Classical theory of radiating electrons

By P. A. M. Dirac, F.R.S., *St John's College, Cambridge*

Introduction

The Lorentz model of the electron as a small sphere charged with electricity, possessing mass on account of the energy of the electric field around it, has proved very valuable in accounting for the motion and radiation of electrons in a certain domain of problems, in which the electromagnetic field does not vary too rapidly and the accelerations of the electrons are not too great. Beyond this domain it will not go unless supplemented by further assumptions about the forces that hold the charge on an electron together. No natural way of introducing such further assumptions has been discovered, and it seems that the Lorentz model has reached the limit of its usefulness and must be abandoned before we can make further progress.

One of the most attractive ideas in the Lorentz model of the electron, the idea that all mass is of electromagnetic origin, appears at the present time to be wrong, for two separate reasons. First, the discovery of the neutron has provided us with a form of mass which it is very hard to believe could be of electromagnetic nature. Secondly, we have the theory of the positron—a theory in agreement with experiment so far as is known—in which positive and negative values for the mass of an electron play symmetrical roles. This cannot be fitted in with the electromagnetic idea of mass, which insists on all mass being positive, even in abstract theory.

The departure from the electromagnetic theory of the nature of mass removes the main reason we have for believing in the finite size of the electron. It seems now an unnecessary complication not to have the field equations holding all the way up to the electron's centre, which would then appear as a point of singularity. In this way we are led to consider a point model for the electron. Further reasons for preferring the point-electron have been given by Frenkel (1925).*

We are now faced with the difficulty that, if we accept Maxwell's theory, the field in the immediate neighbourhood of the electron has an infinite mass. This difficulty has recently received much prominence in quantum mechanics (which uses a point model of the electron), where it appears as

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* See pp. 526 and 527 of Frenkel's paper. The reason on p. 518 is not valid according to present-day knowledge.

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a divergence in the solution of the equations that describe the interaction of an electron with an electromagnetic field and prevents one from applying quantum mechanics to high-energy radiative processes. One may think that this difficulty will be solved only by a better understanding of the structure of the electron according to quantum laws. However, it seems more reasonable to suppose that the electron is too simple a thing for the question of the laws governing its structure to arise, and thus quantum mechanics should not be needed for the solution of the difficulty. Some new physical idea is now required, an idea which should be intelligible both in the classical theory and in the quantum theory, and our easiest path of approach to it is to keep within the confines of the classical theory.

A possible line of attack is to modify Maxwell's theory so as to make the energy of the field around the singularity that represents an electron finite. This method has been tried by Born, but it leads to great complexity, and its present outlook is not very hopeful.

We shall here proceed from the opposite point of view. We shall retain Maxwell's theory to describe the field right up to the point-singularity which represents our electron and shall try to get over the difficulties associated with the infinite energy by a process of direct omission or subtraction of unwanted terms, somewhat similar to what has been used in the theory of the positron. Our aim will be not so much to get a model of the electron as to get a simple scheme of equations which can be used to calculate all the results that can be obtained from experiment. The scheme must be mathematically well-defined and self-consistent, and in agreement with well-established principles, such as the principle of relativity and the conservation of energy and momentum. Provided these conditions are satisfied, it should not be considered an objection to the theory that it is not based on a model conforming to current physical ideas.

We shall be working all the time with the ordinary Maxwell theory, and thus we shall not be led to any essentially new equations. We shall be concerned rather with getting a satisfactory physical interpretation for equations which are already well known. It may be wondered why this problem was not solved long ago. A great deal of work has been done in the past in examining the general implications of Maxwell's theory, but it was nearly all done before the discovery of quantum mechanics in 1925, when people gave all their attention to the question of how an electron could remain in an atomic orbit without radiating—a question we now know can be answered only by going outside classical theory—and were thus not interested in simply looking for the most natural interpretation their equations will allow.

THE FIELDS ASSOCIATED WITH AN ELECTRON

We shall deal first with the problem of a single electron moving in an electromagnetic field. We shall use relativistic notation throughout, taking the velocity of light to be unity. When we want to pass from the contravariant form K_μ to the covariant form K^μ of a vector, we shall use the fundamental tensor $g^{\mu\nu}$ appropriate for flat space-time, with components $g^{00} = 1$, $g^{11} = g^{22} = g^{33} = -1$, the other components vanishing. We shall also sometimes use the scalar product notation

$$(\mathbf{K}, \mathbf{L}) = K_\mu L^\mu = K_0 L_0 - K_1 L_1 - K_2 L_2 - K_3 L_3. \tag{1}$$

Let us suppose the world-line of our electron in space-time to be known and to be described by the equations

$$z_\mu = z_\mu(s), \tag{2}$$

where the $z_\mu(s)$ are given functions of the proper-time s. We have, of course, $dz_0/ds > 0$. The electromagnetic potentials A_μ at the point x_μ satisfy the equations

$$\frac{\partial A_\mu}{\partial x_\mu} = 0, \tag{3}$$

$$\Box A_\mu = 4\pi j_\mu, \tag{4}$$

where j_μ is the charge-current density vector. With our present model of the electron, j_μ vanishes everywhere except on the world-line of the electron, where it is infinitely great. The singularity in j_μ may conveniently be expressed in terms of δ-functions, and is easily seen to be

$$j_\mu = e \int \frac{dz_\mu}{ds} \delta(x_0 - z_0)\, \delta(x_1 - z_1)\, \delta(x_2 - z_2)\, \delta(x_3 - z_3)\, ds \tag{5}$$

for an electron of charge e. From the potentials A_μ the field quantities $F^{\mu\nu}$ can be derived according to

$$F^{\mu\nu} = \frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu}. \tag{6}$$

Equations (3) and (4) have many solutions and thus do not suffice to fix the field. One solution is provided by the well-known retarded potentials of Liénard and Wiechert. We shall call the field derived from these potentials $F^{\mu\nu}_{\text{ret}}$. We can obtain other solutions by adding to this one any solution of (3) and

$$\Box A_\mu = 0, \tag{7}$$

representing a field of radiation. The particular solution that represents the actual conditions in our one-electron problem will be the sum of the retarded potentials and the potentials satisfying (7) that represent the incoming electromagnetic waves incident on our electron. Thus, calling the actual field $F^{\mu\nu}_{\text{act}}$ and the incident field $F^{\mu\nu}_{\text{in}}$, we have

$$F^{\mu\nu}_{\text{act}} = F^{\mu\nu}_{\text{ret}} + F^{\mu\nu}_{\text{in}}. \tag{8}$$

Another important solution of (3) and (4) is that provided by the advanced potentials. We call the field derived from it $F^{\mu\nu}_{\text{adv}}$. We should expect $F^{\mu\nu}_{\text{adv}}$ to play a symmetrical role to $F^{\mu\nu}_{\text{ret}}$ in all questions of general theory. Let us therefore put, corresponding to (8),

$$F^{\mu\nu}_{\text{act}} = F^{\mu\nu}_{\text{adv}} + F^{\mu\nu}_{\text{out}}, \tag{9}$$

defining in this way a new field $F^{\mu\nu}_{\text{out}}$. This field is derivable from potentials satisfying (7) and thus represents some field of radiation, and we should expect it to play a symmetrical role in general theory to $F^{\mu\nu}_{\text{in}}$, so that it should be interpretable as the field of outgoing radiation leaving the neighbourhood of the electron. The difference

$$F^{\mu\nu}_{\text{rad}} = F^{\mu\nu}_{\text{out}} - F^{\mu\nu}_{\text{in}} \tag{10}$$

would then be the field of radiation produced by the electron. This difference may, from (8) and (9), be expressed as

$$F^{\mu\nu}_{\text{rad}} = F^{\mu\nu}_{\text{ret}} - F^{\mu\nu}_{\text{adv}}, \tag{11}$$

which shows that it is completely determined by the world-line of the electron. In the appendix it is calculated near the world-line and found to be free from singularity and to have the value on the world-line

$$F_{\mu\nu\,\text{rad}} = \frac{4e}{3}\left(\frac{d^3z_\mu}{ds^3}\frac{dz_\nu}{ds} - \frac{d^3z_\nu}{ds^3}\frac{dz_\mu}{ds}\right). \tag{12}$$

Let us compare our result (11) with the usual one for the radiation produced by an accelerating electron. For definiteness, suppose we have an electron initially moving with constant velocity, then undergoing an acceleration and finally getting again into a state of constant velocity. According to the usual theory, the radiation emitted by the electron while it is accelerating will be given by the value of $F^{\mu\nu}_{\text{ret}}$ at great distances from the electron and at correspondingly great times after the time of the acceleration. This will be equal to the value of the right-hand side of (11) at these distances and times, since $F^{\mu\nu}_{\text{adv}}$ will be zero in this part of space-time. *Thus our result (11) for the radiation produced by an accelerating electron is*

in agreement with the usual one in that region of space-time where the usual one is defined. Our result goes beyond the usual one in providing a definite value for the field of radiation throughout space-time, which is an advantage in giving us a meaning for the radiation field close to the electron. (The usual theory gives this field inextricably mixed up with the Coulomb field.) Our result also gives a meaning for the radiation field before its time of emission, when it can have no physical significance. This, however, is unavoidable if we are to have the radiation field well defined near the electron.

Our theory has attained symmetry between the use of retarded and advanced potentials. This symmetry will be maintained in all general theoretical discussions, but it will not apply to practical applications, since we then have $F_{\text{in}}^{\mu\nu}$ given (very often it is zero), while its counterpart $F_{\text{out}}^{\mu\nu}$ is unknown.

A field that we shall need in the future is $f^{\mu\nu}$, defined as the actual field minus the mean of the advanced and retarded fields,

$$f^{\mu\nu} = F_{\text{act}}^{\mu\nu} - \tfrac{1}{2}(F_{\text{ret}}^{\mu\nu} + F_{\text{adv}}^{\mu\nu}). \tag{13}$$

It is derivable from potentials satisfying (7) and is free from singularity on the world-line of the electron. It is, in fact, from (8) and (9), just the mean of the ingoing and outgoing fields of radiation,

$$f^{\mu\nu} = \tfrac{1}{2}(F_{\text{in}}^{\mu\nu} + F_{\text{out}}^{\mu\nu}).$$

THE EQUATIONS OF MOTION OF AN ELECTRON

To complete our theory of the interaction between an electron and the electromagnetic field we require equations of motion for the electron—equations to determine the world-line which we assumed known in the preceding section. We can get information on this question from the laws of conservation of energy and momentum. We surround the singular world-line in space-time by a thin tube, whose radius is much smaller than any length of physical importance in our one-electron problem, and we calculate the flow of energy and momentum across the (three-dimensional) surface of this tube, using the stress tensor $T_{\mu\rho}$ of Maxwell's theory, calculated from the actual field $F_{\text{act}}^{\mu\nu}$ according to the formula*

$$4\pi T_{\mu\rho} = F_{\mu\nu} F_\rho{}^\nu + \tfrac{1}{4} g_{\mu\rho} F_{\alpha\beta} F^{\alpha\beta}. \tag{14}$$

* The usual derivation of the stress-tensor is valid only for continuous charge distributions and we are here using it for point charges. This involves adopting as a fundamental assumption the point of view that energy and momentum are localized in the field in accordance with Maxwell's and Poynting's ideas.

We then have the information that the total flow of energy (or momentum) out from the surface of any finite length of tube must equal the difference in the energy (or momentum) residing within the tube at the two ends of this length, and must thus depend only on conditions at the two ends of this length. In mathematical language, the rate of flow of energy (or momentum) out from the surface of the tube must be a perfect differential.

It is easily seen that the information obtained in this way is independent of what shape and size we give to our tube (provided it is sufficiently small for the Taylor expansions used in the calculations to be valid). If we take two tubes surrounding the singular world-line, the divergence of the stress tensor $\partial T_{\mu\rho}/\partial x_\rho$ will vanish everywhere in the region of space-time between them, since there are no singularities in this region and (7) is satisfied throughout it. Expressing the integral

$$\iiint \partial T_{\mu\rho}/\partial x_\rho \,.\, dx_0\, dx_1\, dx_2\, dx_3$$

over the region of space-time between a certain length of the two tubes as a surface integral over the (three-dimensional) surface of this region, we obtain immediately that the difference in the flows of energy (or momentum) across the surfaces of the two tubes depends only on conditions at the two ends of the length considered. Thus the information provided by the conservation laws is well defined.

The calculations involved in getting this information are rather long and are given in the appendix. We choose the simplest shape of tube, a tube which is spherical, of constant radius ϵ, for each value of the proper-time, in that Lorentz frame of reference in which the electron is then at rest. To save writing we put v_μ for dz_μ/ds and use dots to denote differentiations with respect to s. We note for future use the elementary equations

$$\mathbf{v}^2 = 1 \tag{15}$$

$$(\mathbf{v}\dot{\mathbf{v}}) = 0 \tag{16}$$

$$(\mathbf{v}\ddot{\mathbf{v}}) + \dot{\mathbf{v}}^2 = 0 \tag{17}$$

in the vector notation (1).

The result of the appendix is now that, with $f^{\mu\nu}$ defined by (13), the flow of energy and momentum out from the surface of any length of tube is given by the vector

$$\int [\tfrac{1}{2} e^2 \epsilon^{-1} \dot{v}_\mu - e v_\nu f_\mu{}^\nu]\, ds,$$

integrated over the length of tube, where terms that vanish with ϵ are

neglected. This must depend only on conditions at the two ends of the length of tube, so that the integrand must be a perfect differential, i.e.

$$\tfrac{1}{2}e^2\epsilon^{-1}\dot{v}_\mu - ev_\nu f_\mu{}^\nu = \dot{B}_\mu. \tag{18}$$

This is as far as we can get from the laws of conservation of energy and momentum. To fix our equations of motion for the electron, we must make some further assumption to fix the vector B_μ. From (18) we see that

$$(\mathbf{v}\dot{\mathbf{B}}) = \tfrac{1}{2}e^2\epsilon^{-1}(\mathbf{v}\dot{\mathbf{v}}) = 0, \tag{19}$$

with the help of (16), which restricts our choice of B_μ. Apart from this, B_μ may be any vector function of v_μ and its derivatives. The simplest B_μ satisfying the restriction (19) is

$$B_\mu = kv_\mu, \tag{20}$$

where k is a constant. There are other possible expressions for B_μ satisfying (19), for example,

$$B_\mu = k'[\dot{\mathbf{v}}^4 v_\mu + 4(\dot{\mathbf{v}}\ddot{\mathbf{v}})\dot{v}_\mu],$$

k' being another constant, but they are all much more complicated than (20), so that one would hardly expect them to apply to a simple thing like an electron. We therefore assume (20).

Substituting (20) into the right-hand side of (18), we see that the constant k must be of the form

$$k = \tfrac{1}{2}e^2\epsilon^{-1} - m, \tag{21}$$

where m is another constant independent of ϵ, in order that our equations may have a definite limiting form when ϵ tends to zero. We then get

$$m\dot{v}_\mu = ev_\nu f_\mu{}^\nu, \tag{22}$$

as our equations of motion for the electron. They are of the usual form of the equations of motion of an electron in an external electromagnetic field, with m playing the part of the rest-mass of the electron and $f_\mu{}^\nu$, the actual field minus the mean of the advanced and retarded fields, playing the part of the external field.

Equations (22) are not in a form suitable for application to practical problems where we are given, not $f_\mu{}^\nu$, but the incident field $F_\mu{}^\nu{}_{\text{in}}$. The connexion between these two fields is, from (13), (8) and (11),

$$\begin{aligned} f_\mu{}^\nu &= F_\mu{}^\nu{}_{\text{in}} + \tfrac{1}{2}F_\mu{}^\nu{}_{\text{rad}} \\ &= F_\mu{}^\nu{}_{\text{in}} + \tfrac{2}{3}e(\ddot{v}_\mu v^\nu - \ddot{v}^\nu v_\mu) \end{aligned} \tag{23}$$

with the help of (12). Substituting this into (22) and using (15) and (17), we get

$$m\dot{v}_\mu - \tfrac{2}{3}e^2\ddot{v}_\mu - \tfrac{2}{3}e^2\dot{\mathbf{v}}^2 v_\mu = ev_\nu F_{\mu\,\text{in}}^{\ \nu}. \tag{24}$$

These are the same as the equations of motion obtained from the Lorentz theory of the extended electron by equating the total force on the electron to zero, if one neglects terms involving higher derivatives of v_μ than the second. But whereas these equations, as derived from the Lorentz theory, are only approximate, we now see that *there is good reason for believing them to be exact, within the limits of the classical theory.*

To discuss the significance of equations (24), let us take the equation with $\mu = 0$, describing the energy balance. The right-hand side, which gives the rate at which the incident field does work on the electron, is equated to the sum of the three terms $m\dot{v}_0$, $-\tfrac{2}{3}e^2\ddot{v}_0$ and $-\tfrac{2}{3}e^2\dot{\mathbf{v}}^2 v_0$. The first two of these are perfect differentials and the things they are the differentials of, namely mv_0 and $-\tfrac{2}{3}e^2\dot{v}_0$, may be considered as intrinsic energies of the electron. The former is just the usual expression for the kinetic energy of a particle of rest-mass m, while the latter is what is called the "acceleration energy" of the electron (Schott 1915). Changes in the acceleration energy correspond to a reversible form of emission or absorption of field energy, which never gets very far from the electron. The third term $-\tfrac{2}{3}e^2\dot{\mathbf{v}}^2 v_0$ corresponds to irreversible emission of radiation and gives the effect of radiation damping on the motion of the electron. It is necessarily positive, since, according to (16), $\dot{v}_\mu$ is orthogonal to the time-like vector v_μ and is thus a space-like vector, and hence its square is negative.

Let us see how the kinetic energy term arises. The B_μ introduced in (18) can be interpreted as minus the vector of energy and momentum residing within the tube at any value of the proper-time. Thus, from (20) and (21), the energy within the tube must be negative and must tend to $-\infty$ as ϵ tends to zero. This negative energy is needed to compensate for the large positive energy of the Coulomb field just outside the tube, to keep the total energy down to the value appropriate to the rest-mass m. If we want a model of the electron, we must suppose that there is an infinite negative mass at its centre such that, when subtracted from the infinite positive mass of the surrounding Coulomb field, the difference is well defined and is just equal to m. Such a model is hardly a plausible one according to current physical ideas but, as discussed in the Introduction, this is not an objection to the theory provided we have a reasonable mathematical scheme.

APPLICATION OF THE EQUATIONS OF MOTION

Let us now accept equations (24) as giving an exact description of the motion of an electron in a specified incident field and investigate some of their consequences. The equations involve $\ddot{v}_\mu$, which means that the motion will not be determined if we are given only the position and velocity of the electron at one instant of time. We must also be given its acceleration. The equations can then be used to determine the rate of change of the acceleration and the whole motion will then be fixed.

As an interesting special case let us suppose there is no incident field, so that we have the equations of motion

$$a\dot{v}_\mu - \ddot{v}_\mu - \dot{\mathbf{v}}^2 v_\mu = 0, \tag{25}$$

where $a = 3m/2e^2$. In general the electron will not now be moving with constant velocity, as it would according to ordinary ideas, since we may suppose it to be started off with a non-zero acceleration and it cannot then suddenly lose its acceleration. Let us pass over to the notation x, y, z, t instead of z_1, z_2, z_3, z_0 and choose a system of co-ordinates so that the initial velocity and acceleration are in the direction of the x-axis, or, in relativistic language, the initial velocity and acceleration four-vectors lie in the (xt) plane. Then from symmetry considerations the motion must lie entirely in this plane.

The equations (25) now become

$$a\ddot{x} - \ddot{x} - \dot{x}(\ddot{t}^2 - \ddot{x}^2) = 0, \tag{26}$$

$$a\ddot{t} - \ddot{t} - \dot{t}(\ddot{t}^2 - \ddot{x}^2) = 0. \tag{27}$$

Equations (15), (16) and (17) give

$$\dot{t}^2 - \dot{x}^2 = 1, \tag{28}$$

$$\dot{t}\ddot{t} - \dot{x}\ddot{x} = 0, \tag{29}$$

$$(\dot{t}\dddot{t} - \dot{x}\dddot{x}) + (\ddot{t}^2 - \ddot{x}^2) = 0,$$

from which we see that $\dot{x}$ times equation (26) minus $\dot{t}$ times equation (27) vanishes identically, so that either of these equations is equivalent to the other. Eliminating $\dot{t}$ from (28) and (29), we get

$$\ddot{t} = \ddot{x}\frac{\dot{x}}{(1+\dot{x}^2)^{\frac12}},$$

which gives, on being substituted into (26),

$$a\dddot{x} - \ddot{x} + \frac{\dot{x}\ddot{x}^2}{1+\dot{x}^2} = 0.$$

If $\ddot{x}$ does not vanish, we may write this as

$$a - \frac{\dddot{x}}{\ddot{x}} + \frac{\dot{x}\ddot{x}}{1 + \dot{x}^2} = 0,$$

and integrate immediately, to get

$$as - \log \ddot{x} + \tfrac{1}{2} \log (1 + \dot{x}^2) = c,$$

where c is a constant of integration. It is convenient to choose the origin from which s is measured to make $c = -\log a$. We then have

$$\frac{\ddot{x}}{(1 + \dot{x}^2)^{\frac{1}{2}}} = ae^{as},$$

which can be integrated again, to give

$$\log [\dot{x} + (1 + \dot{x}^2)^{\frac{1}{2}}] = e^{as} + b,$$

where b is another constant of integration. We may write this result as

$$\left.\begin{aligned} \dot{x} &= \sinh (e^{as} + b), \\ \dot{t} &= \cosh (e^{as} + b). \end{aligned}\right\} \tag{30}$$

We now have that, as s tends to $-\infty$, the velocity tends to a constant value, given by $\dot{x} = \sinh b$ or $dx/dt = \tanh b$. As s increases from $-\infty$, the motion departs from the asymptotic motion of constant velocity, and the velocity steadily increases. For large values of s the velocity tends to the velocity of light according to an extremely rapid law.

At this stage one would be inclined to say that there is a mistake in sign in our equations and that we ought to have e^{-as} instead of e^{as} in (30). With this alteration we should have a theory in which, if an electron is disturbed in any way and then left alone, it would rapidly settle down into a state of constant velocity, with emission of radiation while it is settling down. This would be a reasonable behaviour for an electron according to our present-day physical ideas. However, it is not possible to tamper with the signs in our theory in any relativistic way to obtain this result, without getting equations of motion which would make the electron in the hydrogen atom spiral outwards, instead of spiralling inwards and ultimately falling into the nucleus, as it should in the classical theory. We are therefore forced to keep the signs in (30) as they are and to see what interpretation we can give to the equations as they stand. This will lead us to the most beautiful feature of the theory.

The motion (30) is certainly never observed for an electron in the absence of incident radiation. This does not give a contradiction between our theory

and observation, however, since our equations of motion (25) admit of alternative solutions, namely $v_\mu = $ constant, which are in agreement with observation. We must merely impose the condition that these solutions are the ones that occur in Nature. Let us see how this condition works in a practical problem.

Suppose we have an electron which is disturbed by some electromagnetic radiation incident on it and is then left alone. We may use our general equations of motion (24) and must restrict ourselves to those solutions for which the velocity is constant during the final period when the electron is left alone. This will in general prevent us from having the velocity exactly constant during the initial period before the disturbance has arrived and will compel us then to have a motion of the form (30), with the electron gradually building up a small acceleration. The difference of such a motion from one of constant velocity is too small to show itself as a contradiction with observation, so that we can get agreement between the solutions of our equations and observed motions on these lines.

We now have a striking departure from the usual ideas of mechanics. *We must obtain solutions of our equations of motion for which the initial position and velocity of the electron are prescribed, together with its final acceleration, instead of solutions with all the initial conditions prescribed.*

MOTION OF AN ELECTRON DISTURBED BY A PULSE

To study the rather unexpected results of the preceding section more closely, let us take a typical special case, which is sufficiently simple to be worked out completely. Suppose we have an electron initially at rest, disturbed by a pulse of electromagnetic radiation passing over it. To simplify the mathematics as much as possible, let us suppose the duration of the pulse to be infinitely short, so that the electric force is represented by a δ-function, say

$$\mathscr{E}_x = k\,\delta(t-y) \quad \mathscr{E}_y = \mathscr{E}_z = 0, \tag{31}$$

in the $xyzt$ notation, where k is a constant, giving us a pulse polarized in the direction of the x-axis and moving in the direction of the y-axis. We shall consider a motion for which the electron remains always in the plane $y = 0$, $z = 0$ in space-time (neglecting the magnetic effects of the pulse).

Substituting (31) into our equations of motion (24), we get

$$a\dddot{x} - \ddot{x} - \dot{x}(\dot{t}^2 - \ddot{x}^2) = \kappa\,\delta(t)\,\dot{t}, \tag{32}$$

where $\kappa = 3k/2e$. The equation involving $\ddot{t}$ is not independent of this one and may be ignored. Let us now suppose k to be sufficiently small for the

velocity acquired by the electron to be small compared with the velocity of light, so that we may neglect relativistic effects. Equation (32) then reduces to

$$a\dddot{x} - \ddot{x} = \kappa\,\delta(t), \tag{33}$$

in which we may take the dot to denote differentiation with respect to t. Equation (33) shows that, at the time $t = 0$, $\ddot{x}$ increases discontinuously by an amount $-\kappa$, and before and after this time we have

$$a\dddot{x} - \ddot{x} = 0. \tag{34}$$

According to the conclusions of the preceding section, we must take a motion for which, after $t = 0$, $\dot{x}$ is a constant, q say. We now have $\ddot{x}$ zero just after $t = 0$, so it must have the value κ just before. The general solution of (34) is

$$\dot{x} = c_1 e^{at} + c_2,$$

where c_1 and c_2 are constants of integration. To obtain the motion of our electron before $t = 0$, we must choose these constants of integration so that $\dot{x} = 0$ for $t = -\infty$ and $\ddot{x} = \kappa$ for $t = 0$, the former condition taking into account that the electron is initially at rest. This fixes $c_2 = 0$ and $c_1 = \kappa/a$. Finally, we have the condition that $\dot{x}$ must be continuous at $t = 0$ (since there is no δ-function in $\ddot{x}$), which gives us $q = c_1$. Thus the solution of our equations of motion is

$$\begin{aligned} \dot{x} &= \kappa/a\,.\,e^{at}, & t &< 0, \\ &= \kappa/a, & t &> 0. \end{aligned} \tag{35}$$

We can describe the motion by saying that the electron is, to a high approximation, at rest for large negative values of t, but as t approaches zero it acquires a velocity and acceleration, in accordance with equations (30), of such amounts that just before $t = 0$ the acceleration has the right value to be exactly cancelled by the effect of the pulse, so that after $t = 0$ the electron is left moving with constant velocity. It would appear here that we have a contradiction with elementary ideas of causality. The electron seems to know about the pulse before it arrives and to get up an acceleration (as the equations of motion allow it to do), just sufficient to balance the effect of the pulse when it does arrive. The electron will, of course, radiate all the time it is accelerating and will thus be radiating before $t = 0$.

The behaviour of our electron can be interpreted in a natural way, however, if we suppose the electron to have a finite size. There is then no need for the pulse to reach the centre of the electron before it starts to accelerate. It starts to accelerate and radiate as soon as the pulse meets

its outside. Mathematically, the electron has no sharp boundary and must be considered as extending to infinity, but for practical purposes it may be considered to have a radius of order a^{-1}, this being the distance within which the pulse must arrive before the acceleration and radiation are appreciable.

The above interpretation leaves one further point to be examined. Suppose we have a pulse sent out from a place A and a receiving apparatus for electromagnetic waves set up at a place B, and suppose there is an electron on the straight line joining A to B. Then the electron will be radiating appreciably at a time a^{-1} before the pulse has reached its centre and this emitted radiation will be detectable at B at a time a^{-1} earlier than when the pulse, which travels from A to B with the velocity of light, arrives. In this way a signal can be sent from A to B faster than light. This is a fundamental departure from the ordinary ideas of relativity and is to be interpreted by saying that *it is possible for a signal to be transmitted faster than light through the interior of an electron. The finite size of the electron now reappears in a new sense, the interior of the electron being a region of failure, not of the field equations of electromagnetic theory, but of some of the elementary properties of space-time.* In spite of this departure from ordinary relativistic ideas, our whole theory is the Lorentz invariant.

Some interesting points about the solution (35) of our problem may be noted. In the first place, the total momentum acquired by the electron is

$$m\kappa/a = ek.$$

This is precisely the same as the momentum the electron would acquire on an elementary theory which ignores the radiation damping completely.

Secondly, let us determine the spectral distribution of the radiation emitted by the electron. The rate of emission of energy is $\frac{2}{3}e^2\ddot{x}^2$ and its spectral distribution is given by the Fourier resolution of $\ddot{x}$. Thus the energy emitted per unit frequency range will be

$$
\begin{aligned}
E_\nu &= \tfrac{2}{3}e^2 \left| \int_{-\infty}^{\infty} \ddot{x}e^{2\pi i\nu t}\,dt \right|^2 \\
&= \tfrac{2}{3}e^2\kappa^2 \left| \int_{-\infty}^{0} e^{(a+2\pi i\nu)t}\,dt \right|^2 \\
&= \frac{2e^2\kappa^2}{3(a^2+4\pi^2\nu^2)} = \frac{2e^4k^2}{3m^2(1+4\pi^2\nu^2/a^2)}.
\end{aligned}
\tag{36}
$$

If we resolve the incident pulse into its Fourier components and suppose each component to be scattered in accordance with Thomson's formula, we should obtain for the energy scattered per unit frequency range the

value $2e^4k^2/3m^2$. This is the same as (36) for values of ν small compared with a.

It has been suggested by Oppenheimer (1935), as a possible escape from the discrepancy between theory and cosmic ray observations for the penetrating power of fast electrons, that it may not be permissible to calculate the scattering of the various Fourier components of the radiation incident on an electron independently, but that the presence of the high-frequency components may reduce the scattering of the low-frequency components. We now see that this suggestion is not supported by our theory and some other explanation of the discrepancy, such as the recently proposed heavy electron, is needed.

Extension to several electrons

The foregoing theory may easily be extended to apply to any number of electrons interacting with each other and with a field of radiation. Let us label the electrons by a suffix n or m. The nth electron will have its own retarded field $F^{\mu\nu}_{n,\,\mathrm{ret}}$ and advanced field $F^{\mu\nu}_{n,\,\mathrm{adv}}$, determined by its world-line, and the difference

$$F^{\mu\nu}_{n,\,\mathrm{rad}} = F^{\mu\nu}_{n,\,\mathrm{ret}} - F^{\mu\nu}_{n,\,\mathrm{adv}} \tag{37}$$

may be considered as the radiation field produced by the nth electron. Equation (12) will still apply for the value of this field on the world-line of the electron producing it.

Generalizing (8), we have that the actual field is equal to the incident field plus the sum of the retarded fields of all the electrons,

$$F^{\mu\nu}_{\mathrm{act}} = F^{\mu\nu}_{\mathrm{in}} + \Sigma_n F^{\mu\nu}_{n,\,\mathrm{ret}}. \tag{38}$$

Similarly, generalizing (9), we have

$$F^{\mu\nu}_{\mathrm{act}} = F^{\mu\nu}_{\mathrm{out}} + \Sigma_n F^{\mu\nu}_{n,\,\mathrm{adv}}. \tag{39}$$

Subtracting, we find that the total radiation field produced by the assembly of electrons is

$$F^{\mu\nu}_{\mathrm{rad}} = F^{\mu\nu}_{\mathrm{out}} - F^{\mu\nu}_{\mathrm{in}} = \Sigma_n (F^{\mu\nu}_{n,\,\mathrm{ret}} - F^{\mu\nu}_{n,\,\mathrm{adv}}),$$

which is thus the sum of all the above-defined radiation fields produced by the electrons individually. Of course only the total radiation field is observable, that for an individual electron being merely a mathematical concept.

To get the field $f^{\mu\nu}_n$ which must be inserted into the equations of motion (22)

for the nth electron we must, generalizing (13), subtract the mean of the retarded and advanced fields of this electron from the actual field, thus

$$f_n^{\mu\nu} = F_{\text{act}}^{\mu\nu} - \tfrac{1}{2}(F_{n,\,\text{ret}}^{\mu\nu} + F_{n,\,\text{adv}}^{\mu\nu}). \tag{40}$$

With the help of (38) we may write this as

$$f_n^{\mu\nu} = F_{\text{in}}^{\mu\nu} + \Sigma_{m\neq n} F_{m,\,\text{ret}}^{\mu\nu} + \tfrac{1}{2}(F_{n,\,\text{ret}}^{\mu\nu} - F_{n,\,\text{adv}}^{\mu\nu}).$$

Substituting this into (22) and proceeding along the same lines as those which led to (24), we get the final equations of motion

$$m\dot{v}_{\mu n} - \tfrac{2}{3}e^2\ddot{v}_{\mu n} - \tfrac{2}{3}e^2\dot{\mathbf{v}}_n^2 v_{\mu n} = ev_{\nu n}\{F_{\mu,\,\text{in}}^{\,\nu} + \Sigma_{m\neq n} F_{\mu m,\,\text{ret}}^{\,\nu}\}. \tag{41}$$

The field on the right-hand side here is just the incident field plus the retarded fields of all the other electrons, which is in agreement with current practice. It is interesting to remember that this result has been obtained in a theory which is fundamentally symmetrical between retarded and advanced potentials.

From (40), (38) and (39) we have

$$f_n^{\mu\nu} = \tfrac{1}{2}(F_{\text{in}}^{\mu\nu} + F_{\text{out}}^{\mu\nu}) + \tfrac{1}{2}\Sigma_{m\neq n}(F_{m,\,\text{ret}}^{\mu\nu} + F_{m,\,\text{adv}}^{\mu\nu}). \tag{42}$$

Suppose now we are concerned with a problem in which

$$F_{\text{in}}^{\mu\nu} + F_{\text{out}}^{\mu\nu} = 0. \tag{43}$$

Then the field $f_n^{\mu\nu}$ to be used in the equations of motion (22) for the nth electron is just the sum of the mean of the retarded and advanced fields of all the other electrons. The interaction of the electrons for this case has been studied by Fokker (1929), who finds that the motion of all the electrons may now be described by a variation principle

$$\delta I = 0. \tag{44}$$

In our present notation I would have the form

$$I = \Sigma_n \int m\,ds_n + \tfrac{1}{2}\Sigma_n \Sigma_{m\neq n} \int \frac{e^2(\mathbf{v}_m,\mathbf{v}_n)}{(\mathbf{z}_n - \mathbf{z}_m, \mathbf{v}_m)}\,ds_n + \tfrac{1}{2}\Sigma_n \Sigma_{m\neq n} \int \frac{e^2(\mathbf{v}_m,\mathbf{v}_n)}{(\mathbf{z}_m - \mathbf{z}_n, \mathbf{v}_m)}\,ds_n, \tag{45}$$

where the two double sums are to be taken over all pairs of electrons, and in the first we must take for $\mathbf{z}_m$ and $\mathbf{v}_m$ their retarded values (with respect

to the point on s_n considered), and in the second we must take their advanced values.

When the condition (43) does not apply, Fokker's variation principle (44) will still hold provided we add on to I the terms

$$\Sigma_n e \int \tfrac{1}{2}(A_{in}^\mu + A_{out}^\mu) \, v_{\mu n} \, ds_n, \tag{46}$$

involving the potentials of the mean of the ingoing and outgoing fields. The variation principle will not be of much use in practical problems, where this mean field is unknown, but it may be of value in suggesting how the quantum generalization of our theory is to be made.

Appendix

We shall consider an electron whose world-line is given by equations (2) and shall evaluate field quantities at the point x_μ. The retarded potentials at this point are, in the scalar product notation (1),

$$A_{\mu, \, ret} = \frac{e\dot{z}_\mu}{(\dot{z}, \mathbf{x} - \mathbf{z})}, \tag{47}$$

taken at the retarded proper-time, namely that value of s for which

$$(\mathbf{x} - \mathbf{z}, \mathbf{x} - \mathbf{z}) = 0 \tag{48}$$

with $x_0 - z_0$ positive. We can get $A_{\mu, \, ret}$ in a form suitable for deriving the field quantities (6) from it with the help of the δ-function. Using the general property of the δ-function, that with $f(s)$ and $g(s)$ any two continuous functions of s and $\dot{g}(s) > 0$,

$$\int f(s) \, \delta\{g(s)\} \, ds = \int \frac{f(s)}{\dot{g}(s)} \, \delta\{g(s)\} \, dg(s) = \frac{f(s)}{\dot{g}(s)}, \tag{49}$$

taken at the value of s in the range of integration that satisfies $g(s) = 0$, we can express (47) in the form

$$A_{\mu, \, ret} = 2e \int \dot{z}_\mu \, \delta(\mathbf{x} - \mathbf{z}, \mathbf{x} - \mathbf{z}) \, ds, \tag{50}$$

where the range of integration is from $-\infty$ to some value of s intermediate between the retarded and advanced times. We now have

$$\frac{\partial A_{\mu,\,\text{ret}}}{\partial x_\nu} = 4e\int \dot{z}_\mu \{x^\nu - z^\nu\}\,\delta'(\mathbf{x-z, x-z})\,ds$$

$$= -2e\int \frac{\dot{z}_\mu\{x^\nu - z^\nu\}}{(\dot{z},\mathbf{x-z})}\frac{d}{ds}\delta(\mathbf{x-z, x-z})\,ds$$

$$= 2e\int \frac{d}{ds}\left[\frac{\dot{z}_\mu\{x^\nu-z^\nu\}}{(\dot{z},\mathbf{x-z})}\right]\delta(\mathbf{x-z, x-z})\,ds,$$

so that
$$F_{\mu\nu,\,\text{ret}} = -2e\int \frac{d}{ds}\left[\frac{\dot{z}_\mu\{x_\nu - z_\nu\} - \dot{z}_\nu\{x_\mu - z_\mu\}}{(\dot{z},\mathbf{x-z})}\right]\delta(\mathbf{x-z, x-z})\,ds. \quad (51)$$

In the form of (47), this result becomes

$$F_{\mu\nu,\,\text{ret}} = -\frac{e}{(\dot{z},\mathbf{x-z})}\frac{d}{ds}\frac{\dot{z}_\mu\{x_\nu - z_\nu\} - \dot{z}_\nu\{x_\mu - z_\mu\}}{(\dot{z},\mathbf{x-z})} \quad (52)$$

taken at the retarded proper-time.

Let us now suppose the field point x_μ to lie very close to the world-line of the electron, so that we can put

$$x_\mu = z_\mu(s_0) + \gamma_\mu, \quad (53)$$

where the γ_μ's are very small. We shall evaluate the field quantities by means of Taylor expansions in which the coefficients depend on z_μ and its successive derivatives with respect to s, namely v_μ, $\dot{v}_\mu$, ..., taken at the proper-time s_0. Similar expansions have already been made by Page (1918, 1924), working with the Lorentz model, but the coefficients in his expansions involve the d/dt derivatives instead of the d/ds ones, which is not so convenient for a relativistic theory. For brevity, the value s_0 of the proper-time will not be mentioned explicitly in the coefficients, but will be understood.

Without loss of generality we may assume the point s_0 to be chosen so that

$$(\gamma\mathbf{v}) = 0 \quad (54)$$

for the value of $\mathbf{v}$ at the proper-time s_0. This equation will be much used in the future, together with the equations (15), (16) and (17).

Let the retarded proper-time, to be used in the right-hand side of (47) or (52), be $s_0 - \sigma$, σ being a small positive quantity of the same order as the γ_μ. We shall obtain expansions for the retarded field quantities, given by the right-hand side of (52), in powers of σ. These field quantities are of the order σ^{-2} and we shall need to have them correctly to the order of terms independent of σ, so we must retain terms of the first three orders in all

our expansions. To make the calculations clearer, commas will be inserted to separate terms of different orders of magnitude.

We have the Taylor expansions

$$x_\mu - z_\mu(s_0 - \sigma) = \gamma_\mu + \sigma v_\mu, \ -\tfrac{1}{2}\sigma^2 \dot{v}_\mu, \ +\tfrac{1}{6}\sigma^3 \ddot{v}_\mu, \tag{55}$$

$$\dot{z}_\mu(s_0 - \sigma) = v_\mu, \ -\sigma \dot{v}_\mu, \ +\tfrac{1}{2}\sigma^2 \ddot{v}_\mu. \tag{56}$$

These give, with the help of (15), (16), (17) and (54),

$$(\dot{z}, \mathbf{x} - \mathbf{z}) = \sigma, \ -\sigma(\gamma\dot{v}), \ +\tfrac{1}{2}\sigma^2(\gamma\ddot{v}) - \tfrac{1}{6}\sigma^3 \dot{v}^2,$$

and hence $(\dot{z}, \mathbf{x} - \mathbf{z})^{-1} = \sigma^{-1}[1 - (\gamma\dot{v})]^{-1}[1,, \ -\tfrac{1}{2}\sigma(\gamma\ddot{v}) + \tfrac{1}{6}\sigma^2 \dot{v}^2].$

It saves a little writing if we do not expand the factor $[1 - (\gamma\dot{v})]^{-1}$. We have again from (55) and (56)

$$\dot{z}_\mu\{x_\nu - z_\nu\} - \dot{z}_\nu\{x_\mu - z_\mu\}$$
$$= v_\mu\gamma_\nu, \ -\sigma\dot{v}_\mu\gamma_\nu - \tfrac{1}{2}\sigma^2\ddot{v}_\mu v_\nu, \ +\tfrac{1}{2}\sigma^2\ddot{v}_\mu\gamma_\nu + \tfrac{1}{3}\sigma^3\ddot{v}_\mu v_\nu -,$$

where the minus sign at the end indicates that we must subtract the terms obtained by interchanging μ and ν, so as to make the whole expression antisymmetrical in μ and ν. We now get

$$\frac{\dot{z}_\mu\{x_\nu - z_\nu\} - \dot{z}_\nu\{x_\mu - z_\mu\}}{(\dot{z}, \mathbf{x} - \mathbf{z})}$$
$$= [1 - (\gamma\dot{v})]^{-1}[\sigma^{-1}v_\mu\gamma_\nu, \ -\dot{v}_\mu\gamma_\nu - \tfrac{1}{2}\sigma\dot{v}_\mu v_\nu, \ -\tfrac{1}{2}(\gamma\ddot{v})v_\mu\gamma_\nu$$
$$+ \tfrac{1}{6}\sigma\dot{v}^2 v_\mu\gamma_\nu + \tfrac{1}{2}\sigma\ddot{v}_\mu\gamma_\nu + \tfrac{1}{3}\sigma^2\ddot{v}_\mu v_\nu -].$$

We have to differentiate this expression with respect to s. This is equivalent to differentiating with respect to σ and changing the sign, as we have not yet used the condition that fixes σ. Thus

$$\frac{d}{ds}\frac{\dot{z}_\mu\{x_\nu - z_\nu\} - \dot{z}_\nu\{x_\mu - z_\mu\}}{(\dot{z}, \mathbf{x} - \mathbf{z})}$$
$$= -[1 - (\gamma\dot{v})]^{-1}[-\sigma^{-2}v_\mu\gamma_\nu, \ -\tfrac{1}{2}\dot{v}_\mu v_\nu, \ +\tfrac{1}{6}\dot{v}^2 v_\mu\gamma_\nu + \tfrac{1}{2}\ddot{v}_\mu\gamma_\nu + \tfrac{2}{3}\sigma\ddot{v}_\mu v_\nu -],$$

and hence

$$F_{\mu\nu, \text{ret}} = e[1 - (\gamma\dot{v})]^{-2}[-\sigma^{-3}v_\mu\gamma_\nu, \ -\tfrac{1}{2}\sigma^{-1}\dot{v}_\mu v_\nu, \ +\tfrac{1}{2}\sigma^{-2}(\gamma\ddot{v})v_\mu\gamma_\nu$$
$$+ \tfrac{1}{2}\sigma^{-1}\ddot{v}_\mu\gamma_\nu + \tfrac{2}{3}\ddot{v}_\mu v_\nu -]. \tag{57}$$

We must now determine the value of σ. From (48) and (55)

$$\gamma^2 + \sigma^2, \ -\sigma^2(\gamma\dot{v}), \ +\tfrac{1}{3}\sigma^3(\gamma\ddot{v}) - \tfrac{1}{12}\sigma^4\dot{v}^2 = 0. \tag{58}$$

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Since γ is a space-like vector, γ^2 is negative. We put it equal to $-\epsilon^2$, ϵ being a positive number. To solve equation (58) for σ, we note that $\sigma = \epsilon$ to the first order, so that, without spoiling the accuracy of the equation, we may rewrite it

$$-\epsilon^2 + \sigma^2, \; -\sigma^2(\gamma\dot{v}), \; +\tfrac{1}{3}\epsilon^3(\gamma\ddot{v}) - \tfrac{1}{12}\epsilon^4\ddot{v}^2 = 0.$$

Hence
$$\sigma^2 = [1 - (\gamma\dot{v})]^{-1}[\epsilon^2,, \; -\tfrac{1}{3}\epsilon^3(\gamma\ddot{v}) + \tfrac{1}{12}\epsilon^4\ddot{v}^2],$$

so that
$$\sigma = \epsilon[1 - (\gamma\dot{v})]^{-\frac{1}{2}}[1,, \; -\tfrac{1}{6}\epsilon(\gamma\ddot{v}) + \tfrac{1}{24}\epsilon^2\ddot{v}^2]. \tag{59}$$

Substituting this expression for σ in (57), we get

$$F_{\mu\nu,\,\text{ret}} = e[1 - (\gamma\dot{v})]^{-\frac{1}{2}}\{-\epsilon^{-3}v_\mu\gamma_\nu, \; -\tfrac{1}{2}\epsilon^{-1}\dot{v}_\mu v_\nu[1, +(\gamma\dot{v})]$$
$$+ \tfrac{1}{8}\epsilon^{-1}\dot{v}^2 v_\mu\gamma_\nu + \tfrac{1}{2}\epsilon^{-1}\ddot{v}_\mu\gamma_\nu + \tfrac{2}{3}\ddot{v}_\mu v_\nu - \}. \tag{60}$$

This gives us the retarded field close to the world-line of the electron. To obtain the advanced field at the same field point, it is easily seen that we must change the sign of ϵ in the right-hand side of (60) and also change the sign of the whole expression. Thus the mean of $F_{\mu\nu,\,\text{ret}}$ and $F_{\mu\nu,\,\text{adv}}$ will be given by the terms of odd powers of ϵ in (60), and half their difference, $\tfrac{1}{2}(F_{\mu\nu,\,\text{ret}} - F_{\mu\nu,\,\text{adv}})$, will be given by the terms of even powers of ϵ. The only terms of even powers are the last term written down explicitly and the term obtained from it by interchanging μ and ν. These remain finite on the world-line of the electron and give the result (12) for the value of $F_{\mu\nu,\,\text{ret}} - F_{\mu\nu,\,\text{adv}}$ on the world-line.

With $f_{\mu\nu}$ defined by (13), we now have for the actual field close to the world-line

$$F_{\mu\nu} = e[1 - (\gamma\dot{v})]^{-\frac{1}{2}}\{[\epsilon^{-3} - \tfrac{1}{8}\epsilon^{-1}\dot{v}^2][\gamma_\mu v_\nu - \gamma_\nu v_\mu],$$
$$+ \tfrac{1}{2}\epsilon^{-1}[1 + (\gamma\dot{v})][v_\mu\dot{v}_\nu - v_\nu\dot{v}_\mu], + \tfrac{1}{2}\epsilon^{-1}[\ddot{v}_\mu\gamma_\nu - \ddot{v}_\nu\gamma_\mu]\} + f_{\mu\nu}.$$

We shall evaluate $F_{\mu\nu}F^\nu_\rho$ to the accuracy of ϵ^{-2}. We note first that

$$[\gamma_\mu v_\nu - \gamma_\nu v_\mu][\gamma^\nu v_\rho - \gamma_\rho v^\nu] = -\gamma_\mu\gamma_\rho + \epsilon^2 v_\mu v_\rho$$
$$[\gamma_\mu v_\nu - \gamma_\nu v_\mu][v^\nu\dot{v}_\rho - v_\rho\dot{v}^\nu] = \gamma_\mu\dot{v}_\rho + (\gamma\dot{v})v_\mu v_\rho$$
$$[\gamma_\mu v_\nu - \gamma_\nu v_\mu][\ddot{v}^\nu\gamma_\rho - \ddot{v}_\rho\gamma^\nu] = -\ddot{v}^2\gamma_\mu\gamma_\rho - (\gamma\ddot{v})v_\mu\gamma_\rho - \epsilon^2 v_\mu\ddot{v}_\rho$$
$$[v_\mu\dot{v}_\nu - v_\nu\dot{v}_\mu][v^\nu\dot{v}_\rho - v_\rho\dot{v}^\nu] = -\dot{v}^2 v_\mu v_\rho - \dot{v}_\mu\dot{v}_\rho.$$

With the help of these results we obtain

$$F_{\mu\nu}F^\nu_\rho = e^2[1 - (\gamma\dot{v})]^{-1}\{[\epsilon^{-6} - \tfrac{1}{4}\epsilon^{-4}\dot{v}^2][-\gamma_\mu\gamma_\rho + \epsilon^2 v_\mu v_\rho]$$
$$+ \tfrac{1}{2}\epsilon^{-4}[1 + (\gamma\dot{v})][\gamma_\mu\dot{v}_\rho + \gamma_\rho\dot{v}_\mu + 2(\gamma\dot{v})v_\mu v_\rho] - \tfrac{1}{2}\epsilon^{-4}[2\dot{v}^2\gamma_\mu\gamma_\rho$$
$$+ (\gamma\ddot{v})(v_\mu\gamma_\rho + v_\rho\gamma_\mu) + \epsilon^2(v_\mu\ddot{v}_\rho + v_\rho\ddot{v}_\mu)] - \tfrac{1}{4}\epsilon^{-2}[\dot{v}^2 v_\mu v_\rho + \dot{v}_\mu\dot{v}_\rho]\}$$
$$+ e\epsilon^{-3}\{[\gamma_\mu v_\nu - \gamma_\nu v_\mu]f^\nu_\rho + [\gamma^\nu v_\rho - \gamma_\rho v^\nu]f_{\mu\nu}\}.$$

Contracting, we find

$$F_{\mu\nu}F^{\nu\mu} = e^2[1-(\boldsymbol{\gamma}\dot{\mathbf{v}})]^{-1}\{[\epsilon^{-6}-\tfrac{1}{4}\epsilon^{-4}\dot{\mathbf{v}}^2]\,2\epsilon^2 + \tfrac{1}{2}\epsilon^{-4}[1+(\boldsymbol{\gamma}\dot{\mathbf{v}})]\,4(\boldsymbol{\gamma}\dot{\mathbf{v}})$$
$$-\tfrac{1}{2}\epsilon^{-4}[-2\epsilon^2\dot{\mathbf{v}}^2 + 2\epsilon^2(\mathbf{v}\ddot{\mathbf{v}})] - \tfrac{1}{2}\epsilon^{-2}\dot{\mathbf{v}}^2\} + 2e\epsilon^{-3}[\gamma_\mu v_\nu - \gamma_\nu v_\mu]f^{\nu\mu}$$
$$= e^2[1-(\boldsymbol{\gamma}\dot{\mathbf{v}})]^{-1}\{2\epsilon^{-4}[1+(\boldsymbol{\gamma}\dot{\mathbf{v}})+(\boldsymbol{\gamma}\dot{\mathbf{v}})^2]+\epsilon^{-2}\dot{\mathbf{v}}^2\} + 4e\epsilon^{-3}\gamma_\mu v_\nu f^{\nu\mu}.$$

We can now get the stress tensor $T_{\mu\rho}$ defined by (14). However, we shall not require the complete tensor, but only its component in the direction of $\boldsymbol{\gamma}$, namely $-\epsilon^{-1}T_{\mu\rho}\gamma^\rho$. We have

$$4\pi T_{\mu\rho}\gamma^\rho = F_{\mu\nu}F^\nu_{\ \rho}\gamma^\rho - \tfrac{1}{4}F_{\alpha\beta}F^{\beta\alpha}\gamma_\mu$$
$$= e^2[1-(\boldsymbol{\gamma}\dot{\mathbf{v}})]^{-1}\{[\epsilon^{-4}-\tfrac{1}{4}\epsilon^{-2}\dot{\mathbf{v}}^2]\gamma_\mu + \tfrac{1}{2}\epsilon^{-4}[1+(\boldsymbol{\gamma}\dot{\mathbf{v}})][(\boldsymbol{\gamma}\dot{\mathbf{v}})\gamma_\mu - \epsilon^2\dot{v}_\mu]$$
$$+ \epsilon^{-2}\dot{\mathbf{v}}^2\gamma_\mu - \tfrac{1}{4}\epsilon^{-2}(\boldsymbol{\gamma}\dot{\mathbf{v}})\dot{v}_\mu - [\tfrac{1}{2}\epsilon^{-4} + \tfrac{1}{2}\epsilon^{-4}(\boldsymbol{\gamma}\dot{\mathbf{v}}) + \tfrac{1}{2}\epsilon^{-4}(\boldsymbol{\gamma}\dot{\mathbf{v}})^2 + \tfrac{1}{4}\epsilon^{-2}\dot{\mathbf{v}}^2]\gamma_\mu\}$$
$$+ e\epsilon^{-3}\{\gamma_\mu v_\nu f^\nu_{\ \rho}\gamma^\rho + \epsilon^2 v^\nu f_{\mu\nu} - \gamma_\rho v_\nu f^{\nu\rho}\gamma_\mu\}$$
$$= e^2[1-(\boldsymbol{\gamma}\dot{\mathbf{v}})]^{-1}\{[\tfrac{1}{2}\epsilon^{-4}+\tfrac{1}{2}\epsilon^{-2}\dot{\mathbf{v}}^2]\gamma_\mu - \tfrac{1}{2}\epsilon^{-2}[1+\tfrac{3}{2}(\boldsymbol{\gamma}\dot{\mathbf{v}})]\dot{v}_\mu\} + e\epsilon^{-1}v^\nu f_{\mu\nu}.$$
$$(61)$$

We must now determine the flow of energy and momentum out from the tube introduced on page 153. The surface of this tube is described by that equation which is the result of eliminating s from the two equations

$$(\mathbf{x}-\mathbf{z},\mathbf{x}-\mathbf{z}) = -\epsilon^2 \tag{62}$$

$$(\mathbf{x}-\mathbf{z},\mathbf{v}) = 0, \tag{63}$$

in which $\mathbf{z}$ and $\mathbf{v}$ are considered as functions of s. Let us make a variation of the point x_μ on the surface to the point $x_\mu + dx_\mu$, also on the surface, and let us suppose this corresponds to s varying by an amount ds. We then have, varying (62) and (63),

$$(\mathbf{x}-\mathbf{z},d\mathbf{x}-\mathbf{v}\,ds) = 0,$$

$$(d\mathbf{x}-\mathbf{v}\,ds,\mathbf{v}) + (\mathbf{x}-\mathbf{z},\dot{\mathbf{v}}\,ds) = 0,$$

which reduce, with the help of (63) and (15), to

$$(\mathbf{x}-\mathbf{z},d\mathbf{x}) = 0$$

$$(d\mathbf{x},\mathbf{v}) = [1-(\mathbf{x}-\mathbf{z},\dot{\mathbf{v}})]\,ds.$$

Putting the $\mathbf{x}-\mathbf{z}$ here equal to $\boldsymbol{\gamma}$, in agreement with equation (53), we have

$$(\boldsymbol{\gamma},d\mathbf{x}) = 0 \tag{64}$$

$$(d\mathbf{x},\mathbf{v}) = [1-(\boldsymbol{\gamma}\dot{\mathbf{v}})]\,ds. \tag{65}$$

Equation (64) shows that the normal to the surface is in the direction of the vector γ, and equation (65) shows that if we split up $\mathbf{dx}$ into a part $\mathbf{d_1 x}$ orthogonal to $\mathbf{v}$ and a part $\mathbf{d_2 x}$ parallel to $\mathbf{v}$, then the magnitude of $\mathbf{d_2 x}$ is

$$|\mathbf{d_2 x}| = [1 - (\gamma \dot{\mathbf{v}})]\, ds. \tag{66}$$

Now the three-dimensional "area" of an element of the surface is equal to the two-dimensional area, dS say, of an element of a section of the surface by a three-dimensional plane orthogonal to $\mathbf{v}$, multiplied by the element $|\mathbf{d_2 x}|$ parallel to $\mathbf{v}$. Hence the flow of energy and momentum out through the surface is

$$\iint -\epsilon^{-1} T_{\mu\rho}\gamma^\rho \, dS \, |\mathbf{d_2 x}|$$
$$= -(4\pi)^{-1}\iint \left[e^2\{(\tfrac{1}{2}\epsilon^{-5} + \tfrac{1}{2}\epsilon^{-3}\dot{\mathbf{v}}^2)\gamma_\mu - \tfrac{1}{2}\epsilon^{-3}[1 + \tfrac{3}{2}(\gamma\dot{\mathbf{v}})]\dot{v}_\mu\} + e\epsilon^{-2}v^\nu f_{\mu\nu} \right] dS\, ds, \tag{67}$$

from (61) and (66), with neglect of terms that vanish with ϵ. The integration with respect to dS here is just integration over the surface of the sphere in three-dimensional space given by (62) and (63) for a particular s. The terms linear in γ in the integrand will contribute nothing to the integral, while the remaining terms will get multiplied by the area of the sphere, namely $4\pi\epsilon^2$. Thus (67) becomes

$$\int \{\tfrac{1}{2}e^2\epsilon^{-1}\dot{v}_\mu - ev^\nu f_{\mu\nu}\}\, ds, \tag{68}$$

which is the result used on page 153.

Summary

The object of the paper is to set up in the classical theory a self-consistent scheme of equations which may be used to calculate all the results that can be obtained from experiment about the interaction of electrons and radiation. The electron is treated as a point charge and the difficulties of the infinite Coulomb energy are avoided by a procedure of direct omission or subtraction of unwanted terms, somewhat similar to what has been used in the theory of the positron. The equations obtained are of the same form as those already in current use, but in their physical interpretation the finite size of the electron reappears in a new sense, the interior of the electron being a region of space through which signals can be transmitted faster than light.

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Page, L. 1918 *Phys. Rev.* **11**, 376–400.
— 1924 *Phys. Rev.* **24**, 296–305.
Schott, G. A. 1915 *Phil. Mag.* **29**, 49–62.

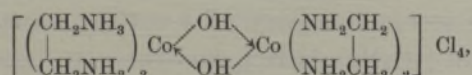# The crystal structure of certain bridged palladium compounds

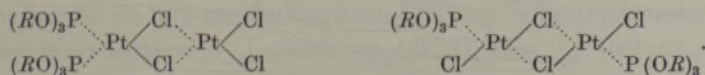### By A. F. Wells

Crystallographic Laboratory, Cambridge

Introduction

Many compounds have been formulated with metal atoms "bridged" by non-metallic atoms or groups such as halogen or hydroxyl. The poly-nuclear cobaltammines provide many examples, e.g.

$$\left[\left(\begin{matrix}CH_2NH_3\\ |\\ CH_2NH_2\end{matrix}\right)_2 Co \begin{matrix}OH\\ OH\end{matrix} Co \left(\begin{matrix}NH_2CH_2\\ |\\ NH_2CH_2\end{matrix}\right)_2\right] Cl_4,$$

and many years ago the phosphite ester derivatives of palladous and platinous chlorides, $(RO)_3P . Pd(Pt)Cl_2$, were shown to have molecular weights twice those of the empirical formulae. This led to their formulation (Werner and Pfeiffer 1923) as bridged derivatives with possible isomerism:

$$\begin{matrix}(RO)_3P\\ (RO)_3P\end{matrix} Pt \begin{matrix}Cl\\ Cl\end{matrix} Pt \begin{matrix}Cl\\ Cl\end{matrix} \qquad \begin{matrix}(RO)_3P\\ Cl\end{matrix} Pt \begin{matrix}Cl\\ Cl\end{matrix} Pt \begin{matrix}Cl\\ P(OR)_3\end{matrix}.$$

Such molecules have a definite existence in the dissolved state and undergo substitution in much the same way as mononuclear cobaltammines. There also exist salts of the type of caesium thallic enneachloride $Cs_3Tl_2Cl_9$ and the corresponding $K_3W_2Cl_9$, and here it is found that a complex ion $(Tl_2Cl_9^{3-}$