ECE 269- Lecture 7: Tuesday Jan 28th

- Inner product spaces
- Examples
- Properties
- Banach and Hilbert spaces
- QR decomposition and Gram-Schmidt Proc.

Anouncement: HW 3 is due next Wed 8:00 pm

Review: For a vector space V over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

$\|\cdot\| : V \to [0, \infty)$ is a norm if

(i) $\|v\| = 0$ iff $v = 0$

(ii) $\|\alpha v\| = |\alpha| \, \|v\|$ $\forall \alpha, \; v \in V$

(iii) $\|v + u\| \leq \|v\| + \|u\|$

For a given norm $\|\cdot\|$ over V, we can define $d(u, v) := \|u - v\|$.

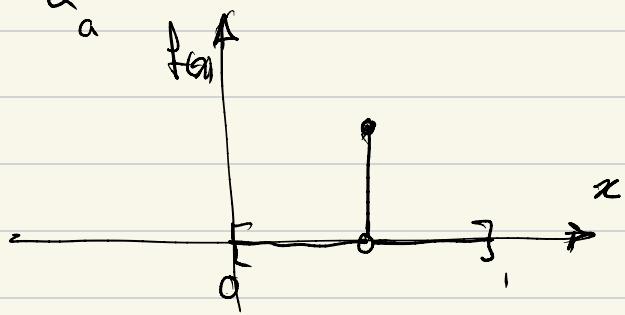
$a<b$

Let us define $L_2([a,b]) := \left\{ f:[a,b] \to \mathbb{R} \,(\mathbb{C}) \;\middle|\; \int_a^b |f(x)|^2 dx < \infty \right\}$

$$L_p([a,b]) := \left\{ f:[a,b] \to \mathbb{R} \;\middle|\; \int_a^b |f(x)|^p dx < \infty \right\} \quad (p \geq 1)$$

Fact: $L_p([a,b])$ is a Normed* Vector spaces with

$$\|f\|_p := \left(\int_a^b |f(x)|^p dx \right)^{\frac{1}{p}} \quad \text{for} \quad p \geq 1.$$



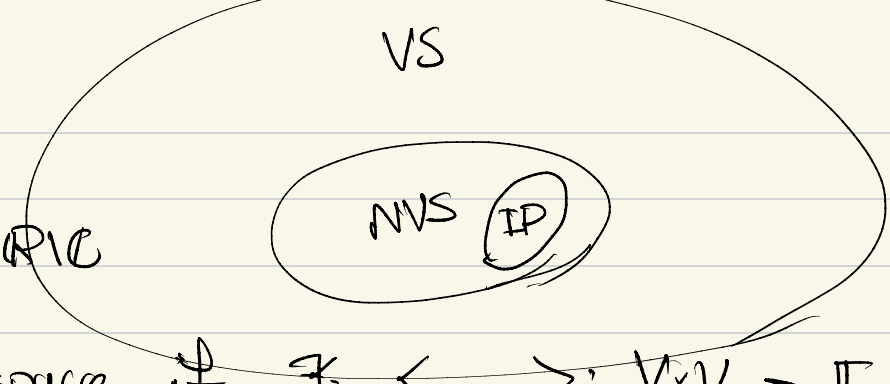
- Example: $\mathbb{R}^{m \times n}$ over $\mathbb{R}$ is a NVS with

$$\|A\|_F = \left(\sum_{j=1}^{n} \sum_{i=1}^{m} A_{ij}^2 \right)^{\frac{1}{2}} = \left(tr(A^T A) \right)^{\frac{1}{2}}$$

Frobenius

$$A^T = \begin{bmatrix} — & v_1^T & — \\ — & v_2^T & — \\ & \vdots & \\ — & v_n^T & — \end{bmatrix} \qquad A = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_m \\ | & & | \end{bmatrix}$$

Inner Product Spaces:



A vector space V over $\mathbb{F} = \mathbb{R} \backslash \mathbb{C}$ is called an inner product space if $\exists$: $\langle -, - \rangle : V \times V \to \mathbb{F}$:

(1) $\langle v, v \rangle > 0$ for $v \neq \underline{0}$

(2) $\langle u, v \rangle = \overline{\langle v, u \rangle}$ $\forall\, u, v \in V$

(3) $\langle u + \alpha v, w \rangle = \langle u, w \rangle + \alpha \langle v, w \rangle$

Example: - $\mathbb{R}^n$, with $\langle x, y\rangle = x^T y = \sum_{i=1}^{n} x_i y_i$.

- $\mathbb{R}^{m\times n}$ with $\langle A, B\rangle = \operatorname{tr}(B^T A)$

- $\mathbb{C}^n$ over $\mathbb{C}$: $\langle x, y\rangle = x^H y = \sum_{i=1}^{n} x_i^* y_i$

- $L_2([a,b])$: $\langle f, g\rangle = \int_a^b f(x) g(x)\, dx$

  $\downarrow$ $p=2$

Non-Example: $\mathbb{Z}_2^n$ with $\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i$

$$\langle x, x\rangle \overset{?}{=} 0$$

Non-Example: L_p for $p \neq 2$

Properties of a inner product:

(1) $\langle \underline{0}, v \rangle = \langle v, \underline{0} \rangle = 0$

Pf: $\langle \underline{0}, v \rangle = \langle w - w, v \rangle \overset{\text{Property (3) of IP}}{=} \langle w, v \rangle - \langle w, v \rangle = 0$

(2) Inner product is conjugate linear in terms of the second argument.

i.e., $\langle u, \alpha v + w \rangle = \bar{\alpha} \langle u, v \rangle + \langle u, w \rangle$

(3) Cauchy-Schwartz Inequality:

$$\langle u, v \rangle^2 \leq \langle u, u \rangle \langle v, v \rangle$$

Implications of C-S inequality: $\langle u,v\rangle^2 \leq \langle u,u\rangle \langle v,v\rangle$

- $\mathbb{R}^n$, $\langle u,v\rangle = \sum_{i=1}^n u_i v_i \Rightarrow \left(\sum_{i=1}^n u_i v_i\right)^2 \leq \left(\sum u_i^2\right)\left(\sum v_i^2\right)$

$$\equiv \left|\sum_{i=1}^n u_i v_i\right| \leq \|u\| \, \|v\|$$

- $L_2([a,b])$: $\left|\int_a^b f(x) g(x)\, dx\right| \leq \left(\int_a^b (f(x))^2 dx\right)^{\frac{1}{2}} \left(\int_a^b |g(x)|^2 dx\right)^{\frac{1}{2}}$

probability space

(*)

- $L_2((\Omega, \mathcal{F}, P(\cdot)))$ = set of random variables X with finite variance

$X, Y \in L_2$; $\langle X, Y\rangle = E[XY]$

C-S: $|E[XY]| \leq (E[X^2])^{\frac{1}{2}} (E[Y^2])^{\frac{1}{2}}$

C-S

$$-\ \mathrm{Cov}(X,Y) = E[(X-\underset{\downarrow\, E[X]}{\bar{X}})(Y-\underset{\downarrow\, E[Y]}{\bar{Y}})] \leq \underbrace{\left(E[(X-\bar{X})^2]\right)^{\frac{1}{2}}}_{\mathrm{Var}(X)^{\frac{1}{2}}} \underbrace{E[(Y-\bar{Y})^2]^{\frac{1}{2}}}_{\mathrm{Var}(Y)^{\frac{1}{2}}}$$

$$\Downarrow$$

$$-1 \leq \frac{\mathrm{Cov}(X,Y)}{\sigma_X\ \sigma_Y} \leq 1$$

Pf of C-S inequality: $|\langle u,v\rangle|^2 \leq \langle u,u\rangle\langle v,v\rangle$

$(\mathbb{F}=\mathbb{R})$

Pf: $0 \leq \langle u-\alpha v, u-\alpha v\rangle = \overset{c}{\langle u,u\rangle} - \overbrace{2\alpha\langle u,v\rangle}^{b} + \alpha^2 \underbrace{\langle v,v\rangle}_{a}$

$\alpha \in \mathbb{R}$

$\alpha^* = -\frac{b}{2a}$

Let $\alpha^* \underset{v\neq 0}{=} \frac{2\langle u,v\rangle}{2\langle v,v\rangle} = \frac{\langle u,v\rangle}{\langle v,v\rangle}$.

$$\Rightarrow 0 \leq \langle u,u\rangle - \frac{2\langle u,v\rangle^2}{\langle v,v\rangle} + \frac{\langle u,v\rangle^2}{\langle v,v\rangle^{\cancel{2}}}\cancel{\langle v,v\rangle}$$

$$= \langle u,u\rangle - \frac{\langle u,v\rangle^2}{\langle v,v\rangle} \Rightarrow \langle u,v\rangle^2 \leq \langle u,u\rangle\langle v,v\rangle$$

$\blacksquare$

Fact: Any inner product space V with the IP $\langle -,- \rangle$ is a NVS with:

$$\|u\|_2 := \langle u, u \rangle^{\frac{1}{2}}.$$

Pf: (i) $\|u\|_2 \geq 0$ and $\|u\|_2 = 0 = \langle u, u \rangle \iff u = \underline{0}$. ✓

(ii) $\|\alpha u\|_2 = \langle \alpha u, \alpha u \rangle^{\frac{1}{2}} = (\alpha \langle u, \alpha u \rangle)^{\frac{1}{2}} = (\alpha \overline{\langle \alpha u, u \rangle})^{\frac{1}{2}}$

$$= (\alpha \bar{\alpha} \overline{\langle u, u \rangle})^{\frac{1}{2}} = (|\alpha|^2 \overline{\langle u, u \rangle})^{\frac{1}{2}} = (\alpha^2 \langle u, u \rangle)^{\frac{1}{2}}$$

$$= |\alpha| \, \|u\|$$

$\mathbb{F} = \mathbb{R}$

$\leq 2\langle u,u\rangle^{\frac{1}{2}}\langle v,v\rangle^{\frac{1}{2}}$

(iii) $\|u+v\|^2 = \langle u+v, u+v\rangle = \underbrace{\langle u,u\rangle} + 2\langle u,v\rangle + \underbrace{\langle v,v\rangle}$

CS inequality

$\leq \|u\|^2 + 2\|u\|\,\|v\| \quad \|v\|^2$

$= (\|u\| + \|v\|)^2$

Orthogonality: We say that $u, v \in V$ with an inner-product $\langle \cdot, \cdot \rangle$ are orthogonal if $\langle u, v \rangle = 0$.

In fact, we define the angle between u, v to be

$$\boxed{-1 \leq \frac{\langle u, v \rangle}{\|u\| \, \|v\|} \leq 1 \qquad \text{C-S} \Rightarrow \quad v, u \neq 0}$$

$$\angle(u, v) := \arccos\left(\frac{\langle u, v \rangle}{\|u\| \, \|v\|} \right).$$