ECE 269- Lecture 15: Tuesday Feb 25th

- Eigenvalue Decomposition
- Jordan Form
- Cayley – Hamilton thm

- HW 5 is due Friday.

- Material on det(·) and multilinear form are from Last Chapter of Linear Algebra Done Right.

Fact: If $v_1, \dots, v_k$ (non-zero) are eigenvectors corresponding to distinct eigenvalues $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_k$ of $A \in \mathbb{R}^{n \times n}$, then $v_1, \dots, v_k$ are independent.

Pf: Suppose that $v_1, \dots, v_\ell$ are dependent and ℓ is smallest integer that this happens

$$\boxed{\alpha_1 v_1 + \dots + \alpha_\ell v_\ell = \underline{0}}$$

$A\alpha$

$$\alpha_1 \lambda_1 v_1 + \alpha_2 \lambda_2 v_2 + \dots + \alpha_\ell \lambda_\ell v_\ell = \underline{0}$$

$\boxed{}$ $\times \lambda_1$

$$\alpha_1 \lambda_1 v_1 + \alpha_2 \lambda_1 v_2 + \dots + \alpha_\ell \lambda_1 v_\ell = 0$$

$$\alpha_2(\lambda_2 - \lambda_1) v_2 + \alpha_3(\lambda_3 - \lambda_1) v_3 + \dots + \alpha_\ell(\lambda_\ell - \lambda_1) v_\ell = 0$$

$\times$

Implication: If $\lambda_1, \dots, \lambda_n$ are distinct, with eigenvects $v_1, \dots, v_n$

$$A \underbrace{\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}}_{V} = \underbrace{\begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}}_{V} \begin{bmatrix} \lambda_1 & 0 & & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

$$AV = V \underbrace{\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}}_{D} = VD$$

$$\Rightarrow \quad V^{-1}AV = D \quad \equiv \quad A = VDV^{-1}$$

Def: We say $A, B \in \mathbb{R}^{n\times n}$ are similar if $\exists$ inv $T \in \mathbb{R}^{n\times n}$:

$$A = TBT^{-1}.$$

Notation: $A \sim B$

Fact: Similarity is an equivalence relationship:

(i) $A \sim A$

(ii) $A \sim B \iff B \sim A$

(iii) $A \sim B$ and $B \sim C \implies A \sim C$.

If $A \sim B$: If we can find B^k easily, then

$$A^k = (TBT^{-1})^k = TBT^{-1}TBT^{-1} \cdots TBT^{-1}$$

$$= TB^kT^{-1}.$$

A class of matrices that B^k is easily computable, is the diagonal matrices.

Which matrices A are similar to diagonal matrices?

$\exists V$? $\quad A = VDV^{-1} = V \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix} V^{-1}$

Note that $d_1, \dots, d_n$ can only be the eigenvalues of A as

$[v_1 \cdots v_n]$

$$AV = VD$$

$$\parallel$$

$$Av_1 = d_1 v_1$$

$$Av_2 = d_2 v_2$$

$$\vdots$$

$$Av_n = d_n v_n$$

Any matrix A that is similar to $D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$ where λ_i's are eigenvalues of A is said to admit an eigenvalue decomposition

EVD : $A = VDV^{-1}$

Define: $V_A(\lambda) := \{ u \mid Au = \lambda u \}$. For an eigenvalue λ_i, we define the geometric multiplicity of λ_i as $g_i = \dim(V_A(\lambda_i))$.

Fact: $g_i \leq b_i$ where b_i is the algebraic multiplicity of λ_i.

$$C_A(\lambda) = \det(\lambda I - A) = (\lambda - \lambda_1)^{b_1} \cdots (\lambda - \lambda_k)^{b_k}.$$

Fact: A is diagonalizable (admitting EVD) $\iff$ $g_i = b_i$ for all i.

$$\Rightarrow \quad D = \begin{bmatrix} \underbrace{\lambda_1 \; \ddots \; \lambda_1}_{g_1} & & & \\ & \underbrace{\lambda_2 \; \ddots \; \lambda_2}_{g_2} & & \\ & & \ddots & \\ & & & \underbrace{\lambda_k \; \ddots \; \lambda_k}_{g_k} \end{bmatrix}$$

To find EVD: find $c_A(\lambda) = (\lambda-\lambda_1)^{b_1} \cdots (\lambda-\lambda_k)^{b_k}$.

$= \det(\lambda I - A)$

then for $i = 1, \ldots, k$; find independent eigenvectors $v_{i1}, \ldots, v_{ib_i} \in \mathbb{R}^n$ for λ_i

if succeed: $V = \left[\begin{array}{c|c|c|c} v_{11} \cdots v_{1b_1} & v_{21} \cdots v_{2b_2} & \cdots & v_{k1} \cdots v_{kb_k} \end{array} \right]$

General Classes of Matrices Admitting EVD:

- Matrices with n distinct evalues.

(TBA) - Symmetric / Hermitian matrices

$$A = VDV^{-1} \qquad A^k = VD^kV^{-1}$$

$$e^A := I + A + \frac{A^2}{2!} + \cdots + \frac{A^k}{k!} + \cdots$$

if $A = VDV^{-1}$

$$e^A = VIV^{-1} + VDV^{-1} + \frac{(VDV^{-1})^2}{2!} + \cdots + \frac{(VDV^{-1})^k}{k!} + \cdots$$

$$= V\left(I + D + \frac{D^2}{2!} \cdots + \frac{D^k}{k!} + \cdots\right)V^{-1}$$

$$= V\begin{pmatrix} e^{\lambda_1} & & & \\ & e^{\lambda_2} & & \\ & & \ddots & \\ & & & e^{\lambda_n} \end{pmatrix}V^{-1}$$

Example of a non-diagonalizable matrix A:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow c_A(\lambda) = \det(\lambda I - A) = \lambda^2 \Rightarrow \lambda_1 = 0$$

$$b_1 = 2$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\equiv \begin{pmatrix} y \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow y = 0$$

$$\Rightarrow V_A(\lambda = 0) = \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} \;\middle|\; x \in \mathbb{R} \right\} \Rightarrow \overset{\dim(V_A(0))}{\overset{\shortparallel}{g_1}} = 1 < b_1 = 2$$

$\Rightarrow$ not diagonalizable.

Fact: For any matrix A, $\exists$ an invertible V:

$$A = V \begin{bmatrix} J_1 & & \underline{0} & \\ & J_2 & & \\ & & \ddots & \\ \underline{0} & & & J_k \end{bmatrix} V^{-1}$$

where $J_i = \begin{bmatrix} \lambda_j & 1 & & \underline{0} \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ \underline{0} & & & \lambda_j \end{bmatrix}_{n_i \times n_i}$ is called a Jordan block of size n_i.

$\equiv \forall A \in \mathbb{R}^{n\times}$, we have

$$A \sim \begin{pmatrix} \begin{matrix} \lambda_1 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_1 \end{matrix} & & & \\ & \begin{matrix} \lambda_2 & 1 & & \\ & \lambda_2 & 1 & \\ & & \ddots & \ddots \\ & & & \lambda_k \end{matrix} & & \\ & & \begin{matrix} \lambda_k & 1 & & \\ & \lambda_k & 1 & \\ & & \ddots & \ddots \\ & & & \lambda_k \end{matrix} & \\ & & & \ddots \end{pmatrix}$$

Furthermore, $\forall$ eigenvalue λ_e with geometric multiplicity g_e and alg mult b_e:

we have g_e Jordan blocks corresponding to λ_e

and the summation of these blocks = # time λ_e appear on diagonal of $J = b_i$.

If $g_i = b_i \ \forall i \Rightarrow$ we have g_i blocks of length 1.

$$J_i = [\lambda_i] \Rightarrow JD \equiv EVD$$