Homework 3

ECE 269: Linear Algebra and Applications
Homework #3-Solution
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1. *A hands on experience!*. For the following matrix (over $\mathbb{R}$),

$$A = \begin{pmatrix} 8 & -1 & 2 \\ 8 & 2 & 0 \\ 0 & 3 & -2 \end{pmatrix}, \tag{1}$$

obtain the following.

(a) Find $\mathcal{R}(A)$.

(b) Find rank A.

(c) Find $\mathcal{N}(A)$.

(d) Perform a rank decomposition $A = BC$.

(e) Find the QR decomposition of A.

Solution:

(a) Notice that $(8\ 8\ 0)'$ and $(-1\ 2\ 3)'$ are linearly independent. We try

$$\begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} = a \begin{pmatrix} 8 \\ 8 \\ 0 \end{pmatrix} + b \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}.$$

Calculations show that $a = 1/6$ and $b = -2/3$. This implies $(2\ 0\ -2)'$ is contained in the span of the two vectors. Therefore,

$$\begin{aligned} \mathcal{R}(A) &= \operatorname{span}\left\{ \begin{pmatrix} 8 \\ 8 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} \right\} \\ &= \operatorname{span}\left\{ \begin{pmatrix} 8 \\ 8 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} \right\} \\ &= \left\{ (8a - b\ \ 8a + 2b\ \ 3b)' \mid a, b \in \mathbb{R} \right\}. \end{aligned}$$

We can obtain a simpler form of $\mathcal{R}(A)$ by change of variable $x = 8a - b$, $z = 3b$. Then

$$\mathcal{R}(A) = \{(x\ \ x+z\ \ z)' \mid x, z \in \mathbb{R}\} = \{(x\ \ y\ \ z)' \in \mathbb{R}^3 \mid y = x + z\}.$$

(b) Since $\mathcal{R}(A)$ has a basis of two vectors, the dimension of $\mathcal{R}(A)$ is two and $\operatorname{rank} A = 2$.

(c) Consider the equation

$$\begin{pmatrix} 8 & -1 & 2 \\ 8 & 2 & 0 \\ 0 & 3 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Observe that $v_1 = 1/6$, $v_2 = -2/3$, $v_3 = -1$ is a solution of this equation. So, we have $(1/6 \ -2/3 \ -1) \in \mathcal{N}(A)$. On the other hand, by the rank-nullity theorem, the dimension of $\mathcal{N}(A)$ should be $3 - \operatorname{rank} A = 1$. Hence

$$\mathcal{N}(A) = \operatorname{span}\{(1/6 \ \ -2/3 \ \ -1)'\}$$

.

(d) Recall that we have

$$\begin{pmatrix} 8 & -1 \\ 8 & 2 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1/6 \\ -2/3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}.$$

Therefore, we have

$$A = \begin{pmatrix} 8 & -1 \\ 8 & 2 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1/6 \\ 0 & 1 & -2/3 \end{pmatrix} = BC.$$

(e) For the QR decomposition, we already showed that the first two independent columns are $v_1 = \begin{pmatrix} 8 \\ 8 \\ 0 \end{pmatrix}$ and $v_2 = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$. So, following the procedure in the class, we use the G-S procedure to find the columns of (reduced) Q as follows:

$$q_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\tilde{q}_2 = v_2 - \langle v_2, q_1 \rangle q_1 = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{3}{2} \\ \frac{3}{2} \\ 3 \end{pmatrix} \Rightarrow q_2 = \frac{\tilde{q}_2}{\|\tilde{q}_2\|} = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}.$$

Letting $Q = [q_1 \quad q_2] = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{2}{\sqrt{6}} \end{pmatrix}$, the reduced QR decomposition, would be $A = QU$ with

$$U = Q'A = \begin{pmatrix} 8\sqrt{2} & \frac{1}{\sqrt{2}} & \sqrt{2} \\ 0 & \frac{9}{\sqrt{6}} & -\sqrt{6} \end{pmatrix}.$$

For the full QR decomposition, we need to find a vector u that is not in $R(A) = R(Q)$ and continue with the Gram-Schmidt procedure. In this case, $v_3 = e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is not in the range (why). Therefore, we have

$$\tilde{q}_3 = e_3 - \langle e_3, q_1\rangle q_1 - \langle e_3, q_2\rangle q_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{3}\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$$

$$\Rightarrow q_3 = \frac{\tilde{q}_3}{\|\tilde{q}_3\|} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

Therefore, the full QR decomposition, would be $A = \tilde{Q}\begin{pmatrix} U \\ 0 \end{pmatrix}$ with

$$\tilde{Q} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}.$$

2. *Orthogonal complement of a subspace.* Suppose that $\mathcal{V}$ is a subspace of $\mathbb{F}^n$. Let

$$\mathcal{V}^\perp = \{x \in \mathbb{F}^n : x'y = 0, \forall y \in \mathcal{V}\}$$

be the set of vectors orthogonal to every element in $\mathcal{V}$.

(a) Verify that $\mathcal{V}^\perp$ is a subspace of $\mathbb{F}^n$.

(b) Suppose that $\mathcal{V} = \text{span}(v_1, v_2, \ldots, v_k)$ for some $v_1, v_2, \ldots, v_k \in \mathbb{F}^n$. Express $\mathcal{V}$ and $\mathcal{V}^\perp$ as subspaces induced by the matrix $A = \begin{pmatrix} v_1 & v_2 & \cdots & v_k \end{pmatrix} \in \mathbb{F}^{n\times k}$ and its transpose A'.

(c) Show that $(\mathcal{V}^\perp)^\perp = \mathcal{V}$.

(d) Show that $\dim(\mathcal{V}) + \dim(\mathcal{V}^\perp) = n$.

(e) Show that $\mathcal{V} \subseteq \mathcal{W}$ for another subspace $\mathcal{W}$ implies $\mathcal{W}^\perp \subseteq \mathcal{V}^\perp$.

(f) Suppose that $\mathbb{F} = \mathbb{R}$. Show that every $x \in \mathbb{F}^n$ can be expressed uniquely as $x = v + v^\perp$, where $v \in \mathcal{V}$ and $v^\perp \in \mathcal{V}^\perp$. (Hint: Let v be the projection of x on $\mathcal{V}$.)

Solution:

(a) We have, for all $y \in \mathcal{V}$, $0'y = 0$ and therefore, $0 \in \mathcal{V}^\perp$. Now, let $a, b \in \mathbb{F}$ and $u_1, u_2 \in \mathcal{V}^\perp$. Then, we have, for all $y \in \mathcal{V}$,

$$\begin{aligned}(au_1 + bu_2)'y &= au_1'y + bu_2'y \\ &= 0,\end{aligned}$$

since $u_1'y = u_2'y = 0$. Thus, $\mathcal{V}^\perp$ is a subspace of $\mathbb{F}^n$.

(b) We have

$$\begin{aligned}\mathcal{V} &= \{x_1v_1 + \cdots + x_kv_k : x_1, x_2, \ldots, x_k \in \mathbb{F}\} \\ &= \left\{A \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{pmatrix} : x_1, x_2, \ldots, x_k \in \mathbb{F}\right\} \\ &= \{Ax : x \in \mathbb{F}^k\} \\ &= \mathcal{R}(A),\end{aligned}$$

and

$$\begin{aligned}\mathcal{V}^{\perp} &= \{y \in \mathbb{F}^n : y'(x_1v_1 + x_2v_2 + \cdots + x_kv_k) = 0, \forall x_1, \ldots, x_k \in \mathbb{F}\}\\ &= \{y \in \mathbb{F}^n : (x_1v_1' + x_2v_2' + \cdots + x_kv_k')y = 0, \forall x_1, \ldots, x_k \in \mathbb{F}\}\\ &= \{y \in \mathbb{F}^n : x'A'y = 0, \forall x \in \mathbb{F}^k\}\\ &= \{y \in \mathbb{F}^n : A'y = 0\}\\ &= \mathcal{N}(A').\end{aligned}$$

(c) We will first show that $\mathcal{V} \subseteq (\mathcal{V}^{\perp})^{\perp}$. Let $x \in \mathcal{V}$. Then, by the definition of $\mathcal{V}^{\perp}$, $x'y = 0$ for all $y \in \mathcal{V}^{\perp}$ and therefore, $x \in (\mathcal{V}^{\perp})^{\perp}$. Thus, $\mathcal{V} \subseteq (\mathcal{V}^{\perp})^{\perp}$. Now, by part (d), $(\mathcal{V}^{\perp})^{\perp}$ has dimension $\dim(\mathcal{V})$ and by what we just proved, includes $\mathcal{V}$. This implies that $(\mathcal{V}^{\perp})^{\perp} = \mathcal{V}$.

(d) From part (b), we have

$$\begin{aligned}\dim(\mathcal{V}) &= \dim(\mathcal{R}(A))\\ &= \mathrm{rank}(A)\\ &= \mathrm{rank}(A').\end{aligned}$$

Also,

$$\begin{aligned}\dim(\mathcal{V}^{\perp}) &= \dim(\mathcal{N}(A'))\\ &= \mathrm{nullity}(A').\end{aligned}$$

Therefore, since $A' \in \mathbb{F}^{k\times n}$, we have, by the rank–nullity theorem, $\dim(\mathcal{V})+\dim(\mathcal{V}^{\perp}) = n$.

(e) Let $\mathcal{V} \subseteq \mathcal{W}$. We have

$$\begin{aligned}x \in \mathcal{W}^{\perp} &\implies x'w = 0 \text{ for all } w \in \mathcal{W}\\ &\overset{(a)}{\implies} x'w = 0 \text{ for all } w \in \mathcal{V}\\ &\implies x \in \mathcal{V}^{\perp}.\end{aligned}$$

Here, implication (a) follows since every $w \in \mathcal{V}$ is also included in $\mathcal{W}$, by assumption.

(f) Consider v, the projection of x on $\mathcal{V}$. Then, by the property of a projection, $v^{\perp} := x - v$ is orthogonal to $\mathcal{V}$ and hence to all vectors in $\mathcal{V}$. Therefore, $v^{\perp} \in \mathcal{V}^{\perp}$. Now, suppose $x = v + v^{\perp} = \tilde{v} + \tilde{v}^{\perp}$, where $v, \tilde{v} \in \mathcal{V}$ and $v^{\perp}, \tilde{v}^{\perp} \in \mathcal{V}^{\perp}$. Then, we have $v - \tilde{v} = \tilde{v}^{\perp} - v^{\perp}$. But $v - \tilde{v} \in \mathcal{V}$ and $\tilde{v}^{\perp} - v^{\perp} \in \mathcal{V}^{\perp}$, therefore $v - \tilde{v} \in \mathcal{V} \cap \mathcal{V}^{\perp} = \{0\}$, implying that $v = \tilde{v}$, demonstrating the uniqueness of the representation.

Note that $\mathcal{V} \cap \mathcal{V}^{\perp} = \{0\}$ may not hold for subspaces over all fields $\mathbb{F}$. Consider, for example, the subspace $\mathcal{V}$ of $\mathbb{F}_2^4$ spanned by $\begin{pmatrix}1 & 1 & 1 & 1\end{pmatrix}'$. Then,

$$\mathcal{V} = \left\{\begin{pmatrix}0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\\1\end{pmatrix}\right\},$$

and as can be verified easily,

$$\mathcal{V}^{\perp} = \left\{\begin{pmatrix}0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\1\end{pmatrix}\right\}.$$

Now consider the vector $\begin{pmatrix}1 & 1 & 0 & 0\end{pmatrix}'$. We have

$$\begin{pmatrix}1\\1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix} + \begin{pmatrix}1\\1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\1\\1\\1\end{pmatrix} + \begin{pmatrix}0\\0\\1\\1\end{pmatrix},$$

demonstrating the non-uniqueness of the representation.

3. *Halfspace.* Suppose that $a, b \in \mathbb{R}^n$ are two given points. Show that the set of points in $\mathbb{R}^n$ that are closer to a than b is a halfspace, i.e.,

$$\{x : \|x - a\| \leq \|x - b\|\} = \{x : c'x \leq d\}$$

for appropriate $c \in \mathbb{R}^n$ and $d \in \mathbb{R}$.

(a) Find c and d explicitly in terms of a and b.

(b) Draw a picture showing a, b, c, and the halfspace.

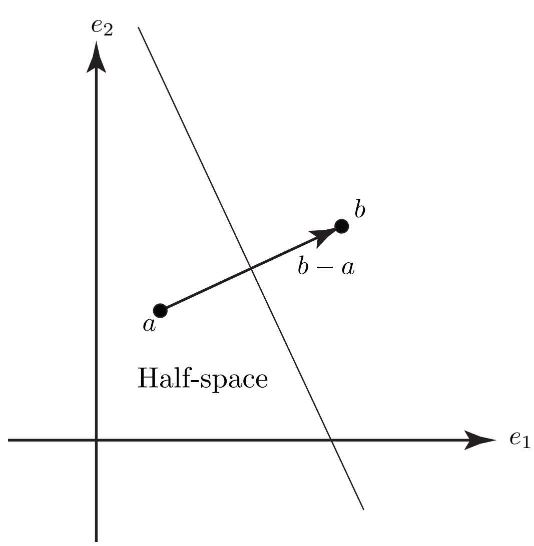
Solution:

(a) We have

$$\begin{aligned}
\{x : \|x - a\| \leq \|x - b\|\} &= \{x : \|x - a\|^2 \leq \|x - b\|^2\} \\
&= \{x : (x - a)'(x - a) \leq (x - b)'(x - b)\} \\
&= \{x : x'x - a'x - x'a + a'a \leq x'x - b'x - x'b + b'b\} \\
&= \{x : -2a'x + a'a \leq -2b'x + b'b\} \\
&= \{x : 2(b - a)'x \leq b'b - a'a\} \\
&= \{x : c'x \leq d\},
\end{aligned}$$

where $c := b - a$ and $d := (|b|^2 - |a|^2)/2$.

(b)



4. *Inner product of polynomials.* Let $\mathcal{P}_3$ be the vector space of all polynomials of degree ≤ 3 with real coefficients, that is,

$$\mathcal{P}_3 = \{\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 : \alpha_0, \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}\}.$$

Let $K : \mathcal{P}_3 \times \mathcal{P}_3 \to \mathbb{R}$ be defined as

$$K(p, q) = \int_{-1}^{1} p(x)q(x)dx.$$

(a) Show that $K(\cdot, \cdot)$ represents an inner product for $\mathcal{P}_3$.

(b) Find an orthogonal basis for $\mathcal{P}_3$ using Gram–Schmidt orthogonalization.

Solution:

(a) We will show that the three properties that an inner product is required to satisfy hold for $K(\cdot, \cdot)$.

- *Linearity in the first argument.*

$$\begin{aligned}
K(\alpha p_1 + \beta p_2, q) &= \int_{-1}^{1} (\alpha p_1(x) + \beta p_2(x))q(x)dx \\
&= \int_{-1}^{1} (\alpha p_1(x)q(x) + \beta p_2(x)q(x))d(x) \\
&= \int_{-1}^{1} \alpha p_1(x)q(x)dx + \int_{-1}^{1} \beta p_2(x)q(x)dx \\
&= \alpha \int_{-1}^{1} p_1(x)q(x)dx + \beta \int_{-1}^{1} p_2(x)q(x)dx \\
&= \alpha K(p_1, q) + \beta K(p_2, q).
\end{aligned}$$

- *Conjugate symmetry.*

$$\begin{aligned}
K(q, p) &= \int_{-1}^{1} q(x)p(x)dx \\
&= \int_{-1}^{1} \overline{q(x)p(x)}dx \\
&= \int_{-1}^{1} \overline{p(x)q(x)}dx \\
&= \overline{\int_{-1}^{1} p(x)q(x)dx} \\
&= \overline{K(p, q)}.
\end{aligned}$$

- *Positive definiteness.* Note that $p(x)^2 \geq 0 \ \forall p \in \mathcal{P}_3, x \in [-1, 1]$. Therefore,

$$K(p, p) = \int_{-1}^{1} p(x)p(x)dx = \int_{-1}^{1} p(x)^2 dx \geq 0.$$

Moreover, if we have $K(p, p) = \int_{-1}^{1} p(x)^2 dx = 0$, since $p(x)^2$ is non-negative, we necessarily need $p(x)$ to be identically 0.

(b) First of all, recall that $1, x, x^2, x^3$ form a basis for $\mathcal{P}_3$ (cf. HW 1, Problem 8(a)). We will use this basis to construct an orthonormal basis using Gram–Schmidt orthogonalization.

$$\tilde{p}_0(x) = 1,$$
$$p_0(x) = \frac{1}{\|1\|} = \boxed{\frac{1}{\sqrt{2}}}$$

$$\tilde{p}_1(x) = x - \left(\int_{-1}^{1} \frac{x}{\sqrt{2}} dx\right) \cdot \frac{1}{\sqrt{2}} = x,$$
$$p_1(x) = \frac{x}{\|x\|} = \boxed{\sqrt{\frac{3}{2}}x}$$

$$\tilde{p}_2(x) = x^2 - \left(\int_{-1}^{1} \sqrt{\frac{3}{2}} x^3 dx\right) \cdot \sqrt{\frac{3}{2}}x - \left(\int_{-1}^{1} \frac{x^2}{\sqrt{2}} dx\right) \cdot \frac{1}{\sqrt{2}} = x^2 - \frac{1}{3},$$
$$p_2(x) = \frac{x^2 - 1/3}{\|x^2 - 1/3\|} = \boxed{\sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right)}$$

$$\begin{aligned}\tilde{p}_3(x) &= x^3 - \left(\int_{-1}^{1} \sqrt{\frac{45}{8}} x^5 dx\right) \cdot \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right) - \left(\int_{-1}^{1} \sqrt{\frac{3}{2}} x^4 dx\right) \cdot \sqrt{\frac{3}{2}}x \\ &\qquad - \left(\int_{-1}^{1} \frac{x^3}{\sqrt{2}} dx\right) \cdot \frac{1}{\sqrt{2}} \\ &= x^3 - \frac{3}{5}x,\end{aligned}$$
$$p_3(x) = \frac{x^3 - (3/5)x}{\|x^3 - (3/5)x\|} = \boxed{\sqrt{\frac{175}{8}}\left(x^3 - \frac{3}{5}x\right)}.$$

5. *Bessel's inequality.* Suppose that the columns of $U \in \mathbb{R}^{n\times k}$ are orthonormal. Show that

$$\|U'x\| \leq \|x\|.$$

Solution: Let $u_1, u_2, \ldots, u_k$ denote the columns of U, and let

$$\tilde{x} := \sum_{j=1}^{k} (u_j' x) u_j = UU'x$$

be the projection of x onto $\mathcal{R}(U)$. Then

$$\begin{aligned} 0 &\le \|x-\tilde{x}\|^2 \\ &= \|(I-UU')x\|^2 \\ &= x'(I-UU')'(I-UU')x \\ &= x'(I-UU')(I-UU')x \\ &= x'(I-UU')x \\ &= x'x - x'UU'x \\ &= \|x\|^2 - \|U'x\|^2. \end{aligned}$$

6. *Wonders of Infinite Dimensional Spaces.*

   (a) Recall that $C^0([a,b])$ is the set of continuous functions $f:[a,b]\to\mathbb{R}$. Let $[a,b]=[0,2]$.

      i. Show that $\|f\|_2 = (\int_0^2 |f(x)|^2dx)^{1/2}$ is well-defined, i.e., $\|f\|_2 < \infty$ for all $f \in C^0([0,2])$. As a result of this, $C^0([0,2]) \subset L_2([0,2])$ and $(C^0([0,2]), \|\cdot\|_2)$ is a normed-vector space.

      ii. Show that this normed-vector space is not complete/Banach. **hint:** Show that the sequence $\{f_k\}$ in $C^0([0,2])$ defined by

$$f_k(x) = \begin{cases} x^k & x\in[0,1] \\ 1 & x\in(1,2] \end{cases}$$

      is a Cauchy sequence, but the sequence does not have a limit in $C^0([0,2])$.

   (b) We defined the space $\ell_\infty(\mathbb{N})$ to be the space of all sequences $(x_n)_{n\ge1}$ with $x_n\in\mathbb{R}$ such that $\sup_{n\ge1}|x_n|<\infty$, and we defined the norm $\|\cdot\|_\infty$ in this space by $\|(x_n)_{n\ge1}\|_\infty = \sup_{n\ge1}|x_n|<\infty$.

      i. For a normed-vector space $(V,\|\cdot\|)$, we can define the ball of radius $r>0$ around a point $x\in V$, to be $B_r(x)=\{y \mid \|y-x\|<r\}$. Identify, the unite ball $B_1(\mathbf{0})$ in $\ell_\infty(\mathbb{N})$ where $\mathbf{0}$ is the zero of $\ell_\infty(\mathbb{N})$.

      ii. Construct a sequence of vectors $\{v_n\}_{n\ge1}$ in $B_1(\mathbf{0})$ such that the distance of any two points is greater than or equal to one. In other words, not only $\{v_n\}_{n\ge1}$ is not Cauchy, but none of its subsequences is Cauchy.

Solution:

(a) i. Since $[0,2]$ is a bounded closed interval and $f\in C^0([0,2])$ is continuous, there exists $c\in\mathbb{R}$ such that $|f(x)|<c$ for all $x\in[0,2]$. Therefore, $\|f\|_2 = (\int_0^2|f(x)|^2dx)^{1/2} < (\int_0^2 c^2dx)^{1/2} = (2c^2)^{1/2} = \sqrt{2}c < \infty$.

ii. For any ϵ, let $N = \lceil \frac{2}{\epsilon^2} \rceil$. For any $n, m > N$,

$$
\begin{aligned}
\|f_n - f_m\|_2 &= \Big(\int_0^2 |f_n(x) - f_m(x)|^2 dx \Big)^{1/2} \\
&= \Big(\int_0^1 |x^n - x^m|^2 dx \Big)^{1/2} \\
&\leq \Big(\int_0^1 (x^{2n} + 2x^{n+m} + x^{2m}) dx \Big)^{1/2} \\
&= \Big(\frac{1}{2n+1} + \frac{2}{n+m+1} + \frac{1}{2m+1} \Big)^{1/2} \\
&< \Big(\frac{1}{2N} + \frac{2}{2N} + \frac{1}{2N} \Big)^{1/2} \\
&= \Big(\frac{2}{N} \Big)^{1/2} \\
&< \epsilon.
\end{aligned}
$$

Hence, f_k is a Cauchy sequence.
From this point on, we essentially argue that if the limit function exists, it cannot be continuous at $x = 1$, as one would expect the limit have the form of:

$$
\lim_{k \to \infty} f_k(x) = \begin{cases} 0 & x \in [0, 1) \\ 1 & x \in [1, 2] \end{cases}.
$$

To formally show this, suppose that $f_k \to g(x)$ in L_2 for some $g(x) \in C^0([0, 2])$ and assume $g(1) = \alpha$. One can argue that we need to have $\alpha \in [0, 1]$. If $\alpha > 0$, then by the right continuity of $g(x)$ at $x = 1$, there exists some $\delta > 0$ such that $0 < \frac{\alpha}{2} \leq g(x) \leq \alpha$ for $x \in (1 - \delta, 1]$. Therefore, for large enough n (more precisely, when $(1 - \delta/2)^n < \alpha/4$),

$$
\begin{aligned}
\|g(x) - f_n(x)\|^2 &= \int_0^2 |g(x) - f_n(x)|^2 dx \geq \int_{1-\delta}^{1-\delta/2} |g(x) - x^n|^2 dx \\
&\geq \int_{1-\delta}^{1-\delta/2} (\frac{\alpha}{4})^2 dx = (\frac{\alpha}{4})^2 \frac{\delta}{2} = \frac{\alpha^2 \delta}{32}.
\end{aligned}
$$

Therefore, the distance does not go to zero and hence, f_n does not converge to g. Therefore, we need to have $\alpha = 0$. But if $\alpha = 0$, then using a similar argument, and the right continuity of g, you can show that

$$
\|g(x) - f_n(x)\|^2 = \int_0^2 |g(x) - f_n(x)|^2 dx \geq \int_{1+\delta/2}^{1+\delta} |g(x) - x^n|^2 dx \geq \frac{\delta}{32},
$$

for large enough n and hence, we cannot have convergence.

(b) i. $B_1(\mathbf{0}) = \{(x_n)_{n\geq 1} \in V \mid \|(x_n)_{n\geq 1}\|_\infty < 1\} = \{(x_n)_{n\geq 1} \in V \mid \sup_{n\geq 1} |x_n| < 1\}$.

ii. For example, we have $\{v_n\}_{n\geq 1}$ in $B_1(\mathbf{0})$ such that

$$\begin{aligned}
v_1 &= (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \cdots),\\
v_2 &= (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \cdots),\\
v_3 &= (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \cdots),\\
&\vdots
\end{aligned}$$

The distance of any two points in this sequence is equal to one.

7. *Projection matrices.* A symmetric matrix $P = P' \in \mathbb{R}^{n\times n}$ is said to be a *projection matrix* if $P = P^2$.

   (a) Show that if P is a projection matrix, then so is $I - P$.

   (b) Suppose that the columns of $U \in \mathbb{R}^{n\times k}$ are orthonormal. Show that UU' is a projection matrix.

   (c) Suppose that $A \in \mathbb{R}^{n\times k}$ is full-rank with $k \leq n$. Show that $A(A'A)^{-1}A'$ is a projection matrix.

   (d) The point $y \in \mathcal{S} \subseteq \mathbb{R}^n$ closest to $x \in \mathbb{R}^n$ is said to be the *orthogonal projection* (or *projection* in short) of x onto $\mathcal{S}$. Show that if P is a projection matrix, then $y = Px$ is the projection of x onto $\mathcal{R}(P)$.

   (e) Let u be a unit vector. Find the projection matrix P such that $y = Px$ is the projection of x onto $\text{span}(u)$.

   Solution:

   (a) Note that $(I - P)' = I - P' = I - P$ and so $I - P$ is symmetric. Also, $(I - P)^2 = (I - P)(I - P) = I - P - P + P^2 = I - 2P + P = I - P$. Hence, $I - P$ is symmetric and $(I - P)^2 = (I - P)$, thus it is a projection matrix.

   (b) Note that $(UU')' = (U')'U = UU'$ and

$$\begin{aligned}
(UU')^2 &= UU'UU'\\
&= UIU'\\
&= UU'.
\end{aligned}$$

   (c) First of all, $(A'A)$ is invertible for a full-rank tall matrix A. For symmetry, consider

$$\begin{aligned}
(A(A'A)^{-1}A')' &= (A')'((A'A)^{-1})'A'\\
&= A((A'A)^{-1})'A'\\
&= A((A'A)')^{-1}A'\\
&= A(A'A)^{-1}A'.
\end{aligned}$$

   Also, consider

$$\begin{aligned}
(A(A'A)^{-1}A')^2 &= A(A'A)^{-1}(A'A)(A'A)^{-1}A'\\
&= A(A'A)^{-1}A'.
\end{aligned}$$

(d) It suffices to show that $\|x - v\|$ is minimized over all $v \in \mathcal{R}(P)$ by $v^* = Px$. For any $v \in \mathcal{R}(P)$,

$$\begin{aligned}
\|x - v\|^2 &= \|x - Px + Px - v\|^2 \\
&= \|x - Px\|^2 + \|Px - v\|^2 + 2(x - Px)'(Px - v) \\
&= \|x - Px\|^2 + \|Px - v\|^2 + 2(x' - x'P)(Px - v) \\
&= \|x - Px\|^2 + \|Px - v\|^2 + 2(x'Px - x'v - x'P^2x + x'Pv) \\
&\overset{(1)}{=} \|x - Px\|^2 + \|Px - v\|^2 + 2(x'Px - x'v - x'Px + x'v) \\
&= \|x - Px\|^2 + \|Px - v\|^2 \\
&\overset{(2)}{\leq} \|x - Px\|^2,
\end{aligned}$$

where (1) follows using $P^2 = P$ and $Pv = v \ \forall v \in \mathcal{R}(P)$. To achieve equality in (2), we need $\|Px - v\|^2 = 0 \implies v = Px$. Thus, $\arg\min_{v \in \mathcal{R}(P)} \|x - v\| = Px$.

(e) Consider $P = uu'$. Since $P = P'$ and $P = P^2$, P is a projection matrix. Since $\mathcal{R}(P) = \text{span}(u)$, by part (d) Px is the projection of x onto $\text{span}(u)$. This can be also directly verified since $(u'x)u = uu'x = Px$ is the component of x in the direction of the unit vector u.

8. *Reflection and projection with an affine hyperplane.* Let a be a nonzero vector in $\mathbb{R}^n$, $b \in \mathbb{R}$, and

$$\mathcal{A} = \{x \in \mathbb{R}^n : a'x = b\}.$$

be an *affine hyperplane*, namely, a shifted version of the hyperplane $\mathcal{H} = \{x : a'x = 0\}$ by b, with the same normal vector a.

(a) Find the projection of the zero vector 0 onto $\mathcal{A}$.

(b) Find the reflection of 0 through $\mathcal{A}$.

(c) Find the projection of x onto $\mathcal{A}$.

(d) Find the reflection of x through $\mathcal{A}$.

Solution: If we have an affine hyperplane $\mathcal{A}$, $p_{\mathcal{A}}(x)$ for any x will be such that the vector $p_{\mathcal{A}}(x) - x$ is perpendicular to the plane $\mathcal{A}$. This can be seen by elementary geometric arguments—there always exists a point $y \in \mathcal{A}$ such that $y - x \perp \mathcal{A}$. If $p_{\mathcal{A}}(x) = z \neq y$, then the vectors $y - x, z - y, x - z$ form a right-angled triangle with $z - x$ as the hypotenuse and we have $\|x - y\| \leq \|x - z\|$ contradicting the minimality in distance from x, that is implicit in a projection. Similarly, the reflection of x, $r_{\mathcal{A}}(x)$ is equidistant from $p_{\mathcal{A}}(x)$ as x, but in the exact opposite direction.

(a) The vector $(p_{\mathcal{A}}(0) - 0)$ must be in the direction of a. Thus, $p_{\mathcal{A}}(0) = \alpha a$. But, we also know that $p_{\mathcal{A}}(0)$ lies on the affine hyperplane $\mathcal{A}$. Thus, $a'p_{\mathcal{A}}(0) = b$. Putting these two together, we get that $a'\alpha a = b \implies \alpha = \frac{b}{a'a}$. Thus, $p_{\mathcal{A}}(0) = \frac{b}{a'a}a$.

(b) Through geometric arguments, we can see that the reflection of 0 through $\mathcal{A}$ will be such that the vector $r_{\mathcal{A}}(0) - p_{\mathcal{A}}(0) = (p_{\mathcal{A}}(0) - 0)$. Thus $r_{\mathcal{A}} = 2p_{\mathcal{A}}(0) = 2\frac{b}{a'a}a$.

(c) We follow the same reasoning as in (a) to argue that $(p_{\mathcal{A}}(x) - x)$ must be in the direction of a, and $p_{\mathcal{A}}(x)$ must lie on the affine hyperplane $\mathcal{A}$. Thus,

$$p_{\mathcal{A}}(x) - x = \alpha a$$

and

$$a'p_{\mathcal{A}}(x) = b,$$

which implies that $a'(\alpha a + x) = b$, or equivalently, $\alpha = \frac{b-a'x}{a'a}$. Therefore,

$$p_{\mathcal{A}}(x) = x - \frac{1}{\|a\|^2}(a'x - b)a.$$

(d) Again, we use the same reasoning as in (b) to argue that the reflection $r_{\mathcal{A}}(x)$ of x must satisfy $r_{\mathcal{A}}(x) - p_{\mathcal{A}}(x) = p_{\mathcal{A}}(x) - x$, which implies that

$$r_{\mathcal{A}}(x) = 2p_{\mathcal{A}}(x) - x = x - 2\frac{1}{\|a\|^2}(a'x - b)a.$$