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GENERAL ENVELOPE EQUATION FOR CYLINDRICALLY SYMMETRIC CHARGED-PARTICLE BEAMS†

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An envelope equation, which includes the effects of a solenoidal field, acceleration, self-induced forces, and scattering by a background medium, is derived for the rms radius of a relativistic beam.

The system is assumed to be cylindrically symmetric with high enough energy that the paraxial approximation is applicable. The solenoidal field is taken to be uniform normal to the direction of propagation but the beam current profile is arbitrary.

The well-known equations of propagation are recovered in their respective domains of applicability (i.e., vacuum transport in a solenoid, equilibrium conditions, the Nordsieck equation, free expansion, and the sausage-mode equation). A treatment is also given of the matching conditions for a beam injected into gas through a foil in the presence of a solenoidal field.

The derivation of the envelope equation differs from previous work in making use of the scalar virial moment of the single-particle equation of motion. The beam emittance appears in a natural way as a constant of integration and is shown to be proportional to the effective phase area occupied by the particles. No distribution function is specified for the transverse velocities, but the beam is assumed to pulsate in a self-similar fashion.

1 INTRODUCTION

Recent interest in propagating intense relativistic electron beams in gas and the continued interest in the transport of accelerated charged-particle beams has prompted the development and use of envelope equations. The usual treatments generally assume special particle distributions or do not relate particle and envelope motion in a consistent manner. Vladimirsky and Kapchinsky,[1] for example, assumed a special four-dimensional distribution function in their work. Sacherer[2] derives envelope equations involving an emittance term that is assumed constant. Poukey and Toepfer[3] consider nonparaxial flow in a relativistic fluid-flow model in which the flow equations are derived assuming a Maxwellian distribution; treatment is limited to the case in which there is no externally applied magnetic field. The present treatment is based on the single-particle equations of motion with no assumption made *ab-initio* for the form of either the distribution function or the emittance. An equation is derived for the rms radius with the aid of the energy, virial, and angular momentum moments of the particle equations. However, to evaluate the contribution to the envelope equation made by the self-induced fields of the beam, it is necessary to introduce some assumption about the form of the mean transverse flow of particles. We use the simplest expression consistent with the macroscopic observables of the theory, namely, selfsimilar expansion.

The approximations made in this work are minimal, the principal ones being

1) The paraxial ray approximation is adopted; i.e., transverse particle velocity is small compared with longitudinal velocity.

2) The beam current density and the fields are azimuthally symmetric.

3) There is no mass spread within a transverse segment of the beam.

4) The solenoidal field is uniform across the beam profile.

5) Only small-angle multiple scattering is considered.

It is further assumed that beam segments do not overtake one another in the direction of propagation. Particles interact with each other only via

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the collective fields they induce, and there is no treatment of synchrotron radiation losses.

To treat new examples of beam propagation as well as recover the old results, we have derived an envelope equation with considerably generality. The most significant effects missing are the possible reshaping of the current profile due to scattering and pulsations, and the damping of pulsations by the phase mixing among particle orbits. It is emphasized that the results are applicable only to a beam of circular cross section, e.g., the derived emittance is not constant if this condition is violated.

2 PARTICLE EQUATIONS

We begin with the equations of motion of a single particle (rest mass m, charge q) traveling through a system of lenses, gas cells, foils, etc. The axis of symmetry is taken as the z axis of a system of cylindrical coordinates (r, ϕ, z); all particle velocities are directed approximately in the positive z direction. The motion is governed by the Lorentz force law with an added term representing the scattering field of the medium:

$$\dot{\mathbf{p}} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) + \boldsymbol{\delta}\mathbf{F}, \tag{1}$$

where $\boldsymbol{\delta}\mathbf{F}$ is the very rapidly fluctuating force due to the individual ions and molecules of the gas. (The dot notation for total time differentiation is employed for simplicity; the d/dt notation will also be used to avoid ambiguity in certain expressions.) The relativistic momentum is

$$\mathbf{p} = \gamma m \mathbf{v} = \gamma m \dot{\mathbf{r}}, \tag{2}$$

with

$$\gamma = (1 - \beta^2)^{-1/2}, \quad \beta = \frac{|\mathbf{v}|}{c}. \tag{3}$$

The symbols $\mathbf{E}$ and $\mathbf{B}$ represent the macroscopic fields generated by external coils, charges, and currents of the beam and those induced in the gas. From Eqs. (1) through (3) we can obtain the equation for the rate of change of the relativistic energy $W \equiv \gamma mc^2$:

$$\dot{W} = q\mathbf{v} \cdot \mathbf{E} + \mathbf{v} \cdot \delta\mathbf{F}. \tag{4}$$

The paraxial approximation greatly simplifies the equations of motion. The velocity is decomposed into longitudinal and transverse components,

$$\mathbf{v} = v_z \hat{z} + \mathbf{v}_\perp, \tag{5}$$

with $v_\perp \ll v_z$ by assumption. For a self-pinched beam this ordering requires the beam current I_b to be small compared with the Alfven limit,[4]

$$I_A = \beta\gamma \frac{mc4\pi}{q\mu_0} = \frac{\beta\gamma qc}{r_0} \approx 17000\beta\gamma A,$$

where r_0 is the classical radius of the electron. A particle moving longitudinally at the edge of a beam having a current equal to I_A passes through the axis with $v_z = 0$. We can thus characterize the motion with the small parameter† ϵ:

$$\epsilon^2 = I_b/I_A \ll 1,$$
$$v_\perp/v_z = O(\epsilon).$$

The depth of the electrostatic potential well of the beam divided by the particle kinetic energy, is also of order ϵ^2.

From Eq. (3) we have

$$v_z^2 = c^2\left(1 - \frac{1}{\gamma^2}\right) - v_\perp^2 = \beta^2 c^2 + O(\epsilon^2). \tag{6}$$

Similarly, the energy equation (4) becomes

$$\begin{aligned} \dot{W} &= q(v_z E_z + \mathbf{v}_\perp \cdot \mathbf{E}_\perp) + \mathbf{v} \cdot \boldsymbol{\delta}\mathbf{F} \\ &= q v_z E_z + \frac{\delta W}{\delta t} + O(\epsilon^2), \end{aligned} \tag{7}$$

where $\delta W/\delta t$ is the known drag due to ionization and excitation of the medium. A segment of beam at given z and t is characterized here by a single value of W (equivalently γ), so Eq. (7) is averaged over the current profile of the segment to get

$$\dot{\gamma} = \frac{q v_z}{mc^2} \bar{E}_z + \frac{1}{mc^2} \frac{\delta W}{\delta t}. \tag{8}$$

Variables v_z and γ are thus regarded as known functions of z and t, independent of the transverse motion. The dynamics are thereby simplified but there can be no treatment of an electrostatic lens. The transverse components of the equations of motion (1) and (2) are

$$\frac{1}{\gamma}\dot{\gamma}\mathbf{v}_\perp + \dot{\mathbf{v}}_\perp = \frac{q}{\gamma m}(\mathbf{E}_\perp + \beta c\hat{z} \times \mathbf{B}_\perp + \mathbf{v}_\perp \times B_z \hat{z}) + \frac{1}{\gamma m}\boldsymbol{\delta}\mathbf{F}_\perp \tag{9}$$

† It is also assumed that B_z is not large compared with B_θ/ϵ. There are special cases in which the paraxial approximation applies even through $\epsilon \gtrsim 1$, for example, a counter-flowing plasma current may cancel some of the beam current.

and

$$\dot{\mathbf{r}}_\perp = \mathbf{v}_\perp. \tag{10}$$

It is advantageous to use the cylindrically resolved field components in Eq. (9). The following components are observed:

1) B_z is a known function of z and t, generated by coils surrounding the region of propagation. Since B_z is assumed not to depend on r, the force $q\mathbf{v}_\perp \times B_z\hat{z}$ is linear. This property always holds near the system axis and no significant error is made in extending it over the beam profile if the scale length of change is large compared with the beam radius. The beam itself may be a source of B_z, but the contribution is $O(\epsilon^2)$ compared with B_θ and may be neglected.

2) B_r is determined from the Maxwell equation $\nabla \cdot \mathbf{B} = 0$. Recalling that by symmetry $\partial B_\theta/\partial\theta = 0$, we have

$$B_r = -\frac{r}{2}\frac{\partial B_z}{\partial z}. \tag{11}$$

3) E_θ is similarly determined (from Faraday's law) to be

$$E_\theta = -\frac{r}{2}\frac{\partial B_z}{\partial t}. \tag{12}$$

4) E_r and B_θ are generated by the charges and currents of the beam and background gas. These components need not be specified in detail at this point.

5) The field E_z is generated by the beam, gas, and external sources and, in general, is $O(cB_\theta)$. However, it only acts directly to change γ and v_z rather than $\mathbf{v}_\perp$, so its apparent effect is small, of order ϵ^2. A cumulative energy change results from the time average E_z during propagation over a long distance; this effect has been taken into account in Eq. (8). The use of the spatial as well as time average $\bar{E}_z$ in Eq. (8) is justified by the fact that, due to betatron oscillations, a particle is not localized at a single value of r during propagation.

In terms of the above field components, the transverse equation of motion is

$$\frac{1}{\gamma}\dot{\gamma}\mathbf{v}_\perp + \dot{\mathbf{v}}_\perp = \frac{q}{\gamma m}\left(E_r\hat{r} - \frac{r}{2}\frac{\partial B_z}{\partial t}\hat{\theta} - \beta c\frac{r}{2}\frac{\partial B_z}{\partial z}\hat{z}\times\hat{r} + \beta c B_\theta \hat{z} \times \hat{\theta} - B_z\hat{z}\times\mathbf{v}_\perp\right) + \frac{1}{\gamma m}\boldsymbol{\delta}\mathbf{F}_\perp. \tag{13}$$

Noting that

$$\frac{\partial B_z}{\partial t} + \beta c\frac{\partial B_z}{\partial z} = \frac{\mathrm{d}B_z}{\mathrm{d}t} = \dot{B}_z \tag{14}$$

(where $\dot{B}_z$ is the convective derivative, i.e., the variation of B_z experienced by a moving particle), we may write

$$\frac{1}{\gamma}\dot{\gamma}\mathbf{v}_\perp + \dot{\mathbf{v}}_\perp = \frac{q}{\gamma m}[(E_r - \beta c B_\theta)\hat{r} - \tfrac{1}{2}\dot{B}_z\hat{z} \times \mathbf{r}_\perp - B_z\hat{z}\times\mathbf{v}_\perp] + \frac{1}{\gamma m}\boldsymbol{\delta}\mathbf{F}_\perp. \tag{15}$$

At this point we define the relativistic cyclotron and betatron frequencies,

$$\omega_c = \frac{qB_z}{\gamma m} \quad \text{and} \quad \omega_\beta^2 = \frac{q}{\gamma m}\frac{\beta c B_\theta - E_r}{r}. \tag{16}$$

In general ω_β^2 is a function of r, although for a flat current and charge profile it is a constant. In a vacuum, ω_β^2 is negative (see Appendix B). Employing these definitions, Eq. (15) may be written

$$\frac{1}{\gamma}\dot{\gamma}\mathbf{v} + \dot{\mathbf{v}} + \omega_\beta^2\mathbf{r} + \omega_c\hat{z}\times\mathbf{v} + \frac{1}{2\gamma}\frac{\mathrm{d}}{\mathrm{d}t}(\gamma\omega_c)\hat{z}\times\mathbf{r} = \frac{1}{\gamma m}\boldsymbol{\delta}\mathbf{F}. \tag{17}$$

Here and henceforth we drop the $\perp$ notation for convenience.

3 MOMENT EQUATIONS

By taking appropriate moments of Eq. (17) we obtain coupled equations for the single particle variables v^2, r^2, and rv_θ. These contain the necessary information, along with the ansatz of self-similar expansion (to be introduced in Section IV) to derive an envelope equation.

First an energy equation is found by forming the scalar product of Eq. (17) with $\mathbf{v}$:

$$\frac{1}{\gamma}\dot{\gamma}v^2 + \frac{\mathrm{d}}{\mathrm{d}t}\frac{v^2}{2} + \omega_\beta^2\frac{\mathrm{d}}{\mathrm{d}t}\frac{r^2}{2} + \frac{1}{2\gamma}\frac{\mathrm{d}}{\mathrm{d}t}(\gamma\omega_c)l = \frac{1}{\gamma m}\mathbf{v}\cdot\boldsymbol{\delta}\mathbf{F}, \tag{18}$$

where

$$l = (\mathbf{r}\times\mathbf{v})\cdot\hat{z} = rv_\theta. \tag{19}$$

Similarly, the virial equation is obtained from the scalar product with $\mathbf{r}$:

$$\frac{1}{\gamma}\dot{\gamma}\frac{\mathrm{d}}{\mathrm{d}t}\frac{r^2}{2} + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{r^2}{2} - v^2 + \omega_\beta^2 r^2 - \omega_c l = \frac{1}{\gamma m}\mathbf{r}\cdot\boldsymbol{\delta}\mathbf{F}. \tag{20}$$

The z component of the cross product of $\mathbf{r}$ with Eq. (17) is the angular momentum equation,

$$\frac{1}{\gamma}\dot{\gamma}l + \dot{l} + \omega_c\frac{\mathrm{d}}{\mathrm{d}t}\frac{r^2}{2} + \frac{1}{2\gamma}\frac{\mathrm{d}}{\mathrm{d}t}(\gamma\omega_c)r^2 = \frac{1}{\gamma m}\hat{z}\cdot(\mathbf{r}\times\boldsymbol{\delta}\mathbf{F}). \tag{21}$$

We now consider a transverse segment of the beam containing a fixed set of N particles labeled by their time of injection t_0. Mean quantities for the segment are defined by summing over the N particles as follows:

$$R^2 = \frac{1}{N}\sum_{i=1}^{N} r_i^2, \tag{22a}$$

$$V^2 = \frac{1}{N}\sum_{i=1}^{N} v_i^2, \tag{22b}$$

$$L = \frac{1}{N}\sum_{i=1}^{N} l_i, \tag{22c}$$

$$\epsilon' = \frac{1}{\gamma N}\sum_{i=1}^{N} \gamma\mathbf{v}_i\cdot\boldsymbol{\delta}\mathbf{F}_i. \tag{22d}$$

R^2 is the mean square radius of the segment ($R = a/\sqrt{2}$ for a beam of flat profile with radius a). The quantity ϵ' is the rate of energy transfer to the transverse motion via scattering (see Appendix A).

Taking the average of Eqs. (18), (20), and (21) over the particles of a segment, we have

$$\frac{1}{\gamma}\dot{\gamma}V^2 + \frac{\mathrm{d}}{\mathrm{d}t}\frac{V^2}{2} + \overline{\omega_\beta^2\frac{\mathrm{d}}{\mathrm{d}t}\frac{r^2}{2}} + \frac{1}{2\gamma}\frac{\mathrm{d}}{\mathrm{d}t}(\gamma\omega_c)L = \frac{\epsilon'}{\gamma m} \tag{23}$$

$$\frac{1}{\gamma}\dot{\gamma}\frac{\mathrm{d}}{\mathrm{d}t}\frac{R^2}{2} + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{R^2}{2} - V^2 + \overline{\omega_\beta^2 r^2} - \omega_c L = 0, \tag{24}$$

$$\frac{1}{\gamma}\dot{\gamma}L + \frac{\mathrm{d}L}{\mathrm{d}t} + \omega_c\frac{\mathrm{d}}{\mathrm{d}t}\frac{R^2}{2} + \frac{1}{2\gamma}\frac{\mathrm{d}}{\mathrm{d}t}(\gamma\omega_c)R^2 = 0. \tag{25}$$

The bar appearing in Eqs. (23) and (24) denotes the averages that depend on the velocity ansatz (see Section IV and Appendix B). Note that no terms involving $\boldsymbol{\delta}\mathbf{F}$ appear in Eqs. (24) and (25); this is a consequence of the mean isotropy of the scattering process.

Our treatment of the beam envelope extends previous work in making use of all three moment equations. For example, Bennett[5] employs a virial equation similar to Eq. (24) to find the condition of equilibrium of a self-pinched beam, but he does not attempt to obtain an envelope equation for R by eliminating V^2.

4 SELF-SIMILAR EXPANSION

Our aim is to derive an equation for R alone, without reference to the detailed internal state of the beam. In characterizing the beam by this single variable we neglect internal distortions that may be of physical consequence. However, Eqs. (23) through (25) are quite general and it is seen that the unknown dynamics enter only through the quantities $\overline{\omega_\beta^2 r^2}$ and $\overline{\omega_\beta^2(\mathrm{d}r^2/\mathrm{d}t)}$. To make further progress these quantities must also be expressed in terms of R. We adopt as a model self-similar expansion (or contraction) of the current profile J_b. That is, the shape of J_b is fixed as the rms radius R changes:

$$J_b = \frac{I_b}{2\pi R^2}F\left(\frac{r}{R}\right), \tag{26}$$

where F is some given function and I_b is the beam current of the slice. The time dependence of J_b is contained in R and possibly I_b. The normalization of F is

$$1 = \int_0^\infty \mathrm{d}u\, u\, F(u). \tag{27}$$

It is useful to write the single particle velocity in the form

$$\mathbf{v} = \frac{\mathbf{r}}{R}\dot{R} + \frac{Lr}{R^2}\hat{\theta} + \boldsymbol{\delta}\mathbf{v}. \tag{28}$$

Then, averaging over all the particles in a segment, we find by the definitions of R and L,

$$0 = \overline{r\delta v_r} \tag{29a}$$

and

$$0 = \overline{r\delta v_\theta}. \tag{29b}$$

The ansatz of self-similar expansion is equivalent to the requirement that condition (29a) hold when the average is taken over any narrow annulus of arbitrary radius r; i.e., $\bar{v}_r = r\dot{R}/R$ in the annulus. Hence for any function of $f(r)$,

$$\overline{f(r)\delta v_r} = 0. \tag{30}$$

There is no further restriction imposed on $\boldsymbol{\delta}\mathbf{v}$. It may be shown from Eqs. (28) through (30) that its mean square value is related to V as

$$V^2 = \overline{v^2} = \dot{R}^2 + \frac{L^2}{R^2} + \overline{|\boldsymbol{\delta}\mathbf{v}|^2}. \tag{31}$$

Thus $\boldsymbol{\delta}\mathbf{v}$ is the residual velocity that is not part of the mean flows.

Since ω_β^2 is not a function of $\dot{r}$, we now have a specific prescription for calculating the quantity,

$$U \equiv \overline{\omega_\beta^2 r^2} = \frac{1}{I_b}\int_0^\infty \mathrm{d}r\, 2\pi r J_b \omega_\beta^2 r^2$$

$$= \int_0^\infty \mathrm{d}r\, r\frac{F(r/R)}{R^2}\omega_\beta^2 r^2. \tag{32}$$

Appendix B presents an evaluation of U for the case in which current and charge densities of the background are related to those of the beam by simple neutralization fractions. We note here that for a charge neutral, self pinch U does not depend on R or self-similarity:

$$U = U_b = \frac{q\beta c\mu_0 I_b}{4\pi\gamma m}. \tag{33}$$

Using Eqs. (28) and (30) we also calculate

$$\overline{\omega_\beta^2\frac{\mathrm{d}}{\mathrm{d}t}r^2} = \overline{\omega_\beta^2 2\mathbf{r}\cdot\mathbf{v}} = \overline{\omega_\beta^2 2\mathbf{r}\cdot\left(\frac{\mathbf{r}\dot{R}}{R}\right)} \tag{34}$$

$$= \frac{2}{R}\dot{R}\overline{\omega_\beta^2 r^2} = \frac{1}{R^2}\frac{\mathrm{d}R^2}{\mathrm{d}t}U.$$

If ω_β is known to be independent of r over the beam profile, then Eq. (34) is rigorously correct; however, this is the exceptional case. Equation (34) has been derived here for an arbitrary dependence $\omega_\beta^2(r)$ with the simplest possible assumption for the beam kinematics (self-similar expansion). The consequent form of $\mathbf{v}$ is very general and related in a natural way to $\dot{R}$, L and V.

5 THE ENVELOPE EQUATION AND EMITTANCE

The envelope equation may now be derived. First, observe that Eq. (25) is integrable and yields the mean canonical angular momentum as a constant of integration:

$$\dot{P}_\theta = 0, \tag{35}$$

where

$$P_\theta = \gamma L + \gamma\omega_c\frac{R^2}{2}. \tag{36}$$

Canonical angular momentum is not conserved for individual particles in the presence of scattering, but it is conserved in the average taken over many particles because $\delta\mathbf{F}$ is distributed isotropically.

The virial equation (24) is used to eliminate V^2 from the energy equation (23) to get

$$\left(\frac{1}{\gamma}\dot{\gamma} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\right)\left[\frac{1}{\gamma}\dot{\gamma}\frac{\mathrm{d}}{\mathrm{d}t}\frac{R^2}{2} + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{R^2}{2} + \overline{\omega_\beta^2 r^2} - \omega_c L\right]$$

$$+ \overline{\omega_\beta^2\frac{\mathrm{d}}{\mathrm{d}t}\frac{r^2}{2}} + \frac{1}{2\gamma}\frac{\mathrm{d}}{\mathrm{d}t}(\gamma\omega_c)L = \frac{\epsilon'}{\gamma m}. \tag{37}$$

The terms involving L may be combined with the aid of Eqs. (35) and (36); they become

$$\left(\frac{1}{\gamma}\dot{\gamma} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\right)(-\omega_c L) + \frac{1}{2\gamma}\frac{\mathrm{d}}{\mathrm{d}t}(\gamma\omega_c)L$$

$$= \frac{-\omega_c}{2\gamma}\frac{\mathrm{d}}{\mathrm{d}t}(\gamma L) = -\frac{\omega_c}{2\gamma}\frac{\mathrm{d}}{\mathrm{d}t}\left(P_\theta - \frac{\gamma\omega_c R^2}{2}\right)$$

$$= \frac{\omega_c}{4\gamma}\frac{\mathrm{d}}{\mathrm{d}t}(\gamma\omega_c R^2). \tag{38}$$

Substituting this expression into Eq. (37) along with the expressions (32) and (34) yields

$$\left(\frac{1}{\gamma}\dot{\gamma} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\right)\left[\frac{1}{\gamma}\dot{\gamma}\frac{\mathrm{d}}{\mathrm{d}t}\frac{R^2}{2} + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{R^2}{2} + U\right]$$

$$+ \frac{U}{R^2}\frac{\mathrm{d}}{\mathrm{d}t}\frac{R^2}{2} + \frac{\omega_c}{4\gamma}\frac{\mathrm{d}}{\mathrm{d}t}(\gamma\omega_c R^2) = \frac{\epsilon'}{\gamma m}. \tag{39}$$

An integrating factor $(2\gamma^2R^2)$ has been found for the entire left-hand side of Eq. (39); we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(\gamma^2R^3\ddot{R} + \gamma\dot{\gamma}R^3\dot{R} + \gamma^2R^2U + \tfrac{1}{4}\gamma^2R^4\omega_c^2)$$

$$= \frac{2\gamma^2R^2\epsilon'}{\gamma m}. \tag{40}$$

Integrating once in time and dividing both sides by $\gamma^2 R^3$ gives the envelope equation,

$$\ddot{R} + \frac{1}{\gamma}\dot{\gamma}\dot{R} + \frac{U}{R} + \frac{\omega_c^2 R}{4} - \frac{C^2}{\gamma^2 R^3} = \frac{1}{\gamma^2 R^3}\int_{t_0}^{t} \mathrm{d}t' \left(\frac{2\gamma R^2 \epsilon'}{m}\right)_{t'}, \quad (41)$$

where C^2 is the constant of integration and t_0 is the time of injection of the particular beam slice under consideration.

The constant of integration C^2 is immediately related to the conditions at injection by

$$C^2 = \left[\gamma^2 R^3\left(\ddot{R} + \frac{1}{\gamma}\dot{\gamma}\dot{R} + \frac{U}{R} + \frac{\omega_c^2 R}{4}\right)\right]_{t_0}. \quad (42)$$

The collection of terms on the right-hand side of Eq. (42), evaluated at any time $t > t_0$, is itself constant in the absence of scattering. In the presence of scattering, it increases according to Eq. (40). A more useful and physically revealing form of this expression is readily obtained. Using the virial equation (24) to eliminate the term proportional to $\ddot{R}$, we find

$$\begin{aligned}\gamma^2 R^3\left(\ddot{R} + \frac{1}{\gamma}\dot{\gamma}\dot{R} + \frac{U}{R} + \frac{\omega_c^2 R}{4}\right) &= \gamma^2 R^2\left[V^2 - (\dot{R})^2 + \omega_c L + \frac{\omega_c^2 R^2}{4}\right], \\ &= \gamma^2 R^2\left[V^2 - (\dot{R})^2 - \left(\frac{L}{R}\right)^2\right] + P_\theta^2. \quad (43)\end{aligned}$$

The emittance (squared) is now defined as

$$E^2 \equiv \gamma^2 R^2\left[V^2 - (\dot{R})^2 - \left(\frac{L}{R}\right)^2\right]. \quad (44)$$

E^2 is clearly a constant in the absence of scattering. Since P_θ is always constant, Eq. (40) gives

$$\frac{\mathrm{d}E^2}{\mathrm{d}t} = \frac{2\gamma R^2 \epsilon'}{m}. \quad (45)$$

The constant C is related to E by

$$C^2 = E^2(t_0) + P_\theta^2. \quad (46)$$

Some insight into the nature of the emittance is gained by writing it in terms of the residual particle velocities $\boldsymbol{\delta}\mathbf{v}$. Eliminating V^2 from Eq. (44) with expression (31), we get

$$E^2 = \gamma^2 R^2 \overline{|\boldsymbol{\delta}\mathbf{v}|^2}. \quad (47)$$

Thus, E^2 is proportional to the *effective transverse phase area* occupied by the segment. This is in the area in $(\mathbf{r}_\perp, \mathbf{p}_\perp)$ space occupied by the beam particles, with small-scale gaps and peaks of density smoothed over. As might be expected, the mean flow velocity, $\dot{R}\mathbf{r}/R + Lr\hat{\theta}/R^2$, makes no contribution to phase area. Scattering increases E^2 by adding directly to the residual velocities without a compensating change in the particle positions.

Our expression [Eq. (44)] for emittance is a generalization of that given by Emigh[6] for variables (x, v_x) or (y, v_y):

$$E_x^2 = \overline{x^2}[\overline{v_x^2} - (\overline{v_x})^2].$$

E_x^2 is proven to be constant in time for a nonrelativistic beam of particles subject to a linear force $(\ddot{x} = K(t)x)$.

In the absence of scattering the *true, microscopically defined phase area* of the segment is conserved since Eq. (17) may be cast into Hamiltonian form. In reality the *effective* phase area defined above is *not* expected to be conserved when the beam pulsates. Instead, the directed flows representing pulsation are converted, via phase mixing of particle orbits, to microscopic gaps in phase space. These gaps are smoothed over to obtain the effective areas, i.e., E^2 increases. It is a weakness of our model (and others) that no account of this process is given, i.e., we find E^2 conserved in the absence of scattering. The source of this loss is the assumption of self-similar expansion and, in particular, the use of Eq. (34), which allowed integration of the moment equations. For the case of ω_β vanishing or independent of r, Eq. (34) is exact; this is precisely the circumstance of no phase mixing.

6 APPLICATIONS

The envelope equation (41) is a generalization of previous work, allowing for scattering, nonzero canonical angular momentum, and nonuniform J_b; however, its basic form is well known. Lawson[7] has given a summary of deductions one can make from envelope equations, including the effects of emittance spread, Child's law, and Brillouin flow. We will describe here some applications of particular interest to ourselves, concerning propagation of a beam into and through a gas cell.

6.1 *Free Expansion*

As a first application we consider a beam subject to no macroscopic transverse force, i.e., $\omega_c = \omega_\beta = 0$. In this case it is convenient to use the envelope equation in the form given by Eq. (39):

$$\left(\frac{1}{\gamma}\dot{\gamma} + \frac{1}{2}\frac{d}{dt}\right)\left[\frac{1}{\gamma}\dot{\gamma}\frac{d}{dt}\frac{R^2}{2} + \frac{d^2}{dt^2}\frac{R^2}{2}\right] = \frac{\epsilon'}{\gamma m}. \quad (48)$$

Multiplying both sides by $4\gamma^2$, the left-hand side of Eq. (48) becomes integrable, giving

$$\frac{d}{dt}\left[\gamma\frac{d}{dt}\left(\gamma\frac{d}{dt}R^2\right)\right] = \frac{4\gamma\epsilon'}{m}. \quad (49)$$

The formal solution is

$$R^2 = \int_{t_0}^{t}\frac{dt'}{\gamma}\int_{t_0}^{t'}\frac{dt''}{\gamma}\int_{t_0}^{t''}dt'''\left(\frac{4\gamma\epsilon'}{m}\right) + a_1 + a_2\int_{t_0}^{t}\frac{dt'}{\gamma} + a_3\int_{t_0}^{t}\frac{dt'}{\gamma}\int_{t_0}^{t'}\frac{dt''}{\gamma}. \quad (50)$$

The constants of integration a_1, a_2, and a_3 are related to conditions at the time (t_0) when the beam passes through a waist $(\dot{R} = 0)$ with known radius R_0, emittance E_0, and angular momentum $P_\theta = \gamma_0 L_0$; that is,

$$a_1 = R_0^2,\ a_2 = 0,\ a_3 = 2\gamma_0^2 V_0^2, \quad (51)$$

with initial mean square velocity,

$$V_0^2 = \frac{E_0^2}{\gamma_0^2 R_0^2} + \frac{L_0^2}{R_0^2} = \frac{C^2}{\gamma_0^2 R_0^2}. \quad (52)$$

$V(t)$ and $L(t)$ are determined directly from the moment equations (23) and (25) to be

$$\gamma^2 V^2 = \gamma_0^2 V_0^2 + \int_{t_0}^{t} dt'\,\frac{2\gamma\epsilon'}{m} \quad (53)$$

and

$$\gamma L = \gamma_0 L_0. \quad (54)$$

From Eqs. (50) and (51),

$$R^2 = R_0^2 + 2\gamma_0^2 V_0^2\int_{t_0}^{t}\frac{dt'}{\gamma}\int_{t_0}^{t'}\frac{dt''}{\gamma} + \int_{t_0}^{t}\frac{dt'}{\gamma}\int_{t_0}^{t'}\frac{dt''}{\gamma}\int_{t_0}^{t''}dt'''\left(\frac{2\gamma\epsilon'}{m}\right). \quad (55)$$

Setting $\dot{\gamma} = \epsilon' = 0$, we have $V(t) = V_0$ and

$$R^2 = R_0^2 + V_0^2(t - t_0)^2. \quad (56)$$

If a constant scattering rate ϵ' is imposed but $\dot{\gamma}$ is neglected, then

$$V^2 = V_0^2 + \left(\frac{2\epsilon'}{\gamma m}\right)(t - t_0), \quad (57)$$

$$R^2 = R_0^2 + V_0^2(t - t_0)^2 + \left(\frac{\epsilon'}{3\gamma m}\right)(t - t_0)^3. \quad (58)$$

Scattering changes the asymptotic dependence of $R(t)$ from t to $t^{3/2}$.

6.2 *Equilibrium Conditions*

The introduction of transverse fields allows the possibility of macroscopic equilibrium, i.e., R is constant in time. It is well known that the condition for a particle orbit to be stable, subject to constant frequencies ω_β and ω_c, is

$$\omega_\beta^2 + \frac{\omega_c^2}{4} > 0. \quad (59)$$

A condition of this type, derived below, is satisfied in equilibrium by the beam segment as a whole. In addition the equilibrium radius $(\tilde{R})$ is related to E and P_θ.

By assumption, $\dot{R} = \ddot{R} = 0$ and we neglect the effects of scattering and energy change at this point; so the envelope equation (41) becomes

$$\frac{\tilde{U}}{\tilde{R}} + \frac{\omega_c^2\tilde{R}}{4} - \frac{C^2}{\gamma^2\tilde{R}^3} = 0. \quad (60)$$

The constants C^2 and ω_c are known explicitly. However, U may depend in general on both R and details of the background medium, necessitating numerical solution of Eq. (60). Here we consider several simple cases that may be treated analytically. Once a solution has been found (by any means), we define the equilibrium betatron frequency by

$$\tilde{\omega}_\beta^2 \equiv \frac{\tilde{U}}{\tilde{R}^2}. \quad (61)$$

Now Eq. (60) may be written

$$\tilde{\omega}_\beta^2 + \frac{\omega_c^2}{4} = \frac{C^2}{\gamma^2\tilde{R}^4} > 0. \quad (62)$$

Consider first a self-pinched beam with E_r and B_θ equal to the fields of the beam reduced respectively with neutralization fractions f_c and f_m (see Appendix B):

$U = \Gamma U_b$, $\omega_c = 0$,

$$\Gamma = \frac{I}{I_b}\left[1 - \frac{1}{\beta^2}\frac{(1-f_e)}{(1-f_m)}\right], \; U_b = \frac{q\beta c\mu_0 I_b}{\gamma m 4\pi}, \quad (63)$$

where I is the net current, I_b is the beam current, and Γ must be positive; then by Eqs. (60) and (61),

$$\tilde{R}^2 = \frac{C^2}{\gamma^2 \Gamma U_b}, \tilde{\omega}_\beta^2 = \left(\frac{\gamma \Gamma U_b}{C}\right)^2. \quad (64)$$

Note that from the definition of emittance (Eq. (44)) the equilibrium mean square velocity is

$$\tilde{V}^2 = \frac{E^2}{\gamma^2 \tilde{R}^2} + \frac{L^2}{\tilde{R}^2} = \frac{C^2}{\gamma^2 \tilde{R}^2} = \Gamma U_b. \quad (65)$$

Hence the mean transverse kinetic energy of the particles is

$$T \equiv \frac{\gamma m V^2}{2} = \frac{\gamma m}{2}\Gamma U_b = \Gamma T_B, \quad (66)$$

where

$$T_B = \frac{q\beta c}{2}\frac{\mu_0 I_b}{4\pi} = \frac{\gamma m \beta^2 c^2}{2}\frac{I_b}{I_A} \quad (67)$$

is the "Bennett temperature" in units of energy.[5]

When a B_z field is applied in addition to the self-induced forces, an equilibrium exists even for negative Γ, allowing vacuum transport in a solenoid. This is a consequence of the dependence $\omega_\beta \propto R^{-1}$, so ω_c can be made to dominate in the inequality (62). Solving Eq. (60) for $\tilde{R}$, we have

$$\tilde{R}^2 = -\frac{2\Gamma U_b}{\omega_c^2} + \left[\left(\frac{2\Gamma U_b}{\omega_c}\right)^2 + \left(\frac{2C}{\omega_c \gamma}\right)^2\right]^{1/2}. \quad (68)$$

The equilibrium radius may be compressed with a solenoidal field to make a more intense beam; this must be done in adiabatic fashion or by careful matching with a lens at injection to avoid pulsations.

6.3 *Quasi-Static Limit*

The beam variables and system parameters are now allowed to change on a time scale τ, which is long compared with the period of particle oscillation, and the scattering rate is similarly assumed to be weak:

$$\frac{\dot{\gamma}\tau}{\gamma}, \frac{\dot{U}\tau}{U}, \frac{\dot{\omega}_c\tau}{\omega_c} = O(1),$$
$$(\omega_\beta^2 + \tfrac{1}{4}\omega_c^2)\tau^2 \gg 1,$$
$$\epsilon'\tau \ll \gamma m(\omega_\beta^2 + \tfrac{1}{4}\omega_c^2)R^2. \quad (69)$$

R is expected to be close to some equilibrium value $\tilde{R}(t)$; the terms $\dot{\gamma}\dot{R}$ and $\ddot{R}$ in Eq. (40) are $O(\tau^{-2})$ and may be neglected to obtain

$$\frac{d}{dt}(\gamma^2 R^2 U + \tfrac{1}{4}\gamma^2 R^4 \omega_c^2) = \frac{2\gamma R^2 \epsilon'}{m}. \quad (70)$$

We define

$$\eta^2 \equiv \gamma^2\left(\frac{U}{R^2} + \frac{\omega_c^2}{4}\right)R^4, \quad (71)$$

which is an adiabatic invariant in the absence of scattering. Within the present approximations, $\eta^2 = (E^2 + P_\theta^2)$.

Equation (70) may be written

$$\frac{1}{\eta}\frac{d\eta}{dt} = \frac{\gamma R^2 \epsilon'}{m\eta^2}. \quad (72)$$

The standard form of the Nordsiéck[8] equation for the rate of expansion of a self-pinched beam is obtained immediately. Substituting $\omega_c = 0$ and $U = U_b$ in Eq. (71), we have

$$\eta^2 = \gamma^2 \frac{q\beta c}{\gamma m}\frac{\mu_0 I_b}{4\pi}R^2 \propto \gamma\beta I_b R^2. \quad (73)$$

Inserting this result in Eq. (72) yields

$$\frac{d}{dt}\log[(\beta\gamma I_b)^{1/2}R] = \frac{4\pi\epsilon'}{q\beta c\mu_0 I_b} = \frac{\epsilon'}{2T_B}. \quad (74)$$

The variable R increases exponentially in time as a result of scattering and varies as $(\beta\gamma I_b)^{-1/2}$ as parameters change. Note that $\epsilon'/T_B \propto (\gamma I_b)^{-1}$, so the rate of expansion is reduced proportionately as the beam power is raised. If a beam characterized by neutralization fractions is considered, Eq. (74) applies, with ΓI_b substituted for I_b.

In the case $U = 0$, $\omega_c \neq 0$, the adiabatic invariant is proportional to the flux enclosed by the beam, as might be expected:

$$\eta = \frac{\gamma\omega_c R^2}{2} = \frac{qB_z R^2}{2m}. \quad (75)$$

Equation (72) gives

$$\frac{d}{dt}\left(\frac{\gamma\omega_c R^2}{2}\right) = \frac{2\epsilon'}{m\omega_c}. \quad (76)$$

The radius R now increases only as $t^{1/2}$ due to gas scattering.

6.4 *Pulsation Induced by a Mismatch*

The injection parameters R_0 and $\dot{R}_0$ are in general specified independently of the emittance. The beam radius then pulsates around the equilibrium position $\tilde{R}$ as described by Eq. (41) (unless of course $R_0 = \tilde{R}$ and $\dot{R} = 0$). To predict the frequency and amplitude of this motion, we study the case $\dot{\gamma} = \dot{\omega}_c = \epsilon' = 0$, with a small mismatch from equilibrium. We denote the amplitude of the motion by

$$\delta R = R - \tilde{R} \ll \tilde{R}. \tag{77}$$

The equilibrium condition (61) is recovered to the order zero in δR. In the first order we have

$$\delta \ddot{R} + \frac{\delta U}{\tilde{R}} - \frac{\tilde{U}}{\tilde{R}^2}\delta R + \frac{\omega_c^2 \delta \tilde{R}}{4} + \frac{3C^2}{\gamma^2 \tilde{R}^4}\delta R = 0. \tag{78}$$

Substituting for $\tilde{U}$ and C^2/γ^2 from Eqs. (60) and (61), Eq. (78) becomes

$$\delta \ddot{R} + \frac{\delta U}{\tilde{R}} + (2\tilde{\omega}_\beta^2 + \omega_c^2)\delta R = 0. \tag{79}$$

The quantity δU may depend on the background properties in a complicated way; here we consider two special limits. First, U is a constant equal to U_b; i.e., the conductivity is low enough that only the magnetic field of the beam is significant. Then $\delta U = 0$, and the solution of Eq. (79) is

$$\delta R = \delta R_0 \cos \Omega t + \frac{\delta \dot{R}_0}{\Omega} \sin \Omega t, \tag{80}$$

$$\Omega = (2\tilde{\omega}_\beta^2 + \omega_c^2)^{1/2}, \tag{81}$$

$$(\delta R)_{\max} = \left[(\delta R_0)^2 + \frac{(\delta \dot{R}_0)^2}{\Omega^2}\right]^{1/2}. \tag{82}$$

The opposite limit of very high conductivity is also easily treated. The magnetic field is "frozen-in" equal to its equilibrium value. It is shown in Appendix B that in this case,

$$\delta U = 2\tilde{\omega}_\beta^2 \psi \tilde{R} \delta R, \tag{83}$$

where

$$\psi = \frac{4\pi^2}{I_b^2}\int_0^\infty \mathrm{d}r\, r^3 J_b^2 \tag{84}$$

is a parameter characterizing the *shape* of the beam current profile. For monotonely decreasing $J_b(r)$, it may be shown that $0 < \psi < 1$, with $\psi = 1$ for a flat profile. The effect of expression (83) is to increase the frequency of oscillation to

$$\Omega = [2\tilde{\omega}_\beta^2(1 + \psi) + \omega_c^2]^{1/2}. \tag{85}$$

6.5 *Transport in a Solenoidal Field*

A useful exact solution of the envelope equation describes the vacuum transport of a beam in a solenoid. The betatron frequency may be neglected if the beam is sufficiently diffuse. Setting $\dot{\gamma} = \dot{\omega}_c = \epsilon' = \omega_\beta^2 = 0$, Eq. (41) becomes

$$\ddot{R} + \frac{\omega_c^2}{4}R - \frac{C^2}{\gamma^2 R^3} = 0. \tag{86}$$

There is a simple constant of the motion,

$$\frac{(\dot{R})^2}{2} + \frac{\omega_c^2 R^2}{8} + \frac{C^2}{2\gamma^2 R^2} = \frac{W}{4}, \tag{87}$$

whose value may be determined from E, P_θ and the initial conditions, R_0 and $\dot{R}_0$. It is useful to form the virial moment of Eq. (86); multiplying both sides by $R/2$ we get

$$\frac{1}{4}\frac{\mathrm{d}^2 R^2}{\mathrm{d}t^2} - \frac{(\dot{R})^2}{2} + \frac{\omega_c^2 R^2}{8} - \frac{C^2}{2\gamma^2 R^2} = 0. \tag{88}$$

Adding Eqs. (87) and (88) yields

$$\frac{\mathrm{d}^2 R^2}{\mathrm{d}t^2} + \omega_c^2 R^2 = W, \tag{89}$$

which can also be derived immediately as a first integral of Eq. (39).

The solution of Eq. (89) is

$$R^2 = \frac{W}{\omega_c^2} + \alpha \sin(\omega_c t + \phi), \tag{90}$$

with

$$\alpha^2 = \frac{W^2}{\omega_c^4} - \frac{4C^2}{\gamma^2 \omega_c^2},$$

$$\tan \phi = \frac{R_0^2 - (W/\omega_c^2)}{2R_0 \dot{R}_0/\omega_c},$$

$$W = 2(\dot{R}_0)^2 + \tfrac{1}{2}\omega_c^2 R_0^2 + \frac{2C^2}{\gamma^2 R_0^2}. \tag{91}$$

The mean square radius oscillates at the cyclotron frequency between maximum and minimum values, R_+^2 and R_-^2:

$$R_\pm^2 = \frac{W}{\omega_c^2} \pm \alpha. \tag{92}$$

These satisfy the relations,

$$\frac{R_+^2 + R_-^2}{2} = \frac{W}{\omega_c^2}, \quad \frac{R_+^2 - R_-^2}{2} = \alpha,$$

$$R_+^2 R_-^2 = \frac{W^2}{\omega_c^4} - \alpha^2 = \frac{4C^2}{\gamma^2 \omega_c^2}. \tag{93}$$

The role of the constant C^2 is elucidated by writing the solution [Eq. (90)] using half angles:

$$R^2 = \left[R_0 \cos\left(\frac{\omega_c t}{2}\right) + \frac{2\dot{R}_0}{\omega_c} \sin\left(\frac{\omega_c t}{2}\right)\right]^2 + \frac{4C^2}{\gamma^2 R_0^2 \omega_c^2} \sin^2\left(\frac{\omega_c t}{2}\right). \qquad (94)$$

If $C^2 = 0$, then

$$R = \left|R_0 \cos\left(\frac{\omega_c t}{2}\right) + \frac{2\dot{R}_0}{\omega_c} \sin\left(\frac{\omega_c t}{2}\right)\right|, \qquad (95)$$

i.e., R has the appearance of a *rectified* sinusoid that touches zero. The frequency defined with respect to the peaks is thus ω_c.

With finite values of C the radius may be matched to the equilibrium value:

$$\tilde{R} = \left(\frac{4C^2}{\gamma^2 \omega_c^2}\right)^{1/4}. \qquad (96)$$

A small mismatch then yields oscillations:

$$\delta R = \delta R_0 \cos \omega_c t + \frac{\delta \dot{R}_0}{\omega_c} \sin \omega_c t. \qquad (97)$$

6.6 *Matching at a Foil in a Solenoid*

When passing a beam from vacuum to gas, an appreciable increment in emittance may occur as a result of scattering in the entrance foil. It is desirable to minimize this effect by focusing to a small radius at the foil while simultaneously matching to an equilibrium configuration in the gas cell. These dual purposes are achieved with the aid of a strong solenoidal field. Here we determine the equilibrium radius $\tilde{R}$ in terms of the parameters obtained prior to passage through the foil. A similar calculation has been made by Eugene Lauer.[9]

If the foil has thickness $\beta c \Delta t$ and scatters at the rate ϵ', then the squared emittance is increased by the amount

$$\Delta(E^2) = \frac{2\gamma \tilde{R}^2}{m} \epsilon' \Delta t. \qquad (98)$$

Since the beam is in equilibrium after passing the foil, Eq. (41) gives

$$\frac{U}{\tilde{R}} + \frac{\omega_c^2 \tilde{R}}{4} - \frac{C^2}{\gamma^2 \tilde{R}^3} = \frac{1}{\gamma^2 \tilde{R}^3} \frac{2\gamma \tilde{R}^2}{m} \epsilon' \Delta t. \qquad (99)$$

In the gas, U is set equal U_b; then a convenient parameterization of Eq. (99) may be given. Let

$$R_1 = \frac{C}{\gamma U_b^{1/2}}, \quad R_2 = \left(\frac{2C}{\gamma \omega_c}\right)^{1/2}. \qquad (100)$$

These are the equilibrium radii in the absence of scattering in the limits of pure self-field or solenoidal field. Multiplying Eq. (99) through by $\gamma^2 \tilde{R}^3 / C^2$ and recalling the Bennett temperature $T_B = \gamma m U_b / 2$, we get

$$\frac{\tilde{R}^4}{R_2^4} + \frac{\tilde{R}^2}{R_1^2} - \frac{\epsilon' \Delta t}{T_B} \frac{\tilde{R}^2}{R_1^2} - 1 = 0. \qquad (101)$$

Setting

$$A = \left(\frac{R_2}{R_1}\right)^4 = \frac{4\gamma^2 U_b^2}{\omega_c^2 C^2},$$

$$B = \frac{\epsilon' \Delta t}{T_B} = \frac{2\epsilon' \Delta t}{\gamma m U_b}, \qquad (102)$$

the solution of Eq. (101) is

$$\left(\frac{\tilde{R}}{R_1}\right)^2 = -\frac{A(1-B)}{2} + \frac{1}{2}[A^2(1-B)^2 + 4A]^{1/2}. \qquad (103)$$

A plot of $(\tilde{R}/R_1)^2$ versus B is given in Figure 1 for several values of A. Note that for $\omega_c = 0$ $(A \to \infty)$

$$\left(\frac{\tilde{R}}{R_1}\right)^2 = \frac{1}{1-B}. \qquad (104)$$

No physical solution exists if $B \geq 1$. On the other hand, an equilibrium is always found when $\omega_c \neq 0$.

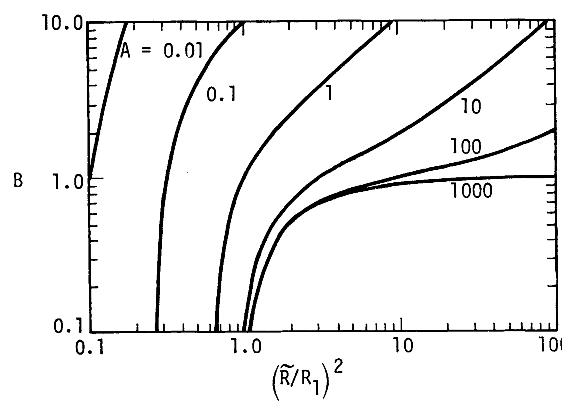

FIGURE 1 Equilibrium radius graphs for several values of A.

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Appendix A

The Scattering Coefficient

We evaluate here the quantity

$$\epsilon' = \frac{1}{\gamma m} \overline{\mathbf{p}_\perp \cdot \boldsymbol{\delta}\mathbf{F}_\perp}, \tag{A1}$$

which represents the effect of gas scattering in the envelope equation.

First consider a single beam particle of net momentum $\mathbf{P}$, moving initially parallel to the z axis. It suffers a small angle scattering event during the short time Δt, and the momentum correlated with the scattering force satisfies

$$\dot{\mathbf{p}}_\perp = \boldsymbol{\delta}\mathbf{F}_\perp. \tag{A2}$$

Hence

$$\overline{\mathbf{p}_\perp \cdot \delta\mathbf{F}_\perp}\ \Delta t = \int_{-\infty}^{\infty} \mathrm{d}t\mathbf{p}_\perp \cdot \dot{\mathbf{p}}_\perp = \frac{|\boldsymbol{\delta}\mathbf{p}_\perp|^2}{2}, \tag{A3}$$

where $\delta P_\perp$ is the net increment of transverse momentum.

Expression (A3) also applies, on the average, to a collection of beam particles with nonzero initial $\mathbf{P}_\perp = O(\epsilon P)$. It is sufficient that

$$\int_{-\infty}^{\infty} p_z\, \delta F_z\, \mathrm{d}t = O(|\boldsymbol{\delta}\mathbf{P}_\perp|^2). \tag{A4}$$

This ordering holds unless the scattering event is highly inelastic; e.g., for elastic small-angle scattering from a gas atom of mass M at rest,

$$\int_{-\infty}^{\infty} p_z\, \delta F_z\, \mathrm{d}t = -\frac{|\boldsymbol{\delta}\mathbf{P}_\perp|^2}{2}\left(1 + \frac{\gamma m}{M}\right). \tag{A5}$$

To connect ϵ' with the literature concerning multiple small-angle scattering we define the scattered angle

$$\boldsymbol{\delta\theta} \simeq \boldsymbol{\delta}\mathbf{P}_\perp / P. \tag{A6}$$

Then for a series of events in time $\Delta t = \Delta l/\beta c$, Eqs. (A1), (A3), and (A6) give

$$\epsilon' = \frac{\beta c P^2}{2\gamma m \Delta l} \sum |\boldsymbol{\delta\theta}|^2. \tag{A7}$$

The sum appearing in this expression has been the subject of considerable study[10–12] and its proper evaluation depends (nontrivially) on Δl, as well as γ, atomic number Z, etc. Here we mention only the well-known approximate result[9] for a thick-gas target of number density N:

$$\sum |\boldsymbol{\delta\theta}|^2 = \frac{8\pi Z(Z+1) q^2 e^2 N \Delta l}{(4\pi\epsilon_0)^2 P^2 \beta^2 c^2} \log\left(\frac{\lambda\!\!\!^{-} Z^{1/3}}{a_0}\right), \tag{A8}$$

where a_0 is the Bohr radius and $\lambda\!\!\!^{-} = \hbar/p$ is the DeBroglie wavelength of the beam particle. Combining expressions (A7) and (A8), we have

$$\epsilon' = \frac{4\pi N Z(Z+1) q^2 e^2}{(4\pi\epsilon_0)^2 P} \log\left(\frac{\lambda\!\!\!^{-} Z^{1/3}}{a_0}\right). \tag{A9}$$

Appendix B

Evaluation of $\overline{\omega_\beta^2 r^2}$

The betatron frequency is a function only of the spatial coordinates and time; hence the mean value of $\omega_\beta^2 r^2$ is a simple spatial average weighted with the beam current density:

$$U = \overline{\omega_\beta^2 r^2} = \frac{1}{I_b}\int_0^\infty \mathrm{d}r\, r 2\pi J_b(r)\omega_\beta^2 r^2. \tag{B1}$$

To evaluate U it is useful to define the mean beam current flowing *within* radius r:

$$I_b(r) = \int_0^r \mathrm{d}r' 2\pi r' J_b(r'), \tag{B2}$$

thus

$$U = \frac{1}{I_b}\int_0^\infty \mathrm{d}r\, \frac{\partial I_b(r)}{\partial r}\,\omega_\beta^2 r^2. \tag{B3}$$

The simplest case is a self-pinched beam, where we neglect space charge, plasma current, and displacement current. We have for Ampere's Law

$$\frac{1}{r}\frac{\partial}{\partial r} r B_\theta = \mu_0 J_b. \tag{B4}$$

Integrating over r and using the definition (B2) we get

$$B_\theta = \frac{\mu_0 I_b(r)}{2\pi r}. \tag{B5}$$

Equation (16) becomes

$$\omega_\beta^2 = \frac{q\beta c B_\theta}{\gamma m r} = \frac{q\beta c \mu_0 I_b(r)}{2\pi\gamma m r^2}. \tag{B6}$$

Inserting this expression in Eq. (B3), we have

$$\begin{aligned} U &= \frac{1}{I_b}\int_0^\infty \mathrm{d}r\, \frac{\partial I_b(r)}{\partial r}\frac{q\beta c\mu_0 I_b(r)}{2\pi\gamma m r^2} r^2 \\ &= \frac{1}{I_b}\left(\frac{q\beta c\mu_0}{4\pi\gamma m}\right)\int_0^\infty \mathrm{d}r \frac{r}{\partial r}[I_b(r)]^2 \\ &= \frac{q\beta c\mu_0 I_b}{4\pi\gamma m} \equiv U_b, \end{aligned} \tag{B7}$$

which is independent of the beam radius and details of its profile.

Next we consider a beam in a vacuum. Its space charge $\rho_b = J_b/\beta c$ is the source of a transverse electric field:

$$\frac{1}{r}\frac{\partial}{\partial r} r E_r = \frac{\rho_b}{\epsilon_0} = \frac{J_b}{\epsilon_0 \beta c}. \tag{B8}$$

It is convenient to write the solution of Eq. (B8) as

$$E_r = \frac{I_b(r)}{2\pi\epsilon_0\beta c r} = \frac{B_\theta}{\epsilon_0\mu_0\beta c} = \frac{cB_\theta}{\beta}. \tag{B9}$$

The betatron frequency is

$$\begin{aligned} \omega_\beta^2 &= \frac{q}{\gamma m}\frac{(\beta c B_\theta - E_r)}{r} = \frac{q\beta c B_\theta}{\gamma m r}\left(1 - \frac{1}{\beta^2}\right) \\ &= -\frac{q\beta c B_\theta}{\gamma m r}\frac{1}{\beta^2\gamma^2} < 0. \end{aligned} \tag{B10}$$

Equations (B3), (B7), and (B10) give

$$U = -\frac{U_b}{\beta^2\gamma^2}. \tag{B11}$$

Let the space charge and current densities have the same radial profile as those of the beam itself, i.e.,

$$J = (1 - f_m)J_b \text{ and } \rho = (1 - f_c)\rho_b, \tag{B12}$$

where f_m and f_c are the neutralization fractions. Then

$$E_r = \frac{1 - f_c}{1 - f_m}\frac{cB_\theta}{\beta}, \tag{B13}$$

and the betatron frequency is

$$\omega_\beta^2 = \frac{q\beta c B_\theta}{\gamma m r}\left[1 - \frac{(1 - f_c)}{\beta^2(1 - f_m)}\right]. \tag{B14}$$

Defining

$$\Gamma \equiv \frac{I}{I_b}\left[1 - \frac{1}{\beta^2}\frac{(1 - f_c)}{(1 - f_m)}\right], \tag{B15}$$

with I the net current, we have

$$U = \Gamma U_b. \tag{B16}$$

It is necessary to derive an expression for the perturbed value of U when the beam radius changes. If the beam continues to be a self pinch, then by Eq. (B7), $\delta U = 0$. However, under conditions of high conductivity it may be more appropriate to assume the magnetic field is "frozen in." Then, ω_β^2 is constant and

$$\delta U = \frac{1}{I_b}\int_0^\infty \mathrm{d}r 2\pi r(\delta J_b)\omega_\beta^2 r^2. \tag{B17}$$

For self-similar expansion the perturbed beam current takes the form,

$$\begin{aligned}\delta J_b &= \delta R\left[\frac{\partial}{\partial R}\frac{I_b}{2\pi R^2}F\left(\frac{r}{R}\right)\right]_r \\ &= -\frac{I_b}{2\pi}\left[\frac{F}{R^2}\frac{2\delta R}{R}+\frac{F'}{R^2}\frac{r}{R}\frac{\delta R}{R}\right] \\ &= -\frac{\delta R}{R}\frac{1}{r}\frac{\partial}{\partial r}r^2 J_b.\end{aligned} \tag{B18}$$

Inserting this result in Eq. (B17), along with expression (B6), yields

$$\begin{aligned}\delta U &= \frac{1}{I_b}\int_0^\infty \mathrm{d}r 2\pi r\left[-\frac{\delta R}{R}\frac{1}{r}\frac{\partial}{\partial r}r^2 J_b\right]\left[\frac{q\beta c\mu_0 I_b(r)}{2\pi\gamma m}\right] \\ &= \frac{\delta R}{R}\frac{q\beta c\mu_0 I_b}{4\pi\gamma m}\frac{1}{I_b^2}8\pi^2\int_0^\infty \mathrm{d}r r^3 J_b^2 \\ &= \frac{\delta R}{R}U_b 2\psi,\end{aligned} \tag{B19}$$

where

$$\psi = \frac{4\pi^2}{I_b^2}\int_0^\infty \mathrm{d}r r^3 J_b^2. \tag{B20}$$

For a beam of flat profile it is seen that $\psi = 1$. If the profile is rounded and decreases with increasing r, then $\psi < 1$. This is shown by noting that an integration by parts in Eq. (B2) gives

$$I_b(r) \geq \pi r^2 J_b. \tag{B21}$$

Hence

$$\psi \leq \frac{4\pi}{I_b^2}\int_0^\infty \mathrm{d}r r J_b I_b(r) = \frac{2}{I_b^2}\int_0^\infty \mathrm{d}r \frac{\partial I_b}{\partial r} I_b = 1. \tag{B22}$$