Homework 4

(Due: Friday February 21st, 2025 at 8 pm)

1. *Parallelogram identity.* For any Hilbert space $\mathcal{H}$ show that

$$2(\|x\|^2 + \|y\|^2) = \|x - y\|^2 + \|x + y\|^2,$$

holds for any $x, y \in \mathcal{H}$.

2. *LU Decomposition.* Find the solution $x \in \mathbb{R}^3$ for $Ax = b$ by obtaining the $LPA = U$ decomposition for the following matrix

$$A = \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{bmatrix}, \tag{1}$$

where $b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

3. *A numerical problem.* Let $A = \begin{pmatrix} -1 & 0 \\ 1 & 3 \\ 1 & 2 \\ 0 & 2 \end{pmatrix}$, $b = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$, and $\bar{b} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

   (a) Find the least-square solution for $Ax = b$.

   (b) Find the least-norm solution for $A'x = \bar{b}$.

4. *Almost orthonormal basis.* Let $u_1, u_2, \ldots, u_n$ form an orthonormal basis for an inner product space $\mathcal{V}$ and let $v_1, v_2, \ldots, v_n$ be a set of vectors in $\mathcal{V}$ such that

$$\|u_j - v_j\| < \frac{1}{\sqrt{n}}, \quad j = 1, 2, \ldots, n.$$

Show that $v_1, v_2, \ldots, v_n$ form a basis for $\mathcal{V}$.

5. *Projection onto a halfspace.* Let a be a nonzero vector in $\mathbb{R}^n$, $b \in \mathbb{R}$, and

$$\mathcal{S} = \{x \in \mathbb{R}^n : a'x \geq b\}$$

be a halfspace. Find the projection of $x \in \mathbb{R}^n$ onto $\mathcal{S}$.

6. *Inverse.* Let $A \in \mathbb{F}^{m \times n}$. Show that if A has a unique left inverse, then A is square and non-singular.

7. *A Matrix inversion lemmas.* Let $A \in \mathbb{F}^{n\times n}$, $B \in \mathbb{F}^{n\times k}$, $C \in \mathbb{F}^{k\times n}$, and $D \in \mathbb{F}^{k\times k}$. Suppose that A, D, and $D - CA^{-1}B$ are invertible. Show that

$$A^{-1}B(D - CA^{-1}B)^{-1} = (A - BD^{-1}C)^{-1}BD^{-1}.$$

(Hint: Consider the block matrix

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

and its inverse.)

8. *Moore–Penrose pseudoinverse.* A pseudoinverse of $A \in \mathbb{R}^{m\times n}$ is defined as a matrix $A^+ \in \mathbb{R}^{n\times m}$ that satisfies

$$AA^+A = A,$$
$$A^+AA^+ = A^+,$$

and AA^+ and A^+A are symmetric.

   (a) Find (with proof) the pseudoinverse of AA' in terms of A^+.
   hint: Show that $(AA')^+ = (A^+)'A^+$ and $(A'A)^+ = A^+(A^+)'$.

   (b) Suppose that A has a rank decomposition $A = BC$, for example, $B = Q \in \mathbb{R}^{m\times r}$ and $C = R \in \mathbb{R}^{r\times n}$ as in the QR decomposition. Find A^+ in terms of B and C.
   hint: Show that $(BC)^+ := C'(CC')^{-1}(B'B)^{-1}B'$.

   (c) Show that $\mathcal{R}(A^+) = \mathcal{R}(A')$ and $\mathcal{N}(A^+) = \mathcal{N}(A')$.

   (d) Show that $y = AA^+x$ and $z = A^+Ax$ are the orthogonal projections of x onto $\mathcal{R}(A)$ and $\mathcal{R}(A')$, respectively.

   (e) Show that
   $$A^+ = \lim_{\delta\to 0}(A'A + \delta I)^{-1}A' = \lim_{\delta\to 0} A'(AA' + \delta I)^{-1}.$$

   (f) Show that $x^* = A^+b$ is a least-squares solution to the linear equation $Ax = b$, i.e., $\|Ax^* - b\| \le \|Ax - b\|$ for every other x.

   (g) Show that $x^* = A^+b$ is the least-norm solution to the linear equation $Ax = b$, i.e., $\|x^*\| \le \|x\|$ for every other solution x, provided that a solution exists.

9. *Projection over convex set.* Let V be a an inner-product vector space over $\mathbb{R}$ with the inner-product $\langle\cdot,\cdot\rangle$ and let S be a convex set in V, i.e., a set such that for any two $x, y \in S$, any point $\alpha x + (1-\alpha)y$ in between x, y, where $\alpha \in [0, 1]$, belongs to S. Let $x \notin S$ be an arbitrary vector and suppose that for $\hat{x} \in S$, we have:

$$\langle x - \hat{x}, \hat{x} - v\rangle \ge 0, \quad \text{for all } v \in S.$$

Show that $\|x - \hat{x}\|^2 = \min_{v\in S} \|x - v\|^2$. Is such an $\hat{x}$ unique?