Honam Mathematical J. **37** (2015), No. 4, pp. 441–455
http://dx.doi.org/10.5831/HMJ.2015.37.4.441

A COMPARISON STUDY OF EXPLICIT AND IMPLICIT NUMERICAL METHODS FOR THE EQUITY-LINKED SECURITIES

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Abstract. In this paper, we perform a comparison study of explicit and implicit numerical methods for the equity-linked securities (ELS). The option prices of the two-asset ELS are typically computed using an implicit finite difference method because an explicit finite difference scheme has a restriction for time steps. Nowadays, the three-asset ELS is getting popularity in the real world financial market. In practical applications of the finite difference methods in computational finance, we typically use relatively large space steps and small time steps. Therefore, we can use an accurate and efficient explicit finite difference method because the implementation is simple and the computation is fast. The computational results demonstrate that if we use a large space step, then the explicit scheme is better than the implicit one. On the other hand, if the space step size is small, then the implicit scheme is more efficient than the explicit one.

1. Introduction

Equity-linked securities (ELS) are auto-callable options whose return on investment is dependent upon the path of the underlying equities linked to the securities. ELS can be made from a few number of stocks or stock indexes such as the KOSPI200 in Korea. ELS is a derivative product in the market. ELS guarantees a debt and is similar to a barrier option. ELS comprises a large portion of exchange volume in Korea financial market. A distinguishing feature of ELS is the automatic early-redemption condition before its maturity. Generally, in order to get price

Received April 15, 2015. Accepted October 6, 2015.
2010 Mathematics Subject Classification. 91G60, 65M06.
Key words and phrases. Black–Scholes partial differential equation, log transformation, explicit finite difference method, equity-linked securities, non-uniform grid.
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of ELS, Monte Carlo simulation (MCS) and finite difference method (FDM) are used. Typically, we use the implicit scheme with operator split method (OSM) or alternating direction implicit (ADI) because they are stable. Although the implicit scheme with OSM has an advantage in stability, it is costly to solve the tridiagonal matrix implicitly. In particular, we need to solve the system three times in each time step for the three-asset problems. In practice, we calculate one time step with one day, which is about 1/365 and discretize the asset by one unit. Therefore, the time step is small enough and space step is large enough. Furthermore, if we want to calculate the Greeks of the option price, especially theta which is the rate of change of the option value with respect to changes in the time to maturity, then we have to use much smaller time step. For these considerations, it is better to use an explicit scheme with smaller time step, which gives much accurate solutions. The program implementation is simple and fast. The main purpose of this paper is to develop an explicit scheme on a non-uniform grid to solve value of ELS.

The paper is organized as follows. We introduce Black–Scholes model in Section 2. In Section 3, we present three-asset step-down ELS. Numerical methods are presented in Section 4. In Section 5, we perform numerical experiments. In Section 6, we take conclusion for this paper.

2. Black–Scholes model

To evaluate value of the ELS option, we consider the standard Black–Scholes model [1], which can be written as

$$\frac{\partial u(\mathbf{x},t)}{\partial t} = -\frac{1}{2}\sum_{i,j=1}^{d}\rho_{ij}\sigma_i\sigma_j x_i x_j \frac{\partial^2 u(\mathbf{x},t)}{\partial x_i \partial x_j} - r\sum_{i=1}^{d} x_i \frac{\partial u(\mathbf{x},t)}{\partial x_i} + ru(\mathbf{x},t),$$
$$\text{for } (\mathbf{x},t) \in \mathbb{R}_+^d \times [0,T].$$

Here, $u(\mathbf{x},t)$ is the value of the option, where $\mathbf{x} = (x_1, x_2, \ldots, x_d)$, d is the total number of underlying assets, x_i is the value of the i-th underlying assets, and t is the time. Also, r represents the riskless interest rate, σ_i is the volatility of i-th the underlying assets, ρ_{ij} is the correlation coefficient between x_i and x_j, and T is the maturity time of the option. Switching to the new coordinate $X = \log x$ [4] and using the transformation $\tau = T - t$, the standard BS equation can be rewritten as

$$(1) \quad \frac{\partial U(\mathbf{X},\tau)}{\partial \tau} = \sum_{i=1}^{d}(r - \frac{\sigma_i^2}{2})\frac{\partial U(\mathbf{X},\tau)}{\partial X_i} + \frac{1}{2}\sum_{i,j=1}^{d}\rho_{ij}\sigma_i\sigma_j\frac{\partial^2 U(\mathbf{X},\tau)}{\partial X_i \partial X_j} - rU(\mathbf{X},\tau),$$

where $U(\mathbf{X},\tau)$ is the value of the option and $\mathbf{X} = (X_1, X_2, \ldots, X_d)$.

3. Initial condition

In this section, we briefly describe the concept of step-down ELS option and we introduce a three-asset step-down ELS option as an example.

3.1. Step-down ELS

The payoff of ELS is determined by early redemption or final maturity redemption. In one-asset step-down ELS, if an underlying asset is in the predetermined exercise price at certain maturity dates according to contract, then ELS gives designated return and is exterminated. However, if the underlying asset is not in a certain price, the contract is not exterminated and will be continued until next maturity. If the contract continues at final maturity and the underlying asset is not in final exercise barrier, the payoff is determined whether the contract hit knock in barrier. If the underlying asset did not hit the knock in barrier, the ELS gives predetermined return, dummy. Otherwise, the ELS will make a loss in face value. The step-down means that the designated strike price decreases.

In this paper, we consider a three-asset step-down ELS which is similar to a one asset step-down ELS, as we explained above. The difference with a one asset ELS is that the base price is referred from the minimum of three underlying assets at certain maturity dates. The payoff structure of three-asset step-down ELS is as follows [3]:

- Early redemption occurs, and the contract is exterminated with predetermined return if the value of the worst performer, which means the minimum value of underlying assets is greater than or equal to a given exercise price on a given date.
- If the early redemption do not occur until the final maturity, the return depends upon whether Knock-In occurs or not.

3.2. Example for three-asset step-down ELS

To help readers' understanding of the step-down ELS option, we include the example for three-asset step-down ELS. We consider the parameters as the reference price $E = 100$, the riskless interest rate $r = 0.03$, the volatilities of the underlying assets $\sigma_1 = \sigma_2 = \sigma_3 = 0.3$, the correlations of underlying assets $\rho_{12} = \rho_{13} = \rho_{23} = 0.5$, the face value $F = 100$, the Knock-In barrier level KIb $= 0.65E$, the dummy rate $d = 0.3$, and the final maturity time $T = 1$. At the early redemption, observation date (τ_i), exercise price (K_i), and return rate (c_i) are described in Table 1.

Observation date (τ_i)	Exercise price (K_i)	Return Rate (c_i)
$\tau_6 = T$	$K_6 = 0.85E$	$c_6 = 0.30$
$\tau_5 = 5T/6$	$K_5 = 0.85E$	$c_5 = 0.25$
$\tau_4 = 4T/6$	$K_4 = 0.90E$	$c_4 = 0.20$
$\tau_3 = 3T/6$	$K_3 = 0.90E$	$c_3 = 0.15$
$\tau_2 = 2T/6$	$K_2 = 0.95E$	$c_2 = 0.10$
$\tau_1 = T/6$	$K_1 = 0.95E$	$c_1 = 0.05$

TABLE 1. Observation date (τ_i), exercise price (K_i), and return rate (c_i) used in example for three-asset step-down ELS.

Figure 1 illustrates the payoff of early obligatory redemption before final maturity.

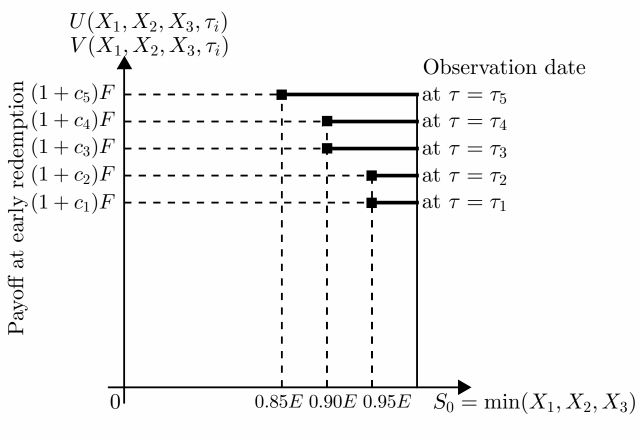

FIGURE 1. Payoff at early redemption before final maturity.

Along with the characteristics of the step-down ELS, we must take two considerations of the payoffs at maturity. According to whether or not the value of ELS hits the Knock-In barrier (KIb) during the contract, we define two values $V(\mathbf{X},\tau)$ and $U(\mathbf{X},\tau)$. Then, the initial conditions of U and V are set to

$$U(X_1, X_2, X_3, 0) = \begin{cases} (1+c_6)F & \text{if } S_0 \geq K_6 \\ (1+d)F & \text{if KIb} < S_0 < K_6 \\ S_0F/E & \text{otherwise,} \end{cases} \tag{2}$$

and

$$V(X_1, X_2, X_3, 0) = \begin{cases} (1+c_6)F & \text{if } S_0 \geq K_6 \\ S_0F/E & \text{otherwise,} \end{cases} \tag{3}$$

where $S_0 = \min(X_1, X_2, X_3)$ which is the value of the worst performer. In Fig. 2, we can see the corresponding payoffs of U and V when $c_6 = d$.

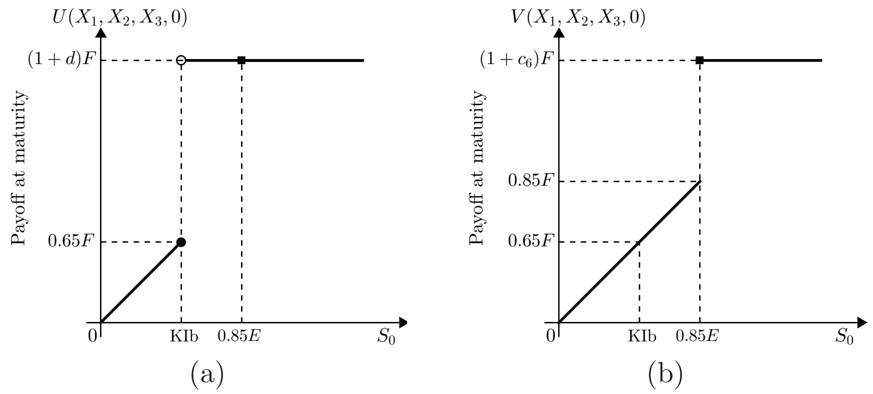

FIGURE 2. Payoffs of (a) U and (b) V at final maturity.

4. Numerical solution

In this section, we describe the numerical discretization of Eq. (1) using explicit scheme on the computational domain $\Omega = [1, S_{\max}]^3$, which is log-transformed by the three-dimensional finite domain $\bar{\Omega} = [e, e^{S_{\max}}]$. Also, to prevent the spurious oscillatory solution by explicit scheme, we derive the condition for time step size.

4.1. Discretization of log-transformed BS equation

Let X, Y, and Z denote the log-transform of x-, y-, and z-variables, respectively. Now, we discretize the log-transformed computational domain $\Omega = [1, S_{\max}]^3$ with non-uniform spatial step size $h_{i-1} = X_i - X_{i-1} = Y_i - Y_{i-1} = Z_i - Z_{i-1}$ (see Fig. 3) and temporal step size $\Delta\tau = T/N_\tau$. Here, $X_0 = 1$, $X_{N_x} = S_{\max}$, where N_x and N_τ are the numbers of grid points in the X- and τ-directions, respectively.

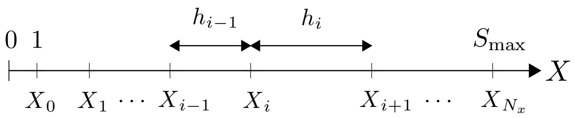

FIGURE 3. Nonuniform mesh on log-transformed grid X.

We denote the numerical solution by $U^n_{ijk} \equiv U(X_i, Y_j, Z_k, \tau^n)$ for $i = 1, 2, \cdots, N_x$, $j = 1, 2, \cdots, N_y$, $k = 1, 2, \cdots, N_z$, and $n = 0, 1, \cdots, N_\tau$.

Applying the explicit finite difference scheme to Eq. (1) gives

$$
\begin{aligned}
(4)\quad \frac{U^{n+1}_{ijk} - U^n_{ijk}}{\Delta\tau} &= \frac{\sigma_1^2}{2}\left(\frac{\partial^2 U}{\partial X^2}\right)^n_{ijk} + \frac{\sigma_2^2}{2}\left(\frac{\partial^2 U}{\partial Y^2}\right)^n_{ijk} + \frac{\sigma_3^2}{2}\left(\frac{\partial^2 U}{\partial Z^2}\right)^n_{ijk} \\
&+ \left(r - \frac{\sigma_1^2}{2}\right)\left(\frac{\partial U}{\partial X}\right)^n_{ijk} + \left(r - \frac{\sigma_2^2}{2}\right)\left(\frac{\partial U}{\partial Y}\right)^n_{ijk} + \left(r - \frac{\sigma_3^2}{2}\right)\left(\frac{\partial U}{\partial Z}\right)^n_{ijk} \\
&+ \rho_{12}\sigma_1^2\sigma_2^2\left(\frac{\partial^2 U}{\partial X \partial Y}\right)^n_{ijk} + \rho_{13}\sigma_1^2\sigma_3^2\left(\frac{\partial^2 U}{\partial X \partial Z}\right)^n_{ijk} \\
&+ \rho_{23}\sigma_2^2\sigma_3^2\left(\frac{\partial^2 U}{\partial Y \partial Z}\right)^n_{ijk} - rU^n_{ijk}.
\end{aligned}
$$

Here, spatial differences on the non-uniform grid are defined by

$$
\begin{aligned}
\left(\frac{\partial U}{\partial X}\right)^n_{ijk} &= -\frac{h_i}{h_{i-1}(h_{i-1}+h_i)}U^n_{i-1,jk} + \frac{h_i - h_{i-1}}{h_{i-1}h_i}U^n_{ijk} \\
&\quad + \frac{h_{i-1}}{h_i(h_{i-1}+h_i)}U^n_{i+1,jk}, \\
\left(\frac{\partial^2 U}{\partial X^2}\right)^n_{ijk} &= \frac{2}{h_{i-1}(h_{i-1}+h_i)}U^n_{i-1,jk} - \frac{2}{h_{i-1}h_i}U^n_{ijk} \\
&\quad + \frac{2}{h_i(h_{i-1}+h_i)}U^n_{i+1,jk},
\end{aligned}
$$

$$\left(\frac{\partial^2 U}{\partial X \partial Y}\right)^n_{ijk} = \frac{U_{i+1,j+1,k} + U_{i-1,j-1,k} - U_{i+1,j-1,k} - U_{i-1,j-1,k}}{h_i h_j + h_i h_{j-1} + h_{i-1} h_j + h_{i-1} h_{j-1}}.$$

Other spatial differences can be similarly defined. Equation (4) is rewritten as follows:

$$\begin{aligned}
(5)\qquad U^{n+1}_{ijk} &= U^n_{ijk} + \Delta\tau\Bigg[\frac{\sigma_1^2 - h_i C_1}{h_{i-1}(h_{i-1}+h_i)} U^n_{i-1,jk} \\
&+ \frac{-\sigma_1^2 + (h_i - h_{i-1})C_1}{h_{i-1}h_i} U^n_{ijk} + \frac{\sigma_1^2 + h_{i-1}C_1}{h_i(h_{i-1}+h_i)} U^n_{i+1,jk} \\
&+ \frac{\sigma_2^2 - h_j C_2}{h_{j-1}(h_{j-1}+h_j)} U^n_{ij-1,k} + \frac{-\sigma_2^2 + (h_j - h_{j-1})C_2}{h_{j-1}h_j} U^n_{ijk} \\
&+ \frac{\sigma_2^2 + h_{j-1}C_2}{h_j(h_{j-1}+h_j)} U^n_{ij+1,k} + \frac{\sigma_3^2 - h_k C_3}{h_{k-1}(h_{k-1}+h_k)} U^n_{ijk-1} \\
&+ \frac{-\sigma_3^2 + (h_k - h_{k-1})C_3}{h_{k-1}h_k} U^n_{ijk} + \frac{\sigma_3 + h_{k-1}C_3}{h_k(h_{k-1}+h_k)} U^n_{ijk+1} - rU^n_{ijk} \\
&+ \rho_{12}\sigma_1^2\sigma_2^2 \left(\frac{U^n_{i+1,j+1,k} + U^n_{i-1,j-1,k} - U^n_{i+1,j-1,k} - U^n_{i-1,j+1,k}}{h_i h_j + h_i h_{j-1} + h_{i-1}h_j + h_{i-1}h_{j-1}}\right) \\
&+ \rho_{13}\sigma_1^2\sigma_3^2 \left(\frac{U^n_{i+1,jk+1} + U^n_{i-1,jk-1} - U^n_{i+1,jk-1} - U^n_{i-1,jk+1}}{h_i h_k + h_i h_{k-1} + h_{i-1}h_k + h_{i-1}h_{k-1}}\right) \\
&+ \rho_{23}\sigma_2^2\sigma_3^2 \left(\frac{U^n_{ij+1,k+1} + U^n_{ij-1,k-1} - U^n_{ij+1,k-1} - U^n_{ij-1,k+1}}{h_j h_k + h_j h_{k-1} + h_{j-1}h_k + h_{j-1}h_{k-1}}\right)\Bigg],
\end{aligned}$$

where $C_1 = r - 0.5\sigma_1^2$, $C_2 = r - 0.5\sigma_2^2$, and $C_3 = r - 0.5\sigma_3^2$.

4.2. Condition for the non-oscillatory solution

Now, we derive the conditions under which the explicit scheme for Eq. (4) will not make spurious oscillations by using the idea in reference [2]. Then, we rewrite Eq. (4) as

$$\begin{aligned}
(6)\qquad U^{n+1}_{ijk} &= \Delta\tau\Bigg(\frac{-\sigma_1^2 + (h_i - h_{i-1})C_1}{h_i h_{i-1}} + \frac{-\sigma_2^2 + (h_j - h_{j-1})C_2}{h_j h_{j-1}} \\
&+ \frac{-\sigma_3^2 + (h_k - h_{k-1})C_3}{h_k h_{k-1}} - r + \frac{1}{\Delta\tau}\Bigg) U^n_{ijk} \\
&+ \frac{\Delta\tau(\sigma_1^2 - h_i C_1)}{h_{i-1}(h_i + h_{i-1})} U^n_{i-1,jk} + \frac{\Delta\tau(\sigma_1^2 + h_{i-1}C_1)}{h_i(h_i + h_{i-1})} U^n_{i+1,jk} \\
&+ \frac{\Delta\tau(\sigma_2^2 - h_j C_2)}{h_{j-1}(h_j + h_{j-1})} U^n_{ij-1,k} + \frac{\Delta\tau(\sigma_2^2 + h_{j-1}C_2)}{h_j(h_j + h_{j-1})} U^n_{ij+1,k}
\end{aligned}$$

$$+\frac{\Delta\tau(\sigma_3^2 - h_k C_3)}{h_{k-1}(h_k + h_{k-1})}U_{ijk-1}^n + \frac{\Delta\tau(\sigma_3^2 + h_{k-1} C_3)}{h_k(h_k + h_{k-1})}U_{ijk+1}^n$$
$$+\Delta\tau\rho_{12}\sigma_1\sigma_2\left(\frac{U_{i+1,j+1k}^n + U_{i-1,j-1,k}^n - U_{i+1,j-1,k}^n - U_{i-1,j+1,k}^n}{h_ih_j + h_ih_{j-1} + h_{i-1}h_j + h_{i-1}h_{j-1}}\right)$$
$$+\Delta\tau\rho_{13}\sigma_1\sigma_3\left(\frac{U_{i+1,jk+1}^n + U_{i-1,jk-1}^n - U_{i+1,jk-1}^n - U_{i-1,jk+1}^n}{h_ih_k + h_ih_{k-1} + h_{i-1}h_k + h_{i-1}h_{k-1}}\right)$$
$$+\Delta\tau\rho_{23}\sigma_2\sigma_3\left(\frac{U_{ij+1,k+1}^n + U_{ij-1,k-1}^n - U_{ij+1,k-1}^n - U_{ij-1,k+1}^n}{h_jh_k + h_jh_{k-1} + h_{j-1}h_k + h_{j-1}h_{k-1}}\right).$$

Next, we substitute $U_{ijk}^{n+1} = \beta_{ijk}^{n+1}(1 - r\Delta\tau)^n$ into Eq. (6), where the superscript n for $(1 - r\Delta\tau)$ represents an exponent. Then, we obtain

$$(7)\quad \beta_{ijk}^{n+1} = \Delta\tau\left[\frac{-\sigma_1^2 + (h_i - h_{i-1})C_1}{(1 - r\Delta\tau)h_ih_{i-1}} + \frac{-\sigma_2^2 + (h_j - h_{j-1})C_2}{(1 - r\Delta\tau)h_jh_{j-1}}\right.$$
$$\left.+\frac{-\sigma_3^2 + (h_k - h_{k-1})C_3}{(1 - r\Delta\tau)h_kh_{k-1}} - \frac{r}{(1 - r\Delta\tau)} + \frac{1}{\Delta\tau(1 - r\Delta\tau)}\right]\beta_{ijk}^n$$
$$+\frac{\Delta\tau(\sigma_1^2 - h_iC_1)}{(1 - r\Delta\tau)h_{i-1}(h_i + h_{i-1})}\beta_{i-1,jk}^n + \frac{\Delta\tau(\sigma_1^2 + h_{i-1}C_1)}{(1 - r\Delta\tau)h_i(h_i + h_{i-1})}\beta_{i+1,jk}^n$$
$$+\frac{\Delta\tau(\sigma_2^2 - h_jC_2)}{(1 - r\Delta\tau)h_{j-1}(h_j + h_{j-1})}\beta_{ij-1,k}^n + \frac{\Delta\tau(\sigma_2^2 + h_{j-1}C_2)}{(1 - r\Delta\tau)h_j(h_j + h_{j-1})}\beta_{ij+1,k}^n$$
$$+\frac{\Delta\tau(\sigma_3^2 - h_kC_3)}{(1 - r\Delta\tau)h_{k-1}(h_k + h_{k-1})}\beta_{ijk-1}^n + \frac{\Delta\tau(\sigma_3^2 + h_{k-1}C_3)}{(1 - r\Delta\tau)h_k(h_k + h_{k-1})}\beta_{ijk+1}^n$$
$$+\frac{\Delta\tau\rho_{12}\sigma_1\sigma_2}{(1 - r\Delta\tau)}\left(\frac{\beta_{i+1,j+1k}^n + \beta_{i-1,j-1,k}^n - \beta_{i+1,j-1,k}^n - \beta_{i-1,j+1,k}^n}{h_ih_j + h_ih_{j-1} + h_{i-1}h_j + h_{i-1}h_{j-1}}\right)$$
$$+\frac{\Delta\tau\rho_{13}\sigma_1\sigma_3}{(1 - r\Delta\tau)}\left(\frac{\beta_{i+1,jk+1}^n + \beta_{i-1,jk-1}^n - \beta_{i+1,jk-1}^n - \beta_{i-1,jk+1}^n}{h_ih_k + h_ih_{k-1} + h_{i-1}h_k + h_{i-1}h_{k-1}}\right)$$
$$+\frac{\Delta\tau\rho_{23}\sigma_2\sigma_3}{(1 - r\Delta\tau)}\left(\frac{\beta_{ij+1,k+1}^n + \beta_{ij-1,k-1}^n - \beta_{ij+1,k-1}^n - \beta_{ij-1,k+1}^n}{h_jh_k + h_jh_{k-1} + h_{j-1}h_k + h_{j-1}h_{k-1}}\right).$$

Since all coefficients of β_{ijk}^n in Eq. (7) are positive, the following conditions should be satisfied.

$$(8)\quad \frac{-\sigma_1^2 + (h_i - h_{i-1})C_1}{(1 - r\Delta\tau)h_ih_{i-1}} + \frac{-\sigma_2^2 + (h_j - h_{j-1})C_2}{(1 - r\Delta\tau)h_jh_{j-1}}$$
$$+\frac{-\sigma_3^2 + (h_k - h_{k-1})C_3}{(1 - r\Delta\tau)h_kh_{k-1}} - \frac{r}{(1 - r\Delta\tau)} + \frac{1}{\Delta\tau(1 - r\Delta\tau)} > 0.$$

Also, when we solve the log-transformed BS equation on the adaptive grid, we will use non-decreasing spatial step size, that is, $h_{i-1} \le h_i$ for all i. By setting the lower bound of spatial step size as $h_{\min}$ satisfying $h_{\min} \leq h_1$, condition (8) gives

$$-r\Delta\tau h_{\min}^2 + h_{\min}^2 - \Delta\tau\sigma_1^2 - \Delta\tau\sigma_2^2 - \Delta\tau\sigma_3^2 > 0. \tag{9}$$

Therefore, we have the following restriction condition for time step size

$$\Delta\tau < \frac{h_{\min}^2}{rh_{\min}^2 + \sigma_1^2 + \sigma_2^2 + \sigma_3^2}. \tag{10}$$

5. Numerical experiments

In this section, we implement numerical tests with our numerical method. For numerical tests, we consider three-asset step-down ELS option as the example described in section 3.2. We compare non-oscillatory explicit FDM and implicit FDM with respect to computational costs and errors.

5.1. Numerical treatment for three-asset step-down ELS option pricing

Before evaluating the option value of the test problem which is stated in section. 3.2, we first have to consider two cases according to the initial payoff. Let U and V be the numerical solutions with payoffs which knock-in event does not happen and happen, respectively. With the initial payoffs (3) and (2) which are described in Fig. 2, we solve Eq. (5). After solving Eq. (5) once, we replace the values of less than KIb in U with the values of V [3]. And then, we update the value of U by using the FDM scheme. These processes are repeated from 0 to T every time step. Also, at the early redemption before $\tau = T$, U and V follow the conditions which are described in Table 1 and Fig. 1.

5.2. Numerical test

We perform numerical experiments for pricing three-asset step-down ELS. The parameters we have used are listed in section 3.2. Also, the computation domain is used as $\Omega = [1, 220] \times [1, 220] \times [1, 220]$. In Figs. 4-5, we can see the initial payoff function and numerical result at $\tau = 1$ of U and V, respectively.

Now, we compare the results from MCS and the FDM, the implicit scheme with OSM and non-oscillatory explicit scheme. We focus on the value of $U(100, 100, 100)$. In this test, MCS is performed 10^6 samples with $\Delta\tau = 1/1440$ using antithetic variates of variance reduction, and results from MCS are used as a reference value [7]. In order to

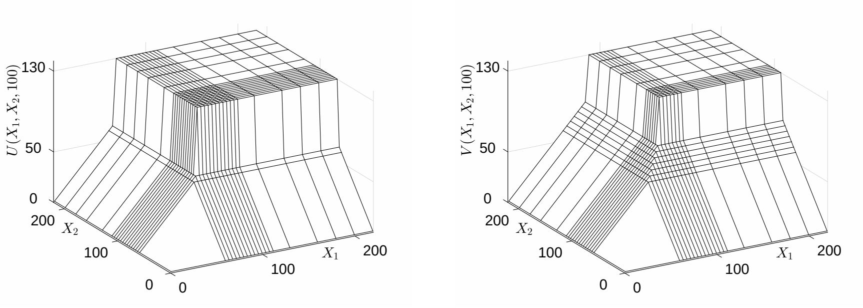

FIGURE 4. Initial payoff functions of U and V.

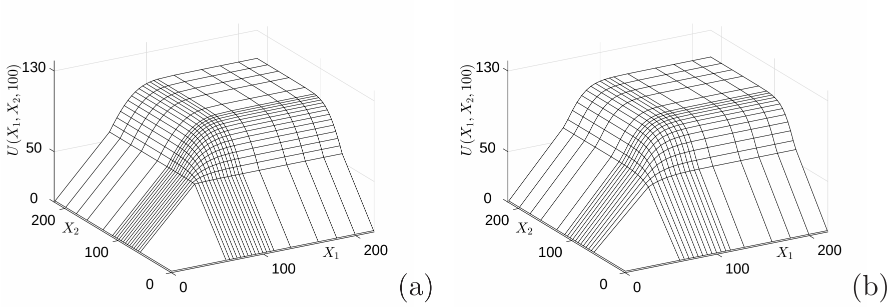

FIGURE 5. Final solution of U at $X_3 = 100$ and $\tau = 1$ using the (a) explicit and (b) implicit scheme.

reduce errors of simulation, we calculate the average of the 100 MCS cases. The option value obtained from MCS is 99.39883385 as a reference value, and the MCS takes 1425 seconds at a time. For FDM tests, a various of non-uniform mesh for each direction is used. We fix $\Delta\tau$ for 1/360 in the implicit scheme and adjust $\Delta\tau$ to prevent from spurious oscillation of explicit scheme satisfying the number of early redemptions. $[\cdots a : h : b \cdots]$ in Tables 2–6 means that computational mesh is $[\cdots a, a+h, a+2h, \cdots, b-h, b, \cdots]$. All tests were performed on Intel(R) Core(TM)2 Duo E8400 CPU@3.00GHZ with 3.46GB of RAM loaded MATLAB 2014a [6].

The option value of $U(100, 100, 100)$ and absolute relative percent error with the value of MCS and each FDM are shown in Tables 2–6, where h is spatial step size for non-uniform mesh. Table 2 shows that non-oscillatory explicit scheme is more superior than implicit scheme with OSM in terms of computational cost and the error with MCS in every spatial step. Note that the more space steps are taken, the faster computational time is taken in our scheme. On the other hand, Tables

3–6 present that explicit scheme has similar error with implicit scheme. Overall, the explicit scheme is better than the implicit if we use a small number of mesh points. However, if the spatial step size is small, then the implicit scheme is more efficient than the explicit scheme.

h	N_x	N_τ		$U(100,100,100)$		$\|Error\|$		CPU time (s)	
		Ex	Im	Ex	Im	Ex	Im	Ex	Im
1	53	2952	360	99.5127665	99.7972266	0.115	0.401	356.31	380.28
2	30	720	360	100.1143215	100.3768134	0.720	0.984	14.77	70.59
2.5	26	468	360	99.5889818	99.7535851	0.191	0.357	5.97	47.05
4	19	180	360	99.9335293	100.2201858	0.538	0.826	0.87	17.82
5	17	114	360	99.4939679	99.8484540	0.096	0.452	0.39	12.84

TABLE 2. Error and the value of $U(100,100,100)$ with non-uniform mesh [1 60:h:105 120 140 160 180 200 220] and various values of h.

h	N_x	N_τ		$U(100,100,100)$		$\|Error\|$		CPU time (s)	
		Ex	Im	Ex	Im	Ex	Im	Ex	Im
1	58	3240	360	99.3007165	99.5076075	0.099	0.109	521.20	497.00
2	33	804	360	99.8402938	99.9929402	0.444	0.598	21.81	92.21
2.5	28	516	360	99.3343135	99.4721610	0.065	0.074	8.38	57.03
4	20	192	360	99.7223972	99.9125133	0.326	0.517	1.09	20.25
5	18	126	360	99.3331324	99.5999701	0.066	0.202	0.51	15.07

TABLE 3. Error and the value of $U(100,100,100)$ with non-uniform mesh [1 60:h:110 120 140 160 180 200 220] and various values of h.

h	N_x	N_τ		$U(100,100,100)$		$\|Error\|$		CPU time (s)	
		Ex	Im	Ex	Im	Ex	Im	Ex	Im
1	62	3540	360	99.2736916	99.4535276	0.126	0.055	698.88	567.67
2	35	864	360	99.8282844	99.9718300	0.432	0.576	25.77	100.68
2.5	29	564	360	99.3139919	99.4392444	0.085	0.041	10.26	56.00
4	21	210	360	99.8289262	99.8798644	0.433	0.484	1.22	18.69
5	18	138	360	99.3367150	99.5773001	0.062	0.180	0.59	14.77

TABLE 4. Error and the value of $U(100,100,100)$ with non-uniform mesh [1 60:h:115 130 160 180 200 220] and various values of h.

Next, we similarly examine the option value $U(100,100,100)$ with respect to changing volatility and riskless interest rate. We use the solution of MCS with variance reduction. Tables 9 and 10 show the option price and error with changing the volatility and riskfree interest

h	N_x	N_τ		$U(100,100,100)$		$\lvert Error\rvert$		CPU time (s)	
		Ex	Im	Ex	Im	Ex	Im	Ex	Im
1	67	3858	360	99.2574268	99.4495997	0.142	0.051	968.88	772.22
2	37	960	360	99.8051615	99.9397719	0.409	0.544	37.46	128.58
2.5	31	612	360	99.3026870	99.4189247	0.097	0.020	13.68	73.94
4	22	240	360	99.6713333	99.8074135	0.274	0.411	1.87	28.56
5	19	150	360	99.3420879	99.5610479	0.057	0.163	0.80	18.100

TABLE 5. Error and the value of $U(100,100,100)$ with non-uniform mesh [1 60:h:120 140 160 180 200 220] and various values of h.

h	N_x	N_τ		$U(100,100,100)$		$\lvert Error\rvert$		CPU time (s)	
		Ex	Im	Ex	Im	Ex	Im	Ex	Im
1	76	4530	360	99.2486586	99.4362524	0.151	0.038	1680.97	1182.66
2	41	1128	360	99.7997143	99.9265827	0.403	0.531	62.67	180.23
2.5	34	720	360	99.3013973	99.4068576	0.098	0.008	21.54	119.73
4	23	270	360	99.6763125	99.7967114	0.279	0.400	2.37	32.32
5	20	180	360	99.3701114	99.5512304	0.029	0.153	1.03	21.45

TABLE 6. Error and the value of $U(100,100,100)$ with non-uniform mesh [1 60:h:130 160 180 200 220] and various values of h.

rate. Changes of volatility and interest rate require the different number of time steps on non-oscillatory explicit scheme. On the other hand, changes of interest rate have less of an effect on the number of time step than the changes of volatility. The reference values when $\sigma = 0.2$ and $\sigma = 0.4$ are 107.9563003 and 91.26505667, respectively. Also, 99.2936492 and 99.3253221 are reference values for $r = 0.01$ and $r = 0.05$.

h	N_x	N_τ		$U(100,100,100)$		$\lvert Error\rvert$		CPU time (s)	
		Ex	Im	Ex	Im	Ex	Im	Ex	Im
1	76	2016	360	107.5307753	107.5918473	0.394	0.338	743.42	1181.89
2	41	504	360	108.1769439	108.1757133	0.204	0.203	27.36	181.53
2.5	34	324	360	106.9965461	106.9837869	0.889	0.901	9.73	120.25
4	23	120	360	107.6745746	107.5635125	0.261	0.364	1.08	32.54
5	20	84	360	106.1369382	106.0404300	1.685	1.775	0.50	21.48

TABLE 7. Error and the value of $U(100,100,100)$ with non-uniform mesh [1 60:h:130 160 180 200 220] and various values of h ($\sigma_1 = \sigma_2 = \sigma_3 = 0.2$).

h	N_x	N_τ		$U(100,100,100)$		$\|Error\|$		CPU time (s)	
		Ex	Im	Ex	Im	Ex	Im	Ex	Im
1	76	8052	360	91.2959294	91.3952965	0.034	0.143	2967.50	1182.43
2	41	1998	360	91.4154188	91.4821136	0.165	0.238	108.08	181.15
2.5	34	1278	360	91.6447605	91.7211964	0.416	0.500	38.30	120.23
4	23	480	360	91.4875394	91.5294626	0.244	0.290	4.34	32.37
5	20	318	360	92.1941121	92.3904074	1.018	1.233	1.81	21.49

TABLE 8. Error and the value of $U(100,100,100)$ with non-uniform mesh [1 60:h:130 160 180 200 220] and various values of h ($\sigma_1 = \sigma_2 = \sigma_3 = 0.4$).

h	N_x	N_τ		$U(100,100,100)$		$\|Error\|$		CPU time (s)	
		Ex	Im	Ex	Im	Ex	Im	Ex	Im
1	76	4530	360	99.2356163	99.4211794	0.058	0.128	1693.28	1185.89
2	41	1128	360	99.8251264	99.9435156	0.535	0.654	61.34	182.53
2.5	34	720	360	99.2864845	99.3809322	0.007	0.088	21.92	123.13
4	23	270	360	99.7009799	99.8016579	0.410	0.512	2.37	32.35
5	20	180	360	99.3540393	99.5158842	0.061	0.224	1.03	22.50

TABLE 9. Error and the value of $U(100,100,100)$ with non-uniform mesh [1 60:h:130 160 180 200 220] and various values of h ($r = 0.01$).

h	N_x	N_τ		$U(100,100,100)$		$\|Error\|$		CPU time (s)	
		Ex	Im	Ex	Im	Ex	Im	Ex	Im
1	76	4530	360	99.2343631	99.4185326	0.092	0.094	2967.50	1180.53
2	41	1128	360	99.7458733	99.8758164	0.423	0.554	108.08	181.59
2.5	34	720	360	99.2900989	99.4013105	0.035	0.077	38.30	123.88
4	23	270	360	99.6248616	99.7595838	0.302	0.437	4.34	32.48
5	20	180	360	99.3615993	99.5569768	0.037	0.233	1.81	22.58

TABLE 10. Error and the value of $U(100,100,100)$ with non-uniform mesh [1 60:h:130 160 180 200 220] and various values of h ($r = 0.05$).

6. Conclusions

In general, the implicit scheme is used with Thomas algorithm. However, there are several drawbacks for practical calculation. First, the implicit scheme has more time complexity in terms of computations for solving option values. Next, it is hard to use the methodology for multi-dimensional problem since it is necessary to apply the OSM or ADI. Hence we compared the explicit scheme satisfying non-oscillatory condition and the implicit method for multi-dimensional financial options.

Throughout the paper, the non-oscillatory explicit scheme is simple to implement and superior if we use a small number of mesh points. On the other hand, the implicit scheme is more efficient than the explicit scheme if the spatial step size is small.

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