ECE 269- Lecture 6: Thursday Jan 23rd

- rank decomposition
- an implication
- Normed vector spaces
- Inner product spaces

Review: $A \in \mathbb{F}_{m \times n}$

$$\text{rank}(A) = \dim(\mathcal{R}(A))$$

Fact: $\text{rank}(A) = \text{rank}(A')$ for any $A \in \mathbb{F}_{m \times n}$

Pf: if $\text{rank}(A) = k$, $\exists\, [B]_{m \times k}$, $[C]_{k \times n}$:

$$A = BC$$

This is called a rank-decomposition of A!

Application: to store A we require $m \times n$ units of memory.

To store B and C, we need $k \times (m+n)$ units of memory.

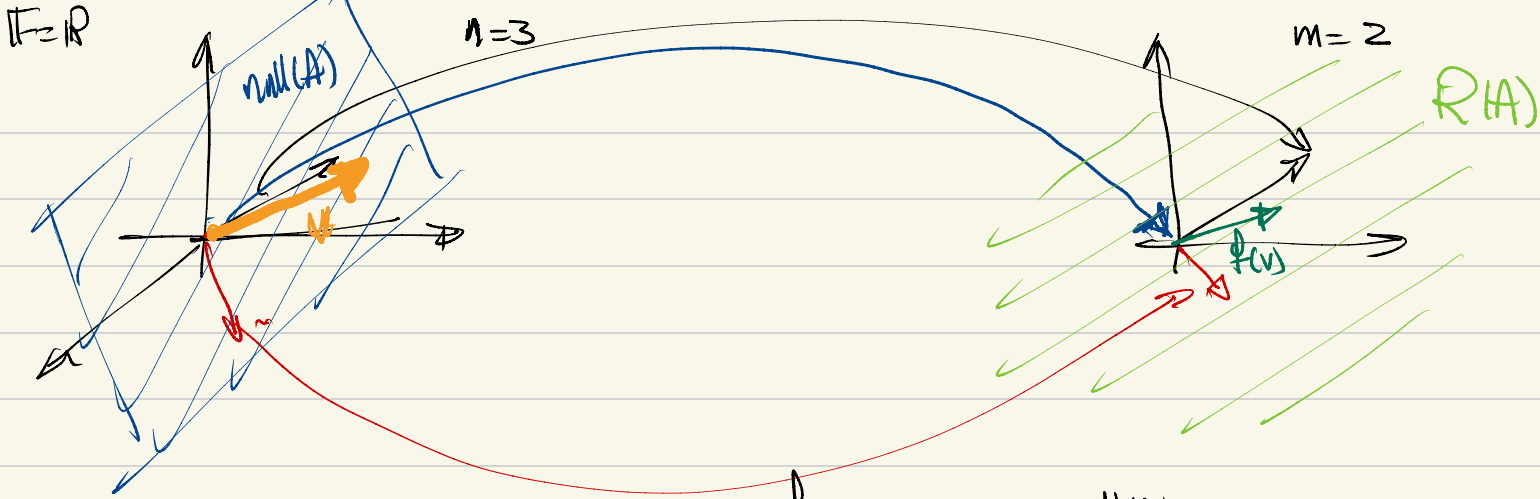
If $k \ll \min(m,n) \Rightarrow$ storing B, C requires much less memory!

Rank-nullity theorem: For any $A \in \mathbb{F}^{m \times n}$, then

$$\text{rank}(A) + \text{nullity}(A) = n.$$

View 1: Ax can be viewed as linear combinations of the columns of A with the coefficients coming from the vector x.

View 2: $f(x) = Ax$ with $f: \mathbb{F}^n \to \mathbb{F}^m$. In fact f is linear iff f can be written this way.



$$u = u_k + \alpha v \qquad \text{for} \qquad u_k \in \text{null}(A)$$

$$f(u) = f(u_k) + \alpha f(v) = \alpha f(v)$$

Rank-nullity theorem: For any $A \in \mathbb{F}^{m \times n}$, then

$$\text{rank}(A) + \text{nullity}(A) = n.$$

Pf: Let $S = \{u_1, \dots, u_k\} \subseteq \mathbb{F}^m$ be a basis for $R(A)$

and let $T = \{v_1, \dots, v_\ell\} \subseteq \mathbb{F}^n$ be a basis for null(A). Let $q_1, \dots, q_k$:

$$u_i = A q_i$$

Claim: $\{q_1, \dots, q_k, v_1, \dots, v_\ell\}$ is a basis for $\mathbb{F}^n$.

Suppose that $\underbrace{\alpha_1 q_1 + \dots + \alpha_k q_k}_{\tilde{q}} + \underbrace{\beta_1 v_1 + \dots + \beta_\ell v_\ell}_{\tilde{v}} = \underline{0}$

$\Rightarrow A\tilde{q} + A\tilde{v} = \underline{0} \equiv A\tilde{q} = \underline{0}$ but $A\tilde{q} = \alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_k u_k = \underline{0}$

$A\tilde{q} + A\tilde{v} = \underline{0} \equiv A\tilde{q} = \underline{0}$ but $A\tilde{q} = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_k u_k = \underline{0}$

But since $u_1, \ldots, u_k$ are independent $\Rightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$.

$\Rightarrow \alpha_1 q_1 + \cdots + \alpha_k q_k + \beta_1 v_1 + \cdots + \beta_e v_e = \underline{0} \Rightarrow \beta_1 v_1 + \cdots + \beta_e v_e = \underline{0}$

$\Rightarrow \beta_1 = \cdots = \beta_e = 0$.

$\Rightarrow W = \{ \underbrace{q_1, \ldots, q_k}, \underbrace{v_1, \ldots, v_e} \}$ is an independent set.

Next we show that $\text{span}(W) = \mathbb{F}^n$. Let $z \in \mathbb{F}^n$.

$Az \in R(A)$; $\exists \alpha_1, \ldots, \alpha_k$; $Az = \alpha_1 u_1 + \cdots + \alpha_k u_k$

$= \alpha_1 A q_1 + \cdots + \alpha_k A q_k = A(\overbrace{\alpha_1 q_1 + \cdots + \alpha_k q_k}^{q})$

$$\Rightarrow Az = A\hat{q} \equiv A(z-\hat{q}) = \underline{0} \Rightarrow z-\hat{q} \in \text{null}(A)$$

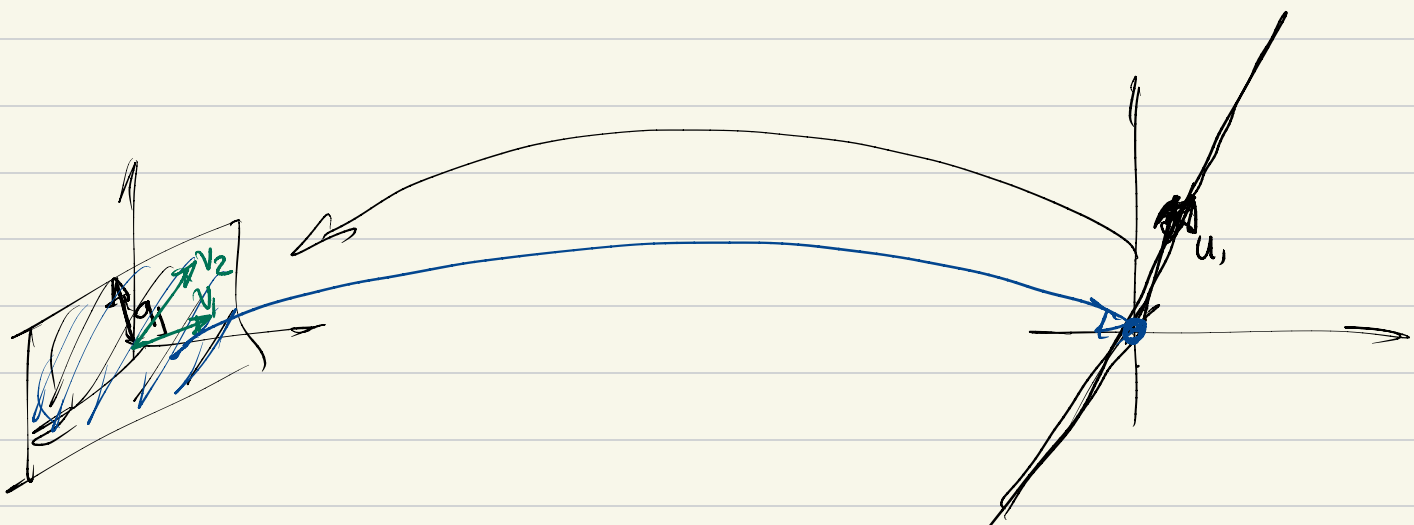
$\exists\, \beta_1, \dots, \beta_e$

$$\Rightarrow \quad z-\hat{q} = \beta_1 v_1 + \dots + \beta_e v_e$$

$$\equiv \quad z = \alpha_1 q_1 + \dots + \alpha_k q_k + \beta_1 v_1 + \dots + \beta_e v$$

$$\Rightarrow z \in \text{span}(W).$$

$\blacksquare$



(Banach Spaces) (Hilbert Spaces)

Normed vector spaces, inner product spaces

Throghout this discussion $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

Def: We say V over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ is a Normed Vector Space; if

a vector space

there exists: $\|\cdot\| : V \longrightarrow [0,\infty) = \mathbb{R}^+$ satifying;

(i) $\|v\| \geq 0$ and $\|v\| = 0$ iff $v = \underline{0}$

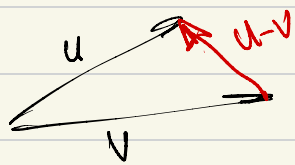
(ii) $\|\alpha v\| = |\alpha| \, \|v\|$ $\forall \alpha \in \mathbb{F}$ $\forall v \in V$

(iii) $\|v+u\| \leq \|v\| + \|u\|$ $\forall u, v \in V$

Any such function $\|\cdot\|$ is called a norm.

Short notation: $(V, \|\cdot\|)$ is a NVS

Note that a norm results is a natural notion of distance between two vectors u, v given by $d(u,v) := \|u-v\|$



Examples: $\mathbb{R}^n$, and $\|x\|_2 := \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}$

This is called the ℓ_2 norm, Euclidean norm, etc.

ℓ_1 norm

Example: $\mathbb{R}^n$ is NVS with $\|x\|_1 = \sum_{i=1}^{n} |x_i|$ (why?)

$\mathbb{R}^n$ is " " $\|x\|_\infty = \max\{|x_i| \mid i=1,\dots,n\}$

ℓ_∞ norm ove $\mathbb{R}^n$

In fact we have that:

Fact: For any $p \geq 1$, ℓ_p norm (p-norm) defined by

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}$$

Examples: $C^0([0,1])$ is NVS with any of the following norms:

$$\|f\|_2 := \left(\int_0^1 f^2(x)\,dx\right)^{\frac{1}{2}} \qquad \text{(aka } L_2\text{-norm)}$$

$$\|f\|_1 := \int_0^1 |f(x)|\,dx \qquad L_1\text{-norm}$$

$$\|f\|_\infty := \max_{x\in[0,1]} |f(x)| \qquad L_\infty\text{-norm}$$

and in fact for any $p \geq 1$: $\|f\|_p := \left(\int_0^1 |f(x)|^p\,dx\right)^{\frac{1}{p}}$ is a norm over $C^0([0,1])$.