Homework 5

(Due: Friday Feb 28th 2025 at 8 pm)

1. Show that for any alternating linear form $f : (\mathbb{R}^n)^m \to \mathbb{R}$, we have the following:
   (a) for any $i \neq j$,
   $$f(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_m) = -f(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_m).$$
   (b) if $m = n$, and $f(e_1, \ldots, e_n) \neq 0$ (where $e_1, \ldots, e_n$ are the standard basis elements) then $f(v_1, \ldots, v_n) = 0$ only if $v_1, \ldots, v_n$ are linearly dependent (of course, this is if and only if statement, as we have shown the if part in the class).

2. *Practical Determinant.* In practice, one never goes over the extensive formula discussed in lecture for computing determinant, but rather the transformations involving the matrices. One of them being LU decomposition.
   (a) For a $n \times n$ lower-triangular matrix Q, show that $\det(Q) = Q_{11} \cdots Q_{nn}$. (Note that since $\det(A) = \det(A')$, same result holds for upper triangular matrices)
   (b) Using an LU decomposition of A (of the form $LPA = U$ or $PA = LU$), find $\det(A)$ for the following matrix.
   $$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 5 & 3 & 1 \\ 1 & 3 & 3 & 2 \\ 0 & 1 & 2 & 4 \end{bmatrix}$$

3. *Eigenvalues.* Suppose that A has $\lambda_1, \lambda_2, \ldots, \lambda_n$ as its eigenvalues.
   (a) Show that $\det(A) = \lambda_1 \cdot \lambda_2 \cdots \lambda_n$.
   (b) Show that the eigenvalues of A' are $\lambda_1, \lambda_2, \ldots, \lambda_n$, that is, A and A' have the same set of eigenvalues.
   (c) Show that the eigenvalues of A^k are $\lambda_1^k, \lambda_2^k, \ldots, \lambda_n^k$ for $k = 1, 2, \ldots$ and if A is invertible, the result holds for all $k \in \mathbb{Z}$.
   (d) Show that A is invertible if and only if it does not have a zero eigenvalue.
   (e) For an invertible matrix $T \in \mathbb{R}^{n\times n}$, show that A and $T^{-1}AT$ have the same set of eigenvalues, that is, eigenvalues are invariant under a similarity transformation $A \mapsto T^{-1}AT$.
   (f) Let us define the set of eigenvectors corresponding to eigenvalue λ, to be $v_\lambda(A) = \{v \in \mathbb{R}^n \mid Av = \lambda v\}$. Show that $v_\lambda(A)$ is a subspace of $\mathbb{R}^n$.
   (g) The *trace* of $A \in \mathbb{R}^{n\times n}$ is defined by sum of its diagonal elements, i.e.,
   $$\text{trace}(A) = A_{11} + A_{22} + \cdots + A_{nn}.$$
   Show that
   $$\text{trace}(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n.$$

4. *Gershgorin circles.* Let v be an eigenvector of $A \in \mathbb{C}^{n\times n}$ associated with eigenvalue λ such that $\|v\|_\infty = |v_i| = 1$.

   (a) Show that $(\lambda - A_{ii})v_i = \sum_{j\neq i} A_{ij}v_j$.

   (b) Let the *Gershgorin circles* of A be defined as

   $$\mathcal{G}_i = \{\xi \in \mathbb{C} : |A_{ii} - \xi| \leq \rho_i\}, \quad i = 1, 2, \ldots, n,$$

   where the radius of the i-th circle centered at A_{ii} is

   $$\rho_i = \sum_{j\neq i} |A_{ij}|.$$

   Show that all eigenvalues of A are contained in the union of the Gershgorin circles.

   (c) We say that $A \in \mathbb{C}^{n\times n}$ is *diagonally dominated* if

   $$A_{ii} > \sum_{j\neq i} |A_{ij}|, \quad i = 1, 2, \ldots, n.$$

   Show that a diagonally dominated matrix A is nonsingular.

5. *Nilpotent matrices.* We say that a square matrix A is *nilpotent* if $A^k = 0$ for some $k \geq 1$. We define the smallest k for which $A^k = 0$ to be its (nilpotent) index. For example,

   $$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

   is nilpotent of index 3.

   (a) Show that every nilpotent matrix $A \in \mathbb{F}^{n\times n}$ has no nonzero eigenvalue and thus that its characteristic function is $\chi(\lambda) = \det(\lambda I - A) = \lambda^n$.

   (b) Show that the index of a nilpotent matrix $A \in \mathbb{F}^{n\times n}$ is always $\leq n$.

   (c) Suppose that $A \in \mathbb{F}^{n\times n}$ is nilpotent of index n. Show that if $A^{n-1}x \neq 0$, then $x, Ax, A^2x, \ldots, A^{n-1}x$ form a basis of $\mathbb{F}^n$.

   (d) Continuing part (c), let

   $$T = \begin{bmatrix} x & Ax & A^2x & \cdots & A^{n-1}x \end{bmatrix} \in \mathbb{F}^{n\times n}.$$

   Show that the similarity transformation of A by T is

   $$T^{-1}AT = \begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}.$$