ECE 269- Lecture 4: Thursday Jan 14th

- Vector spaces cont.:
  - span
  - independent set
  - basis
  - dimension

Announcements: HW1 deadline $\longrightarrow$ tomorrow (Friday) at 8:00pm

Office hours: 11:00 – 12:30 pm Tue. through Friday

When dealing with vector spaces, and their subspaces, we assume they are non-empty.

Review:

V : Vector space over $\mathbb{F}$

$U \subseteq V$ is a subspace if : (a) $\forall u, v \in U$, $u+v \in U$

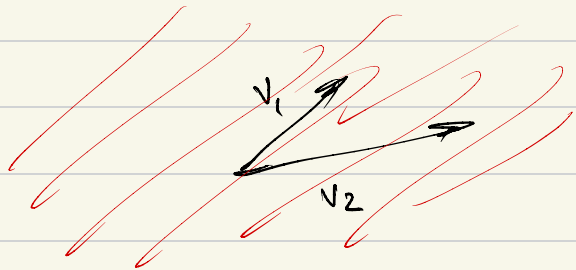
(b) $\forall v \in U$ and $a \in \mathbb{F}$, $av \in U$

For $S \subseteq V$,

$$\operatorname{span}(S) := \{ a_1 v_1 + \cdots + a_n v_n \mid \forall n;\ \forall a_i \in \mathbb{F},\ \forall v_i \in S \}$$

Fact: $\forall S \subseteq V$; $\operatorname{span}(S)$ is a subspace of V.

Pf (?)



$\mathrm{span}(\{v_1, v_2\})$

We say $v_1, \dots, v_k \in V$ is independent if

- $\alpha_1 v_1 + \dots + \alpha_k v_k = \underline{0} \Rightarrow \alpha_1 = \dots = \alpha_k = 0$

$\equiv$

- $v_i \neq \alpha_1 v_1 + \dots + \alpha_{i-1} v_{i-1} + \alpha_{i+1} v_{i+1} + \dots + \alpha_k v_k \quad \forall i \in \{1, \dots, k\}$

We say Q is an independent set if any finitely many vectors in Q are linearly independent.

Def: We say a $S \subseteq V$ is a spanning/generating set for V (a subspace of U) if $\operatorname{span}(S) = V$. ($\operatorname{span}(S) = U$)

Def: We say $S \subseteq V$ is a basis (pl. bases) for V if

(i) S is linearly independent

(ii) S is a spanning set for V ($\operatorname{span}(S) = V$).

(pl. bases)

Def: We say $S \subseteq V$ is a basis for a subspace $U \subseteq V$ if

(i) S is linearly independent

(ii) S is a spanning set for U ($\mathrm{span}(S) = U$).

Example: $V = \mathbb{R}^2$ over $\mathbb{R}$, then

- $S = \{(1,0), (0,1)\}$ is a bases for V

- $S = \{(1,0), (0,1), (1,1)\}$ is not a basis for V

We say that V is a finite dimensional space if it has a spanning set with finitely many elements.

Otherwise, we say V is an infinite dimensional space.

For $V = \mathbb{R}^n$ over $\mathbb{R}$, we say that $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$ are orthogonal if $\sum_{i=1}^{n} x_i y_i = 0$.

Finite Dimensional Spaces

Q: How to find a bases?

Let $v_1 \neq \underline{0}$ in V and let $D = \{v_1\}$

While $\text{span}(D) \neq V$:

let $u \in V - \text{span}(D)$ and let $D^+ \leftarrow D \cup \{u\}$.

Fact: If D is an independent set and $v \in V - \mathrm{span}(D)$, then $D \cup \{v\}$ is an independent set.

Pf: Suppose that

$$\alpha v + \alpha_1 v_1 + \cdots + \alpha_k v_k = \underline{0}$$

for some $v_1, \ldots, v_k \in D$ and $\alpha, \alpha_1, \ldots, \alpha_k \in \mathbb{F}$

If $\alpha = 0$, and since D is independent $\Rightarrow$

$$\alpha_1 = \alpha_2 = \cdots = \alpha_k = 0 \quad \checkmark$$

If $\alpha \neq 0$, $\alpha v = -\alpha_1 v_1 - \alpha_2 v_2 - \cdots - \alpha_k v_k$

$$v = \underbrace{-\alpha^{-1}\alpha_1}_{\beta_1} v_1 - \underbrace{\alpha^{-1}\alpha_2}_{\beta_2} v_2 - \cdots - \underbrace{\alpha^{-1}\alpha_k}_{\beta_k} v_k \Rightarrow v \in \mathrm{span}(D) \quad \times$$

For a set A, we define its cardinality (denoted by $|A|$) to be the number of elements in A.
(size)

Example: $A = \{\triangle, \square\} \Rightarrow |A| = 2$

if $A = \mathbb{R} \Rightarrow |A| = \infty$

Next step: show that for any bases S, T for V, $|S| = |T|$.

Fact: Suppose that S is an independent set and T is a generating/spanning set for V. Then $|S| \leq |T|$.

Pf: Wlog assume $S = \{v_1, \dots, v_k\}$, $T = \{w_1, \dots, w_m\}$.

$\Rightarrow \exists\, \alpha_1, \dots, \alpha_m$: $v_1 = \alpha_1 w_1 + \alpha_2 w_2 + \dots + \alpha_m w_m$

Since S is independent $\exists i$; $\alpha_i \neq 0$ otherwise $v_1 = \underline{0}$ ※

Wlog assume $\alpha_1 \neq 0 \Rightarrow w_1 = \alpha_1^{-1} v - \alpha_1^{-1}\alpha_2 w_2 - \dots - \alpha_1^{-1}\alpha_m w_m$

$\Rightarrow T'' = \{v_1, w_2, \dots, w_m\}$ is a spanning set

$\Rightarrow v_2 = \tilde{\alpha}_1 v_1 + \tilde{\alpha}_2 w_2 + \dots + \tilde{\alpha}_m w_m$

Note that we cannot have $\tilde{\alpha}_2 = \dots = \tilde{\alpha}_m = 0$

as it contradicts $\{v_1, v_2\}$ being independent.

wlog $\tilde{\alpha}_2 \neq 0$, $\Rightarrow w_2$ can be written as a linear combination

of $v_1, v_2, w_3, \dots, w_m \Rightarrow T^{(2)} = \{v_1, v_2, w_3, \dots, w_m\}$ is a

spanning set for V.

Using this procedure and induction, you can show that

$$T^{(k)} := \{v_1, \dots, v_k, w_{k+1}, \dots, w_m\}$$

is a spanning set for $V \Rightarrow k \leq m$.

Essentially what we showed: $\exists\, i_1, i_2, \dots, i_k$:

$$T - \{w_{i_1}, \dots, w_{i_k}\} \cup S \text{ is a generating set!}$$

Corollary: If two basis $S, T \subseteq V$, $|S| = |T|$.

Pf: Since S is independent and T is a spanning set

$$|S| \leq |T|$$

Since S is spanning and T is independent $|S| \geq |T|$

$$\Rightarrow |S| = |T|.$$

Def: $|S|$ is called the dimension of V for a basis S.

Notation: $\dim(V) = |S|$.

Similarily we refer to $|S|$ as the dimension of a subspace $U \subseteq V$ if S is basis for U.

Notation: $\dim(U) = |S|$.

We let $\dim(\{\underline{0}\}) = 0$

(advanced)

Fact: Every vector space has a basis. (Zorn's Lemma)

(easy)

Fact: If W is a subspace of V, then $\dim(W) \leq \dim(V)$.

Fact: For any spanning set S for V; $\exists\, Q \subseteq S$ that is a basis.

Fact: Any linearly independent set is contained in a basis.