ECE 269- Lecture 19: Tuesday March 11th

- SVD
  - low rank approximation
  - connection to PCA

- Final logistics:
  - one cheat sheet
  - blue book
  - venue
  - bonus problem: need 180 (155 right now)

Sylvester Criterion:

- A matrix A is p.d. iff all the leading principal minors are positive.

- A matrix A is p.s.d. iff <u>all</u> principal minors are non-negative (Horn Johnson).

Portfolio theory: $X_1, \dots, X_n$ are the r.v.s representing returns for stocks $i = 1, \dots, n$.

$$\mu_i = E[X_i], \quad \Sigma_{ij} = E[(X_i - \bar{X}_i)(X_j - \bar{X}_j)] \quad 1 \le i, j \le n.$$

w_i = portion wealth invested on i

Return: $Y = \sum_{i=1}^{n} w_i X_i$

Objective: find $\arg\min \; Var(Y) = w^T \Sigma w$

s.t. $\mu^T w = R$

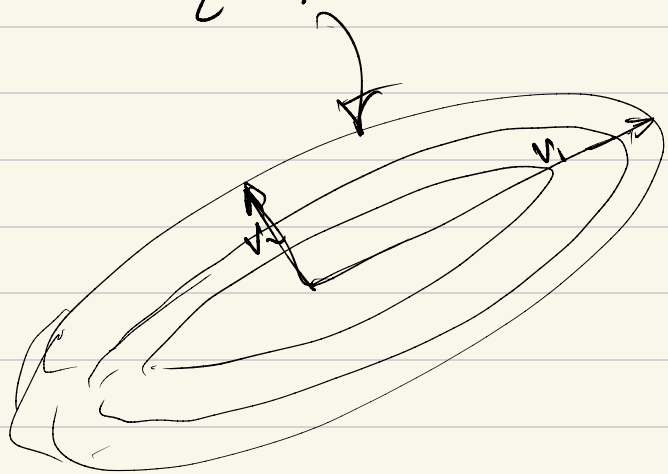
$\mathbb{1}^T w = 1$

Assume Σ is P.D.

Markowitz Portfolio Theory

$$\min \quad w^T \Sigma w$$
$$\text{s.t.} \quad \mu^T w = R$$
$$\mathbb{1}^T w = 1$$

$$V(r) = \{ x \mid x^T \Sigma x \leq r \}$$



$$\min \; w^T \Sigma w \qquad\qquad \min \; \|w\|_C^2$$

$$\text{s.t.} \quad \begin{aligned} \mu^T w &= R \\ 1^T w &= 1 \end{aligned} \quad \equiv \quad \text{s.t.} \quad \begin{aligned} \mu^T w &= R \\ 1^T w &= 1 \end{aligned}$$

Since $\Sigma \succ 0$; $\Sigma = C^T C$ for an invertible $C \in \mathbb{R}^{n \times n}$.

$$\langle x, y \rangle_C = x^T C^T C y.$$

Use orthogonality principle to find the solution to this.

Singular Value Decomposition (SVD):

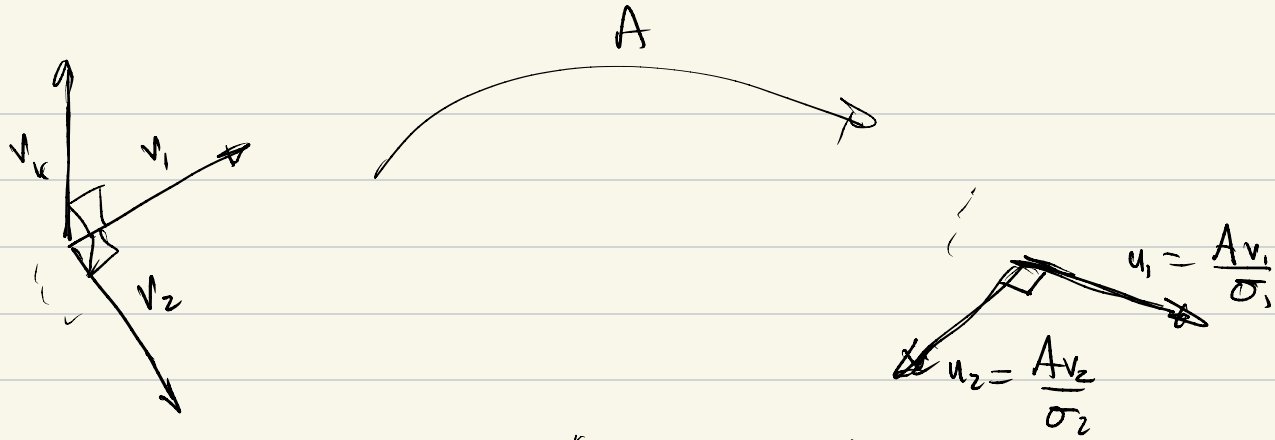
Let $A \in \mathbb{C}^{m \times n}$. Then $A^* A \succeq 0$

$$\Rightarrow \quad A^* A = V \begin{pmatrix} \sigma_1^2 & & & & & \\ & \sigma_2^2 & & & & \\ & & \ddots & & & \\ & & & \sigma_k^2 & & \\ & & & & 0 & \\ & & & & & \ddots \; 0 \end{pmatrix} V^*.$$

Assume that $\sigma_1^2 \geq \sigma_2^2 \cdots \geq \sigma_k^2 > 0$. Note that $k = \operatorname{rank}(A) \leq \min(m,n)$.

Let $\tilde{y}_i = A v_i \quad i = 1, \ldots, k. \quad \|\tilde{y}_i\|^2 = \tilde{y}_i^* y_i = v_i^* A^* A v_i = \sigma_i^2.$

Let $y_i = \frac{1}{\sigma_i} A v_i$



$$\langle u_i, u_j \rangle = u_j^* u_i = \frac{1}{\sigma_j \sigma_i} v_j^* A^* A v_i = \frac{\sigma_i^2}{\sigma_j \sigma_i} v_j^* v_i = 0.$$

Let $B = U \Sigma \tilde{V}^* = \begin{bmatrix} | & & | \\ u_1 & \cdots & u_k \\ | & & | \end{bmatrix} \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_k \end{bmatrix} \begin{bmatrix} - & v_1^* & - \\ - & v_2^* & - \\ & \vdots & \\ - & v_k^* & - \end{bmatrix}$

$x \in \mathbb{C}^n$; $\quad x = V \underbrace{V^* x}_{\begin{bmatrix} \langle x, v_1 \rangle \\ \langle x, v_2 \rangle \\ \vdots \\ \langle x, v_n \rangle \end{bmatrix}} = \langle x, v_1 \rangle v_1 + \cdots + \langle x, v_n \rangle v_n$

$u_i = \dfrac{A v_i}{\sigma_i}$

$i > k$; $\quad A v_i = 0 \quad$ as $\quad \|A v_i\|^2 = v_i^* A^* A v_i = 0$

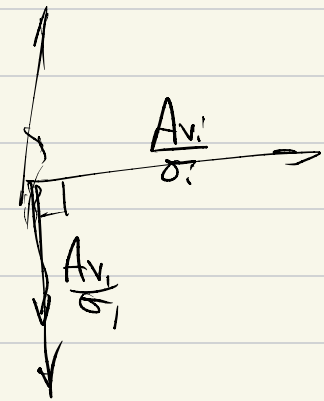
$$\Rightarrow \quad A x = \sum_{i=1}^{n} \langle x, v_i \rangle A v_i = \sum_{i=1}^{k} \langle x, v_i \rangle A v_i$$

$$= \sum_{i=1}^{k} \langle x, v_i \rangle \sigma_i u_i$$

$$\Rightarrow \quad Ax = \sum_{i=1}^{n} \langle x, v_i \rangle A v_i = \sum_{i=1}^{k} \langle x, v_i \rangle A v_i$$

$$= \sum_{i=1}^{k} \langle x, v_i \rangle \sigma_i u_i = \overbrace{\begin{bmatrix} | & & | \\ u_1 & \cdots & u_k \\ | & & | \end{bmatrix}}^{\tilde{U}} \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_k \end{bmatrix} \tilde{V}^* x = Bx$$

$$\xrightarrow{\quad A \quad}$$



$$Ax = \tilde{U} \Sigma \tilde{V}^* x$$

Therefore, $$Ax = \tilde{U} \Sigma \tilde{V}^* x$$ where.

- $\tilde{V} = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_k \\ | & & | \end{bmatrix}$
- σ_i^2 is the ith positive eigenvalue of A^*A
- $\tilde{U} = \begin{bmatrix} | & & | \\ \frac{Av_1}{\sigma_1} & \cdots & \frac{Av_k}{\sigma_k} \\ | & & | \end{bmatrix}$
- $\tilde{U}^* \tilde{U} = I$

Since $\quad A x = \tilde{U} \Sigma \tilde{V}^* x \quad$ for all $x \in \mathbb{C}^n$

$$\Rightarrow \underset{m \times n}{A} = \underset{m \times k}{\tilde{U}} \underset{k \times k}{\Sigma} \underset{k \times n}{V^*}$$

where $k = \text{rank}(A)$.

This is called singular value decomposition of A.

- $\sigma_i > 0$: the ith singular value of A.
- u_i : ith left singular vector of A.
- v_j : jth right " " " A.

Note that

$$A = \tilde{U} \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_k \end{pmatrix} \tilde{V}^*$$

$$= \begin{bmatrix} | & & | \\ u_1 & \cdots & u_k \\ | & & | \end{bmatrix} \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_k \end{pmatrix} \begin{pmatrix} — & v_1^* & — \\ & \vdots & \\ — & v_k^* & — \end{pmatrix}$$

$$= \left[\begin{array}{ccc|ccc} | & & | & | & & | \\ u_1 & \cdots & u_k & u_{k+1} & & u_m \\ | & & | & | & & | \end{array}\right] \left[\begin{array}{ccc|ccc} \sigma_1 & & & & & \\ & \ddots & & & 0 & \\ & & \sigma_k & & & \\ \hline & & & & & \\ & 0 & & & 0 & \end{array}\right] \left[\begin{array}{c} — v_1^* — \\ \vdots \\ — v_k^* — \\ \hline — v_{k+1}^* — \\ \vdots \\ — v_n^* — \end{array}\right]$$

$\tilde{U}$ $\hat{U}$ $\hat{V}^*$

$$= \left[\begin{array}{c|c} \underbrace{\begin{matrix} | & & | \\ u_1 & \cdots & u_k \\ | & & | \end{matrix}}_{\tilde{U}} & \underbrace{\begin{matrix} | & & | \\ u_{k+1} & \cdots & u_m \\ | & & | \end{matrix}}_{\hat{U}} \end{array}\right] \left(\begin{array}{c|c} \begin{matrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_k \end{matrix} & 0 \\ \hline 0 & 0 \end{array}\right) \left(\begin{array}{c} \text{---} \; v_1^* \; \text{---} \\ \vdots \\ \text{---} \; v_k^* \; \text{---} \\ \text{---} \; v_{k+1}^* \; \text{---} \\ \vdots \\ \text{---} \; v_n^* \; \text{---} \end{array}\right)$$

where columns of $\hat{U}$ is an orthonormal basis $R(\tilde{U})^{\perp}$

" " " $\hat{V}$ " " " " $R(\tilde{V})^{\perp}$

Writing $\underset{m\times n}{A} = \underset{m\times m}{U} \; \underset{m\times n}{\overline{\Sigma}} \; \underset{n\times n}{V^*}$ where

U, V are unitary matrices and $\overline{\Sigma}$ is a diagonal matrix is called a full SVD of A.

Properties and application:

- Moore-Penrose of A: let $A = \tilde{U}\tilde{\Sigma}\tilde{V}^*$ be a compact SVD of A. Then

$$A^\dagger = \tilde{V}\tilde{\Sigma}^{-1}\tilde{U}^*$$ is the Moore-Penrose inverse of A.

Pf: e.g. $$AA^\dagger A = \tilde{U}\tilde{\Sigma}\underbrace{\tilde{V}^*\tilde{V}}_{I}\tilde{\Sigma}^{-1}\tilde{U}^*\tilde{U}\tilde{\Sigma}\tilde{V}^*$$

$$= \tilde{U}\underbrace{\tilde{U}^*\tilde{U}}\tilde{\Sigma}\tilde{V}^* = \tilde{U}\tilde{\Sigma}\tilde{V}^* = A$$

(DIY other properties)

Low rank approximation through SVD:

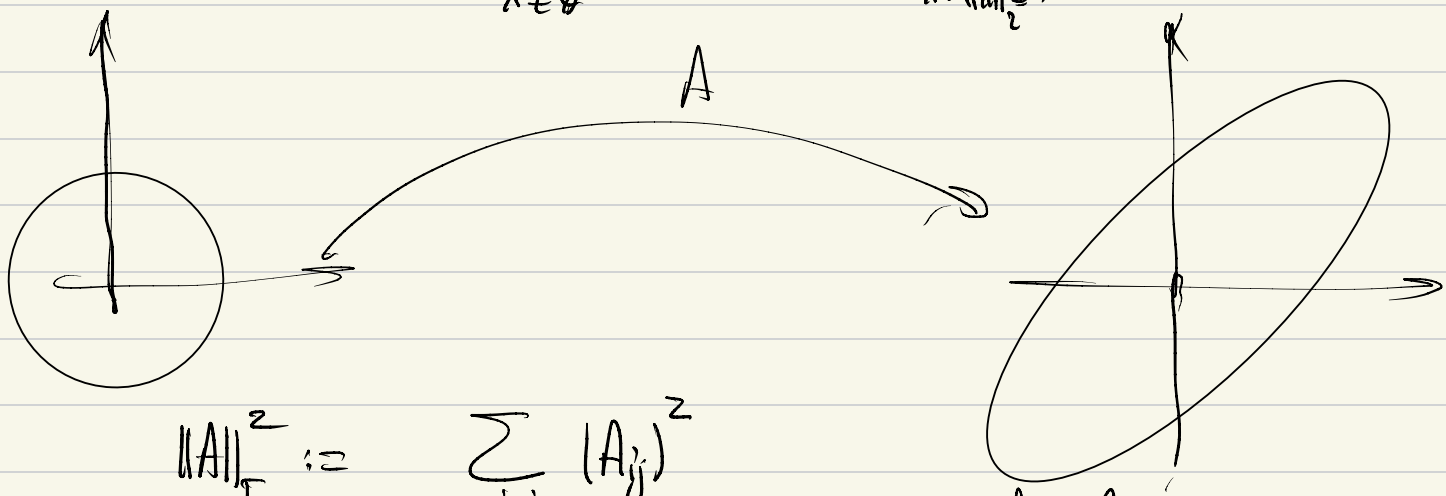
$$A = \tilde{U} \tilde{\Sigma} \tilde{V}^*$$

$$\equiv A = \sum_{i=1}^{k} \sigma_i u_i v_i^* \qquad \sigma_1 \geq \sigma_2 \cdots \geq \sigma_k > 0.$$

Q: What is the best rank 1 approximation of A?

$$\min_{B \in \mathbb{C}^{m \times n} : \operatorname{rank}(B) = 1} \|A - B\|$$

Define: $\|A\| = \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \max_{x : \|x\|_2 = 1} \|Ax\|_2$



$$\|A\|_F^2 := \sum_{i,j} (A_{ij})^2$$

ith column of A

$$= \sum_{j=1}^{n} \|A^{(i)}\|_2^2 = \sum_{i=1}^{m} \|A_{(i)}\|_2^2$$

ith row of A.