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Kinetic theory of a relativistic beam

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A Fokker–Planck equation is derived to study the evolution of a stable low-current beam propagating in a gas-plasma medium. Small-angle scattering of the beam particles by the medium causes diffusion in the phase space projected transverse to the direction of propagation. The projected components of dynamical friction vanish. As a result, there is a continued input of energy into the transverse particle motions, which is taken up in expansion against the pinch field. A quasi-static Bennett equilibrium, with isothermal distribution of transverse momenta, is shown to be a similarity solution of the Fokker–Planck equation with scale radius increasing in accord with Nordsiéck's formula. An H theorem is proved and the Bennett distribution is shown to minimize both H and $-dH/dt$; hence, it is the time-dependent asymptotic state. The predicted current profile and radius are shown to be in fair agreement with experiment.

I. INTRODUCTION

The evolution of equilibrium properties of a beam is governed by its interaction with the background medium. This is a consideration of some importance because the growth rate of the hose instability, which limits pulse length and propagation distance, is sensitive to the details of the current profile.[1,2] We treat a low-current beam which is subject to small-angle scattering by a gas-plasma medium. Scattering causes the beam to expand and reshape, but the effects are strongly modified by the pinching field.

To study the coupled effects of pinch and scattering in detail, a Fokker–Planck equation is derived which treats beam coordinates and momenta in two-dimensional phase space projected transverse to the direction of propagation. The derivation of the Fokker–Planck equation and its elementary properties is carried out with maximum generality within the low-current, long-wave-length approximation. Detailed application is then made to the evolution of a beam-plasma system characterized by azimuthal symmetry, quasi-static equilibrium, zero angular momentum density, a homogeneous scattering medium, and background current and charge density described only by neutralization fractions.

Under these restrictions it is shown that the isothermal (Bennett) equilibrium is a slowly expanding, self-similar solution of the Fokker–Planck equation. Nordsiéck's[2,3] formula for the rate of increase of the scale radius appears as a consistency condition. An H theorem is proved and the Bennett distribution is shown to minimize H; the quantity $(-dH/dt)$ is simultaneously minimized. The Bennett distribution is thus singled out as the final (expanding) state independent of initial conditions.

Interestingly enough, even though dynamical friction vanishes in this model, the beam is driven to an isothermal distribution. A role similar to that of friction is played by the pinch field; the particles are cooled by doing work against the expanding field. However, the mean transverse kinetic energy is fixed as a condition of equilibrium (the Bennett pinch condition[4]). Thus, the beam expands at just the rate required to absorb the excess kinetic energy generated by scattering. It is also remarkable that a time-dependent analytic solution is found in view of the inhomogeneity of the current profile and the large-scale gyroradius of the particle orbits.

In Sec. II, the basic model of particle motion and self-consistent fields is outlined. It is much simplified by the low-current approximation. Sections III and IV are devoted to the derivation of the Fokker–Planck equation and its general properties. The energy and virial equations are derived and the H theorem is proved. In Sec. V the global properties of a slowly expanding beam are determined in a manner independent of the details of the distribution function. In addition, an envelope equation is derived to describe rapid expansion and pulsation. The Bennett distribution is examined in Secs. VI and VII. It is shown to minimize H and $(-dH/dt)$ subject to the restrictions outlined here. Section VIII contains a brief comparison of results with available experimental data.

II. PARTICLE AND FIELD EQUATIONS

The beam of charged particles (rest mass m, charge q) propagates along the $+z$ axis of a cylindrical coordinate system which is stationary in the laboratory frame. The relativistic equations for single-particle momentum and position are

$$\mathring{\mathbf{P}} = q(\mathbf{E} + \boldsymbol{\beta} \times \mathbf{B}) + \mathbf{S} , \tag{1}$$

$$\dot{\mathbf{R}} = \mathbf{V} = c\boldsymbol{\beta} = \mathbf{P}/\gamma m \tag{2}$$

with $\gamma \equiv (I - \beta^2)^{-1/2}$. Here, $\mathbf{E}$ and $\mathbf{B}$ are the collective fields of the beam-plasma system and $\mathbf{S}$ is the fluctuating force causing the small-angle scattering in the background medium. Occasional large-angle scattering, synchrotron radiation, and collisions among beam particles are neglected. The fields satisfy Maxwell's equations with sources $\mathbf{J}$ and ρ produced by both the beam and the plasma.

The particle and field equations are simplified in the low current or paraxial limit.[1,5] When particle orbits deviate only slightly from parallel straight lines, transverse momentum effectively decouples from the longitudinal motion. This is a valid assumption if total current (I) is small compared with the Alfvén limit ($I_A = \beta\gamma m c^3/q = 17\,000\beta\gamma A$).[6] A particle gyroradius is then large compared with the scale beam radius (a). Thus, we define the smallness parameter

$$\epsilon^2 = I/I_A \ll 1 , \tag{3}$$

$$\frac{V_\perp}{V_z} , \frac{a}{\beta c} \frac{d}{dt} = O(\epsilon) . \tag{4}$$

The Bennett condition for pinch equilibrium is, for example, mean $V_\perp^2 = \epsilon^2 V_z^2$.

For fields, the scale of change is also ordered with ϵ; the consistent scheme[1] is

$$\frac{J_\perp}{J_z} , \frac{B_z}{B_\perp} , \frac{E_z}{E_\perp} , \frac{a\partial}{\partial z} , \frac{a}{c}\frac{\partial}{\partial t} = O(\epsilon) , \tag{5}$$

$$B_\perp , E_\perp , S_\perp/q , a\rho , aJ_z/c = O(1) . \tag{6}$$

Transverse (x-y) components are denoted by the subscript $\perp$; henceforth, these quantities are denoted by the lower case

$$\begin{aligned} &\mathbf{P}_\perp \to \mathbf{p} , \quad \mathbf{R}_\perp \to \mathbf{r} , \quad \nabla_\perp \to \nabla , \\ &\mathbf{B}_\perp \to \mathbf{b} , \quad \mathbf{E}_\perp \to \mathbf{e} , \quad \mathbf{S}_\perp \to \mathbf{s} . \end{aligned} \tag{7}$$

From Eqs. (2) and (4)

$$V_z = \beta c + O(\epsilon^2) , \tag{8}$$

so the transverse equations of motion become, to order ϵ^2,

$$\dot{\mathbf{p}} = q(\mathbf{e} + \beta \hat{e}_z \times \mathbf{b}) + \mathbf{s} , \tag{9}$$

$$\dot{\mathbf{r}} = \mathbf{v} = \mathbf{p}/\gamma m . \tag{10}$$

Total energy γmc^2 has rapid variations of order ϵ^2 since the particle is moving in a weak electrostatic well; this neither changes γ in the time average nor effects Eq. (9). A slow secular change of γ does result from the mean drag of the medium and E_z. This effect is incorporated as the implicitly known change of the parameters $Z(t)$, $V_z(t)$, $\gamma(t)$.

In similar fashion the transverse fields decouple from the small components B_z, E_z. In lowest order

$$\nabla \times \mathbf{b} = \frac{4\pi}{c} J\hat{e}_z + O(\epsilon^2) , \tag{11}$$

$$\nabla \cdot \mathbf{b} = O(\epsilon^2) , \tag{12}$$

$$\nabla \times \mathbf{e} = O(\epsilon^2) , \tag{13}$$

$$\nabla \cdot \mathbf{e} = 4\pi\rho + O(\epsilon^2) , \tag{14}$$

where subscript z is dropped from J_z.

It is very convenient to introduce the scalar potential (ϕ) and z component of the vector potential (A):

$$\mathbf{b} = \nabla \times A\hat{e}_z = -\hat{e}_z \times \nabla A , \tag{15}$$

$$\mathbf{e} = -\nabla\phi . \tag{16}$$

Equations (12) and (13) are automatically satisfied, and Eqs. (11) and (14) yield

$$\nabla^2 A = -\frac{4\pi}{c} J , \quad \nabla^2\phi = -4\pi\rho . \tag{17}$$

In addition,

$$E_z = -\frac{1}{c}\left(\frac{\partial A}{\partial t}\right)_z - \frac{\partial\phi}{\partial z} . \tag{18}$$

B_z can be derived from the transverse components of the Ampere–Maxwell law if desired. To satisfy the continuity equation all components of $\mathbf{J}$ are required in lowest order. However, total current and charge per unit length (Q) are related by

$$\frac{\partial I}{\partial z} + \left(\frac{\partial Q}{\partial t}\right)_z = 0 . \tag{19}$$

This relation is satisfied separately by beam and plasma contributions. The beam particles are assumed to be characterized by a single value of β at given x and z, hence $I_b = \beta c Q_b$.

To avoid confusion over infinite potentials, the system is surrounded by a conducting shell at radius (R) which is very large compared with the scale beam radius. A and ϕ vanish at R, but consequent effects of order a^2/R^2 are neglected. The formal Green's function solution of Eqs. (17) is

$$\phi = -2 \int d^2r' \rho(\mathbf{r}') \log \left| \frac{\mathbf{r} - \mathbf{r}'}{R} \right| , \tag{20}$$

$$A = -\frac{2}{c} \int d^2r' J(\mathbf{r}') \log \left| \frac{\mathbf{r} - \mathbf{r}'}{R} \right| . \tag{21}$$

A charged particle beam injected into gas in the density range of primary interest here (where scattering is important) rapidly becomes charge neutralized. A reduction of the pinch magnetic field by induced plasma currents may occur, and their amplitude depends on the details of the conductivity and beam pulse rise time.

Let us assume the plasma currents and charges are related to those of the beam by neutralization fractions (independent of $\mathbf{r}$)

$$\begin{aligned} &J = J_b + J_p = (1 - f_m) J_b , \\ &\rho = \rho_b + \rho_p = (1 - f_c)\rho_b = (1 - f_c) J_b/\beta c . \end{aligned} \tag{22}$$

With this common simplification, often used in beam studies, we may incorporate many parameter regimes into the calculation. Equations (20) and (21) then give the relation

$$\phi = \frac{1}{\beta}\left(\frac{1 - f_c}{1 - f_m}\right) A . \tag{23}$$

Substituting the potentials into the particle force law (9) we have

$$\dot{\mathbf{p}} = q\nabla(-\phi + \beta A) + \mathbf{s} . \tag{24}$$

For the special case described by neutralization fractions (22)

$$\dot{\mathbf{p}} = q\beta\lambda\nabla A + \mathbf{s} , \tag{25}$$

where

$$\lambda = 1 - \frac{1}{\beta^2}\left(\frac{1 - f_c}{1 - f_m}\right) . \tag{26}$$

We treat a magnetically pinched beam here, i.e., $f_m < 1$. It is also necessary that λ be positive so that the attractive magnetic force is stronger than the repulsive electric force.

III. THE FOKKER–PLANCK EQUATION

We now examine a thin transverse segment of the beam containing a fixed set of particles. By assumption, every particle in the segment has the same relativistic mass γm and velocity $V_z = \beta c$, and there is no "overtaking" among segments. V_z, γ, and Z are regarded as known functions of time and initial conditions. These assumptions, although common in the study of beams, are idealizations which fail for propagation over large distances. For example, small differences in V_z exist within a segment as a result of the differing amplitudes of transverse motion, hence, the segment gradually elongates and mixes with its neighbors. In addition, a spread distribution by γ develops if E_z and background density have radial profiles. At very high energies ($\gamma \gtrsim 10^3$ in H_2) emission of bremsstrahlung dominates over ionization in determining energy loss. This process is of a pronounced statistical character and results in spread γ for propagation over distances of the order of a radiation length, so the theory is imprecise in this limit. In the treatment that follows $\gamma(t)$ is retained as a slowly changing parameter at almost no cost in effort. This allows the average effect of energy change to be determined.

To make a kinetic description of the beam segment we define $f(\mathbf{r}, \mathbf{p}, t)\, d^2r d^2p$ to be the fraction of its particles which lie in phase space volume $d^2r d^2p$ at time t. If many segments are studied, we distinguish them by their time of injection (ξ) at $Z=0$; that is, $f(\mathbf{r}, \mathbf{p}, t) \to f(\mathbf{r}, \mathbf{p}, t, \xi)$. The spatial weight is

$$\eta(\mathbf{r}, t, \xi) = \int d^2p f\,, \tag{27}$$

with normalization $1 = N = \int d^2r\eta$. Thus, the beam current density is $J_b = \eta I_b$. The transport of particles in phase space is governed by the continuity equation

$$\left(\frac{\partial f}{\partial t}\right)_\xi = -\nabla_r \cdot (\dot{\mathbf{r}} f) - \nabla_p \cdot (\dot{\mathbf{p}} f) + \left(\frac{\delta f}{\delta t}\right)'\,, \tag{28}$$

with $\dot{\mathbf{p}}$ determined by the collective fields and $(\delta f/\delta t)'$ representing transport due to scattering. Substituting from Eqs. (10) and (24) for $\dot{\mathbf{r}}$ and the collective part of $\dot{\mathbf{p}}$, Eq. (28) becomes

$$\left(\frac{\partial f}{\partial t}\right)_\xi + \frac{\mathbf{p}}{\gamma m} \cdot \nabla f + q(-\nabla\phi + \beta\nabla A) \cdot \nabla_p f = \left(\frac{\delta f}{\delta t}\right)'\,. \tag{29}$$

Equation (29), in the absence of scattering and electric potential, is essentially that employed in the first study of the resistive hose instability.[7]

The time scale τ_E measures the transfer of energy from longitudinal to transverse motion via scattering. In the present work we are interested in phenomena associated with the τ_E time scale rather than the dynamic time scale, although they may be comparable. The dynamic scale is associated with the particle oscillation (or betatron) frequency, which for a beam of flat current profile and radius (a) has the unique value

$$\omega_\beta = \left(\frac{2q\beta\lambda I}{\gamma m c a^2}\right)^{1/2}\,. \tag{30}$$

The time scale for significant transfer of energy to the transverse motion is

$$\tau_E \approx \left(\frac{2E'}{\gamma m \omega_\beta^2 a^2}\right)^{-1}\,, \tag{31}$$

where E' is the mean rate of energy transfer per particle. A quantitative definition of E' is based on the formalism developed in this section.

The primary interest is the interaction of scattering with the focusing forces. For our purpose the Fokker–Planck formalism[8] is an adequate approximation. In this formalism the scattering term in Eq. (29) is of the form

$$\left(\frac{\delta f}{\delta t}\right)' = -\frac{\partial}{\partial \mathbf{p}}\left(f\left\langle\frac{\Delta p}{\Delta t}\right\rangle\right) + \frac{1}{2}\frac{\partial^2}{\partial\mathbf{p}\partial\mathbf{p}}\left(f\left\langle\frac{\Delta p\Delta p}{\Delta t}\right\rangle\right)\,. \tag{32}$$

$\boldsymbol{\Delta} p$ is the cumulative change in transverse momentum suffered by a beam particle in a short time Δt due to many small-angle scattering events. The angular brackets denote the expectation value. The coefficients of friction and diffusion in Eq. (32) are readily evaluated using well-known results for the multiple scattering of a high energy particle.[9,10] First, scattering is isotropic in the plane normal to the velocity vector of the particle. Thus, the transverse frictional coefficient vanishes in the paraxial approximation

$$\left\langle\frac{\Delta p}{\Delta t}\right\rangle = 0\,. \tag{33}$$

Another consequence of this symmetry is that the momentum space diffusion tensor takes the form

$$\left\langle\frac{\Delta p\Delta p}{\Delta t}\right\rangle = \tfrac{1}{2}\mathbf{I}\left\langle\frac{|\Delta p|^2}{\Delta t}\right\rangle \tag{34}$$

with $\mathbf{I}$ the (two-dimensional) unit dyadic.

The squared momentum transfer is proportional to the square of the angle of defection, which is computed in the references

$$\left\langle\frac{|\Delta p|^2}{\Delta t}\right\rangle = P_z^2\left\langle\frac{(\Delta\theta)^2}{\Delta t}\right\rangle \simeq \frac{8\pi N_z Z(Z+1)q^2e^2}{\beta c}\log\left(\frac{\delta\theta_1}{\delta\theta_2}\right)\,. \tag{35}$$

N_z is the number density of nuclei with atomic number Z and $\delta\theta_1/\delta\theta_2$ is the maximum-to-minimum ratio of scattering angles included in the mean. The formula for this ratio is dependent on the range of γ, N_z, nuclear charge and distance propagated, it is not specified here. For our purpose, it is most convenient to define the rate of energy transfer

$$E' = \frac{1}{2m\gamma}\left\langle\frac{|\Delta p|^2}{\Delta t}\right\rangle \cong \frac{4\pi N_z Z(Z+1)q^2e^2}{m\gamma\beta c}\log\left(\frac{\delta\theta_1}{\delta\theta_2}\right)\,, \tag{36}$$

which may depend on $\mathbf{r}$ through N_z but is independent of $\mathbf{p}$. The diffusion tensor is written

$$\left\langle\frac{\Delta p\Delta p}{\Delta t}\right\rangle = \mathbf{I}\gamma m E'\,. \tag{37}$$

Substituting expressions (32), (33), and (37) in Eq. (29) yields the principal result

$$\left(\frac{\partial f}{\partial t}\right)_\xi + \frac{\mathbf{p}}{\gamma m}\cdot\nabla f + q(-\nabla\phi + \beta\nabla A)\cdot\nabla_p f = \frac{\gamma m E'}{2}\nabla_p^2 f\,. \tag{38}$$

It is elementary to show that the scattering operator conserves local particle number, position, momentum and angular momentum,

$$0=\int d^2p\left(\frac{\delta f}{\delta t}\right)'\begin{pmatrix}1\\ \mathbf{r}\\ \mathbf{p}\\ \mathbf{r}\times\mathbf{p}\end{pmatrix}. \tag{39}$$

As expected, kinetic energy is not conserved, rather

$$\int d^2p\left(\frac{\delta f}{\delta t}\right)'\frac{p^2}{2m\gamma}=\eta E'. \tag{40}$$

The absence of friction in the transverse variables may be viewed as a consequence of the extreme non-equilibrium state represented by a beam. Conversely, change in P_z is dominated by the frictional effect. Energy is transferred from longitudinal motion to the transverse variables, driving the beam toward a more isotropic state; friction plays an important role only as that final state is closely approached. However, the initial stages of this process are examined here, $V_\perp/V_z$ is always assumed small. If beam particles are not pinched, isotropization is rapid. On the other hand, if the pinch forces are strong enough for equilibrium to hold ($\omega_\beta>\tau_E^{-1}$), then transverse kinetic energy is fixed by the Bennett condition, and the anisotropic (beam-like) character is maintained for long times.

IV. GLOBAL PROPERTIES

H theorem

The H function, defined for the segment as a whole, is

$$H(\xi,t)=\int d^2r\,d^2p\,f\log f. \tag{41}$$

The rate of change of H is then calculated from the Fokker–Planck equation (38) to be

$$\begin{aligned}\frac{dH}{dt}&=\int d^2r\,d^2p\left(\frac{\partial}{\partial f}f\log f\right)\frac{\partial f}{\partial t}\\ &=\int d^2r\,d^2p\left(\frac{\partial}{\partial f}f\log f\right)\\ &\quad\times\left(-\frac{p}{\gamma m}\circ\nabla f-q(-\nabla\phi+\beta\nabla A)\cdot\nabla_p f+\frac{\gamma mE'}{2}\nabla_p^2 f\right).\end{aligned} \tag{42}$$

Convective terms in Eq. (42) may be converted to surface integrals that vanish and the diffusion term yields, after an integration by parts,

$$\frac{dH}{dt}=-\int d^2r\,d^2p\,\frac{\gamma mE'}{2}\frac{|\nabla_p f|^2}{f}<0. \tag{43}$$

H decreases in time.

Moment equations

We will use equations for the flow of mass, momentum, and energy densities in real space. Defining averages in momentum space

$$\eta\begin{bmatrix}\overline{\mathbf{p}}\\ \overline{\mathbf{pp}}\\ \overline{p^2\mathbf{p}}\end{bmatrix}=\int d^2pf\begin{bmatrix}\mathbf{p}\\ \mathbf{pp}\\ p^2\mathbf{p}\end{bmatrix}, \tag{44}$$

the lowest moments of the Fokker–Planck equation yield

$$\frac{\partial\eta}{\partial t}+\nabla\circ\left(\eta\frac{\overline{\mathbf{p}}}{\gamma m}\right)=0, \tag{45}$$

$$\frac{\partial}{\partial t}(\eta\overline{\mathbf{p}})+\nabla\circ\left(\eta\frac{\overline{\mathbf{pp}}}{\gamma m}\right)-q\eta(-\nabla\phi+\beta\nabla A)=0, \tag{46}$$

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{\eta\overline{p^2}}{2m\gamma}\right)+\frac{\eta\overline{p^2}}{2m\gamma}\frac{1}{\gamma}\frac{d\gamma}{dt}+\nabla\circ\left(\eta\frac{\overline{p^2\mathbf{p}}}{2\gamma^2m^2}\right)&\\ -q\eta\frac{\overline{\mathbf{p}}}{\gamma m}\cdot(-\nabla\phi+\beta\nabla A)=\eta E'.&\end{aligned} \tag{47}$$

Conservation laws

Except for conservation of particles, which is built into the theory, no conservation law can be demonstrated without a treatment of the gas-plasma medium. The medium can absorb momentum, energy, etc., so it is necessary to introduce further equations for its dynamics. Here, we adopt the neutralization fractions (22). Henceforth,

$$q(-\nabla\phi+\beta\nabla A)=q\beta\lambda\nabla A \tag{48}$$

with λ given by Eq. (26). Equation (21) becomes

$$A(r,t,\xi)=-\frac{2I}{c}\int d^2r'\eta(\mathbf{r}',t,\xi)\log\left|\frac{\mathbf{r}-\mathbf{r}'}{R}\right|. \tag{49}$$

I is a known function of t. The transverse momentum of the beam segment is now found to be conserved:

$$\begin{aligned}\frac{d}{dt}\int d^2r\eta\,\overline{\mathbf{p}}=\int d^2r\frac{\partial}{\partial t}(\eta\overline{\mathbf{p}})=-\int d^2r\nabla\cdot\frac{\eta\overline{\mathbf{pp}}}{\gamma m}&\\ +\int d^2r\eta q\beta\lambda\nabla A.&\end{aligned} \tag{50}$$

The first term on the right is converted to a surface integral which vanishes. The second term on the right also vanishes since we may substitute from expression (49) for $A(\mathbf{r})$ to get

$$\begin{aligned}\frac{d}{dt}\int d^2r\eta\mathbf{p}&=\int d^2r\eta q\beta\lambda\left(-\frac{2I}{c}\right)\nabla\int d^2r'\eta'\log\left|\frac{\mathbf{r}-\mathbf{r}'}{R}\right|\\ &=-\frac{2q\beta\lambda I}{c}\int d^2r\,d^2r'\eta\eta'\,\frac{(\mathbf{r}-\mathbf{r}')}{|r-r'|^2}=0.\end{aligned} \tag{51}$$

In similar fashion the net angular momentum of the beam segment may be shown to be conserved, i.e.,

$$\frac{dL}{dt}=\frac{d}{dt}\int d^2r\eta(\mathbf{r}\times\overline{\mathbf{p}})\cdot\hat{e}_z=0. \tag{52}$$

The motion of the center of mass is related to the mean momentum by

$$\frac{d}{dt}\int d^2r\,\mathbf{r}\eta=\int d^2r\,\mathbf{r}\frac{\partial\eta}{\partial t}=-\int d^2r\,\mathbf{r}\nabla\circ\frac{\eta\overline{\mathbf{p}}}{\gamma m}=\int d^2r\frac{\eta\overline{\mathbf{p}}}{\gamma m}. \tag{53}$$

Hence we assume, without further loss of generality, that mean momentum and the displacement of the center of mass from the coordinate axis both vanish. The conservation of *beam* momentum and angular momentum are not fundamental properties at all but only a peculiarity of the neutralization fraction model. What has really been shown is that the assumption of these conservation laws, which is convenient for our analysis, is consistent with the neutralization fraction model.

Virial equation

A very powerful tool for macroscopic treatment of the system is provided by the virial equation. Forming the scalar product of Eq. (46) with $\mathbf{r}$ and integrating over the segment yields

$$0=\int d^2r\,\mathbf{r}\cdot\left(\frac{\partial}{\partial t}(\eta\overline{\mathbf{p}})+\nabla\cdot\frac{\eta\overline{\mathbf{pp}}}{\gamma m}-\eta q\beta\lambda\nabla A\right)$$
$$=\frac{d}{dt}\int d^2r\,\eta\mathbf{r}\cdot\overline{\mathbf{p}}-\int d^2r\,\frac{\eta\overline{p^2}}{\gamma m}-q\beta\lambda\int d^2r\,\eta\mathbf{r}\cdot\nabla A\ . \tag{54}$$

The first integral on the right is, by Eq. (45),

$$\int d^2r\eta\mathbf{r}\cdot\overline{\mathbf{p}}=\frac{\gamma m}{2}\int d^2r(\nabla r^2)\cdot\mathbf{I}\cdot\frac{\eta\overline{\mathbf{p}}}{\gamma m}=-\frac{\gamma m}{2}\int d^2r r^2\nabla\cdot\frac{\eta\overline{\mathbf{p}}}{\gamma m}$$
$$=\frac{\gamma m}{2}\int d^2r r^2\frac{\partial\eta}{\partial t}=\frac{\gamma m}{2}\frac{d}{dt}\int d^2r\,\eta r^2\ . \tag{55}$$

The second term is just twice the mean kinetic energy

$$T\equiv\int d^2r\eta\,\frac{\overline{p^2}}{2m\gamma}\ . \tag{56}$$

For the third term we use Eq. (49) to calculate

$$-q\beta\lambda\int d^2r\eta\mathbf{r}\cdot\nabla A=q\beta\lambda\frac{2I}{c}\int d^2r\eta\mathbf{r}\cdot\int d^2\mathbf{r}'\eta'\,\frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^2}$$
$$=\frac{2q\beta\lambda I}{c}\int d^2r\,d^2r'\,\frac{\eta\eta'}{2}\,\frac{(\mathbf{r}-\mathbf{r}')\cdot(\mathbf{r}-\mathbf{r}')}{|r-r'|^2}$$
$$=+\frac{q\beta\lambda I}{c}\int d^2r\,d^2r'\eta\eta'=\frac{q\beta\lambda I}{c}\ . \tag{57}$$

Inserting the expressions (55), (56), and (57) in Eq. (54) finally gives

$$0=\frac{d}{dt}\left(\frac{\gamma m}{2}\,\frac{d}{dt}\int d^2r\eta r^2\right)-2T+\frac{q\beta\lambda I}{c}\ . \tag{58}$$

Equation (58), which is the scalar virial equation, immediately yields a simple necessary condition for dynamical equilibrium:

$$T=\frac{q\beta\lambda I}{2c}\equiv T^*\ . \tag{59}$$

The results (58) and (59) are essentially similar to those of Bennett[4]; they are more general here in the consideration of scattering and the time dependence of parameters. Condition (59) also holds when the mean square radius is only changing slowly because the second time derivative is involved. When an imbalance between T and I is imposed at $Z=0$, the mean square radius pulsates. However, if the pulsations are bounded, the property (59) still holds in the time average.

Energy equation

Transverse energy is not conserved, but an equation can be derived for its rate of change. The total energy (W) is the sum of kinetic and potential contributions

$$W=T+U=\int d^2r\eta\,\frac{p^2}{2m\gamma}-\frac{1}{2}\int d^2r\eta\,q\beta\lambda A\ . \tag{60}$$

Then, differentiating in time, we have

$$\frac{dW}{dt}=-\frac{T}{\gamma}\,\frac{d\gamma}{dt}+\frac{U}{(\beta\lambda I)}\,\frac{d}{dt}(\beta\lambda I)$$
$$+\int d^2r\left(\frac{1}{2m\gamma}\,\frac{\partial}{\partial t}(\eta\overline{p^2})-q\beta\lambda I\,\frac{\partial}{\partial t}\,\frac{\eta A}{2I}\right)\ . \tag{61}$$

The energy flow equation (47) is used to substitute for the first term of the integral in (61), giving

$$\frac{dW}{dt}=-\frac{T}{\gamma}\,\frac{d\gamma}{dt}+\frac{U}{(\beta\lambda I)}\,\frac{d}{dt}(\beta\lambda I)+\int d^2r\eta E'+q\beta\lambda I$$
$$\times\int d^2r\left(\frac{\eta\overline{p}}{\gamma m}\cdot\nabla\frac{A}{I}-\frac{\partial}{\partial t}\,\frac{\eta A}{2I}\right)\ . \tag{62}$$

Equations (45) and (49) are now used to show the second integral of Eq. (62) vanishes:

$$\int d^2r\left(\frac{\eta\overline{p}}{\gamma m}\cdot\nabla\frac{A}{I}-\frac{\partial}{\partial t}\,\frac{\eta A}{2I}\right)=\int d^2r\left(\frac{A}{I}\,\frac{\partial\eta}{\partial t}-\frac{1}{2}\,\frac{\partial}{\partial t}\,\frac{\eta A}{I}\right)$$
$$=\frac{1}{2}\int d^2r\left(\frac{A}{I}\,\frac{\partial\eta}{\partial t}-\eta\,\frac{\partial}{\partial t}\,\frac{A}{I}\right)$$
$$=-\frac{1}{c}\int d^2r\int d^2r'\left(\eta'\,\frac{\partial\eta}{\partial t}-\eta\,\frac{\partial\eta'}{\partial t}\right)\log\left|\frac{\mathbf{r}-\mathbf{r}'}{R}\right|=0\ . \tag{63}$$

Hence, the energy equation is

$$\frac{dW}{dt}\equiv\frac{dT}{dt}+\frac{dU}{dt}=-\frac{T}{\gamma}\,\frac{d\gamma}{dt}+\frac{U}{(\beta\lambda I)}\,\frac{d}{dt}(\beta\lambda I)+\int d^2r\eta E'\ . \tag{64}$$

V. BEAM EXPANSION

If γ and the quantity $(\beta\gamma\lambda)$ are fixed and there is no scattering, then transverse energy is conserved. In addition if the beam is in dynamic equilibrium, the kinetic energy is known from the virial theorem to be $T=T^*=q\beta\lambda I/2c$. These conditions are now relaxed to study the macroscopic properties of a beam evolving slowly enough in time that it stays close to equilibrium. It was shown above that $T=T^*$ holds for slow change; this allows the integration of the energy equation to determine U. Dividing Eq. (64) through by T^*, we have

$$\frac{I}{T^*}\,\frac{dT^*}{dt}+\frac{1}{T^*}\,\frac{dU}{dt}=-\frac{1}{\gamma}\,\frac{d\gamma}{dt}+\frac{U}{(T^*)^2}\,\frac{dT^*}{dt}+\frac{1}{T^*}\int d^2r\eta E'\ , \tag{65}$$

or

$$\frac{d}{dt}\left(\log(\gamma T^*)+\frac{U}{T^*}\right)=\frac{1}{T^*}\int d^2r\eta\,E'\ . \tag{66}$$

The quantity

$$\Gamma=\log(\gamma T^*)+U/T^* \tag{67}$$

is an adiabatic invariant that is conserved in the absence of scattering. Γ may be considered the entropy associated with macroscopic configuration ($\Gamma\neq-H$, in general). For example, T^* is the temperature and $\int d^2r\eta E'$ is the rate of heat input ($\delta Q/dt$); then, Eq. (66) takes the well-known form

$$T^*\,d\Gamma=\delta Q\ . \tag{68}$$

The Nordsiéck equation for the rate of beam expansion may be derived from Eq. (66) by assuming a specific form for the current profile. We let

$$J(r)=\begin{cases}\dfrac{I}{\pi a^2}, & 0<r<a\ ,\\ 0, & a<r<R\ ,\end{cases} \tag{69}$$

with consistent potential

$$A=\begin{cases}\dfrac{I}{a^2c}(a^2-r^2)-\dfrac{2I}{c}\log\left(\dfrac{a}{R}\right), & 0<r<a\ ,\\ -\dfrac{2I}{c}\log\left(\dfrac{r}{R}\right), & a<r<R\ .\end{cases} \tag{70}$$

The potential energy is calculated to be

$$U=-\frac{1}{2}\int d^2r\eta\, q\beta\lambda A=-\frac{q\beta\lambda I}{c}\left[\frac{1}{4}-\log\left(\frac{a}{R}\right)\right]$$
$$=T^*\left[\log\left(\frac{a^2}{R^2}\right)-\frac{1}{2}\right]. \tag{71}$$

From Eq. (67)

$$\Gamma=\log(\gamma T^*)+\left[\log\left(\frac{a^2}{R^2}\right)-\frac{1}{2}\right]=\log\left(\frac{\gamma T^* a^2}{R^2}\right)-\tfrac{1}{2}\ ; \tag{72}$$

so Eq. (66) becomes

$$\frac{d}{dt}\log(\gamma T^* a^2)=\frac{1}{T^*}\int d^2r\eta E'\ . \tag{73}$$

This is essentially Nordsiéck's equation[2,3] for beam expansion. The expansion rate derived by Nordsiéck was lower than that given in this paper by a factor of two in the exponential. This is because he considered the motion of a particle at the edge of a beam with a flat current profile rather than taking a mean over all particles.

The radius is seen to increase exponentially in time, but at a rate that varies inversely with I. This is an appreciable improvement over the free scattering result (for which $a^2\propto t^3$), but only when $\omega_\beta>\tau_E^{-1}$. In fact, Eq. (73) clearly gives a gross overestimate of expansion as $I\to 0$; this is a consequence of the assumption of quasi-static equilibrium, which fails if the pinch field is weak. Expansion to large enough radius also weakens the pinch field, so that the beam ultimately is subject to free scattering.

Equation (73), although derived with a flat current profile, may be shown to be correct for any profile if it expands in self-similar fashion with the single scale radius $a(t)$. However, scattering also causes reshaping of the profile, so, in general, the Nordsiéck equation is only a useful approximation. In the next section we show that it is exact for the special case of a Bennett profile $J(r)\propto(a^2+r^2)^{-2}$.

A single equation for the beam radius which covers both the free scattering and quasi-static limits is derived by returning to the general forms of the virial and energy equations. Note that for the flat profile (69) the mean square radius is $a^2/2$. Thus, the virial equation (58) may be written

$$\frac{d}{dt}\left[\frac{\gamma m}{2}\frac{d}{dt}\left(\frac{a^2}{2}\right)\right]=2(T-T^*)\ . \tag{74}$$

With the aid of expression (71), the energy equation (64) takes the form

$$\frac{d}{dt}(\gamma T)=-\gamma T^*\frac{d}{dt}\log(a^2)+\gamma\int d^2r\eta E'\ . \tag{75}$$

Eliminating T between Eqs. (74) and (75), we find the third order equation for a^2:

$$\frac{d}{dt}\left\{\gamma\frac{d}{dt}\left[\frac{\gamma m}{2}\frac{d}{dt}\left(\frac{a^2}{2}\right)\right]\right\}+2\gamma T^*\frac{d}{dt}\log(a^2)+2\frac{d}{dt}(\gamma T^*)$$
$$=2\gamma\int d^2r\eta E'\ . \tag{76}$$

The Nordsiéck equation is recovered from Eq. (76) by dropping the first term on the left. In the opposite limit, to describe free scattering we set $T^*=0$ to get

$$\frac{d}{dt}\left\{\gamma\frac{d}{dt}\left[\frac{\gamma m}{2}\frac{d}{dt}\left(\frac{a^2}{2}\right)\right]\right\}=2\gamma\int d^2r\eta E'\ . \tag{77}$$

If the medium is homogeneous and γ is assumed not to change, then at large time

$$a^2\approx\frac{4}{3}\frac{E'}{\gamma m}t^3\ . \tag{78}$$

Equation (76) may also be used to study pulsation of the beam when it is injected in a nonequilibrium state. However, the possibility of relaxation via phase mixing of orbits of differing betatron frequency has been lost in the assumption of a flat current profile.

To gain contact with the standard form of beam envelope equations[11] we note that Eq. (76) may be written

$$\frac{d}{dt}\frac{m}{2}\left(\gamma^2a^3\frac{d^2a}{dt^2}+\gamma\frac{d\gamma}{dt}a^3\frac{da}{dt}\right)=-2\frac{d}{dt}(\gamma T^*a^2)+2\gamma a^2$$
$$\times\int d^2r\eta E'\ . \tag{79}$$

Integrating once in time and dividing by $m\gamma^2a^3/2$ gives

$$\frac{d^2a}{dt^2}+\frac{1}{\gamma}\frac{d\gamma}{dt}\frac{da}{dt}+\frac{(\omega_\beta a)^2}{a}-\frac{2C^2}{\gamma^2a^3}$$
$$=\frac{1}{\gamma^2a^3}\int_{\xi}^{t}dt'\,\frac{4\gamma a^2}{m}\int d^2r\eta E'\ , \tag{80}$$

where we have substituted the squared betatron frequency

$$\frac{4T^*}{m\gamma a^2}=\frac{2q\beta\lambda I}{\gamma mca^2}=\omega_\beta^2\ . \tag{81}$$

The constant of integration is $2C^2$, which is essentially related to the "quality" at injection. For example, it is readily shown that

$$C^2=\left\{\gamma^2a^2\left[\frac{2T}{m\gamma}-\frac{1}{2}\left(\frac{da}{dt}\right)^2\right]\right\}_{t=\xi}>0\ . \tag{82}$$

Equation (80) is essentially similar to that of Ref. 11 with the addition of terms to account for scattering and energy loss.

C^2 is generally determined by conditions in the accelerator producing the beam and is regarded as known here. The equilibrium radius consistent with C^2 is produced by focusing the beam to a neck at injection ($da/dt=0$) with

$$a^2=\frac{m}{2\gamma T^*}C^2\ . \tag{83}$$

Note that $\Gamma = \log(mC^2/2R^2) - \frac{1}{2}$ for an equilibrium flat profile.

VI. THE BENNETT DISTRIBUTION

The condition $T = T^*$ is only a necessary condition of dynamic equilibrium; if it is satisfied, the rms radius does not oscillate. In general, the distribution function for axisymmetric equilibrium must be a physically reasonable function of the particle integrals of motion, which are angular momentum (l) and energy (w). The potential A must be consistent with η. Hence,

$$f(\mathbf{r}, \mathbf{p}) = F(l, w) , \tag{84}$$

with

$$l = r p_\theta , \tag{85}$$

$$w = \frac{p^2}{2m\gamma} - q\beta\lambda A . \tag{86}$$

The Bennett distribution is the special case of isothermal momenta without rotation,

$$F_B = K e^{-w/T^*} , \tag{87}$$

with normalization constant K. The density and field associated with F are[12]

$$\eta = \frac{1}{\pi a^2} \frac{1}{(1 + r^2/a^2)^2} , \tag{88}$$

$$A = -\frac{I}{c} \log\left(\frac{1 + r^2/a^2}{1 + R^2/a^2}\right) , \tag{89}$$

with cutoff (R) used only when needed. By direct calculation

$$K^{-1} = (2\pi^2 m\gamma T^* R^4/a^2)\left(1 + \frac{a^2}{R^2}\right)^2 , \tag{90}$$

$$\begin{aligned} U &= T^*\left[1 + \log\left(\frac{a^2}{R^2}\right)\right] , \\ W &= T^*[2 + \log(a^2/R^2)] . \end{aligned} \tag{91}$$

The rms radius is infinite.

F_B is shown to minimize H subject to constraints of fixed probability mass (N), angular momentum (L), and energy (W). The variation includes distributions which are far from equilibrium, so potential and kinetic energy are only known in a time average sense. This is sufficient to fix the value of W by integrating Eq. (64) in time, assuming E' is independent of r. N and L equal their values at injection since they are conserved quantities. Using the method of Lagrange multipliers, we set

$$0 = \delta(H + \lambda_1 N + \lambda_2 L + \lambda_3 W) \tag{92}$$

$$\begin{aligned} &= \delta \int d^2r\, d^2p \left[f \log f + \lambda_1 f + \lambda_2 f r p_\theta + \lambda_3 f\left(\frac{p^2}{2m\gamma} - \frac{q\beta\lambda A}{2}\right)\right] \\ &= \int d^2r\, d^2p (\delta f)\left[(\log f + 1) + \lambda_1 + \lambda_2 r p_\theta + \lambda_3 \right. \\ &\qquad \left. \times \left(\frac{p^2}{2m\gamma} - q\beta\lambda A\right)\right] . \end{aligned} \tag{93}$$

The solution is

$$f = \exp[-(1 + \lambda_1) - \lambda_2 l - \lambda_3 w] . \tag{94}$$

Since net angular momentum vanishes, $\lambda_2 = 0$. Comparison with the Bennett distribution then yields

$$K^{-1} = \exp(1 + \lambda_1) , \quad T^* = \lambda_3^{-1} . \tag{95}$$

F gives the unique stable minimum value of H. To prove this, it is necessary to show that, in addition to taking form (94), minimal f must satisfy the condition $T = T^*$. Suppose $T \neq T^*$. Then, by the virial equation (58) macroscopic radial motion must follow. H is conserved in this motion but f cannot retain form (94), which is a function of v_r^2. This is a contradiction because these new distributions also minimize H.

The case of $L \neq 0$ and f given by (94) cannot be normalized and thus is never realized physically.

We now turn to the effects of scattering and parameter change occurring on a slow time scale. It is found that F_B is a similarity solution of the Fokker–Planck equation with $a(t)$ governed by the Nordsiéck equation. Specifically,

$$f_B = \left(1 + \frac{\mathbf{r}\cdot\mathbf{p}}{T^*}\frac{1}{a}\frac{da}{dt}\right) F_B \tag{96}$$

satisfies the Fokker–Planck equation to first order in E' with all parameters considered as slowly changing functions of time. To zeroth order in E', f_B is a solution because it is then a function of $\mathbf{r}$ and $\mathbf{p}$ only through dependence on w. We must show that in the first order (see Eq. [38])

$$\frac{\partial F_B}{\partial t} + F_B\left(\frac{\mathbf{p}\cdot\nabla}{\gamma m} + q\beta\lambda\nabla A\cdot\nabla_p\right)\left(\frac{\mathbf{r}\cdot\mathbf{p}}{T^*}\frac{1}{a}\frac{da}{dt}\right) = \frac{\gamma m E'}{2}\nabla_p^2 F_B . \tag{97}$$

Substituting from Eq. (87) for F_B gives (the requirement)

$$\begin{aligned} &\frac{1}{K}\frac{dK}{dt} - p^2\frac{d}{dt}(2m\gamma T^*)^{-1} + \frac{\partial}{\partial t}\left(\frac{q\beta\lambda A}{T^*}\right) + \frac{p^2}{\gamma m T^*}\frac{1}{a}\frac{da}{dt} \\ &\quad + q\beta\lambda\nabla A\cdot\frac{\mathbf{r}}{T^*}\frac{1}{a}\frac{da}{dt} = \frac{\gamma m E'}{2}\left(\frac{p^2}{\gamma^2 m^2 T^{*2}} - \frac{2}{\gamma m T^*}\right) . \end{aligned} \tag{98}$$

Note that

$$\begin{aligned} &\frac{\partial}{\partial t}\left(\frac{q\beta\lambda A}{T^*}\right) + q\beta\lambda\nabla A\cdot\left(\frac{\mathbf{r}}{T^*}\frac{1}{a}\frac{da}{dt}\right) \\ &\quad = \left(\frac{\partial}{\partial t} + \frac{1}{a}\frac{da}{dt}\mathbf{r}\cdot\nabla\right)\left[-2\log\left(\frac{1 + r^2/a^2}{1 + R^2/a^2}\right)\right] \\ &\quad = 2\frac{d}{dt}\log\left(1 + \frac{R^2}{a^2}\right) . \end{aligned} \tag{99}$$

Thus, Eq. (98) reduces to

$$\begin{aligned} &\left[\frac{1}{K}\frac{dK}{dt} + 2\frac{d}{dt}\log\left(1 + \frac{R^2}{a^2}\right) + \frac{E'}{T^*}\right] \\ &\quad - p^2\left[\frac{d}{dt}(2m\gamma T^*)^{-1} - \frac{1}{\gamma m T^*}\frac{1}{a}\frac{da}{dt} + \frac{E'}{2\gamma m T^{*2}}\right] = 0 . \end{aligned} \tag{100}$$

The bracketed quantities of Eq. (100) must vanish separately; they each yield the Nordsiéck equation , e.g., the first bracketed quantity yields

$$\frac{E'}{T^*}=-\frac{1}{K}\frac{dK}{dt}-2\frac{d}{dT}\log\left(1+\frac{R^2}{a^2}\right)=\frac{d}{dt}\log\left(\frac{K^{-1}}{(1+R^2/a^2)^2}\right)$$
$$=\frac{d}{dt}\log(\gamma T^* a^2)\ , \tag{101}$$

which is identical to Eq. (73) when E' is independent of $\mathbf{r}$. For the Bennett equilibrium the Nordsiéck equation is exact.

The quantities Γ and $-H$ are found to differ only by a constant:

$$-H_B=-\int d^2r\,d^2p F_B\log(F_B)=-\int d^2r\,d^2pF_B\left(\log(K)-\frac{w}{T^*}\right)$$
$$=-\log(K)+\frac{T^*+2U}{T^*}=\log(\gamma T^*a^2)+3+\log(2\pi^2 m)\ , \tag{102}$$

which may be compared with Eq. (72). The H theorem thus provides an alternative derivation of the Nordsiéck equation for F_B by Eqs. (43) and (102),

$$\frac{d}{dt}\log(\gamma T^*a^2)=\frac{-dH_B}{dt}=\int d^2r\,d^2p\,\frac{\gamma mE'}{2}\frac{|\nabla_p F_B|^2}{F_B}$$
$$=\int d^2r\,d^2p\frac{\gamma mE'}{2}\frac{p^2}{(\gamma m T^*)^2}F_B$$
$$=\frac{E'}{T^{*2}}\int d^2r\,d^2pF_B\frac{p^2}{2m\gamma}=\frac{E'}{T^*}. \tag{103}$$

H_B is seen to decrease linearly with time.

VII. MINIMIZATION OF $-dH/dt$

It has been shown that H decreases in time for any distribution and at fixed time it has a unique minimum for the Bennett distribution, which is thus the natural candidate for an asymptotic state. To prove this, it is sufficient to show that H actually decreases more rapidly than H_B for any other comparable state. The proof requires some further restrictions. Since many assumptions and approximations have already been introduced, it is useful to list them all here in addition to the new ones.

(a) The basic approximations of the theory apply: paraxial approximation for particles and fields, small-angle scattering, integrity of segment is preserved, γ is unique within the segment.

(b) The additional assumptions of convenience made in Secs. III–V hold: neutralization fractions, slowly changing parameters, E' is independent of $\mathbf{r}$ and small in the sense $\tau_E \gg \omega_\beta^{-1}$.

(c) The system is axisymmetric.

(d) The system is close to dynamical equilibrium, that is, f is a function of the particle integrals up to terms of order E' as in Eq. (96). However, it is *not* assumed that f is close to "thermal equilibrium".

(e) To the zeroth order in E' f is a function of w and $l^2\equiv h$; there may be anisotropy but no azimuthal flow. This is a new and severe constraint in the analysis.

(f) As in the minimization of H, the quantities, N, L, and W are known at any given time and are thus held fixed in the variation.

(g) Mean kinetic energy $T=T^*$ is also fixed because the system is close to dynamic equilibrium.

To find the function(s) that minimize $(-dH/dt)$ we again use the method of Lagrange multipliers:

$$0=\delta\left(\frac{dH}{dt}+\alpha_1N+\alpha_2W+\alpha_3T\right)=\delta\int d^2r\,d^2p$$
$$\times\left[\frac{-\gamma mE'}{2}\frac{|\nabla_pf|^2}{f}+\alpha_1f+\alpha_2f\left(\frac{p^2}{2m\gamma}-\frac{q\beta\lambda A}{2}\right)+\alpha_3f\frac{p^2}{2m\gamma}\right]$$
$$=\int d^2r\,d^2p(\delta f)\left[+\frac{\gamma mE'}{2}\left(\frac{|\nabla_pf|^2}{f^2}+2\nabla_p\cdot\frac{\nabla_pf}{f}\right)\right.$$
$$\left.+\alpha_1+\alpha_2\left(\frac{p^2}{2m\gamma}-q\beta\lambda A\right)+\alpha_3\frac{p^2}{2m\gamma}\right]. \tag{104}$$

Minimal f must satisfy

$$0=+\frac{\gamma mE'}{2}\left(\frac{|\nabla_pf|^2}{f^2}+2\nabla_p\cdot\frac{\nabla_pf}{f}\right)+\alpha_1+\alpha_2\left(\frac{p^2}{2m\gamma}-q\beta\lambda A\right)$$
$$+\alpha_3\frac{p^2}{2m\gamma}. \tag{105}$$

The most general solution of Eq. (105) has not yet been found; here, we consider only the distributions of the form

$$f=F(w,h)\ ,\ \ h=l^2\ . \tag{106}$$

Inserting this form into Eq. (105) yields

$$0=\frac{\gamma mE'}{2}\left\{-\frac{1}{F^2}\left[\left(\frac{\partial F}{\partial w}\right)^2\frac{p^2}{\gamma^2m^2}+2\frac{\partial F}{\partial w}\frac{\partial F}{\partial h}\frac{2h}{\gamma m}+\left(\frac{\partial F}{\partial h}\right)^2 4r^2h\right]\right.$$
$$+\frac{2}{F}\left[\frac{\partial^2F}{\partial w^2}\frac{p^2}{\gamma^2m^2}+\frac{\partial^2F}{\partial h\partial w}\frac{2h}{\gamma m}+\frac{\partial F}{\partial w}\frac{2}{\gamma m}+\frac{\partial^2F}{\partial w\partial h}\frac{2h}{\gamma m}\right.$$
$$\left.\left.+\frac{\partial^2F}{\partial h^2}4r^2h+\frac{\partial F}{\partial h}2r^2\right]\right\}+\alpha_1+\alpha_2w+\alpha_3\frac{p^2}{2m\gamma}. \tag{107}$$

Equation (107) contains terms of three types; those that are integrals of the motion (functions of w and h), terms that are of the form r^2 times an integral, and terms of the form p^2 times an integral. Since r^2 and p^2 cannot be combined linearly to form an integral, their coefficients must vanish separately and we have

$$0=-\frac{1}{F^2}\left(\frac{\partial F}{\partial w}\right)^2+\frac{2}{F}\frac{\partial^2F}{\partial w^2}+\frac{\alpha_3}{E'}\ , \tag{108}$$

$$0=-\frac{1}{F^2}\left(\frac{\partial F}{\partial h}\right)^2h+\frac{2}{F}\frac{\partial^2F}{\partial h^2}h+\frac{1}{F}\frac{\partial F}{\partial h}\ , \tag{109}$$

$$0=-\frac{1}{F^2}\frac{\partial F}{\partial w}\frac{\partial F}{\partial h}h+\frac{2}{F}\frac{\partial^2F}{\partial h\partial w}h+\frac{1}{F}\frac{\partial F}{\partial w}+\frac{\alpha_1}{2E'}+\frac{\alpha_2}{2E'}w\ . \tag{110}$$

F_B satisfies Eqs. (108)–(110) with

$$\alpha_1=\frac{2E'}{T^*},\quad \alpha_2=0\ ,\quad \alpha_3=-\frac{E'}{T^{*2}}\ . \tag{111}$$

It is interesting that F_B is not the only physically acceptable solution of Eqs. (108)–(110); there exists one other solution, denoted $F_{\hat{B}}$, which will be derived. However, $F_{\hat{B}}$ gives only a local extremum of $(-dH/dt)$; F_B gives the absolute minimum. To show this we find the general acceptable solution of Eqs. (108)–(110). To satisfy (108), the solution must be of the form

$$F = [\psi(h)\exp(-gw/2) + \chi(h)\exp(+gw/2)]^2 , \tag{112}$$

with ψ and χ arbitrary functions of h and

$$\tilde{g} = (-\alpha_3/E')^{1/2} . \tag{113}$$

For F to be physically acceptable, g is taken to be positive real and $\chi = 0$:

$$F = \psi^2 e^{-gw} . \tag{114}$$

Inserting this expression in Eq. (110) yields

$$0 = \frac{-g}{\psi^2} h \frac{d\psi^2}{dh} - g + \frac{\alpha_1}{2E'} + \frac{\alpha_2}{2E'} w . \tag{115}$$

Hence, α_2 must vanish and

$$\psi^2 \propto h^{(\alpha_1/2gE'-1)} . \tag{116}$$

Finally, Eq. (109) is only satisfied for the two cases

$$\psi^2 \propto 1,\ h . \tag{117}$$

The first case (117) corresponds to the Bennett distribution. The second case is

$$F_{\hat{B}} = \hat{K} l^2 e^{-\hat{g}w} \tag{118}$$

with $\hat{g}$ and $\hat{K}$ to be determined. The Lagrange multipliers are then given by

$$\hat{g}^2 = -\frac{\alpha_3}{E'} , \quad \left(\frac{\alpha_1}{2gE'} - 1\right) = 1 . \tag{119}$$

It may be verified that $F_{\hat{B}}$ is characterized by

$$\hat{g} = \frac{2}{T^*} , \quad \hat{K}^{-1} = \frac{\pi^2 a^4}{2}\left(1 + \frac{R^4}{a^4}\right)^2 T^{*2} m^2 \gamma^2 , \tag{120}$$

$$J_{\hat{B}} = \frac{2I}{\pi a^2}\frac{r^2}{a^2}\left(1 + \frac{r^4}{a^4}\right)^{-2} , \tag{121}$$

$$A_{\hat{B}} = -\frac{I}{2c}\log\left(\frac{a^4 + r^4}{a^4 + R^4}\right) . \tag{122}$$

The mean square radius is $\pi a^2/2$. Note that J_B is hollowed out on axis; this is a consequence of the anisotropic distribution of velocities. Using Eqs. (43) and (118) we calculate directly

$$\begin{aligned}\frac{dH_{\hat{B}}}{dt} &= -\int d^2r\, d^2p \frac{\gamma m E'}{2}\frac{|\nabla_p F_{\hat{B}}|^2}{F_{\hat{B}}} \\ &= -\frac{\gamma m E'}{2}\int d^2r\, d^2p\, F_B\left(\frac{4}{P_\theta^2} - \frac{4\hat{g}}{\gamma m} + \frac{\hat{g}^2 p^2}{\gamma^2 m^2}\right) \\ &= -\left(\frac{\gamma m E'}{2}\right)\left(\frac{4\hat{g}}{\gamma m}\right) = -\frac{4E'}{T^*} ,\end{aligned} \tag{123}$$

which is four times the rate calculated for the Bennett distribution [Eq. (103)]. If λ, I, and γ are varied slowly and there is no scattering, $F_{\hat{B}}$ is a similarity solution with $a(t)$ fixed by the adiabatic invariant (67). However, unlike F_B, scattering destroys the form of $F_{\hat{B}}$.

As a final point it is necessary to show that the quantities

$$\frac{H - H_B}{H_B} \quad \text{and} \quad \frac{dH/dt - dH_B/dt}{dH_B/dt} \tag{124}$$

actually go to zero at large t. This is clear from the following arguments. H_B decreases linearly in time at the rate

$$\frac{dH_B}{at} = -\frac{E'}{T^*} , \tag{125}$$

and for any other distribution

$$-\frac{dH}{dt} > \frac{E'}{T^*} . \tag{126}$$

However, dH/dt cannot be bounded away from dH_B/dt since, otherwise, $H < H_B$ at large t; in fact, by Eq. (126) the rates must converge monotonically

$$\frac{dH}{dt} \to \frac{dH_B}{dt} = -\frac{E'}{T^*} . \tag{127}$$

Similarly, the difference $H - H_B$ must approach some limit since it can neither increase nor fall below zero.

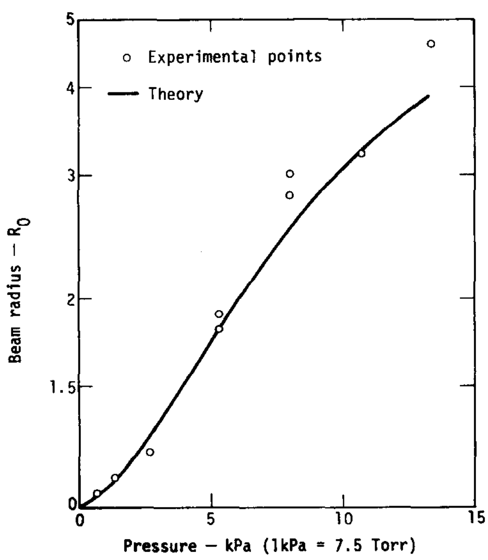

FIG. 1. Comparison of calculated and measured rms beam radius in nitrogen.

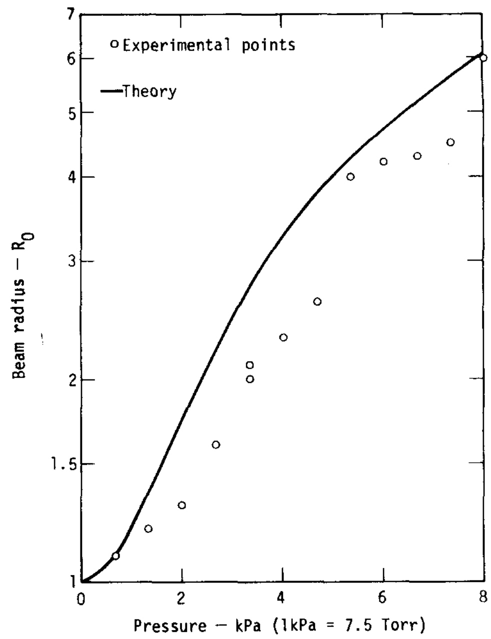

FIG. 2. Comparison of calculated and measured rms beam radius in argon.

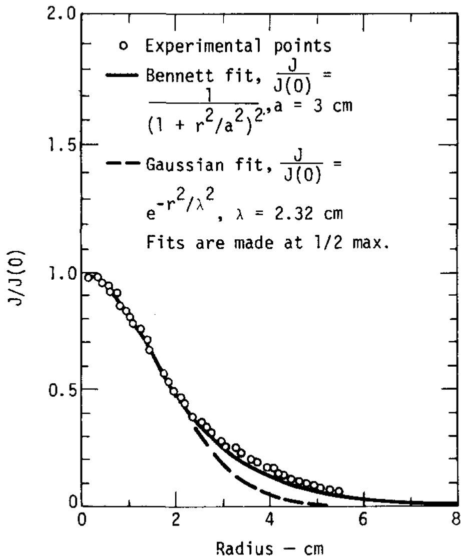

FIG. 3. Beam profile after scattering by nitrogen. Energy, 5.2 MeV, current 220 A, pressure, 5.3 kPa (40 Torr.) N_2, length, 2m.

However, as $H_B \to -\infty$, dH_B/dt is bound away from zero so the ratios (124) vanish at large t.

VIII. COMPARISON WITH EXPERIMENT

Experimental verification of the dependence of rate of beam expansion on gas pressure and the generation of the Bennett profile by scattering has recently been reported.[13]

In brief, a low-current beam from the astron accelerator ($E = 5.2$ MeV, $I = 220$ A, $a \approx 1$ cm, $\tau_p = 300$ nsec) is injected into a tank of nitrogen or argon. Pressures range from 0.7–13 kPa (5–100 Torr). At injection the beam radius is matched to the equilibrium value to avoid pulsations. The current density at any instant during the pulse is then monitored at a distance of 2 m with a small Faraday cup. The radial current profile is measured by varying the transverse position of the Faraday cup and utilizing a large number of (highly reproducible) beam pulses. The radius and profile are inferred from this trace.

Most of the conditions of the kinetic theory developed here are met in the experiment: The 2 m distance is short enough to bypass the hose instability,[14] which has a growth distance of $3c/\omega_\beta \approx 62$ cm. No plasma current is observed at the time of measurement and charge neutrality is complete at these pressures; thus, the validity of the neutralization fraction models does not arise. The gas is of uniform density, so scattering is homogeneous. Neither beam energy nor velocity changes significantly in the 2 m range. There is no evidence, either positive or negative, that the beam is rotating. Finally, the scattering is not always weak, particularly at the high pressures of argon.

To calculate the scattering coefficient the "thin target" limit treated by Williams[10] is appropriate. This gives

$$E' = \frac{4\pi N_g Z(Z+1)\, e^4}{m\gamma\beta c} \log\left(\frac{Z^{4/3} N_g l h^2}{\pi m^2 \beta^2 c^2}\right)^{1/2}, \tag{128}$$

where $l = 2$ m, the propagation distance, and h is Planck's constant. The envelope equation (80) is integrated to give a prediction of the radius as a function of pressure. The result is plotted along with the experimental points in Figs. 1 and 2. Agreement is excellent for the case of nitrogen and good (within 40%) for argon.

Figure 3 presents a detailed comparison between the measured profile at 5.3 kPa (40 Torr) of nigrogen and calculated Bennett and Gaussian fits. Again the agreement with experiment is excellent.

A publication describing this experiment and the results in greater detail is planned.

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