① $q_s := q_1$ & $q_v := [q_2, q_3, q_4]^T \in \mathbb{R}^3$

$\|x\| := \sqrt{x^T x}$, $\|q\| \neq 0$

a) $f(q) := \log \|q\|$ $\quad g(q) := \frac{q_v}{\|q_v\|}$ $\quad h(q) := \arccos \frac{q_s}{\|q\|}$

$\frac{\partial f}{\partial q} = \frac{\partial f}{\partial q}(\log\|q\|) = \frac{\partial f}{\partial q}(\log(\sqrt{q^T q})) = \frac{1}{\|q\|}\frac{\partial f}{\partial q}((q^T q)^{1/2}) = \frac{1}{\|q\|}\left(\frac{1}{2}(q^T q)^{-1/2} \cdot \frac{\partial}{\partial q}(q^T q)\right)$

$\|q\| = \sqrt{q_1^2 + q_2^2 + q_3^2 + q_4^2}$

$\frac{1}{\|q\|}\left(\frac{1}{2} \cdot \frac{1}{\sqrt{q^T q}} \frac{\partial}{\partial q}(q^T q)\right)$ (Note $\|q\| = \sqrt{q^T q}$)

$\rightarrow \frac{1}{\|q\|} \cdot \frac{1}{2\|q\|} \cdot \frac{\partial}{\partial q}(q^T q)$

$q^T q = q_1^2 + q_2^2 + q_3^2 + q_4^2 \Rightarrow \frac{\partial}{\partial q}(q^T q) = 2q$

[Substitute] $\frac{1}{\|q\|}\frac{1}{2\|q\|} 2q$

$$\boxed{\frac{\partial}{\partial q} f(q) = \frac{q}{\|q\|^2}}$$

$g(q) = \frac{q_v}{\|q\|}$ derivative

$q_v = \begin{bmatrix} q_2 \\ q_3 \\ q_4 \end{bmatrix}$

$g(q) = \begin{bmatrix} \frac{q_2}{\|q\|} \\ \frac{q_3}{\|q\|} \\ \frac{q_4}{\|q\|} \end{bmatrix}$

$\frac{\partial}{\partial q} g(q) = \frac{\|q\| \frac{\partial}{\partial q} q_v - q_v \frac{\partial}{\partial q}\|q\|}{\|q\|^2}$

$$\begin{bmatrix} \frac{\partial}{\partial q_1}(q_2) & \cdots & \frac{\partial}{\partial q_4}(q_2) \\ \vdots & \vdots & \frac{\partial}{\partial q_4}(q_3) \\ \frac{\partial}{\partial q_1}(q_4) & \cdots & \frac{\partial}{\partial q_4}(q_4) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \frac{\partial q_v}{\partial q}$$

$\frac{\partial \|q\|}{\partial q} = \frac{q}{\|q\|} = \frac{q}{\sqrt{q_1^2 + \dots q_4^2}}$ & Numerator $= \|q\| \frac{\partial q_v}{\partial q} - q_v \frac{\partial \|q\|}{\partial q}$

$\|q\| \frac{\partial q_v}{\partial q}$: $\|q\| \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \|q\| & 0 & 0 \\ 0 & 0 & \|q\| & 0 \\ 0 & 0 & 0 & \|q\| \end{bmatrix}$

$q_v \frac{\partial \|q\|}{\partial q}$: $q_v = \begin{bmatrix} q_2 \\ q_3 \\ q_4 \end{bmatrix}$, $\frac{\partial \|q\|}{\partial q} = \frac{q}{\|q\|} \rightarrow q_v \frac{\partial \|q\|}{\partial q} = \frac{1}{\|q\|}\begin{bmatrix} q_2 \\ q_3 \\ q_4 \end{bmatrix}[q_1\ q_2\ q_3\ q_4] =$

$\frac{1}{\|q\|}\begin{bmatrix} q_2 q_1 & q_2^2 & q_2 q_3 & q_2 q_4 \\ q_3 q_1 & q_3 q_2 & q_3^2 & q_3 q_4 \\ q_4 q_1 & q_4 q_2 & q_4 q_3 & q_4^2 \end{bmatrix} \xrightarrow[\text{SUB}]{\text{Sum}} \begin{bmatrix} 0 & \|q\| & 0 & 0 \\ 0 & 0 & \|q\| & 0 \\ 0 & 0 & 0 & \|q\| \end{bmatrix} - \frac{1}{\|q\|}\begin{bmatrix} q_2 q_1 & q_2^2 & q_2 q_3 & q_2 q_4 \\ q_3 q_1 & q_3 q_2 & q_3^2 & q_3 q_4 \\ q_4 q_1 & q_4 q_2 & q_4 q_3 & q_4^2 \end{bmatrix}$

$$\frac{\partial g(q)}{\partial q} = \frac{1}{\|q\|^2}\left(\begin{bmatrix} 0 & \|q\| & 0 & 0 \\ 0 & 0 & \|q\| & 0 \\ 0 & 0 & 0 & \|q\| \end{bmatrix} - \frac{1}{\|q\|}\begin{bmatrix} q_2 q_1 & q_2 & \dots \end{bmatrix} \text{(SAME AS BEFORE)}\right)$$

$$\boxed{\frac{\partial g(q)}{\partial q} = \frac{1}{\|q\|^2}\left(\|q\|\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \frac{q \cdot q^T}{\|q\|}\right)}$$

$$\cos^{-1}\left(\frac{q_3}{\|q\|}\right) \rightarrow \frac{\partial h(q)}{\partial q} = -\frac{1}{\sqrt{1-\left(\frac{q_3}{\|q\|}\right)^2}}\frac{\partial}{\partial q}\left(\frac{q_3}{\|q\|}\right)$$

Quotient Rule: $\frac{\partial}{\partial q}\left(\frac{q_3}{\|q\|}\right) = \frac{\|q\|\frac{\partial q_3}{\partial q} - q_3\frac{\partial \|q\|}{\partial q}}{\|q\|^2}$

$\frac{\partial q_3}{\partial q} = [1 \quad 0 \quad 0 \quad 0]$ (be $q_3 = q_1$)

$\frac{\partial \|q\|}{\partial q} = \frac{q}{\|q\|}$, $q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}$

$$\frac{\partial}{\partial q}\left(\frac{q_3}{\|q\|}\right) = \frac{\|q\|[1\ 0\ 0\ 0] - q_3\frac{q}{\|q\|}}{\|q\|^2}$$

$$\frac{\partial}{\partial q}\left(\frac{q_3}{\|q\|}\right) = \frac{[\|q\|\ 0\ 0\ 0] - \frac{q_3}{\|q\|}q^T}{\|q\|^2}$$

$$\frac{\partial h(q)}{\partial q} = -\frac{1}{\sqrt{1-\left(\frac{q_3}{\|q\|}\right)^2}} \cdot \frac{[\|q\|\ 0\ 0\ 0] - \frac{q_3}{\|q\|}q^T}{\|q\|^2}$$

$$= \frac{-1}{\sqrt{1-\left(\frac{q_3}{\|q\|}\right)^2}} \cdot \frac{[1\ 0\ 0\ 0]}{\|q\|} - \frac{q_3}{\|q\|^3\sqrt{1-\left(\frac{q_3}{\|q\|}\right)^2}}q^T$$

$$\boxed{\frac{\partial h(q)}{\partial q} = \frac{-1}{\sqrt{1-\left(\frac{q_3}{\|q\|}\right)^2}} \cdot \left(\frac{[1\ 0\ 0\ 0]}{\|q\|} - \frac{q_3}{\|q\|^3}q^T\right)}$$

b)

```python
import jax.numpy as jnp
from jax import grad, jacfwd
import math

class VectorOperations:
    @staticmethod
    def computeLogNorm(inputVec):
        vecMagnitude = jnp.linalg.norm(inputVec)
        return jnp.log(vecMagnitude)

    @staticmethod
    def computeNormalizedSubvector(inputVec):
        subVec = inputVec[1:]
        subMagnitude = jnp.linalg.norm(subVec)
        return subVec / subMagnitude

    @staticmethod
    def computeFirstComponentAngle(inputVec):
        firstComp = inputVec[0]
        vecMagnitude = jnp.linalg.norm(inputVec)
        return jnp.arccos(firstComp / vecMagnitude)

logNormGrad = grad(VectorOperations.computeLogNorm)
normSubvectorJacobian = jacfwd(VectorOperations.computeNormalizedSubvector)
angleGrad = grad(VectorOperations.computeFirstComponentAngle)

testVec = jnp.array([1.0, 2.0, 3.0, 4.0])

print("Gradient of f(q):", logNormGrad(testVec))
print("Jacobian of g(q):", normSubvectorJacobian(testVec))
print("Gradient of h(q):", angleGrad(testVec))
```

PROBLEMS OUTPUT DEBUG CONSOLE TERMINAL PORTS

```
(base) yusufmorsi@CoolBeanMachine Desktop % /usr/local/bin/python3 /Users/yusufmorsi/Desktop/main.py
Gradient of f(q): [0.03333333 0.06666666 0.09999999 0.13333333]
Jacobian of g(q): [[ 0.          0.16008219 -0.03841972 -0.0512263 ]
 [ 0.         -0.03841972  0.12806575 -0.07683945]
 [ 0.         -0.0512263  -0.07683945  0.08324274]]
Gradient of h(q): [-0.17950548  0.01237969  0.01856953  0.02475937]
```

(2) $p(x;\mu,\Sigma) = \frac{1}{\sqrt{(2\pi)^d \det(\Sigma)}} e^{(-\frac{1}{2}\|x-\mu\|^2_2}$

- MAP

- $\min_{\mu \in \mathbb{R}^d} f(\mu) := \frac{1}{2}\left(\sum_{i=1}^{n} \|z_i - \mu\|^2_{\Sigma}\right) + \frac{1}{2}\|\mu - \mu_0\|^2_{\Sigma_0}$

$\|z_i - \mu\|^2_{\Sigma} = (z_i - \mu)^T \Sigma^{-1} (z_i - \mu)$

$\|\mu - \mu_0\|^2_{\Sigma_0} = (\mu - \mu_0)^T \Sigma_0^{-1} (\mu - \mu_0)$

$f(\mu) = \frac{1}{2}\sum_{i=1}^{n} (z_i - \mu)^T \Sigma^{-1} (z_i - \mu) + \frac{1}{2}(\mu - \mu_0)^T \Sigma_0^{-1} (\mu - \mu_0)$

$\nabla f(\mu) = \frac{\partial}{\partial \mu}\left[\frac{1}{2}\sum_{i=1}^{n} (z_i - \mu)^T \Sigma^{-1} (z_i - \mu) + \frac{1}{2}(\mu - \mu_0)^T \Sigma_0^{-1} (\mu - \mu_0)\right]$

$\frac{\partial}{\partial \mu}\left(\sum_{i=1}^{n} (z_i - \mu)^T \Sigma^{-1} (z_i - \mu)\right) = -\Sigma^{-1}\sum_{i=1}^{n} z_i + n\Sigma^{-1}\mu$

$\frac{\partial}{\partial \mu}\left((\mu - \mu_0)^T \Sigma_0^{-1} (\mu - \mu_0)\right) = \Sigma_0^{-1}(\mu - \mu_0)$

$\nabla f(\mu) = -\Sigma^{-1}\sum_{i=1}^{n} z_i + n\Sigma^{-1}\mu + \Sigma_0^{-1}(\mu - \mu_0) \rightarrow$ $\nabla^2 f(\mu) = \frac{\partial^2}{\partial \mu^2} f(\mu)$

$\nabla^2 f(\mu) = n\Sigma^{-1} + \Sigma_0^{-1}$

$\mu^{(t+1)} = \mu^{(t)} - (\nabla^2 f(\mu^{(t)}))^{-1} \nabla f(\mu^{(t)}) \longrightarrow$ w/ GVB

$\mu^{(t+1)} = \mu^{(t)} - (n\Sigma^{-1} + \Sigma_0^{-1})^{-1}\left(-\Sigma^{-1}\sum_{i=1}^{n} z_i + n\Sigma^{-1}\mu^{(t)} + \Sigma_0^{-1}(\mu^{(t)} - \mu_0)\right)$

$= (n\Sigma^{-1} + \Sigma_0^{-1})^{-1}\left(\Sigma^{-1}\sum_{i=1}^{n} z_i + \Sigma_0^{-1}\mu_0\right)$

$$\boxed{\mu_1 = \mu_0 + (\Sigma_0^{-1} + n\Sigma^{-1})^{-1}\left(\sum_{i=1}^{n} \Sigma^{-1}(z_i - \mu_0)\right)}$$

DATE

③ a) $R = [\text{In the PDF}]$

We'll use Python.

- Gets us $[.577 \quad .577 \quad .577]^T = \alpha$

$\theta = 1.813$

```python
import numpy as np
import math

rotationMatrix = np.asarray([
    [0.1729, -0.1468, 0.9739],
    [0.9739, 0.1729, -0.1468],
    [-0.1468, 0.9739, 0.1729]
])

rotationAngle = math.acos((np.matrix.trace(rotationMatrix) - 1) / 2)
rotationAxis = 1/(2 * math.sin(rotationAngle)) * np.asarray([
    rotationMatrix[2, 1] - rotationMatrix[1, 2],
    rotationMatrix[0, 2] - rotationMatrix[2, 0],
    rotationMatrix[1, 0] - rotationMatrix[0, 1]
])

print(rotationAngle)
print(rotationAxis)
```

PROBLEMS OUTPUT DEBUG CONSOLE TERMINAL PORTS

```
(base) yusufmorsi@CoolBeanMachine Desktop % /usr/local/bin/python3 /Users/yus
1.8138318028969072
[0.57731619 0.57731619 0.57731619]
```

b) $\hat{\alpha} = \begin{bmatrix} 0 & -.577\ldots & .577\ldots \\ .577\ldots & 0 & -.577\ldots \\ -.577\ldots & .577\ldots & 0 \end{bmatrix}$

$$\det(\hat{\alpha} - \lambda I) = \begin{vmatrix} -\lambda & -.57\ldots & .577\ldots \\ .57\ldots & -\lambda & -.577\ldots \\ -.57\ldots & .577\ldots & -\lambda \end{vmatrix} = -\lambda(\lambda^2 + \underbrace{.999}_{\approx 1})$$

$= 0 \rightarrow -\lambda(\lambda^2 + 1) = 0$

thus ... $\boxed{\lambda = 0}$ & $\lambda^2 + 1 = 0$

$\lambda^2 = -1$

$\boxed{\lambda = \sqrt{-1} = \pm i}$

So $\boxed{\lambda = 0, \pm i}$

c) $R - \lambda I = \begin{bmatrix} .1729 - \lambda & -.1468 & .9739 \\ .9739 & .1729 - \lambda & -.1468 \\ -.1468 & .9739 & .1729 - \lambda \end{bmatrix}$

$\det(R - \lambda I) = \ldots$ used Python!

To find Eigenvector associated w/ $\lambda = 1$ is $\begin{bmatrix} .57735 \\ .57735 \\ .57735 \end{bmatrix}$ (same w/ α's in pt a)

```python
import numpy as np

rotationMatrix = np.array([[0.1729, -0.1468, 0.9739],
                           [0.9739, 0.1729, -0.1468],
                           [-0.1468, 0.9739, 0.1729]])

evalues, evectors = np.linalg.eig(rotationMatrix)

print("Eigenvalues:", evalues)
print("Eigenvectors:", evectors)

unitIndex = np.argmin(np.abs(evalues - 1))
rotationAxis = evectors[:, unitIndex]
print("Eigenvector for Lambda=1:", rotationAxis)
```

PROBLEMS OUTPUT DEBUG CONSOLE TERMINAL PORTS

```
(base) yusufmorsi@CoolBeanMachine Desktop % /usr/local/bin/python3 /Users/yusu
Eigenvalues: [-0.24065+0.97055467j -0.24065-0.97055467j  1.     +0.j        ]
Eigenvectors: [[-0.28867513-0.5j -0.28867513+0.5j  0.57735027+0.j ]
 [-0.28867513+0.5j -0.28867513-0.5j  0.57735027+0.j ]
 [ 0.57735027+0.j   0.57735027-0.j   0.57735027+0.j ]]
Eigenvector for Lambda=1: [0.57735027+0.j 0.57735027+0.j 0.57735027+0.j]
```

d) $\gamma = \alpha \times \beta$ — $R\beta \cos(\theta)\beta + \sin(\theta)\gamma$

$\beta \cdot (R\beta) = \beta \cdot (a\beta + b\gamma)$ $\boxed{\text{since } \beta \perp \gamma \ \& \ \beta \cdot \gamma = 0}$

$\beta \cdot (R\beta) = a(\beta \cdot \beta) + b(\beta \cdot \gamma) \longrightarrow \beta \cdot (R\beta) = a(\beta \cdot \beta)$

$\|\beta\|^2 = 1 \longrightarrow \beta \cdot \beta = 1 \Rightarrow \beta \cdot (R\beta) = a$

$\beta \times (R\beta) = \beta \times (a\beta + b\gamma) \rightarrow \beta \times (R\beta) = a\underbrace{(\beta \times \beta)}_{=0} + b(\beta \times \gamma)$

$\beta \times (R\beta) = b(\beta \times \gamma)$ & we know that

$\gamma = \alpha \times \beta$ & $\beta \times \gamma = \|\beta\|\|\gamma\| \sin\theta$

& we also know that $\|\beta\| = 1$ & $\|\gamma\| = 1$

$\underline{b = \sin(\theta)}$ $\xrightarrow[\text{(after substituting)}]{} \boxed{R\beta = \cos(\theta)\beta + \sin(\theta)\gamma}$