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Classical limit of quantum electrodynamics

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Employing the path-integral formulation of quantum electrodynamics, we perform a functional integration to eliminate the photon coordinates. The classical limit (including radiation reaction) emerges when this is done subject to the conditions that the initial photon state is the vacuum while the final photon state is the coherent radiation field.

I. INTRODUCTION

After mass and charge renormalizations are performed, the predictions of QED are in remarkable agreement with experimental results. These accomplishments, however, should not allow physicists to ignore problems that still exist at the foundations of the theory. This paper deals with one such problem: Is classical electrodynamics (including radiation-reaction effects) the correspondence limit of QED?

Why is it important to establish the classical limit of QED? One obvious justification is that the problem remains unsolved.[1] Secondly, one may expect to learn something new from the classical theory about the possible modifications of the quantum field theory.

This paper is an application of some ideas presented by Feynman on eliminating field oscillators in QED.[2] The following sections use the same notation as used by Feynman in that reference, with the exception that we have set $\hbar = c = 1$. In Sec. II, the path-integral form of quantum mechanics is applied to the problem of one particle interacting with one oscillator. Section III completes the work by functionally integrating out the photon coordinates (subject to initial vacuum and final radiation) and displaying the correspondence limit. The final section, IV, contains some concluding remarks.

II. ELIMINATION OF OSCILLATOR COORDINATES

Suppose a particle of coordinates $x(t)$ and Lagrangian $L_p = L_p(\dot{x}, x)$ interacts with an oscillator of coordinates $q(t)$ and Lagrangian $L_0 = \frac{1}{2}(\dot{q}^2 - \omega^2 q^2)$ through an interaction Lagrangian $L_I = \gamma(t) q(t)$. We want the transition amplitude for a particle being in state $\chi_{t''}$ at time t'' with the oscillator in eigenstate $\phi_m(q_{t''})$ [energy $E_m = (m+\frac{1}{2})\omega$], when it is known that at a previous time t', the particle was in state $\psi_{t'}$ and the oscillator in $\phi_n(q_{t'})$. Using the path-integral formulation of quantum mechanics,[3] the amplitude for this transition is

$$\langle \chi_{t''}\phi_m | 1 | \psi_{t'}\phi_n\rangle_{S_p+S_0+S_I} = \langle \chi_{t''} | G_{mn} | \psi_{t'}\rangle_{S_p}, \tag{1}$$

where G_{mn} is a functional of the particle path $x(t)$ given by

$$\begin{aligned} G_{mn} &= \langle \phi_m | \exp[i\textstyle\int q(t)\gamma(x(t))dt]\,\phi_n\rangle_{S_0} \\ &= \lim_{\epsilon=0} \int \phi_m^*(q_j) \exp\left[i\epsilon \sum_{i=0}^{j-1} \frac{1}{2\epsilon^2}(q_{i+1}-q_i)^2 - \tfrac{1}{2}\omega^2 q_i^2 + q_i\gamma(x(t_i))\right] \phi_n(q_0)\,dq_0 a^{-1} dq_1 a^{-1}\cdots a^{-1}dq_j \\ &\equiv \int \phi_m^*(q_{t''})\exp[i(S_0+S_I)]\,\phi_n(q_{t'})\,dq_{t'}dq_{t''}D(q), \end{aligned} \tag{2}$$

where $a = (2\pi i\epsilon h)^{1/2}$, $t'' - t' = j\epsilon$, $q_{t'} = q_0$, and $q_{t''} = q_j$; in the last line we have explicitly indicated what is meant by a functional integration [denoted by $D(q)$].

The evaluation of G_{mn} was completed by Feynman.[2] His result is

$$G_{mn} = (G_{00})\frac{1}{(m!\,n!)^{1/2}} \sum_r \frac{m!}{(m-r)!\,r!}\frac{n!}{(n-r)!\,r!}\, r!\,(i\beta^*)^{m-r}(i\beta)^{n-r}, \tag{3}$$

where

$$\beta = \frac{1}{\sqrt{2\omega}}\int_{t'}^{t''}\gamma(t')e^{-i\omega t'}dt', \quad \beta^* = \frac{1}{\sqrt{2\omega}}\int_{t'}^{t''}\gamma(t')e^{i\omega t'},$$

and

$$G_{00} = \exp\left[-\frac{1}{2\omega}\int_{t'}^{t''}\int_{t'}^{t}\exp[-i\omega(t-s)]\gamma(t)\gamma(s)\,dt\,ds\right]$$

for arbitrary t', t''. The sum on r goes from 0 to n or m, whichever is the smaller.

For what follows, we will be interested in transition amplitudes from $t' = -\infty$ to $t'' = +\infty$. For this situation,

$$G_{mn} = (G_{00})\frac{1}{(m!\,n!)^{1/2}}\sum_r \frac{m!}{(m-r)!\,r!}\,\frac{n!}{(n-r)!\,r!}\,r!(\alpha)^{m-r}(-\alpha^*)^{n-r}, \tag{4a}$$

where

$$G_{00} = \exp\left[-\frac{1}{4\omega}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp(-i\omega|t-s|)\gamma(t)\gamma(s)\,dt\,ds\right] \tag{4b}$$

and

$$\alpha = \frac{i}{\sqrt{2\omega}}\int_{-\infty}^{\infty}\gamma(t)e^{i\omega t}\,dt. \tag{4c}$$

With this last expression, we have written the amplitude for a particle transition from $\psi(t=-\infty)$ to $\chi(t=+\infty)$ when it is specified that the interacting oscillator goes from ϕ_n to ϕ_m. We will explicitly work out two cases.

(1) The oscillator goes from its ground state ϕ_0 to its ground state ϕ_0. This amplitude is given by

$$\begin{aligned}\langle \chi | G_{00} | \psi\rangle_{S_p} &= \int \chi_{t''}(x_{t''})\exp(iS_p)G_{00}\psi_{t'}(x_{t'})D(x(t))\,dx_{t''}\,dx_{t'}\\ &= \int \chi_{t''}(x_{t''})\exp[i(S_p + I_{00})]\psi_{t'}(x_{t'})D(x)\,dx_{t''}\,dx_{t'},\end{aligned} \tag{4d}$$

where $t' = -\infty$, $t'' = \infty$, and

$$I_{00} = \frac{i}{4\omega}\int\int\exp(-i\omega|t-s|)\gamma(t)\gamma(t)\,ds\,dt. \tag{5}$$

The effect of the coupling to the oscillator is contained in I_{00}; the particle's coupling to the oscillator (oscillator goes from ground state to ground state) requires that we compute particle transition amplitudes using an effective action $S_{\text{eff}} = S_p + I_{00}$. It describes the particle's ability at one time (t) to affect itself at a different time (s) by means of a temporary storage of energy in the oscillator (notice that the regions $S>t$, $S<t$ both contribute to I_{00}).

(2) The oscillator goes from its ground state, ϕ_0, to the coherent state generated by the external force $\gamma(t)$.[4] This coherent state is given by

$$|\alpha(t)\rangle \equiv e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\phi_n e^{-i(n+1/2)\omega t}$$

with $\{\phi_n\}$ eigenstates of the harmonic oscillator and α defined in (4c). This particle transition amplitude is given as

$$\begin{aligned}\langle\chi|G_{\alpha 0}|\psi\rangle_{S_p} &= \langle \chi|\textstyle\sum\langle\alpha|n\rangle G_{n0}|\psi\rangle_{S_p}\\ &= \int \chi_{t''}(x_{t''})\\ &\quad\times\exp[i(S_p + I_{\alpha 0})]\psi_{t'}(x_{t'})D(x)\,dx_{t''}\,dx_{t'}\end{aligned}$$

with

$$I_{\alpha 0} = \frac{1}{2\omega}\int_{-\infty}^{\infty}ds\int_{-\infty}^{s}dt\,\sin[\omega(s-t)]\gamma(s)\gamma(t) \tag{6}$$

supplementing the particle action S_p. Once again, particle transition amplitudes are computed using $S'_{\text{eff}} = S_p + I_{\alpha 0}$; this describes the particle's ability at time (t) to affect its motion at later times (s) by temporarily storing energy in and permanently transferring energy to the interacting oscillator (notice that now only the region $s \geq t$ contributes to $I_{\alpha 0}$).

If several independent oscillators (coordinates $\{q_r\}$) interact with the particle, and if each oscillator makes a ground-state to ground-state transition, then particle transition amplitudes are calculated with

$$S_{\text{eff}} = S_p + \sum_r I_{00}^r,$$

where each independent oscillator contributes a term I_{00}^r. When each oscillator makes a transition from ground state to a coherent state, then

$$S_{\text{eff}} = S_p + \sum_r I_{\alpha_r,0}^r.$$

It is even possible to consider a situation in which some oscillators (index t) go from ground state to ground state while others (index r) go from ground state to coherent states. For this case,

$$S_{\text{eff}} = S_p + \sum_r I_{\alpha_r,0}^r + \sum_t I_{0,0}^t.$$

III. ELIMINATION OF FIELD OSCILLATIONS IN QED

The Lagrangian for the QED of point charged particles[2] is $L = (L_p + L_c) + (L_{tr} + L_I)$ and the corresponding action is $S = (S_p + S_c) + (S_{tr} + S_I)$, where

$$L_p = \frac{1}{2}\sum_n m\dot{x}_n^2 \tag{7a}$$

$$L_c = -\frac{1}{2}\sum_{n,m}\frac{e_n e_m}{r_{nm}} \tag{7b}$$

$$(L_{tr} + L_I) = \sum_{k,r}\frac{1}{2}[\dot{q}_k^{(r)2} - k^2 q_k^{(r)2}] + \gamma_k^{(r)} q_k^{(r)}. \tag{7c}$$

Feynman has shown[2] that once the transverse (tr) field oscillator coordinates q_k^r are functionally integrated out, the particle transition amplitudes should be calculated using an effective action given by

$$S_{\text{eff}} = (S_p + S_c) + \sum_{k,r} I_k^r,$$

where there is one term I_k^r for each transverse field mode. The exact form for each I_k^r depends on the specification of initial and final photon states; we will work out two cases.

(1) The transverse field oscillators go from their initial ground state to a final ground state;[2] all photons are virtual. In this situation, each I_k^r is given by [see Eq. (5)]

$$I_k^r = \frac{i}{4|k|}\int\int \exp(-i\omega|t-s|)\gamma_k^r(s)\gamma_k^r(t)\,ds\,dt$$

and

$$S_{\text{eff}} = (S_p + S_c) + \sum_k\sum_r I_k^r .$$

Substituting in the expressions for γ_k^r and performing the summations over k and r,

$$\sum_k \to \frac{1}{2}\int\frac{d^3k}{(2\pi)^3},$$

we find

$$\sum_k\sum_r I_k^r = I_{tr} = i\sum_{n\,m}\int_{t'\to-\infty}^{t''\to\infty} dt\int_{t'\to-\infty}^{t''\to\infty} ds\int\frac{d^3k}{(8\pi^2)k}\, e_n e_m \exp(-ik|t-s|)$$
$$\times[\dot{x}_n(t)\cdot\dot{x}_m(s) - k^{-2}k\cdot\dot{x}_n(t)k\cdot\dot{x}_m(s)]\cos\{\vec{k}\cdot[\vec{x}_n(t)-\vec{x}_m(s)]\}. \tag{8}$$

Following Feynman[2] we break up the expression by performing an integration by parts on the term in $k\cdot\dot{x}_n(t)$:

$$I_{tr} = R - I_c + I_{\text{transient}},$$

where

$$R = -i\sum_n\sum_m\int_{t'=-\infty}^{t''=-\infty}\int_{t'=-\infty}^{t''=\infty} e_n e_m \exp(-ik|t-s|)[1-\dot{x}_n(t)\cdot\dot{x}_m(s)]\cos\{\vec{k}\cdot[\vec{x}_n(t)-\vec{x}_m(s)]\}\frac{d^3k}{8\pi^2 k}, \tag{9}$$

and

$$I_c = -\sum_n\sum_m\int_{t'\to-\infty}^{t''\to\infty} dt\int e_n e_m \cos\{\vec{k}\cdot[\vec{x}_n(t)-\vec{x}_m(t)]\}\frac{d^3k}{4\pi^2k^2} \tag{10}$$

comes from the discontinuity in the slope $\exp(-ik|t-s|)$ at $t=s$. Using the identity

$$\int\cos(\vec{k}\cdot\vec{R})\frac{d^3k}{4\pi^2k^2} = \frac{1}{2|R|},$$

it is easy to show that I_c just cancels the term S_c in the effective action. The final term, $I_{\text{transient}}$, is a boundary term left from the parts integration; Feynman has shown this term to be zero as $t'\to-\infty$ and as $t''\to\infty$.

With these results, the effective action for particles interacting with virtual transitions in the transverse field becomes

$$S_{\text{eff}} = (S_p + S_c) + (R - I_c + I_{\text{transient}}) = S_p + R.$$

Feynman[2] simplifies the expression for R and eventually arrives at

$$R = -\frac{1}{2}\sum_n\sum_m\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e_n e_m[1-\dot{x}_n(t)\cdot\dot{x}_m(s)]\{\delta_+((t-s)^2 - [x_n(t)-x_m(s)]^2)\}dt\,ds, \tag{11}$$

with

$$\delta_+(x) \equiv \delta(x) - \frac{i}{\pi}\frac{P}{x}\quad\text{(P denotes principal value).}$$

To calculate particle transition elements with (virtual) electromagnetic interaction included, we use

$$\langle \chi_{t''\to\infty}| \exp(iR)|\psi_{t'\to-\infty}\rangle_{S_p}. \qquad (12)$$

Feynman has shown that this last expression contains the effects of virtual quanta on a particle system according to QED. As it stands, it is an exact expression to all orders of e^2; if it is expanded in a power series in e^2, the conventional diagrammatical methods emerge.

(2) Rather than constrain the final state for the electromagnetic field to be the vacuum, we now calculate transition amplitudes for a situation where initially there is a photon vacuum and finally there is a "coherent radiation field of photons" generated by the particle currents.[4] This situation corresponds to what actually happens in any real experiment with accelerating charges. With these initial and final conditions specified,

$$S_{\text{eff}} = (S_p + S_c) + \sum_k \sum_r I^r_{k\,\text{rad}}$$

where [see Eq. (6)]

$$I^r_{k_{\text{rad}}} = \frac{1}{2|k|} \int_{t'\to-\infty}^{t''\to\infty} ds \int_{t'\to-\infty}^{s} dt \, \sin[\,|k|(s-t)]\gamma^r_k(s)\gamma^r_k(t).$$

Substituting in the expressions[2] for γ^r_k and performing the k and r summations

$$\sum_k \to \frac{1}{2} \int \frac{d^3k}{(2\pi)^3}$$

yields

$$\sum_k\sum_r I^r_{k,\text{rad}} = I_{\text{tr, rad, 0}}$$

$$= 2\sum_n\sum_m \int_{t'\to-\infty}^{t''\to\infty} ds \int_{t'\to-\infty}^{s} dt \; e_n e_m \sin[k(s-t)]$$

$$\times[\dot{x}_n(t)\cdot\dot{x}_m(t) - k^{-2}\, k\cdot\dot{x}_n(t)k\cdot\dot{x}_m(s)]\,\frac{\cos\{\vec{k}\cdot[\vec{x}_n(t)-\vec{x}_m(s)]\}}{8\pi^2 k}\, d^3k. \qquad (13)$$

Once again it proves convenient to break $I_{\text{tr, rad, 0}}$ into three parts:

$$I_{\text{tr, rad, 0}} = R_{\text{rad, 0}} - I_c + I_{\text{transient, rad, 0}},$$

where

$$R_{\text{rad, 0}} = -2\sum_n\sum_m \int_{t'\to-\infty}^{t''\to\infty} ds \int_{t'\to-\infty}^{s} dt \; e_n e_m \sin[k(s-t)]\,[1-\dot{x}_n(t)\cdot\dot{x}_m(s)]\cos\{\vec{k}\cdot[\vec{x}_n(t)-\vec{x}_m(s)]\}\,\frac{d^3k}{8\pi^2 k} \qquad (14)$$

and

$$(-I_c + I_{\text{transient, rad, 0}}) = -2\sum_n\sum_m \int_{t'}^{t''} ds \int_{t'}^{t''} dt \; e_n e_m \frac{\sin[k(s-t)]}{k^2}\,[k\cdot\dot{x}_n(t)k\cdot\dot{x}_m(s)]\,\frac{\cos\{\vec{k}\cdot[\vec{x}_n(t)-\vec{x}_m(s)]\}}{8\pi^2 k}$$

$$+ 2\sum_n\sum_m \int_{t'}^{t''} ds \int_{t'}^{s} dt \; e_n e_m \sin[k(s-t)]\,\cos\{\vec{k}\cdot[\vec{x}_n(t)-\vec{x}_m(s)]\}\,\frac{d^3k}{8\pi^2 k}. \qquad (15)$$

Integrating the last term by parts allows us to identify the boundary term at $t = s$ as

$$-I_c = \sum_{n,m}\sum e_n e_m \int_{t'\to-\infty}^{t''\to\infty} ds \int d^3k \frac{\cos\{\vec{k}\cdot[\vec{x}_n(s)-\vec{x}_m(s)]\}}{4\pi^2 k^2}. \qquad (16)$$

This terms cancels S_c in S_{eff}. The remaining terms are easily simplified to give

$$I_{\text{transient, rad, 0}} = \sum_{n,m}\sum e_n e_m\Big[-\int_{t'\to-\infty}^{t''\to\infty} ds \frac{\cos[k(s-t')]}{k} \cos\{\vec{k}\cdot[\vec{x}_n(t')-\vec{x}_m(s)]\}\,\frac{d^3k}{4\pi^2 k}$$

$$+ \int_{t'\to-\infty}^{t''\to\infty} dt \frac{\sin[k(t''-t)]}{k^2}\,\vec{k}\cdot\dot{\vec{x}}_n(t)\frac{\sin\{\vec{k}\cdot[\vec{x}_n(t)-\vec{x}_m(t'')]\}}{4\pi^2 k}\, d^3k\Big]. \qquad (17)$$

Paralleling the treatment of Feynman,[2] we argue that for t' and t'' exceedingly far in the past and future "there is no correlation to be expected between these temporally distant coordinates and the present ones, so the effects of $I_{\text{transient}}$ will cancel out quantum mechanically by interference." Feynman was referring to a different $I_{\text{transient}}$, but his reasoning is appropriate for our $I_{\text{transient, rad, 0}}$. (These results are also supported by a method of solution employing an ordered operator calculus instead of functional integration methods; the details of the ordered operator approach will be given in a subsequent paper.)

Combining these results, the total complex action S_{eff} for the system becomes

$$S_{\text{eff}} = S_p + R_{\text{rad, 0}},$$

where

$$R_{\text{rad, 0}} = -2\sum_m\sum_n e_n e_m \int_{t'\to-\infty}^{t''\to\infty} ds \int_{t'\to-\infty}^{s} dt \sin[k(s-t)][1-\dot{x}_n(t)\cdot\dot{x}_m(s)]\cos\{\vec{k}\cdot[\vec{x}_n(t)-\vec{x}_m(s)]\}\frac{d^3k}{8\pi^2 k}. \tag{18}$$

This last expression is simplified by using

$$\int \sin[k(s-t)]\cos(\vec{k}\cdot\vec{x})\frac{d^3k}{8\pi^2 k} = \int_0^\infty dk\,\frac{\sin[k(s-t)]\sin(k|x|)}{2\pi|x|} = \frac{1}{4|x|}[\delta(s-t-|x|)-\delta(s-t+|x|)].$$

This result gives the effective action to be used in calculating transition amplitudes for particles as

$$\begin{aligned} S_{\text{eff}} &= S_p + R_{\text{rad, 0}} \\ &= S_p - \frac{1}{2}\sum_n\sum_m e_n e_m \int_{-\infty}^{\infty} ds \int_{-\infty}^{s} dt\,\frac{1-\dot{x}_n(t)\cdot\dot{x}_m(s)}{|x_n(t)-x_m(s)|} \\ &\qquad \times\{\delta(t-[s-|x_n(t)-x_m(s)|])-\delta(t-[s+|x_n(t)-x_m(s)|])\}. \end{aligned} \tag{19}$$

This is just the expression which should arise in a classical electrodynamics calculation. When $|x_n(t)-x_m(s)|\neq 0$, only the first δ function will turn on, and it is evident that different particles separated from one another will interact via retarded Liénard-Wiechert potentials. For $m=n$ the possibility $|x_n(t)-x_n(s)|=0$ exists. In this case of radiation reaction we have to be more careful in interpreting the action

$$\begin{aligned} \{\text{self-action, } m=n \text{ term}\} &\equiv -\frac{1}{2}\sum_n e_n^2 \int_{-\infty}^{\infty} ds \int_{-\infty}^{s} dt\,\frac{1-\dot{x}_n(t)\cdot\dot{x}_n(s)}{|x_n(t)-x_n(s)|} \\ &\qquad \times\{\delta(t-[s-|x_n(t)-x_n(s)|])-\delta(t-[s+|x_n(t)-x_n(s)|])\}, \end{aligned} \tag{20}$$

since both δ functions can turn on at the end point s. Careful analysis of the integrand shows that the difficulty of a self-energy divergence is still present.[5]

IV. CONCLUSIONS

Employing the path-integral formulation of quantum electrodynamics, it is possible to functionally integrate out the photon coordinates. The classical limit emerges if this is done subject to the conditions that the initial photon state is the vacuum, while the final photon state is the coherent radiation field generated by particle currents. The resulting interactions of the particle currents are identifiable as different charged point particles interacting via retarded Liénard-Wiechert potentials. The radiation-reaction terms ($m=n$ in the sum) are difficult to interpret, but it appears that the self-fields close to the world line give a self-energy divergence.

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[5]A particular term in the self-action takes the form

$$I=\int_{-\infty}^{\infty} ds \int_{-\infty}^{s} dt\, f(s,t)$$

with $f(s,t)=-f(t,s)$. The integral can be written with a step function:

$$\begin{aligned} I &= \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} dt\, \theta(s-t) f(s,t) \\ &= \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} ds\, \theta(t-s) f(t,s) \\ &= -\int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} dt\, \theta(t-s) f(s,t) \end{aligned}$$

and therefore,

$$I=\frac{1}{2}\int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} dt [\theta(s-t)-\theta(t-s)] f(s,t).$$

In this form, a self-energy divergence from the region near $t=s$ is obvious.