ECE 269- Lecture 5: Tuesday Jan 21st

- $\mathbb{F}^m$
- Range, Null-space, rank, nullity
- $\text{rank}(A) = \text{rank}(A')$
- rank-nullity thrm

Anouncements:

- Use the updated HW file
- You can miss upto two quizzes without penalty
- Notation: $A' = A^T$ is A transpose i.e., $[A']_{ij} = A_{ji}$

For a complex matrix: $A^* = [A']^* \equiv A^*_{ij} = \text{conj}(A_{ji})$

Example: $A = \begin{bmatrix} i & i+1 \\ 0 & 2 \end{bmatrix}$, $A^T = \begin{bmatrix} i & 0 \\ i+1 & 2 \end{bmatrix}$ $\quad i = j = \sqrt{-1}$

$$A^* = \begin{bmatrix} -i & 0 \\ 1-i & 2 \end{bmatrix}$$

Last quiz: How many element span$(\{v_1, v_2\})$ has for $V = \mathbb{Z}_3^2$ over $\mathbb{Z}_3$ with

$v_1 = (1,2)$, $v_2 = (2,1)$?

$$\text{span}(\{v_1, v_2\}) = \{\alpha v_1 + \beta v_2 \mid \alpha, \beta \in \underbrace{\mathbb{Z}_3}_{\{0,1,2\}}\} = \text{span}(\{v_1\})$$

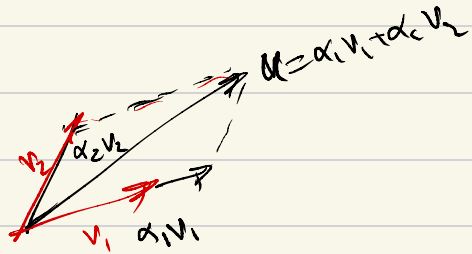
Here, $v_2 = -v_1$ as $v_1 + v_2 = (1,2) + \overbrace{(2,1)}^{-v_1} = (0,0)$

Recall: $\dim(U) = |\mathcal{B}|$ where $\mathcal{B}$ is a basis for U.

Let V be a finite dimensional space over $\mathbb{F}$ with a basis $\mathcal{B} = \{v_1, \dots, v_d\}$.

Then for $u \in V$, $\exists!\ \alpha_1, \dots, \alpha_d \in \mathbb{F}$:

$$u = \alpha_1 v_1 + \dots + \alpha_d v_d = \beta_1 v_1 + \dots + \beta_d v_d$$



$$\Rightarrow (\alpha_1 - \beta_1) v_1 + (\alpha_2 - \beta_2) v_2 + \dots + (\alpha_d - \beta_d) v_d = \underline{0} \Rightarrow \alpha_1 - \beta_1 = 0, \dots, \alpha_d - \beta_d = 0$$

$$\Rightarrow \alpha_i = \beta_i$$

$\alpha_1, \dots, \alpha_d$ are called the coordinates of u over B.

Note that $(\alpha_1, \dots, \alpha_d) \in \mathbb{F}^d$.

In fact if $(\alpha_1, \dots, \alpha_d)$ is the coordinate vector of u

$(\beta_1, \dots, \beta_d)$ " " " " of w over B

then $(\alpha_1+\beta_1, \dots, \alpha_d+\beta_d)$ is the coordinate vector of $u+w$ over B.

and $(\gamma\alpha_1, \dots, \gamma\alpha_d)$ " " " " " γu for any $\gamma \in \mathbb{F}$

$\Rightarrow$ Fact: Any d-dimensional vector space V over $\mathbb{F} \equiv \mathbb{F}^d$ over $\mathbb{F}$.

Example: $V = C^0([0,1])$ over $\mathbb{R}$ is a vector space

$$P^n([0,1]) = \{ \text{all polynomials } g(x) : [0,1] \to \mathbb{R} \text{ with } \deg(g) \leq n \}$$

is an $(n+1)$-dimensional subspace of $C^0([0,1])$ with a basis

$$B = \{ \overset{v_1}{1}, \overset{v_2}{x}, x^2, \ldots, \overset{v_n}{x^n} \} \equiv \mathbb{R}^{n+1}$$

$$g(x) = 1 + x + x^5 \equiv (1,1,0,0,0,1)$$

$(n=5)$

$+ \quad \mathbb{F}^m$ over $\mathbb{F}$

Notation: $v \in \mathbb{F}^m$, we view it $\begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix}$

$$\begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

For $v_1, \dots, v_n \in \mathbb{F}^m$:

Let $A = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}$, for $x \in \mathbb{F}^n$

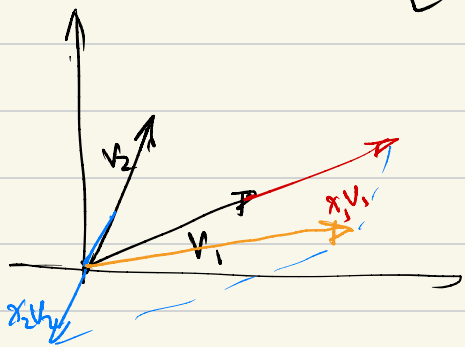
$$y = Ax \in \mathbb{F}^m; \quad y_i = [Ax]_i = \sum_{j=1}^{n} A_{ij} x_j$$

$$\equiv y = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n$$

$\mathbb{F} = \mathbb{R}$; $\quad \mathbb{R}^2$

$$y = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} x$$

v_1 $\quad$ v_2



Similarily; we $\quad y_{1\times n} = x_{1\times m} A_{m\times n}$ $\quad$ where $\quad x \in \mathbb{F}^{1\times m}$

$$y = x_1 u_1 + \cdots + x_m u_m \qquad \text{where} \qquad A = \begin{bmatrix} \text{---} & u_1 & \text{---} \\ \text{---} & u_2 & \text{---} \\ & \vdots & \\ \text{---} & u_m & \text{---} \end{bmatrix}$$

For a matrix $[A]_{m\times n} = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}$, we define the range space (column) of A:

$$R(A) := \{ Ax \mid x \in \mathbb{F}^n \} = \operatorname{span}\{v_1, \dots, v_n\} \subseteq \mathbb{F}^m$$

$\dim(R(A))$ is called the (column) rank of A. Notation: $\operatorname{rank}(A)$.

We define the null (Kernel) space of A:

$$\operatorname{null}(A) := \{ x \in \mathbb{F}^n \mid Ax = \underline{0} \} \subseteq \mathbb{F}^n$$

You can show that $\operatorname{null}(A)$ is a subspace of $\mathbb{F}^n$. We define the nullity of A: $\operatorname{nullity}(A) = \dim(\operatorname{null}(A))$.

Similar to $R(A)$, we define the row-space of A to be

$$\{ yA \mid y \in \mathbb{F}^{1\times m} \} = \operatorname{span}(u_1, \dots, u_m)$$

$$A = \begin{bmatrix} — & u_1 & — \\ — & u_2 & — \\ & \vdots & \\ — & u_m & — \end{bmatrix}$$

Note that this is essentially the same as $R(A^T)$.

We refer to $\dim(R(A^T))$ as the row rank of A.

Fact: For any $A \in \mathbb{F}^{m \times n}$, $\operatorname{rank}(A) = \operatorname{rank}(A')$.

Suppose that $\operatorname{rank}(A) = k \leq m \Rightarrow \exists\, 1 \leq i_1 < i_2 < \cdots < i_k \leq n$:

$\{v_{i_1}, \ldots, v_{i_k}\}$ is a basis for $R(A)$ (why?).

$$A = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$$

Let $[B]_{m \times k} = \begin{bmatrix} | & | & & | \\ v_{i_1} & v_{i_2} & \cdots & v_{i_k} \\ | & | & & | \end{bmatrix}$

Then $\exists\, [C]_{k \times n}$: $A = BC$, since columns of B form a basis for $R(A)$

$$v_1 = \alpha_{11} v_{i_1} + \alpha_{21} v_{i_2} + \cdots + \alpha_{k1} v_{i_k} = B \begin{bmatrix} \alpha_{11} \\ \vdots \\ \alpha_{k1} \end{bmatrix}$$

in fact $\exists\ \alpha_{1\ell}, \dots, \alpha_{k\ell}$; $v_\ell = \alpha_{1\ell} v_{i_1} + \dots + \alpha_{k\ell} v_{i_k}$

$$= \begin{bmatrix} — u_1 — \\ — u_2 — \\ \vdots \\ — u_m — \end{bmatrix} = A = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix} = \underset{m \times k}{B}\ \underset{k \times n}{C} = \begin{bmatrix} — b_1 — \\ \vdots \\ — b_m — \end{bmatrix} \begin{bmatrix} — c_1 — \\ — c_2 — \\ \vdots \\ — c_k — \end{bmatrix}$$

with $c_{ij} = \alpha_{ij}$

$\Rightarrow \{c_1, \dots, c_k\}$ generates all the rows of A.

$\Rightarrow \operatorname{rank}(A') \leq k = \operatorname{rank}(A)$.

Using a similar argument, we can show that $\operatorname{rank}(A') \geq \operatorname{rank}(A)$

$\Rightarrow \operatorname{rank}(A) = \operatorname{rank}(A')$. $\blacksquare$

Properties of rank :

(i) $\text{rank}(A) = \text{rank}(A') \leq \min(m, n)$

(ii) $\text{rank}(AB) \leq \min(\text{rank}(A), \text{rank}(B))$

(iii) $\text{rank}(A+B) \leq \text{rank}(A) + \text{rank}(B)$

(iv) rank-nullity thrm:

$$\text{rank}(A) + \text{nullity}(A) = n$$