ECE 269- Lecture 8: Thursday Jan 30th

- Inner product spaces
  - Orthonormal Set
  - Gram-Schmidt Proc.
  - QR decomposition

A

- Announcements: "Prof. Kim's notes" is updated.

Correction to Lecture 7:

an inner product $\langle -, - \rangle$ is <u>antilinear</u> function of the second argument, i.e.,

conjugate linear

$$\langle u, \alpha w + \beta v \rangle = \alpha^* \langle u, w \rangle + \beta^* \langle u, v \rangle$$

NVP, IP spaces;

Def: For a NVS $(V, \|\cdot\|)$, we say that a sequence $\{v_k\}_{k\geq 1}$ is Cauchy if $\forall \varepsilon > 0; \exists N_\varepsilon; \forall m, n \geq N_\varepsilon;$

$$\|v_m - v_n\| < \varepsilon$$

The NVS that every Cauchy sequence is convergent is called a Banach space.
(complete)

Example: - $\mathbb{R}^n$ and any norm $\|\cdot\|$ is a Banach space.

- L_p spaces $(1 \leq p < \infty)$

(wrt $\|u\| := \langle u, u \rangle^{\frac{1}{2}}$)

An inner product space that is Banach is called a <u>Hilbert space</u>.

Examples: - $\mathbb{R}^n$; $\langle x, y \rangle := x^T y$

$L_2(I)$; $\langle f, g \rangle := \int_I f(x) g(x) dx$

Orthonormal sets and basis, ...:

Setup: V is an inner product space, with the IP $\langle -, - \rangle$.
and $\|v\| := \langle v, v \rangle^{\frac{1}{2}}$

We say a set S is an orthonormal set if:

(i) $\forall v \in S$, $\|v\| = 1$

(ii) $\forall u \neq v \in S$, $\langle u, v \rangle = 0$.

Properties of an orthormal set S:

(i) S is independent:

(ii) $u \in \text{span}(S)$: i.e., $u = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k$ for $v_i \in S$.

then $\alpha_i = \langle u, v_i \rangle$!

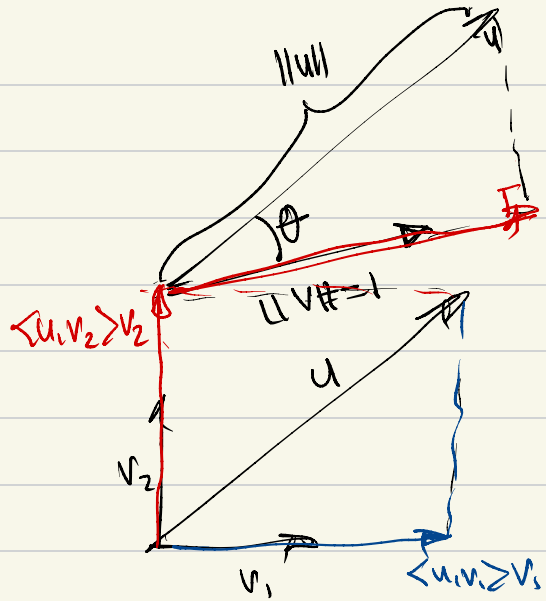
Pf: $u = \alpha_1 v_1 + \cdots + \alpha_k v_k$

investigate: $\langle u, v_i \rangle = \langle \alpha_1 v_1 + \cdots + \alpha_k v_k, v_i \rangle =$

linearity of IP

$$= \alpha_1 \cancel{\langle v_1, v_i \rangle}^{\emptyset} + \alpha_2 \cancel{\langle v_2, v_i \rangle}^{\emptyset} + \cdots + \alpha_i \underline{\langle v_i, v_i \rangle} + \cdots + \alpha_k \cancel{\langle v_k, v_i \rangle}^{\emptyset}$$

$\Rightarrow \alpha_i = \langle u, v_i \rangle$!



$\langle u, v\rangle = \|u\|\|v\| \cos\theta$

$\Rightarrow \langle u, v\rangle v$ when v is a unit length vector is simply projection of u along v!

Implication of (ii) is (i): if $u \in S$, then in the above decomposition $\alpha_i = \langle u, v_i\rangle = 0$ for all $v_i \neq u$.

Assume: $\dim < \infty$ and

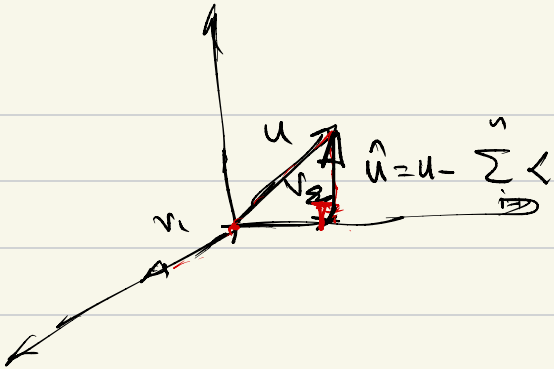
(iii) $u \notin \operatorname{span}(S)$: then $\hat{u} = u - \sum_{i=1}^{n} \langle u, v_i \rangle v_i \neq \underline{0}$

$S = \{v_1, \dots, v_n\}$

(a) would be orthogonal to all v_i's and hence, (b) $\hat{u} \perp v$ where $v \in \operatorname{span}(S)$.

Pf of (a): $\langle \hat{u}, v_j \rangle = \langle u, v_j \rangle - \underbrace{\sum_{i=1}^{n} \langle u, v_i \rangle \langle v_i, v_j \rangle}_{\langle u, v_j \rangle \times 1} = 0.$

(b): Note that $v \in \operatorname{span}(S) \Leftrightarrow v = \sum_{i=1}^{n} \alpha_i v_i$, then by linearity of the inner product, $\langle v, \hat{u} \rangle = 0$.



$$\hat{u} = u - \sum_{i=1}^{n} \langle u, v_i \rangle v_i$$

$\Rightarrow$ $\hat{S} = S \cup \left\{ \frac{\hat{u}}{\|\hat{u}\|} \right\}$ is a larger orthormal set.

Fact: Suppose that $T = \{u_1, \dots, u_n\}$ is an independent set in V.

Then there exists an orthonormal set $S = \{v_1, \dots, v_n\}$:

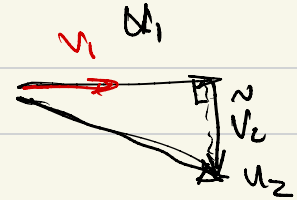
$$\operatorname{span}(\{v_1, \dots, v_k\}) = \operatorname{span}(\{u_1, \dots, u_k\}) \quad \text{for } 1 \leq k \leq n.$$

Pf: Constructive through a procedure called the Gram-Schmidt procedure:

Start with $v_1 := \dfrac{u_1}{\|u_1\|}$.

Let $\tilde{v}_2 = u_2 - \langle u_2, v_1 \rangle v_1$

$$v_2 = \frac{\tilde{v}_2}{\|\tilde{v}_2\|}$$



inductively: if for $k \geq 1$, we have the orthonormal set $\{v_1, \dots, v_k\}$

$\text{span}\{v_1, \dots, v_k\} = \text{span}\{u_1, \dots, u_k\}$;

let $\tilde{v}_{k+1} = u_{k+1} - \sum_{i=1}^{k} \langle u_{k+1}, v_i \rangle v_i$,

and we let $v_{k+1} = \dfrac{\tilde{v}_{k+1}}{\|\tilde{v}_{k+1}\|}$

Def: An orthonormal set that is a basis is called an orthonormal basis.

Let $U = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$ for an orthonormal basis $\{v_1, \ldots, v_n\}$ of $\mathbb{R}^n$.

Note that $\{v_1, \ldots, v_n\}$ is an orthonormal basis iff $U^T U = I$

Orthogonal matrices: Any $U \in \mathbb{R}^{n \times n}$; $U^T U = U U^T = I$ is called an orthogonal matrix.

Unitary matrices. $U \in \mathbb{C}^{n \times n}$; $U^* U = U U^* = I$.

For an orthogonal U; for $y = Ux$, we have $\|y\| = \|x\|$.

(Baby version of Parseval's equality)

QR decomposition: Suppose that $A \in \mathbb{R}^{m \times n}$ is a full rank matrix.

Performing the G-S procedure on $A = [u_1, \dots, u_n]$, we get

$$[u_1, \dots, u_n] = \underbrace{\begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}}_{Q} \times \underbrace{\begin{bmatrix} * & * & & * \\ 0 & * & & * \\ \vdots & 0 & \ddots & \\ 0 & 0 & & * \end{bmatrix}}_{R}$$

Definition: $[A]_{m \times n} = Q_{m \times m} R_{m \times n}$ is called a (full) QR decomposition if Q is an orthogonal matrix and R is an upper triangular matrix.

Fact: QR decomposition always exist

Pf: $A \in \mathbb{R}^{n \times k}$; $A = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_k \\ | & & | \end{bmatrix}$ with $v_1, \ldots, v_k \in \mathbb{R}^n$.

$r = \text{rank}(A)$

Let i_1 be the index of the first non-zero column,

$i_2 > i_1$, " " " " first column $\notin \text{span}(v_{i_1})$

$\vdots$

$i_r > i_{r-1}$ " " " " " " " $\notin \text{span}(v_{i_1}, \ldots, v_{i_{r-1}})$

$$A = \left[\emptyset \cdots v_{i_1} \cdots v_{i_2} \cdots v_{i_3} \cdots v_{i_r} \cdots \right]$$

Note that, $\{v_{i_1}, \ldots, v_{i_r}\}$ is an independent set, let $\{u_1, \ldots, u_r\}$ be an orthonormal set obtained throught the G-S procedure.

$$A = \hat{Q} \left[\emptyset \right]$$