# ECE 269- Lecture 20: Thursday March 13th

- SVD and low rank approximation
- SVD and PCA
- Perron-Frobenius Theory

Announcement:

- Sample exam difficulty
- Reminder about bonus points: fill instructor feedback form

Recap: $\forall A \in \mathbb{C}^{m \times n}$ with rank $k \leq \min(m,n)$;

$$A = \underset{m \times k}{U} \begin{pmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_k \end{pmatrix} V^* \qquad V \in \mathbb{C}^{k \times n}$$

(compact SVD)

Spectral norm: $\|A\| = \max\limits_{\{x;\, \|x\|=1\}} \|Ax\|_2 = \max\limits_{x:\, x \neq 0} \dfrac{\|Ax\|_2}{\|x\|_2}$

Frobenius norm: $\|A\|_F = \left( \sum\limits_{i,j} (A_{ij})^2 \right)^{\frac{1}{2}}$

Spectral: $\|A\|^2 = \max_{x:\|x\|=1} \|Ax\|^2 = \max_{x:\|x\|=1} x^* A^* A x = \lambda_{max}(A^* A) = \sigma_1^2$

$\Rightarrow \|A\| = \sigma_1$

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Nuclear norm:

$\|A\|_N := \sigma_1 + \cdots + \sigma_k$

With $A = U \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_k \end{pmatrix} V^* \Rightarrow A = \sum_{i=1}^{k} \sigma_i u_i v_i^*$

where $u_i, v_i$ are the ith columns of $U$ and $V$, respectively.

Define $A_d = \sum_{i=1}^{d} \sigma_i u_i v_i^*$ for $d = 1, \ldots, k$.

(Young–Eckart thrm)

Thrm: For any rank $d$ matrix $B \in \mathbb{C}^{m \times n}$,

$$A_d = \sum_{i=1}^{d} \sigma_i u_i v_i^*$$

(i) $\|A - B\| \geq \|A - A_d\|$ ✓

(ii) $\|A - B\|_F \geq \|A - A_d\|_F$ ✓

(iii) $\|A - B\|_N \geq \|A - A_d\|_N$

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Pf of (i): For a rank $d$ matrix $B$, let $Q = \mathrm{span}(v_1, \dots, v_{d+1})$,

we claim: $\exists \overset{\emptyset}{\overset{\neq}{w}} \in Q$; $Bw = \emptyset$.

Let $w = \alpha_1 v_1 + \dots + \alpha_{d+1} v_{d+1}$ for $\alpha_1, \dots, \alpha_{d+1} \in \mathbb{C}$, with $\sum_{i=1}^{d+1} \alpha_i^2 = 1$

$$\|A-B\| = \max_{x:\|x\|=1} \|(A-B)x\| \geq \|(A-B)w\| = \|Aw\|$$

$$= \|A \sum_{i=1}^{d+1} \alpha_i v_i\| = \|\sum_{i=1}^{d+1} \alpha_i u_i \sigma_i\| \geq \|\sum_{i=1}^{d+1} \alpha_i u_i\| \sigma_{d+1}$$

$$= \sigma_{d+1} = \|A - A_d\|$$

$$\|A - A_k\| = \left\| U \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_k \end{pmatrix} V^* - U \begin{pmatrix} \sigma_1 & & & & \\ & \ddots & & & \\ & & \sigma_d & & \\ & & & 0 & \\ & & & & \ddots \; 0 \end{pmatrix} V^* \right\|$$

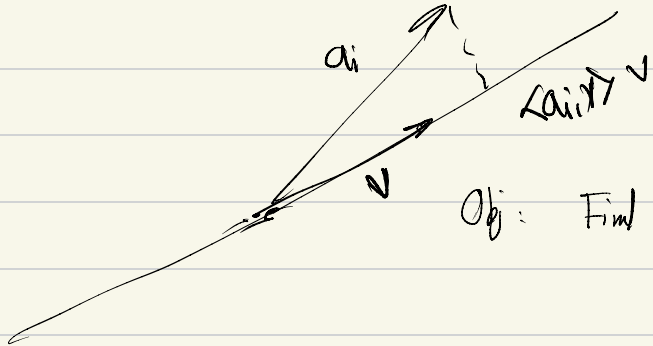
$$= \left\| U \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & 0 & & \\ & & & \sigma_{d+1} & \\ & & & & \ddots \; \sigma_k \end{pmatrix} V^* \right\| = \sigma_{d+1}$$

Assume: $\mathbb{1}^T A = \underline{0}$

Frobenius norm analysis:

$$
\begin{array}{cc}
 & \begin{array}{ccc} \alpha_1 & \cdots & \alpha_n \end{array} \\
A = & \begin{array}{c} o_1 \rightarrow \\ o_2 \rightarrow \\ \\ \\ \\ o_m \rightarrow \end{array}
\begin{bmatrix}
\text{---} & a_1^T & \text{---} \\
\text{---} & a_2^T & \text{---} \\
 & \vdots & \\
\text{---} & a_m^T & \text{---}
\end{bmatrix}
\end{array}
$$

Obj: Find $v$, $\|v\| = 1$ and $\gamma_1, \ldots, \gamma_m$; $\min \sum_{i=1}^{m} \|\gamma_i v - a_i\|^2$




Obj: Find $v$, $\|v\|=1$ and $\gamma_1, \dots, \gamma_m$: $\min \sum_{i=1}^{m} \|\gamma_i v - a_i\|^2$

For a given $v$; $\gamma_i = \langle v, a_i \rangle$ minimizes the above sum.

$$\Rightarrow \underset{v:\|v\|=1}{\operatorname{argmin}} \sum_{i=1}^{m} \|(a_i^T v)v - a_i\|^2 = \sum_{i=1}^{m} (a_i^T v)^2 - 2(a_i^T v)^2 + \|a_i\|^2$$

$$\equiv \underset{v:\|v\|=1}{\arg\min} - \sum_{i=1}^{m} (a_i^T v)^2 = \operatorname{argmax} \sum_{i=1}^{m} (a_i^T v)^2$$

$$= \underset{v:\|v\|=1}{\operatorname{argmax}} \|Av\|^2$$

The last quntity is maximized at the right singular vector $v_1$ of $A$. Recall

$$\gamma \nearrow \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{pmatrix} = Av_1 = U \begin{pmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_k \end{pmatrix} V^* v_1 = U \begin{pmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_k \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$= \sigma_1 u_1 .$$

That is why $u_1$ is also called the leading principal component of $A$.

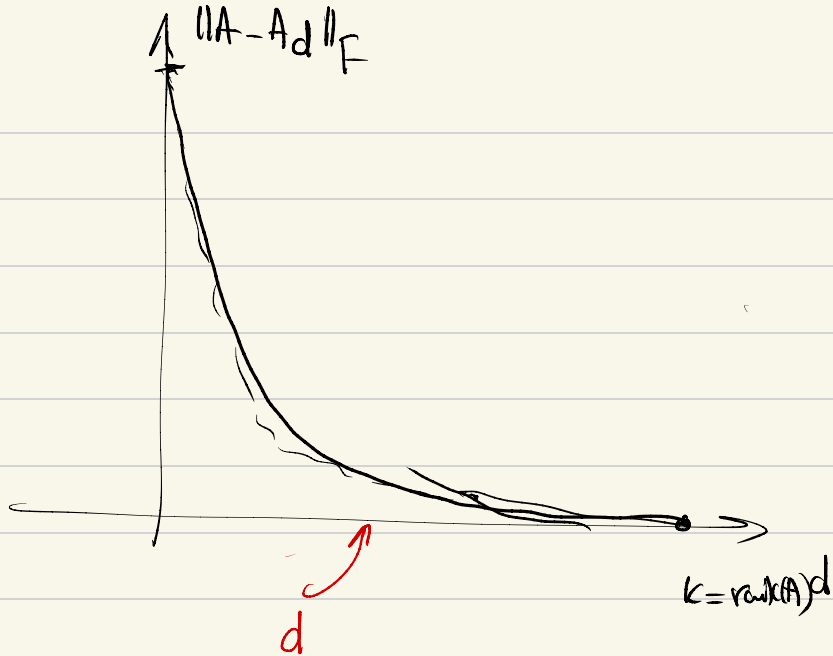
Starting point: $\arg\min \sum_{i=1}^{m} \|\gamma_i v^T - a_i^T\|^2 = v_1$

Notice that $\gamma_i v^T$ is the ith row of $\gamma v^T$.

$$\Rightarrow \quad \|\gamma v^T - A\|_F^2 = \sum_{i=1}^{m} \|\gamma_i v^T - a_i^T\|^2$$

$$\Rightarrow \quad \min_{\gamma, v;\, \|v\|=1} \sum_{i=1}^{m} \|\gamma_i v^T - a_i^T\|^2 = \min_{B:\, \mathrm{rank}(B)=1} \|B - A\|_F^2$$

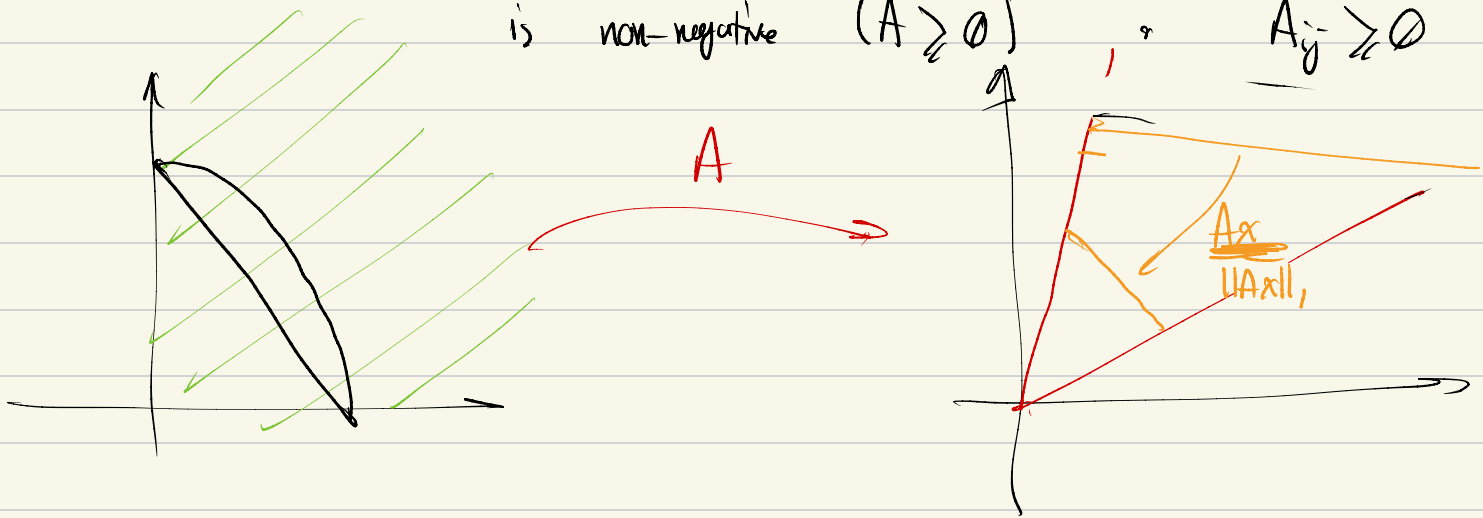
$\gamma = Av_1 \Rightarrow B = \underset{\sim}{A}v_1 v_1^T = \sigma_1 u_1 v_1^T.$


$\|A - A_d\|_F$
$k = \mathrm{rank}(A)$
$d$

(*) Perron-Frobenius Theory: Concerned with positive / non-negative matrix theory:

We say that $A$ is positive, $(A > 0)$ if $A_{ij} > 0$ for all $i,j$

is non-negative $(A \geq 0)$, $A_{ij} \geq 0$



For a positive matrix $A$, $\exists\ v > 0$; $\lambda > 0$; $Av = \lambda v$

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Pf. Let $B = \left\{ x \geq 0 \;\middle|\; \sum x_i = 1 = \|x\|_1 \right\}$.

Let $C = \left\{ \dfrac{Ax}{\|Ax\|_1} \;\middle|\; x \in B \right\}$

Note, $y \in C$, $\|y\|_1 = \dfrac{\|Ax\|_1}{\|Ax\|_1} = 1$.

Note that $B$ is a closed, bounded and convex set in $\mathbb{R}^n$.

(Closed = limit of every convergent sequence is in the set)

Schauder's Fixed Point Thrm:

$\Rightarrow$ if $f(x) = \dfrac{Ax}{\|Ax\|_1}$, $f: B \to C \subseteq B$. (every cont's mapping from a closed bounded convex set to itself has a fixed point)

$\Rightarrow \exists x^* \in B \cap C$; $\dfrac{Ax^*}{\|Ax^*\|_1} = x^* \equiv Ax^* = \underbrace{\|Ax^*\|_1}_{\lambda} x^*$