First Name: Last Name:

ECE 269: Linear Algebra and Applications, Sample Midterm Solution
Instructor: Behrouz Touri

1. For each of the following statements, determine whether it is true or false. Explain your answer. Correct answers without explanation carry no point.

   a. There exists a 2×3 matrix $A \in \mathbb{R}^{2\times 3}$, such that there are two matrices B and C with $BA = I_{3\times 3}$ and $AC = I_{2\times 2}$ (here, $I_{n\times n}$ is the $n \times n$ identity matrix).

   Solution: False.

   We know that, $\operatorname{rank}(BA) \leq \min\{\operatorname{rank}(A), \operatorname{rank}(B)\}$.
   Here, $\operatorname{rank}(B) \leq \min\{2, 3\} = 2$ whereas $\operatorname{rank}(BA) = 3$.

   b. If columns of the matrix $A \in \mathbb{F}^{n\times n}$ are independent, then the columns of A^2 are also independent.

   Solution: True. We have, $\operatorname{rank}(A) = n$, since the columns of $A \in \mathbb{F}^{n\times n}$ are independent.
   We know that, $\operatorname{rank}(A)+\operatorname{rank}(B)-k \leq \operatorname{rank}(AB) \leq \min\{\operatorname{rank}(A), \operatorname{rank}(B)\}$, where k is the number of columns in A. Therefore,

   $$2\operatorname{rank}(A) - n \leq \operatorname{rank}(A^2) \leq \operatorname{rank}(A) \implies \operatorname{rank}(A^2) = n.$$

   c. For an inner-product space V with the inner product $\langle \cdot, \cdot \rangle$, if S is a basis and $\langle x, v \rangle = 0$ for all $v \in S$, then $x = 0$.

   Solution: True.(If you assume $x \in V$)
   Since $x \in V$ and S is a basis of V. We can express x as $x = \sum_{k=1}^{n} a_k v_k$, where $a_v \in \mathbb{F}$ and $v_k \in S$ for some $n \geq 1$.
   We know that $\langle x, x \rangle = 0$ if and only if $x = 0$ and since $\langle x, v \rangle = 0 \forall v \in S$, we have $\langle x, x \rangle = \sum_{v\in S} a_v \overline{\langle x, v \rangle} = 0$ implying $x = 0$.

   False(if no assumption for x is made). x being orthogonal to the vector space V.

   d. For all matrices $A \in \mathbb{F}^{n\times n}$, $\operatorname{rank}(A) = \operatorname{rank}(A^T A)$.

   Solution: False.
   Consider $\mathbb{F}_2^{2\times 2}$ and let $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$. We have $A^T A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.
   Therefore, $\operatorname{rank}(A) \neq \operatorname{rank}(A^T A)$.

2. Prove the following statements.

(a) We say that a matrix $A \in \mathbb{F}^{n\times n}$ is a lower-triangular matrix if $A_{ij} = 0$ for $j > i$. Show that if A is an invertible lower-triangular matrix, then all its diagonal elements are non-zero. **Solution:** Suppose that $A = [a_1|\cdots|a_n]$ is an invertible lower-triangular matrix with the inverse $B = A^{-1}$. For $k \leq n$, let $A^{(k)}$ be the $k \times k$ top-left submatrix of A, i.e.,

$$A^{(k)}_{ij} = A_{ij}, \quad \text{for all } 1 \leq i, j \leq k.$$

Similarly, define $B^{(k)}$ to be the $k \times k$ top-left sub-matrix of B. You can verify that indeed $A^{(k)}B^{(k)} = I$, i.e., $A^{(k)}$ is invertible for all $1 \leq k \leq n$ with the inverse $B^{(k)}$.

Note that $A_{nn} \neq 0$ as otherwise, the last column of A would be a zero vector and hence, $[BA]_{nn} = 0$ which is contradiction with $BA = AB = I$. Since, all $A^{(k)}$ are lower triangular matrices, and they are all invertible, the same argument holds for them, which implies that $A_{kk} \neq 0$ for all $1 \leq k \leq n$.

(b) Suppose that $\mathbb{F}$ is a finite-field. Show that if A is invertible, then $A^k = I$ for some $k \geq 1$. **Solution:**

Let $|\mathbb{F}| = p$, so there are p^{n^2} possible $n \times n$ matrices over $\mathbb{F}$. Let $q = p^{n^2}$ and consider $\{A^k | k = 1, \ldots, q+1\}$. Notice that its cardinality is less than q but there are $q+1$ possible values of k. Thus, this means that $\exists i, j \in \{1, ..., q+1\}, i < j$ such that $A^i = A^j$. Recall that A is invertible, and:

$$((A^{-1})^i \cdot A^i = ((A^{-1})^{i-1} \cdot (A^{-1}A)A^{i-1} = \ldots = A^{-1}A = I,$$

meaning $(A^i)^{-1} = (A^{-1})^i$.

Using the above observation and multiplying both sides of $A^i = A^j$ by $(A^{-1})^i$, we get that: $A^{j-i} = I$, thus for some $k = j - i, A^k = I$.

(c) Is the statement in Part (b) still true for an infinite field? Justify your answer. **Solution:**

It is not true. Consider invertible $A = 2I = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ over $\mathbb{R}$. Assume in contradiction there exists $k \geq 1$ such that $A^k = I$.

So we require $A^k = (2I)^k = 2^k I = I$, so there must be $k \geq 1$ such that $2^k = 1$, which is a contradiction.

3. Consider the space of finite-energy real-valued functions $L_2([0,1])$, i.e., the space of functions $f : [0,1] \to \mathbb{R}$ with $\int_0^1 f^2(x)dx < \infty$.

   (a) Show that the mapping $\langle f(x), g(x)\rangle := \int_0^1 f(x)g(x)dx$ is an inner-product in this space.
   Solution: This is an inner product because it has the following three properties.

   i. For any $f, g, h \in L_2([0,1])$ and $a, b \in \mathbb{R}$,

$$\begin{aligned}\langle af(x) + bg(x), h(x)\rangle &= \int_0^1 (af(x) + bg(x))h(x)dx \\ &= a\int_0^1 f(x)h(x)dx + b\int_0^1 g(x)h(x)dx \\ &= a\langle f(x), h(x)\rangle + b\langle g(x), h(x)\rangle.\end{aligned}$$

   ii. For any $f, g \in L_2([0,1])$,

$$\begin{aligned}\langle f(x), g(x)\rangle &= \int_0^1 f(x)g(x)dx \\ &= \int_0^1 g(x)f(x)dx \\ &= \overline{\int_0^1 g(x)f(x)dx} \\ &= \overline{\langle g(x), f(x)\rangle}.\end{aligned}$$

   iii. For any $f \in L_2([0,1])$,

$$\langle f(x), f(x)\rangle = \int_0^1 f(x)f(x)dx \geq 0.$$

   The equation holds only if $f(x) = 0$ (almost everywhere) on $[0,1]$.

   (b) Consider the three polynomials $p_1(x) = 1$, $p_2(x) = x$, and $p_3(x) = x^2$. Using the Gram-Schmidt procedure, find out three orthonormal polynomials $q_1(x), q_2(x), q_3(x)$, such that $\text{span}(\{p_1(x), \ldots, p_i(x)\}) = \text{span}(\{q_1(x)), \ldots, q_i(x)\}$ for $i = 1, 2, 3$.

Solution: Using the Gram-Schmidt procedure,

$$
\begin{aligned}
q_1' &= p_1 = 1. \\
\|q_1'\| &= \sqrt{\int_0^1 1 \cdot 1 dx} = 1. \\
q_1 &= q_1'/\|q_1'\| = 1. \\
q_2' &= p_2 - \langle p_2, q_1 \rangle q_1 = x - \int_0^1 x \cdot 1 dx \cdot 1 = x - \frac{1}{2}. \\
\|q_2'\| &= \sqrt{\int_0^1 (x - 1/2) \cdot (x - 1/2) dx} = \sqrt{\frac{1}{12}}. \\
q_2 &= q_2'/\|q_2'\| = \sqrt{12}(x - \frac{1}{2}). \\
q_3' &= p_3 - \langle p_3, q_1 \rangle q_1 - \langle p_3, q_2 \rangle q_2 \\
&= x^2 - \int_0^1 x^2 \cdot 1 dx \cdot 1 - \int_0^1 x^2 \cdot \sqrt{12}(x - \frac{1}{2}) dx \cdot \sqrt{12}(x - \frac{1}{2}) \\
&= x^2 - x + \frac{1}{6}. \\
\|q_3'\| &= \sqrt{\int_0^1 (x^2 - x + \frac{1}{6}) \cdot (x^2 - x + \frac{1}{6}) dx} = \sqrt{\frac{1}{180}}. \\
q_3 &= q_3'/\|q_3'\| = \sqrt{180}(x^2 - x + \frac{1}{6}).
\end{aligned}
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