Midterm-A

ECE 269: Linear Algebra and Applications
Midterm-A Solution
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1. For each of the following statements, determine whether it is true or false. Explain your answer. Correct answers without explanation carry no point.

   a. For $n \geq 1$, $\mathbb{R}^n$ is a vector space over the field $\mathbb{Q}$ with the standard vector addition and scalar multiplication.

   b. If for two matrices $A, B \in \mathbb{R}^{n\times m}$, we have $\mathcal{R}(A) = \mathcal{R}(B)$, then $\mathcal{N}(A) = \mathcal{N}(B)$.

   c. For any $A \in \mathbb{R}^{n\times n}$, $x \in \mathcal{R}(A)$ iff $x \in \mathcal{N}(A')$.

   d. The vector space $\mathbb{Q}^2$ (2-dimensional rational numbers) over $\mathbb{Q}$ (with the usual addition and scalar multiplication) is a Banach space with the Euclidean norm $\|x - y\| = (\sum_{i=1}^{n} |x_i - y_i|^2)^{1/2}$.

   Solution:

   a. True. For any $v_1, v_2 \in \mathbb{R}^n$ and $\alpha \in \mathbb{Q}$, we have

   $$\alpha v_1 + v_2 \in \mathbb{R}^n.$$

   Therefore, $\mathbb{R}^n$ over the field $\mathbb{Q}$ is closed under the standard addition and scalar multiplication. Besides, the standard vector addition and scalar multiplication satisfy all the vector axioms (associativity, commutativity, identity, compatibility, distributivity). Therefore, this is a vector space.

   b. False. Let $A = [1\ 2]$ and $B = [1\ 1]$. Then $\mathcal{R}(A) = \mathcal{R}(B) = \mathbb{R}$, but $\mathcal{N}(A)$ is the span of $[2\ -1]^T$, while $\mathcal{N}(B)$ is the span of $[1\ -1]^T$.

   c. False. Let $A = 1$. Then $\mathcal{R}(A) = \mathbb{R}$, but $\mathcal{N}(A') = \{0\}$.

   d. False. Let $(a_n)_{n\in\mathbb{N}} \subset \mathbb{Q}^2$ be a sequence defined by

   $$a_n = (0, q_n),$$

   where q_n is the first n digits of π, i.e., $q_1 = 3$, $q_2 = 3.1$, $q_3 = 3.14$ and so on. Then we know

   $$\lim_{m,n\to\infty} \|a_m - a_n\| = \lim_{m,n\to\infty} |q_m - q_n| = 0.$$

   Therefore, (a_n) is a Cauchy sequence. However, this sequence converges (in $\mathbb{R}^2$) to $(0, \pi)$, which is not an element of $\mathbb{Q}^2$. So, $\mathbb{Q}^2$ over $\mathbb{Q}$ is not a Banach space.

2. Consider the following matrix

$$A = \begin{pmatrix} b & a & 1 \\ a & b & -1 \\ b & a & 1 \end{pmatrix},$$

where a, b is the parameters from your PID (see the cover page).

(a) Find $\mathcal{R}(A)$.

(b) Find $\mathcal{N}(A)$.

(c) Find the reduced QR decomposition for A.

Solution: Note that $a, b \geq 0$ and either $a > 0$ or $b > 0$ for all students.

(a) *For general* $a, b \geq 0$: Let $v_1, v_2, v_3 \in \mathbb{R}^3$ be the first, second, and the thirs column of A, respectively. Note that the first and the last rows are the same and non-zero, so $1 \leq rank(A) \leq 2$. In addition, since $a, b \geq 0$, the second row and the first row are independent unless $a, b = 0$. Therefore,

$$rank(A) = \begin{cases} 1 & a = b = 0 \\ 2 & \text{otherwise.} \end{cases}$$

If $a = b = 0$, the the first two columns would be zero and hence, $\mathcal{R}(A) = \text{span}(v_3) = \{\alpha(1, -1, 1)^T \mid \alpha \in \mathbb{R}\}$.

When either of the parameters is non-zero, the first column and the 3rd column would be independent and

$$\mathcal{R}(A) = \text{span}(\{v_1, v_3\}) = \{(\alpha, \beta, \alpha) \mid \alpha, \beta \in \mathbb{R}\}.$$

(b) For $\mathcal{N}(A)$, we have $\mathcal{N}(A) = \{(x, y, z)' \mid bx + ay + z = 0, ax + by - z = 0\}$. This system of equation holds iff $(a + b)(x + y) = 0$ and $z = ax + by$. If

$$a+b \neq 0 : x = -y, z = (a-b)x, \mathcal{N}(A) = \text{span}\{(1, -1, a-b)'\} = \{(x, -x, (a-b)x) \mid x \in \mathbb{R}\}.$$

If $a+b = 0$: the above holds for all $x, y \in \mathbb{R}$. Therefore, $\mathcal{N}(A) = \text{span}\{(1, 0, a)', (0, 1, b)'\}$. Note that since $a, b \geq 0$, this latter condition only holds when $a = b = 0$, in which case clearly $\mathcal{N}(A) = \text{span}\{(1, 0, 0)', (0, 1, 0)'\}$.

(c) again, when either parameter is nonzero: $v_1 \neq 0$ and hence,

$$q_1 = \frac{v_1}{\|v_1\|} \Rightarrow q_1 = \begin{pmatrix} \frac{b}{\sqrt{a^2+2b^2}} \\ \frac{a}{\sqrt{a^2+2b^2}} \\ \frac{b}{\sqrt{a^2+2b^2}} \end{pmatrix}$$

Now, unless $a = b$, $\{v_1, v_2\}$ would be independent. So, if $a \neq b$:

$$\tilde{q}_2 = v_2 - \langle v_2, q_1 \rangle q_1 = \frac{1}{a^2 + 2b^2} \begin{pmatrix} a^3 - ab^2 \\ 2b^3 - 2a^2 b \\ a^3 - ab^2 \end{pmatrix} = \frac{a^2 - b^2}{a^2 + 2b^2} \begin{pmatrix} a \\ -2b \\ a \end{pmatrix} \Rightarrow$$

$$q_2 = \frac{\tilde{q}_2}{\|\tilde{q}_2\|} = \frac{1}{\sqrt{2(a^2 + 2b^2)}} \begin{pmatrix} a \\ -2b \\ a \end{pmatrix}.$$

Therefore,

$$Q = \begin{pmatrix} \frac{b}{\sqrt{a^2+2b^2}} & \frac{a}{\sqrt{2(a^2+2b^2)}} \\ \frac{a}{\sqrt{a^2+2b^2}} & \frac{-2b}{\sqrt{2(a^2+2b^2)}} \\ \frac{b}{\sqrt{a^2+2b^2}} & \frac{a}{\sqrt{2(a^2+2b^2)}} \end{pmatrix}.$$

And $R = Q^T A = \begin{pmatrix} \sqrt{a^2+2b^2} & \frac{3ab}{\sqrt{a^2+2b^2}} & \frac{2b-a}{\sqrt{a^2+2b^2}} \\ 0 & \frac{\sqrt{2}(a^2-b^2)}{\sqrt{a^2+2b^2}} & \frac{\sqrt{2}(a+b)}{\sqrt{a^2+2b^2}} \end{pmatrix}$.

If $a = b$, then we get the same Q matrix (why?) with $a = b = 1$, i.e.,

$$Q = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

And

$$R = Q^T A = \begin{pmatrix} \frac{1}{\sqrt{3}}a & \frac{1}{\sqrt{3}}a & \frac{2}{\sqrt{3}} \\ 0 & 0 & \frac{\sqrt{2}}{\sqrt{3}} \end{pmatrix}.$$

3. Let $A \in \mathbb{F}^{m\times n}$ for an arbitrary field $\mathbb{F}$. Prove that the following statements are equivalent.

   (a) $\mathcal{N}(A) = \{0\}$.

   (b) $\mathcal{R}(A') = \mathbb{F}^n$.

   (c) The columns of A are independent.

   (d) A is tall (i.e., $n \leq m$) and full-rank (i.e., $\text{rank}(A) = \min(m, n) = n$).

   Solution: *(Copied from HW2-Solution)* We will show the chain of equivalences (a) $\implies$ (b) $\implies$ (c) $\implies$ (d) $\implies$ (a).

   (a) $\implies$ (b): By the rank–nullity theorem, we have $\dim(\mathcal{N}(A)) + \text{rank}(A) = n$, which implies $\text{rank}(A) = n$ (since $\dim(\mathcal{N}(A)) = 0$). Since $\text{rank}(A) = \text{rank}(A')$, we then have $\text{rank}(A') = n$. Since rank is equivalent to the dimension of the column space, the dimension of the column space of A' is n. Because each column vector in A' is of length n, this means that $\mathcal{R}(A') = \mathbb{F}^n$.

   (b) $\implies$ (c): Since A' is onto, $\text{rank}(A') = \dim(\mathcal{R}(A')) = n$. Because $\text{rank}(A) = \text{rank}(A') = n$, the $\dim(\mathcal{R}(A)) = n$. Note now that A has n column vectors and for them to span a space of dimension n, all of these column vectors have to be independent.

   (c) $\implies$ (d): If the columns of A are independent, since each column vector is of length m, there cannot be more than m of them (since more than m vectors of length m necessarily need to be dependent). Thus $n \leq m$. Since n independent vectors span a space of dimension n, we know that $\dim(\mathcal{R}(A)) = n \implies \text{rank}(A) = n = \min(m, n)$.

   (d) $\implies$ (a): By the rank–nullity theorem, $\text{rank}(A) + \dim(\mathcal{N}(A)) = n$. Since $\text{rank}(A) = n$, we have $\dim(\mathcal{N}(A)) = 0$, which implies that $\mathcal{N}(A) = \{0\}$.

4. Let $K \in \mathbb{R}^{n\times n}$ be an invertible matrix.

   (a) Show that the mapping $\langle x, y\rangle_K := (Kx)^T(Ky) = x^TK^TKy$ is an inner-product for the vector space $\mathbb{R}^n$ (over $\mathbb{R}$).

   (b) Let $e_1 = \begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}, e_2 = \begin{pmatrix}0\\1\\\vdots\\0\end{pmatrix}, \ldots, e_n = \begin{pmatrix}0\\0\\\vdots\\1\end{pmatrix}$, be the standard basis of $\mathbb{R}^n$. Show that $\{e_1, \ldots, e_n\}$ is an orthonormal set in $(\mathbb{R}^n, \langle x, y\rangle_K)$ if and only if K is an orthogonal matrix.

   (c) Let $Q = \begin{pmatrix}1 & a & 1\\0 & 1 & a\\0 & 0 & 1\end{pmatrix}$ (where a is the last digit of your PID). Find an orthonormal basis $\{b_1, b_2, b_3\}$ in $(\mathbb{R}^3, \langle x, y\rangle_Q)$. Hint: you can use the Gram-Schmidt procedure on e_1, e_2, e_3.

Solution:

(a) i. $\langle \alpha x + z, y\rangle_K = (K(\alpha x + z))^T(Ky) = \alpha(Kx)^TKy + (Kz)^TKy = \alpha\langle x, y\rangle_K + \langle z, y\rangle_K$

ii. $\langle x, y\rangle_K = x^TK^TKy = (y^TK^TKx)^T = \overline{\langle y, x\rangle}_K$

iii. Since K is invertible, $rank(K) = n \Rightarrow dim(\mathcal{N}(K)) = 0 \Rightarrow Kx = 0$ only when $x = 0$ and vice-versa, therefore we have $(Kx)^TKx > 0$ for all $x \neq 0$, and $x^TK^TKx = 0$ if and only if $x = 0$.

(b) $\Rightarrow$ If K is orthogonal, then $K^TK = I$ which means $\langle x, y\rangle_K = x^TK^TKy = x^Ty = \langle x, y\rangle$ (standard inner product). Therefore, $\{e_1, \ldots, e_n\}$ are going to be orthogonal in $(\mathbb{R}^n, \langle x, y\rangle_K)$ as $\langle e_i, e_j\rangle = 0 \Rightarrow \langle e_i, e_j\rangle_k = 0$ s.t. $i \neq j$ and $\langle e_i, e_i\rangle = 1 \Rightarrow \langle e_i, e_i\rangle_k = 1$.

$\Leftarrow$ If $\langle e_i, e_j\rangle_k = 0$ and $\langle e_i, e_i\rangle_k = 1$ s.t. $i \neq j$, then $e_i^TK^TKe_j = 0$ as $\{e_1, \ldots, e_n\}$ are orthonormal in $(\mathbb{R}^n, \langle x, y\rangle)$, $e_i^Te_j = 0 \Rightarrow e_i^TIe_j = 0$. Similarly, $e_i^TK^TKe_i = 1$ and $e_i^TIe_i = 1$ This is only possible if $K^TK = I \Rightarrow K^T = K^{-1}$, which means K is orthogonal.

(c)

$$Q^TQ = \begin{bmatrix}1 & a & 1\\a & 1+a^2 & 2a\\1 & 2a & 2+a^2\end{bmatrix}.$$

First vector b_1: Set $b_1 = e_1$, normalize it:

$$b_1 = \frac{e_1}{\|e_1\|_Q},\ \|e_1\|_Q = \sqrt{\langle e_1, e_1\rangle_Q} = 1$$

$$b_1 = e_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}.$$

Second vector b_2: Start with e_2 and subtract its projection onto b_1:

$$b_2' = e_2 - \frac{\langle e_2, b_1\rangle_Q}{\langle b_1, b_1\rangle_Q}b_1.$$

Compute $\langle e_2, b_1\rangle_Q$:

$$\langle e_2, b_1\rangle_Q = e_2^T Q^T Q e_1 = \begin{bmatrix}0 & 1 & 0\end{bmatrix} \begin{bmatrix}1 & a & 1\\ a & 1+a^2 & 2a\\ 1 & 2a & 2+a^2\end{bmatrix} \begin{bmatrix}1\\0\\0\end{bmatrix} = a$$

$$b_2' = e_2 - ab_1.$$

Normalize b_2:

$$b_2 = \frac{b_2'}{\|b_2'\|_Q},\ \|b_2'\|_Q = \sqrt{\begin{bmatrix}-a & 1 & 0\end{bmatrix} Q^T Q \begin{bmatrix}-a & 1 & 0\end{bmatrix}^T} = 1.$$

$$b_2 = \begin{bmatrix}-a\\1\\0\end{bmatrix}.$$

Third vector b_3: Start with e_3 and subtract projections onto b_1 and b_2:

$$b_3' = e_3 - \frac{\langle e_3, b_1\rangle_Q}{\langle b_1, b_1\rangle_Q} b_1 - \frac{\langle e_3, b_2\rangle_Q}{\langle b_2, b_2\rangle_Q} b_2.$$

Compute $\langle e_3, b_1\rangle_Q$ & $\langle e_3, b_2\rangle_Q$:

$$\langle e_3, b_1\rangle_Q = e_3^T Q^T Q e_1 = \begin{bmatrix}0 & 0 & 1\end{bmatrix} \begin{bmatrix}1 & a & 1\\ a & 1+a^2 & 2a\\ 1 & 2a & 2+a^2\end{bmatrix} \begin{bmatrix}1\\0\\0\end{bmatrix} = 1$$

$$\langle e_3, b_2\rangle_Q = e_3^T Q^T Q b_2 = \begin{bmatrix}0 & 0 & 1\end{bmatrix} \begin{bmatrix}1 & a & 1\\ a & 1+a^2 & 2a\\ 1 & 2a & 2+a^2\end{bmatrix} \begin{bmatrix}-a\\1\\0\end{bmatrix} = a$$

$$b_3' = \begin{bmatrix}a^2-1\\-a\\1\end{bmatrix},\ \|b_3'\|_Q = \sqrt{\begin{bmatrix}a^2-1 & -a & 1\end{bmatrix} Q^T Q \begin{bmatrix}a^2-1 & -a & 1\end{bmatrix}^T} = 1.$$

$$b_3 = b_3'$$

Thus, the orthonormal basis is:

$$\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}-a\\1\\0\end{bmatrix}, \begin{bmatrix}a^2-1\\-a\\1\end{bmatrix} \right\}.$$

Alternatively: Orthonormal basis for $z = Qx$ is $\{e_1, e_2, e_3\}$. As Q is invertible, it is a bijective mapping with every z having a unique x. Therefore, the orthonormal basis over $(\mathbb{R}^n, \langle x, y\rangle_K)$ will be $\{Q^{-1}e_1, Q^{-1}e_2, Q^{-1}e_3\}$

$$Q^{-1} = \begin{bmatrix}1 & -a & a^2-1\\ 0 & 1 & -a\\ 0 & 0 & 1\end{bmatrix} \Rightarrow Basis = \left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}-a\\1\\0\end{bmatrix}, \begin{bmatrix}a^2-1\\-a\\1\end{bmatrix} \right\}$$