ECE 269- Lecture 9: Tuesday Feb 4th

- QR decomp cont.
- Solving $Ax=b$

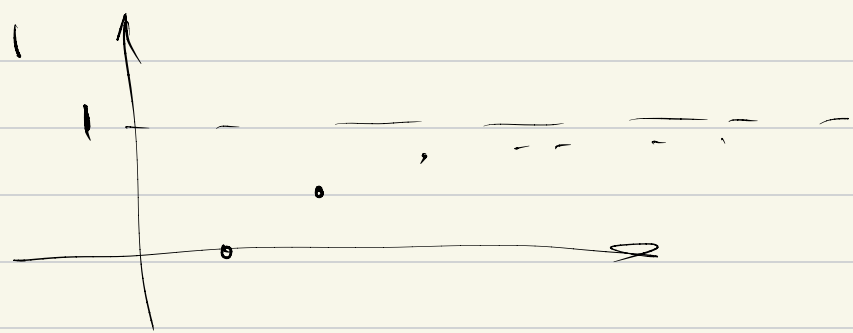
- sup
- reflection

For a sequence $\{x_k\}_{k\geq 1}$ in $\mathbb{R}$, we say that B is an upper bound if $x_k \leq B$. The smallest of such upper bounds is called the $\sup\limits_{k\to\infty} x_k$.

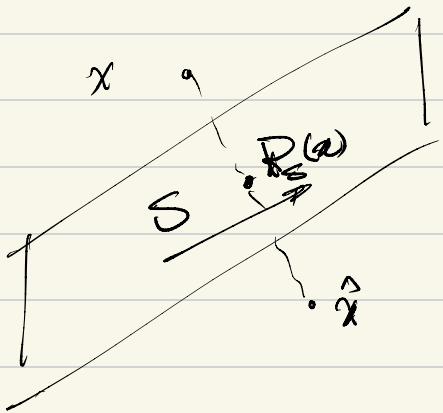
($B=\infty$ is an upper bound for any sequence)

Example: $x_k = (-1)^k \longrightarrow \sup x_k = 1$

$$\sup_{k\to\infty} \underbrace{\left(1-\frac{1}{k}\right)}_{y_k}$$



(affine subspace)



$\hat{x}$ is the reflection of x wrt S if

$$\frac{x+\hat{x}}{2} = P_S(x)$$

- Midterm:
  - Closed book / Closed note
  - During the class time next Thursday
  - Topics: Every thing up to and including QR decomposition
  - Sample exam is posted
  - no electronics.

QR decomposition:

For any $n \times k$ matrix A; $\exists$ an orthogonal matrix $[Q]_{n \times n}$ and an upper triangular matrix R ; $A = QR$.

Pf: First we show that $A = \tilde{Q}R$ where $\tilde{Q}$ is an $n \times r$ matrix with orthonormal columns and $[R]_{r \times k}$ being an upper triangular matrix.

Let $\underset{n\times k}{A} = \overset{v_{i_1}, v_{i_2}, \dots, v_{i_r}}{\left[v_1 \,\middle|\, \cdots \,\middle|\, v_k \right]}$ and $\{v_{i_1}, \dots, v_{i_r}\}$ be the "first" r independent columns of A where $r = \text{rank}(A)$. Through the G-S procedure we can turn $\{v_{i_1}, \dots, v_{i_r}\}$ into an orthonormal set $\{q_1, \dots, q_r\}$

$$A = \begin{array}{c} \overbrace{}^{\emptyset} \\ \left[\begin{array}{ccccc} | & | & & | & \\ v_1 & \cdots & v_{i_1} & \cdots & v_{i_2} \cdots \quad v_{i_r} \cdots \\ | & | & & | & \end{array} \right] \end{array}$$

$$= \underbrace{\left[\begin{array}{ccc} | & & | \\ q_1 & \cdots & q_r \\ | & & | \end{array} \right]}_{\tilde{Q}_{n\times r}} \underbrace{\left[\begin{array}{cccccccc} * & * & * & * & * & & & \\ 0 & 0 & 0 & \cdots & 0 & * & & \\ \vdots & & & & & 0 & & \\ 0 & & & & & 0 & & \\ & & & & & 0 & & \end{array} \right]}_{U}$$

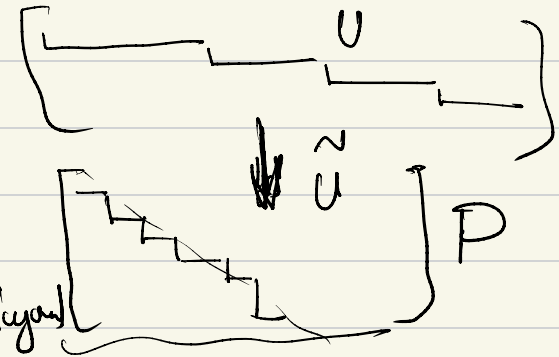
$\text{rank}(A) = r < n$: Let $\bar{A} = [A \mid \underbrace{B}_{n-r}]$ where $\text{rank}(\bar{A}) = n$

Then by the above result:

$$[A \mid B] = \bar{A} = \bar{Q}\,\bar{U} = [\tilde{Q} \mid Q_2] \begin{array}{c} \begin{array}{cc} k & k \end{array} \\ \left[\begin{array}{c|c} U_{rk} & \\ \hline 0 & \end{array}\right] \end{array}$$

But $A = [\tilde{Q} \;\; Q_2] \left[\begin{array}{c} U_{r\times k} \\ \hline 0 \end{array}\right]$

$\Rightarrow$ thin / reduced QR decomposition: $r = \text{rank}(A)$



$$A = \underset{n\times r}{\tilde{Q}} \; \underset{r\times k}{U}$$

$$A = \tilde{Q}\tilde{U}P$$

where $\tilde{U}$ is upper triangular with all the main diagonal element $\neq 0$

where $P_{k\times k}$ is a permutation matrix, i.e.,

$$\boxed{P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

- a binary matrix
- with each row and each column have exactly one element $= 1$.

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_2 \\ v_1 \\ v_3 \end{pmatrix}$$

Full QR: $A = Q_{n \times n} \begin{bmatrix} U_{r \times m} \\ \emptyset_{(n-r) \times m} \end{bmatrix}$

Note that for $Q = [\tilde{Q} \mid Q_2]$, we know that

$$N(A') = \{ x \mid x^T A = 0 \} \overset{?}{=} R^{\perp}(A) = \{ z \mid \langle z, y \rangle = 0 \quad y \in R(A) \}.$$

As if $x \in N(A')$, $\forall\, y \in R(A) \Rightarrow y = Au \Rightarrow x^T y = x^T A u = 0$

$\Rightarrow x \in R^{\perp}(A)$

$\Rightarrow N(A') \subseteq R^{\perp}(A)$

Also if $z \in R^{\perp}(A)$, $(z^T A u) = 0 \quad \forall u \Rightarrow z^T A = 0$

$$R^{\perp}(A) = R^{\perp}(\tilde{Q}) = R(Q_2)$$

$$\| $$

$$N(A')$$

Solving $Ax=b$: Gaussian Elimination

Suppose that $A_{n\times n} \in \mathbb{R}^{n\times n}$.

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{L_1} \overset{A^{(0)}}{\begin{bmatrix} 1 & 0 & 3 \\ 2 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{L_1} \overset{b_1}{\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}}$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}}_{L_2} \overset{A^{(1)}}{\begin{bmatrix} 1 & 0 & 3 \\ 0 & 0 & -5 \\ 1 & -1 & 1 \end{bmatrix}} x = L_2 L_1 b$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \overset{U\ A^{(2)}}{\begin{bmatrix} 1 & 0 & 3 \\ 0 & 0 & -5 \\ 0 & -1 & -2 \end{bmatrix}} x = P L_2 L_1 b$$

For any $A \in \mathbb{R}^{n \times n}$; $PA = LU$

where L is ~~is~~ a lower triangular matrix

U is an upper " "

P is a permutation matrix.