# ECE 269- Lecture 17: Tuesday March 4th

- Eigenstructure of (Hermitian) symmetric matrices.
- Rayleigh Quotient.
- Courant - Fischer Thm and interlacing prop.

Announcements:

- Final format:
  - Only <u>one</u> page (can be double sided) cheat sheet allowed.
  - Need to bring a blue book
  - No electronics
  - Covers the whole course.
  - A bonus problem (of at least 15 points) if 80% or more fill out the instructor reviews
- New HW posted

Focus: Symmetric/Hermitian Matrices

Eigenstructures:

Fact: A Hermitian matrix $A$ has only real eigenvalues.

Pf: $Av = \lambda v \Rightarrow v^* A v = \lambda v^* v = \lambda \|v\|^2$

$$\| $$

$$(A^* v)^* v = (Av)^* v = (\lambda v)^* v = \bar{\lambda} v^* v = \bar{\lambda} \|v\|^2$$

$\Rightarrow \lambda \in \mathbb{R}.$

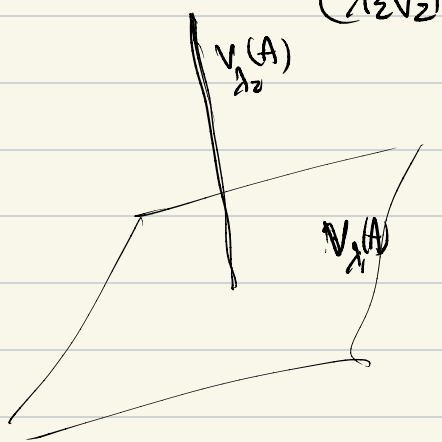
Fact: Eigenspaces of a Hermitian matrix $A$ are mutually orthogonal.

Pf: Let $\lambda_1 \neq \lambda_2$ are two eigenvalues of $A$ and $Av_1 = \lambda_1 v_1$, $Av_2 = \lambda_2 v_2$

$$v_2^* A v_1 = \lambda_1 v_2^* v_1 \qquad \lambda_1 \neq \lambda_2$$

$$\Downarrow \qquad\qquad \| \qquad \Longrightarrow \quad v_2^* v_1 = 0 \equiv \langle v_1, v_2 \rangle = 0.$$

$$(\lambda_2 v_2)^* v_1 = \lambda_2 v_2^* v_1$$



Fact 3: For any Hermitian matrix $A \in \mathbb{C}^{n\times n}$, $\exists$ a unitary (orthogonal) matrix $U \in \mathbb{C}^{n\times n}$ $(U^*U = I)$;

$$A = U \begin{pmatrix} \lambda_1 & & \emptyset \\ & \ddots & \\ \emptyset & & \lambda_n \end{pmatrix} U^*.$$

(for real matrices)

Pf: Consider the Jordan form of $A$

$$A = V J V^{-1}$$

Since $V$ is invertible, let $V = QU$ be a QR decomposition for $V \Rightarrow$ $A = QUJ(QU)^{-1} = Q\underbrace{UJU^{-1}}Q^T$.

$$U = \begin{bmatrix} * & & \\ & * & \\ & & \ddots & \\ \emptyset & & & * \end{bmatrix}$$

Also, $A = A^T = Q(UJU^{-1})^T Q^T \Rightarrow Q(UJU^{-1})^T Q^T = Q(UJU^{-1})Q^T$.

$Q(UJU^{-1})^T Q^T = Q(UJU^{-1})Q^T \Rightarrow (UJU^{-1})^T = UJU^{-1} \Rightarrow UJU^{-1} = D$ is a diagonal matrix $\Rightarrow$ $A = QDQ^T$ while $QQ^T = Q^TQ = I$. $\square$

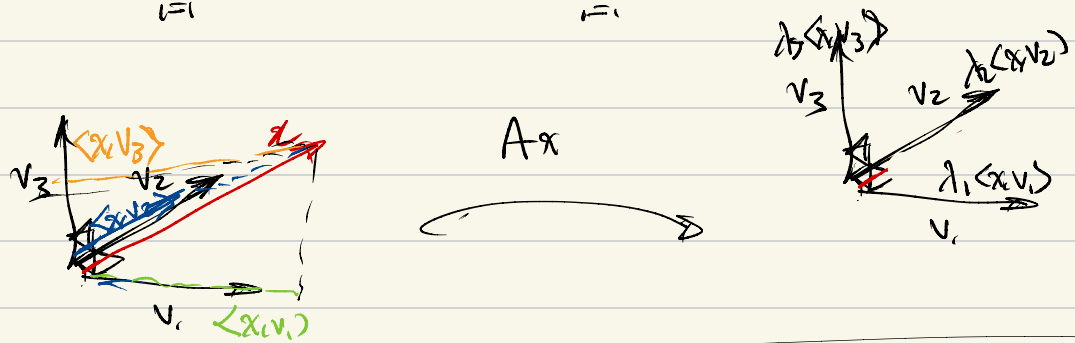
Fact 3 $\equiv$ For any symmetric/Hermitian matrix, $\exists$ an orthonormal basis from its eigenvectors.

Implication: Since $A = Q\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} Q^T$

$$= \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix} \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \begin{pmatrix} — & v_1^T & — \\ — & v_2^T & — \\ & \vdots & \\ — & v_n^T & — \end{pmatrix},$$

we have $A = \sum_{i=1}^{n} \lambda_i v_i v_i^T$. $\quad$ (#: $A = \sum_{i=1}^{n} \lambda_i v_i v_i^T$)

$$\Rightarrow Ax = \sum_{i=1}^{n} \lambda_i v_i v_i^T x = \sum_{i=1}^{n} \lambda_i \langle x, v_i \rangle v_i .$$

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Pf 2:

$$x^T A x = \sum_{i=1}^{n} \lambda_i \langle x, v_i \rangle^2 \leq \lambda_1 \sum_{i=1}^{n} \langle x, v_i \rangle^2 = \lambda_1 \|x\|^2$$

$(A \in \mathbb{R}^{n \times n})$

Consider $\left( \frac{x^T}{\|x\|} A \frac{x}{\|x\|} \right)$ $\qquad A = QDQ^T$

$$\max_{x \neq 0} \frac{x^T A x}{x^T x} = \max_{x : \|x\| = 1} x^T A x$$

$$= \max_{x : \|x\| = 1} \underbrace{x^T Q}_{y^T} D \underbrace{Q^T x}_{y} = \max_{y : \|y\| = 1} y^T \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} y$$

(Assume $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$)

$$= \max_{y : \|y\| = 1} \left( \sum \lambda_i y_i^2 \leq \lambda_1 \sum_{i=1}^{n} y_i^2 \right)$$

$$\leq \max_{y : \|y\| = 1} \lambda_1 = \lambda_1 \, !$$

Fact: $\lambda_1 = \max_{x \neq 0} \frac{x^T A x}{x^T x} = \max_{x : \|x\| = 1} x^T A x$.

largest eigenvalue of $A$

$R_A(x) := \frac{x^T A x}{x^T x}$ is called the Rayleigh Quotient of $A$.

Similarily you can show that $\lambda_n = \min_{x : x \neq 0} R_A(x) = \min_{x : x \neq 0} \frac{x^T A x}{x^T x}$.

Courant - Fischer Thrm: For a symmetric matrix $A$ with evs $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$

$$\lambda_d = \max_{V:\, \dim(V)=d} \min_{0 \neq x \in V} R_A(x) = \max_{V:\, \dim(V)=d} \min_{0 \neq x \in V} \frac{x^T A x}{x^T x}$$

$$\|$$

$$\min_{V:\, \dim(V)=n-d+1} \max_{0 \neq x \in V} \frac{x^T A x}{x^T x}.$$

This is acheived for $V = \text{span}(v_1, \ldots, v_d)$

This one is achieved for $V = \text{span}(v_d, \ldots, v_n)$.

Eigenvalue

Implication: Interlacing Property of symmetric matrices:

$$B = \begin{bmatrix} A_{n\times n} & c \\ c^T & \alpha \end{bmatrix}_{(n+1)\times(n+1)}$$

Suppose that eigenvalues of $B$ are $\beta_1 \leq \cdots \leq \beta_n \leq \beta_{n+1}$, and eigenvalues of $A$ are $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Then

$$\beta_1 \leq \lambda_1 \leq \beta_2 \leq \lambda_2 \cdots \leq \lambda_n \leq \beta_{n+1},$$

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Pf: Using C-F theorem with $V = \text{span}(v_1, \cdots, v_d)$ (max min), $V = \text{span}(v_d, \cdots, v_n)$ (min max)

where $A = (v_1 \cdots v_n) D \begin{pmatrix} - v_1^T - \\ \vdots \\ - v_n^T - \end{pmatrix}$.

Positive (semi) definite matrices: We say that $A \in \mathbb{C}^{n \times n}$ is p.d. (or p.s.d)

if $x^* A x > 0 \quad \forall \quad x \neq 0$

(p.s.d $x^* A x \geq 0 \quad \forall \quad x \neq 0$)

Example: Cov matrices

Control: Quadratic Lyapunov functions

Note that for e.vector $v$: $v^* A v = \lambda \|v\|^2$

$\Rightarrow$ A p.d. (p.s.d.) $\Rightarrow$ all eigenvalues are positive (non-negative).

Reverse implication is also true: if all eigenvalues are positive (non-neg)

$\Rightarrow$ A is pd (p.s.d.):

$$0 < \lambda_1 \leq \frac{x^* A x}{x^* x} \Rightarrow 0 < x^* A x.$$

Test 1: $A$ is p.d. (p.s.d) iff all eigenvalues are positive (non-negative).

Notation: $A \succeq B \equiv A - B$ is psd

$A \succeq 0 \equiv A$ is p.s.d.

$A \succ B \equiv A - B$ is p.d.

$A \succ 0 \equiv A$ is p.d.