Homework 6

(Due: March 7th 2025 at 8 pm)

1. For any matrix $A \in \mathbb{R}^{n\times n}$ show that

$$\dim(\text{span}\{A^k \mid k \geq 1\}) = \dim(\text{span}\{I, A, A^2, \ldots, A^k, \ldots\}) \leq n.$$

   Hint: Show that for any $k \geq 0$, A^k is a linear combination of $I, \ldots, A^{n-1}$.

2. Show that if A and B are similar, then not only the eigenvalues of the two matrices are the same, but also the algebraic and geometric multiplicity of them are the same for the two matrices.

3. *A computational problem.*

   (a) Find the eigenvalues of the matrix:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

   (b) Show that A does not have an eigenvalue decomposition.

   (c) Provide the Jordan Decomposition of A.

4. *Properties of symmetric matrices.* Let $A = A' \in \mathbb{R}^{n\times n}$ and $B = B' \in \mathbb{R}^{n\times n}$. Prove or provide a counterexample to each of the following statements.

   (a) If $A \succeq 0$, then $X'AX \succeq 0$ for every $X \in \mathbb{R}^{n\times k}$.

   (b) If $A \succeq 0$ and $B \succeq 0$, then $\text{trace}(AB) \geq 0$.

   (c) If $A \succeq 0$, then $A + B \succeq B$.

   (d) If $A \succeq B$, then $-B \succeq -A$.

   (e) If $A \succeq I$, then $I \succeq A^{-1}$.

   (f) If $A \succeq B \succ 0$, then $B^{-1} \succeq A^{-1} \succ 0$.

   (g) If $A \succeq B \succeq 0$, then $A^2 \succeq B^2$.

5. *Induced matrix norms.* We define the induced p-norm of $A \in \mathbb{C}^{m\times n}$ for $p \in [1, \infty]$ as

$$\|A\|_p = \max_{x\neq 0} \frac{\|Ax\|_p}{\|x\|_p}.$$

   When $p = 2$, $\|A\|_2$ is called the *spectral norm* of the matrix. One can view such norms as the maximum attenuation of the corresponding linear mapping on the unit ball.

   (a) Show that $\|A\|_p$ satisfies the axioms of matrix norms.

(b) Show that

$$\|A\|_1 = \max_j \sum_i |A_{ij}|.$$

(c) Show that

$$\|A\|_\infty = \max_i \sum_j |A_{ij}| = \|A^*\|_1.$$

6. *Properties of the spectral norm.*

(a) Show that $\|A^*A\| = \|A\|^2$.

(b) Show that the spectral norm is *unitarily invariant*, namely, $\|UAV\| = \|A\|$ for any unitary matrices U and V.

(c) Show that

$$\left\| \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \right\| = \max(\|A\|, \|B\|).$$