ECE 269: Linear Algebra and Applications, Sample Final Solution
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1. Prove or disprove each of the following statements (to disprove you need to provide an example and explain why the example disproves the statement):

   (a) The matrix $A = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$ is diagonalizable.

   (b) For an $n \times n$ real-valued matrix A, $\max_{x \neq 0} \frac{x^T A x}{x^T x} = \lambda_{max}(A)$ where $\lambda_{max}(A)$ is the eigenvalue with the maximum modulus (absolute value).

   (c) For a real-valued $n \times n$ and full-rank matrix A, $\text{span}(\{A^k \mid k \geq 1\})$ has dimension n^2. Here, $\{A^k \mid k \geq 1\}$ is the set of all powers of A.

   (d) If the $n \times n$ matrix A is a positive-definite matrix then the diagonal elements $a_{ii} > 0$ for all $i = 1, \ldots, n$.

   Solution:

   (a) **False:** Compute the characteristic polynomial:

   $$\det(A - \lambda I) = \det \begin{pmatrix} 2-\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = (2-\lambda)(-\lambda) + 1 = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2.$$

   The eigenvalue is $\lambda = 1$ (with algebraic multiplicity 2). The eigenspace is found by solving $(A - I)\mathbf{x} = \mathbf{0}$:

   $$A - I = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix},$$

   which reduces to $x - y = 0$. The geometric multiplicity is 1, distinct from the algebraic multiplicity. Since the geometric multiplicity < 2, A is defective **and not diagonalizable**.

   (b) **False:** Let $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Its eigenvalues are $\pm i$, so $\lambda_{\max}(A) = 1$ (modulus). However, for any real vector $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$,

   $$\mathbf{x}^T A \mathbf{x} = x_1 x_2(-1) + x_2 x_1(1) = 0.$$

   Thus, $\max_{x \neq 0} \frac{x^T A x}{x^T x} = 0 \neq 1$, disproving the statement.

   (c) **False:** As a consequence of the Cayley-Hamilton theorem, we know that the rank of this subspace is at most n and hence, it is false.

   (d) **True:** For the i-th standard basis vector $\mathbf{e}_i$, positive-definiteness implies:

   $$\mathbf{e}_i^T A \mathbf{e}_i = a_{ii} > 0.$$

   This holds for all i, confirming that all diagonal elements are positive.

2. Let $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & -1 \end{pmatrix}$.

   (a) Find a QR decomposition for A.

   (b) Find a PLU decomposition for A.

   (c) Find the (compact) SVD for A.

   (d) Find a full SVD for A.

(e) Determine whether a solution to $Ax = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ exists for an $x \in \mathbb{R}^3$. If so, find the least norm solution to this problem. If the solution does not exist, find the least square solution to the problem.

(f) Determine the spectral norm and Frobenius norm of A.

Solution:

(a) Let $A = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}$. By Gram-Schmidt orthogonalization,

$$\begin{aligned}
q_1 &= \frac{a_1}{\|a_1\|} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \\
\tilde{q_2} &= a_2 - (a_2' q_1) q_1 = a_2 - \frac{3}{\sqrt{2}} q_1 = \frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \\
q_2 &= \frac{\tilde{q_2}}{\|\tilde{q_2}\|} = \sqrt{2}\tilde{q_2} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \\
\tilde{q_3} &= a_3 - (a_3' q_1) q_1 - (a_3' q_2) q_2 = a_3 - 0 q_1 - \sqrt{2} q_2 = 0.
\end{aligned}$$

Therefore, QR decomposition of A is

$$A = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} \sqrt{2} & 3/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & \sqrt{2} \end{pmatrix}.$$

(b) By Gaussian Elimination,

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & -2 \end{pmatrix}.$$

Therefore, LU decomposition of A is

$$\begin{aligned}
A &= \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & -2 \end{pmatrix}. \\
&= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & -2 \end{pmatrix}.
\end{aligned}$$

(c) We have

$$AA^* = \begin{pmatrix} 6 & 2 \\ 2 & 3 \end{pmatrix}.$$

By eigenvalue decomposition,

$$AA^* = \begin{pmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{pmatrix} \begin{pmatrix} 7 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{pmatrix}.$$

Let Σ be $\begin{pmatrix} \sqrt{7} & 0 \\ 0 & \sqrt{2} \end{pmatrix}$ and U be $\begin{pmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{pmatrix}$. Also, let

$$V^* = \Sigma^{-1} U^* A = \begin{pmatrix} 3/\sqrt{35} & 5/\sqrt{35} & 1/\sqrt{35} \\ -1/\sqrt{10} & 0 & 3/\sqrt{10} \end{pmatrix}.$$

Then, $A = U\Sigma V^*$ is the compact SVD of A.

(d) We will find a vector $\tilde{v_3}$ orthogonal to column vectors of $V = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$. For example, $\tilde{v_3} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$.

By normalizing this, $v_3 = \frac{1}{\sqrt{14}} \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$. Therefore, the full SVD of A is

$$A = \begin{pmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{pmatrix} \begin{pmatrix} \sqrt{7} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} 3/\sqrt{35} & 5/\sqrt{35} & 1/\sqrt{35} \\ -1/\sqrt{10} & 0 & 3/\sqrt{10} \\ 3/\sqrt{14} & -2/\sqrt{14} & 1/\sqrt{14} \end{pmatrix}.$$

(e) Since A is full row-rank, there exists a solution to the given system of linear equations. To find that, we have:

$$\begin{aligned} A^\dagger &= A^*(AA^*)^{-1} \\ &= A^* \begin{pmatrix} 6 & 2 \\ 2 & 3 \end{pmatrix}^{-1} \\ &= A^* \frac{1}{14} \begin{pmatrix} 3 & -2 \\ -2 & 6 \end{pmatrix} \\ &= \frac{1}{14} \begin{pmatrix} 1 & 4 \\ 4 & 2 \\ 5 & -8 \end{pmatrix}. \end{aligned}$$

The least norm solution $\hat{x}$ is

$$\hat{x} = A^\dagger \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{14} \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix}.$$

(f) The singular values of A are $\sigma_1 = \sqrt{7}, \sigma_2 = \sqrt{2}$. Therefore, the spectral norm is $\sigma_1 = \sqrt{7}$, Frobenius norm is $\sqrt{\sigma_1^2 + \sigma_2^2} = \sqrt{7+2} = 3$.

3. Let $A, B \in \mathbb{R}^{m \times n}$ be matrices such that $R(B) \perp R(A)$, i.e., any vector in the range space of B is orthogonal to any vector in the range space of A.

(a) Show that $\|A+B\|_F^2 = \|A\|_F^2 + \|B\|_F^2$, where $\|\cdot\|_F$ is the Frobenius-norm.

(b) Provide an example to show that part (a) does not hold for the spectral norm.

Solution:

(a) Recall that $\|A\|_F^2 = \text{trace}(A'A)$. Also notice that $R(B) \perp R(A)$ implies $A'B = B'A = 0$. Therefore, using linearity of trace:
$\|A+B\|_F^2 = \text{trace}((A+B)'(A+B)) = \text{trace}(A'A + A'B + B'A + B'B) = \text{trace}(A'A + B'B) = \text{trace}(A'A) + \text{trace}(B'B) = \|A\|_F^2 + \|B\|_F^2$.

(b) Take $A = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}$. Notice that $R(A) \perp R(B)$. It is not hard to see that $\|A\|^2 = 4, \|B\|^2 = 9, \|A+B\|^2 = 9$, therefore $\|A\|^2 + \|B\|^2 \neq \|A+B\|^2$.

4. For all $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \in \mathbb{R}^3$, define $\langle x, y \rangle = x_1 y_1 + \frac{1}{2} x_1 y_3 + \frac{1}{2} x_3 y_1 + 2 x_2 y_2 + x_3 y_3$.

(a) Prove that $\langle x, y \rangle$ is a valid inner product (over $\mathbb{R}$).

(b) Find an orthonormal basis for $\mathbb{R}^3$ using the above inner product.

5. (a) Let $A = \begin{pmatrix} 5 & -6 & 3 \\ -6 & 4 & -6 \\ 3 & -6 & -4 \end{pmatrix}$. Is A diagonalizable?

(b) Let B be a complex-valued square matrix with $rank(B) = 2$. The characteristic polynomial of B is given by $c_B(\lambda) = (\lambda^2 + 4)\lambda^3$.

i. Find the dimension of B and all its eigenvalues.

ii. Is B diagonalizable? Explain.

iii. Show that $B + I$ is invertible and find the eigenvalues of $(B+I)^{-1}$.

Solution:

(a) Notice that $A = A^H$, meaning A is Hermitian, therefore by a theorem taught in class, it is diagonalizable. Note: Attempting to solve this by finding the eigenvalues directly is difficult, as it would require to find the roots of a cubic polynomial. If you chose to take this route, you must show all roots are different to receive full credit. You can do so either explicitly by solving the equation or by showing the function is zero at three different points using various methods.

(b) i. The degree of the characteristic polynomial is 5, therefore B is a square matrix of dimension 5. The eigenvalues are the roots of the characteristic polynomial, meaning the eigenvalues are $0, \pm 2i$.

ii. Note that for $\pm 2i$, the algebraic multiplicity is 1. Recall that for each eigenvalue, $1 \leq$ geometric multiplicty $\leq$ algebraic multiplicty, therefore these eigenvalues have geometric multiplicity 1. Using rank nullity theorem: $nullity(B) + rank(B) = 5 \implies nullity(B) = 3$. By definition of nullity $nullity(B) = dim(N(B)) = dim(N(B - 0I)) = 3$, meaning the geometric multiplicty of eigenvalue 0 is 3. Since for each eigenvalue, the geometric multiplicty is equal to the algebraic multiplicity, B is diagonalizable by a theorem taught in class.

iii. Since B is diagonalizable, we can write $B = PDP^{-1}$, where $D = diag(0, 0, 0, 2i, -2i)$, where $diag()$ means it is a diagonal matrix with the given elements on its diagonal. We can therefore write: $B + I = PDP^{-1} + PIP^{-1} = P(D + I)P^{-1}$ where $D + I = diag(1, 1, 1, 1 + 2i, 1 - 2i)$. Since $B + I$ does not have 0 as an eigenvalue, it is invertible by a theorem taught in class. Using the result of Q4 (e) in Homework 5, the eigenvalues of $(B + I)^{-1}$ are therefore 1 and $\frac{1}{1 \pm 2i}$.