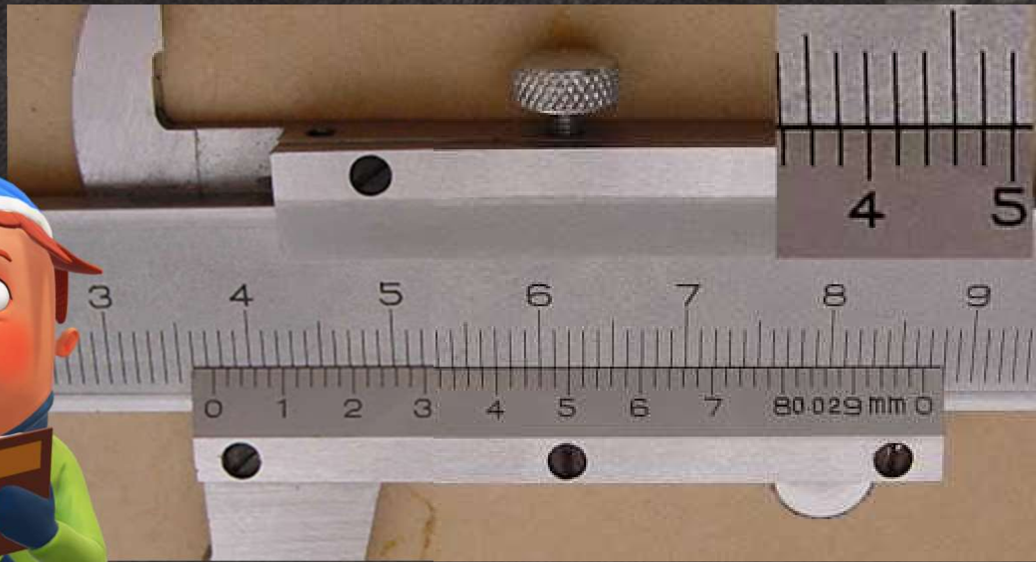
- m is a real number ($10 > m \geq 1$) and n is an integer

Significant Figures

4.2 cm

2 significant figures

3.746 cm



4 significant figures

Rules of Significant Figures

Four rules:

1. **All non-zero digits are significant digits**
   - 4 has one significant digit
   - 1.3 has two significant digits
   - 4,325.334 has seven significant digits

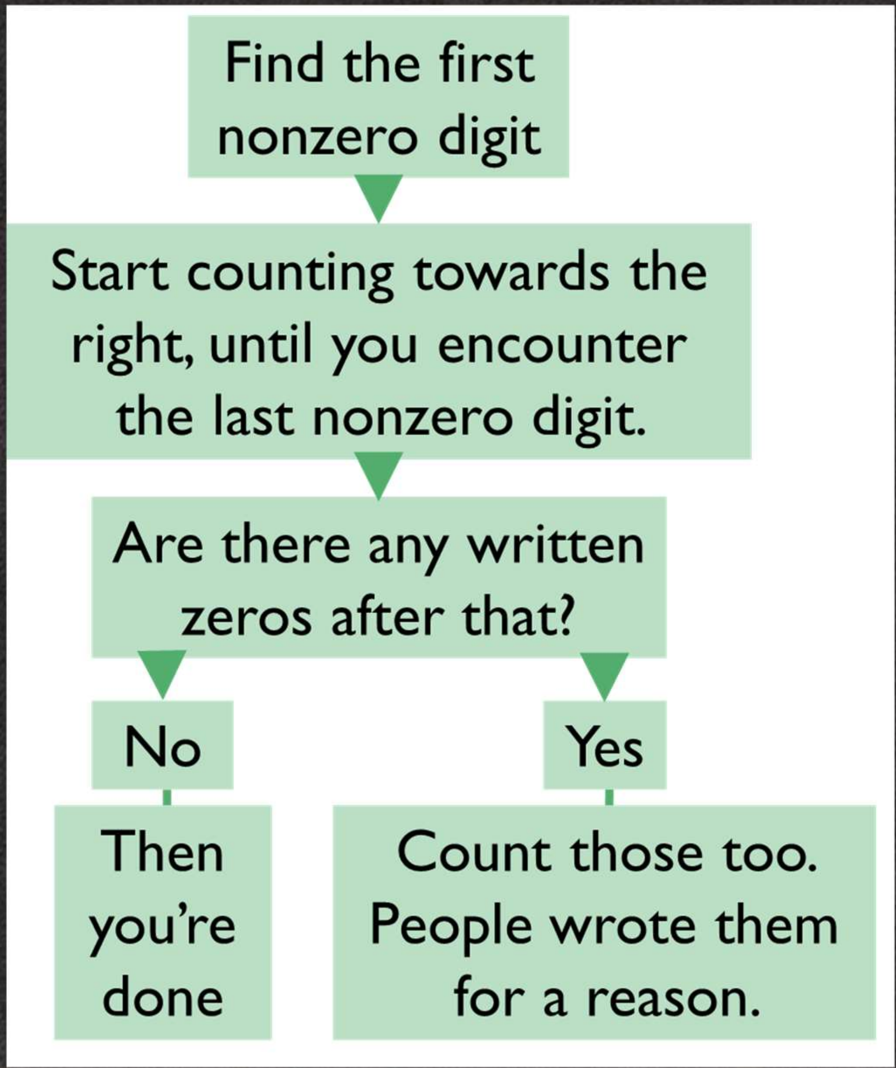
2. **Zeros that occur between the significant digits are significant**
   - 109 has three significant digits
   - 3.005 has four significant digits

Significant Figures

3. Zeros to the right of a non-zero digit (trailing zero) are significant
    - 0.0080 has two significant digits
    - 0.0000700 has three significant digits
    - 8000 has four significant digits – use scientific notation (8.0×10^3 has two significant digits & 8.00×10^3 has three significant digits)

4. Zero to the left of the first non-zero digit (leading zero) are not significant
    - 0.0080 has two significant digits
    - 0.0000700 has three significant digits

Pause the video and Try the following exercises.



Arithmetic Operations

ADDITION AND SUBTRACTION

Find the **left-most decimal place**. Round your answer to this decimal place.

$$
\begin{array}{r}
86.34\ \text{cm} \\
-\ 9.1\ \text{cm} \\
\hline
77.24\ \text{cm} = 77.2\ \text{cm}
\end{array}
$$

$$
\begin{array}{r}
1.6\ \text{m} \\
14.32\ \text{m} \\
+\ 8.014\ \text{m} \\
\hline
23.934\ \text{m} = 23.9\ \text{m}
\end{array}
$$

MULTIPLICATION AND DIVISION

The answer should contain the **same number of significant figures** as the value with the least number of significant digits.

$$
\begin{array}{r}
24.3\ \text{m} \\
\times\ 2.3\ \text{m} \\
\hline
729 \\
486 \\
\hline
55.89\ \text{m}^2 = 56\ \text{m}^2
\end{array}
$$

Part II : Systematic errors and Random errors

Learning Outcomes

1. To determine Systematic and Random uncertainties
2. To calculate the mean value and standard error.
3. To determine the accuracy and precision of a measurement by calculating the percentage error and the uncertainty.
4. To calculate the propagation errors for indirect measurement.

Systematic & Random errors

- Systematic errors will shift measurements from their true value by the same amount or fraction in the same direction all the time. It will not affect the precision but affect the accuracy.

- Random errors will shift each measurement from its true value by a random amount and in random direction. It will affect the precision but may not affect the overall accuracy.

Systematic Errors

Systematic Errors

The result of

1. A **<u>mis-calibrated device</u>**, or
2. A **<u>measuring technique</u>** which always makes the measured value larger (or smaller) than the "true" value.

Example: Using a steel ruler at liquid nitrogen temperature to measure the length of a rod.

- ➢ The ruler will contract at low temperatures and therefore **<u>overestimate the true length.</u>**

<u>Careful design of an experiment</u> will allow us to eliminate or to correct for systematic errors.

Instrument Tolerances

Measuring Tools （Instrument）	Measuring Range	Smallest Scale Division	Tolerances (Factory Default)
Meter Stick	30-50cm 60-100cm	1mm 1mm	±1.0mm ±1.5mm
Steel Ruler	150mm 500mm 1000mm	1mm 1mm 1mm	±1.0mm ±1.5mm ±2.0mm
Metal Tape Ruler	1m 2m	1mm 1mm	±0.8mm ±1.2mm
Vernier Caliper	125mm 300mm	0.02mm 0.05mm	±0.02mm ±0.05mm
Micrometer Screw Gauge	0-25mm	0.01mm	±0.004mm

Measuring Tools (Instrument)	Measuring Range	Smallest Scale Division	Tolerances (Factory Default)
Digital Balance	500g	0.05g	0.08g（close to full scale） 0.06g（around 1/2 full scale） 0.04g（around & lower than 1/3full scale）
Analytical Balance	200g	0.1mg	1.3mg（close to full scale） 1.0mg（around 1/2 full scale） 0.7mg（around & lower than 1/3full scale）
Thermometer (Mercury or Organic Solvents)	0-100^{0}C	1^{0}C	± 1^{0}C
Resistance Box			k%×Reading (k is the exact degree of accuracy)
Analog Meter			k%×Full Scale
Digital Multimeter			A_mk%+Number of Digit

Rule of Thumb to determine the uncertainty from measurement tools.

- Digital instrument

The uncertainty = the minimum scale

- Analog instrument

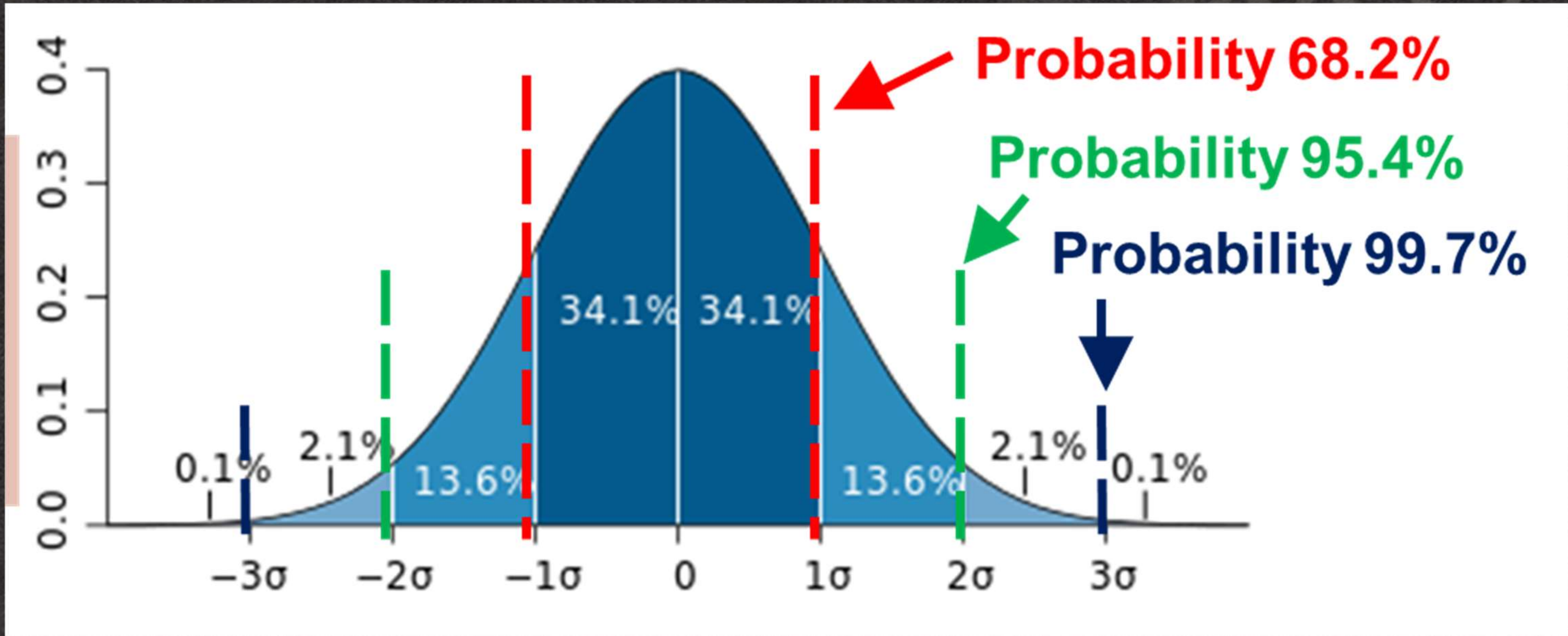
Both

The uncertainty = the minimum scale

The uncertainty = the minimum scale/2

Are acceptable.

Random Error



$$P(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Normal distribution

Error and uncertainty

The error of a measured value is 0.1,
The uncertainty of a measured value is 0.1

- Error is the discrepancy of a measured value from the true value.

- Uncertainty is the range of the possible errors for a measured value.

Result of a measurement, X can be expressed as the following form

$$x = x_{best} \pm \Delta x$$

Reading value from instrument

Uncertainty of X

$$x = (4.2 \pm 0.1)\ \text{cm}$$

$$x_{actual} \in (x_{best} - \Delta x,\ x_{best} + \Delta x)$$

Measurement

Single Measurement

- measure one time only

$$x = x_{best} \pm \Delta x_{sys}$$

Reading of x from instrument

Minimum scale of instrument.

Repeated Measurements

- repeat the same measurement many times

$$x = x_{measured} \pm \Delta x$$

Mean value of all the reading values

Combined uncertainty

$$\Delta x = \sqrt{\Delta x_{sys}^2 + (standard\ error)^2}$$

Mean value and standard error

For a repeated measurement $x = \{x_1, x_2, x_3, \ldots x_n\}$

Sample Mean $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$

Standard deviation $\sigma_x = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{(n-1)}}$

Standard error $\sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{n}} = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n(n-1)}}$

Combined uncertainty $\Delta x = \sqrt{\Delta x_{sys}^2 + \sigma_{\bar{x}}^2}$

Important Rules for Stating Uncertainty

- Commonly, uncertainty can be expressed in one significant figure. If the first digit is less than 3, can express in two significant figures. (example: Δx = 0.12 cm, Δx = 0.26 cm & Δx = 0.7 cm)
- When rounding, the uncertainty is always round up (example: Δx = 0.32 cm becomes Δx = 0.4 cm)
- The last digit of the measured value is aligned with the last digit of the uncertainty. The measured value follows the rounding rules.

Example:

$\overline{x} = 2.1445$ cm, $\Delta x = 0.0124$ cm

Measurement result = (2.144 ± 0.013) cm

Example :

The expression of (72.6 ± 0.382) kg is wrong because the uncertainty has more than one significant figure. The correct expression is (72.6 ± 0.4) kg.

Example

- A student measured the length of a pen 5 times using ruler.

Number of measurement	1	2	3	4	5
Length, *d* ± 0.05 (cm)	1.40	1.35	1.40	1.40	1.35

Uncertainty of instrument

The first step when dealing with a series of data is to find the mean.

$$\text{Mean}, \bar{d} = \left(\frac{1.40 + 1.35 + 1.40 + 1.40 + 1.35}{5}\right) cm = 1.38 \text{ cm}$$

Example

- A student is measuring the length of a pen 5 times using ruler.

Number of measurement	1	2	3	4	5
Length, *d* ± 0.05 (cm)	1.40	1.35	1.40	1.40	1.35

Uncertainty of instrument

Mean = 1.38 cm

$$\text{Standard Deviation, } \sigma_d = \sqrt{\frac{\sum_1^n (d_i - \bar{d})^2}{n-1}}$$

$$\sigma_d = \left(\sqrt{\left[\frac{(1.40-1.38)^2 + (1.35-1.38)^2 + (1.40-1.38)^2 + (1.40-1.38)^2 + (1.35-1.38)^2}{(5-1)} \right]} \right) \text{cm}$$

$$= 0.027386127\ cm \approx 0.0274\ cm$$

Example

- A student is measuring the length of a pen 5 times using ruler.

Number of measurement	1	2	3	4	5
Length, d ± 0.05 (cm)	1.40	1.35	1.40	1.40	1.35

Uncertainty of instrument

$$\text{Standard Error}, \sigma_{\bar{d}} = \sqrt{\frac{\sum_{i=1}^{n}(d_i - \bar{d})^2}{n(n-1)}} = \frac{\sigma_d}{\sqrt{n}} = \frac{0.0274\ \text{cm}}{\sqrt{5}} = 0.01252\ \text{cm} \approx 0.0125\ \text{cm}$$

Measurement Uncertainty

$$\text{System uncertainty} = \pm 0.05\ \text{cm}$$

Combined Uncertainty

$$= \pm\sqrt{(\text{Measurement Uncertainty})^2 + (\text{System Uncertainty})^2}$$

$$= \pm\sqrt{(0.0125\ \text{cm})^2 + (0.05\ \text{cm})^2}$$

$$= \pm\, 0.0515\ \text{cm} \approx 0.06\ cm$$

$$\boxed{\text{Length of pen}, d = (1.38\ \pm 0.06)\ \text{cm}}$$

Mean

Combined Uncertainty

Summary

Find mean

↓

Standard deviation

↓

Standard error

↓

Combined uncertainty

Not required for single measurements

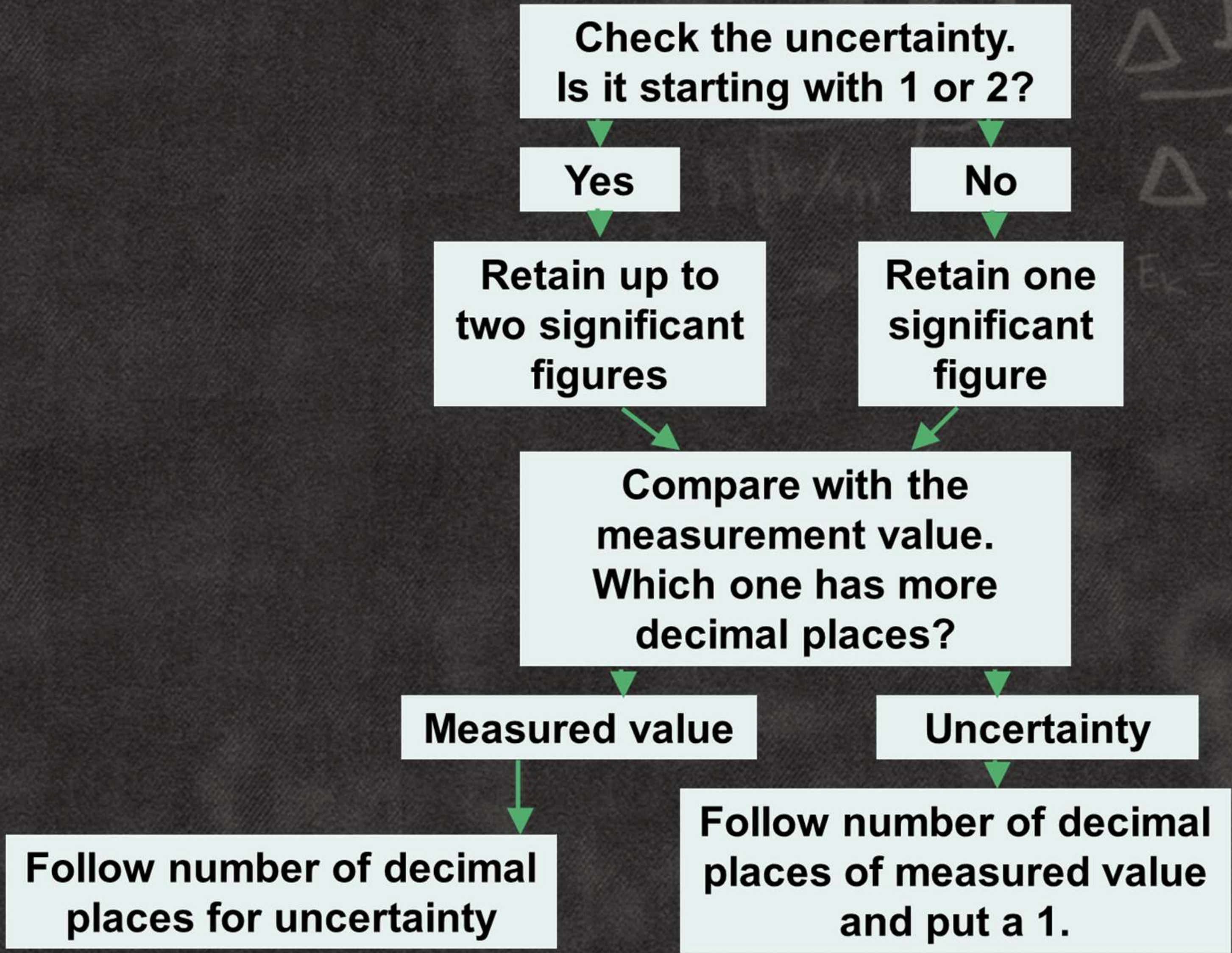
↓

Reduce uncertainty to the correct number of s.f. (change to scientific notation whenever necessary)

↓

Express the measurement value to the same number of decimal places as the uncertainty

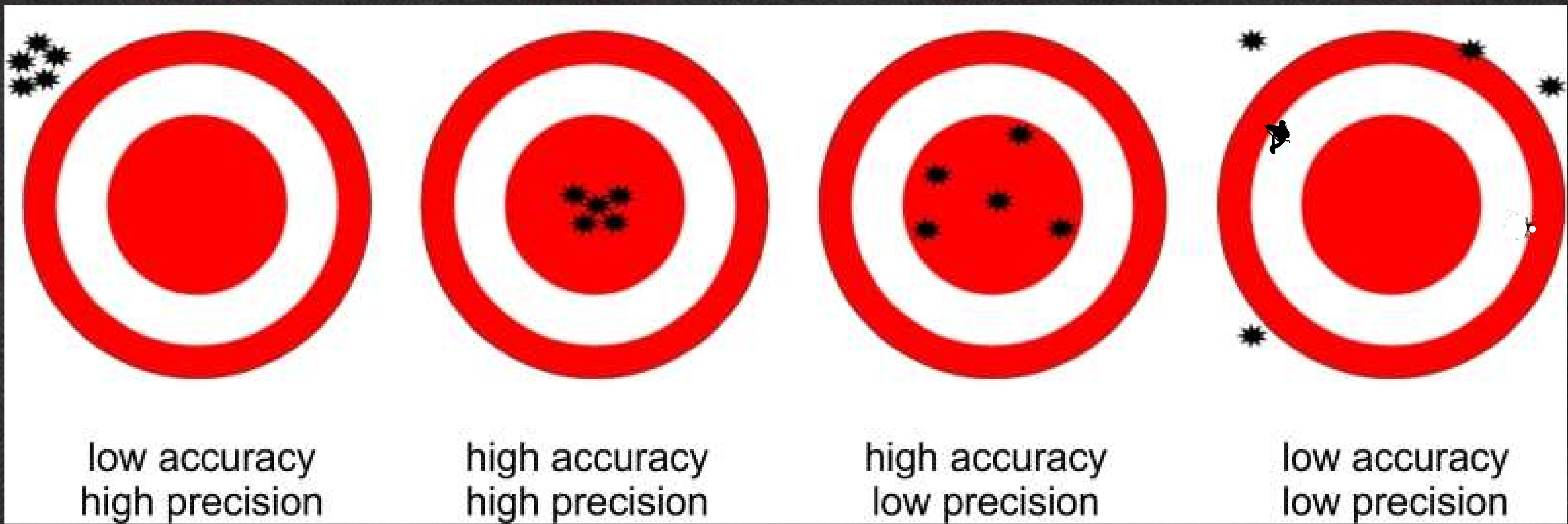
You can pause the video and try to express the following measurement result correctly



Accuracy & Precision

- **Accuracy** is the closeness of agreement between a measured value and a true or accepted value.

- **Precision** is the degree of consistency and agreement among independent measurements of the same quantity. It is sometimes referred to as “repeatability” or “reproducibility”.



Percentage of error

- Measures the accuracy of a measurement by the difference between a measured/experimental value and a true/theoretical value.

$$Percentage\ error(\%) = \left|\frac{x_{measured} - x_{true}}{x_{true}}\right| \times 100\%$$

- Example: if the true value of wavelength of a diode laser is 650nm and your experimental result is 665 nm.

Percentage of error = $\frac{\mathbf{(665-650)\ nm}}{\mathbf{650\ nm}} \times \mathbf{100\%} = \mathbf{2.41\%}$

Relative uncertainty

- Measures the precision of a measurement by the ratio of uncertainty and the measured value.

$$Relative\ uncertainty = \frac{\Delta x}{x_{measured}} \times 100\%$$

- Example: if we measure a diameter with Micrometer screw gauge as the picture shown. Suppose the uncertainty of the measurement is subjected to the minimum scale of the gauge. Then

$$\text{Relative uncertainty} = \frac{\mathbf{0.01\ mm}}{\mathbf{5.78\ mm}} \times 100\% = 0.2\%$$

Percentage difference

- Measures precision of two measurements by the difference between the measured or experimental values E1 and E2 expressed as a fraction the average of the two values.
- Comparison between two experimental values.

$$\boldsymbol{Percentage\ of\ Difference}\ (\%) = \left| \frac{\boldsymbol{E1 - E2}}{\left(\frac{\boldsymbol{E1 + E2}}{\boldsymbol{2}}\right)} \right| \times \boldsymbol{100\%}$$

- Example: You obtained two measurements of 20 ml and 22 ml. What is the percentage of difference?

$$\left| \frac{20 - 22}{\frac{20 + 22}{2}} \right| \times 100\% = 9.5\%$$

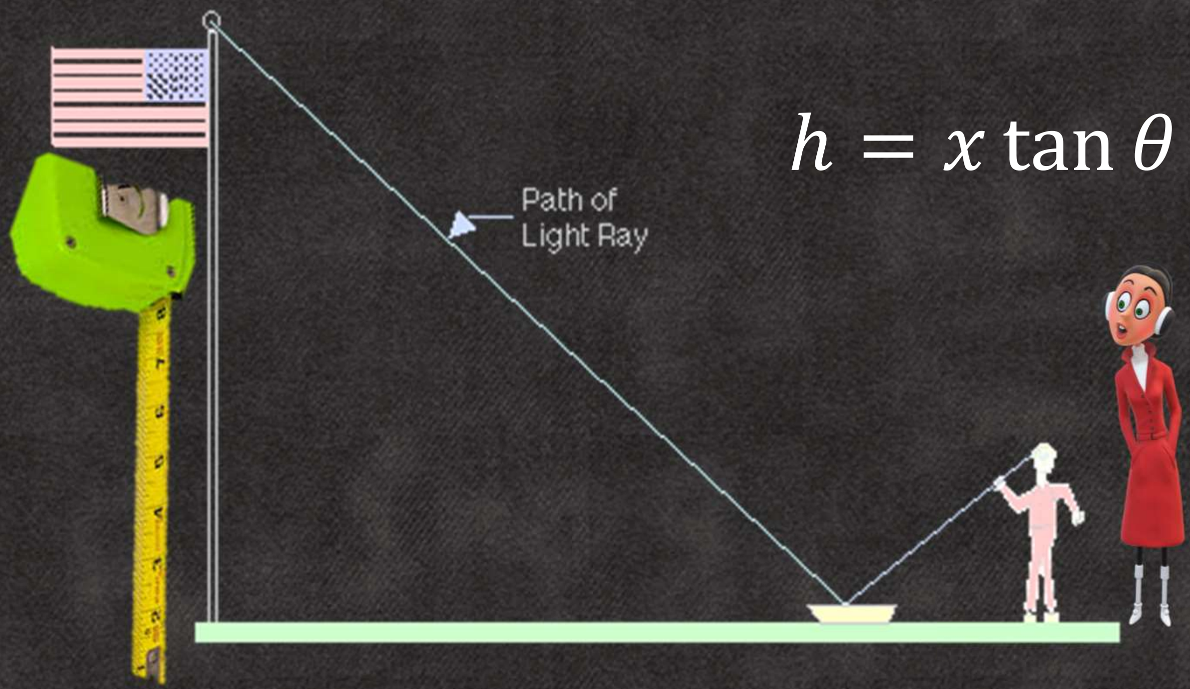
Propagation of Errors

DIRECT MEASUREMENT

- measuring exactly the physical quantity that you're looking to measure

INDIRECT MEASUREMENT

- measuring the actual physical quantity by measuring something else
- calculated using formula



$$h = x \tan \theta$$

Uncertainty of errors propagation

A physical quantity f is a function of $x_1, x_2, x_3, \ldots x_n$

If the uncertainty of each independent variables are given as

$$\Delta x_1, \Delta x_2, \Delta x_3, \ldots \Delta x_n$$

The uncertainty of f is

$$\Delta f = \sqrt{\left(\frac{\partial f}{\partial x_1}\right)^2 \Delta x_1^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 \Delta x_2^2 + \left(\frac{\partial f}{\partial x_3}\right)^2 \Delta x_3^2 + \cdots + \left(\frac{\partial f}{\partial x_n}\right)^2 \Delta x_n^2}$$

1) f = x + y
2) f = x - y

$$\frac{\partial f}{\partial x} = 1, \frac{\partial f}{\partial y} = 1 \qquad \therefore \sigma_f = \sqrt{\sigma_x^2 + \sigma_y^2}$$

3) f = xy

$$\frac{\partial f}{\partial x} = y, \frac{\partial f}{\partial y} = x \quad \therefore \sigma_f = \sqrt{y^2\sigma_x^2 + x^2\sigma_y^2} \quad \therefore \frac{\sigma_f}{|f|} = \sqrt{\left(\frac{\sigma_x}{x}\right)^2 + \left(\frac{\sigma_y}{y}\right)^2}$$

4) f = x/y

$$\therefore \sigma_f = \sqrt{\left(\frac{1}{y}\right)^2 \sigma_x^2 + \left(\frac{x}{y^2}\right)^2 \sigma_y^2}$$

$$\frac{\partial f}{\partial x} = \frac{1}{y}, \frac{\partial f}{\partial y} = -\frac{x}{y^2}$$

$$\therefore \frac{\sigma_f}{|f|} = \sqrt{\left(\frac{\sigma_x}{x}\right)^2 + \left(\frac{\sigma_y}{y}\right)^2}$$

Example:

Find relative uncertainty of speed *v*, where *v* = *at* with *a* = (5.8 ± 0.1) m/s^2, *t* = (1.25 ± 0.01) s and then express the speed with it uncertainty.

$$\therefore \frac{\Delta v}{|v_b|} = \sqrt{\left(\frac{\Delta a}{a_b}\right)^2 + \left(\frac{\Delta t}{t_b}\right)^2} = \sqrt{\left(\frac{0.1}{5.8}\right)^2 + \left(\frac{0.01}{1.25}\right)^2}$$

$$= 0.02 \; or \; 2\%$$

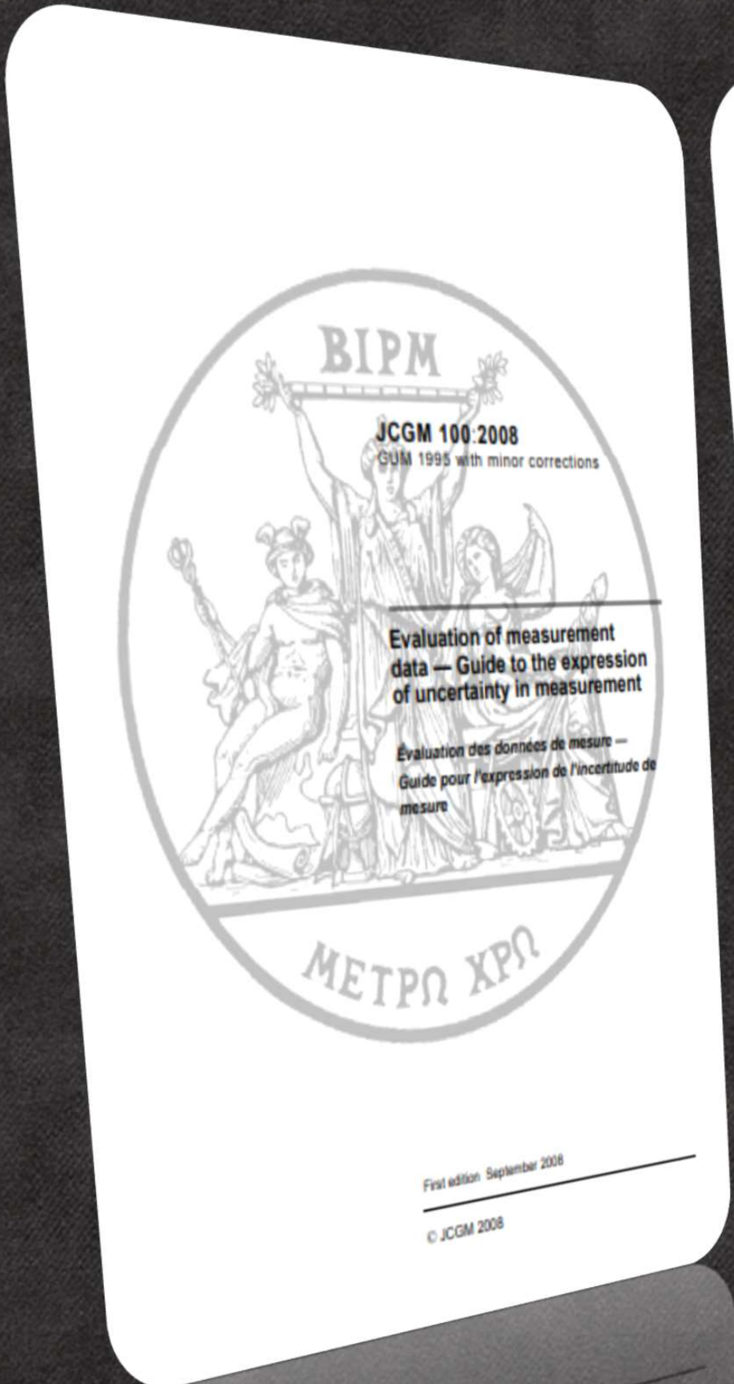
The best value for the speed is $v_b = 5.8 \times 1.25 = 7.25 \approx 7.2 \; m/s$

$$\Delta v = 7.2 \; \times 0.02 = 0.144 \approx 0.15 \; m/s$$

$$v = (7.2 \pm 0.15) \; m/s$$

$$v = (7.2 \pm 0.2) \; m/s$$

References – Check on Moodle



BIPM

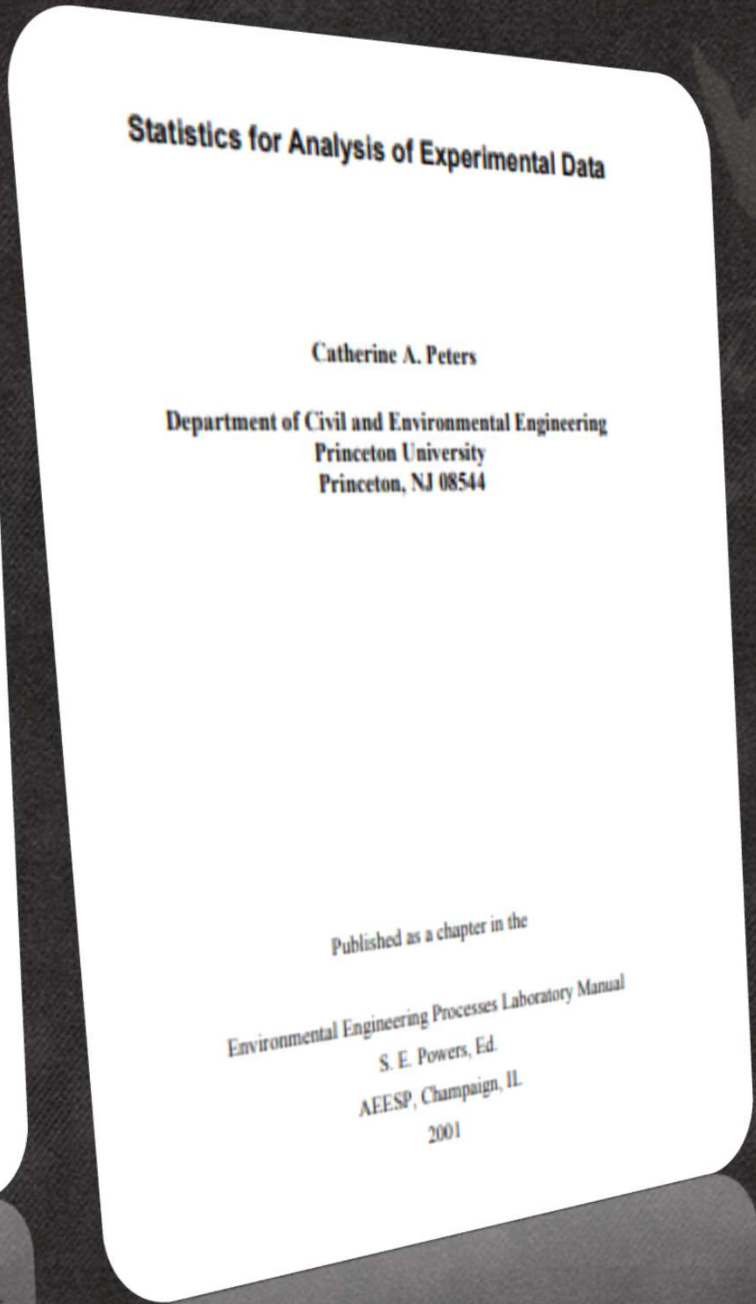
JCGM 100:2008
GUM 1995 with minor corrections

Evaluation of measurement data — Guide to the expression of uncertainty in measurement

Évaluation des données de mesure — Guide pour l'expression de l'incertitude de mesure

ΜΕΤΡΩ ΧΡΩ

First edition September 2008



Statistics for Analysis of Experimental Data

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Published as a chapter in the

Environmental Engineering Processes Laboratory Manual
S. E. Powers, Ed.
AEESP, Champaign, IL
2001

Thank you for your attention…