Homework 2

ECE 269: Linear Algebra and Applications
Homework #1
Instructor: Behrouz Touri
Due Date: Friday Jan 24th, 8 pm (submission through Gradescope)

1. *Integrator!* Let $U = \{u_1, \ldots, u_m\}$ be a set of m vectors in a vector space V and $W = \{w_1, \ldots, w_m\}$ where
$$w_k = u_1 + \cdots + u_k$$
for $k = 1, 2, \ldots, m$.
   (a) Show that $\text{span}(U) = \text{span}(W)$.
   (b) Show that U is a basis if and only if W is a basis.

2. Suppose U, W are both five-dimensional subspaces of $\mathbb{R}^9$. Show that $U \cap W \neq \{\underline{0}\}$.

3. *Properties of Matrices over Fields.* Let $(\mathbb{F}, +, \cdot)$ be a field.
   (a) ~~Show that $0 \cdot a = 0$ for all $a \in \mathbb{F}$.~~
   (b) We define a left inverse of a matrix $A \in \mathbb{F}^{n\times n}$ (if exists), to be a matrix $B \in \mathbb{F}^{n\times n}$ such that $BA = I$ (I is the identity matrix). Similarly, we define the right inverse of A to be a matrix $C \in \mathbb{F}^{n\times n}$ such that $AC = I$. Show that left and right inverse of a matrix are equal. (we denote that matrix by A^{-1})
   (c) We say that a matrix $A \in \mathbb{F}^{n\times n}$ is a lower-triangular matrix if $A_{ij} = 0$ for $j > i$. Show that for such a matrix if A is an invertible matrix, its inverse is also a lower-triangular matrix.
   (d) Suppose that $\mathbb{F}$ is a finite-field. Show that if A is invertible, then $A^k = I$ for some $k \geq 1$.

4. *Linear functions over* $\mathbb{F}^n$. A function (operator) $L : V \to W$ from a vector space V to a vector space W (both on a common field $\mathbb{F}$) is called *linear* if (i) $L(x + y) = L(x) + L(y)$, and (ii) $L(\alpha x) = \alpha L(x)$ for all $x, y \in V$ and $\alpha \in \mathbb{F}$.
   (a) Show that the function $f : \mathbb{F}^n \to \mathbb{F}^m$ defined by $f(x) = Ax$, where $A \in \mathbb{F}^{m\times n}$, is linear.
   (b) Show than any linear function $f : \mathbb{F}^n \to \mathbb{F}^m$ has a representation $f(x) = Ax$ for some $A \in \mathbb{F}^{m\times n}$.
   (c) Show that the representation in part (b) is unique by proving that $Ax = Bx$ for every x implies that $A = B$.

5. *Differentiation of polynomials.* Let $\mathcal{P}_n$ be the vector space consisting of all polynomials of degree $\leq n$ with real coefficients.
   (a) Show that the monomials x^i, $i = 0, 1, \ldots, n$, form a basis for $\mathcal{P}_n$.

(b) Consider the transformation $T : \mathcal{P}_n \to \mathcal{P}_n$ defined by

$$T(p(x)) = \frac{dp(x)}{dx}.$$

For example, $T(1 + 3x + x^2) = 3 + 2x$. Show that T is linear.

(c) Using $\{1, x, \dots, x^n\}$ as a basis, represent the transformation in part (b) by a matrix $A \in \mathbb{R}^{(n+1)\times(n+1)}$. Find the rank of A.

(d) Characterize the nullspace of A.

6. *Zero nullspace.* Let $A \in \mathbb{R}^{m\times n}$. Prove that the following statements are equivalent.

   (a) $\mathcal{N}(A) = \{0\}$.

   (b) $\mathcal{R}(A') = \mathbb{R}^n$.

   (c) The columns of A are independent.

   (d) A is tall (i.e., $n \le m$) and full-rank (i.e., $\text{rank}(A) = \min(m, n) = n$).

7. *Rank of AA'.* Let $A \in \mathbb{F}^{m\times n}$.

   (a) Suppose that $\mathbb{F} = \mathbb{R}$. Prove that $\text{rank}(AA') = \text{rank}(A)$ or provide a counterexample.

   (b) Suppose that $\mathbb{F} = \mathbb{Z}_2$. Repeat part (a).

   (c) Suppose that $\mathbb{F} = \mathbb{C}$. Repeat part (a).

   (d) Suppose that $\mathbb{F} = \mathbb{C}$. Prove that $\text{rank}(AA^*) = \text{rank}(A)$ or provide a counterexample.

8. *Rank of a sum.* Let $A, B \in \mathbb{F}^{m\times n}$. Show that

$$\text{rank}(A + B) \le \text{rank}(A) + \text{rank}(B).$$

9. *Rank of a product.* Let $A \in \mathbb{R}^{6\times 4}$ has rank 2 and $B \in \mathbb{R}^{4\times 5}$ has rank 3.

   (a) Find the smallest possible value r_{min} of $\text{rank}(AB)$. Find specific A and B such that $\text{rank}(AB) = r_{\text{min}}$.

   (b) Find the largest possible value r_{max} of $\text{rank}(AB)$. Find specific A and B such that $\text{rank}(AB) = r_{\text{max}}$.

10. *Parity check codes.* Let

$$G = \begin{bmatrix} I \\ A \end{bmatrix} \in \mathbb{Z}_2^{n\times k},$$

where $A \in \mathbb{Z}_2^{(n-k)\times k}$ and $n \ge k$, where I is the $k \times k$ identity matrix. Suppose that a k-bit message $x \in \mathbb{Z}_2^k$ is encoded into an n-bit codeword $y = Gx \in \mathbb{Z}_2^n$. This is an example of an (n, k) *binary linear parity check code.* In this context, G is referred to as a *generator* matrix of the code and its range $\mathcal{R}(G)$ is referred to as the set of codewords or the *codebook.* The additional $n - k$ bits, or *parity bits*, provide redundant information that can be used for correction (or detection) of errors that occur to the codewords.

   (a) Find $|\mathcal{R}(G)|$ and interpret this value in terms of the codewords of the (n, k) code.

(b) Let $H = \begin{bmatrix} A & I \end{bmatrix} \in \mathbb{Z}_2^{(n-k)\times n}$. Show that $HG = 0$.

(c) Show that $\mathcal{N}(H) = \mathcal{R}(G)$, namely, y is a codeword if and only if $Hy = 0$. For this reason, H is referred to as a *parity check matrix* of the code.

(d) Consider the code with generator matrix H' that encodes $(n-k)$-bit messages into n-bit codewords. This $(n, n-k)$ code is said to be *dual* to the original (n, k) code with generator matrix G. Find a parity check matrix P of the dual code, that is, a matrix P that satisfies $Py = 0$ if and only if y is a codeword of the dual code.