ECE 269- Lecture 13: Tuesday Feb 18th

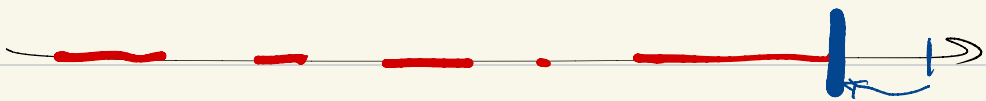
- Least norm solution to $Ax=b$
- Moore-Penrose inverse
- Multilinear forms and determinants

$A \subseteq \mathbb{R}$

$\sup A$: is the lowest of all upper bounds for A.

b is an upper bound for A if $a \leq b \quad \forall a \in A$

($b = \infty$ is an upper bound for all A)



Anouncement: Midterm solution is postad

$A \in \mathbb{R}^{m \times n}$

Recall: for overdetermined system $Ax = b$ (when $b \notin R(A)$)

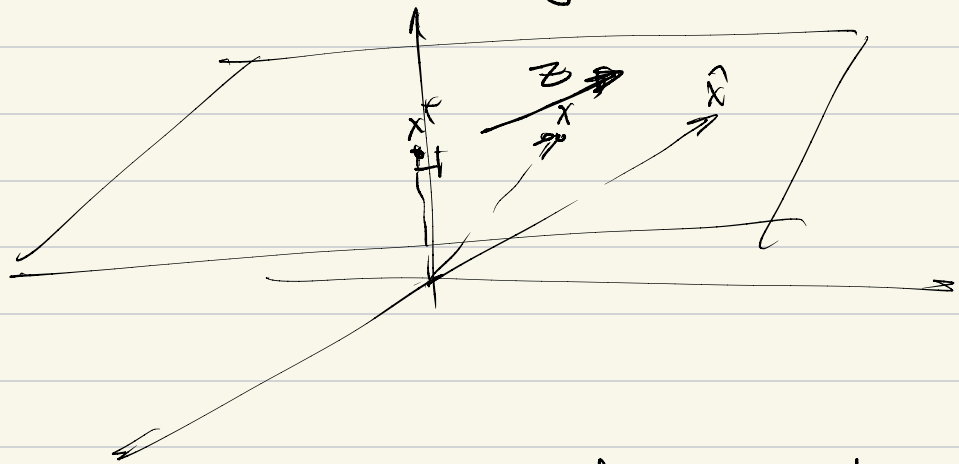
if $(A^T A)$ is invertible, $\Rightarrow$ $x^* = (A^T A)^{-1} A^T b$ is the least square solution.

i.e., $x^* = \arg\min_{x \in \mathbb{R}^n} \|Ax - b\|$

(Read Prof. Kim's note on Tykhonov / Ridge ~~regul~~ regularization)

$A \in \mathbb{R}^{m \times n}$

Suppose $Ax = b$ has many solutions $S \subseteq \mathbb{R}^n$.



$$\begin{cases} A\tilde{x} = b \\ A\hat{x} = b \end{cases} \Rightarrow A(\tilde{x} - \hat{x}) = 0$$

In this case we are often interested in the least norm solution

$$x^* = \arg\min_{x \in S} \|x\|$$

By the orthogonality principle, x^* is the unique point:

$$\langle x^*, z \rangle = 0 \quad \forall\ z \in N(A) \qquad \forall z : Az = 0$$

$$\equiv$$

$$x^* \in R(A^T) \Rightarrow \underline{x^* = A^T q} \quad \text{for some } q \in \mathbb{R}^m$$

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if AA^T is invertible

$$x^* \in S \Rightarrow Ax^* = b \equiv AA^T q = b \Rightarrow q = (AA^T)^{-1} b \Rightarrow x^* = A^T (AA^T)^{-1} b$$

Note that (AA^T) is invertible iff A is fat and A is full rank

If A is not full rank

$$v = r^T \tilde{A}$$

$$A = \begin{bmatrix} \tilde{A} \\ \hline v \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_{m-1} \\ b_m \end{bmatrix} \leftarrow \tilde{b}$$

Note that if $\tilde{A}x = \tilde{b}$

$$A = \begin{bmatrix} \tilde{A} \\ \hline v \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix} \leftarrow \tilde{b} \qquad v = q^T \tilde{A}$$

$$\Downarrow$$

$$vx = r^T \tilde{A} \tilde{x} = r^T \tilde{b} \Rightarrow b_n = r^T \tilde{b}$$

$\Rightarrow$ $Ax = b$ has a solution iff $\begin{cases} \tilde{A}x = \tilde{b} \\ b_n = r^T \tilde{b} \end{cases}$

$\Rightarrow$ if A is not full rank (i.e., $\text{rank}(A) = k < m$) let $\tilde{A}$ be a matrix with k independent rows of A; let $\tilde{b}$ be the corresponding elements of b; then $Ax = b$

$$\Rightarrow \tilde{A}x = \tilde{b}$$

- least square solution:

(overdetermined) $x^* = (A^T A)^{-1} A^T b$ (if $A^T A$ is invertible)

- least norm solution:

(underdetermined) $x^* = A^T (A A^T)^{-1} b$ (if $A A^T$ is ")

"right" - determined: $x^* = A^{-1} b$

Moore - Penrose inverse of $A \in \mathbb{R}^{m \times n}$:

For any non-zero matrix A, we define $\underline{A^{\dagger} \in \mathbb{R}^{n \times m}}$ to be a Moore - Penrose pseudoinverse of A if:

(i) $AA^{\dagger}A = A$

(ii) $A^{\dagger}AA^{\dagger} = A^{\dagger}$

(iii) $(AA^{\dagger})^{T} = AA^{\dagger}$

(iv) $(A^{\dagger}A)^{T} = A^{\dagger}A$

Fact: M-P pseudo-inverse of any $m \times n$ A, exists and it is unique!

Pf: Uniqueness: Suppose that we have B and C satisfying i-iv

(i) $AA^{\dagger}A = A$

(ii) $A^{\dagger}AA^{\dagger} = A^{\dagger}$

(iii) $(AA^{\dagger})^T = AA^{\dagger}$

(iv) $(A^{\dagger}A)^T = A^{\dagger}A$

$$AB = ACAB = C^TA^TB^TA^T = C^T(ABA)^T$$

$$= C^TA^T = AC.$$

$$\boxed{(FG)^T = G^TF^T}$$

Similarily, $BA = CA$.

$$B = BAB = BAC = CAC = C.$$

Existence: $Q = \lim_{\mu \to 0} \left((A^T A) + \mu I\right)^{-1} A^T$.

Fun Fact: if $A^T A$ invertible $A^+ = (A^T A)^{-1} A^T$. (if $A^T A$ is invertible $Ax = b$)

if AA^T invertible $A^+ = A^T (AA^T)^{-1}$

if A is invertible $A^+ = A^{-1}$

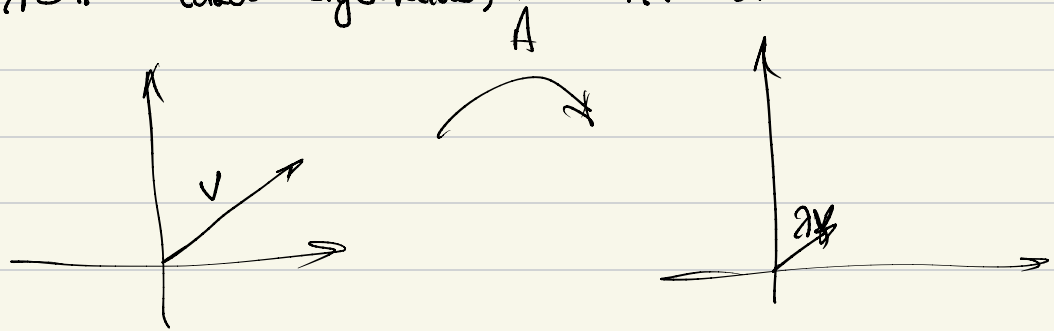
For $Ax = b$, $x^+ = A^+ b$ is what to look for!

$\boxtimes$

Eigenvalue, eigenvectors, determinant, multilinear form:

For $A \in \mathbb{F}^{n \times n}$, we say $0 \neq v \in \mathbb{F}^n$ is called an eigenvector of A

if $\exists\ \lambda \in \mathbb{F}$ (aka eigenvalue): $Av = \lambda v$.



Focus: $\mathbb{F} = \mathbb{R}$.

$A \in \mathbb{R}^{n \times n}$

$$Av = \lambda v \equiv \underbrace{(A - \lambda I)}_{A} v = \underline{0} \equiv (A - \lambda I) \text{ is not invertible.}$$

Q: How to determine whether $(A - \lambda I)$ is not invertible?

Multilinear forms: fix $V = \mathbb{R}^n$ $n \geq 1$. We say that $f: V \times V \to \mathbb{R}$ is a bilinear form if

(i) $f(\bullet, v)$ is linear for any fixed v

(ii) $f(u, \circ)$ is linear for any fixed u.

Fact: f is bilinear (in $\mathbb{R}^n$) iff

$$f(u,v) = u^T A v \quad \text{for some } A \in \mathbb{R}^{n \times n}.$$

For a bilinear form $f(u,v) = u^T A v$, $f(u,u) = u^T A u$ is called a quadratic form.

$$u^T A u = f(u,u) = \sum_{i,j} A_{ij} u_i u_j$$

Similarily, we define an m-multilinear form over $\mathbb{R}^n$ to be a function $f: \underbrace{V \times V \times \cdots \times V}_{m \text{ times}} \to \mathbb{R}$, such that f is a linear function of each variable, when other variables are fixed.

We say that f is a alternating (alt.) multilinear form, if f is a multilinear form and $f(v_1, v_2, \ldots, v_i, \ldots, v_j, \ldots, v_m) = 0$

$$\parallel$$

$$f(v_1, \ldots, v_m) = 0 \quad \text{if} \quad v_i = v_j \quad \text{for some } i \neq j.$$

For alt multilinear forms:

Fact: (i) $f(v_1, v_2, \dots, v_m) = -f(v_2, v_1, \dots, v_m)$

in general $f(v_1, \dots, v_i, \dots, v_j, \dots, v_m) = -f(v_1, \dots, v_j, \dots, v_i, \dots, v_m)$

(ii) if $v_1, \dots, v_m$ are dependent:

$f(v_1, \dots, v_m) = 0$!

$v_i \in \mathbb{R}^n$

(iii) An alternating multilinear form in $\mathbb{R}^n$ with $m=n$ is

essentially unique!