Homework 3

ECE 269: Linear Algebra and Applications
Homework #3
Instructor: Behrouz Touri
Due Date: Wedneseday Feb 5th, 8 pm (submission through Gradescope)

In this homework, unless otherwise is mentioned, for problems involving the vector space $\mathbb{R}^n$, $\|\cdot\|$ is the ℓ_2 norm.

1. *DIY!.* For the following matrix (over $\mathbb{R}$),

$$A = \begin{pmatrix} 8 & -1 & 2 \\ 8 & 2 & 0 \\ 0 & 3 & -2 \end{pmatrix}, \tag{1}$$

   obtain the following.

   (a) Find $\mathcal{R}(A)$.

   (b) Find $\text{rank}(A)$.

   (c) Find $\mathcal{N}(A)$.

   (d) Perform a rank decomposition $A = BC$.

   (e) Find the QR decomposition of A.

2. *Orthogonal complement of a subspace.* Suppose that $\mathcal{V}$ is a subspace of $\mathbb{F}^n$. Let

$$\mathcal{V}^\perp = \{x \in \mathbb{F}^n : x^T y = x'y = 0, \forall y \in \mathcal{V}\}.$$

   (a) Show that $\mathcal{V}^\perp$ is a subspace of $\mathbb{F}^n$.

   (b) Suppose that $\mathcal{V} = \text{span}(v_1, v_2, \ldots, v_k)$ for some $v_1, v_2, \ldots, v_k \in \mathbb{F}^n$. Express $\mathcal{V}$ and $\mathcal{V}^\perp$ as subspaces induced by the matrix $A = \begin{bmatrix} v_1 & v_2 & \cdots & v_k \end{bmatrix} \in \mathbb{F}^{n\times k}$ and its transpose A'.

   (c) Show that $(\mathcal{V}^\perp)^\perp = \mathcal{V}$.

   (d) Show that $\dim(\mathcal{V}) + \dim(\mathcal{V}^\perp) = n$.

   (e) Show that $\mathcal{V} \subseteq \mathcal{W}$ for another subspace $\mathcal{W}$ implies $\mathcal{W}^\perp \subseteq \mathcal{V}^\perp$.

   (f) Suppose that $\mathbb{F} = \mathbb{R}$. Show that every $x \in \mathbb{F}^n$ can be expressed uniquely as $x = v + v^\perp$, where $v \in \mathcal{V}$ and $v^\perp \in \mathcal{V}^\perp$. (Hint: Let v be the projection of x on $\mathcal{V}$.)

3. *Halfspace.* Suppose that $a, b \in \mathbb{R}^n$ are two given points. Show that the set of points in $\mathbb{R}^n$ that are closer to a than b is a halfspace, i.e.,

$$\{x : \|x - a\| \leq \|x - b\|\} = \{x : c'x \leq d\}$$

   for appropriate $c \in \mathbb{R}^n$ and $d \in \mathbb{R}$.

   (a) Find c and d explicitly in terms of a and b.

(b) Draw a picture showing a, b, c, and the halfspace.

4. *Inner product of polynomials.* Let $\mathcal{P}_3$ be the vector space of all polynomials of degree ≤ 3 with real coefficients, that is,

$$\mathcal{P}_3 = \{\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 : \alpha_0, \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}\}.$$

Let $K : \mathcal{P}_3 \times \mathcal{P}_3 \to \mathbb{R}$ be defined as

$$K(p, q) = \int_{-1}^{1} p(x)q(x)dx.$$

(a) Show that $K(\cdot, \cdot)$ represents an inner product for $\mathcal{P}_3$.

(b) Find an orthogonal basis for $\mathcal{P}_3$ using Gram–Schmidt orthogonalization.

5. *Bessel's inequality.* Suppose that the columns of $U \in \mathbb{R}^{n\times k}$ are orthonormal. Show that

$$\|U'x\| \leq \|x\|.$$

6. *Wonders of Infinite Dimensional Spaces.*

(a) Recall that $C^0([a, b])$ is the set of continuous functions $f : [a, b] \to \mathbb{R}$. Let $[a, b] = [0, 2]$.

i. Show that $\|f\|_2 = (\int_0^2 |f(x)|^2 dx)^{1/2}$ is well-defined, i.e., $\|f\|_2 < \infty$ for all $f \in C^0([0, 2])$. As a result of this, $C^0([0, 2]) \subset L_2([0, 2])$ and $(C^0([0, 2]), \|\cdot\|_2)$ is a normed-vector space.

ii. Show that this normed-vector space is not complete/Banach. **hint:** Show that the sequence $\{f_k\}$ in $C^0([0, 2])$ defined by

$$f_k(x) = \begin{cases} x^k & x \in [0, 1] \\ 1 & x \in (1, 2] \end{cases}$$

is a Cauchy sequence, but the sequence does not have a limit in $C^0([0, 2])$.

(b) We defined the space $\ell_\infty(\mathbb{N})$ to be the space of all sequences $(x_n)_{n\geq 1}$ with $x_n \in \mathbb{R}$ such that $\sup_{n\geq 1} |x_n| < \infty$, and we defined the norm $\|\cdot\|_\infty$ in this space by $\|(x_n)_{n\geq 1}\|_\infty = \sup_{n\geq 1} |x_n| < \infty$.

i. For a normed-vector space $(V, \|\cdot\|)$, we can define the ball of radius $r > 0$ around a point $x \in V$, to be $B_r(x) = \{y \mid \|y - x\| < r\}$. Identify, the unit ball $B_1(\mathbf{0})$ in $\ell_\infty(\mathbb{N})$ where $\mathbf{0}$ is the zero of $\ell_\infty(\mathbb{N})$.

ii. Construct a sequence of vectors $\{v_n\}_{n\geq 1}$ in $B_1(\mathbf{0})$ such that the distance of any two points is greater than or equal to one. In other words, not only $\{v_n\}_{n\geq 1}$ is not Cauchy, but none of its subsequences is Cauchy.

7. *Projection matrices.* A symmetric matrix $P = P' \in \mathbb{R}^{n\times n}$ is said to be a *projection matrix* if $P = P^2$.

(a) Show that if P is a projection matrix, then so is $I - P$.

(b) Suppose that the columns of $U \in \mathbb{R}^{n\times k}$ are orthonormal. Show that UU' is a projection matrix.

(c) Suppose that $A \in \mathbb{R}^{n\times k}$ is full-rank with $k \le n$. Show that $A(A'A)^{-1}A'$ is a projection matrix.

(d) The point $y \in \mathcal{S} \subseteq \mathbb{R}^n$ closest to $x \in \mathbb{R}^n$ is said to be the *orthogonal projection* (or *projection* in short) of x onto $\mathcal{S}$. Show that if P is a projection matrix, then $y = Px$ is the projection of x onto $\mathcal{R}(P)$.

(e) Let u be a unit vector. Find the projection matrix P such that $y = Px$ is the projection of x onto $\text{span}(u)$.

8. *Reflection and projection with an affine hyperplane.* Let a be a nonzero vector in $\mathbb{R}^n$, $b \in \mathbb{R}$, and

$$\mathcal{A} = \{x \in \mathbb{R}^n : a'x = b\}.$$

be an *affine hyperplane*, namely, a shifted version of the hyperplane $\mathcal{H} = \{x : a'x = 0\}$ by b, with the same normal vector a.

(a) Find the projection of the zero vector $\mathbf{0}$ onto $\mathcal{A}$, i.e., find the vector $u \in \mathcal{A}$ with $(\mathbf{0} - u)^T(x - y) = 0$ for all $x, y \in \mathcal{H}$.

(b) Find the reflection of $\mathbf{0}$ through $\mathcal{A}$.

(c) Find the projection of x onto $\mathcal{A}$.

(d) Find the reflection of x through $\mathcal{A}$.