ECE 269- Lecture 11: Tuesday Feb 11th

- Least square solution to $Ax=b$
- Orthogonality principle
- Least norm solution to $Ax=b$

- Midterm on Thursday (Bring student ID)

$Ax = b$

Orthogonality principle:

Let $\mathcal{H}$ be a Hilbert space and V be a closed subspace of $\mathcal{H}$.

Then for every $y \in V$;

(a) $\min_{v \in V} \|y - v\|$ exists let $y^* = \arg\min_{v \in V} \|y - v\|$

(b) it is unique

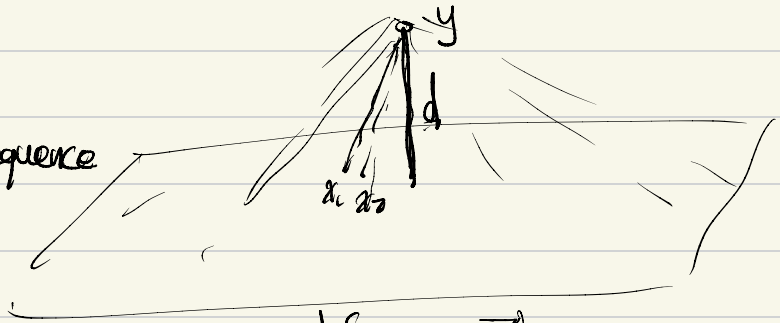
(c) $\langle y - y^*, z \rangle = 0 \quad \forall z \in V.$

Def: We say that $S \subseteq \mathcal{H}$ is closed if every Cauchy sequence in S converges to a point in S.

We say b is a lower bound for $D \subseteq \mathbb{R}$ if $\forall d \in D$ $b \leq d$.
Largest such b is called $\inf_{d \in D} d$

Pf: Suppose that $d = \inf_{v \in V} \|y - v\|$

And let $\{x_k\} \in V$; $\lim_{k \to \infty} \|y - x_k\| \to d$.

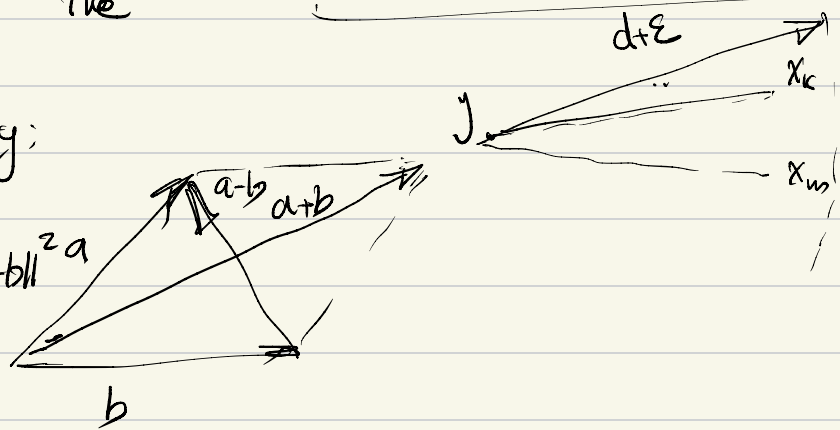


Claim: $\{x_k\}$ is a Cauchy sequence

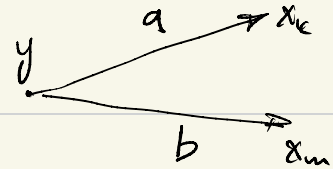
For this we use the

Parallelogram identity:

$$2\|a\|^2 + 2\|b\|^2 = \|a+b\|^2 + \|a-b\|^2$$



$$\Rightarrow \|x_k - x_m\|^2 + \underline{\underline{\|y - x_k + y - x_m\|^2}}$$



$$= 2\|y - x_k\|^2 + 2\|y - x_m\|^2$$

$$\Rightarrow \|x_k - x_m\|^2 = 2\|y - x_k\|^2 + 2\|y - x_m\|^2 - 4\left\|y - \frac{x_k + x_m}{2}\right\|^2$$

Now for $\varepsilon > 0$; $\exists N_\varepsilon : d \leq \|y - x_k\| < (d+\varepsilon) \quad \forall k \geq N_\varepsilon$.

if $k, m \geq N_\varepsilon$,

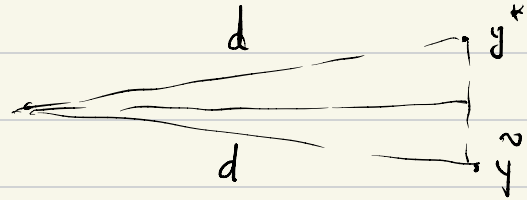
$$\|x_k - x_m\|^2 = 2\|y - x_k\|^2 + 2\|y - x_m\|^2 - 4\left\|y - \frac{x_k + x_m}{2}\right\|^2$$

$$\leq 2(d+\varepsilon)^2 + 2(d+\varepsilon)^2 - 4d^2 = 8d\varepsilon + 4\varepsilon^2$$

$\Rightarrow \{x_k\}$ is a Cauchy sequence in $V \Rightarrow \lim_{k \to \infty} x_k = y^*$

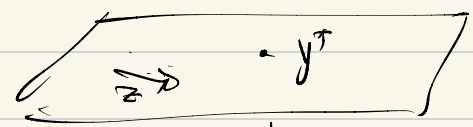
exists and $\|y - y^*\| = \lim_{k\to\infty} \|y - x_k\| = d = \inf_{x\in V} \|y - x\|$.

(b) Using the parallelogram



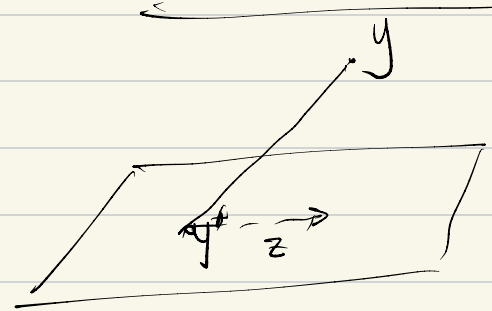
identity you can show that if $y^*, \tilde{y} \in \arg\min_{x\in V} \|y - x\|$

$\Rightarrow$ $\frac{y^* + \tilde{y}}{2} \in V$ is a closer point to y.



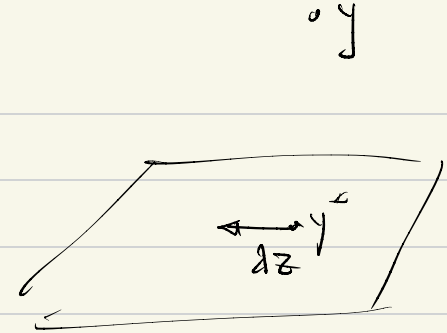
(c) Want to show that $\langle y - y^*, z \rangle = 0$

$\forall z \in V$;



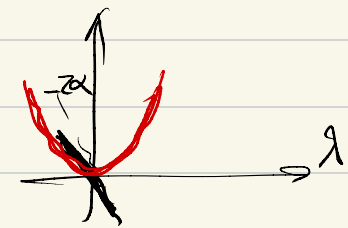
Since y^* is the closest point to y:

$$\|y-y^*\|^2 \leq \|y-(y^*+\lambda z)\|^2 = \|(y-y^*)-\lambda z\|^2$$



$$= \underbrace{\|y-y^*\|^2 - 2\lambda \underbrace{\langle y-y^*, z\rangle}_{\alpha} + \lambda^2 \underbrace{\|z\|^2}_{\beta}}_{f(\lambda)}$$

$$\equiv \quad 0 \leq -2\alpha\lambda + \beta\lambda^2 \qquad \text{for all } \lambda$$

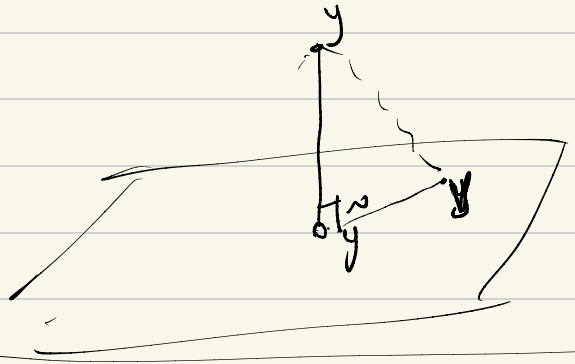


$\Rightarrow$ Whether $2\alpha > 0$ or $2\alpha < 0$; $\exists\, \lambda^* \approx 0$; $f(\lambda^*) < 0$

$\Rightarrow$ in order to have $f(\lambda) \geq 0 \quad \forall \lambda$, we need to have $\alpha = \langle y-y^*, z\rangle = 0$.

$\Rightarrow \quad \langle y - y^{+}, z \rangle = 0 \quad \forall z \in V.$

Conversely if $\langle y - \tilde{y}, z \rangle = 0 \quad \forall z \in V;$



$$\|y - v\|^2 = \|y - \tilde{y}\|^2 + \|\tilde{y} - v\|^2 + 0 \geq \|y - \tilde{y}\|^2.$$

$$\equiv \quad 0 \leq -2\alpha\lambda + \beta\lambda^2 \quad \Rightarrow \quad \frac{2\alpha}{\lambda} \leq \beta \quad \forall \lambda$$

$A \in \mathbb{R}^{m \times n}$

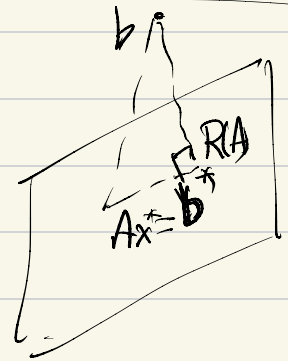
$Ax = b$ suppose that $b \notin R(A)$

Recall: $b \in R(A)$; $A = Q \begin{bmatrix} \tilde{R} & U \end{bmatrix} P^T$

then $x^* = P^{-1} \begin{bmatrix} \tilde{R}^{-1} \tilde{x} \\ 0 \end{bmatrix} = P^{-1} \begin{bmatrix} \tilde{R}^{-1} Q^T b \\ 0 \end{bmatrix}$ is a solution

Let's find

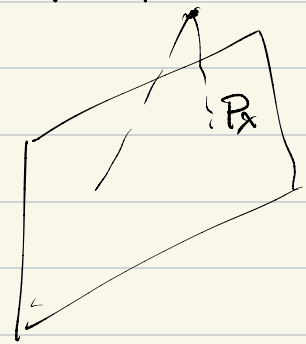
$$x^* = \arg\min_{x \in \mathbb{R}^n} \|Ax - b\|^2$$



By the orthogonality principle we know that

$$\langle b - \underbrace{Ax^*}_{b^*}, z \rangle = 0 \quad \forall z \in R(A)$$

P is a projection matrix if $P = P^T$ and $P^2 = P$



We know if Q is a matrix with orthonormal columns, QQ^T is a projection matrix.

Note that $R(QQ^T) = R(Q)$ for such matrices.

$\Rightarrow$ if $b \notin R(A)$, if we know $b^* = \mathrm{Proj}_{R(A)}(b)$, then

$$x^* = P^{-1}\begin{bmatrix} \tilde{R}^{-1}\tilde{x} \\ 0 \end{bmatrix} = P^{-1}\begin{bmatrix} \tilde{R}^{-1}\tilde{Q}^T b^* \\ 0 \end{bmatrix}$$ is a solution to $\min_{x \in \mathbb{R}^n} \|Ax - b\|^2$

$$\Rightarrow b^* = QQ^T b \Rightarrow x^* = P^{-1}\begin{bmatrix} \tilde{R}^{-1}\tilde{Q}^T Q Q^T b \\ 0 \end{bmatrix} = P^{-1}\begin{bmatrix} \tilde{R}^{-1}\tilde{Q}^T b \\ 0 \end{bmatrix}.$$

Note that $A'A$ is invertible; $A(A'A)^{-1}A'$ is a projection matrix into $R(A(A'A)^{-1}A') = R(A)$.

projection into $R(A)$

$$\Rightarrow \quad A\underbrace{(A'A)^{-1}A'b} = b^*$$

$$\equiv \quad A(x^*) = b^* \quad \text{for } x^* = (A'A)^{-1}A'b.$$

$$\min \quad \|Ax-b\|^2 + \mu\|x\|^2$$