ECE 269: Linear Algebra and Applications, Sample-Final Exam
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1. Prove or disprove each of the following statements (to disprove you need to provide an example and explain why the example disproves the statement):

   (a) The matrix $A = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$ is diagonalizable.

   (b) For an $n \times n$ real-valued matrix A, $\max_{x \neq 0} \frac{x^T A x}{x^T x} = \lambda_{max}(A)$ where $\lambda_{max}(A)$ is the eigenvalue with the maximum modulus (absolute value).

   (c) For a real-valued $n \times n$ and full-rank matrix A, $\text{span}(\{A^k \mid k \geq 1\})$ has dimension n^2. Here, $\{A^k \mid k \geq 1\}$ is the set of all powers of A.

   (d) If the $n \times n$ matrix A is a positive-definite matrix then the diagonal elements $a_{ii} > 0$ for all $i = 1, \ldots, n$.

2. Let $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & -1 \end{pmatrix}$.

   (a) Find a QR decomposition for A.

   (b) Find a PLU decomposition for A.

   (c) Find the (compact) SVD for A.

   (d) Find a full SVD for A.

   (e) Determine whether a solution to $Ax = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ exists for an $x \in \mathbb{R}^3$. If so, find the least norm solution to this problem. If the solution does not exist, find the least square solution to the problem.

   (f) Determine the spectral norm and Frobenius norm of A.

3. Let $A, B \in \mathbb{R}^{m \times n}$ be matrices such that $R(B) \perp R(A)$, i.e., any vector in the range space of B is orthogonal to any vector in the range space of A.

   (a) Show that $\|A + B\|_F^2 = \|A\|_F^2 + \|B\|_F^2$, where $\|\cdot\|_F$ is the Frobenius-norm.

   (b) Provide an example to show that part (a) does not hold for the spectral norm.

4. For all $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \in \mathbb{R}^3$, define $\langle x, y \rangle = x_1 y_1 + \frac{1}{2} x_1 y_3 + \frac{1}{2} x_3 y_1 + 2x_2 y_2 + x_3 y_3$.

   (a) Prove that $\langle x, y \rangle$ is a valid inner product (over $\mathbb{R}$).

   (b) Find an orthonormal basis for $\mathbb{R}^3$ using the above inner product.

5. (a) Let $A = \begin{pmatrix} 5 & -6 & 3 \\ -6 & 4 & -6 \\ 3 & -6 & -4 \end{pmatrix}$. Is A diagonalizable?

   (b) Let B be a complex-valued square matrix with $rank(B) = 2$. The characteristic polynomial of B is given by $c_B(\lambda) = (\lambda^2 + 4)\lambda^3$.

      i. Find the dimension of B and all its eigenvalues.

      ii. Is B diagonalizable? Explain.

      iii. Show that $B + I$ is invertible and find the eigenvalues of $(B + I)^{-1}$.