ECE 269- Lecture 16: Thursday Feb 27th

- Jordan Form Decom
- Cayley-Hamilton Thm
- Symmetric / Hermitian matrices

Fact: $\forall A \in \mathbb{R}^{n\times n}$; $\exists$ an invertible V;

$$AV = V \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_k \end{bmatrix}$$

$$\underbrace{}_{n_1} \quad \underbrace{}_{n_2} \quad \underbrace{}_{n_k}$$

$$J_i = \begin{bmatrix} \lambda_i & 1 & & \emptyset \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ \emptyset & & & \lambda_i \end{bmatrix}$$

Jordan blocks with corresponding to $\lambda_i = g_i$

appearance of $\lambda_i = b_i$

$$AV = V \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_k \end{bmatrix}$$

$$A \left[\overbrace{V_1}^{n_1} \middle| \overbrace{V_2}^{n_2} \middle| \cdots \middle| \overbrace{V_k}^{n_k} \right] = \left[V_1 \middle| V_2 \middle| \cdots \middle| V_k \right] \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_k \end{bmatrix}$$

$$\Rightarrow \begin{cases} AV_1 = V_1 J_1 \\ AV_2 = V_2 J_2 \\ \vdots \\ AV_k = V_k J_k \end{cases}$$

Let's $\quad AV_1 = V_1 J = V_1 \begin{bmatrix} \lambda_1 & 1 & 0 & \cdots & 0 \\ 0 & \lambda_1 & 1 & 0 & \cdots \\ & & \ddots & \ddots & 0 \\ & & & \lambda_1 & 1 \\ & & & & \lambda_1 \end{bmatrix} \quad V_1 = \begin{bmatrix} | & | & & | \\ v_{11} & v_{12} & \cdots & v_{1n_1} \\ | & | & & | \end{bmatrix}$

$$\Leftrightarrow \begin{cases} Av_{11} = \lambda_1 v_{11} \Leftrightarrow (A-\lambda_1 I)v_{11} = 0 & (v_{11} \text{ is an eigenvector for } \lambda_1) \\ Av_{12} = v_{11} + \lambda v_{12} \Leftrightarrow (A-\lambda_1 I)v_{12} = v_{11} \\ \vdots \\ A_{1n_1} = v_{1(n_1-1)} + \lambda v_{1n_1} \Leftrightarrow (A-\lambda_1 I)v_{1n_1} = v_{1(n_1-1)} \end{cases}$$

To find the columns of V_1: (1) find an eigenvector v_{11}

(2) $v_{1i} \in R(A-\lambda I)$ → yes: $v_{1i} = (A-\lambda I)v_{1(i+1)}$

→ No: i is the last column of V_1

v_{ij}'s are called generalized eigenvectors.

$$A = \begin{bmatrix} 2 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix}, \quad c_A(\lambda) = \det(\lambda I - A) = (\lambda - 2)^3$$

$$\lambda_1 = 2, \quad b_1 = 3$$

Solving eigenvectors of $\lambda_1 = 2$:

$$\begin{bmatrix} 2 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix}$$

$$\Downarrow$$

$$\begin{cases} 2x + 2y + 3z = 2x \\ 2y + 4z = 2y \\ \underline{2z = 2z} \end{cases}$$

$\rightarrow y = \emptyset$

$\rightarrow z = \emptyset$

$$V_A(\lambda_1) = \left\{ \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} \middle| \; x \in \mathbb{R} \right\} \Rightarrow g_1 = 1$$

$$A = \begin{bmatrix} 2 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix},$$

$\Rightarrow$ since $g_1 = 1$, we have one Jordan block $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

$$A \begin{bmatrix} | & | & | \\ 0 & v_2 & v_3 \\ 0 & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ 0 & v_2 & v_3 \\ 0 & | & | \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$(A - \lambda I) v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$A = \begin{bmatrix} 2 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix},$$

$$\underline{(A-\lambda I)} v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 2 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \longrightarrow z = 0 \Rightarrow 2y = 1$$

$$y = \frac{1}{2}, \quad x = 0$$

$$v_2 = \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \end{pmatrix}$$

$$(A-\lambda I) v_3 = v_2 = \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \end{pmatrix}$$

Left-eigenvector: $v^T A = \lambda v^T \equiv A^T v = \lambda v \equiv (A^T - \lambda I) v = \emptyset$

If A has EVD:

$$A = V \Lambda V^{-1}$$

$$\equiv V^{-1} A = \Lambda V^{-1}$$

$$M = V^{-1} = \begin{bmatrix} - & u_1 & - \\ & & \\ - & u_n & - \end{bmatrix}$$

$$\equiv \begin{bmatrix} - & u_1 & - \\ - & u_2 & - \\ & & \\ - & u_n & - \end{bmatrix} A = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} - & u_1 & - \\ - & u_2 & - \\ & \vdots & \\ - & u_n & - \end{bmatrix}$$

$$\equiv \qquad u_i A = \lambda_i u_i \qquad \forall i$$

$\Rightarrow$ rows of V^{-1} are left eigenvectors of A.

Cayley-Hamilton Thrm: Note that $\dim(\mathbb{R}^{n\times n}) = n^2$.

Q: For a matrix A, can $\operatorname{span}\{A^k \mid k \geq 0\} = \mathbb{R}^{n\times n}$?

A: No!

C-H Thrm: For any $A \in \mathbb{R}^{n\times n}$, if $c_A(\lambda) = \det(\lambda I - A) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_1\lambda + \alpha_0$,

then $$A^n + \alpha_{n-1}A^{n-1} + \cdots + \alpha_1 A + \alpha_0 I = \underline{0}\ !$$

Implication: $A^n = (-\alpha_{n-1}A^{n-1} - \cdots - \alpha_0 I)$

Implication 2: $\dim(\operatorname{span}\{A^k \mid k \geq 0\}) \leq n$.

Implication: If A is invertible: $A^{-1} = \beta_0 I + \cdots + \beta_{n-1}A^{n-1}$

Pf (the case of A diagonalizable)

$$A = V \Lambda V^{-1}, \quad \Lambda \to \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}, \quad \Lambda \to \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

$$A^n + \alpha_{n-1} A^{n-1} + \cdots + \alpha_1 A + \alpha_0 I = V\left(\Lambda^n + \alpha_{n-1}\Lambda^{n-1} + \cdots + \alpha_1 \Lambda + \alpha_0 I\right)V^{-1}$$

$$= V \underbrace{\begin{pmatrix} C_A(\lambda_1) & & \emptyset \\ & \ddots & \\ \emptyset & & C_A(\lambda_n) \end{pmatrix}}_{\emptyset} V^{-1} = \emptyset.$$

Implication 4: $\forall$ any analytic function $f(x) = \sum_{k=0}^{\infty} a_k x^k$,

$$f(A) := \alpha_0 I + \alpha_1 A + \cdots + \alpha_k A^k + \cdots$$

(Example: $f(x) = e^x$)

$$\therefore \quad f(At) = \alpha_0(t) I + \alpha_1(t) A + \cdots + \alpha_{n-1}(t) A^{n-1}$$

Symmetric / Hermitian Matrices:

Symmetric: $A^T = A$ (real valued matrice)

Hermitian: $A^H = A$ (complex valued matrice)

Quadratic form on $\mathbb{R}^n$; $f(x,y)$ bilinear form

if $\begin{pmatrix} f(\cdot, y) \\ f(x, \cdot) \end{pmatrix}$ are linear

$q(x) = f(x,x)$ is called quadratic.

$\equiv$ $q(x) = x^T A x = \sum_{ij} A_{ij} x_i x_j$ for some $A \in \mathbb{R}^{n \times n}$

Note that $q(x) = x^T A x = (x^T A x)^T$

$$\Rightarrow q(x) = x^T \left(\frac{A + A^T}{2}\right) x$$

Note that $\frac{A + A^T}{2}$ is symmetric.

Any quadratic form $q(x)$ has a unique represention

$$q(x) = x^T S x$$

where S is a symmetric matrix.

Q: What can we say about the eigenvalues and eigenvectors of Symmetric (Hermitian) matrices?

A: eigenvalues $\equiv$ all real

Any symmetric/Hermitian matrix can be written as

$$A = V \Lambda V^T$$

with $VV^T = I$.