Advanced Industrial Organization 2 – Problem Set 1

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Problem 1.

1. Note that since $\{X_i\}_{i=1}^N$ are independent, we have that

$$\begin{aligned}\mathbb{P}\left(Y \leq x\right) &= \mathbb{P}(X_i \leq x \forall i)\\ &= \prod_{i=1}^N \mathbb{P}\left(X_i \leq x\right)\\ &= \prod_{i=1}^N \exp\left(-\exp\left(-\frac{x-\mu_i}{\sigma}\right)\right)\\ &= \exp\left(-\sum_{i=1}^N \exp\left(-\frac{x-\mu_i}{\sigma}\right)\right)\\ &= \exp\left(-\exp\left(-\frac{x}{\sigma}\right)\sum_{i=1}^N \exp\left(\frac{\mu_i}{\sigma}\right)\right).\end{aligned}$$

Let $\mu = \log\left(\sum_{i=1}^N \exp\left(\frac{\mu_i}{\sigma}\right)\right)$ then we have that

$$\mathbb{P}(Y \leq x) = \exp\left(-\exp\left(-\frac{x-\sigma\mu}{\sigma}\right)\right)$$

which is T1EV$(\sigma\mu, \sigma)$

2. We know that the characteristic function of X is $\phi_X(t) = \Gamma(1 - i\sigma t)e^{i\mu_x t}$ and that the characteristic function of Y is $\phi_Y(t) = \Gamma(1 - i\sigma t)e^{i\mu_y t}$. We then get that the characteristic function for $X - Y$ (utilizing properties of the gamma function) is

$$\begin{aligned}\mathbb{E}[e^{it(X-Y)}] &= \mathbb{E}[e^{itX}]\,\mathbb{E}[e^{-itY}]\\ &= \phi_X(t)\phi_Y(-t)\\ &= \Gamma(1-i\sigma t)e^{i\mu_x t}\Gamma(1+i\sigma t)e^{i\mu_y t}\\ &= e^{(\mu_x-\mu_y)it}\Gamma(1-i\sigma t)\Gamma(1+i\sigma t)\\ &= e^{(\mu_x-\mu_y)it}\Gamma(1-i\sigma t)\cdot i\sigma t\cdot\Gamma(i\sigma t)\\ &= e^{(\mu_x-\mu_y)it}\cdot i\sigma t\cdot\frac{\pi}{\sin(\pi i\sigma t)}\\ &= e^{(\mu_x-\mu_y)it}\cdot\frac{\pi\sigma t}{\sinh(\pi\sigma t)}.\end{aligned}$$

This is the characteristic function of a logistic distribution with location parameter $\mu_x - \mu_y$ and scale parameter σ. Hence we have that $X - Y \sim Logistic(\mu_x - \mu_y, \sigma)$.

3. We have that

$$\begin{aligned}
\mathbb{P}(i \text{ chooses } j) &= \mathbb{P}(u_j > u_k, \forall k \neq j) \\
&= \mathbb{P}(\mu_j + \varepsilon_j > \mu_k + \varepsilon_k, \forall k \neq j) \\
&= \mathbb{P}(\mu_j + \varepsilon_j > \max_{k \neq j}\{\mu_k + \varepsilon_k\}).
\end{aligned}$$

We know from problem 1.1 that $\max_{k\neq j}\{\mu_k+\varepsilon_k\} \sim T1EV(\ln[\sum_{k\neq j}\exp(\mu_k)], 1)$. Let $\mu_{-j} = \ln[\sum_{k\neq j}\exp(\mu_k)]$. Now we have that

$$\begin{aligned}
\mathbb{P}(i \text{ chooses } j) &= \mathbb{P}(\mu_j + \varepsilon_j > \max_{k \neq j}\{\mu_k + \varepsilon_k\}) \\
&= \mathbb{P}(\max_{k \neq j}\{\mu_k + \varepsilon_k\} - (\mu_j + \varepsilon_j) < 0).
\end{aligned}$$

This is now the difference of two T1EV random variables, which we showed in problem 1.2 is distributed logistically. We thus have that $\max_{k\neq j}\{\mu_k + \varepsilon_k\} - (\mu_j + \varepsilon_j) \sim Logistic(\mu_{-j} - \mu_j, 1)$. We then use the logistic distribution's CDF to get that

$$\begin{aligned}
\mathbb{P}(i \text{ chooses } j) &= \mathbb{P}(\max_{k \neq j}\{\mu_k + \varepsilon_k\} - (\mu_j + \varepsilon_j) < 0) \\
&= \frac{\exp(\mu_j)}{\exp(\mu_{-j}) + \exp(\mu_j)} \\
&= \frac{\exp(\mu_j)}{\exp(\ln[\sum_{k\neq j}\exp(\mu_k)]) + \exp(\mu_j)} \\
&= \frac{\exp(\mu_j)}{\sum_{k=1}^{N}\exp(\mu_k)}.
\end{aligned}$$

4. (a) Consumer i's probability of choosing j is

$$\begin{aligned}
\mathbb{P}(i \text{ chooses } j) &= \mathbb{P}(\alpha(y_i - p_j) + \varepsilon_{ij} > \alpha(y_i - p_k) + \varepsilon_{ik}, \forall k \neq j) \\
&= \mathbb{P}(-\alpha p_j + \varepsilon_{ij} > -\alpha p_k + \varepsilon_{ik}, \forall k \neq j).
\end{aligned}$$

We can now use the result in problem 1.3 (noting that in this situation, $\mu_j = -\alpha p_j$ and $p_0 = 0$), we get that

$$s_j(i) = \frac{\exp(-\alpha p_j)}{\sum_{k=0}^{J}\exp(-\alpha p_k)}.$$

This means that

$$\frac{\partial s_j(i)}{\partial y_i} = 0,$$

suggesting that the choice probabilities are independent of income. In other words, there are no income effects in this model.

(b) We have that

$$s_j = \frac{1}{\mu(\mathcal{I})}\int_{i\in\mathcal{I}} s_j(i)dF(i),$$

where $\mu(\mathcal{I})$ is the measure of set $\mathcal{I}$. Thus we have that

$$\begin{aligned}
s_j &= \frac{1}{\mu(\mathcal{I})}\int_{i\in\mathcal{I}} \frac{\exp(-\alpha p_j)}{\sum_{k=0}^{J}\exp(-\alpha p_k)}dF(i) \\
&= \frac{\mu(\mathcal{I})}{\mu(\mathcal{I})}\frac{\exp(-\alpha p_j)}{\sum_{k=0}^{J}\exp(-\alpha p_k)}
\end{aligned}$$

because $s_j(i)$ does not depend on i. Finally, we get that

$$s_j = \frac{\exp(-\alpha p_j)}{\sum_{k=0}^{J} \exp(-\alpha p_k)}.$$

This means that the own-price derivative is

$$\begin{aligned}\frac{\partial s_j}{\partial p_j} &= -\alpha \frac{\exp(-\alpha p_j)}{\sum_{k=1}^{N} \exp(-\alpha p_k} + \alpha \frac{\exp(-\alpha p_j)^2}{\left(\sum_{k=1}^{N} \exp(-\alpha p_k)\right)^2} \\ &= -\alpha s_j(1-s_j)\end{aligned}$$

and so the own price elasticity is

$$-\alpha p_j(1-s_j).$$

The cross price elasticities can similarly be computed by observing that the derivative is

$$\frac{ds_j}{dp_k} = \alpha s_j s_k$$

and so the elasticity is given

$$\alpha s_k p_k.$$

Note that this cross price elasticity is the same for all j which implies that a percent increase in the price of product k leads to the same percent decrease in the market shares of all the other products, a manifestation of the IIA problem.

(c) (i) Note here that we have no non-trivial choice probabilities since all elements of the indirect utility function are deterministic. Thus we have that

$$s_j(i) = \prod_{k \neq j} \mathbb{1}\{\alpha(p_k - p_j) + \beta(x_j - x_k) > 0\}$$

and the market share $s_j = s_j(i)$ assuming a finite measure of consumers in $\mathcal{I}$.

This model assumes no random heterogeneity in utility, which means that choices are *ex-ante* deterministic: consumers either choose a product when its deterministic payoff is higher than that of every other product. Otherwise they do not choose the product.

(ii) Note that we can now write the down the market share equation

$$\begin{aligned}s_j &= \mathbb{P}\left(\beta_i > \alpha \frac{p_j - p_k}{x_j - x_k} \forall k \neq j\right) \\ &= 1 - \frac{\alpha}{\overline{\beta}} \max_{k \neq j} \frac{p_j - p_k}{x_j - x_k}.\end{aligned}$$

The own price derivative is therefore

$$\frac{ds_j}{dp_j} = -\frac{\alpha}{\overline{\beta}} \frac{1}{x_j - x_{\tilde{k}}},$$

where

$$\tilde{k} = \text{argmax}_{k \neq j} \frac{p_j - p_k}{x_j - x_k}.$$

Hence, the own price elasticity is

$$-\frac{\alpha}{\overline{\beta}} \frac{1}{x_j - x_{\tilde{k}}} \frac{p_j}{s_j}.$$

The cross price derivative is therefore

$$\frac{ds_j}{dp_k} = \begin{cases} 0, & k \neq \tilde{k} \\ \frac{\alpha}{\beta} \frac{1}{x_j - x_{\tilde{k}}}, & k = \tilde{k}, \end{cases}$$

giving us a cross-price elasticity of

$$\begin{cases} 0, & k \neq \tilde{k} \\ \frac{\alpha}{\beta} \frac{1}{x_j - x_{\tilde{k}}} \frac{p_k}{s_j}, & k = \tilde{k}. \end{cases}$$

We find that in this case we have avoided the IIA problem. The cross price elasticity as defined above does depend on j, which is more reasonable than what we found in part (b). On the other hand, this introduces a new problem: namely, only one of the cross-price elasticities is non-zero.

(d) Let $\mu_{ij} = -\alpha p_j + \beta_i x_j$ so that $u_{ij} = \mu_{ij} + \nu_{ij}$. We can ignore y_i since it's fixed for an individual and only differences in utility matter across products. Then

$$\begin{aligned} s_j &= \mathbb{E}_i \left[\mathbb{P} \left(\mu_{ij} + \nu_{ij} > \max_{k \neq j} \mu_{ik} + \nu_{ik} \mid \beta_i \right) \right] \\ &= \mathbb{E} \left[\frac{\exp(\mu_{ij})}{\sum_{k=1}^N \exp(\mu_{ik})} \right] \\ &= \int_{\mathbb{R}} \frac{\exp(-\alpha p_j + \beta_i x_i)}{\sum_{k=1}^N \exp(-\alpha p_k + \beta_i x_k)} dF(\beta_i) \end{aligned}$$

where the first equality is by iterated expectations and the second by problem 1.3 since β_i is conditioned out. Note that the absolute value of the derivative of the integrand (call the integrand $L_{ij}(\beta_i)$) with respect to p_j is [1]

$$|-\alpha L_{ij}(\beta_i)(1 - L_{ij}(\beta_i))| \leq \alpha$$

which is bounded in L^1 for any p_j and any β_i and so by the Leibniz rule,

$$\frac{ds_j}{dp_j} = -\alpha \int_{\mathbb{R}} L_{ij}(\beta_i)(1 - L_{ij}(\beta_i)) dF(\beta_i).$$

The own price elasticity can then be computed

$$\frac{ds_j}{dp_j} \frac{p_j}{s_j} = -\frac{\alpha p_j}{s_j} \int_{\mathbb{R}} L_{ij}(\beta_i)(1 - L_{ij}(\beta_i)) dF(\beta_i).$$

Similarly, we can compute the cross price derivative as

$$\frac{ds_j}{dp_k} = \alpha \int_{\mathbb{R}} L_{ij}(\beta_i) L_{ik}(\beta_i) dF(\beta_i)$$

which yields the cross-price elasticity

$$\frac{ds_j}{dp_k} \frac{p_k}{s_j} = \frac{\alpha p_k}{s_j} \int_{\mathbb{R}} L_{ij}(\beta_i) L_{ik}(\beta_i) dF(\beta_i).$$

Note that unlike in part 1 (b), the terms described product j's share don't cancel out in this setting due to the mixing in the numerator. Therefore in this setting, the cross price elasticities vary by product j.

[1] See problem 1.4.(b) for the detailed derivation

(e) (i) Let $\delta_j = -\alpha p_j$ (we are again ignoring y_i because only differences in utility matter!) so that $u_{ij} = \delta_j + \epsilon_{ij}$ where $\epsilon_{ij} \sim \text{T1EV}(0,1)$. Then we can show using arguments analogous to part 1.1 that

$$\mathbb{P}(\max_j u_{ij} \leq x) = \exp\left(-\exp(-x+\delta)\right)$$

where $\delta := \log\left(\sum_j \exp(\delta_{ij})\right)$ and so $\max_j u_{ij} \sim \text{T1EV}(\delta, 1)$. Therefore,

$$\begin{aligned}\mathbb{E}[\max_j u_{ij}] &= \delta + k \\ &= \log\left(\sum_j \exp(\delta_j)\right) + k\end{aligned}$$

where k is The Euler-Mascheroni constant. Note that $p_{i0} = 0$ and so

$$\log s_0 = \log\left(\frac{1}{\sum_j \exp(\delta_j)}\right) = -\delta$$

which implies that

$$\mathbb{E}[\max_j u_{ij}] = k - \log s_0$$

(ii) Note that the entry of a new good means that now

$$s_0^1 = \frac{1}{\sum_{j=1}^{J+1} \exp\left(\delta_j\right)} \leq \frac{1}{\sum_{j=1}^{J} \exp\left(\delta_j\right)} = s_0^0$$

where the superscripts denote time periods before and after entry. Note that then W is increasing since s_0 (and hence $\log s_0$) is falling. Having more options is always welfare improving.

Problem 2.

1. The coefficients in β capture the tastes for product characteristics that are shared across individuals, whereas Γ contains coefficients describing tastes that vary with interaction of household/individual characteristics and product characteristics.

2. Note that our likelihood function is given by

$$\prod_{i=1}^{I}\prod_{j=1}^{J}\left(\frac{\exp(\delta_j + d_i^T \Gamma x_j)}{1 + \sum_{k=1}^{J}\exp(\delta_k + d_i^T \Gamma x_k)}\right)^{y_{ij}}$$

where y_{ij} is an indicator of whether consumer i bought product j.

Taking logs, we have that

$$LL(\delta, \Gamma) = \sum_i \sum_j y_{ij}\left(\delta_j + d_i^T \Gamma x_j\right) - \sum_i \sum_j y_{ij} \log\left(1 + \sum_{k=1}^{J} \exp(\delta_k + d_i^T \Gamma x_k)\right).$$

3. The first order condition with respect to δ are given

$$\begin{aligned}\frac{\partial LL(\delta,\Gamma)}{\partial \delta_l} &= \sum_i y_{il} - \sum_i \sum_j y_{ij} \frac{\exp(\delta_l + d_i^T \Gamma x_l)}{1 + \sum_k \exp(\delta_k + d_i^T \Gamma x_k)} \\ &= \sum_i \left(y_{il} - L_{ik}(\delta, \Gamma)\right).\end{aligned}$$

where $L_{ik} = \frac{\exp(\delta_l + d_i^T \Gamma x_l)}{1+\sum_k \exp(\delta_k + d_i^T \Gamma x_k)}$ for every $l \in \{1, 2, \ldots, J\}$. Setting this derivative to zero yields the moment condition that equates the observed market share for good l to the predicted market share for good l.

4. The first-order conditions with respect to Γ can be recovered by enumerating the various partial derivatives

$$\begin{aligned}\frac{\partial LL(\delta,\Gamma)}{\partial \gamma_{lm}} &= \sum_i \sum_j y_{ij} d_{il} x_{jm} - \sum_i \sum_j y_{ij} \sum_k L_{ik}(\delta,\Gamma) d_{il} x_{km} \\ &= \sum_i \sum_j y_{ij} d_{il} x_{jm} - \sum_i \sum_k L_{ik}(\delta,\Gamma) d_{il} x_{km} \underbrace{\sum_j y_{ij}}_{=1} \\ &= \sum_i \sum_j \left(y_{ij} - L_{ij}(\delta,\Gamma)\right) d_{il} x_{jm}\end{aligned}$$

where d_{il} is the lth entry of the $D \times 1$ vector of household characteristics for individual d_i and x_{jm} is the mth characteristic of product j.

5. While the problem asks us to recover the entire parameter vector (δ, Γ) with maximum likelihood, the large number of parameters makes the solver behave erratically and and thus we fail to find a global maximum. In order to circumvent this problem, we concentrate out the mean utilities δ_j using the BLP contraction and recover Γ. We can then recover back the δ_j as a function of our estimate of Γ. The estimates are given in Table 1.

6. The moment condition that would allow us to consistently estimate β would be that $\mathbb{E}[\xi_j \mid x_j] = 0$.

7. See the *Linear parameters* section of Table 1

Table 1: MLE estimates of demand parameters

	MLE
Linear parameters	
x.1	0.14
x.2	1.01
x.3	0.43
Interact hh and prod. char	
$d.1 \times x.1$	0.69
$d.1 \times x.2$	0.30
$d.1 \times x.3$	0.71
$d.2 \times x.1$	0.50
$d.2 \times x.2$	0.26
$d.2 \times x.3$	0.82

Problem 3.

1. The moment condition that allows us to consistently estimate (α, β) is as follows:

$$\mathbb{E}[\xi_{jt} \mid z] = 0.$$

We get the following estimates for α and β. Note that we get a negative coefficient on prices which corresponds to a positive value of α in the language of the problem. One caveat here is that in the estimation of this model, we include a constant; without the constant, we do not observe negative price coefficients or positive marginal costs.

Table 2: Logit 2SLS estimates of demand parameters

	2SLS
Linear parameters	
Prices	-0.47
x	0.30

2. We compute own and cross-price elasticities in each market and report the average (over markets) in Table 3. We notice that for each product, the cross-price elasticities are all equivalent. This is a manifestation of the IIA problem.

Table 3: Logit estimates of average cross-price elasticities

1	2	3	4	5	6
-1.25	0.32	0.13	0.13	0.13	0.13
0.32	-1.25	0.13	0.13	0.13	0.13
0.32	0.32	-1.29	0.13	0.13	0.13
0.32	0.32	0.13	-1.29	0.13	0.13
0.32	0.32	0.13	0.13	-1.29	0.13
0.32	0.32	0.13	0.13	0.13	-1.29

3. As shown in Nevo (2001), under Nash-Bertrand compeition and no multi-product firms, we have the following condition

$$p_{jt} - c_{jt} = \frac{-s_{jt}}{\frac{\partial s_{jt}}{\partial p_{jt}}} \implies c_{jt} = p_{jt} + \frac{s_{jt}}{\frac{\partial s_{jt}}{\partial p_{jt}}}.$$

We use the above equation to calculate marginal costs over all products and markets and then average over all markets. The results are reported below. We see that the two products (product 1 and 2) with the highest average prices and highest average market shares also have the highest marginal costs. This implies that differences in marginal costs can't explain all of the differences in prices and market shares. If it could, we would expect that those with the highest marginal costs have the highest prices and lowest market shares. Therefore, there must be some product quality (observed or unobserved) that is driving the differences in prices and market shares.

Table 4: Average Marginal Costs by Product

	1	2	3	4	5	6
Marginal Costs	3.198	3.205	2.963	2.971	2.963	2.968

4. To simulate the counterfactual of product 1 leaving the market, we use the following contraction on prices found in Conlon and Gortmaker (2020) for each product j:

$$p_{jt} \leftarrow c_{jt} + \frac{1}{\alpha} + s(p_{jt})(p_{jt} - c_{jt}).$$

We get the following average prices and market shares after product 1 leaves the market.

Table 5: Counterfactual Average Prices and Market Shares

	2	3	4	5	6
Prices	5.748	5.275	5.280	5.270	5.280
Market Shares	0.157	0.074	0.073	0.072	0.074

5. We next calculate profits and consumer welfare before and after product 1 leaves the market. We know that profits for product j and market t are given by $s_{jt}(p_{jt} - c_{jt})$. We also know from problem 1 that consumer welfare can be calculated as $W = k - \log s_0$. We find that consumer welfare decreases by 0.845 units after product 1 leaves the market. We show the average changes in profits for each of the products in the below table. We see that consumers lose as a result of this product exit and that firms producing products 2-6 win as a result of this product exit.

Table 6: Change in Profits after Product Exit

	2	3	4	5	6
Change in Profits	0.366	0.165	0.162	0.161	0.165

Problem 4.

1. Note that our specification, the linear component of the model

$$\delta_{tj} = x_1\bar{\beta} + \xi$$

contains a constant, and thus the results here may differ from those estimated with other specifications which do not include a constant. Another difference from standard models is that we do not estimate a two-step GMM estimator since efficiency is not of concern in this problem set.

Table 7: BLP GMM estimates of demand parameters

	GMM
Linear parameters	
p	-1.86
x	1.26
Random coefficients	
$\gamma_{1,1}$	4.31
$\gamma_{1,2}$	0.00
$\gamma_{2,1}$	5.05
$\gamma_{2,2}$	-1.99

2. Table 8 reports the average cross-price elasticites from the BLP specification and data. These elasticities do not exhibit the IIA problem we observed in the logit elasticities.

Table 8: BLP estimates of average cross-price elasticities

1	2	3	4	5	6
-0.003	0.000	0.037	0.027	0.160	0.140
0.001	-0.003	0.041	0.027	0.099	0.188
0.001	0.000	-2.329	0.123	0.674	1.137
0.001	0.000	0.161	-2.229	0.742	0.844
0.000	0.001	0.300	0.149	-1.917	1.267
0.000	0.000	0.190	0.155	1.023	-1.534

3. We report the average prices, shares, and observed quality for the 6 products in Table 9. We see that the most important factor driving the differences in prices and market shares is the observed product quality. The products with the highest observed quality have the highest market shares and the highest prices. This observed measure of product differentiation is therefore likely what's generating dispersion in prices and market shares.

Table 9: Summary statistics of product data across markets

	Price	Quality	Share
1	0.002	-0.019	0.099
2	0.002	-0.026	0.089
3	2.019	-0.081	0.043
4	1.752	-0.180	0.039
5	3.577	1.693	0.152
6	4.443	2.002	0.193