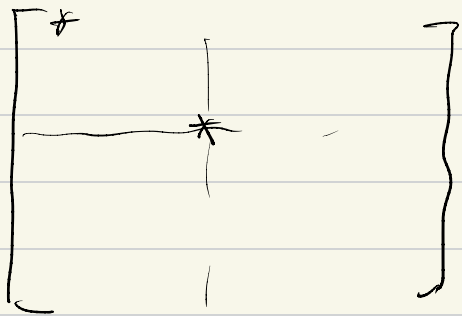
ECE 269- Lecture 10: Thursday Feb 6th

- LU decomposition cont.
- $Ax = b$
  - least square solution

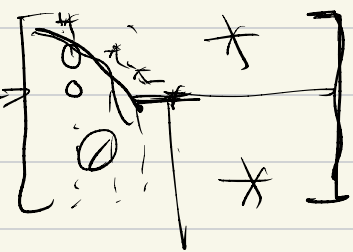
- No blue book for the exams

LU decomposition: $A \in \mathbb{R}^{n \times n}$

$$Ax = b$$



for $i = 1:n$: if $A^{(i)}_{i,i} \neq 0$:

if $A^{(i)} =$  $x = b$

by successively multiplying from left by

$$L = \begin{array}{c} \\ \\ \\ \\ \end{array}\begin{array}{c} i \\ \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \ddots & & & \\ & & & 1 & & \\ & & & l_{k,i} & \ddots & \\ & & & & & 1 \end{bmatrix} \end{array} \qquad L_{pq} = \begin{cases} 1 & p=q \\ \neq 0 & p=k,\ q=i \\ 0 & \text{ow} \end{cases}$$

$$-\frac{A^{(i)}_{k,i}}{A^{(i)}_{i,i}}$$

we arrive at

$$A^{(i+1)} = \begin{array}{c} i \\ \begin{bmatrix} * & & & & \\ & * & & & \\ & & \ddots & & \\ & & * & & \\ & & 0 & * & \\ & & 0 & & \\ & & 0 & & \end{bmatrix} \end{array}$$

let $i \to i+1$

$$L^{-1} = \begin{bmatrix} 1 & & & \\ & 1 & & \emptyset \\ & & \ddots & \\ & -l_{k,i} & & 1 \end{bmatrix}$$

if $A^{(i)}_{ii} = 0$

if $\exists\, k > i : A^{(i)}_{ki} \neq 0$, we swap the kth and the ith row

and proceed as before

$$\Rightarrow \underbrace{L^{(n)}}\, P^{(n)} L^{(n-1)} P^{(n-1)} \cdots \underbrace{L^{(1)} P^{(1)}} A = U = \begin{bmatrix} * & * & * \\ 0 & \ddots & \\ & & \ddots \end{bmatrix}$$

We can show that with the obtained $L^{(i)}$ and $P^{(i+1)}$, you can show that

$$\boxed{LPA = U \equiv PA = \tilde{L}U}$$

$$P^{(i+1)} L^{(i)} = L^{(i)} P^{(i+1)T} = L^{(i)} P^{(i+1)}$$

$$\Rightarrow LPA = U \quad \text{where } L = L^{(n)} L^{(n-1)} \cdots L^{(1)},\ P = P^{(n)} \cdots P^{(1)}$$

$$Ax = b$$

$$\equiv LPAx = LPb$$

$$\equiv Ux = c$$

$$\underbrace{\begin{bmatrix} * & & \\ & * & \\ & & \ddots \\ 0 & & & u_{nn} \end{bmatrix}}_{U} x = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

$u_{nn} \neq 0$

Note that $u_{nn} \cdot x_n = c_n \Rightarrow x_n = \dfrac{c_n}{u_{nn}}$

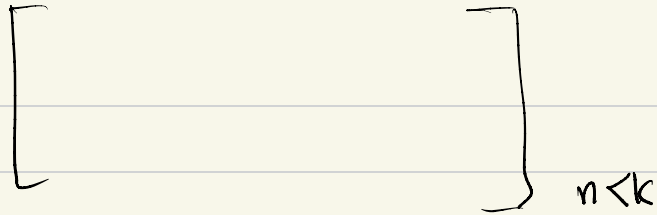
$$u_{n-1,n-1} x_{n-1} + u_{n-1,n} x_n = c_{n-1} \Rightarrow x_{n-1} = \frac{1}{u_{n-1,n-1}} \left(c_{n-1} - u_{n-1,n} x_n \right)$$

$\vdots$

recursively

from the ith equation, we obtain x_i.

$\underset{n\times k}{Ax} = b$ has a solution



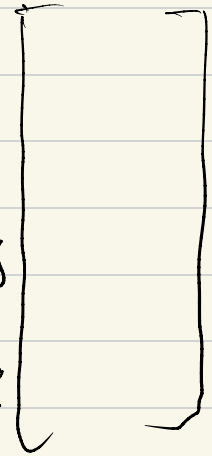
fat matrix

$n < k$

iff $b \in R(A)$.

Uniqueness: if $b \in R(A)$ and $u \in N(A)$,
with $A\hat{x} = b$

$A(\hat{x} + u) = b \Rightarrow$ Uniqueness $\equiv N(A) = \{\underline{0}\}$



tall matrix

$\Rightarrow$ For a fat matrix A, $\text{rank}(A) + \text{nullity}(A) = k$

$\text{rank}(A) \leq n$

$\Rightarrow \text{nullity}(A) \geq k - n \geq 1$.

$n \times n$: $Ax = b$ has a unique solution for all b

$\Leftrightarrow$ $\text{rank}(A) = n$ $\overset{r-n}{\Leftrightarrow}$ $\text{Nullity}(A) = \emptyset$ $\Leftrightarrow$ A is invertible

$\Rightarrow$ $Ax = b$ has a unique solution for all b or

$\left\{ Ax = b \text{ has no solution for some } b \text{ and } \infty \text{ many solutions for some } b \right\}$

Def: We say that $A_{k\times n}$ has a left inverse if

$\exists B_{n\times k}$; $BA = I_{n\times n}$

" " " " " " right inverse if

" $C_{m\times k}$; $AC = I_{k\times k}$

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m, k are arbitrary; $\quad Ax = b \quad (*)$

if A has a left inverse, then $(*)$ has at most one solution.

$$\begin{bmatrix} & \\ & \end{bmatrix} \begin{bmatrix} & \end{bmatrix} \overset{?}{=} \begin{bmatrix} I & \\ & \end{bmatrix}$$

Pf: $Ax = b \Rightarrow \underbrace{BA}_{I}x = Bb \Rightarrow x = \tilde{b}$

Note: $x = \tilde{b}$ is not necessarily a solution!

Fact: If A has a right inverse $(AC = I)$ $\Rightarrow$ there is at least one solution to $Ax = b$.

Pf: If $\tilde{x} = Cb \Rightarrow A\tilde{x} = ACb = b$.

Suppose $b \in R(A)$: i.e., we have a solution $Ax = b$.

Objective: find such an x through (reduced) QR decomposition.

$$\underset{n\times k}{A} = \underset{n\times n}{Q}\left[\underset{r\times r}{\tilde{R}} \;\middle|\; \underset{r\times(k-r)}{S}\right]\underset{k\times k}{P}$$

where all the diagonal elements of $\tilde{R}$ is non-zero.

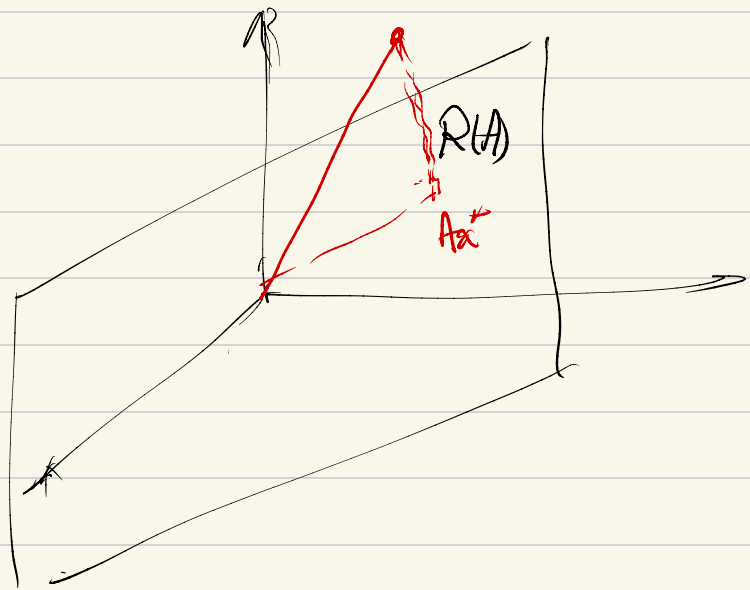
Let $x^* = P^{-1}\begin{bmatrix}\tilde{R}^{-1}\tilde{x} \\ 0\end{bmatrix}$ with $\tilde{x} = Q^T b$

$QQ^T b = b$?

as $b \in R(A) = R(Q)$

$$\Rightarrow Ax^* = Q\left[\tilde{R} \;\middle|\; S\right]\cancel{P}\left(\cancel{P^{-1}}\begin{bmatrix}\tilde{R}^{-1}\tilde{x} \\ 0\end{bmatrix}\right) = Q\tilde{R}\tilde{R}^{-1}\tilde{x} = Q\tilde{x} = QQ^T b = b$$

In fact even if $b \notin R(A)$; $x^* = P^{-1}\begin{bmatrix} \tilde{R}^{-1}\tilde{x} \\ 0 \end{bmatrix}$ would give the least square solution to $\min_x \|Ax-b\|$



Orthogonality principle:

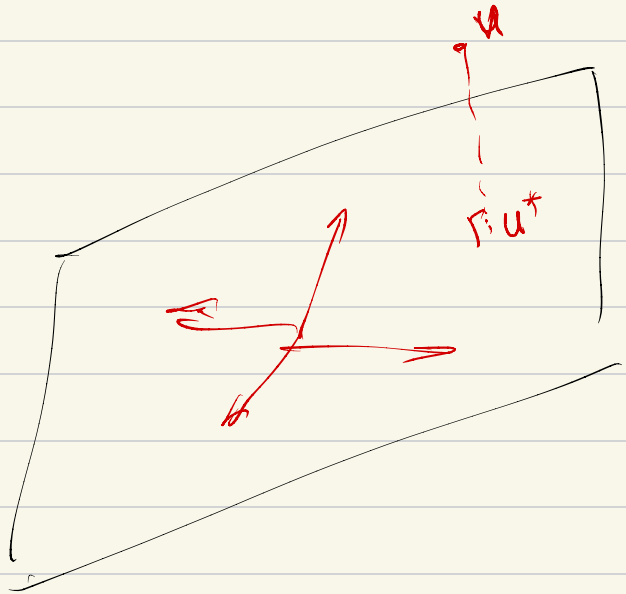
Let H be a Hilbert space and let $V \subseteq H$ be a linear and closed subspace of H. Then for any $u \notin V$

(i) $u^* = \min_{v \in V} \|v - u\|$ exists

(ii) it is unique

(iii) $\langle u^* - u, z \rangle = 0 \quad \forall z \in V$.

is the unique point satisfying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$\Gamma_i u^*$