OVA cheat sheet

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目录

1 重要参考

- 曲线坐标系, 胡友秋, 2010. 中国科学技术大学地球和空间科学学院
- Flux coordinates and magnetic field structure: a guide to a fundamental tool of plasma theory, D'haeseleer, William D and Hitchon, William NG and Callen, James D and Shohet, J Leon, 2012, Springer Science & Business Media
- http://euclid.mas.ucy.ac.cy/ georgios/courses/eljadidaU/Chapter1.pdf
- Theory Of Toroidally Confined Plasmas, White, 2013, World Scientific Publishing Company

2 cheat sheet

2.1 基本概念

1. 引入曲线坐标系的目的
    - 边界和坐标曲面重合, 便于处理边界条件。
    - 引入对称性, 降维。
    - 便于处理介质的各向异性问题。
2. 爱因斯坦求和约定 (Einstein summation convention)
    - **求和指标总是一上一下** (一逆一协)
    - 同指标求和
3. 度规张量与空间性质
    - 度规系数完全决定空间的度量性质和几何结构
    - 度规张量只要是非退化的 ($g \neq 0$) 就是合法的
    - 空间包含关系:
        - 一般空间
            * 黎曼空间 (度规张量是对称的)
                · 欧几里得空间, i.e. 平直空间, 即黎曼曲率张量 $R^{\nu}_{\rho\sigma\mu}$ 等于 0
                · 弯曲空间

2.2 通用公式

2.2.1 基矢

$$
\begin{aligned}
&\vec{v} = v^i\vec{e}_i = v_i\vec{e}^i \\
&v^i = \vec{v}\cdot\vec{e}^i = g^{ik}v_k \quad v_i = \vec{v}\cdot\vec{e}_i = g_{ik}v^k \\
&\vec{e}_i = g_{ik}\vec{e}^k \quad \vec{e}^k = g^{ik}\vec{e}_i \\
&\vec{e}_i = \frac{\partial}{\partial\xi^i} = \frac{\partial}{\partial x}\frac{\partial x}{\partial\xi^i} + \frac{\partial}{\partial y}\frac{\partial y}{\partial\xi^i} + \frac{\partial}{\partial z}\frac{\partial z}{\partial\xi^i} = \hat{x}_a\frac{\partial x^a}{\partial\xi^i} = \frac{\partial\vec{R}}{\partial\xi^i} \\
&\vec{e}^i = \nabla\xi^i = \frac{\partial\xi^i}{\partial x}\vec{e}_x + \frac{\partial\xi^i}{\partial y}\vec{e}_y + \frac{\partial\xi^i}{\partial z}\vec{e}_z \\
&\vec{e}^1 = \frac{1}{V}(\vec{e}_2\times\vec{e}_3), \quad \vec{e}^2 = \frac{1}{V}(\vec{e}_3\times\vec{e}_1), \quad \vec{e}^3 = \frac{1}{V}(\vec{e}_1\times\vec{e}_2) \\
&\vec{e}_1 = V(\vec{e}^2\times\vec{e}^3), \quad \vec{e}_2 = V(\vec{e}^3\times\vec{e}^1), \quad \vec{e}_3 = V(\vec{e}^1\times\vec{e}^2)
\end{aligned}
\tag{1}
$$

2.2.2 位移矢量

位移矢量 $d\vec{r}$ 是位置矢量 $\vec{r}$ 的微分

$$
\begin{aligned}
d\vec{r} &= \frac{\partial\vec{r}}{\partial x^i}dx^i = dx^i\vec{e}_i \\
d\vec{r} &= dx_k\vec{e}^k = g_{ik}dx^i\vec{e}^k \\
dx_k &= g_{ik}dx^i \quad dx^i = g^{ik}dx_k
\end{aligned}
\tag{2}
$$

2.2.3 度规

$$
\begin{aligned}
&\overleftrightarrow{G} = g_{ik}\vec{e}^i\vec{e}^k = g^{ik}\vec{e}_i\vec{e}_k \\
&g_{ik} = \vec{e}_i\cdot\vec{e}_k = g_{ki} \\
&g^{ik} = \vec{e}^i\cdot\vec{e}^k = g^{ki} \\
&g_{ik} = \frac{\partial x}{\partial x^i}\frac{\partial x}{\partial x^k} + \frac{\partial y}{\partial x^i}\frac{\partial y}{\partial x^k} + \frac{\partial z}{\partial x^i}\frac{\partial z}{\partial x^k} \\
&g^{ik} = \frac{\partial x^i}{\partial x}\frac{\partial x^k}{\partial x} + \frac{\partial x^i}{\partial y}\frac{\partial x^k}{\partial y} + \frac{\partial x^i}{\partial z}\frac{\partial x^k}{\partial z}
\end{aligned}
\tag{3}
$$

2.2.4 Jacobian

$$
\begin{aligned}
\mathcal{J} &= \frac{\partial(x,y,z)}{\partial(x^1,x^2,x^3)} = \begin{vmatrix} \frac{\partial x}{\partial x^1} & \frac{\partial x}{\partial x^2} & \frac{\partial x}{\partial x^3} \\ \frac{\partial y}{\partial x^1} & \frac{\partial y}{\partial x^2} & \frac{\partial y}{\partial x^3} \\ \frac{\partial z}{\partial x^1} & \frac{\partial z}{\partial x^2} & \frac{\partial z}{\partial x^3} \end{vmatrix} \\
\mathcal{J}^{-1} &= \frac{\partial(x^1,x^2,x^3)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial x^1}{\partial x} & \frac{\partial x^1}{\partial y} & \frac{\partial x^1}{\partial z} \\ \frac{\partial x^2}{\partial x} & \frac{\partial x^2}{\partial y} & \frac{\partial x^2}{\partial z} \\ \frac{\partial x^3}{\partial x} & \frac{\partial x^3}{\partial y} & \frac{\partial x^3}{\partial z} \end{vmatrix} \\
\mathcal{J} &= V = \sqrt{\det|g_{ij}|} = \sqrt{g} = \vec{e}_1 \cdot \vec{e}_2 \times \vec{e}_3 \\
\mathcal{J}^{-1} &= \frac{1}{V} = \frac{1}{\sqrt{\det|g_{ij}|}} = \frac{1}{\sqrt{g}} = \vec{e}^1 \cdot \vec{e}^2 \times \vec{e}^3 \\
g &= \det|g_{ij}| = \mathcal{J}^2 = V^2 = \frac{g_{22}g_{33}-g_{23}^2}{g^{11}} = \frac{g_{11}g_{33}-g_{13}^2}{g^{22}} = \frac{g_{11}g_{22}-g_{12}^2}{g^{33}}
\end{aligned}
\tag{4}
$$

2.2.5 基矢的微分运算

$$
\begin{aligned}
\nabla \cdot \vec{e}_i &= \frac{1}{V}\frac{\partial V}{\partial x^i} = \frac{1}{2g}\frac{\partial g}{\partial x^i}, \quad (i=1,2,3) \\
\nabla \times \vec{e}^i &= 0 \\
(\vec{e}_\sigma \cdot \nabla)\,\vec{e}_\rho &= \frac{1}{2}g^{\mu\lambda}\left(\frac{\partial g_{\rho\lambda}}{\partial x^\sigma} + \frac{\partial g_{\sigma\lambda}}{\partial x^\rho} - \frac{\partial g_{\rho\sigma}}{\partial x^\lambda}\right)\vec{e}_\mu = \Gamma^\mu_{\rho\sigma}\vec{e}_\mu, \quad (\rho,\sigma=1,2,3) \\
\Gamma^\mu_{\rho\sigma} &= \frac{1}{2}g^{\mu\lambda}\left(\frac{\partial g_{\rho\lambda}}{\partial x^\sigma} + \frac{\partial g_{\sigma\lambda}}{\partial x^\rho} - \frac{\partial g_{\rho\sigma}}{\partial x^\lambda}\right), \quad (\rho,\sigma=1,2,3)
\end{aligned}
\tag{5}
$$

2.2.6 Dot, cross

$$
\begin{aligned}
\vec{a} \cdot \vec{b} &= a^i b_i = a_i b^i = g_{ij}a^i b^j = g^{ij}a_i b_j \\
\vec{a} \times \vec{b} &= \epsilon^{ijk} V a^i b^j \vec{e}^k = \epsilon_{ijk}\frac{1}{V}a_i b_j \vec{e}_k
\end{aligned}
\tag{6}
$$

2.2.7 梯度

$$
\begin{aligned}
\nabla \phi &= \frac{\partial \phi}{\partial x^i} \vec{e^i} \\
\nabla \vec{f} &= f_{\rho;\sigma} \vec{e^\sigma} \vec{e^\rho} \\
f_{\rho;\sigma} &= \vec{e}_\rho \vec{e}_\sigma : \nabla \vec{f} = \frac{\partial f_\rho}{\partial x^\sigma} - \Gamma^\mu_{\rho\sigma} f_\mu \\
\nabla\nabla \vec{f} &= f_{\rho;\sigma;\mu} \vec{e^\mu} \vec{e^\sigma} \vec{e^\rho} \\
f_{\rho;\sigma;\mu} &= \frac{\partial f_{\rho;\sigma}}{\partial x^\mu} - \Gamma^\lambda_{\rho\mu} f_{\lambda;\sigma} - \Gamma^\lambda_{\sigma\mu} f_{\rho;\lambda}
\end{aligned} \tag{7}
$$

2.2.8 散度

$$
\nabla \cdot \vec{f} = \nabla \cdot \left(f^i \vec{e_i} \right) = \vec{e}_i \cdot \nabla f^i + f^i \nabla \cdot \vec{e}_i = \frac{\partial f^i}{\partial x^i} + \frac{f^i}{2g} \frac{\partial g}{\partial x^i} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(\sqrt{g} f^i \right) \tag{8}
$$

2.2.9 旋度

$$
\begin{aligned}
\nabla \times \vec{f} &= \frac{1}{\sqrt{g}} \frac{\partial f_i}{\partial x^j} \varepsilon_{kji} \vec{e}_k \\
\nabla \times \vec{f} &= \frac{1}{\sqrt{g}} \left\{ \left(\frac{\partial f_3}{\partial x^2} - \frac{\partial f_2}{\partial x^3} \right) \vec{e}_1 + \left(\frac{\partial f_1}{\partial x^3} - \frac{\partial f_3}{\partial x^1} \right) \vec{e}_2 + \left(\frac{\partial f_2}{\partial x^1} - \frac{\partial f_1}{\partial x^2} \right) \vec{e}_3 \right\}
\end{aligned} \tag{9}
$$

2.3 正交曲线坐标系

正交曲线坐标系的性质

- 三个基矢彼此正交
- 度规张量为对角矩阵 (主对角线外的元素为 0), $g_{ik} = 0 \quad (i \neq k)$

2.3.1 矢量表示

$$
\begin{aligned}
\vec{f} &= F_i \hat{e}_i, \quad F_i = \hat{e}_i \cdot \vec{f} = f_i / h_i = h_i f^i, \quad (i = 1, 2, 3) \\
\overleftrightarrow{K} &= K_{ij} \hat{e}_i \hat{e}_j, \quad K_{ij} = k_{ij} / \left(h_i h_j \right) = h_i h_j k^{ij}, \quad (i, j = 1, 2, 3)
\end{aligned} \tag{10}
$$

2.3.2 Lamé 系数

$$
\begin{aligned}
& h_1=\sqrt{g_{11}}=\frac{1}{\sqrt{g^{11}}}, \quad h_2=\sqrt{g_{22}}=\frac{1}{\sqrt{g^{22}}}, \quad h_3=\sqrt{g_{33}}=\frac{1}{\sqrt{g^{33}}} \\
& h_i=|\vec{e}_i|=\left[\left(\frac{\partial x}{\partial x^i}\right)^2+\left(\frac{\partial y}{\partial x^i}\right)^2+\left(\frac{\partial z}{\partial x^i}\right)^2\right]^{1/2} \\
& \quad=\left[\left(\frac{\partial x^i}{\partial x}\right)^2+\left(\frac{\partial x^i}{\partial y}\right)^2+\left(\frac{\partial x^i}{\partial z}\right)^2\right]^{-1/2}, \quad (i=1,2,3) \\
& V=\sqrt{g}=h_1h_2h_3 \\
& \hat{e}_i=\frac{\vec{e}_i}{h_i}=h_i\vec{e}^i \\
& F_i=F^i=\frac{f_i}{h_i}=f^ih_i
\end{aligned}
\tag{11}
$$

2.3.3 弧元、面元和体元

$$
\begin{aligned}
(ds)^2 &= \left(h_1dx^1\right)^2+\left(h_2dx^2\right)^2+\left(h_3dx^3\right)^2 \\
ds_1 &= h_1dx^1\hat{e}_1, \quad ds_2=h_2dx^2\hat{e}_2, \quad ds_3=h_3dx^3\hat{e}_3 \\
da_1 &= h_2h_3dx^2dx^3\hat{e}_1 \\
da_2 &= h_1h_3dx^1dx^3\hat{e}_2 \\
da_3 &= h_1h_2dx^1dx^2\hat{e}_3 \\
d\tau &= V dx^1dx^2dx^3=h_1h_2h_3dx^1dx^2dx^3
\end{aligned}
\tag{12}
$$

2.3.4 基矢的微分运算

$$
\begin{aligned}
\nabla\cdot\hat{e}_i &= \frac{1}{Vh_i}\frac{\partial V}{\partial x_i}-\frac{1}{h_i^2}\frac{\partial h_i}{\partial x_i}=\frac{1}{V}\frac{\partial}{\partial x_i}\left(\frac{V}{h_i}\right), \quad (i=1,2,3) \\
\nabla\times\hat{e}_i &= -h_i\nabla\left(\frac{1}{h_i}\right)\times\hat{e}_i=\frac{1}{h_i}\sum_{j=1}^{3}\left(\frac{1}{h_j}\frac{\partial h_i}{\partial x_j}\hat{e}_j\right)\times\hat{e}_i \\
\left(\hat{e}_k\cdot\nabla\right)\hat{e}_i &= -\frac{1}{h_k}\left(\nabla h_i\right)\delta_k^i+\frac{1}{h_ih_k}\frac{\partial h_k}{\partial x_i}\hat{e}_k, \quad (i,k=1,2,3) \\
\frac{\partial\hat{e}_i}{\partial x_k} &= -\left(\nabla h_i\right)\delta_k^i+\frac{1}{h_i}\frac{\partial h_k}{\partial x_i}\hat{e}_k \quad (i,k=1,2,3)
\end{aligned}
\tag{13}
$$

$$
\begin{cases}
\left(\hat{e}_1\cdot\nabla\right)\hat{e}_1=-\frac{1}{h_1}\nabla h_1+\frac{1}{h_1^2}\frac{\partial h_1}{\partial x_1}\hat{e}_1=-\frac{1}{h_1h_2}\frac{\partial h_1}{\partial x_2}\hat{e}_2-\frac{1}{h_1h_3}\frac{\partial h_1}{\partial x_3}\hat{e}_3, \\
\left(\hat{e}_2\cdot\nabla\right)\hat{e}_2=-\frac{1}{h_2}\nabla h_2+\frac{1}{h_2^2}\frac{\partial h_2}{\partial x_2}\hat{e}_2=-\frac{1}{h_1h_2}\frac{\partial h_2}{\partial x_1}\hat{e}_1-\frac{1}{h_2h_3}\frac{\partial h_2}{\partial x_3}\hat{e}_3, \\
\left(\hat{e}_3\cdot\nabla\right)\hat{e}_3=-\frac{1}{h_3}\nabla h_3+\frac{1}{h_3^2}\frac{\partial h_3}{\partial x_3}\hat{e}_3=-\frac{1}{h_1h_3}\frac{\partial h_3}{\partial x_1}\hat{e}_1-\frac{1}{h_2h_3}\frac{\partial h_3}{\partial x_2}\hat{e}_2.
\end{cases}
\tag{14}
$$

$$
\begin{cases}
\frac{\partial \hat{e}_1}{\partial x_1} = -\nabla h_1 + \frac{1}{h_1}\frac{\partial h_1}{\partial x_1}\hat{e}_1 = -\frac{1}{h_2}\frac{\partial h_1}{\partial x_2}\hat{e}_2 - \frac{1}{h_3}\frac{\partial h_1}{\partial x_3}\hat{e}_3 \\
\frac{\partial \hat{e}_2}{\partial x_2} = -\nabla h_2 + \frac{1}{h_2}\frac{\partial h_2}{\partial x_2}\hat{e}_2 = -\frac{1}{h_1}\frac{\partial h_2}{\partial x_1}\hat{e}_1 - \frac{1}{h_3}\frac{\partial h_2}{\partial x_3}\hat{e}_3 \\
\frac{\partial \hat{e}_3}{\partial x_3} = -\nabla h_3 + \frac{1}{h_3}\frac{\partial h_3}{\partial x_3}\hat{e}_3 = -\frac{1}{h_1}\frac{\partial h_3}{\partial x_1}\hat{e}_1 - \frac{1}{h_2}\frac{\partial h_3}{\partial x_2}\hat{e}_2
\end{cases}
\tag{15}
$$

圆柱坐标系和球坐标系的坐标、拉梅系数及其偏导数、基矢及其偏导数 [1]

圆柱坐标系				球坐标系			
曲线坐标	r	θ	z	曲线坐标	r	θ	φ
拉梅系数	1	r	1	拉梅系数	1	r	$r\sin\theta$
$\partial/\partial r$	0	1	0	$\partial/\partial r$	0	1	$\sin\theta$
$\partial/\partial\theta$	0	0	0	$\partial/\partial\theta$	0	0	$r\cos\theta$
$\partial/\partial z$	0	0	0	$\partial/\partial\varphi$	0	0	0
单位基矢	$\hat{e}_r$	$\hat{e}_\theta$	$\hat{e}_z$	单位基矢	$\hat{e}_r$	$\hat{e}_\theta$	$\hat{e}_\varphi$
$\partial/\partial r$	0	0	0	$\partial/\partial r$	0	0	0
$\partial/\partial\theta$	$\hat{e}_\theta$	$-\hat{e}_r$	0	$\partial/\partial\theta$	$\hat{e}_\theta$	$-\hat{e}_r$	0
$\partial/\partial z$	0	0	0	$\partial/\partial\varphi$	$\sin\theta\hat{e}_\varphi$	$\cos\theta\hat{e}_\varphi$	$-\sin\theta\hat{e}_r - \cos\theta\hat{e}_\theta$

2.3.5 Dot, cross

$$
\begin{aligned}
\vec{a}\cdot\vec{b} &= a^i b_i = \frac{A_i}{h_i}B_i h_i = A_i B_i \\
\vec{a}\cdot\vec{\vec{T}} &= A_k\hat{e}_k \cdot T_{ij}\hat{e}_i\hat{e}_j = A_i T_{ij}\hat{e}_j \\
\vec{\vec{T}}\cdot\vec{a} &= T_{ij}\hat{e}_i\hat{e}_j \cdot a_k\hat{e}_k = A_j T_{ij}\hat{e}_i \\
\vec{a}\times\vec{b} &= A_i B_j \hat{e}_i \times \hat{e}_j = \epsilon_{ijk}A_i B_j \hat{e}_k
\end{aligned}
\tag{16}
$$

2.3.6 梯度

$$
\nabla\phi = \frac{\partial\phi}{\partial x^i}\vec{e^i} = \frac{1}{h_i}\frac{\partial\phi}{\partial x_i}\hat{e}_i \tag{17}
$$

$$
\hat{e}_i\cdot\nabla\phi = \frac{1}{h_i}\frac{\partial\phi}{\partial x_i} \tag{18}
$$

$$
\begin{aligned}
\nabla\vec{u} = \hat{e}_i\hat{e}_j\frac{1}{h_i}\frac{\partial U_j}{\partial x^i} + \hat{e}_i\frac{\partial\hat{e}_j}{\partial x^i}\frac{U_j}{h_i} &= \hat{e}_i\hat{e}_j\frac{1}{h_i}\frac{\partial U_j}{\partial x^i} + \hat{e}_i\frac{U_j}{h_i}\left(-(\nabla h_j)\delta_i^j + \frac{1}{h_j}\frac{\partial h_i}{\partial x_j}\hat{e}_i\right) \\
&= \hat{e}_i\hat{e}_j\frac{1}{h_i}\frac{\partial U_j}{\partial x^i} - \hat{e}_i\hat{e}_k\frac{U_i}{h_i}\frac{1}{h_k}\frac{\partial h_i}{\partial x_k} + \hat{e}_i\hat{e}_i\frac{U_j}{h_i}\frac{1}{h_j}\frac{\partial h_i}{\partial x_j}
\end{aligned}
\tag{19}
$$

2.3.7 散度

$$\nabla \cdot \vec{f} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(\sqrt{g} f^i \right) = \frac{1}{V} \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \left(\frac{V F_i}{h_i} \right) \tag{20}$$

$$\nabla^2 \phi = \frac{1}{V} \frac{\partial}{\partial x_i} \left(\frac{V}{h_i^2} \frac{\partial \phi}{\partial x_i} \right) \tag{21}$$

$$\begin{aligned} \nabla \cdot \vec{\vec{T}} &= \frac{1}{h_i} \frac{\partial T_{ik}}{\partial x_i} \hat{e}_k + \frac{T_{ik}}{V} \frac{\partial (\frac{V}{h_i})}{\partial x_i} \hat{e}_k + \frac{T_{ik}}{h_i} \frac{\partial \hat{e}_k}{\partial x_i} \\ &= \frac{1}{h_i} \frac{\partial T_{ik}}{\partial x_i} \hat{e}_k + \frac{T_{ik}}{V} \frac{\partial (\frac{V}{h_i})}{\partial x_i} \hat{e}_k + \frac{T_{ik}}{h_i} \left[-(\nabla h_k) \delta_i^k + \frac{1}{h_k} \frac{\partial h_i}{\partial x_k} \hat{e}_i \right] \\ &= \frac{1}{h_i} \frac{\partial T_{ik}}{\partial x_i} \hat{e}_k + \frac{T_{ik}}{V} \frac{\partial (\frac{V}{h_i})}{\partial x_i} \hat{e}_k + \frac{T_{ik}}{h_i} \frac{1}{h_k} \frac{\partial h_i}{\partial x_k} \hat{e}_i - \frac{T_{ii}}{h_i} \frac{1}{h_j} \frac{\partial h_i}{\partial x_j} \hat{e}_j \end{aligned} \tag{22}$$

2.3.8 旋度

$$\nabla \times \vec{f} = \frac{1}{\sqrt{g}} \frac{\partial f_i}{\partial x_j} \varepsilon_{kji} \vec{e}_k = \sum_{i,j,k=1}^{3} \frac{h_k}{V} \frac{\partial \left(h_i F_i \right)}{\partial x_j} \varepsilon_{kji} \hat{e}_k \tag{23}$$

$$\begin{aligned} \nabla \times \vec{f} = & \frac{1}{h_2 h_3} \left[\frac{\partial \left(h_3 F_3 \right)}{\partial x_2} - \frac{\partial \left(h_2 F_2 \right)}{\partial x_3} \right] \hat{e}_1 + \\ & \frac{1}{h_1 h_3} \left[\frac{\partial \left(h_1 F_1 \right)}{\partial x_3} - \frac{\partial \left(h_3 F_3 \right)}{\partial x_1} \right] \hat{e}_2 + \\ & \frac{1}{h_1 h_2} \left[\frac{\partial \left(h_2 F_2 \right)}{\partial x_1} - \frac{\partial \left(h_1 F_1 \right)}{\partial x_2} \right] \hat{e}_3 \end{aligned} \tag{24}$$

2.3.9 Laplace

$$\nabla^2 \phi = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j} \left(g^{ij} \sqrt{g} \frac{\partial \phi}{\partial x^i} \right) = \frac{1}{V} \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \left(\frac{V}{h_i^2} \frac{\partial \phi}{\partial x_i} \right) \tag{25}$$

$$\nabla^2 \phi = \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial x_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial \phi}{\partial x_1} \right) + \frac{\partial}{\partial x_2} \left(\frac{h_1 h_3}{h_2} \frac{\partial \phi}{\partial x_2} \right) + \frac{\partial}{\partial x_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial \phi}{\partial x_3} \right) \right\} \tag{26}$$

2.4 柱坐标系 (r, θ, z)

$$h_1 = 1 \quad h_2 = r \quad h_3 = 1 \tag{27}$$

$$\mathcal{J} = V = \sqrt{g} = h_1 h_2 h_3 = r \tag{28}$$

$$\begin{aligned}
\vec{a} \cdot \vec{b} &= a^i b_i = \frac{A_i}{h_i} B_i h_i = A_i B_i \\
\vec{a} \cdot \vec{\vec{T}} &= A_k \hat{e}_k \cdot T_{ij} \hat{e}_i \hat{e}_j = A_i T_{ij} \hat{e}_j \\
\vec{\vec{T}} \cdot \vec{a} &= T_{ij} \hat{e}_i \hat{e}_j \cdot a_k \hat{e}_k = A_j T_{ij} \hat{e}_i \\
\vec{a} \times \vec{b} &= A_i B_j \hat{e}_i \times \hat{e}_j = \epsilon_{ijk} A_i B_j \hat{e}_k
\end{aligned} \tag{29}$$

$$\vec{a} \times \vec{b} = \hat{e}_r (A_\theta B_z - A_z B_\theta) + \hat{e}_\theta (A_z B_r - A_r B_z) + \hat{e}_z (A_r B_\theta - A_\theta B_r) \tag{30}$$

$$\begin{aligned}
\nabla \cdot \hat{e}_r &= \frac{1}{r} \\
\nabla \cdot \hat{e}_\theta &= 0 \\
\nabla \cdot \hat{e}_z &= 0
\end{aligned} \tag{31}$$

$$\begin{aligned}
\nabla \times \hat{e}_r &= 0 \\
\nabla \times \hat{e}_\theta &= \frac{1}{r} \\
\nabla \times \hat{e}_z &= 0
\end{aligned} \tag{32}$$

$$\begin{cases}
(\hat{e}_r \cdot \nabla)\, \hat{e}_r = 0 \\
(\hat{e}_\theta \cdot \nabla)\, \hat{e}_\theta = -\frac{1}{r} \hat{e}_r \\
(\hat{e}_z \cdot \nabla)\, \hat{e}_z = 0
\end{cases} \tag{33}$$

$$\begin{cases}
\frac{\partial \hat{e}_r}{\partial r} = 0 \\
\frac{\partial \hat{e}_\theta}{\partial \theta} = -\hat{e}_r \\
\frac{\partial \hat{e}_z}{\partial z} = 0
\end{cases} \tag{34}$$

$$\begin{aligned}
\nabla &= \vec{e^i} \frac{\partial}{\partial x^i} = \hat{e}_i \frac{1}{h_i} \frac{\partial}{\partial x^i} = \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{e}_z \frac{\partial}{\partial z} \\
\nabla^2 &= \Delta = \frac{1}{V} \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \left(\frac{V}{h_i^2} \frac{\partial}{\partial x_i} \right) = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \\
\vec{u} \cdot \nabla &= U_i \hat{e}_i \cdot \hat{e}_j \frac{1}{h_j} \frac{\partial}{\partial x^j} = U_i \frac{1}{h_i} \frac{\partial}{\partial x^i} = U_r \frac{\partial}{\partial r} + \frac{U_\theta}{r} \frac{\partial}{\partial \theta} + U_z \frac{\partial}{\partial z}
\end{aligned} \tag{35}$$

$$\nabla\phi=\hat{e}_r\frac{\partial\phi}{\partial r}+\hat{e}_\theta\frac{1}{r}\frac{\partial\phi}{\partial\theta}+\hat{e}_z\frac{\partial\phi}{\partial z}$$

$$\nabla\cdot\vec{U}=\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\sqrt{g}U^i\right)=\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\sqrt{g}\frac{U^i}{h_i}\right)=\frac{1}{r}\frac{\partial}{\partial r}(rU_r)+\frac{1}{r}\frac{\partial U_\theta}{\partial\theta}+\frac{\partial U_z}{\partial z}$$

$$\begin{aligned}\nabla\times\vec{u}&=\frac{1}{h_i}\sum_{j=1}^{3}\left(\frac{1}{h_j}\frac{\partial h_i}{\partial x_j}\hat{e}_j\right)\times\hat{e}_i\\&=\left(\frac{1}{r}\frac{\partial U_z}{\partial\theta}-\frac{\partial U_\theta}{\partial z}\right)\hat{e}_r+\left(\frac{\partial U_r}{\partial z}-\frac{\partial U_z}{\partial r}\right)\hat{e}_\theta+\left[\frac{1}{r}\frac{\partial}{\partial r}(rU_\theta)-\frac{1}{r}\frac{\partial U_r}{\partial\theta}\right]\hat{e}_z\end{aligned}$$

$$\begin{aligned}\nabla\vec{u}=\hat{e}_i\hat{e}_j\frac{1}{h_i}\frac{\partial U_j}{\partial x^i}+\hat{e}_i\frac{\partial\hat{e}_j}{\partial x^i}\frac{U_j}{h_i}&=\frac{\partial U_r}{\partial r}\hat{e}_r\hat{e}_r+\frac{\partial U_\theta}{\partial r}\hat{e}_r\hat{e}_\theta+\frac{\partial U_z}{\partial r}\hat{e}_r\hat{e}_z\\&+\left(\frac{1}{r}\frac{\partial U_r}{\partial\theta}-\frac{U_\theta}{r}\right)\hat{e}_\theta\hat{e}_r+\left(\frac{1}{r}\frac{\partial U_\theta}{\partial\theta}+\frac{U_r}{r}\right)\hat{e}_\theta\hat{e}_\theta+\frac{1}{r}\frac{\partial U_z}{\partial\theta}\hat{e}_\theta\hat{e}_z\\&+\frac{\partial U_r}{\partial z}\hat{e}_z\hat{e}_r+\frac{\partial U_\theta}{\partial z}\hat{e}_z\hat{e}_\theta+\frac{\partial U_z}{\partial z}\hat{e}_z\hat{e}_z\end{aligned}\tag{36}$$

$$\begin{aligned}\vec{u}\cdot\nabla\vec{u}&=\left[U_r\frac{\partial U_r}{\partial r}+U_\theta\left(\frac{1}{r}\frac{\partial U_r}{\partial\theta}-\frac{U_\theta}{r}\right)+U_z\frac{\partial U_r}{\partial z}\right]\hat{e}_r\\&+\left[U_r\frac{\partial U_\theta}{\partial r}+U_\theta\left(\frac{1}{r}\frac{\partial U_\theta}{\partial\theta}+\frac{U_r}{r}\right)+U_z\frac{\partial U_\theta}{\partial z}\right]\hat{e}_\theta\\&+\left[U_r\frac{\partial U_z}{\partial r}+U_\theta\frac{1}{r}\frac{\partial U_z}{\partial\theta}+U_z\frac{\partial U_z}{\partial z}\right]\hat{e}_z\end{aligned}$$

$$\begin{aligned}\nabla\cdot\vec{\vec{T}}=&\left[\frac{1}{r}\frac{\partial}{\partial r}\left(rT_{rr}\right)+\frac{1}{r}\frac{\partial T_{\theta r}}{\partial\theta}+\frac{\partial T_{zr}}{\partial z}-\frac{T_{\theta\theta}}{r}\right]\hat{e}_r\\&+\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2T_{r\theta}\right)+\frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial\theta}+\frac{\partial T_{z\theta}}{\partial z}-\frac{T_{r\theta}-T_{\theta r}}{r}\right]\hat{e}_\theta\\&+\left[\frac{1}{r}\frac{\partial}{\partial r}\left(rT_{rz}\right)+\frac{1}{r}\frac{\partial T_{\theta z}}{\partial\theta}+\frac{\partial T_{zz}}{\partial z}\right]\hat{e}_z\end{aligned}$$

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