ECE 269- Lecture 14: Thursday Feb 20th

- Multilinear forms and determinants
- Eigenvectors' geometry
- Eigenvalue decomposition

Announcements:

- HW 4 due tomorrow at 8 pm
- Regrad requests available 8 am – 12 pm.

Recall: An m-multilinear form on $\mathbb{R}^n$:

$$f : V \times V \times \cdots \times V \longrightarrow \mathbb{R}$$

A multilinear m-form can be viewed as a function from $n \times m$ matrices to $\mathbb{R}$:

$$f\left(\left.\begin{bmatrix} | & | & & | \\ v_1 & v_2 & A & v_m \\ | & | & & | \end{bmatrix}\right\} n\right) \in \mathbb{R}$$

m-columns

An alternating m-linear form (alt) is a multilinear form that $f(A) = 0$ if A has two identical columns.

(the inputs to the multilinear form)

Facts: (i) if columns of A are linearly dependent, then

$f(A) = 0$ for all $\widehat{\text{alt}}$ multilinear forms.

(ii) $f(v_1, v_2, v_{i+1}, \ldots, v_j, \ldots, v_n) = -f(v_1, v_2, \ldots, v_j, \ldots, v_i, \ldots, v_n)$

Pf of (i): wlg $v_1 = \alpha_2 v_2 + \cdots + \alpha_m v_m$.

$$f(v_1, v_2, \ldots, v_m) = f(\alpha_2 v_2 + \cdots + \alpha_m v_m, \underline{v_2}, \ldots, \underline{v_m})$$

$$\overset{\text{multilinear form}}{=} \alpha_2 f(v_2, v_2, \ldots, v_m) + \alpha_3 f(v_3, v_2, v_3, \ldots, v_m)$$

$$+ \cdots + \alpha_m f(v_m, v_2, v_3, \ldots, v_m)$$

$$= 0.$$

$e_1 = \begin{pmatrix}1\\0\end{pmatrix}, \; e_2 = \begin{pmatrix}0\\1\end{pmatrix}$

$m = n = 2$: $\quad f(e_1, e_2) = f\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right) \equiv \gamma.$

$f(e_2, e_1) = f\left(\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\right) = -\gamma, \quad f\left(\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}\right) = f\left(\begin{bmatrix}0 & 0\\1 & 1\end{bmatrix}\right) = 0.$

$$f\left(\begin{bmatrix}a & c\\b & d\end{bmatrix}\right) = f\left(\begin{bmatrix}a & c\\0 & d\end{bmatrix}\right) + f\left(\begin{bmatrix}0 & c\\b & d\end{bmatrix}\right)$$

$$\overset{a,b\neq 0}{=} a\, f\left(\begin{bmatrix}1 & c\\0 & d\end{bmatrix}\right) + b\, f\left(\begin{bmatrix}0 & c\\1 & d\end{bmatrix}\right)$$

$$= a\left(\overbrace{f\left(\begin{bmatrix}1 & c\\0 & 0\end{bmatrix}\right)}^{c f(e_1, e_1)} + \underbrace{f\left(\begin{bmatrix}1 & 0\\0 & d\end{bmatrix}\right)}\right) + \quad '' \quad = ad\, f(I) + bc\, f(e_2, e_1)$$

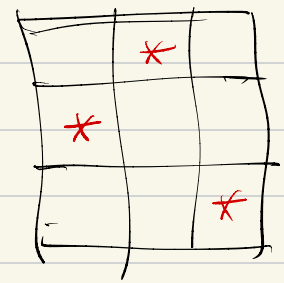
$$= \gamma(ad - bc).$$

Fact: if $m=n$, and $f(I) = f(e_1, \dots, e_n) = \gamma$,

$$f(A) = \gamma \sum_{\substack{P:\ n\times n \text{ permut} \\ \text{matrices}}} (-1)^{\#P} \underbrace{\prod_{i,j \in P} A_{ij}}$$

$$\prod_{i,j \in P} A_{ij} := \prod_{P_{ij}=1} A_{ij}$$

$$\#P = \sum_{i<j} P_{ij}$$



For $A \in \mathbb{R}^{n \times n}$, $\det(A)$ is ~~a~~ the alt multilinear n-form in $\mathbb{R}^n$ that

$$\det(I) = 1!$$

Det is the only function $f: \mathbb{R}^{n \times n} \to \mathbb{R}$ satisfying:

(i) linearity on each column

(ii) $\det(A) = 0$ if two columns are equal

(iii) $\det(I) = 1$.

$A = [v_1 \cdots v_n]$

Properties of $\det(\cdot)$:

i — $\text{rank}(A) < n \Longrightarrow \det(A) = 0$

ii — $\det([v_1 \cdots v_i \cdots v_j \cdots v_n]) = -\det([v_1 \cdots v_j \cdots v_i \cdots v_n])$

iii — $\det(\lambda A) = \lambda^n \det(A)$ $\quad \forall \lambda \in \mathbb{R}$

$$\left(\det(\lambda A) = \det([\lambda v_1 \ \lambda v_2 \cdots \lambda v_n]) = \lambda \det([v_1 \ \lambda v_2 \cdots \lambda v_n]) = \lambda^2 \det([v_1 \ v_2 \ \lambda v_3 \cdots \lambda v_n]) \right.$$

$$\vdots$$

$$= \lambda^n \det(A)$$

v — $\det(A) \neq 0$ if A is invertible.

iv — $\det(AB) = \det(A)\det(B)$

vi — $\det(A) = \det(A^T$

Pf of iv: $\det(AB) = \det(A)\det(B)$: if $\det(B) \neq 0$,

define $f(A) = \dfrac{\det(AB)}{\det(B)}$!

You can show $f(A)$ is ~~an~~ the alternating linear form with $f(I) = 1$.

$\Rightarrow f(A) = \det(A)$!

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vii - $\det(U) = U_{11} U_{22} \cdots U_{nn}$ for a UT matrix U

Pf of v: if A is invertible

$$1 = \det(I) = \det(AA^{-1}) = \det(A)\det(A^{-1}) \Rightarrow v': \det(A^{-1}) = \frac{1}{\det(A)}.$$

Recall: v is an eigenvector with eigenvalue λ, if
(corresponding to)

$$Av = \lambda v \qquad v \neq \emptyset.$$

$$\equiv (\lambda I - A)v = \emptyset \equiv v \in \mathcal{N}(\lambda I - A)$$

$$\equiv \text{nullity}(\lambda I - A) \neq \emptyset.$$

$$\equiv \det(\lambda I - A) = \emptyset.$$

Note that $\det(\lambda I - A)$ is a polynomial of degree n

i.e., $\det(\lambda I - A) = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$

e.g., $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $\det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda-1 & -2 \\ -3 & \lambda-4 \end{bmatrix}\right)$

$$= (\lambda-1)(\lambda-4) - 6 = \lambda^2 \underbrace{-5}_{c_{up}}\lambda \underbrace{-2}_{c_0}$$

$\det(\lambda I - A)$ is called the characteristic polynomial of A, we denote it by $c_A(\lambda) := \det(\lambda I - A)$.

$\Rightarrow$ eigenvalues are the zeros of the char. polynomial.

By the theorem of algebra, we know that this polynomial has exactly n roots, i.e., $c_A(\lambda) = (\lambda-\lambda_1)^{n_1} \cdots (\lambda-\lambda_k)^{n_k}$
(potentially repeated).

with $n_1 + n_2 + \cdots + n_k = n$, $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_k$ being distinct eigenvalues of A.

n_i is called the algebraic multiplicity of λ_i.

$$c_A(\lambda) = (\lambda - \lambda_1)^{n_1} \cdots (\lambda - \lambda_k)^{n_k}$$

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \qquad \det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda & -1 & -2 \\ 0 & \lambda & -3 \\ 0 & 0 & \lambda \end{bmatrix}\right)$$

Fact: eigenvectors $v_1, \ldots, v_k$ corresponding to distinct eigenvalues $\lambda_1, \ldots, \lambda_k$ are linearly independent.

Pf. why $(*)$ $v_k = \alpha_1 v_1 + \cdots + \alpha_{k-1} v_{k-1}$

$A \wedge$

$$\lambda_k v_k = \alpha_1 \lambda_1 v_1 + \cdots + \alpha_{k-1} \lambda_{k-1} v_{k-1}$$

$$\Rightarrow (\#) \quad v_k = \frac{\alpha_1 \lambda_1}{\lambda_k} v_1 + \cdots + \frac{\alpha_{k-1} \lambda_{k-1}}{\lambda_k} v_{k-1}$$

$*$ − $\#$: $\quad 0 = \alpha_1 \left(\frac{\lambda_1}{\lambda_k} - 1\right) v_1 + \cdots + \alpha_{k-1} \left(\frac{\lambda_{k-1}}{\lambda_k} - 1\right) v_{k-1}$