Igor N. Toptygin

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Contents

Preface

This monograph presents the foundations and modern achievements of classical and quantum microscopic electrodynamics, combining classical and current results and classical and quantum mechanical formalisms. It is written for specialists of different levels – bachelors, masters, graduate students, and generally for researchers in various branches of physics, theoreticians, and experimentalists; it can also be useful for those who specialize in engineering. The book is self-sufficient and can used for self-education in various aspects of microscopic electrodynamics; it can serve as a collection of exercises, and it presents solutions to many specific problems described in a unified manner. The table of contents shows that the monograph contains a lot of material. This is achieved by the nontraditional composition of the monograph, which combines the style of a brief textbook and a collection of problems. The style is similar to that of the well-known monographs by Ryogo Kubo, that is, *Statistical Mechanics* (1965) and *Thermodynamics* (1968), but the material is different. Such a style is helpful for self-education and deeper study of the material.

The reader should not worry about the large number of pages. It is not obligatory to read all the material to be familiar with the basic laws of electromagnetic theory. Depending on the specific aims, some parts of the book can be skipped.

Chapter 1 contains the mathematical formalism which is widely used in theoretical physics. This material can be consulted whenever necessary. Physics is presented starting from Chapter 2. Each chapter contains material for theoretical studies and material for solving problems.

The material for theoretical studies takes up the smaller part of each chapter; it contains the formulations of the main principles and presents basic equations and other relations between physical quantities. Derivations of the majority of the mathematical relations are included in examples, and these are supplemented by detailed solutions. It is recommended to read these pages carefully and reproduce the derivations.

The material for solving problems takes up the larger part of each chapter. It contains formulations of problems, as well as answers with the analysis of the results and – in many cases – with solutions. Each chapter includes many problems (typically about 100). It is certainly not obligatory to solve all of them, but it is desirable to solve at least 10–15 problems from each chapter. This is recommended not only for theoreticians but for all physicists for better understanding of the basic relations. Especially useful problems are marked by a bold dot. Asterisk marks are used to give the reader an idea of the complexity of selected problems. High-complexity problems are marked with one asterisk and supercomplex problems with two asterisks.

Depending on the specific interests and level of study, some subjects can be skipped. Future bachelors can exclude from Section 2.3 the subsections on the partial polarization of waves and the analytical signal, and from Section 4.2 the subsections on the dynamics of orbital and spin magnetic moments and approximate methods. In addition, one can completely skip Sections 4.3 and 6.3 and Chapter 7, as well as some parts of Sections 5.3, 5.4, 6.1, and 6.2.

The master's program includes additional sections, for instance, those which combine classical and quantum mechanical approaches. They are Section 4.3 (Lagrangian and Hamiltonian methods in field theory), Sections 6.1 and 6.2 (coherent quantum states and related problems), and Section 6.3 and Chapter 7 (relativistic Dirac equation and invariant perturbation theory). Depending on the direction of the master's program, one may need other sections and problems. It would also be useful to look through additional literature, which is recommended at the end of each section. I have tried to cite the most valuable review articles and monographs.

The recommendations presented above can be useful for other potential readers, for instance, for engineers and young researchers who wish to improve their knowledge of electrodynamics. One should bear in mind that this monograph does not describe electromagnetic phenomena in matter. This important branch of electrodynamics is presented in another monograph: *Electromagnetic Phenomena in Matter. Statistical and Quantum Approaches.* Its publication by Wiley is currently in preparation. It is written in the same style and it also contains material of different levels.

While writing the present monograph, I tried to follow the best traditions of the Theoretical Physics Department of the St. Petersburg Polytechnical University in Russia. They have been formed since the 1920s by the founders of teaching theoretical physics at the Leningrad Polytechnic Institute, particularly, by A.A. Friedmann, V.R. Bursian, V.K. Frederiks, and Ya.I. Frenkel. In addition, I used my own experience of teaching theoretical physics at four physical faculties of the Saint Petersburg State Polytechnical University (the former Leningrad Polytechnic Institute) as well as my experience of writing earlier textbooks and monographs (*Problems in Electrodynamics* by V.V. Batygin and I.N. Toptygin, Fizmatgiz, 1962; *Classical Electrodynamics* by M.M. Bredov, V.V. Rumyantsev, and I.N. Toptygin, Nauka, 1985). Both books were published in several editions in Russian (with extensions and addenda). Moreover, they were translated and published in several languages.

The present monograph is essentially different from these earlier monographs. It is more general, contains a greater amount of material, and is updated with modern achievements, in the attempt to unify various branches of theoretical physics (sometimes expanding beyond electrodynamics itself) and make them most useful for scientists of various specializations. The author is very grateful to John Wiley & Sons for the decision to publish this monograph.

Initially, the present monograph was published in Russian by publishing house "Regular and Chaotic Dynamics" as volume 1 of a two-volume course entitled "Modern Electrodynamics". There were two Russian editions, in 2003 and 2005. In comparison with the Russian editions, the present monograph is greatly updated. Chapter 6 was updated with material concerning quantum memory, atomic spectra and other matter. Chapter 7 is entirely new and describes relativistic invariant perturbation theory; it contains some basic problems of quantum electrodynamics of relativistic systems. Other chapters have also been updated and corrected. Volume 2 of the Russian edition will also be published by Wiley, but as a separate and updated monograph.

The overall design and outline of the book were developed jointly with Professor V.V. Batygin, my good friend and coauthor of many publications, who made an important and valuable contribution to this work. Unfortunately, he passed away in 1998 and could not participate in the practical realization of the project. D.V. Kupriyanov and I.M. Sokolov wrote a considerable part of the theoretical material in Sections 6.1 and 6.2, and suggested many problems. I greatly appreciate the assistance of my colleagues from the A.F. Ioffe Physical Technical Institute and from Saint Petersburg Polytechnical University. Publication of the English version of this book would have been impossible without the active help and financial support by means of grants of A.M. Bykov, A.M. Krasilshchikov, D.G. Yakovlev, V. Zelevinsky, A.I. Tsygan, V.S. Beskin, V.V. Dubov, D.V. Kupriyanov, N.V. Larionov, I.M. Sokolov, and K.Yu. Platonov. I am grateful to the translators of the book– A.B. Nemtsev (Chapters 1 and 2), N.V. Larionov (Chapter 6), and especially Yu.V. Morozov (Chapters 3, 4, 5, and 7). Professor D.V. Kupriyanov edited the translation of Chapters 6 and 7. Undergraduate students A. Egorov and F. Savenkov checked the solutions of the problems; A. Dubov and F. Savenkov improved the presentation of Chapter 7. Special thanks are due to A. Egorov for drawing a large number of the figures. I am grateful to the Russian Ministry of Education and Science (project 11.G34.31.0001) for partial financial support of the translation. Support from the External Fellowship Program of the Russian Quantum Center (reference number 86) is also greatly appreciated.

Saint Petersburg, March 2013 *Igor N. Toptygin*

Fundamental Constants and Frequently Used Numbers

Ratio of the circumference of a circle to its diameter $\pi = 3.142$
Base of natural logarithms $e = 2.718\,282$
Speed of light in vacuum (maximum speed) $c = 299\,792\,458\ \mathrm{m/s} \approx 3 \times 10^{10}\ \mathrm{cm/s}$
Reduced Planck constant $\hbar = 1.055 \times 10^{-27}\ \mathrm{erg\,s}$
Electron rest mass $m_\mathrm{e} = 0.9110 \times 10^{-27}\ \mathrm{g}$
Proton rest mass $m_\mathrm{p} = 1.673 \times 10^{-24}\ \mathrm{g}$
Elementary charge $e_0 = 4.803 \times 10^{-10}\ \mathrm{statC}$
Fine structure constant $\alpha = 0.007\,297$; $\alpha^{-1} = 137.036$
Bohr radius $a_\mathrm{B} = 0.5292 \times 10^{-8}\ \mathrm{cm}$
Bohr magneton $\mu_\mathrm{B} = \frac{e_0\hbar}{2m_\mathrm{e}c} = 0.9274 \times 10^{-20}\ \mathrm{erg/G}$
Nuclear magneton $\mu_\mathrm{N} = \frac{e_0\hbar}{2m_\mathrm{p}c} = 0.5051 \times 10^{-23}\ \mathrm{erg/G}$
Neutron magnetic moment $\mu_\mathrm{n} = 1.913\mu_\mathrm{N}$
Proton magnetic moment $\mu_\mathrm{p} = 2.793\mu_\mathrm{N}$
Classical electron radius $r_0 = \frac{e_0^2}{m_\mathrm{e}c^2} = 2.818 \times 10^{-13}\ \mathrm{cm}$
Compton wavelength of an electron $\Lambda_\mathrm{C} = \frac{\hbar}{m_\mathrm{e}c} = 3.862 \times 10^{-11}\ \mathrm{cm}$
Compton wavelength of a proton $\Lambda_\mathrm{C} = \frac{\hbar}{m_\mathrm{p}c} = 2.103 \times 10^{-14}\ \mathrm{cm}$
1 erg $= 10^{-7}$ J $= 6.24 \times 10^{11}$ eV
1 eV $= 1.602 \times 10^{-12}$ erg
1 dyn $= 10^{-5}$ N
1 year $= 3.156 \times 10^{7}$ s
1 pc (parsec; unit of length used in astronomy) $= 3.086 \times 10^{18}$ cm
1 AU (astronomical unit – the mean distance between the Earth and the Sun) $= 1.496 \times 10^{13}$ cm
Mass of the Earth $M_\oplus = 5.976 \times 10^{27}\ \mathrm{g}$
Mass of the Sun $M_\odot = 1.989 \times 10^{33}\ \mathrm{g}$

Basic Notation

A	work
$\boldsymbol{A}$, $\mathbf{A}$	vector potential
a_{B}	Bohr radius
$\widehat{a}_s^\dagger$, $\widehat{a}_s$	creation and annihilation operators for Dirac particles
$\widehat{b}_s^\dagger$, $\widehat{b}_s$	creation and annihilation operators for Dirac antiparticles
c	speed of light in vacuum (limiting speed)
$\widehat{c}_s^\dagger$, $\widehat{c}_s$	creation and annihilation operators for photons
$\boldsymbol{d}$, $\boldsymbol{p}$	electric dipole moment.
∂_k, $k = 0, 1, 2, 3$	covariant components of a 4-gradient
e, q, Q	charge of a particle or macroscopic body
$\boldsymbol{e}$	unit vector
$\boldsymbol{E}$, $\mathbf{E}$	strength of the electric field
E, W	energy
$\mathcal{E}$	electromotive force; total energy of a relativistic particle
$\boldsymbol{F}$, $\boldsymbol{\mathcal{F}}$	force
$\boldsymbol{f}$	density of force
$\boldsymbol{g}$	field momentum density
G	Green's function
$H(p, q)$	classical Hamiltonian function
$\widehat{H}$, $\widehat{\mathcal{H}}$	Hamiltonian quantum operator
$\boldsymbol{H}$, $\mathbf{H}$	strength of the magnetic field
p, P	pressure
$\boldsymbol{p}$	momentum of a particle
$Q_{\alpha\beta}$	electric quadrupole moment tensor
$\hbar$	reduced Planck constant
$\boldsymbol{i}$	density of surface current
I	intensity of emission
$\boldsymbol{j}$	current density (volumetric)
J	total current passing through the surface
K, T	kinetic energy
$\boldsymbol{k}$	wave vector
L	inductance; length (characteristic scale)

L_{ik}	inductance coefficient
l	length (characteristic scale)
M, m	mass
$\boldsymbol{m}$	magnetic moment
n, N	particle number density
$\boldsymbol{n}$	unit normal vector
N	total number of particles
$\boldsymbol{r}$, $\boldsymbol{R}$	radius vector
S	surface
$\widehat{S}$	scattering matrix
t	time
T	temperature in energy units of the Kelvin scale; kinetic energy
$\widehat{T}$	chronological operator
Tr	trace (sum of diagonal elements)
T_{ik}	energy–momentum tensor of the electromagnetic field
U	potential energy
v, V, u	velocity
V	volume; interaction energy
$\widehat{V}$	quantum interaction operator
$\mathcal{V}$	volume
w	energy density (volumetric)
$dw_{i\to f}$	probability of a transition per unit time
W	energy
Z	charge of an atomic nucleus
$\boldsymbol{Z}$	Hertz vector
α	angle
$\widehat{\alpha}$	Dirac matrix
$\beta = v/c$	ratio of the speed of a particle to the speed of light
$\widehat{\beta}$	Dirac matrix
γ	relativistic (Lorentz) factor; damping constant
$\widehat{\gamma^k}$, $k = 0, 1, 2, 3, 5$	Dirac matrices
Δ	increment; Laplace operator
ϑ, θ	angle
Θ	step function
λ	wavelength
$\Lambda_C = \hbar/(m_e c)$	Compton wavelength
$d\nu$	number of quantum states in a continuous spectrum
$\Pi_\pm$	projection operators
ρ	charge density (volumetric)
$\widehat{\rho}$	quantum density operator
σ	density of surface charge; effective cross-section
$\sigma_{\alpha\beta}$	Maxwell tension tensor of the electromagnetic field
Σ	effective cross-section; surface
ϕ	angle

φ	scalar potential of the electromagnetic field; angle
$\boldsymbol{\Phi}$	magnetic flux
ψ, Ψ	wave function; pseudoscalar potential of the magnetic field
ω, Ω	frequency

1
The Mathematical Methods of Electrodynamics

1.1
Vector and Tensor Algebra

1.1.1
The Definition of a Tensor and Tensor Operations

In three-dimensional space, select a rectangular and rectilinear (Cartesian[1]) system of coordinates x_1, x_2, x_3. Regard the space as *Euclidean*. This means that all axioms of Euclidean geometry[2] and their consequences considered in school courses on mathematics are valid in it. For instance, the square of the distance between two close points is given by the following expression:

$$dl^2 = dx_1^2 + dx_2^2 + dx_3^2 \,.$$

Along with the original system of coordinates, consider some other systems of common origin yet rotated with respect to the original one (Figure 1.1).

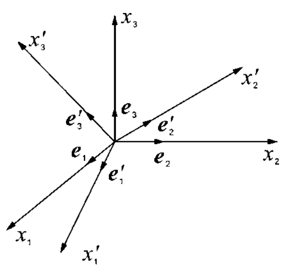

Figure 1.1 The rotation of the Cartesian system of coordinates.

1) René Descartes (Renatus Cartesius) (1596–1650) was a French philosopher and mathematician, the founder of the coordinates method. He introduced a large number of mathematical concepts and notations used even now.

2) Euclid (lived in the third century BC) was an ancient Greek scientist, "the father of geometry." His mathematical treatise entitled *Elements* is the best known. Euclid studied various aspects of geometric optics.

Foundations of Classical and Quantum Electrodynamics, First Edition. Igor N. Toptygin.

A *scalar or invariant* is a quantity that does not change when the system of coordinates is rotated, that is, it is the same in either the original or the rotated system of coordinates

$$S' = S = \text{inv} . \quad (1.1)$$

For instance, $\mathrm{d}l^2 = \mathrm{d}l'^2 = \text{inv}$.

In three-dimensional space, a *vector* is the titality of three quantities V_α ($\alpha = 1, 2, 3$) defined in all coordinate systems and transformed according to the following rule:

$$V'_\alpha = a_{\alpha\beta} V_\beta \quad (1.2)$$

(summing of elements over the repeated symbol β, from 1 to 3 is assumed). Here V_β are the projections of the vector on an axis of the original system of coordinates, V'_α are the projections of the vector on an axis of the rotated system, and $a_{\alpha\beta}$ are the coefficients of the transformation, which are the cosines of the angles between the β axis of the original system and the α axis of the rotated system. They may be written through the single vectors (orts) of the coordinate axes:

$$a_{\alpha\beta} = \boldsymbol{e}'_\alpha \cdot \boldsymbol{e}_\beta . \quad (1.3)$$

In three-dimensional space, a *tensor of rank 2* is a nine-component quantity $T_{\alpha\beta}$ (each index varies independently assuming three values: 1, 2, 3) which is defined in every system of coordinates and, when a coordinate system is rotated, is transformed as the products of the components of the two vectors $A_\alpha V_\beta$, in the following way:

$$T'_{\alpha\beta} = a_{\alpha\mu} a_{\beta\nu} T_{\mu\nu} . \quad (1.4)$$

In three-dimensional space, a tensor of rank s is a 3^s-component quantity $T_{\lambda\ldots\nu}$ that is transformed as the product of s components of vectors:

$$T'_{\beta\ldots\kappa} = a_{\beta\mu} \ldots a_{\kappa\sigma} T_{\mu\ldots\sigma} . \quad (1.5)$$

Scalars and vectors may be regarded as tensors of rank 0 and 1, respectively.

Rotation matrix $\widehat{a}$ has the following properties:

1. Orthogonality

$$a_{\alpha\mu} a_{\beta\mu} = \delta_{\alpha\beta} , \quad a_{\alpha\mu} a_{\alpha\nu} = \delta_{\mu\nu} , \quad (1.6)$$

where

$$\delta_{\alpha\beta} = 1 \quad \text{if} \quad \alpha = \beta \quad \text{and} \quad \delta_{\alpha\beta} = 0 \quad \text{if} \quad \alpha \neq \beta \quad (1.7)$$

is *Kronecker symbol*[3];

3) Leopold Kronecker (1823–1891) was a German mathematician, a specialist in algebra and theory of numbers.

2. The determinant of a rotation matrix equals 1:

$$\det \widehat{a} \equiv |\widehat{a}| = 1\ . \tag{1.8}$$

3. The product of two rotation matrices

$$\widehat{c} = \widehat{a}\widehat{g}\ , \quad c_{\alpha\beta} = a_{\alpha\mu} g_{\mu\beta} \tag{1.9}$$

describes the evolution of a system resulting from two consecutive rotations, first with matrix $\widehat{g}$ and then with matrix $\widehat{a}$.[4] In the general case, rotation matrices are noncommutative, that is,

$$\widehat{a}\widehat{g} \neq \widehat{g}\widehat{a}\ . \tag{1.10}$$

As follows from property 1, a reverse matrix defined by the relation

$$\widehat{a}^{-1}\widehat{a} = \widehat{a}\widehat{a}^{-1} = \widehat{1} \quad \text{or} \quad a^{-1}_{\alpha\mu} a_{\mu\beta} = a_{\alpha\mu} a^{-1}_{\mu\beta} = \delta_{\alpha} \tag{1.11}$$

results from the original matrix when the latter is transposed, that is, its columns are substituted for lines and vice versa:

$$\widehat{a}^{-1} = \widehat{\tilde{a}}\ , \quad a^{-1}_{\alpha\beta} = \tilde{a}_{\alpha\beta} = a_{\beta\alpha}\ . \tag{1.12}$$

The reverse transformation (1.2) looks like this:

$$V_\beta = a^{-1}_{\beta\alpha} V'_\alpha\ . \tag{1.13}$$

All vectors are transformed identically according to rule (1.2) when a coordinate system is rotated. But they may behave in one of two ways if a system of coordinates is inverted, that is,

$$x'_\alpha = -x_\alpha\ . \tag{1.14}$$

Here the transformation matrix is $a_{\alpha\beta} = -\delta_{\alpha\beta}$. Vectors whose components, just like coordinates x_a, change their signs during inversions are called *polar* (regular, real) vectors. Vectors whose components do not change sign as the result of inversions of coordinate systems are called *axial* vectors or *pseudovectors* (an angular velocity, a cross-product of two polar vectors $\boldsymbol{A} \times \boldsymbol{B}$, etc.) This definition also includes tensors of arbitrary rank s: when the inversion of coordinates occurs, the components of *polar* (regular) tensors acquire a factor of $(-1)^s$ and the components of *pseudotensors* acquire a factor of $(-1)^{s+1}$.

The *sum* of two tensors of the same rank produces a third tensor of the same rank with components

$$Q_{\alpha\beta} = T_{\alpha\beta} + P_{\alpha\beta}\ . \tag{1.15}$$

4) The family all rotation operations forms makes a group of three-dimensional rotations. See Gel'fand *et al.* (1958).

The *direct products* of the components of two tensors (without summing) constitute a tensor whose rank equals the sum of the ranks of the factors, for instance,

$$Q_{\alpha\beta\lambda} = T_{\alpha\beta} V_\lambda \,, \tag{1.16}$$

where $Q_{\alpha\beta\lambda}$ is a tensor of rank 3.

The *contraction of a tensor* means the formation of a new tensor whose components are produced by the selection of components with two identical symbols and, further, their summing. For instance, $Q_{\alpha\beta\beta} = A_\alpha$ is a vector and $Q_{\alpha\beta\alpha} = B_\beta$ is another vector. Contraction decreases the rank of the tensor by 2, for instance,

$$S = T_{\alpha\alpha} = \text{inv} \tag{1.17}$$

is a scalar.

When an equality between tensors is written, the rule of the same tensor dimensionality must be observed: only tensors of the same rank may be equated. This means that the number of free symbols (over which no summation is done) must be the same in the first and second members of an equality. The number of pairs of "mute" symbols (those over which summing is done) may be any on the right and on the left.

Tensors may be *symmetric (antisymmetric)* with respect to a pair of indices α and β if their components satisfy the conditions

$$Q_{\alpha\beta\mu} = Q_{\beta\alpha\mu} \quad (Q_{\alpha\beta\mu} = -Q_{\beta\alpha\mu}) \,. \tag{1.18}$$

Tensor components may be either real or complex numbers. In the latter case, the concepts of Hermitian[5] and anti-Hermitian tensors play an important role. The definition of a Hermitian tensor is as follows:

$$T^h_{\alpha\beta} = T^{h*}_{\beta\alpha} \,, \tag{1.19}$$

where the asterisk indicates complex conjugation. The definition of an anti-Hermitian tensor is as follows:

$$T^{ah}_{\alpha\beta} = -T^{ah*}_{\beta\alpha} \,. \tag{1.20}$$

In applications, invariant unit tensors $\delta_{\alpha\beta}$ and $e_{\alpha\beta\lambda}$ are very important. The former is a symmetric polar tensor whose components coincide with the Kronecker symbol (1.7), whereas the latter is antisymmetric over any pair of indices, and its components are determined by the following conditions:

$$\text{(a)} \quad e_{123} = 1 \,, \quad e_{\alpha\beta\lambda} = -e_{\beta\alpha\lambda} = -e_{\alpha\lambda\beta} = e_{\lambda\alpha\beta} = e_{\beta\lambda\alpha} = -e_{\lambda\beta\alpha} \,. \tag{1.21}$$

5) Charles Hermite (1822–1901) was a French mathematician, the author of works on classical analysis, algebra, and theory of numbers.

It is called the Levi-Civita tensor.[6] Both tensors, transforming during rotations according to rule (1.7), are peculiar in that their components have the same values in all coordinate systems:

$$\delta'_{\alpha\beta} = \delta_{\alpha\beta} \, , \quad e'_{\alpha\beta\lambda} = e_{\alpha\beta\lambda} \, . \tag{1.22}$$

Problems

1.1. Prove equality (1.8). What is the determinant of the transformation matrix if rotation is accompanied by the inversion of the coordinate axes?

1.2. Prove the equalities $\delta'_{\alpha\beta} = \delta_{\alpha\beta}$ and $e'_{\alpha\mu\nu} = e_{\alpha\mu\nu}$ for an arbitrary rotation of a coordinate system.

1.3. Write down the rule of transformation for the components of a pseudotensor of rank s that would be valid not just for the rotation but also for the mirror reflections of the coordinate axes.

1.4. Represent an arbitrary tensor of rank 2 $T_{\alpha\beta}$ as the sum of a symmetric tensor ($S_{\alpha\beta} = S_{\beta\alpha}$) and an antisymmetric tensor ($A_{\alpha\beta} = -A_{\beta\alpha}$). Make sure that this representation is unique.

1.5. Represent an arbitrary complex tensor of rank 2 $T_{\alpha\beta}$ as the sum of a Hermitian tensor ($S^h_{\alpha\beta} = S^{h*}_{\beta\alpha}$) and an anti-Hermitian tensor ($A^h_{\alpha\beta} = -A^{h*}_{\beta\alpha}$). Make sure that this representation is unique.

1.6. Show that

1. the contraction of a symmetric tensor and an antisymmetric tensor equals zero: $S_{\alpha\beta}A_{\alpha\beta} = 0$.
2. the contraction of two Hermitian tensors or two anti-Hermitian tensors of rank 2 is a real number.
3. the contraction of a Hermitian tensor and an anti-Hermitian tensor of rank 2 is a purely imaginary number.

1.7. Show that the symmetry of a tensor is a property that is invariant with respect to rotations, that is, a tensor that is symmetric (antisymmetric) over a pair of indices in a certain system of reference remains symmetric (antisymmetric) over these indices in every system rotated with respect to the original one.

1.8. Using rules (1.2)–(1.6) of tensor transformation, show that

1. A_α is a vector (pseudovector) if $A_\alpha B_\alpha = \text{inv}$ and B_α is a vector (pseudovector).
2. A_α is a vector if $A_\alpha = T_{\alpha\beta} B_\beta$ in any system of coordinates and $T_{\alpha\beta}$ is a tensor of rank 2, and B_β is a vector;
3. $T_{\alpha\alpha} = \text{inv}$, where $T_{\alpha\beta}$ is a tensor of rank 2.

6) Tullio Levi-Civita (1873–1941) was an Italian mathematician who contributed to the development of tensor analysis.

4. $\varepsilon_{\alpha\beta}$ is a tensor of rank 2 if A_α and B_α are vectors and $A_\alpha = \varepsilon_{\alpha\beta} B_\beta$ in all systems of coordinates. What is $\varepsilon_{\alpha\beta}$ if A_α is a vector and B_α is a pseudovector? What is $\varepsilon_{\alpha\beta}$ if A_α and B_α are both pseudovectors?
5. $A_{\alpha\beta\lambda} B_{\alpha\beta}$ is a vector if $A_{\alpha\beta\lambda}$ and $B_{\alpha\beta}$ are tensors of ranks 3 and 2, respectively.
6. $T_{\alpha\beta} P_{\alpha\beta}$ is a pseudoscalar if $T_{\alpha\beta}$ and $P_{\alpha\beta}$ are a tensor and a pseudotensor of rank 2, respectively.

1.9. Show the rule of the transformation of an aggregate of volumetric integrals $T_{\alpha\beta} = \int x_\alpha x_\beta \mathrm{d}V$ in the cases of rotation and mirror reflection (x_α, x_β are Cartesian coordinates).

1.10. Show that the components of an antisymmetric tensor of rank 2 $A_{\alpha\beta} = -A_{\beta\alpha}$ (either polar or axial) may be identified by the components of a certain vector C_α (either axial or polar) because they are transformed in the same way in the case of rotation or reflection. In this case, C_α is called the vector dual to tensor $A_{\alpha\beta}$.

1.11. Prove the following equalities:

$$[\boldsymbol{A} \times \boldsymbol{B}]_\alpha = e_{\alpha\beta\lambda} A_\beta B_\lambda \ ,$$

$$[\boldsymbol{A} \times \boldsymbol{B}] \cdot \boldsymbol{C} = e_{\alpha\beta\lambda} A_\alpha B_\beta C_\lambda = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} \ . \tag{1.23}$$

How are the vector, the dual vector, and the mixed products transformed in the cases of rotation and reflection if all three vectors are polar?

1.12. Show that if the respective components of two vectors are proportional in a certain system of coordinates, then they are also proportional in any other system of coordinates. Vectors such as these are called parallel vectors.

1.13. The area of an elementary parallelogram constructed on the small vectors $\mathrm{d}\boldsymbol{r}$ and $\mathrm{d}\boldsymbol{r}'$ is represented by vector $\mathrm{d}\boldsymbol{S}$ directed along a normal to the plane of the parallelogram and, by the absolute value, is equal to its area. Write down $\mathrm{d}S_\alpha$ in tensor notation.

1.14. Write down, in tensor notation, the volume $\mathrm{d}V$ of the elementary parallelepiped constructed on the small vectors $\mathrm{d}\boldsymbol{r}, \mathrm{d}\boldsymbol{r}', \mathrm{d}\boldsymbol{r}''$. How is it transformed in the cases of rotation and reflection?

1.15. Prove the identities

$$\begin{aligned}
(\boldsymbol{A} \times \boldsymbol{B}) \cdot (\boldsymbol{C} \times \boldsymbol{D}) - (\boldsymbol{A} \cdot \boldsymbol{C})(\boldsymbol{B} \cdot \boldsymbol{D}) + (\boldsymbol{A} \cdot \boldsymbol{D})(\boldsymbol{B} \cdot \boldsymbol{C}) &= 0 \ , \\
(\boldsymbol{A} \times \boldsymbol{B}) \cdot (\boldsymbol{C} \times \boldsymbol{D}) + (\boldsymbol{B} \times \boldsymbol{C}) \cdot (\boldsymbol{A} \times \boldsymbol{D}) + (\boldsymbol{C} \times \boldsymbol{A}) \cdot (\boldsymbol{B} \times \boldsymbol{D}) &= 0 \ , \\
\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) + \boldsymbol{B} \times (\boldsymbol{C} \times \boldsymbol{A}) + \boldsymbol{C} \times (\boldsymbol{A} \times \boldsymbol{B}) &= 0 \ , \\
(\boldsymbol{A} \times \boldsymbol{B}) \times (\boldsymbol{C} \times \boldsymbol{D}) - (\boldsymbol{A} \cdot [\boldsymbol{B} \times \boldsymbol{D}])\boldsymbol{C} + (\boldsymbol{A} \cdot [\boldsymbol{B} \times \boldsymbol{C}])\boldsymbol{D} &= 0 \ , \\
(\boldsymbol{A} \times \boldsymbol{B}) \times (\boldsymbol{C} \times \boldsymbol{D}) - (\boldsymbol{A} \cdot [\boldsymbol{C} \times \boldsymbol{D}])\boldsymbol{B} + (\boldsymbol{B} \cdot [\boldsymbol{C} \times \boldsymbol{D}])\boldsymbol{A} &= 0 \ .
\end{aligned} \tag{1.24}$$

1.16. In a spherical system of coordinates, the two directions $\boldsymbol{n}$ and n' are determined by the angles ϑ, α and ϑ', α'. Find the cosine of the angle θ between them.

1.17. In certain cases, it may be more convenient to consider the complex cyclic components

$$A_{\pm 1} = \frac{\mp A_x \pm i A_y}{\sqrt{2}}\,, \quad A_0 = A_z\,, \tag{1.25}$$

of the vector $\boldsymbol{A}$ instead of its Cartesian components. Express the scalar and vector products of two vectors through their cyclic components. Also, express the cyclic components of the radius vector through spherical functions.[7]

1.18. Write down the matrix $\widehat{g}$ of the transformation of the components of a vector in the case of the rotation of the Cartesian system of coordinates around the Ox_3 axis by angle α.

1.19. Form the matrices of the transformation of basic orts when changing from Cartesian to spherical coordinates and back and from Cartesian to cylindrical coordinates and back.

1.20. Find the matrix $\widehat{g}$ of the transformation of the components of a vector in the case of the rotation of the coordinate axes determined by the Euler angles[8] α_1, θ, and α_2 (Figure 1.2) by mutually multiplying matrices corresponding to rotation around the Ox_3 axis by angle α_1, around the line of nodes ON by angle θ, and around the Ox_3' axis by angle α_2.

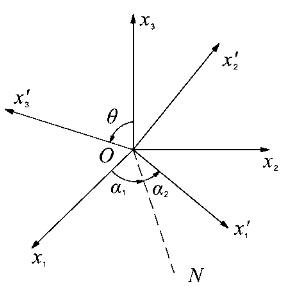

Figure 1.2 The specification of the rotation of Cartesian axes by Euler angles α_1, θ, α_2.

1.21. Find the matrix $\widehat{D}(\alpha_1 \theta \alpha_2)$ used for transforming the cyclic components of vector (1.25) when rotating the system of coordinates. The rotation is determined by the Euler angles α_1, θ, and α_2 (Figure 1.2).

7) The definition of spherical functions is given in Section 1.3; see the answer to Problem 1.118*

8) Leonard Euler (1707–1783) was an outstanding mathematician, astronomer, and physicist who astonished his contemporaries by his efficiency, and range of interests. He was born and studied in Switzerland, but for most of his life worked at the Saint Petersburg Academy of Sciences. Pierre Laplace called him the teacher of all mathematicians of the second half of the eighteenth century.

1.22. Show that the matrix of an infinitesimal rotation of a coordinate system may be written as $\widehat{a} = 1 + \widehat{\varepsilon}$, where $\widehat{\varepsilon}$ is an antisymmetric matrix ($\varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha}$). Find the geometric meaning of $\varepsilon_{\alpha\beta}$.

1.23. Show that the representation of a small rotation by vector $\delta\boldsymbol{\varphi}$ used in the solution of the previous problem is only possible in relation to quantities of the first order of smallness. In the next order, the vector of the resulting rotation is not equal to the sum of the vectors of individual rotations and the relevant matrices do not commute.

1.1.2
The Principal Values and Invariants of a Symmetric Tensor of Rank 2

The selection of a system of coordinates wherein a certain tensor has the simplest structure is of great practical importance. Consider the selection of such a system for a symmetric tensor of rank 2.

If vector $\boldsymbol{n}$ satisfies the condition

$$S_{\alpha\beta} n_\beta = S n_\alpha \, , \quad \alpha, \beta = 1, 2, 3 \, , \tag{1.26}$$

where S is a certain scalar, then the direction that is determined by vector $\boldsymbol{n}$ is called *the principal direction* of the tensor, vector $\boldsymbol{n}$ is called *the proper vector* of the tensor, and S is called its *principal value*.

Example 1.1

Reducing a real ($S_{\alpha\beta} = S^*_{\alpha\beta}$) symmetric ($S_{\alpha\beta} = S_{\beta\alpha}$) tensor of rank 2 to diagonal form means finding such a system of axes wherein only the diagonal components of the tensor are not equal to zero. Specify a way of calculation of the principal values and the principal directions of such tensor.

Solution. Use the system of algebraic equations (1.26) to find the proper vectors and principal values of the tensor in question. Normalize the proper vectors to 1: $n^*_\alpha n_\alpha = 1$. The equations (1.26) and the properties of the tensor $S_{\alpha\beta}$ show us that the proper values of S are real scalars: $S = n^*_\alpha S_{\alpha\beta} n_\beta = S^*$. They follow from the condition of equality to zero of the determinant of the system (1.26):

$$|S_{\alpha\beta} - S\delta_{\alpha\beta}| = 0 \, . \tag{1.27}$$

This is a cubic algebraic equation whose solution, in relation to S, includes three real roots: $S^{(1)}$, $S^{(2)}$, $S^{(3)}$. In the general case, they are different from each other, although multiple roots ($S^{(1)} = S^{(2)} \neq S^{(3)}$ or $S^{(1)} = S^{(2)} = S^{(3)}$) are possible. Here, the bracketed indices are not tensor symbols!

In the case of different roots, inserting the values found for S, one by one, in the system in (1.26) results in two projections of each of the proper vectors $n^{(1)}_\alpha \neq n^{(2)}_\alpha \neq n^{(3)}_\alpha$ through the third one, which is determined by the condition of normalization. All the proper vectors are real because the coefficients of (1.26) are real. They are mutually perpendicular, which follows from the same system of equations: $(S^{(1)} - S^{(2)})(\boldsymbol{n}^{(1)} \cdot \boldsymbol{n}^{(2)}) = 0$. The same goes for the other two pairs. Regarding the proper vectors as the orts of the system of coordinates (they determine the principal axes of the tensor), use (1.26) to find the form of the tensor in this system of axes:

$$\widehat{S}' = \begin{pmatrix} S^{(1)} & 0 & 0 \\ 0 & S^{(2)} & 0 \\ 0 & 0 & S^{(3)} \end{pmatrix} . \tag{1.28}$$

In the case of two repeated roots, $S^{(1)} = S^{(2)}$, the proper vectors $\boldsymbol{n}^{(1)}$ and $\boldsymbol{n}^{(2)}$ are determined ambiguously, that is, any pair of mutually perpendicular directions may be selected in the plane perpendicular to $\boldsymbol{n}^{(3)}$. If all three roots are the same, then any three mutually perpendicular directions may be regarded as the principal axes. □

Problems

1.24. Is it possible to reduce an arbitrary real tensor of rank 2 ($T_{\alpha\beta} \neq T_{\beta\alpha}$) to the diagonal form by rotating its system of coordinates in physical three-dimensional space? What about a Hermitian tensor of rank 2 ($T^h_{\alpha\beta} = T^{h*}_{\beta\alpha}$)?

1.25. Write down a real symmetric tensor of rank 2 $S_{\alpha\beta}$ in an arbitrary system of coordinates through its principal values $S^{(1)}$, $S^{(2)}$, $S^{(3)}$ and the orts $n^{(i)}_\alpha$ of the principal axes.

1.26. Using the characteristic (1.27), compile the invariants relative to rotation from the components of an arbitrary tensor of rank 2 $T_{\alpha\beta}$.

1.27. Using the theorem for the expansion of the determinant in the elements of a row or a column, find the components of the inverse tensor $T^{-1}_{\alpha\beta}$. Its definition coincides with that of (1.11) for the inverse matrix. Indicate the condition of the existence of an inverse tensor.

1.28. Prove the identities

$$e_{\alpha\beta\gamma} e_{\alpha\beta\gamma} = 6 ,$$

$$e_{\alpha\beta\gamma} e_{\alpha\beta\sigma} = 2\delta_{\gamma\sigma} ,$$

$$e_{\alpha\beta\gamma} e_{\alpha\nu\sigma} = \delta_{\beta\nu}\delta_{\gamma\sigma} - \delta_{\beta\sigma}\delta_{\gamma\nu} = \begin{vmatrix} \delta_{\beta\nu} & \delta_{\gamma\nu} \\ \delta_{\beta\sigma} & \delta_{\gamma\sigma} \end{vmatrix} ,$$

$$e_{\alpha\beta\gamma} e_{\mu\nu\sigma} = \delta_{\alpha\mu}\delta_{\beta\nu}\delta_{\gamma\sigma} + \delta_{\alpha\nu}\delta_{\beta\sigma}\delta_{\gamma\mu} + \delta_{\alpha\sigma}\delta_{\beta\mu}\delta_{\gamma\nu}$$
$$-\delta_{\alpha\nu}\delta_{\beta\mu}\delta_{\gamma\sigma} - \delta_{\alpha\mu}\delta_{\beta\sigma}\delta_{\gamma\nu} - \delta_{\alpha\sigma}\delta_{\beta\nu}\delta_{\gamma\mu}$$

$$= \begin{vmatrix} \delta_{\alpha\mu} & \delta_{\beta\mu} & \delta_{\gamma\mu} \\ \delta_{\alpha\nu} & \delta_{\beta\nu} & \delta_{\gamma\nu} \\ \delta_{\alpha\sigma} & \delta_{\beta\sigma} & \delta_{\gamma\sigma} \end{vmatrix} .$$

Using the third identity, prove the formula of vector algebra

$$\boldsymbol{A} \times [\boldsymbol{B} \times \boldsymbol{C}] = \boldsymbol{B}(\boldsymbol{A} \cdot \boldsymbol{C}) - \boldsymbol{C}(\boldsymbol{A} \cdot \boldsymbol{B}) .$$

1.29. Write down the following in the invariant vector form:

1. $e_{\alpha\beta\gamma} e_{\alpha\sigma\kappa} e_{\gamma\nu\varepsilon} e_{\kappa\omega\varepsilon} A_\beta A_\sigma B_\nu C_\omega$,
2. $e_{\alpha\beta\gamma} e_{\rho\sigma\kappa} e_{\gamma\nu\varepsilon} e_{\kappa\omega\varepsilon} A_\sigma A_\beta B_\rho B_\alpha C_\omega C_\nu$.

1.30. Prove the identity

$$T_{\alpha\beta} A_\alpha B_\beta - T_{\alpha\beta} A_\beta B_\alpha = 2\boldsymbol{C} \cdot (\boldsymbol{A} \times \boldsymbol{B}) ,$$

where $T_{\alpha\beta}$ is an arbitrary tensor of rank 2, $\boldsymbol{A}$ and $\boldsymbol{B}$ are vectors, and $\boldsymbol{C}$ is the vector of the dual antisymmetric part of the tensor $T_{\alpha\beta}$.

1.31. Present the product $(\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C}))(\boldsymbol{A}' \cdot (\boldsymbol{B}' \times \boldsymbol{C}'))$ as the sum of members that contain only the scalar products of the vectors.

Hint: Apply the theorem for the multiplication of determinants or use the pseudotensor $e_{\alpha\beta\gamma}$.

1.32. Show that the only vector whose components are the same in all systems of coordinates is a null vector, that any tensor of rank 3 whose components are the same in all systems of coordinates is proportional to $e_{\alpha\beta\gamma}$, and that any tensor of rank 4 whose components are the same in all systems of coordinates is proportional to $(\delta_{\alpha\beta}\delta_{\mu\nu} + \delta_{\alpha\mu}\delta_{\beta\nu} + \delta_{\alpha\nu}\delta_{\beta\mu})$.

1.33. Regard $\boldsymbol{n}$ as a unit vector whose directions in space are equiprobable. Find the mean values of its components and their products – n_α, $n_\alpha n_\beta$, $n_\alpha n_\beta n_\gamma$, $n_\alpha n_\beta n_\gamma n_\nu$ – using the transformational properties of the quantities sought.

1.34. Find the average values for all directions of the expressions $(\boldsymbol{a}\cdot\boldsymbol{n})^2$, $(\boldsymbol{a}\cdot\boldsymbol{n})(\boldsymbol{b}\cdot\boldsymbol{n})$, $(\boldsymbol{a} \cdot \boldsymbol{n})\boldsymbol{n}$, $(\boldsymbol{a} \times \boldsymbol{n})^2$, $(\boldsymbol{a} \times \boldsymbol{n}) \cdot (\boldsymbol{b} \times \boldsymbol{n})$, $(\boldsymbol{a} \cdot \boldsymbol{n})(\boldsymbol{b} \cdot \boldsymbol{n})(\boldsymbol{c} \cdot \boldsymbol{n})(\boldsymbol{d} \cdot \boldsymbol{n})$, if $\boldsymbol{n}$ is a unt vector whose all directions are equiprobable and $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$ are constant vectors.

Hint: Use the results obtained in the previous problem.

1.35. Write down all possible invariants of polar vectors $\boldsymbol{n}$, and $\boldsymbol{n}'$ and pseudovector $\boldsymbol{l}$.

1.36. What independent pseudoscalars may be made of two polar vectors $\boldsymbol{n}$ and $\boldsymbol{n}'$ and one pseudovector $\boldsymbol{l}$? What independent pseudoscalars may be made of three polar vectors $\boldsymbol{n}_1$, $\boldsymbol{n}_2$, and $\boldsymbol{n}_3$?

1.1.3
Covariant and Contravariant Components

In physics, many problems require nonorthogonal and curvilinear systems of coordinates be used so that the relations between the old and new coordinates are nonlinear and different from (1.2). The transition to new coordinates may not come down to just the simple and obvious rotation of axes. One of the most important areas where such a mathematical apparatus needs to be used is special and, especially, general relativity.

Closing this section, we will come up with the definition of tensors with respect to overall transformations of coordinates and consider their basic properties in three-dimensional Euclidean space. This is appropriate because in three-dimensional space the meaning of many concepts and relation is more obvious and transparent than in four-dimensional space–time of the relativistic theory. We will begin by immersing ourselves in these issues by considering a case that is half way between Cartesian rectangular coordinates and common coordinates when the coordinate axes of the reference frame are still rectilinear but become nonorthogonal (oblique or affine coordinates).

Example 1.2

Three noncoplanar and nonorthogonal unit vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$ are selected as the basic vectors in a three-dimensional Euclidean space. Three systems of rectilinear lines passing through every point of the space and parallel to the basic vectors are the coordinate lines. Build a mutual basis $\boldsymbol{e}^1$, $\boldsymbol{e}^2$, $\boldsymbol{e}^3$ which, by definition, is connected to the original basis by the following relations:

$$\boldsymbol{e}^\alpha \cdot \boldsymbol{e}_\beta = \delta^\alpha_\beta = \begin{cases} 0\,, & \alpha \neq \beta\,; \\ 1\,, & \alpha = \beta\,. \end{cases} \tag{1.29}$$

Will the vectors of the mutual basis be unit vectors?

Expand an arbitrary vector $\boldsymbol{A}$ (including also the radius vector $\boldsymbol{r}$) in vectors $\boldsymbol{e}_\alpha$ and $\boldsymbol{e}^\beta$ of the original and mutual bases. Show the geometric meaning of its components in both cases (in the first case, they are called *contravariant* and are labeled with upper indices, A^1, A^2, A^3. In the second case, they are *covariant*, and are labeled with lower indices, A_1, A_2, A_3).

Solution. In accordance with (1.29), $\boldsymbol{e}^1$ must be perpendicular to $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$. Look for it in the form of $\boldsymbol{e}^1 = k\boldsymbol{e}_2 \times \boldsymbol{e}_3$ and, from the condition of normalization $\boldsymbol{e}^1 \cdot \boldsymbol{e}_1 = 1$, find

$$k = \frac{1}{\underline{V}} = \frac{1}{\boldsymbol{e}_1 \cdot (\boldsymbol{e}_2 \times \boldsymbol{e}_3)}\,,$$

where $k^{-1} = \underline{V}$ is the volume of the parallelepiped built on the vectors of the original basis. $\underline{V} > 0$ if the right-hand system of coordinates is selected. Therefore,

$$e^{\alpha} = \frac{e_{\beta} \times e_{\gamma}}{\underline{V}}, \tag{1.30}$$

where α, β, and γ form a cyclic permutation. Radius vector $\boldsymbol{r}$ and any other vectors are expanded in basic vectors in the usual way:

$$\boldsymbol{r} = x_1 \boldsymbol{e}^1 + x_2 \boldsymbol{e}^2 + x_3 \boldsymbol{e}^3 = x^1 \boldsymbol{e}_1 + x^2 \boldsymbol{e}_2 + x^3 \boldsymbol{e}_3 . \tag{1.31}$$

Multiplying the first equality, in a scalar way, by $\boldsymbol{e}_{\alpha}$, we find

$$x_{\alpha} = \boldsymbol{e}_{\alpha} \cdot \boldsymbol{r} . \tag{1.32}$$

Therefore, the geometric meaning of the covariant components is revealed by projecting the radius vector, in the usual way, by lowering perpendiculars from the end of the vector onto the coordinate axes. When this has been done, the directions of the contravariant basic vectors, by which the covariant components of the vector are multiplied, do not coincide with the directions of the coordinate axes (Figure 1.3) and have no unit lengths. For instance, if vector $\boldsymbol{e}_3$ is orthogonal to $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ and the angle between the latter is ϕ, then $|\boldsymbol{e}^1| = |\boldsymbol{e}^2| = 1/\sin\phi$ and the length of the hypotenuse $OB = |x_1 \boldsymbol{e}^1| = x_1/\sin\phi > x_1$. However, the length of the leg $OC = x_1$. As follows from (1.31) and Figure 1.3, the contravariant components result from projecting the vector onto the coordinate axes with segments parallel to the axes. For them, a representation identical to (1.32) is valid:

$$x^{\alpha} = \boldsymbol{e}^{\alpha} \cdot \boldsymbol{r} . \tag{1.33}$$

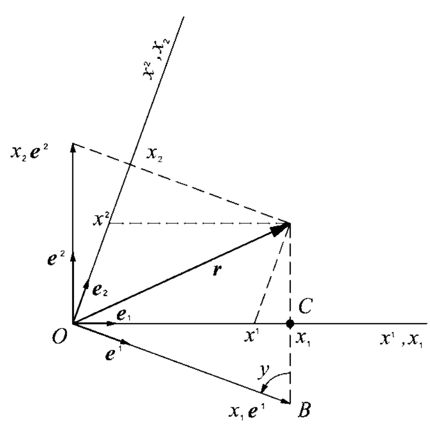

Figure 1.3 The clarification of the geometric meaning of the covariant and contravariant components of a vector.

□

Example 1.3

Determine the nine-component quantities:

$$g_{\alpha\beta} = \boldsymbol{e}_\alpha \cdot \boldsymbol{e}_\beta\ , \quad g^{\alpha\beta} = \boldsymbol{e}^\alpha \cdot \boldsymbol{e}^\beta\ , \tag{1.34}$$

where $\boldsymbol{e}_\alpha$ and $\boldsymbol{e}^\beta$ are the basic vectors of the original and mutual nonorthogonal bases, introduced in Example 1.2. The values $g_{\alpha\beta}$ and $g^{\alpha\beta}$ are called the covariant and contravariant components of a *metric tensor.*

Prove the following relations that connect the covariant and contravariant components of an arbitrary vector (the rules of rasing and lowering indices):

$$\text{(i)} \quad A_\alpha = g_{\alpha\beta} A^\beta\ ; \quad \text{(ii)} \quad A^\alpha = g^{\alpha\beta} A_\beta\ ; \quad \text{(iii)} \quad g_{\alpha\beta} g^{\beta\gamma} = g_\alpha^\gamma \equiv \delta_\alpha^\gamma\ . \tag{1.35}$$

Here, δ_α^γ is a Kronecker symbol.

Find the determinants of a covariant and a contravariant metric tensor and express them through the volumes $\underline{V}$ and $\overline{V}$ of parallelepipeds built on the vectors of the original and mutual bases.

Solution. The expression below follows from expansion (1.31):

$$\boldsymbol{A} = A_\beta \boldsymbol{e}^\beta = A^\beta \boldsymbol{e}_\beta\ .$$

Multiplying it, in a scalar way, by $\boldsymbol{e}_\alpha$ and using the definitions of mutual basis (1.30) and metric tensor (1.34), we get the first expression in (1.35); multiplying this expansion, in a scalar way, by $\boldsymbol{e}^\alpha$, we get the second expression in (1.35); and inserting the second expression in (1.35) in the first expression in (1.35), we get the third expression in (1.35).

If we label $g = |g_{\alpha\beta}|$ and use definition (1.34) and the formula from the first task in Problem 1.29, we find the following:

$$\begin{aligned}
g &= \begin{vmatrix} \boldsymbol{e}_1 \cdot \boldsymbol{e}_1 & \boldsymbol{e}_1 \cdot \boldsymbol{e}_2 & \boldsymbol{e}_1 \cdot \boldsymbol{e}_3 \\ \boldsymbol{e}_2 \cdot \boldsymbol{e}_1 & \boldsymbol{e}_2 \cdot \boldsymbol{e}_2 & \boldsymbol{e}_2 \cdot \boldsymbol{e}_3 \\ \boldsymbol{e}_3 \cdot \boldsymbol{e}_1 & \boldsymbol{e}_3 \cdot \boldsymbol{e}_2 & \boldsymbol{e}_3 \cdot \boldsymbol{e}_3 \end{vmatrix} \\
&= \begin{bmatrix} (\boldsymbol{e}_1 \cdot \boldsymbol{e}_1)(\boldsymbol{e}_2 \cdot \boldsymbol{e}_2)(\boldsymbol{e}_3 \cdot \boldsymbol{e}_3) + (\boldsymbol{e}_2 \cdot \boldsymbol{e}_1)(\boldsymbol{e}_3 \cdot \boldsymbol{e}_2)(\boldsymbol{e}_1 \cdot \boldsymbol{e}_3) \\ +(\boldsymbol{e}_3 \cdot \boldsymbol{e}_1)(\boldsymbol{e}_1 \cdot \boldsymbol{e}_2)(\boldsymbol{e}_2 \cdot \boldsymbol{e}_3) - (\boldsymbol{e}_1 \cdot \boldsymbol{e}_2)(\boldsymbol{e}_2 \cdot \boldsymbol{e}_2)(\boldsymbol{e}_3 \cdot \boldsymbol{e}_1) \\ -(\boldsymbol{e}_2 \cdot \boldsymbol{e}_3)(\boldsymbol{e}_3 \cdot \boldsymbol{e}_1)(\boldsymbol{e}_1 \cdot \boldsymbol{e}_1) - (\boldsymbol{e}_3 \cdot \boldsymbol{e}_3)(\boldsymbol{e}_1 \cdot \boldsymbol{e}_2)(\boldsymbol{e}_2 \cdot \boldsymbol{e}_1) \end{bmatrix} \\
&= e_{\alpha\beta\gamma} (\boldsymbol{e}_1)_\alpha (\boldsymbol{e}_2)_\beta (\boldsymbol{e}_3)_\gamma\, e_{\mu\nu\sigma} (\boldsymbol{e}_1)_\mu (\boldsymbol{e}_2)_\nu (\boldsymbol{e}_3)_\sigma \\
&= [\boldsymbol{e}_1 \cdot (\boldsymbol{e}_2 \times \boldsymbol{e}_3)]^2 = \underline{V}^2 > 0\ .
\end{aligned}$$

In the same way, we get $|g^{\alpha\beta}| = \overline{V}^2$. As follows from (1.35), $|g^{\alpha\beta}| g = 1$; therefore, $|g^{\alpha\beta}| = g^{-1} = \underline{V}^{-2} > 0$ and $\overline{V} = \underline{V}^{-1}$. □

Problems

1.37. When we transition from one oblique rectilinear system of coordinates to another, the basic vectors $\boldsymbol{e}_\alpha$ determining the directions of the coordinate axes are transformed in accordance with the following law:

$$\boldsymbol{e}'_\alpha = a_\alpha{}^\beta \boldsymbol{e}_\beta \ , \tag{1.36}$$

where $a_\alpha{}^\beta$ is the transformation matrix.[9)]

1. Express its elements through the scalar products of the basic vectors of the original and transformed systems.
2. Build the reverse transformation matrix.
3. Show that the same matrices define the transformations of the vectors of the mutual basis.
4. Find the rules of the transformation of the covariant and contravariant components of an arbitrary vector.
5. Find the rules of the transformation of the covariant and contravariant components of a metric tensor.

1.38. Show the laws of the transformation of the vectors of the original and mutual bases in the case of the mirror reflection of the system of coordinates.

1.39. Express the scalar product of two vectors in three different forms: through the covariant and contravariant components and through both of them. Prove its invariance with respect to the transformations (1.36) of the coordinate system. Express, in various forms, the square of the distance $\mathrm{d}l^2$ between two close points.

1.40. Write down the vector product of two vectors $\boldsymbol{C} = \boldsymbol{A} \times \boldsymbol{B}$ in terms the covariant and contravariant components of the factors.

1.41. Write down the cosine of the angle between vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ in terms of their covariant and contravariant components.

1.1.4
Tensors in Curvilinear and Nonorthogonal Systems of Coordinates

We will now consider arbitrary transformations in the case of a transition from a Cartesian to a certain curvilinear and, generally speaking, nonorthogonal system of coordinates or between curvilinear and nonorthogonal systems of coordinates (Borisenko and Tarapov, 1966, Section 2.8). The connection between the coordinates x^α and x'^β $(\alpha, \beta = 1, 2, 3)$ of two coordinate systems described by certain general form relations is

$$x^\alpha = f^\alpha(x'^1, x'^2, x'^3) \tag{1.37}$$

9) The transformation in question is not necessarily limited to the rotation of the oblique system as a whole. It may change the angles between the axes and coordinates scales.

(we will now indicate coordinate numbers with upper indices). The linear homogeneous function $f^{\alpha}(x'^1, x'^2, x'^3)$ with constant coefficients corresponds to the affine transformation (1.36). The rotation of the orthogonal rectilinear coordinate system is determined by the orthogonal matrix of coefficients with a unit determinant.

So that (1.37) can be solved with respect to x'^{β} and the reverse transformation $x'^{\beta} = \varphi^{\beta}(x^1, x^2, x^3)$ can be found, the functional determinant J must be different from zero,

$$J = \left| \frac{\partial x^{\alpha}}{\partial x'^{\beta}} \right| \neq 0 \,, \tag{1.38}$$

which hereafter will be presumed. The differentials of the coordinates are transformed in accordance with

$$\mathrm{d}x^{\alpha} = \frac{\partial x^{\alpha}}{\partial x'^{\beta}} \mathrm{d}x'^{\beta} \,, \tag{1.39}$$

where the coefficients of the transformation $\partial x^{\alpha}/\partial x'^{\beta}$, in the general case, become the functions of the coordinates. The connection between the differentials remains linear, as in the case of affine transformations, which, generally speaking, is not the case for the connection between the coordinates themselves. Although (1.37) describes the transition from the orthogonal Cartesian system of coordinates x^{α} to an arbitrary system q^{β} (to make things clearer, we hereafter will label curvilinear coordinates as q), we will write the square of the distance between close points with the use of (1.39) as

$$\mathrm{d}l^2 = \delta_{\alpha\beta} \mathrm{d}x^{\alpha} \mathrm{d}x^{\beta} = g_{\mu\nu} \mathrm{d}q^{\mu} \mathrm{d}q^{\nu} \,, \tag{1.40}$$

where the values

$$g_{\mu\nu}(q) = \frac{\partial x^{\alpha}}{\partial q^{\mu}} \frac{\partial x^{\beta}}{\partial q^{\nu}} \delta_{\alpha\beta} \,, \quad g_{\mu\nu} = g_{\nu\mu} \tag{1.41}$$

are called the *covariant components of the metric tensor,* and its *contravariant components* $g^{\mu\nu} = g^{\nu\mu}$ are determined by the conditions

$$g^{\alpha\nu} g_{\nu\mu} = g_{\mu\nu} g^{\nu\alpha} = \delta^{\alpha}_{\mu} \,, \tag{1.42}$$

which means that the tensors $g^{\mu\nu}$ and $g_{\mu\nu}$ are mutually inverse. Because the coefficients of transformation (1.39) satisfy the relation

$$\frac{\partial x^{\alpha}}{\partial q^{\beta}} \frac{\partial q^{\beta}}{\partial x^{\nu}} = \frac{\partial x^{\alpha}}{\partial x^{\nu}} = \delta^{\alpha}_{\nu} \,, \tag{1.43}$$

the contravariant components of the metric tensor may be written as[10)]

$$g^{\alpha\beta} = \frac{\partial q^{\alpha}}{\partial x^{\sigma}} \frac{\partial q^{\beta}}{\partial x^{\kappa}} \delta^{\sigma\kappa} \,. \tag{1.44}$$

10) Tensors $\delta_{\mu\nu}$, $\delta^{\mu\nu}$, and δ^{μ}_{ν} correspond to the rectilinear Cartesian system of coordinates, their contravariant and covariant components coincide with each other, and the location of the symbols is indifferent.

The latter relations, just like (1.41), may be regarded as the rule of the transformation of the metric tensor from Cartesian coordinates ($\delta^{\sigma\kappa}$) to arbitrary curvilinear coordinates q^α. It is easy to see that the same rule applies to the transformation of the metric tensor from a curvilinear system q^α to another curvilinear system q'^β:

$$g'^{\kappa\sigma} = \frac{\partial q'^\kappa}{\partial x^\mu}\frac{\partial q'^\sigma}{\partial x^\nu}\delta^{\mu\nu} = \frac{\partial q'^\kappa}{\partial q^\alpha}\frac{\partial q'^\sigma}{\partial q^\beta}g^{\alpha\beta}\,, \tag{1.45}$$

where $g^{\alpha\beta}$ is defined in accordance with (1.44).

One can easily make sure that the relations written above mostly repeat the formulas obtained when considering the oblique-angled (affine) system of coordinates, being their generalizations, in a certain way. For instance, multiplying both parts of (1.39) by the Cartesian orts $\boldsymbol{e}_\alpha^{(D)}$ and relabeling x'^β as q^β, we get the increase of the radius vector

$$\mathrm{d}\boldsymbol{r} = \boldsymbol{e}_\alpha^{(D)}\mathrm{d}x^\alpha = \frac{\partial x^\alpha}{\partial q^\beta}\boldsymbol{e}_\alpha^{(D)}\mathrm{d}q^\beta = \boldsymbol{e}_\beta\mathrm{d}q^\beta\,.$$

This means that the basic vectors $\boldsymbol{e}_\beta$ of the curvilinear system (not unit in the general case) may be written as

$$\boldsymbol{e}_\beta = \frac{\partial x^\alpha}{\partial q^\beta}\boldsymbol{e}_\alpha^{(D)}\,. \tag{1.46}$$

The right-hand side of the latter equality includes Cartesian orthogonal unit vectors. As follows from (1.46), the connection between the basic vectors of the curvilinear systems of coordinates q'^μ and q^β looks the same way as (1.46):

$$\boldsymbol{e}'_\beta = \frac{\partial q^\alpha}{\partial q'^\beta}\boldsymbol{e}_\alpha\,. \tag{1.47}$$

Further on, we will define the vectors of the mutual basis $\boldsymbol{e}^\beta$ of the curvilinear system. As follows from (1.46) and the conditions in (1.29),

$$\boldsymbol{e}^\alpha\cdot\boldsymbol{e}_\beta = \frac{\partial x^\mu}{\partial q^\beta}\boldsymbol{e}^\alpha\cdot\boldsymbol{e}^\mu_{(D)} = \delta^\alpha_\beta\,, \tag{1.48}$$

which means that

$$\boldsymbol{e}^\alpha = \frac{\partial q^\alpha}{\partial x^\nu}\boldsymbol{e}^\nu_{(D)} \tag{1.49}$$

(we use the equality of the lower and upper symbols for Cartesian vectors). Finally, considering (1.41) and (1.44), we see that the relations in (1.34) remain valid for curvilinear coordinates,

$$g_{\alpha\beta} = \boldsymbol{e}_\alpha\cdot\boldsymbol{e}_\beta\,,\quad g^{\alpha\beta} = \boldsymbol{e}^\alpha\cdot\boldsymbol{e}^\beta\,,\quad g^\beta_\alpha = \boldsymbol{e}_\alpha\cdot\boldsymbol{e}^\beta = \delta^\beta_\alpha\,, \tag{1.50}$$

as do the rules of raising and lowering indices (1.35).

We now will give a definition of tensor, as it relates to the general transformations of coordinates.

A *tensor of rank 2* in the three-dimensional space is a nine-component quantity whose contravariant components are transformed as products of the differentials of coordinates, that is, in accordance with the following:

$$T^{\alpha\beta} = \frac{\partial q^\alpha}{\partial q'^\mu}\frac{\partial q^\beta}{\partial q'^\nu} T'^{\mu\nu} \quad \text{or} \quad T'^{\mu\nu} = \frac{\partial q'^\mu}{\partial q^\alpha}\frac{\partial q'^\nu}{\partial q^\beta} T^{\alpha\beta} \,. \tag{1.51}$$

This definition is directly generalized to include tensors of any rank. For instance, scalar S is not transformed, whereas the covariant components of a tensor of rank 1 (vector) are transformed in accordance with

$$A_\alpha = \frac{\partial q'^\beta}{\partial q^\alpha} A_\beta \,. \tag{1.52}$$

The fundamental difference between the above definition of a tensor and the previous ones (for the cases of rotation and affine transformation) is that now the transformation coefficients depend on the locations. This means that the definition of a tensor is of a local nature. For instance, the products of the components of vectors located at different points $q^\alpha \neq p^\alpha$, that is, $A^\alpha(q)B^\beta(p)$, do not form a tensor.

Unlike Cartesian coordinates, the totality of arbitrary curvilinear coordinates q^α, $\alpha = 1, 2, 3$, does not form a vector because the coordinates do not comply with rule of transformation (1.51). Most significantly, these peculiarities manifest themselves in differentiating and integrating tensor operations, which are considered in Section 1.2.

The covariant components of a tensor of any rank are produced from the contravariant ones by the metric tensor as per (1.35). In the general case, the mixed tensor depends on the place, first or second, occupied by the upper and lower symbols, that is, $T_\alpha{}^\beta \neq T^\beta{}_\alpha$. The contraction operation, decreasing the rank of any tensor by 2, is defined as summation over one upper and one lower indices, for instance,

$$A_\alpha B^\alpha = A'_\beta B'^\beta = \text{inv} \,, \quad T_{\alpha\beta}{}^\beta = C_\alpha \tag{1.53}$$

– the covariant vector, and so on.

Problems

1.42. Express the components of a metric tensor through the components of the orthogonal Cartesian orts $\boldsymbol{e}_\alpha^{(D)} = \boldsymbol{e}_{(D)}^\alpha$, $\alpha = 1, 2, 3$ specified in a certain curvilinear system of coordinates.

1.43. Show that the functional determinant (1.39) is expressed through the determinant of a metric tensor $g = |g_{\mu\nu}| : \; J = \sqrt{g}$.

Hint: Following from equality (1.42), express the determinant g through the determinants of the matrices found in the second member of the equality.

1.44. Write down the square of the length of the vector A^2 and the cosine of the angle between two vectors in a arbitrary curvilinear system of coordinates.

1.45. Transform the antisymmetric unit tensor $e^{\alpha\beta\gamma}$ in an curvilinear system of coordinates.

1.46. The metric tensor $g_{\alpha\beta}$ determining the square of the small element of length in curvilinear nonorthogonal coordinates, in accordance with formulas (1.41), is known. Three curvilinear coordinate lines may be drawn through each point of the space, only one coordinate q^1, q^2, or q^3 changing along each of these lines, whereas the other two remain constant.

1. Find the connection between the element of length of a coordinate line and the differential of the respective coordinate.
2. Indicate the three basic vectors tangent to the coordinate curves at the specified point.
3. Find the cosines of the angles between the coordinate curves at that point.
4. Indicate the properties the metric tensor must have to make the curvilinear system orthogonal.

1.47. Write down the covariant and contravariant components of a metric tensor for a spherical and a cylindrical system of coordinates (see the drawing in the solution of Problem 1.18). Also, write down the vectors of the covariant and contravariant bases, expressing them through the basic orts considered in Problem 1.18.

1.48. Show that the volume element in curvilinear coordinates has the following form:

$$\mathrm{d}V = \sqrt{g}\mathrm{d}q^1\mathrm{d}q^2\mathrm{d}q^3 \,, \tag{1.54}$$

where g is the determinant of a metric tensor. Find the volume element in spherical and cylindrical coordinates.

Hint: The volume element sought is the volume of an oblique-angled parallelepiped built on the elementary lengths $\mathrm{d}l^1$, $\mathrm{d}l^2$, and $\mathrm{d}l^3$ of the curvilinear coordinate axes. It may be found with the use of the results obtained in Problems 1.40 and 1.46.

Recommended literature:
Borisenko and Tarapov (1966); Arfken (1970); Rashevskii (1953); Lee (1965); Mathews and Walker (1964). See also Ugarov (1997, Addendum I).

1.2
Vector and Tensor Calculus

Scalar or vector functions representing the distribution of various physical quantities in three-dimensional space are sometimes called the fields of those quantities. This is how one may speak of fields of temperatures $T(x, y, z)$ or pressures $p(x, y, z)$ in the atmosphere, the fields of speeds in moving fluids or gases $\boldsymbol{u}(x, y, z)$, the electromagnetic vector field, and so on. Derivatives and integrals from such scalar and vector functions have certain common mathematical properties, which are very important for physical applications. One should become familiar and comfortable with these properties in advance. Only then, may such areas of physics as the theory of electromagnetic phenomena, the mechanics of fluids, gases, and solid bodies, quantum physics, and quantum field theory be successfully learned and fully understood.

1.2.1
Gradient and Directional Derivative. Vector Lines

We encounter the concept of the gradient of a scalar function in classical mechanics when learning about the properties of potential forces. Let us say there is a differentiable function $U(x, y, z)$ whose partial derivatives are equal to the components of the vector of the force $\boldsymbol{F}(x, y, z)$, which, in this case, is called a *potential*:

$$F_x = -\frac{\partial U}{\partial x}\,, \quad F_y = -\frac{\partial U}{\partial y}\,, \quad F_z = -\frac{\partial U}{\partial z}\,, \quad \text{or} \quad \boldsymbol{F} = -\nabla U(x, y, z)\,, \tag{1.55}$$

where

$$\nabla = \boldsymbol{e}_x \frac{\partial}{\partial x} + \boldsymbol{e}_y \frac{\partial}{\partial y} + \boldsymbol{e}_z \frac{\partial}{\partial z} = \boldsymbol{e}_\alpha \frac{\partial}{\partial x_\alpha} \tag{1.56}$$

is *Hamilton's operator*[11] (nabla).

$$\text{grad}\; U(x, y, z) \equiv \nabla U(x, y, z) = \boldsymbol{e}_x \frac{\partial U}{\partial x} + \boldsymbol{e}_y \frac{\partial U}{\partial x} + \boldsymbol{e}_z \frac{\partial U}{\partial z} \tag{1.57}$$

is called the *gradient* of the scalar function $U(x, y, z)$. The necessary and sufficient conditions for the representation of the vector as a scalar function come in the form of equalities:

$$\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x}\,, \quad \frac{\partial F_y}{\partial z} = \frac{\partial F_z}{\partial y}\,, \quad \frac{\partial F_z}{\partial x} = \frac{\partial F_x}{\partial z}\,. \tag{1.58}$$

They follow from the equality of cross-derivatives, for example,

$$\frac{\partial^2 U}{\partial x \partial y} = \frac{\partial^2 U}{\partial y \partial x}\,.$$

So far, we have been using only Cartesian coordinates. A generalization to include oblique nonorthogonal coordinates will be made in the closing part of this section (also see Problem 1.50 and later).

11) William Rowan Hamilton (1805–1865) was an outstanding Irish mathematician and physicist. He was engaged in mechanics and optics, and created the mathematical apparatus that, after many decades, became the basis of quantum mechanics and quantum field theory.

It is important to understand that a gradient is always directed toward increasing U, along a normal to the surface of the constant value of the scalar field $U(x, y, z) = \text{const}$. This follows from our obtaining, when differentiating the latter equality, $\mathrm{d}\boldsymbol{r} \cdot \nabla U = 0$. Since $\mathrm{d}\boldsymbol{r}$ is here a tangent to the surface $U = \text{const}$, the gradient is perpendicular to that surface.

Example 1.4

Show that the derivative of the scalar function, along the direction determined by the unit vector $\boldsymbol{l}$, is equal to the projection of the gradient onto that direction:

$$\frac{\partial U}{\partial l} = \mathrm{grad}_l\, U \equiv (\boldsymbol{l} \cdot \nabla)\, U \,. \tag{1.59}$$

Solution. Label the derivative, along the specified direction $\boldsymbol{l}$, as $\partial U/\partial l$. When displaced from the point with radius vector $\boldsymbol{r}$ to a distance s along the direction $\boldsymbol{l}$, the function will take the value of $U(x + l_x s, y + l_y s, z + l_z s)$. The derivative in the specified direction is the derivative at distance s:

$$\begin{aligned} \frac{\partial U}{\partial l} &= \frac{\partial}{\partial s} U(x + l_x s, y + l_y s, z + l_z s)|_{s=0} = \frac{\partial U}{\partial x} l_x + \frac{\partial U}{\partial y} l_y + \frac{\partial U}{\partial z} l_z \\ &= (\boldsymbol{l} \cdot \nabla)\, U(\boldsymbol{r}) \,. \end{aligned}$$

□

Expression (1.58) also makes sense when applied to an arbitrary vector $\boldsymbol{A}(x, y, z)$: the quantity $(\boldsymbol{l} \cdot \nabla)\boldsymbol{A}(x, y, z)$ is a *derivative of vector* $\boldsymbol{A}$ *in direction* $\boldsymbol{l}$. This follows from the condition that the operator $(\boldsymbol{l} \cdot \nabla)$ must be applied to every projection of $\boldsymbol{A}$ and will produce the required derivatives, whereas their combination must be construed as a derivative of the whole vector in the specified direction.

A vivid conception of the structure of the vector field $\boldsymbol{A}$ is provided by *vector lines*.[12)] These are lines tangents to which, at any point, indicate the direction of vector $\boldsymbol{A}$ at that point. It is easy to write a system of equations in order to find the vector lines of the specified field $\boldsymbol{A}(x, y, z)$. The condition of the small element $\mathrm{d}\boldsymbol{l} = (\mathrm{d}x, \mathrm{d}y, \mathrm{d}z)$ being parallel to the vector line and vector $\boldsymbol{A}$ may be written as $\boldsymbol{A} \times \mathrm{d}\boldsymbol{l} = 0$. Having written this vector equality in projections on the respective axes, we get differential equations for two families of surfaces whose intersect lines are exactly the vector lines sought.

For instance, using Cartesian coordinates, we will have

$$\frac{\mathrm{d}x}{A_x(x, y, z)} = \frac{\mathrm{d}y}{A_y(x, y, z)} = \frac{\mathrm{d}z}{A_z(x, y, z)} \,. \tag{1.60}$$

12) If $\boldsymbol{A}$ is a vector of a force, the lines are called force lines. Sometimes, the term "force lines" is applied to any vector regardless of its physical meaning.

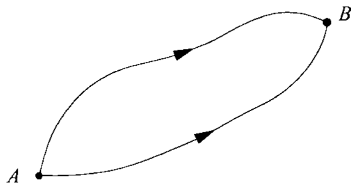

Figure 1.4 The independence of work done by a potential force from the shape of the path of a material point.

The vector lines of any potential vector are perpendicular to the equipotential surfaces $U(x, y, z) = \text{const}$. This follows from the properties of the gradient of a scalar function.

The loop integral of the scalar product of a potential vector and the vector element of the length of the loop has an important property:

$$\int_A^B \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} = \int_A^B (F_x \mathrm{d}x + F_y \mathrm{d}y + F_z \mathrm{d}z) \,, \tag{1.61}$$

where the vector $\mathrm{d}\boldsymbol{s}$ has constituents $\mathrm{d}x$, $\mathrm{d}y$, and $\mathrm{d}z$, that is, the differentials of the coordinates are not independent and are just increments *along the loop*. Such integrals express work done by the force $\boldsymbol{F}$ on a material point moving along a specified trajectory from A to B and many other physical quantities. If the vector is a potential vector, then

$$F_x \mathrm{d}x + F_y \mathrm{d}y + F_z \mathrm{d}z = -\frac{\partial U}{\partial x}\mathrm{d}x - \frac{\partial U}{\partial y}\mathrm{d}y - \frac{\partial U}{\partial z}\mathrm{d}z = -\mathrm{d}U \tag{1.62}$$

is the complete differential of the function $U(x, y, z)$. The computation of the integral gives us

$$\int_A^B \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} = -\int_A^B \mathrm{d}U = U_{\mathrm{A}} - U_{\mathrm{B}} \,, \tag{1.63}$$

where $\mathrm{d}U$ is the increase of the function along the small segment $\mathrm{d}\boldsymbol{s}$ and $\int_A^B \mathrm{d}U$ is the full increase along the distance AB.

In this case, integration along the loop does not depend on the form of the curve, and only depends on the start and end points of the integration (Figure 1.4).

Integrating along a closed loop (Figure 1.5), we get the following:

$$\int_A^B \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} = U_{\mathrm{A}} - U_{\mathrm{B}} \,, \quad \int_{\mathrm{B}}^A \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} = U_{\mathrm{B}} - U_{\mathrm{A}} \,,$$

$$\oint \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} = \int_A^B \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} + \int_{\mathrm{B}}^A \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} = 0 \,. \tag{1.64}$$

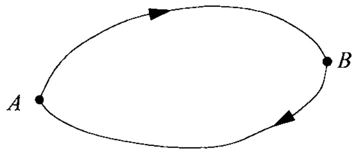

Figure 1.5 Diagram for the computation of the circulation of a vector along a closed loop.

Closed-loop integration over $\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s}$ is called *the circulation of vector* $\boldsymbol{F}$ along the loop. The circulation of a *potential vector* along any closed loop equals zero (however, an arbitrary vector has no such property!).

It is important, however, to note that the condition of the representation of a vector as (1.55) is necessary but not sufficient for equalities (1.63) and (1.64) to be valid. It is also necessary for the potential function $U(\boldsymbol{r})$ to be the unambiguous function of a point. Otherwise, for instance, after the circulation of the loop and return to point A, the potential U may take a different value, and equality (1.64) will be no longer valid.

Problems

1.49. Show that when a Cartesian system of coordinates is rotated, Hamilton's operator (∇) (1.56) is transformed in accordance with rule (1.2) of vector transformation.

1.50. Find the potential energy that corresponds to the force $F_x(x, y) = x + y$, $F_y(x, y) = x - y^2$. Find the work R done by this force between points (0,0) and (a, b).

1.51. Show that in cylindrical and spherical systems of coordinates , Hamilton's operator ∇ is expressed, respectively, as

$$1. \quad \nabla = \boldsymbol{e}_\rho \frac{\partial}{\partial \rho} + \boldsymbol{e}_\varphi \frac{1}{\rho} \frac{\partial}{\partial \varphi} + \boldsymbol{e}_z \frac{\partial}{\partial z}\,, \tag{1.65}$$

$$2. \quad \nabla = \boldsymbol{e}_r \frac{\partial}{\partial r} + \boldsymbol{e}_\vartheta \frac{1}{r} \frac{\partial}{\partial \vartheta} + \boldsymbol{e}_\varphi \frac{1}{r \sin \vartheta} \frac{\partial}{\partial \varphi}\,. \tag{1.66}$$

For that purpose, consider the elementary lengths in the directions of the respective coordinate orts and use formula (1.59), which connects the gradient with the directional derivative.

1.52. Use Cartesian spherical and cylindrical coordinates (see (1.56), (1.65), and (1.66)) to find $\operatorname{grad}(\boldsymbol{l} \cdot \boldsymbol{r}), (\boldsymbol{l} \cdot \nabla)\boldsymbol{r}$, where $\boldsymbol{r}$ is a radius vector and $\boldsymbol{l}$ is a constant vector.

1.53. Show that

$$\operatorname{grad} f(r) = \frac{\mathrm{d}f}{\mathrm{d}r} \frac{\boldsymbol{r}}{r}\,.$$

1.54. Write down a system of equations determining the vector lines in cylindrical and spherical coordinates, respectively.

1.55. Find

$$\text{grad}\,\frac{(\boldsymbol{p}\cdot\boldsymbol{r})}{r^3}\,, \quad \boldsymbol{p} = \text{const}\,.$$

1.56. Use spherical coordinates to draw a family of lines tangent to vector

$$\boldsymbol{E} = \frac{3(\boldsymbol{p}\cdot\boldsymbol{r})\boldsymbol{r}}{r^5} - \frac{\boldsymbol{p}}{r^3}\,, \quad \boldsymbol{p} = \text{const}\,.$$

1.57. Write down the cyclic components of a gradient in spherical coordinates. Find the definition of the cyclic components in the situation in Problem 1.17.

1.2.2
Divergence and Curl. Integral Theorems

Now, we will consider the effect of the ∇ operator on an arbitrary vector $\boldsymbol{A}$. As is known, two vectors may produce two types of products: a scalar

$$\text{div}\,\boldsymbol{A} \equiv \nabla\cdot\boldsymbol{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \frac{\partial A_\alpha}{\partial x_\alpha} \tag{1.67}$$

and a vector

$$\text{curl}\,\boldsymbol{A} \equiv \nabla\times\boldsymbol{A} = \boldsymbol{e}_x\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \boldsymbol{e}_y\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \boldsymbol{e}_z\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\,. \tag{1.68}$$

Both of these quantities are extremely important for vector calculus and are called the *divergence* (scalar!) and the *curl* (vector!). The left-hand-side members of the equalities contain the respective lettering. The right-hand-side members contain their explicit expressions *in Cartesian coordinates only*. For you to better realize their mathematical and physical meanings, we give other definitions of these important quantities, less formal and more obvious, if somewhat more complex. Yet the latter disadvantage is also an advantage in that the definitions in questions, unlike (1.67) and (1.68), do not depend on the selection of a system of coordinates. We will begin with divergence.

Select point M where you would like to define the divergence of vector field $\boldsymbol{A}(\boldsymbol{r})$. Surround that point with a closed smooth surface, enclosing a certain volume ΔV and find at every point of the surface an outside normal $\boldsymbol{n}$. We will call the product $\boldsymbol{n}\mathrm{d}S$ the vector element of the surface. The integral over the closed surface $\oint_S \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{S}$ produces the flux of the vector $\boldsymbol{A}$ through the surface S. Now, we will define divergence in a way different from (1.67):

$$\text{div}\,\boldsymbol{A}(\boldsymbol{r}) = \lim_{\Delta V\to 0}\frac{1}{\Delta V}\oint_S \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{S}\,. \tag{1.69}$$

It is presumed here that the volume ΔV shrinks into point M. The little circle on the integral sign means a closed surface.

Example 1.5

Make sure that the definitions (1.67) and (1.69) are equivalent when Cartesian coordinates are used. In order to do that, select volume $\Delta V = \mathrm{d}V = \mathrm{d}x\mathrm{d}y\mathrm{d}z$ forming a small rectangular parallelepiped with edges $\mathrm{d}x, \mathrm{d}y, \mathrm{d}z$ and find the boundary (1.69).

Solution. Making use of the smallness of the ribs of the parallelepiped, write down the approximate expression for the surface integral:

$$\oint_S \boldsymbol{A} \cdot d\boldsymbol{S} \approx [A_x(x + dx, \bar{y}, \bar{z}) - A_x(x, \bar{y}, \bar{z})]dy\,dz$$

$$+ [A_y(\bar{x}, y + dy, \bar{z}) - A_y(\bar{x}, y, \bar{z})]dx\,dz$$
$$+ [A_z(\bar{x}, \bar{y}, z + dz) - A_z(\bar{x}, \bar{y}, z)]dx\,dy$$
$$\approx \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) dV \,.$$

The mean value theorem was used when evaluating the integrals over the six separate edges, the quantities $\bar{x}$, $\bar{y}$, and $\bar{z}$ being the values of the coordinates at a certain point of a respective edge. Also considered was the fact that normals are directed oppositely at the opposite edges and that when the volume shrinks to point M, all the coordinates take the values they must have at that point. Using the latter result, make sure that the definition of divergence (1.69), when Cartesian coordinates are used, leads to formula (1.67). □

This means that the divergence at a certain point is other than zero if there is a nonzero vector flux through a closed surface surrounding the point in question. Inside the surface, there must be a source of a vector field that creates the flux. This is to say that divergence characterizes the density of field sources.

The above method of computing an integral over a small surface may be used to obtain explicit expressions of divergence in the most often used systems of coordinates, such as spherical, cylindrical, and so on. The shape of the volume should be selected each time so that one of the coordinates remains constant on each of its side surfaces.

Example 1.6

On the basis of the definition of divergence (1.67), produce a relation connecting the integral from div $\boldsymbol{A}$ over a certain volume with vector flux $\boldsymbol{A}$ through the surface bounding the volume in question.

Solution. Select any finite volume V bounded by a smooth closed surface S. Divide it into small cells ΔV_i, each bounded by a respective surface ΔS_i. The surfaces bounding the cells adjacent to the outside surface S will partially coincide with S. All other portions of the surfaces S_i will be shared by pairs of adjacent cells. Making use of the smallness of each cell, use relation (1.69), giving it an approximate form:

$$(\operatorname{div} \boldsymbol{A})_i \Delta V_i \approx \oint_{S_i} \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{S}_i \ . \tag{1.70}$$

Now sum the first and second members of the latter approximate equality over i and pass to a limit, reducing the volume of each cell to zero and expanding the number of cells to infinity. The first member of the equality will now become an integral over the full volume V of divergence $\boldsymbol{A}$: $\int_V \operatorname{div} \boldsymbol{A} \mathrm{d}V$. In the second member of the equality, the integrals over the inner portions of the surface will cancel each other, the outer normals to each pair of adjacent cells being oppositely directed. Only the integral over the outside surface S bounding the full volume V remains. As a result, you will have an exact integral relation,

$$\int_V \operatorname{div} \boldsymbol{A} \mathrm{d}V = \oint_S \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{S} \ , \tag{1.71}$$

called the *Gauss–Ostrogradskii theorem*[13] (in Western literature, the name Ostrogradskii is omitted).

The Gauss–Ostrogradskii theorem is applicable to any tensor of rank $s \geq 1$, for instance,

$$\int_V \frac{\partial T_{\alpha\beta\mu}}{\partial x_\mu} \mathrm{d}V = \oint_S T_{\alpha\beta\mu} \mathrm{d}S_\mu \tag{1.72}$$

(for the proof, refer to Problem 1.70*). □

The curl of a vector field allows a definition similar to that of divergence (1.69). At point M, specify a unit vector $\boldsymbol{n}$, that is, a direction. Make up a small flat area ΔS containing a point M and perpendicular to $\boldsymbol{n}$. Then define the direction of tracing the loop l that bounds the area, coordinated with the direction $\boldsymbol{n}$ as per the right-screw rule. The projection of the rotor onto direction $\boldsymbol{n}$ at point M is defined as follows:

$$\operatorname{curl}_n \boldsymbol{A} = \lim_{\Delta S \to 0} \frac{1}{\Delta S} \oint_l \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{l} \ , \tag{1.73}$$

where the integral represents the circulation of the vector $\boldsymbol{A}$ along the closed loop l.

13) Carl Friedrich Gauss (1777–1855) was an outstanding German mathematician, astronomer, and physicist. Mikhail Ostrogradskii (1801–1862) was a Russian mathematician known for his works in mathematical physics, theoretical mechanics, and probability theory.

Example 1.7

Make sure that the definitions of (1.68) and (1.73) are equivalent when Cartesian coordinates are used. For that purpose, find the projections of the curl on Cartesian axes using (1.73) and by selecting a rectangular area with sides parallel to the coordinate axes.

Solution. Direct $\boldsymbol{n}$ along the Oz axis, select a rectangular area $\Delta S = \mathrm{d}S = \mathrm{d}x\mathrm{d}y$, and use, as in the previous integral calculation, the mean value theorem to get the following:

$$\begin{aligned}\oint_l \boldsymbol{A}\cdot \mathrm{d}\boldsymbol{l} &\approx [A_y(x+\mathrm{d}x, \bar{y}, z) - A_y(x, \bar{y}, z)]\mathrm{d}y \\ &\quad + [A_x(\bar{x}, y, z) - A_y(\bar{x}, y+\mathrm{d}y, z)]\mathrm{d}x \\ &\approx \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\mathrm{d}S\ .\end{aligned}$$

After inserting this result into (1.73) and passing to a limit, we get the exact expression for $\mathrm{curl}_z \boldsymbol{A}$ in Cartesian coordinates, coinciding with (1.68). In the same way, one may find other projections of the curl. The curl will be other than zero if the lines of vector $\boldsymbol{A}$ curved, having either closed or spiral configurations. □

Example 1.8

Using the definition of the curl (1.73), find the integral relation that connects the circulation of any vector along a closed loop with the curl flux of that vector through a nonclosed surface bounded by that loop.

Solution. Find an arbitrary three-dimensional nonclosed surface S bounded by loop l and, at every point of the surface, find normal $\boldsymbol{n}$. Divide the surface into small portions ΔS_i, each bounded by loop l_i. On the basis of (1.73), an approximate value may be written for every such area:

$$\mathrm{curl}_n \boldsymbol{A}\Delta S_i \approx \oint_{l_i} \boldsymbol{A}\cdot \mathrm{d}\boldsymbol{l}_i\ . \tag{1.74}$$

After summing the two members of the approximate equality over i and passing to a limit of the infinitely small areas, we get the exact equality (*Stokes theorem*[14]):

$$\int_S \mathrm{curl}\,\boldsymbol{A}\cdot \mathrm{d}\boldsymbol{S} = \oint_l \boldsymbol{A}\cdot \mathrm{d}\boldsymbol{l}\ . \tag{1.75}$$

14) George Gabriel Stokes (1819–1903) was an Irish physicist and mathematician.

An integral over the outer loop that bounds area S remains in the second member. All integrals over inner loops are canceled. Stokes theorem connects the integral over the curl flux through the surface with the circulation of the vector along the loop that bounds that surface. □

1.2.3
Solenoidal and Potential (Curl-less) Vectors

Let us say that vector field $\boldsymbol{H}(\boldsymbol{r})$, over the whole space, satisfies the condition

$$\operatorname{div} \boldsymbol{H} = 0 \tag{1.76}$$

(in this case, vector $\boldsymbol{H}$ is called a *solenoidal* vector). This, for instance, is a property of a magnetic field. It is possible to prove (we will, for now, abstain from doing that) that condition (1.76) is necessary and sufficient for vector $\boldsymbol{H}$ to be represented as the curl of another vector $\boldsymbol{A}(\boldsymbol{r})$:

$$\boldsymbol{H} = \operatorname{curl} \boldsymbol{A} . \tag{1.77}$$

Using the rules of vector differentiation, we can easily make sure that condition (1.76) is satisfied whatever the value of $\boldsymbol{A}$ is:

$$\operatorname{div} \boldsymbol{H} = \nabla \cdot \boldsymbol{H} = \nabla \cdot [\nabla \times \boldsymbol{A}] = [\nabla \times \nabla] \cdot \boldsymbol{A} = 0 .$$

As noted previously, a *potential vector* is a vector that may be represented as the gradient of a certain scalar function:

$$\boldsymbol{E}(\boldsymbol{r}) = -\operatorname{grad} U(\boldsymbol{r}) \equiv -\nabla U(\boldsymbol{r}) . \tag{1.78}$$

The necessary and sufficient conditions of the potentiality of a vector are expressed by equalities of the kind in (1.58), which, in their vector form, give the following:

$$\operatorname{curl} \boldsymbol{E} = 0 . \tag{1.79}$$

Using the definition of the potential vector (1.78) and expressing the curl operation through the ∇ operator, we make sure that equality (1.79) is equally valid for any $U(\boldsymbol{r})$ functions that have second derivatives.

1.2.4
Differential Operations of Second Order

Differential operations of second order appear when the ∇ operator is applied to expressions of the kind ∇U, $\nabla \cdot \boldsymbol{A}$, and $\nabla \times \boldsymbol{A}$ that already contain this operator. Using the rules of vector algebra, we find that, in Cartesian coordinates, the *Laplace* *operator*[15)]

$$\nabla \cdot \nabla U(\mathbf{r}) = (\nabla \cdot \nabla) U(\mathbf{r}) = \nabla^2 U(\mathbf{r}) = \Delta U(\mathbf{r}) \,, \tag{1.80}$$

$\Delta = \nabla^2$, has the following form:

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \,. \tag{1.81}$$

This is a very important operator used in just about all problems when complex physical phenomena have to be described in the language of mathematics.

Further,

$$\nabla\nabla \cdot \mathbf{A} \equiv \nabla(\nabla \cdot \mathbf{A}) = \operatorname{grad}\operatorname{div}\mathbf{A} \,. \tag{1.82}$$

Even though such a combination of derivatives is hardly rare, no more compact letter notation has been devised for it.

The last operation of this kind is called a double vortex. It is transformed with the use of the following vector algebra formula (one should remember to place the differentiable vector function to the right of any operators that may affect it):

$$\operatorname{curl}\operatorname{curl}\mathbf{A} = \nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \operatorname{grad}\operatorname{div}\mathbf{A} - \Delta \mathbf{A} \,. \tag{1.83}$$

We see, therefore, that all the differential operations involving scalar and vector functions are expressed through the ∇ operator.

Problems

1.58. Show that div $\mathbf{A}$ (1.67) and the Laplace operator (1.81) are invariant with respect to rotations of Cartesian systems of coordinates and that curl $\mathbf{A}$ (1.68) is transformed as an antisymmetric tensor of rank 2 or as a vector that is dual to it.

1.59. Find $\nabla \cdot \mathbf{r}$, $\nabla \times \mathbf{r}$, $\nabla \cdot [\boldsymbol{\omega} \times \mathbf{r}]$, and $\nabla \times [\boldsymbol{\omega} \times \mathbf{r}]$, where $\boldsymbol{\omega}$ is a constant vector.

1.60. Find

$$\mathbf{H} = \operatorname{curl}\frac{(\mathbf{m} \times \mathbf{r})}{r^3} \,, \quad \mathbf{m} = \text{const} \,.$$

Build vector lines for vector $\mathbf{H}$ (draw a picture).

1.61. Using the rules of vector algebra and calculus and without making projections onto the coordinate axes, prove the following important identities frequently used in practical calculations:

$$\operatorname{grad}(\varphi\psi) = \varphi \operatorname{grad}\psi + \psi \operatorname{grad}\varphi \,, \tag{1.84}$$

15) Pierre Simon Laplace (1749–1827) was a French astronomer, mathematician, and physicist who actively expressed the ideas of mechanistic determinism; he was an atheist. His many scientific achievements were outstanding. Laplace repeatedly changed his politics, remaining in favor in republican France as well as in France under the rule of Napoleon Bonaparte and the restored Bourbons.

$$\operatorname{div}(\varphi \boldsymbol{A}) = \varphi \operatorname{div} \boldsymbol{A} + \boldsymbol{A} \cdot \operatorname{grad} \varphi\,, \tag{1.85}$$

$$\operatorname{curl}(\varphi \boldsymbol{A}) = \varphi \operatorname{curl} \boldsymbol{A} - \boldsymbol{A} \times \operatorname{grad} \varphi\,, \tag{1.86}$$

$$\operatorname{div}(\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{B} \cdot \operatorname{curl} \boldsymbol{A} - \boldsymbol{A} \cdot \operatorname{curl} \boldsymbol{B}\,, \tag{1.87}$$

$$\operatorname{curl}(\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{A} \operatorname{div} \boldsymbol{B} - \boldsymbol{B} \operatorname{div} \boldsymbol{A} + (\boldsymbol{B} \cdot \nabla)\boldsymbol{A} - (\boldsymbol{A} \cdot \nabla)\boldsymbol{B}\,, \tag{1.88}$$

$$\operatorname{grad}(\boldsymbol{A} \cdot \boldsymbol{B}) = \boldsymbol{A} \times \operatorname{curl} \boldsymbol{B} + \boldsymbol{B} \times \operatorname{curl} \boldsymbol{A} + (\boldsymbol{B} \cdot \nabla)\boldsymbol{A} + (\boldsymbol{A} \cdot \nabla)\boldsymbol{B}\,. \tag{1.89}$$

Here, φ and ψ are the scalar and $\boldsymbol{A}$, $\boldsymbol{B}$ vector functions of the coordinates.

1.62. Prove the following identities:

$$\boldsymbol{C} \cdot \operatorname{grad}(\boldsymbol{A} \cdot \boldsymbol{B}) = \boldsymbol{A} \cdot (\boldsymbol{C} \cdot \nabla)\boldsymbol{B} + \boldsymbol{B} \cdot (\boldsymbol{C} \cdot \nabla)\boldsymbol{A}\,, \tag{1.90}$$

$$(\boldsymbol{C} \cdot \nabla)(\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{A} \times (\boldsymbol{C} \cdot \nabla)\boldsymbol{B} - \boldsymbol{B} \times (\boldsymbol{C} \cdot \boldsymbol{A}\,, \tag{1.91}$$

$$(\nabla \cdot \boldsymbol{A})\boldsymbol{B} = (\boldsymbol{A} \cdot \nabla)\boldsymbol{B} + \boldsymbol{B} \operatorname{div} \boldsymbol{A}\,, \tag{1.92}$$

$$(\boldsymbol{A} \times \boldsymbol{B}) \cdot \operatorname{curl} \boldsymbol{C} = \boldsymbol{B} \cdot (\boldsymbol{A} \cdot \nabla)\boldsymbol{C} - \boldsymbol{A} \cdot (\boldsymbol{B} \cdot \nabla)\boldsymbol{C}\,, \tag{1.93}$$

$$(\boldsymbol{A} \times \nabla) \times \boldsymbol{B} = (\boldsymbol{A} \cdot \nabla)\boldsymbol{B} + \boldsymbol{A} \times \operatorname{curl} \boldsymbol{B} - \boldsymbol{A} \operatorname{div} \boldsymbol{B}\,, \tag{1.94}$$

$$(\nabla \times \boldsymbol{A}) \times \boldsymbol{B} = -\boldsymbol{A} \operatorname{div} \boldsymbol{B} + (\boldsymbol{A} \cdot \nabla)\boldsymbol{B} + \boldsymbol{A} \times \operatorname{curl} \boldsymbol{B} + \boldsymbol{B} \times \operatorname{curl} \boldsymbol{A}\,. \tag{1.95}$$

1.63. Find $\operatorname{grad} \varphi(r)$, $\operatorname{div} \varphi(r)\boldsymbol{r}$, $\operatorname{curl} \varphi(r)\boldsymbol{r}$, and $(\boldsymbol{l} \cdot \nabla)\varphi(r)\boldsymbol{r}$.

1.64. Find a function $\varphi(r)$ that satisfies the condition $\operatorname{div} \varphi(r)\boldsymbol{r} = 0$.

1.65. Find the divergences and curls of the following vectors:

$$(\boldsymbol{a} \cdot \boldsymbol{r})\boldsymbol{b}, (\boldsymbol{a} \cdot \boldsymbol{r})\boldsymbol{r}\,, \quad \varphi(r)(\boldsymbol{a} \times \boldsymbol{r})\,, \quad \boldsymbol{r} \times (\boldsymbol{a} \times \boldsymbol{r})\,,$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are constant vectors.

1.66. Find $\operatorname{grad} \boldsymbol{r} \cdot \boldsymbol{A}(r)$, $\operatorname{grad} \boldsymbol{A}(r) \cdot \boldsymbol{B}(r)$, $\operatorname{div} \varphi(r)\boldsymbol{A}(r)$, $\operatorname{curl} \varphi(r)\boldsymbol{A}(r)$, and $(\boldsymbol{l} \cdot \nabla)\varphi(r)\boldsymbol{A}(r)$.

1.67. Prove that

$$(\boldsymbol{A} \cdot \nabla)\boldsymbol{A} = -\boldsymbol{A} \times \operatorname{curl} \boldsymbol{A} \quad \text{if} \quad A^2 = \text{const}\,.$$

1.68. Transform the integral over volume $\int_V (\operatorname{grad} \varphi \cdot \operatorname{curl} \boldsymbol{A})\mathrm{d}V$ into the integral over the surface.

1.69. Express the integrals over the closed surface $\oint_S \boldsymbol{r}(\boldsymbol{a} \cdot \mathrm{d}\boldsymbol{S})$ and $\oint_S (\boldsymbol{a} \cdot \boldsymbol{r})\mathrm{d}\boldsymbol{S}$ in terms of the volume bounded by that surface. Here $\boldsymbol{a}$ is a constant vector.

Hint: Multiply each integral by the arbitrary constant vector $\boldsymbol{b}$ and use the Gauss–Ostrogradskii theorem

1.70*. Transform the integrals over a closed surface

$$\oint \boldsymbol{n}\varphi \mathrm{d}S, \oint (\boldsymbol{n} \times \boldsymbol{A})\mathrm{d}S\ , \quad \oint (\boldsymbol{n} \cdot \boldsymbol{b})\boldsymbol{A}\mathrm{d}S\ , \quad \oint T_{\alpha\beta}(\boldsymbol{r})\boldsymbol{n}_\beta \mathrm{d}S$$

into integrals over the volume bounded by that surface. Here $\boldsymbol{b}$ is a constant vector and $\boldsymbol{n}$ is the ort of the normal.

1.71. Using one of the identities proven in the previous problem, formulate the Archimedean law by summing pressures applied to the elements of the surface of a submerged body.

1.72*. Prove the identity

$$\int_V (\boldsymbol{A} \cdot \operatorname{curl}\operatorname{curl} \boldsymbol{B} - \boldsymbol{B} \cdot \operatorname{curl}\operatorname{curl} \boldsymbol{A})\mathrm{d}V = \oint_S (\boldsymbol{B} \times \operatorname{curl} \boldsymbol{A} - \boldsymbol{A} \times \operatorname{curl} \boldsymbol{B}) \cdot \mathrm{d}\boldsymbol{S}\ . \tag{1.96}$$

1.73. Inside volume V, vector $\boldsymbol{A}$ satisfies the condition $\operatorname{div} \boldsymbol{A} = 0$ and at the boundary of the volume (surface S) the condition $A_n = 0$. Prove that $\int_V \boldsymbol{A}\mathrm{d}V = 0$.

1.74*. Prove that

$$\operatorname{div}_{\boldsymbol{r}} \int_V \frac{\boldsymbol{A}(\boldsymbol{r}')\mathrm{d}V'}{|\boldsymbol{r} - \boldsymbol{r}'|} = 0\ ,$$

where $\boldsymbol{A}(\boldsymbol{r})$ is the vector defined in the previous problem.

1.75. Prove the Green's identities[16)]

$$\int_V (\varphi\Delta\psi + \nabla\varphi\nabla\psi)\mathrm{d}V = \oint_S \varphi\nabla\psi \cdot \mathrm{d}\boldsymbol{S}\ , \tag{1.97}$$

$$\int_V (\varphi\Delta\psi - \psi\Delta\varphi)\mathrm{d}V = \oint_S (\varphi\nabla\psi - \psi\nabla\varphi) \cdot \mathrm{d}\boldsymbol{S}\ , \tag{1.98}$$

where φ an ψ are scalar differentiable functions.

1.76. Transform the integral over the closed loop $\oint_l u\mathrm{d}f$ into the integral over a surface bounded by that loop.

1.77*. Prove the integral identities

$$\oint_l \varphi \mathrm{d}\boldsymbol{l} = \int_S (\boldsymbol{n} \times \operatorname{grad} \varphi)\mathrm{d}S\ , \tag{1.99}$$

16) George Green (1793–1841) was an English mathematician and physicist who introduced the concept of potential and contributed to the development of the theory of electrical and magnetic phenomena.

$$\oint_l (\mathrm{d}\boldsymbol{l} \times \boldsymbol{A}) = \int_S ((\boldsymbol{n} \times \nabla) \times \boldsymbol{A})\mathrm{d}S \ , \tag{1.100}$$

$$\oint_l \mathrm{d}\boldsymbol{l} \cdot \boldsymbol{A} = \int_S (\boldsymbol{n} \times \nabla) \cdot \boldsymbol{A}\mathrm{d}S \ . \tag{1.101}$$

Here $\boldsymbol{n}$ is the ort of the normal to the surface, φ and $\boldsymbol{A}$ are functions of the coordinates, l is a closed loop, and S is a nonclosed surface bounded by that loop. These identities may be regarded as special cases of the generalized Stokes theorem

$$\oint_l (\ldots)\mathrm{d}\boldsymbol{l} = \int_S (\boldsymbol{n} \times \nabla)(\ldots)\mathrm{d}S \ , \tag{1.102}$$

where the symbol $(\ldots)$ labels a tensor of any rank.

1.78. Show that if the scalar function ψ is a solution of the Helmholtz equation[17] $\triangle\psi + k^2\psi = 0$ and $\boldsymbol{a}$ is a certain constant vector, then the vector functions $\boldsymbol{L} = \nabla\psi$, $\boldsymbol{M} = \nabla \times (\boldsymbol{a}\psi)$, and $\boldsymbol{N} = \nabla \times \boldsymbol{M}$ satisfy the Helmholtz vector equation $\triangle\boldsymbol{A} + k^2\boldsymbol{A} = 0$.

1.2.5 Differentiating in Curvilinear Coordinates

Unlike in Cartesian rectangular coordinates, when we use curvilinear nonorthogonal coordinates $q^\alpha(\alpha = 1,2,3)$, x^β $(\beta = 1,2,3)$, the derivative over coordinates from a tensor of rank $s \geq 1$ does not produce any tensor, which we will see later. This is due to the local nature of the definition of the tensor (1.51) applicable to a certain point. In the meantime, a derivative is defined through the difference of the values of two vectors at close but still different points. In order to define a *covariant derivative* from a tensor of any rank, that is, such a differential operation that increases the rank of a tensor by one, we will, for simplicity, consider a tensor of rank 1 (vector) and expand it in basic vectors of the curvilinear system of coordinates in question:

$$\boldsymbol{A} = A^\mu \boldsymbol{e}_\mu = A_\mu \boldsymbol{e}^\mu \ . \tag{1.103}$$

Differentiate the equalities in (1.103) and form the covariant derivatives:

$$A_{\mu;\alpha} \equiv \boldsymbol{e}_\mu \cdot \frac{\partial \boldsymbol{A}}{\partial q^\alpha} = \frac{\partial A_\mu}{\partial q^\alpha} + A_\nu \boldsymbol{e}_\mu \cdot \frac{\partial \boldsymbol{e}^\nu}{\partial q^\alpha} \ , \tag{1.104}$$

$$A^\mu_{;\alpha} \equiv \boldsymbol{e}^\mu \cdot \frac{\partial \boldsymbol{A}}{\partial q^\alpha} = \frac{\partial A^\mu}{\partial q^\alpha} + A^\nu \boldsymbol{e}^\mu \cdot \frac{\partial \boldsymbol{e}_\nu}{\partial q^\alpha} \ . \tag{1.105}$$

17) Herman Ludwig Ferdinand Helmholtz (1821–1894) was a German physicist, mathematician, physiologist, and psychologist.

The first members of the equalities use the notation commonly accepted for covariant derivatives of covariant and contravariant vector components, respectively. The sign of the identity is followed by their definitions. The second members include derivatives of the components of the vector and basic vectors. In curvilinear systems of coordinates, unlike in Cartesian coordinates, derivatives of basic vectors are not equal to zero.

Differentiating equality (1.48) over the coordinate, we find that

$$\boldsymbol{e}_\mu \cdot \frac{\partial \boldsymbol{e}^\nu}{\partial q^\alpha} = -\boldsymbol{e}^\nu \cdot \frac{\partial \boldsymbol{e}_\mu}{\partial q^\alpha} . \tag{1.106}$$

Now, add the *Christoffel symbols* of the second kind to our consideration:[18)]

$$\Gamma^\nu_{\mu\alpha} = \boldsymbol{e}^\nu \cdot \frac{\partial \boldsymbol{e}_\mu}{\partial q^\alpha} . \tag{1.107}$$

They allow us to write covariant derivatives in a more compact form:

$$A_{\mu;\alpha} = \frac{\partial A_\mu}{\partial q^\alpha} - A_\nu \Gamma^\nu_{\mu\alpha} , \quad A^\mu_{;\alpha} = \frac{\partial A^\mu}{\partial q^\alpha} - A^\nu \Gamma^\mu_{\nu\alpha} . \tag{1.108}$$

Christoffel symbols are not tensors since they do not satisfy the applicable rules of transformation. They are symmetric as to the two lower symbols: $\Gamma^\nu_{\mu\alpha} = \Gamma^\nu_{\alpha\mu}$. The latter property follows from the representation of basic vectors (1.46):

$$\frac{\partial \boldsymbol{e}_\mu}{\partial q^\alpha} = \frac{\partial \boldsymbol{e}_\alpha}{\partial q^\mu} . \tag{1.109}$$

The rules (1.108) of computing a covariant derivative of a tensor of rank 1 are generalized, in an obvious way, to include tensor T of any rank. Besides the derivative over the coordinate from the tensor in question, one needs to add as many terms with a plus sign as the tensor has upper symbols and as many terms with a minus sign as the tensor has lower symbols.

Example 1.9

Express the Christoffel symbols (1.107) through the components of metric tensor $g_{\mu\nu}$.

Solution. The definition (1.107) of Christoffel symbols allows us to write the following relation:

$$\boldsymbol{e}_\nu \Gamma^\nu_{\mu\alpha} = \frac{\partial \boldsymbol{e}_\mu}{\partial q^\alpha} . \tag{1.110}$$

It follows from the equality $(\boldsymbol{e}_\nu)_\lambda (\boldsymbol{e}^\nu)^\sigma = \delta^\sigma_\lambda$, which follows from the representations of basic vectors (1.46) and (1.49).

18) Elwin Bruno Christoffel (1829–1900) was a German mathematician.

If we use the relation,

$$e^{\nu}\Gamma_{\nu\,\mu\alpha} = \frac{\partial e_{\mu}}{\partial q^{\alpha}}\,, \tag{1.111}$$

Also consider Christoffel symbols of the first kind, $\Gamma_{\nu\,\mu\alpha}$

As follows from (1.110) and (1.111), Christoffel symbols of the first and second kinds may be regarded as the coefficients of the expansion of the quantity $\partial e_{\mu}/\partial q^{\alpha}$ in vectors of covariant and contravariant bases.

Using (1.48), we find from (1.111) that

$$\Gamma_{\nu,\mu\alpha} = e_{\nu}\cdot\frac{\partial e_{\mu}}{\partial q^{\alpha}}\,. \tag{1.112}$$

Multiplying (1.110), in a scalar way, by e_{λ} and (1.111) by e^{λ} and using (1.50), we find the connection between Christoffel symbols of the first and second kinds:

$$\Gamma_{\nu,\mu\alpha} = g_{\nu\lambda}\Gamma^{\nu}_{\mu\alpha}\,,\quad \Gamma^{\nu}_{\mu\alpha} = g^{\nu\lambda}\Gamma_{\nu\,\mu\alpha}\,. \tag{1.113}$$

Then, sequentially using the symmetry of the two symbols separated by a comma and relations (1.109), (1.112), and (1.113), find the following:

$$\begin{aligned}
\Gamma_{\nu,\mu\alpha} &= \frac{1}{2}\left(e_{\nu}\cdot\frac{\partial e_{\mu}}{\partial q^{\alpha}} + e_{\nu}\cdot\frac{\partial e_{\alpha}}{\partial q^{\mu}}\right)\\
&= \frac{1}{2}\left(\frac{\partial g_{\mu\nu}}{\partial q^{\alpha}} + \frac{\partial g_{\alpha\nu}}{\partial q^{\mu}} - e_{\mu}\cdot\frac{\partial e_{\nu}}{\partial q^{\alpha}} - e_{\alpha}\cdot\frac{\partial e_{\nu}}{\partial q^{\mu}}\right)\\
&= \frac{1}{2}\left(\frac{\partial g_{\mu\nu}}{\partial q^{\alpha}} + \frac{\partial g_{\alpha\nu}}{\partial q^{\mu}} - \frac{\partial g_{\alpha\mu}}{\partial q^{\nu}}\right)\,,
\end{aligned} \tag{1.114}$$

$$\Gamma^{\nu}_{\mu\alpha} = \frac{1}{2}g^{\nu\lambda}\left(\frac{\partial g_{\mu\lambda}}{\partial q^{\alpha}} + \frac{\partial g_{\alpha\lambda}}{\partial q^{\mu}} - \frac{\partial g_{\alpha\mu}}{\partial q^{\nu}}\right)\,. \tag{1.115}$$

□

Example 1.10

Find the rules of the transformation of Christoffel symbols of the first and second kinds when they are transferred to another curvilinear coordinate system.

Solution. Do the sequential computations

$$\begin{aligned}
\Gamma'^{\nu}_{\mu\alpha} &= e'^{\nu}\cdot\frac{\partial e'_{\mu}}{\partial q'^{\alpha}} = \frac{\partial q'^{\nu}}{\partial q^{\beta}}e^{\beta}\cdot\frac{\partial}{\partial q'^{\alpha}}\frac{\partial q^{\sigma}}{\partial q'^{\mu}}e_{\sigma}\\
&= \frac{\partial q'^{\nu}}{\partial q^{\beta}}\frac{\partial q^{\lambda}}{\partial q'^{\alpha}}\frac{\partial q^{\sigma}}{\partial q'^{\mu}}e^{\beta}\cdot\frac{\partial e_{\sigma}}{\partial q^{\lambda}} + \frac{\partial q'^{\nu}}{\partial q^{\beta}}\frac{\partial q^{\lambda}}{\partial q'^{\alpha}}\frac{\partial q'^{\kappa}}{\partial q^{\lambda}}\frac{\partial^2 q^{\sigma}}{\partial q'^{\kappa}\partial q'^{\mu}}e^{\beta}\cdot e_{\sigma}
\end{aligned}$$

$$= \frac{\partial q'^{\nu}}{\partial q^{\beta}} \frac{\partial q^{\lambda}}{\partial q'^{\alpha}} \frac{\partial q^{\sigma}}{\partial q'^{\mu}} \Gamma^{\beta}_{\lambda\sigma} + \frac{\partial q'^{\nu}}{\partial q^{\beta}} \frac{\partial^2 q^{\beta}}{\partial q'^{\alpha} \partial q'^{\mu}} , \tag{1.116}$$

$$\Gamma'_{\nu\,\mu\alpha} = \boldsymbol{e}'_{\nu} \cdot \frac{\partial \boldsymbol{e}'_{\mu}}{\partial q'^{\alpha}} = \frac{\partial q^{\beta}}{\partial q'^{\nu}} \boldsymbol{e}_{\beta} \cdot \frac{\partial}{\partial q'^{\alpha}} \frac{\partial q^{\sigma}}{\partial q'^{\mu}} \boldsymbol{e}_{\sigma}$$

$$= \frac{\partial q^{\beta}}{\partial q'^{\nu}} \frac{\partial q^{\lambda}}{\partial q'^{\alpha}} \frac{\partial q^{\sigma}}{\partial q'^{\mu}} \boldsymbol{e}_{\beta} \cdot \frac{\partial \boldsymbol{e}_{\sigma}}{\partial q^{\lambda}} + \frac{\partial q^{\beta}}{\partial q'^{\nu}} \frac{\partial^2 q^{\sigma}}{\partial q'^{\alpha} \partial q'^{\mu}} \boldsymbol{e}_{\beta} \cdot \boldsymbol{e}_{\sigma}$$

$$= \frac{\partial q^{\beta}}{\partial q'^{\nu}} \frac{\partial q^{\lambda}}{\partial q'^{\alpha}} \frac{\partial q^{\sigma}}{\partial q'^{\mu}} \Gamma_{\beta,\lambda\sigma} + \frac{\partial q^{\beta}}{\partial q'^{\nu}} \frac{\partial^2 q^{\beta}}{\partial q'^{\alpha} \partial q'^{\mu}} g_{\beta\sigma} . \tag{1.117}$$

Only the first terms in the second members of the resulting expressions conform to the rules of the transformation of tensors. The second terms violate the said rules, which means that *Christoffel symbols are not tensors.* □

Example 1.11

Prove that the covariant derivatives of the vectors $A_{\nu;\alpha}$ and $A^{\nu}_{;\alpha}$ are transformed as covariant and mixed tensors, respectively, of rank 2.

Solution. Using the definition of covariant derivative (1.104) and the rule of transformation (1.116), sequentially find the following:

$$\begin{aligned} A'_{\mu;\alpha} &= \frac{\partial A'_{\mu}}{\partial q'^{\alpha}} - \Gamma'^{\nu}_{\mu\alpha} A'_{\nu} \\ &= \frac{\partial}{\partial q^{\lambda}} \left(\frac{\partial q^{\sigma}}{\partial q'^{\mu}} A_{\sigma} \right) \frac{\partial q^{\lambda}}{\partial q'^{\alpha}} \\ &\quad - \left(\frac{\partial q'^{\nu}}{\partial q^{\beta}} \frac{\partial q^{\lambda}}{\partial q'^{\alpha}} \frac{\partial q^{\sigma}}{\partial q'^{\mu}} \Gamma^{\beta}_{\lambda\sigma} + \frac{\partial q'^{\nu}}{\partial q^{\beta}} \frac{\partial^2 q^{\beta}}{\partial q'^{\alpha} \partial q'^{\mu}} \right) \frac{\partial q^{\kappa}}{\partial q'^{\nu}} A_{\kappa} \\ &= \frac{\partial q^{\sigma}}{\partial q'^{\mu}} \frac{\partial q^{\lambda}}{\partial q'^{\alpha}} \left(\frac{\partial A_{\sigma}}{\partial q^{\lambda}} - \Gamma^{\beta}_{\lambda\sigma} A_{\beta} \right) = \frac{\partial q^{\sigma}}{\partial q'^{\mu}} \frac{\partial q^{\lambda}}{\partial q'^{\alpha}} A_{\sigma;\lambda} . \end{aligned} \tag{1.118}$$

It has been proven that the quantity in question is transformed as a covariant tensor of rank 2. When considering the second tensor, one must use the following equality:

$$\frac{\partial q'^{\nu}}{\partial q^{\beta}} \frac{\partial^2 q^{\beta}}{\partial q'^{\alpha} \partial q'^{\mu}} = \frac{\partial q^{\beta}}{\partial q'^{\alpha}} \frac{\partial q^{\lambda}}{\partial q'^{\mu}} \frac{\partial^2 q'^{\nu}}{\partial q^{\beta} \partial q^{\lambda}} . \tag{1.119}$$

It follows from differentiating over the coordinate of an equality such as (1.43). □

Problems

1.79. Show that a derivative of a coordinate of the scalar (gradient) $\partial S/\partial q^\mu = S_{;\mu}$ is a covariant vector.

1.80. Show that a covariant curl coincides with a proper curl:

$$A_{\mu;\nu} - A_{\nu;\mu} = \frac{\partial A_\mu}{\partial q^\nu} - \frac{\partial A_\nu}{\partial q^\mu} \, .$$

1.81*. Show that the covariant divergence of a covariant vector (scalar) may be written as

$$A^\mu_{;\mu} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial q^\mu} \left(\sqrt{g} A^\mu\right) \, . \tag{1.120}$$

1.82. In curvilinear coordinates, write down the Laplace operator influencing a scalar function.

1.83. Write down covariant divergence $T^{\mu\nu}{}_{;\mu}$ for any tensor of rank 2.

1.84. Do the same for the antisymmetric tensor $A^{\mu\nu} = -A^{\nu\mu}$.

1.85. Prove the following relation for the covariant components of the antisymmetric tensor $A_{\mu\nu} = -A_{\nu\mu}$:

$$A_{\mu\nu;\lambda} + A_{\lambda\mu;\nu} + A_{\nu\lambda;\mu} = \frac{\partial A_{\mu\nu}}{\partial q^\lambda} + \frac{\partial A_{\lambda\mu}}{\partial q^\nu} + \frac{\partial A_{\nu\lambda}}{\partial q^\mu} \, .$$

1.86. Find the covariant derivatives of the metric tensor $g_{\mu\nu;\lambda}$ and $g^{\nu\mu}{}_{;\lambda}$.

1.87. Prove the identity $\partial g_{\mu\nu}/\partial q^\lambda = \Gamma_{\mu,\nu\lambda} + \Gamma_{\nu\,\mu\lambda}$.

1.2.6
Orthogonal Curvilinear Coordinates

Orthogonal curvilinear coordinates in which $g_{\mu\nu} = 0$ while $\mu \neq \nu$ (see Problem 1.46) are practically used very frequently. In those cases, the following notation is used: $g_{\mu\nu} = h_\mu^2(q)\delta_{\mu\nu}$ (no summing over μ is necessary). The element of length is written as

$$\mathrm{d}l^2 = g_{\mu\nu}\mathrm{d}q^\mu \mathrm{d}q^\nu = h_1^2(\mathrm{d}q^1)^2 + h_2^2(\mathrm{d}q^2)^2 + h_3^2(\mathrm{d}q^3)^2 \, , \tag{1.121}$$

where, in accordance with (1.46), values h_μ (*Lamé coefficients*)[19] have the following form:

$$h_\mu = \sqrt{\left(\frac{\partial x}{\partial q^\mu}\right)^2 + \left(\frac{\partial y}{\partial q^\mu}\right)^2 + \left(\frac{\partial z}{\partial q^\mu}\right)^2} \, . \tag{1.122}$$

19) Gabriel Lamé (1795–1870) was a French mathematician and engineer who conducted research in mathematical physics and the theory of elasticity.

Since $\sqrt{g} = h_1 h_2 h_3$, the invariant volume element (1.54) assumes the following form:

$$dV = h_1 h_2 h_3 dq^1 dq^2 dq^3 \,. \tag{1.123}$$

The characteristic peculiarity of an orthogonal basis is that the vectors of the original and mutual bases have the same directions but different sizes and physical dimensions (because the coordinates x^α and q^β may have different dimensions). This is why the dimensions of different components of the same vector, expanded in the vectors of those bases, may also be different, which creates a certain inconvenience when physical problems are being solved. This is why the introduction of an orthogonal basis of unit vectors $\boldsymbol{e}_{\alpha *}, \boldsymbol{e}_{\alpha *} \cdot \boldsymbol{e}_{\beta *} = \delta_{\alpha\beta}$ is useful (we will label them with lower indices and an asterisk) and through which, in accordance with (1.50), the covariant and contravariant bases will be expressed in the following way:

$$\boldsymbol{e}_\beta = h_\beta \boldsymbol{e}_{\beta *} \,, \quad \boldsymbol{e}^\beta = \frac{1}{h_\beta} \boldsymbol{e}_{\beta *} \,. \tag{1.124}$$

The expansion of an arbitrary vector $\boldsymbol{A}$ in orts $\boldsymbol{e}_{\beta *}$ assumes the following form:

$$\boldsymbol{A} = A_{1*} \boldsymbol{e}_{1*} + A_{2*} \boldsymbol{e}_{2*} + A_{3*} \boldsymbol{e}_{3*} \,, \tag{1.125}$$

where the "physical" components of the vector $A_{\mu *}$ now have the same dimensionality matching that of $\boldsymbol{A}$, that is, the physical quantity in question, and are connected to its covariant and contravariant components by the following relations:

$$A_{\beta *} = \frac{A_\mu}{h_\mu} = A^\mu h_\mu \,. \tag{1.126}$$

Since the use of the basis $\boldsymbol{e}_{\beta *}$ is convenient, hereafter we will use that basis everywhere, omitting the asterisk.

Using relations (1.120)–(1.126), and also (1.25), write down the principal operations of differentiation in orthogonal curvilinear coordinates:

$$\operatorname{grad} S = \frac{1}{h_1} \frac{\partial S}{\partial q^1} \boldsymbol{e}_1 + \frac{1}{h_2} \frac{\partial S}{\partial q^2} \boldsymbol{e}_2 + \frac{1}{h_3} \frac{\partial S}{\partial q^3} \boldsymbol{e}_3 \,; \tag{1.127}$$

$$\operatorname{div} \boldsymbol{A} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial q^1} (h_2 h_3 A_1) + \frac{\partial}{\partial q^2} (h_1 h_3 A_2) + \frac{\partial}{\partial q^3} (h_1 h_2 A_3) \right] \,; \tag{1.128}$$

$$\begin{aligned} \Delta S = \frac{1}{h_1 h_2 h_3} &\left[\frac{\partial}{\partial q^1} \left(\frac{h_2 h_3}{h_1} \frac{\partial S}{\partial q^1} \right) + \frac{\partial}{\partial q^2} \left(\frac{h_1 h_3}{h_2} \frac{\partial S}{\partial q^2} \right) \right. \\ &\left. + \frac{\partial}{\partial q^3} \left(\frac{h_1 h_2}{h_3} \frac{\partial S}{\partial q^3} \right) \right] \,; \end{aligned} \tag{1.129}$$

$$\begin{aligned}\operatorname{curl}\boldsymbol{A} &= \frac{1}{h_2h_3}\left[\frac{\partial}{\partial q^2}(h_3A_3) - \frac{\partial}{\partial q^3}(h_2A_2)\right]\boldsymbol{e}_1 \\ &+ \frac{1}{h_1h_3}\left[\frac{\partial}{\partial q^3}(h_1A_1) - \frac{\partial}{\partial q^1}(h_3A_3)\right]\boldsymbol{e}_2 \\ &+ \frac{1}{h_1h_2}\left[\frac{\partial}{\partial q^1}(h_2A_2) - \frac{\partial}{\partial q^2}(h_1A_1)\right]\boldsymbol{e}_3 \,. \end{aligned} \tag{1.130}$$

Problems

1.88. From the common expressions (1.27)–(1.29), derive the basic differential operations below in the (r, α, z) cylindrical coordinate system where $x = r\cos\alpha$, $y = r\sin\alpha$, and $z = z$:

$$\operatorname{grad} S = \frac{\partial S}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial S}{\partial \alpha}\boldsymbol{e}_\alpha + \frac{\partial S}{\partial z}\boldsymbol{e}_z \,; \tag{1.131}$$

$$\operatorname{div}\boldsymbol{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\alpha}{\partial \alpha} + \frac{\partial A_z}{\partial z} \,; \tag{1.132}$$

$$\Delta S = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial S}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 S}{\partial \alpha^2} + \frac{\partial^2 S}{\partial z^2} \,; \tag{1.133}$$

$$\begin{aligned}\operatorname{curl}\boldsymbol{A} &= \left[\frac{1}{r}\frac{\partial A_z}{\partial \alpha} - \frac{\partial A_\alpha}{\partial z}\right]\boldsymbol{e}_r + \left[\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right]\boldsymbol{e}_\alpha \\ &+ \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\alpha) - \frac{\partial A_r}{\partial \alpha}\right]\boldsymbol{e}_z \,. \end{aligned} \tag{1.134}$$

1.89. Do the same for the (r, ϑ, α) spherical coordinate system where $x = r\sin\vartheta\cos\alpha$, $y = r\sin\vartheta\sin\alpha$, and $z = r\cos\vartheta$:

$$\operatorname{grad} S = \frac{\partial S}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial S}{\partial \vartheta}\boldsymbol{e}_\vartheta + \frac{1}{r\sin\vartheta}\frac{\partial S}{\partial \alpha}\boldsymbol{e}_\alpha \,; \tag{1.135}$$

$$\operatorname{div}\boldsymbol{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r) + \frac{1}{r\sin\vartheta}\frac{\partial}{\partial \vartheta}(A_\vartheta\sin\vartheta) + \frac{1}{r\sin\vartheta}\frac{\partial A_\alpha}{\partial \alpha} \,; \tag{1.136}$$

$$\Delta S = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial S}{\partial r}\right) + \frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial \vartheta}\left(\sin\vartheta\frac{\partial S}{\partial \vartheta}\right) + \frac{1}{r^2\sin^2\vartheta}\frac{\partial^2 S}{\partial \alpha^2} \,; \tag{1.137}$$

$$\begin{aligned}\operatorname{curl}\boldsymbol{A} &= \frac{1}{r\sin\vartheta}\left[\frac{\partial}{\partial \vartheta}(A_\alpha\sin\vartheta) - \frac{\partial A_\vartheta}{\partial \alpha}\right]\boldsymbol{e}_r \\ &+ \left[\frac{1}{r\sin\vartheta}\frac{\partial A_r}{\partial \alpha} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\alpha)\right]\boldsymbol{e}_\vartheta + \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\vartheta) - \frac{\partial A_r}{\partial \vartheta}\right]\boldsymbol{e}_\alpha \,. \end{aligned} \tag{1.138}$$

1.90*. Use identity (1.83) to write the projections of the vector $\triangle \boldsymbol{A}$ onto the axes of a cylindrical coordinate system.

1.91*. Do the same for a spherical coordinate system.

1.92. Find the general form solution of Laplace's equation for a scalar function that depends only on (i) r, (ii) α, and (iii) z (cylindrical coordinates).

1.93. Find the general form solution of Laplace's equation for a scalar function that depends only on (i) r, (ii) ϑ, and (iii) α (spherical coordinates).

Note In Problems 1.94*–1.98*, examples of curvilinear orthogonal systems of coordinates are considered. These systems are more complex than cylindrical and spherical systems. For more information, see Arfken (1970) and Stratton (1948)

1.94*. The equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad (a > b > c)$$

represents an ellipsoid with semiaxes a, b, and c. The equations

$$\frac{x^2}{a^2+\xi} + \frac{y^2}{b^2+\xi} + \frac{z^2}{c^2+\xi} = 1\,, \quad \xi \geq -c^2\,,$$

$$\frac{x^2}{a^2+\eta} + \frac{y^2}{b^2+\eta} + \frac{z^2}{c^2+\eta} = 1\,, \quad -c^2 \geq \eta \geq -b^2\,,$$

$$\frac{x^2}{a^2+\zeta} + \frac{y^2}{b^2+\zeta} + \frac{z^2}{c^2+\zeta} = 1\,, \quad -b^2 \geq \zeta \geq -a^2$$

represent an ellipsoid and one-sheet and two-sheet hyperboloids confocal with the first ellipsoid, respectively. Each point of the space is crossed by a surface characterized by values ξ, η, and ζ. ξ, η, and ζ are called ellipsoidal coordinates of the point x, y, z. Find the formulas of transformation of ellipsoidal to Cartesian coordinates. Make sure that an ellipsoidal system of coordinates is orthogonal. Find the Lamé coefficients and Laplace's operator in ellipsoidal coordinates.

1.95*. When $a = b > c$, the ellipsoidal coordinate system (see the previous problem) degenerates to become a so-called flattened spheroidal coordinate system. When this happens, the coordinate ζ becomes constant, equals $-a^2$, and must be replaced by another coordinate. To serve as such, an azimuthal angle α on the surface xy is selected. The coordinates ξ and η are found from the following equations:

$$\frac{r^2}{a^2+\xi} + \frac{z^2}{c^2+\xi} = 1\,, \quad \xi \geq -c^2\,,$$

$$\frac{r^2}{a^2+\eta} + \frac{z^2}{c^2+\eta} = 1\,, \quad -c^2 \geq \eta \geq -a^2\,,$$

where $r^2 = x^2 + y^2$.

Surfaces $\xi = \text{const}$ are flattened ellipsoids of rotation around the Oz axis. Surfaces $\eta = \text{const}$ are one-sheet hyperboloids of rotation confocal with them (Figure 1.6).

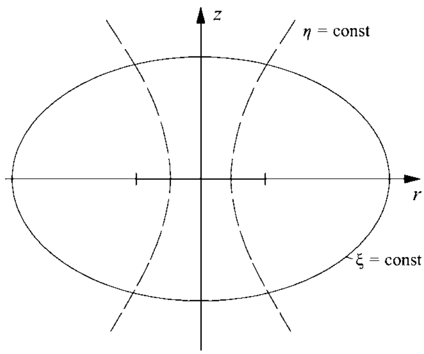

Figure 1.6 A flattened spheroidal system of coordinates.

Find the expressions for r and z in flattened spheroidal coordinates, the Lamé coefficients, and Laplace's operator in those coordinates.

1.96*. An extended spheroidal system of coordinates is derived from an ellipsoidal one (Problem 1.94*) when $a > b = c$. When this happens, the coordinate η becomes constant and must be replaced with an azimuthal angle α marked off on the yz surface by the Oy axis. The coordinates ξ and ζ are found from the following equations:

$$\frac{x^2}{a^2+\xi} + \frac{r^2}{b^2+\xi} = 1\,, \quad \xi \geq -b^2\,,$$

$$\frac{x^2}{a^2+\zeta} + \frac{r^2}{b^2+\zeta} = 1\,, \quad -b^2 \geq \zeta \geq -a^2\,,$$

where $r^2 = y^2 + z^2$.

The surfaces of the constants ξ and η are extended ellipsoids and two-sheet hyperboloids of rotation (Figure 1.7). Express the quantities x and r through ξ and ζ. Find the Lamé coefficients and Laplace's operator in the variables ξ, ζ, and α.

1.97*. Bispherical cooordinates ξ, η, and α are connected to Cartesian coordinates by the following relations:

$$x = \frac{a\sin\eta\cos\alpha}{\cosh\xi - \cos\eta}\,, \quad y = \frac{a\sin\eta\sin\alpha}{\cosh\xi - \cos\eta}\,, \quad z = \frac{a\sinh\xi}{\cosh\xi - \cos\eta}\,,$$

where a is a constant parameter, $-\infty < \xi < \infty$, $0 < \eta < \pi$, and $0 < \alpha < 2\pi$.

Show that the coordinate surfaces $\xi = \text{const}$ are spheres,

$$x^2 + y^2 + (z - a\coth\xi)^2 = \left(\frac{a}{\sinh\xi}\right)^2\,,$$

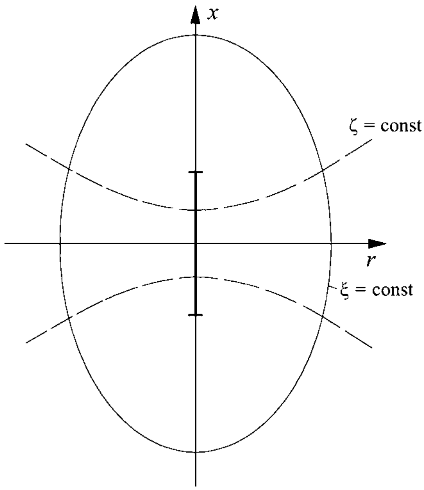

Figure 1.7 An extended spheroidal system of coordinates.

the surfaces $\eta = \text{const}$ are spindle-shaped surfaces of rotation around the Oz axis, whose equation is

$$\left(\sqrt{x^2+y^2} - a\cot\eta\right)^2 + z^2 = \left(\frac{a}{\sin\eta}\right)^2 ,$$

and surfaces $\alpha = \text{const}$ are half-planes diverging from the Oz axis (Figure 1.8). Make sure that these coordinate surfaces are orthogonal with respect to each other. Find the Lamé coefficients and Laplace's operator.

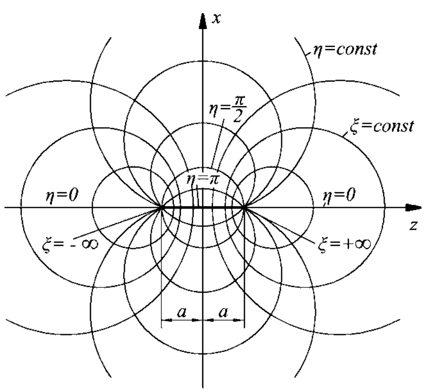

Figure 1.8 A bispherical system of coordinates.

1.98*. The toroidal coordinates ρ, ξ, and α make up an orthogonal system and are connected to Cartesian coordinates by the following relations:

$$x = \frac{a\sinh\rho\cos\alpha}{\cosh\rho - \cos\xi} , \quad y = \frac{a\sinh\rho\sin\alpha}{\cosh\rho - \cos\xi} , \quad z = \frac{a\sin\xi}{\cosh\rho - \cos\xi} ,$$

where a is a constant parameter, $-\infty < \rho < \infty$, $-\pi < \xi < \pi$, and $0 < \alpha < \pi$.

Show that $\rho = \ln(r_1/r_2)$ (see Figure 1.9, displaying the surfaces α = const and $\alpha + \pi$ = const) and the quantity ξ is the angle between r_1 and r_2 ($\xi > 0$ if $z > 0$ and $\xi < 0$ if $z < 0$). What is the form of the coordinate surfaces ρ = const and ξ = const? Find the Lamé coefficients.

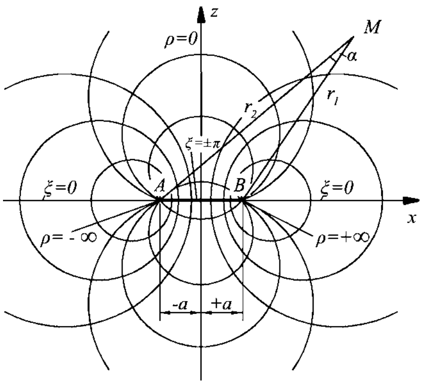

Figure 1.9 A toroidal system of coordinates.

Suggested literature:
Borisenko and Tarapov (1966); Weinberg (1972); Arfken (1970); Mathews and Walker (1964); Lee (1965); Rashevskii (1953); Morse and Feshbach (1953); Stratton (1948); Madelung (1957)

1.3 The Special Functions of Mathematical Physics

1.3.1 Cylindrical Functions

Cylindrical functions are used when solving many specific problems. Of these, the *Bessel functions* are the most commonly used. They may be obtained by expanding a purposely selected exponent (*generating function*) in a power series over u:

$$\exp\left\{\frac{x}{2}\left(u - \frac{1}{u}\right)\right\} = \sum_{n=-\infty}^{\infty} J_n(x)u^n . \tag{1.139}$$

The coefficients $J_n(x)$ of this expansion are called Bessel functions[20] of the first kind and order n. The representation of a Bessel function as a power series may be

20) Friedrich Wilhelm Bessel (1784–1846) was a German astronomer, land surveyor, and mathematician.

obtained from the power series for exponents:

$$\exp\left(\frac{xu}{2}\right)\exp\left(-\frac{x}{2u}\right) = \sum_{r=0}^{\infty}\left(\frac{x}{2}\right)^r \frac{u^r}{r!}\sum_{s=0}^{\infty}(-1)^s\left(\frac{x}{2}\right)^s \frac{u^{-s}}{s!} . \tag{1.140}$$

These expansions are valid for any (including complex) values of x and u, which is due to the unboundedness of the radius of convergence of an exponent. Changing to summing over $n = r - s(-\infty < n < \infty)$, we get, from (1.140)

$$\begin{aligned}\exp\left\{\frac{x}{2}\left(u - \frac{1}{u}\right)\right\} &= \sum_{n=-\infty}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^s}{s!(n+s)!}\left(\frac{x}{2}\right)^{n+2s} u^n \\ &= \sum_{n=-\infty}^{\infty} J_n(x)u^n ,\end{aligned} \tag{1.141}$$

wherefrom it follows that

$$J_n(x) = \sum_{s=0}^{\infty}\frac{(-1)^s}{s!(n+s)!}\left(\frac{x}{2}\right)^{n+2s} . \tag{1.142}$$

The use of this representation for $J_n(x)$ is expedient when $n \geq 0$. When $n < 0$, the following may be written instead of (1.142):

$$J_n(x) = \sum_{s=|n|}^{\infty}\frac{(-1)^s}{s!(n+s)!}\left(\frac{x}{2}\right)^{n+2s} = \sum_{s=0}^{\infty}\frac{(-1)^{s+|n|}}{s!(|n|+s)!}\left(\frac{x}{2}\right)^{|n|+2s} u^n. \tag{1.143}$$

This is because when $s + n < 0$, $(s + n)! \to \infty$. As a result, we get a simple dependence between the Bessel functions of the whole positive and negative orders:

$$J_{-n}(x) = (-1)^n J_n(x) . \tag{1.144}$$

Example 1.12

Obtain recurrent relations between Bessel functions of various orders by differentiating equality (1.139) over u and over x, comparing the second and first members of the equality.

Solution. Differentiating (1.139) over u, we get

$$\begin{aligned}\frac{x}{2}\left(1 + \frac{1}{u^2}\right)\exp\left[\frac{x}{2}\left(u - \frac{1}{u}\right)\right] &= \frac{x}{2}\left(1 + \frac{1}{u^2}\right)\sum_{n=-\infty}^{\infty} J_n(x)u^n \\ &= \sum_{n=-\infty}^{\infty} n J_n(x)u^{n-1} .\end{aligned}$$

Equating the coefficients of u^{n-1} in second and first members of the latter equality, we find the following:

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x) \,. \tag{1.145}$$

Differentiating (1.139) over x, we get in the same way

$$J_{n-1}(x) - J_{n+1}(x) = 2J_n'(x) \,. \tag{1.146}$$

These recurrent relations may be rewritten as follows, in other forms:

$$J_{n\pm1} = \frac{n}{x} J_n(x) \mp J'(x) \,; \quad J_{n\mp1} = \pm x^{\mp n} \frac{d}{dx}[x^{\pm n} J_n(x)] \,. \tag{1.147}$$

Specifically,

$$J_1(x) = -J_0'(x) \,. \tag{1.148}$$

□

Example 1.13

Obtain representations of the Bessel function as integrals from exponential and trigonometric functions. For that purpose, use the substitution $u = \exp(i\varphi)$ in expansion (1.139).

Solution. The substitution leads to the expansion

$$\exp(ix \sin\varphi) = \sum_{n=-\infty}^{\infty} J_n(x) \exp(in\varphi) \,. \tag{1.149}$$

Use the periodicity of the functions $\sin\varphi$ and $\exp(in\varphi)$ and also the easily verifiable equality

$$\int_{\alpha}^{\alpha+2\pi} \exp(i(n-m)\varphi)d\varphi = 2\pi\delta_{mn} \,,$$

where m is an integer and α is any real number. Multiplying both parts of (1.149) by $\exp(-im\varphi)$ and integrating over φ, we get the representation of the Bessel function:

$$\begin{aligned} J_m(x) &= \frac{1}{2\pi} \int_{\alpha}^{\alpha+2\pi} \exp(ix\sin\varphi - im\varphi)d\varphi \\ &= \frac{(-i)^m}{2\pi} \int_{\alpha}^{\alpha+2\pi} \exp(ix\cos\varphi - im\varphi)d\varphi \,. \end{aligned} \tag{1.150}$$

□

Example 1.14

Presume that certain functions of $Z_\nu(x)$, different, generally speaking, from Bessel functions (1.142), satisfy the recurrent relations (1.145)–(1.148) when the value of $n = \nu$ is arbitrary and complex. Produce a differential equation of the second order whose solution is $Z_\nu(x)$.

Solution. Differentiate the second equality (1.147) over x and add a term equaling zero to it (replacing $n \to \nu$, $J_n \to Z_\nu$):

$$\begin{aligned} Z'_{\nu-1} &= [x^{-\nu}(x^\nu Z_\nu)']' + \frac{1}{x}\left(Z'_\nu + \frac{\nu}{x}Z_\nu - Z_{\nu-1}\right) \\ &= Z''_\nu + \frac{\nu+1}{x}Z'_\nu - \frac{1}{x}Z_{\nu-1} \,. \end{aligned}$$

Once again, add a term equaling zero to the second member:

$$\begin{aligned} Z'_{\nu-1} &= Z''_\nu + \frac{\nu+1}{x}Z'_\nu - \frac{1}{x}Z_{\nu-1} + \frac{n}{x}\left[Z_{\nu-1} - \frac{\nu}{x}Z_\nu - Z'_\nu\right] \\ &= Z''_\nu + \frac{1}{x}Z'_\nu - \frac{n^2}{x^2}Z_\nu + \frac{\nu-1}{x}Z_{\nu-1} \,. \end{aligned}$$

Finally, from the second equality in (1.147), if we make the replacement $n + 1 \to \nu$, $J_n \to Z_\nu$, we find

$$Z_\nu = \frac{\nu-1}{x}Z_{\nu-1} - Z'_{\nu-1} \,.$$

Excepting $Z'_{\nu-1}$ from the latter two equalities, we get the *Bessel equation*, satisfied by the function $Z_\nu(x)$:

$$Z''_\nu + \frac{1}{x}Z'_\nu + \left(1 - \frac{\nu^2}{x^2}\right)Z_\nu = 0 \,. \tag{1.151}$$

□

This or a similar equation appears when solving many physical problems. Below, we will briefly summarize the basic information concerning the solutions of this equation. The generation of the necessary formulas is shown in special mathematical texts (Arfken, 1970; Nikiforov and Uvarov, 1988; Gradshtein and Ryzhik, 2007; Lee, 1965; Mathews and Walker, 1964; Abramovitz and Stegun, 1965; Vilenkin, 1988).

A solution of (1.151), limited when $\mathrm{Re}\,\nu \geq 0$, called a Bessel function of the first order when $x \to 0$, may be represented as a power series, which is a generalization of (1.142):

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,\Gamma(\nu+s+1)}\left(\frac{z}{2}\right)^{2s} \,. \tag{1.152}$$

The independent variable is labeled z, because the series remains valid whatever the values of ν and throughout the complex plane z, except for the slit along the negative part of the real axis.

Another linearly independent solution, when $\nu \neq n = 0, \pm 1, \ldots$, may be $J_{-\nu}(x)$. When n is an integer, there is a linear connection (1.144) between the two solutions shown. This is why the Bessel function of the second kind (the same as Neumann's function[21] or Weber's function[22]) is selected as the second solution:

$$Y_\nu(z) = \frac{J_\nu(z)\cos\nu\pi - J_{-\nu}(z)}{\sin\nu\pi} \,. \tag{1.153}$$

This solution has a finite bound when $\nu \to n$.

Also, Bessel functions of the third order, also called Hankel functions[23], may be selected as two linearly independent solutions:

$$H_\nu^{(1)}(z) = J_\nu(z) + i\,Y_\nu(z) \,; \quad H_\nu^{(2)}(z) = J_\nu(z) - i\,Y_\nu(z) \,. \tag{1.154}$$

All these functions are solutions of Bessel's equation. The functions Y_ν and $H_\nu^{(1,2)}$ have singularities when $z \to 0$. All these solutions satisfy the recurrent relations (1.145)–(1.147) (with the replacement of $n \to \nu$, $J_n \to Z_\nu$).

The asymptotic values are as follows: when $z \to 0$,

$$J_\nu(z) \approx \frac{z^\nu}{2^\nu \Gamma(\nu+1)} \,, \quad \nu \neq -1, -2, \ldots \,, \tag{1.155}$$

$$Y_0(z) \approx -i\,H_0^{(1)}(z) \approx i\,H_0^{(2)}(z) \approx \frac{2}{\pi}\ln z \,, \tag{1.156}$$

$$Y_\nu(z) \approx -i\,H_\nu^{(1)}(z) \approx i\,H_\nu^{(2)}(z) \approx -\frac{\Gamma(\nu)}{\pi}\left(\frac{z}{2}\right)^{-\nu} \,, \quad \mathrm{Re}\nu > 0 \,, \tag{1.157}$$

and when $|z| \to \infty$ and ν is arbitrary,

$$J_\nu(z) \approx \sqrt{\frac{2}{\pi z}}\cos\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \,, \quad |\arg z| < \pi \,, \tag{1.158}$$

$$Y_\nu(z) \approx \sqrt{\frac{2}{\pi z}}\sin\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \,, \quad |\arg z| < \pi \,, \tag{1.159}$$

$$H_\nu^{(1)}(z) \approx \sqrt{\frac{2}{\pi z}}\exp\left[i\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)\right] \,, \quad -\pi < \arg z < 2\pi \,, \tag{1.160}$$

$$H_\nu^{(2)}(z) \approx \sqrt{\frac{2}{\pi z}}\exp\left[-i\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)\right] \,, \quad -2\pi < \arg z < \pi \,. \tag{1.161}$$

21) Karl Gottfried Neumann (1832–1925) was a German mathematician.
22) Heinrich Weber (1842–1913) was a German mathematician.
23) Hermann Hankel (1839–1873) was a German mathematician and a historian of mathematics.

Cylindrical functions of purely imaginary arguments are called *modified Bessel functions*. The second of them is also called the Macdonald function.[24] They are described by the following relations:

$$I_\nu(z) = e^{-i\pi\nu/2} J_\nu(iz) \,, \tag{1.162}$$

$$K_\nu(z) = \frac{\pi}{2} e^{i\pi(\nu+1)/2} H_\nu^{(1)}(iz) \tag{1.163}$$

or

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{s=0}^{\infty} \frac{(z/2)^{2s}}{s!\Gamma(\nu+s+1)} \,, \quad K_\nu(z) = \frac{\pi}{2} \frac{I_{-\nu}(z) - I_\nu(z)}{\sin \nu\pi} \,. \tag{1.164}$$

These functions take real values when ν and $z > 0$ are real. Recurrent relation and differentiation formulas are produced from (1.145)–(1.147), (1.162), and (1.163). For instance,

$$I_0'(z) = I_1(z) \,, \quad K_0'(z) = -K_1(z) \,. \tag{1.165}$$

Modified Bessel functions satisfy the equation

$$W_\nu'' + \frac{1}{z} W_\nu' - \left(1 + \frac{\nu^2}{z^2}\right) W_\nu = 0 \,. \tag{1.166}$$

The asymptotic values are as follows: when $z \to 0$,

$$I_\nu(z) \approx \frac{z^\nu}{2^\nu \Gamma(\nu+1)} \,, \quad \nu \neq -1, -2 \ldots \,, \tag{1.167}$$

$$K_0(z) \approx -\ln z \,, \quad K_\nu(z) \approx \frac{1}{2}\Gamma(\nu)\left(\frac{z}{2}\right)^{-\nu} \,, \ \mathrm{Re}\nu > 0 \,, \tag{1.168}$$

and when $|z| \to \infty$,

$$I_\nu(z) \approx \frac{1}{\sqrt{2\pi z}} e^z \,, \quad |\arg z| < \frac{\pi}{2} \,, \tag{1.169}$$

$$K_\nu(z) \approx \sqrt{\frac{\pi}{2z}} e^{-z}, \quad |\arg z| < \frac{3\pi}{2} \,. \tag{1.170}$$

The spherical functions of Bessel, Hankel, and Weber often appear when problems are solved in spherical coordinates. They are of half-integer order and are described by the following equalities:

$$j_l(x) = \sqrt{\frac{\pi}{2x}} J_{l+\frac{1}{2}}(x) \,; \quad h_l^{(1,2)} = \sqrt{\frac{\pi}{2x}} H_{l+\frac{1}{2}}^{(1,2)}(x) \,;$$

$$y_l(x) = \sqrt{\frac{\pi}{2x}} Y_{l+\frac{1}{2}}(x) \,. \tag{1.171}$$

24) Hector Munro Macdonald (1865–1935) was a Scottish physicist and mathematician.

All these functions (their common symbol is $z_l(x)$) satisfy the following equation:

$$z_l'' + \frac{2}{x} z_l' + \left[1 - \frac{l(l+1)}{x^2}\right] z_l = 0 \, .$$

When x is small,

$$j_l(x) \approx \frac{x^l}{1 \cdot 3 \cdots (2l+1)} \, , \quad h_l^{(1,2)} \approx \mp \frac{x^{-l-1}}{1 \cdot 3 \cdots (2l-1)} \, . \tag{1.172}$$

When x is large,

$$\begin{aligned} j_l(x) &\approx \frac{1}{x} \cos\left[x - \frac{(l+1)\pi}{2}\right] \, , \\ h_l^{(1,2)}(x) &\approx \frac{1}{x} \exp\left\{\pm\left[x - \frac{(l+1)\pi}{2}\right]\right\} \, . \end{aligned} \tag{1.173}$$

Problems

1.99. Compute the indefinite integrals
$\int x^{\nu} Z_{\nu-1}(x) \mathrm{d}x$ and $\int x^{-\nu} Z_{\nu+1}(x) \mathrm{d}x$.

1.100. Compute the definite integrals
$\int_0^\infty J_1(x) \mathrm{d}x$, $\int_0^\infty J_2(x) x^{-1} \mathrm{d}x$, and $\int_0^\infty J_n(x) x^{-n} \mathrm{d}x$.

1.101. Prove the equality of the integrals
$\int_0^\infty J_n(x) \mathrm{d}x = \int_0^\infty J_n(x) \mathrm{d}x$, $n = 0, 1, \ldots$

1.102. Obtain the integral representation

$$J_0(x) = \frac{2}{\pi} \int_0^1 \frac{\cos ux}{\sqrt{1-u^2}} \mathrm{d}u \, .$$

Hint: Perform the substitution $u = \sin\varphi$.

1.103*. Compute the integrals

$$\int_0^{\pi/2} J_0(x \cos\varphi) \cos\varphi \mathrm{d}\varphi = \frac{\sin x}{x} \quad \text{and} \quad \int_0^{\pi/2} J_1(x \cos\varphi) \mathrm{d}\varphi = \frac{1 - \cos x}{x} \, .$$

Hint: You may use expansion in power series.

1.104. Produce the formulas

$$\begin{aligned} J_n(x) &= (-1)^k x^n \left(\frac{\mathrm{d}}{x \mathrm{d}x}\right)^k \left(x^{k-n} J_{n-k}(x)\right) \\ &= (-1)^n x^n \left(\frac{\mathrm{d}}{x \mathrm{d}x}\right)^n J_0(x) \, . \end{aligned}$$

1.105. Produce recurrent relations for the modified Bessel functions:

$$I_{\nu-1}(z) - I_{\nu+1}(z) = \frac{2\nu}{z} I_\nu(z) \; ; \quad I_{\nu-1}(z) + I_{\nu+1}(z) = 2I'_\nu(z) \; ; \tag{1.174}$$

$$K_{\nu-1}(z) - K_{\nu+1}(z) = -\frac{2\nu}{z} K_\nu(z) \; ;$$
$$K_{\nu-1}(z) + K_{\nu+1}(z) = -2K'_\nu(z) \; . \tag{1.175}$$

1.106*. Show that

$$J_0(|\boldsymbol{r}_1 - \boldsymbol{r}_2|) = \sum_{n=-\infty}^{\infty} J_n(r_1) J_n(r_2) \exp(i n\vartheta) \; ,$$

where ϑ is the angle between vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$.

1.107*.

1. Write an equation satisfied by the function $u(x) = J_n(ax)$.
2. Compute the integral ($b \neq a$)

$$\int_0^1 x J_n(ax) J_n(bx) \mathrm{d}x = \frac{a J'_n(a) J_n(b) - b J'_n(b) J_n(a)}{b^2 - a^2} \; . \tag{1.176}$$

3. $a \neq b$ are the roots of the equation $J_n(x) = 0$, that is, $J_n(a) = J_n(b) = 0$. Show that

$$\int_0^1 x J_n(ax) J_n(bx) \mathrm{d}x = 0 \quad \text{and} \quad \int_0^1 x J_n^2(ax) \mathrm{d}x = \frac{1}{2}[J'_n(a)]^2 \; . \tag{1.177}$$

Note The first equality (1.177) expresses the property called *the orthogonality of Bessel functions weighted by x.*

1.108*. Produce "summation theorems" for Bessel functions:

$$\sum_{k=-\infty}^{\infty} J_{n-k}(x) J_k(y) = J_n(x+y) \; , \quad n = 0, 1, 2, \ldots \; ; \tag{1.178}$$

$$\sum_{k=-\infty}^{\infty} (-1)^k J_{n-k}(x) J_k(x) = 0 \; , \quad n = 1, 2, \ldots \; ; \tag{1.179}$$

$$J_0(x) J_0(y) + 2\sum_{k=1}^{\infty} (-1)^k J_k(x) J_k(y) = J_0(x+y) \; , \quad n = 1, 2, \ldots \; ; \tag{1.180}$$

$$J_0^2(x) + 2\sum_{k=1}^{\infty} (-1)^k J_k^2(x) = J_0(2x) \; . \tag{1.181}$$

1.3.2
Spherical Functions and Legendre Polynomials

Spherical functions and Legendre polynomials are widely used in many fields of physics, especially in electrodynamics and quantum mechanics. The generating function for Legendre polynomials[25] $P_l(\cos\vartheta)$ is the reverse distance between two points with radius vectors $\boldsymbol{a}$ and $\boldsymbol{r}$, the angle between them equaling ϑ:

$$\frac{1}{|\boldsymbol{r}-\boldsymbol{a}|} = \frac{1}{\sqrt{r^2+a^2-2ar\cos\vartheta}} = \frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{a}{r}\right)^l P_l(\cos\vartheta)\,, \qquad \frac{a}{r}<1\,. \tag{1.182}$$

Designating $x=\cos\vartheta$ and $u=a/r$ and using binomial expansion, which, for the negative exponents is conveniently written as

$$(1-\alpha)^{-q} = \sum_{n=0}^{\infty}\frac{\Gamma(n+q)}{n!\,\Gamma(q)}\alpha^n\,, \quad |\alpha|<1\,,$$

and the binomial expansion for $\alpha^n=(2ux-u^2)^n$, we get a double sum:

$$(1-2ux+u^2)^{-1/2} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{\Gamma(n+1/2)}{\Gamma(1/2)k!(n-k)!}(-1)^k(2x)^{n-k}u^{n+k}\,.$$

Begin summing over k and $n+k=l\geq 0$, which will result in the rearrangement of the terms of the series. In this case, this rearrangement is valid because the infinite series is absolutely convergent, which will be shown below. As the result, we have

$$\begin{aligned}(1-2ux+u^2)^{-1/2} &= \sum_{l=0}^{\infty}\sum_{k=0}^{l}\frac{\Gamma(l-k+1/2)}{\Gamma(1/2)k!(l-2k)!}(-1)^k(2x)^{l-2k}u^l\\ &= \sum_{l=0}^{\infty}P_l(x)u^l\,,\end{aligned}$$

where

$$\begin{aligned}P_l(x) &= \sum_{k=0}^{l}\frac{\Gamma(l-k+1/2)}{\Gamma(1/2)k!(l-2k)!}(-1)^k(2x)^{l-2k}\\ &= \sum_{k=0}^{l}(-1)^k\frac{(2l-2k)!}{2^l k!(l-k)!(l-2k)!}x^{l-2k}\,.\end{aligned} \tag{1.183}$$

In the latter two equalities, the sum over k is actually limited to the value of the integer part of $l/2$ because the infinite factorial of the negative integer in the denominator will eliminate all terms with $l-2k<0$.

25) Adrien-Marie Legendre (1752–1833) was a French mathematician.

Example 1.15

Find the values of the polynomials $P_l(1)$, $P_l(-1)$, and $P_l(0)$ by assigning particular values to the angle ϑ in (1.182) and using the binomial expansion.

Solution. If we assume that $\cos\vartheta = 1$, we find, from (1.182), that

$$\frac{1}{1-u} = \sum_{l=0}^{\infty} u^l = \sum_{l=0}^{\infty} P_l(1)u^l$$

and, therefore, $P_l(1) = 1$ whatever the values of l, and $P_0 = 1$ when $0 \le \vartheta \le \pi$. Similarly, we get the following:

$$P_l(-1) = (-1)^l\,, \quad P_{2l}(0) = (-1)^l\frac{(2l-1)!!}{2^l l!}\,, \quad l \ge 1\,;$$
$$P_{2l+1}(0) = 0\,, \quad l \ge 0\,. \tag{1.184}$$

□

Example 1.16

Acquire limits of the values of Legendre polynomials $|P_l(\cos\vartheta)| \le 1$ by analyzing the expansion of the generating function (1.182) in series over $\cos m\vartheta$.

Solution. Sequentially obtain the following from the generating function:

$$\begin{aligned} 1 - 2u\cos\vartheta + u^2)^{-1/2} &= (1-ue^{i\vartheta})^{-1/2}(1-ue^{i\vartheta})^{-1/2} \\ &= \left\{1 + \frac{1}{2}ue^{i\vartheta} + \frac{3}{8}u^2e^{2i\vartheta} + \cdots\right\} \\ &\quad \times \left\{1 + \frac{1}{2}ue^{-i\vartheta} + \frac{3}{8}u^2e^{-2i\vartheta} + \cdots\right\} \\ &= \sum_{l=0}^{\infty} P_l(\cos\vartheta)u^l\,, \end{aligned}$$

where $P_l(\cos\vartheta) = \sum_{k=0}^{l} a_k \cos k\vartheta$. Coefficients a_k are selected from the values in braces and, importantly, they are all not negative: $a_k \ge 0$. In this case, the sum $\sum a_k \cos k\vartheta$ is maximal when $\vartheta = 0$, which corresponds to $P_l(1) = 1$. Therefore, $|P_l(\cos\vartheta)| \le 1$. □

The estimate we have made allows us to establish that the series (1.182), when $a/r < 1$, is absolutely convergent, that is, what converges is the series $\sum_{l=0}^{\infty} |P_l(\cos\vartheta)|(a/r)^l$. This follows from the established inequality and the fact that the dominating series $\sum_{l=0}^{\infty}(a/r)^l$ is knowingly convergent when $a/r < 1$. It represents the sum of the elements of a decreasing geometric progression.

Expansion (1.183) may yield a more compact representation of the Legendre polynomials if the following transforms are done sequentially:

$$\begin{aligned} P_l(x) &= \sum_{k=0}^{l} (-1)^k \frac{(2l-2k)!}{2^l k!(l-k)!(l-2k)!} x^{l-2k} \\ &= \sum_{k=0}^{l} \frac{(-1)^k}{2^l k!(l-k)!} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^l x^{2l-2k} \\ &= \frac{1}{2^l l!} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^l \sum_{k=0}^{l} \frac{(-1)^k l!}{k!(l-k)!} x^{2l-2k} = \frac{1}{2^l l!} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^l (x^2-1)^l . \end{aligned} \tag{1.185}$$

The latter expression for Legendre polynomials is called the *Rodrigues formula*.[26]

Example 1.17

Using the Rodrigues formula, produce recurrent relations between the Legendre polynomials:

$$P_l'(x) = x P_{l-1}'(x) + l P_{l-1}(x) ; \tag{1.186}$$

$$\begin{aligned} (1-x^2) P_l''(x) &= 2(l+1) P_{l+1}'(x) - 2(l+2)x \\ &\quad \times P_l'(x) - (l+1)(l+2) P_l(x) . \end{aligned} \tag{1.187}$$

Using the said relations, obtain a differential equation of the second order satisfied by $P_l(x)$.

Solution. Use the Leibniz formula[27] to find the derivative of order n from the product of the following functions:

$$(fg)^{(n)} = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} f^{(n-k)} g^{(k)} .$$

26) Benjamin Olinde Rodrigues (1794–1851) was a French mathematician and economist.

27) Gottfried Wilhelm Leibniz (1646–1716) was a German philosopher, jurist, and historian as well as a mathematician, physicist, and inventor. He was one of the founders of classical mathematical analysis.

Compute

$$P_l'(x) = \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^{l+1}(x^2-1)^l = \frac{2l}{2^l l!}\left(\frac{d}{dx}\right)^l\left[x(x^2-1)^{l-1}\right]$$
$$= \frac{2l}{2^l l!}\left[x\left(\frac{d}{dx}\right)^l(x^2-1)^{l-1} + l\left(\frac{d}{dx}\right)^{l-1}(x^2-1)^{l-1}\right].$$

Expression (1.186) follows from this equality and the Rodrigues formula. Obtain (1.187) using the following similar relation:

$$\left(\frac{d}{dx}\right)^{l+2}(x^2-1)^{l+1} = (x^2-1)\left(\frac{d}{dx}\right)^{l+2}(x^2-1)^l$$
$$+ 2(l+2)x\left(\frac{d}{dx}\right)^{l+1}(x^2-1)^l$$
$$+ (l+1)(l+2)\left(\frac{d}{dx}\right)^l(x^2-1)^l .$$

Obviously, the two recurrent relations obtained produce the differential *equation of Legendre* that has the following form:

$$(1-x^2)P_l''(x) - 2xP_l'(x) + l(l+1)P_l(x) = 0 . \tag{1.188}$$

The second linearly independent solution of Legendre's equation has singularities when $x = \pm 1$. □

Example 1.18

The adjoint Legendre polynomials are described by the expression

$$P_l^m(x) = (1-x^2)^{m/2}\left(\frac{d}{dx}\right)^m P_l(x)$$
$$= \frac{(1-x^2)^{m/2}}{2^l l!}\left(\frac{d}{dx}\right)^{l+m}(x^2-1)^l , \quad -l \le m \le l . \tag{1.189}$$

Obtain a differential equation satisfied by the adjoint Legendre polynomials.

Solution. When $m > 0$, differentiate the two parts of Legendre's equation (1.188) m times and get an equation of the form

$$(1-x^2)F'' - 2(m+1)xF' + (l-m)(l+m+1)F = 0$$

for the function

$$F(x) = \left(\frac{d}{dx}\right)^m P_l(x) = (1-x^2)^{-m/2}P_l^m(x) .$$

After inserting the derivatives in the equation obtained,

$$F'(x) = (1 - x^2)^{-m/2} \left[\frac{\mathrm{d} P_l^m}{\mathrm{d}x} + \frac{m x P_l^m}{1 - x^2} \right],$$

$$F''(x) = (1 - x^2)^{-m/2}$$
$$\times \left[\frac{\mathrm{d}^2 P_l^m}{\mathrm{d}x^2} + \frac{2mx}{1 - x^2} \frac{\mathrm{d} P_l^m}{\mathrm{d}x} + \frac{m P_l^m}{1 - x^2} + \frac{m(m+2) x^2 P_l^m}{(1 - x^2)^2} \right],$$

find the required equation:

$$(1 - x^2) \frac{\mathrm{d}^2 P_l^m}{\mathrm{d}x^2} - 2x \frac{\mathrm{d} P_l^m}{\mathrm{d}x} + \left[l(l+1) - \frac{m^2}{1 - x^2} \right] P_l^m = 0 . \tag{1.190}$$

Since the equation is not sensitive to the sign of m, $P_l^{-m}(x)$ and $P_l^m(x)$ may differ only in the factor independent of x (see Problem 1.116*). □

Example 1.19

Use (1.190) to prove the orthogonality of the adjoint Legendre polynomials where symbols m are the same and symbols l are different.

Solution. Write down (1.190) in the form

$$\frac{\mathrm{d}}{\mathrm{d}x} (1 - x^2) \frac{\mathrm{d} P_l^m}{\mathrm{d}x} + \left[l(l+1) - \frac{m^2}{1 - x^2} \right] P_l^m = 0$$

and another similar equation for $P_{l'}^m$. Further, multiply the first equation by $P_{l'}^m$ and the second one by P_l^m, deduct two equations term by term, and integrate over x. This gives us

$$[l(l+1) - l'(l'+1)] \int_{-1}^{1} P_{l'}^m(x) P_l^m(x) \mathrm{d}x = 0 ,$$

wherefrom we obtain the orthogonality of

$$\int_{-1}^{1} P_{l'}^m(x) P_l^m(x) \mathrm{d}x = 0 , \quad l' \neq l . \tag{1.191}$$

□

Example 1.20

The spherical Legendre function $Y_{lm}(\vartheta, \varphi)$ is described as follows:

$$Y_{lm}(\vartheta, \varphi) = C_{lm} P_l^m(\vartheta) e^{im\varphi} , \tag{1.192}$$

where $P_l^m(\vartheta)$ is an adjoint Legendre polynomial, expressed through trigonometric functions and C_{lm} is the normalization factor. Find the law of transition of this function when the coordinate system is inverted. Make sure that the Legendre spherical functions are orthogonal as to their indices when integrated over the whole spatial angle and write, in an explicit form, the condition of their normalization per unit.

Solution. When the coordinate system is inverted (see Section 1.1), the polar angles are transformed as per the rule $\vartheta \to \pi - \vartheta, \varphi \to \pi + \varphi$, $\cos\vartheta \to -\cos\vartheta$, $e^{im\varphi} \to (-1)^m e^{im\varphi}$. On the basis of the definition of $P_l^m(x)$ (1.189) and (1.191), find

$$Y_{lm}(\vartheta, \varphi) \to Y_{lm}(\pi - \vartheta, \pi + \varphi) = (-1)^l Y_{lm}(\vartheta, \varphi) . \tag{1.193}$$

Integrating over the whole spatial angle means that the boundaries of the angle measurement are $0 \le \vartheta \le \pi, 0 \le \varphi \le 2\pi$. Integrating over φ ensures the orthogonality of m-indexed spherical functions:

$$\int_0^{2\pi} e^{i(m-m')\varphi} d\varphi = 2\pi\delta_{mm'} .$$

Orthogonality over index l is ensured by the adjoint Legendre polynomials (see Example 1.19). The condition of orthogonality and per-unit normalization is as follows:

$$\int_0^{\pi} \sin\vartheta d\vartheta \int_0^{2\pi} d\varphi Y^*_{l'm'}(\vartheta, \varphi) Y_{lm}(\vartheta, \varphi) = \delta_{ll'}\delta_{mm'} . \tag{1.194}$$

The normalization factor is found as per the following condition:

$$2\pi|C_{lm}|^2 \int_{-1}^{1} [P_l^m(x)]^2 dx = 1 .$$

For the computation of the latter integral, see Problem 1.118*. □

Here is a rather useful relation called *the summation of spherical functions theorem.* Assume that it is probable, which it actually is. If θ is an angle between two vectors (r, ϑ, φ) and $(r', \vartheta', \varphi)$, that is,

$$\cos\theta = \cos\vartheta\cos\vartheta' + \sin\vartheta\sin\vartheta'\cos(\varphi - \varphi') ,$$

then

$$P_l(\cos\theta) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y_{lm}(\vartheta,\varphi)\, Y_{lm}(\vartheta',\varphi') \,. \tag{1.195}$$

The derivation of this expansion may be found in Arfken (1970). The method of the theory of group representations is described in detail in Vilenkin (1988) and Gel'fand *et al.* (1958). See also Abramovitz and Stegun (1965), Gradshtein and Ryzhik (2007), Kolokolov *et al.* (2000), and Madelung (1957).

Problems

1.109. Show that when $x = \cos\vartheta$, Legendre's equation assumes the following form:

$$\frac{1}{\sin\vartheta}\frac{\mathrm{d}}{\mathrm{d}\vartheta}\sin\vartheta\frac{\mathrm{d}P_l}{\mathrm{d}\vartheta} + l(l+1)P_l = 0 \,. \tag{1.196}$$

1.110. Obtain the recurrent relations

$$(2l+1)x P_l(x) = (l+1)P_{l+1}(x) + l P_{l-1}(x) \,,$$

$$(2l+1)P_l(x) = P'_{l+1}(x) - P'_{l-1}(x) \,,$$

where $l = 1, 2, \ldots$

For that purpose, you may use the Rodrigues formula and the method used in Example 1.12 when considering Bessel functions.

1.111. Using the recurrent relations, find the first five Legendre polynomials.

1.112*. Using the Rodrigues formula, prove the orthogonality of Legendre polynomials with various values of l and find the normalization integral:

$$\int_{-1}^{1} P_l(x)P_{l'}(x)\mathrm{d}x = \frac{2}{2l+1}\delta_{ll'} \,. \tag{1.197}$$

Hint: Express the normalization integral through the Euler beta function.

1.113. Using the generating function for Legendre polynomials, obtain the expansion

$$\frac{1-u^2}{(1-2ux+u^2)^{3/2}} = \sum_{l=0}^{\infty}(2l+1)P_l(x)u^l \,.$$

1.114. Using the results from Example 1.17, obtain the second Legendre polynomial in the form $P_2 = \sum_{k=0}^{2} a_k \cos k\vartheta$.

1.115. Write down (1.190) for adjoint Legendre polynomials in spherical coordinates.

1.116*. Using formula (1.189), show that

$$P_l^{-m}(x) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(x) . \tag{1.198}$$

Hint: Apply the Leibniz formula to the product $(x-1)^l(x+1)^l$ (see Example 1.17).

1.117. Write down, in explicit form, Legendre polynomials P_l^m for $l = 0, 1, 2, 3$.

1.118*. Find the normalization coefficient C_{lm} introduced in Example 1.20. Write down, in explicit form, Legendre's spherical function.

1.119. Write down an equation satisfied by Legendre's spherical function $Y_{lm}(\vartheta, \varphi)$.

1.3.3
Dirac Delta Function

We encounter the concept of the Dirac delta function[28] when trying to describe the charge density $\rho(\boldsymbol{r})$ of a point particle. If a particle with charge e is at the origin, then, obviously, the function $\rho(\boldsymbol{r})$ must have the following properties:

$$\rho(\boldsymbol{r}) = 0 \quad \text{if} \quad \boldsymbol{r} \neq 0 . \tag{1.199}$$

Yet when $r \to 0$, the density of $\rho(\boldsymbol{r})$ must increase fast enough for

$$\int\limits_{\Delta V} \rho(\boldsymbol{r}) \mathrm{d}V = e , \tag{1.200}$$

that is, for an integral over any volume ΔV, containing the point where the particle in question is located, to have the final value that equals the charge e.

Having written $\rho(\boldsymbol{r}) = e\delta(\boldsymbol{r})$, we get, from (1.199) and (1.200), the conditions determining the three-dimensional delta function:

$$\delta(\boldsymbol{r}) = 0 , \quad \boldsymbol{r} \neq 0 ; \quad \delta(\boldsymbol{r}) \to \infty , \quad \boldsymbol{r} \to 0 ; \tag{1.201}$$

$$\int\limits_{\Delta V} \delta(\boldsymbol{r}) \mathrm{d}V = 1 . \tag{1.202}$$

The one-dimensional delta function is described by similar relations:

$$\delta(x) = 0 , \quad x \neq 0 ; \quad \delta(x) \to \infty , \quad x \to 0 ; \quad \int\limits_{\Delta} \delta(x) \mathrm{d}x = 1 , \tag{1.203}$$

28) Paul Adrien Maurice Dirac (1902–1984) was an outstanding English theoretical physicist, a Nobel Prize recipient, one of the founders of quantum mechanics, and the creator of the first quantum field theory (quantum electrodynamics). He formulated the relativist quantum equation for electrons and other leptons and introduced the concept of antiparticles (see Chapter 6).

where Δ is the segment of the x axis that contains the point $x = 0$.

The delta function belongs to the class of singular generalized functions. It acquires its exact meaning under an integral. Consider the integral of the product of the delta function and any continuous and bounded function $f(x)$:

$$\int_{x_1}^{x_2} \delta(x) f(x) \mathrm{d}x \ ,$$

where $x_1 < 0$ and $x_2 > 0$. Since $\delta(x) = 0$ when $x \neq 0$, then only the small neighborhood ϵ of the point $x = 0$ where $f(x)$ is constant, equaling $f(0)$, makes a contribution to the integral:

$$\int_{x_1}^{x_2} \delta(x) f(x) \mathrm{d}x = f(0) \ . \tag{1.204}$$

Further, having replaced variable x with $x - a$ in the argument of the delta function, retracing the previous reasoning, we find the following:

$$\int_{x_1}^{x_2} \delta(x - a) f(x) \mathrm{d}x = f(a) \ , \tag{1.205}$$

if the interval (x_1, x_2) contains the point $x = a$.

Equalities (1.203) and (1.204) show that $\delta(x)$ is an even function of its argument:

$$\delta(x) = \delta(-x) \ . \tag{1.206}$$

Using the latter property and inserting the variable $|\alpha| x = y$, make sure that the relation

$$\int_{x_1}^{x_2} \delta(\alpha x) f(x) \mathrm{d}x = \frac{1}{|\alpha|} f(0) \tag{1.207}$$

is valid. Finally, consider the integral

$$\int_{x_1}^{x_2} \delta(g(x)) f(x) \mathrm{d}x \ ,$$

where a certain smooth function $g(x)$ is in the argument of the delta function. Only points where $g(x) = 0$, that is, the real roots of the function $g(x)$, contribute to the integral. Having labeled them as a_i, we may write

$$\int_{x_1}^{x_2} \delta(g(x)) f(x) \mathrm{d}x = \sum_i \int_{a_i - \epsilon}^{a_i + \epsilon} \delta(g(x)) f(x) \mathrm{d}x \ ,$$

where ϵ is a small number. If $f(x)$ is continuous, then $f(x)$ in the segment $[a_i - \epsilon, a_i + \epsilon]$ may be replaced by $f(a_i)$ and $g(x)$ approximated with the first member of the expansion $g(x) = g'(a_i)(x - a_i)$. As the result, using (1.207), we get

$$\int_{x_1}^{x_2} \delta(g(x)) f(x) \mathrm{d}x = \sum_i \frac{1}{|g'(a_i)|} f(a_i) . \tag{1.208}$$

This property of the delta function may be written as a symbolic equality:

$$\delta(g(x)) = \sum_i \frac{1}{|g'(a_i)|} \delta(x - a_i) . \tag{1.209}$$

If $g'(a_i) = 0$, that is, a_i is a multiple root, relations (1.208) and (1.209) become meaningless. Similarly, the product $\delta(x) f(x)$ is meaningless if the function $f(x)$ has a singularity when $x = 0$.

The derivative from the delta function may also be found. Its exact meaning is in the formula

$$\int_{x_1}^{x_2} f(x) \frac{\partial \delta(x - a)}{\partial x} \mathrm{d}x = -\frac{\partial f(a)}{\partial a} , \tag{1.210}$$

which is produced by integrating by parts. Derivatives of higher orders are found in a similar way:

$$\int_{x_1}^{x_2} f(x) \delta^{(n)}(x - a) \mathrm{d}x = (-1)^n f^{(n)}(a) . \tag{1.211}$$

The function $\delta(x)$ may be regarded as a derivative from the Heaviside step (or staircase) function[29] $\Theta(x)$. This follows from the obvious relation

$$\int_{x_1}^{x} \delta(x) \mathrm{d}x = \Theta(x) = \begin{cases} 1, & x > 0 , \\ \frac{1}{2}, & x = 0 , \\ 0, & x < 0 , \end{cases} \tag{1.212}$$

where the lower bound of integration x_1 is any negative number. Differentiating this equality, we get the following:

$$\Theta'(x) = \delta(x) . \tag{1.213}$$

In equality (1.212), when the bound of integration coincides with the point where the argument of the delta function is reduced to zero, we use *half* of the value of

29) Oliver Heaviside (1850–1925) was an English physicist, engineer, and mathematician. He developed the basics of operational and vector calculus in their present state. For instance, Heaviside introduced the concept of ort, the name "nabla" for Hamilton's operator (∇), and the in-bold notation for labeling vectors (Prokhorov, Yu.,V. (1988) Mathematical Encyclopaedic Dictionary, Sovietskaya Enciklopediya).

the smooth function $f(x) = 1$, that is, we use the integration rule:

$$\int_{x_1}^{a} f(x)\delta(x-a)\mathrm{d}x = \frac{1}{2} f(a) \,. \tag{1.205'}$$

This rule agrees with property (1.206), which is the evenness of the delta function.

The three-dimensional delta function may be regarded as the product of three one-dimensional delta functions:

$$\delta(\boldsymbol{r}-\boldsymbol{a}) = \delta(x-a_x)\delta(y-a_y)\delta(z-a_z) \,. \tag{1.214}$$

This is why all the above properties of one-dimensional delta functions are easily generalized to include the case of three dimensions.

1.3.4 Certain Representations of the Delta Function

One may obtain a visual representation of the delta function and its derivatives by looking at the diagram of a certain continuous function $\delta_\epsilon(x-a)$, such as $\int_\Delta \delta_\epsilon(x-a)\mathrm{d}x = 1$. The parameter ϵ characterizes the width of the interval within which the function in question is other than zero (Figure 1.10).

The delta function and its derivatives are defined as the limits

$$\delta(x-a) = \lim_{\epsilon\to 0} \delta_\epsilon(x-a) \,, \qquad \frac{\partial\delta(x-a)}{\partial x} = \lim_{\epsilon\to 0} \frac{\partial\delta_\epsilon(x-a)}{\partial x} \,,$$

and so on.

Many nonsingular functions depending on a parameter when it has certain limiting values assume the properties of the delta function. The most often used such representations of the delta function areas follows:

$$\delta(x) = \frac{1}{\pi} \lim_{\epsilon\to 0} \frac{\epsilon}{\epsilon^2 + x^2} = \frac{1}{2\pi i} \lim_{\epsilon\to 0} \left(\frac{1}{x-i\epsilon} - \frac{1}{x+i\epsilon} \right) \,; \tag{1.215}$$

$$\delta(x) = \frac{1}{\pi} \lim_{K\to\infty} \left(\frac{\sin Kx}{x} \right) \,; \tag{1.216}$$

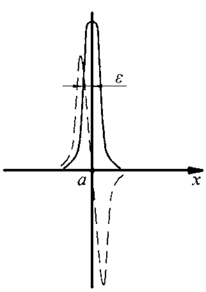

Figure 1.10 The visualization of the delta function and its first derivative.

$$\delta(x) = \frac{1}{\pi} \lim_{K \to \infty} \left(\frac{\sin^2 Kx}{Kx^2} \right) ; \tag{1.217}$$

$$\delta(x) = \lim_{\epsilon \to 0} \frac{1}{\sqrt{\pi\epsilon}} e^{-x^2/\epsilon} . \tag{1.218}$$

Example (1.216) yields the following representations:

$$\delta(x) = \frac{1}{2\pi} \lim_{K \to \infty} \int_{-K}^{K} e^{ikx} \mathrm{d}k = \frac{1}{\pi} \lim_{K \to \infty} \int_{0}^{K} \cos kx \mathrm{d}k . \tag{1.219}$$

They may be regarded as expansions of the delta function in a Fourier integral.[30)] Sometimes, formulas (1.219) are written without the sign for passage to the limit when integrating over infinite limits.

It is easy to make sure that any of the representations (1.215)–(1.219) agrees with all the properties of (1.203)–(1.207) and the definition (1.210) of a derivative from the delta function. *When computing integrals with delta functions with the use of representations such as ((1.215))–((1.219)), one should pass to the limit after integrating.* For instance, when using (1.216), we have

$$\int_{-a}^{b} \delta(x) f(x) \mathrm{d}x = \frac{1}{\pi} \lim_{K \to \infty} \int_{-aK}^{bK} f\left(\frac{y}{K}\right) \frac{\sin y}{y} \mathrm{d}y = f(0) , \tag{1.220}$$

and the limit (1.216) per se does not exist.

1.3.5
The Representation of the Delta Function through Loop Integrals in a Complex Plane

We will now use Cauchy's[31)] integral formula:

$$\frac{1}{2\pi i} \int_{C} \frac{f(z)}{z - a} \mathrm{d}z = f(a) , \tag{1.221}$$

where $f(z)$ is a function without singularities either within the area bounded by the closed loop C or on the loop itself, in the plane of the complex variable z, integration over which is done counterclockwise. As follows from the comparison of (1.221) with (1.205), the quantity

$$\frac{1}{2\pi i} \frac{1}{z - a}$$

may be regarded as a representation of $\delta(z - a)$ if we agree to integrate over the closed loop that surrounds point $z = a$, within which, just as on the loop itself,

30) For Fourier integrals, see Section 1.3.8.

31) Augustin-Louis Cauchy (1789–1857) was an outstanding French mathematician and physicist. Unlike Laplace, Cauchy was a catholic and a royalist.

there are no other singularities of the subintegral expression. For instance, the loop C may be a circle of small radius.

In applications, one frequently encounters an integral over a proper axis:

$$\int_{x_1}^{x_2} \frac{f(x)}{x-a} \mathrm{d}x ,$$

where $f(x)$ has no singularities on the segment $[x_1, x_2]$, whereas the limits x_1, x_2 may be infinite. Such an integral, when a is real, has no particular value because the subintegral expression has a pole on the path of integration. Computing this integral requires additional information, that is, the rule of circumventing the special point must be indicated. Usually, the circumvention rule is established on the basis of physical arguments:

$$\int_{C_{\mathrm{Re}}+C_r} \frac{f(x)}{x-a} \mathrm{d}x = \int_{C_r} \frac{f(x)}{x-a} \mathrm{d}x + \int_{C_{\mathrm{Re}}} \frac{f(x)}{x-a} \mathrm{d}x .$$

This means that the integral in the first member of the above relation, where integration is done over the whole loop, may be represented (see Figure 1.11) as the sum of two integrals. In the first one of these, integration is done over either the top or the bottom semicircle of a small radius C_r, whereas in the second one, it is done over the remaining part of the loop running along the proper axis (this part of the loop is labeled with the symbol C_{Re}).

The integral over the semicircle of radius $\epsilon \to 0$ gives half of the remainder (with a minus sign for the upper loop, as in Figure 1.11, because the pole is circumvented clockwise):

$$\int_{C_r} \frac{f(x)}{x-a} \mathrm{d}x = -i\pi f(a) .$$

The computation of the integral over the proper axis, with the excepted main point, is done so as to find its principal value:

$$\int_{x_1}^{x_2} \frac{f(x)}{x-a} \mathrm{d}x = \lim_{\epsilon \to 0} \left\{ \int_{x_1}^{a-\epsilon} \frac{f(x)}{x-a} \mathrm{d}x + \int_{a+\epsilon}^{x_2} \frac{f(x)}{x-a} \mathrm{d}x \right\} \equiv \mathcal{P} \int_{x_1}^{x_2} \frac{f(x)}{x-a} \mathrm{d}x .$$

When we circumvent the pole along the lower semicircle, the sign of half-remainder changes. As a result, we get the following rule of computing integrals (Sokhotskii

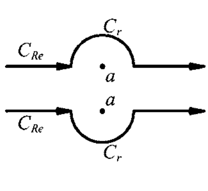

Figure 1.11 The contours of the rounding of poles in the plane of a complex variable.

formulas)[32]:

$$\frac{1}{x-a} = \mp i\pi\delta(x-a) + \mathcal{P}\frac{1}{x-a} \,. \tag{1.222}$$

The symbol $\mathcal{P}$ represents the principal value (the upper sign is for the upper loop and the lower sign is for the lower one; see Figure 1.11).

Instead of deforming the path of integration, one may slightly displace the pole away from the proper axis. This is done by adding a small imaginary part to the number a: $a \to a \mp i\epsilon$, $\epsilon \to 0$. This kind of substitution will give the following form to identity (1.222):

$$\lim_{\epsilon\to 0} \frac{1}{x-a \pm i\epsilon} = \mp i\pi\delta(x-a) + \frac{\mathcal{P}}{x-a} \,. \tag{1.223}$$

Both identities, (1.222) and (1.223), have a symbolic (operator) character and must be understood in the way that the integration of their second and first members with any continuous function gives the same result.

Having separated, in the first member of equality (1.223), the real part of the complex expression from the imaginary one, we get the representation (1.215) for $\delta(x-a)$ (with the substitution $x \to x-a$) and for the principal value

$$\frac{\mathcal{P}}{x-a} = \lim_{\epsilon\to 0} \frac{x-a}{(x-a)^2+\epsilon^2} \,. \tag{1.224}$$

For the rigorous mathematical theory of generalized functions, please see Vladimirov (2002). The applied aspects are described in Zel'dovich and Myshkis (1972). See also Kolokolov *et al.* (2000).

Problems

1.120. Compute the integrals

$$\int_{-2}^{3} (x^2-x-5)\delta(-3x)\mathrm{d}x \,, \quad \int_{-10}^{-3} (x+3)\delta(x+5)\mathrm{d}x \,, \quad \int_{0}^{5} (x+5)\delta(x+5)\mathrm{d}x,$$

and

$$\int_{-\infty}^{\infty} \exp(\alpha x)\delta(x^2+x-2)\mathrm{d}x \,, \quad \alpha = \text{const} \,.$$

1.121. Simplify the expressions $(x-a)\delta(x-a)$, $f(x)\delta(x-a)$, and $(3x^3-7x)\delta(2x^2-6x-4)$.

32) Julian Sokhotskii (1842–1927) was a Russian mathematician who contributed to the development of the theory of functions of complex variables.

1.122. Prove that representations (1.215), (1.217), and (1.218) describe the delta function. For that purpose, compute the integrals of the form $\int_{-\infty}^{\infty} f(x)\delta(x)\mathrm{d}x$ from the continuous function $f(x)$, substituting the second member of the respective representation for $\delta(x)$, and then make sure that, after proceeding to the limit, the said integrals produce $f(0)$.

1.123. Write the three-dimensional delta functions $\delta(\boldsymbol{r})$ and $\delta(\boldsymbol{r}-\boldsymbol{a})$ in cylindrical coordinates, where $\boldsymbol{a} = (a_\perp, \alpha_0, a_z)$ is a constant vector given by its cylindrical coordinates.

1.124. Do the same in spherical coordinates $\boldsymbol{a} = (a, \vartheta_0, \ \alpha_0)$.

1.125. Using the delta function, write down the first derivative from the discontinuous function:

$$f(x) = \begin{cases} x^3\,, & \text{if } x < 1\,, \\ 2\,, & \text{if } x = 1\,, \\ x^2 + 2\,, & \text{if } x > 1\,. \end{cases}$$

1.126. The function $f(x)$ has jump discontinuities (finite jumps) at points a_i, $i = 1, 2, \ldots, n$. Write down its first derivative through the delta function.

1.127. Find the rule for computing the integral from the product $f(x)x^n\delta^{(m)}(x)$, where $f(x)$ is the function differentiated (in the classical sense) when $x = 0$, $\delta^{(m)}(x)$ is the mth derivative of the delta function, and n is a positive integer.

1.128. Show that the function $G(|\boldsymbol{r}-\boldsymbol{r}'|) = 1/|\boldsymbol{r}-\boldsymbol{r}'|$ satisfies a Poisson equation with a delta-like second member:

$$\Delta G(|\boldsymbol{r}-\boldsymbol{r}'|) = -4\pi\delta(\boldsymbol{r}-\boldsymbol{r}')\,. \tag{1.225}$$

1.3.6
Expansion in Total Systems of Orthogonal and Normalized Functions. General Considerations

Let us say there is a certain system of linearly independent functions $\varphi(x, \lambda_n) \equiv \varphi_n(x)$, generally complex valued, defined over a certain interval $[a, b]$ of a real variable x and dependent on the real parameter λ that takes a discrete series of values: $\lambda_1, \lambda_2, \ldots$

Such systems of functions often appear when solving ordinary differential equations or equations in partial derivatives with appropriate boundary conditions, and the number of functions in them is, usually, infinitely large: $n = 0, 1, \ldots$ Let us say the functions have the following properties:

1. They are normalized to unity, that is,

$$\int_a^b |\varphi_n(x)|^2 \mathrm{d}x = 1\,. \tag{1.226}$$

2. They are mutually orthogonal, that is,

$$\int \varphi_m^*(x)\varphi_n(x)\mathrm{d}x = 0 \quad \text{at} \quad m \neq n \, . \tag{1.227}$$

Here, the asterisk marks a complex conjugate. Such systems are called *orthonormalized*, and equalities (1.226) and (1.227) may be written similarly with the use of the Kronecker delta symbol:

$$\int_a^b \varphi_m^*(x)\varphi_n(x)\mathrm{d}x = \delta_{mn} \, . \tag{1.228}$$

Now, we will consider an arbitrary function $f(x)$ with integrable square. That is, a function for which the integral $\int_a^b |f(x)|^2 \mathrm{d}x$ is finite. In the case of the finite interval $[a, b]$, this condition will be satisfied by any piecewise continuous function with a limited number of finite jumps within this interval. Now, we will find out how possible the expansion of such a function is in a series over functions $\varphi_n(x)$. For that purpose, we will, firstly, approximate the function in question as a linear superposition that includes n basic functions:

$$f(x) = \sum_{k=0}^{n} c_n \varphi_k(x) + R_n(x) \, , \tag{1.229}$$

where the remainder of the series is labeled $R_n(x)$. We will select the coefficients c_n of the superposition so as to ensure the smallest approximation error. Our measure of error will be the quantity

$$G_n = \int_a^b |R_n(x)|^2 \mathrm{d}x = \int_a^b \left| f(x) - \sum_{k=0}^{n} c_n \varphi_k(x) \right|^2 \mathrm{d}x \, . \tag{1.230}$$

Opening the square of the module and using the condition of orthonormality (1.228), we will have

$$\begin{aligned} G_n = & \int_a^b |f(x)|^2 \mathrm{d}x - \sum_{k=0}^{n} c_k \int_a^b f^*(x)\varphi_k(x)\mathrm{d}x \\ & - \sum_{k=0}^{n} c_k^* \int_a^b f(x)\varphi_k^*(x)\mathrm{d}x + \sum_{k=0}^{n} c_k^* c_k \, . \end{aligned} \tag{1.231}$$

The necessary condition of the minimum quantity G_n, regarded as the function of coefficients c_k, gives us

$$c_k = \int_a^b f(x)\varphi_k^*(x)\mathrm{d}x \, , \tag{1.232}$$

and the expansion error assumes the form of

$$G_n = \int_a^b |f(x)|^2 \mathrm{d}x - \sum_{k=0}^{n} |c_k|^2 . \tag{1.233}$$

Since $G_n \geq 0$ by definition, the inequality

$$\sum_{k=0}^{n} |c_k|^2 \leq \int_a^b |f(x)|^2 \mathrm{d}x \tag{1.234}$$

is valid whatever the value of n. If the equality

$$\lim_{n \to \infty} G_n = 0 \tag{1.235}$$

is valid for any function with integrable square at the limit, or in another form

$$\int_a^b |f(x)|^2 \mathrm{d}x = \sum_{k=o}^{\infty} |c_k|^2 \tag{1.236}$$

(Parseval's identity)[33], then the system of functions $\varphi_n(x), n = 0, 1, \ldots$ is called *complete* or *closed*. These terms mean that no other functions linearly independent of $\varphi_n(x)$ and orthogonal to them exist: any function of the series in question is expandable in a series:

$$f(x) = \sum_{k=0}^{\infty} c_k \varphi_k(x) , \tag{1.237}$$

where expansion coefficients are given by formula (1.232). We note that the above conditions ensure the convergence "on the average" of series (1.237), that is, the reduction of integral (1.230) to zero. This means that the convergence of the series on the function $f(x)$ in question may be disrupted at certain points whose number is finite. If the system of functions $\varphi_n(x)$ is orthonormalized but not complete, then, instead of Parseval's identity (1.236), *Bessel's inequality* becomes valid:

$$\sum_{k=0}^{\infty} |c_k|^2 \leq \int_a^b |f(x)|^2 \mathrm{d}x . \tag{1.238}$$

Example 1.21

Show that a complete system of orthonormalized functions satisfies the following relation:

$$\sum_{k=0}^{\infty} \varphi_k^*(x')\varphi_k(x) = \sum_{k=0}^{\infty} \varphi_k(x')\varphi_k^*(x) = \delta(x - x') , \tag{1.239}$$

33) Marc-Antoine Parseval (1755–1836) was a French mathematician.

which may be regarded as one more, different from (1.236), form of the condition of completeness (closeness).

Solution. Having inserted the expansion coefficients from (1.232) into (1.237) and changed the order of the operations of summation and integration, we will have

$$f(x) = \int_a^b dx' f(x') \sum_{k=0}^{\infty} \varphi_k^*(x')\varphi_k(x) = \int_a^b K(x, x') f(x') \mathrm{d}x' \, , \qquad (1.240)$$

where

$$K(x, x') = \sum_{k=0}^{\infty} \varphi_k^*(x')\varphi_k(x) \, . \qquad (1.241)$$

Since equality (1.240) must be valid for any function $f(x)$ of a large class, then the nucleus $K(x, x')$ of the integral transformation (11.15) must have the properties of a delta function. Having computed the expansion coefficients $\delta(x - x')$ for the system of functions $\varphi_k(x)$ as per (1.232), we may make sure that this is the case[34]:

$$c_n = \int \delta(x - x')\varphi_n^*(x)\mathrm{d}x = \varphi_n^*(x') \, .$$

Therefore, equality (1.239) is true and is the expansion of the delta function in functions $\varphi_k(x)$. □

In certain physical problems, especially in quantum mechanics, a complete system includes not just a discrete series of functions $\varphi_n(x)$ but also functions $\varphi(x, \lambda)$ dependent on the parameter λ, which assumes continuous values from a certain interval, or just functions with a continuous parameter. In cases such as that, the expansion of any function includes both the sum and the integral over the continuous values of λ or just the integral, whereas the condition of completeness assumes the following form:

$$\delta(x - x') = \sum_{k=0}^{\infty} \varphi_k^*(x')\varphi_k(x) + \int \varphi^*(x', \lambda)\varphi(x, \lambda)\mathrm{d}\lambda \, . \qquad (1.242)$$

1.3.7
Fourier Series

The proof of the completeness of specific systems of functions is a nontrivial mathematical problem whose solutions may be found in particular texts[35] (see,

34) Here we leave the class of proper functions with integrable square and use generalized functions.

35) Jean Baptiste Joseph Fourier (1768–1830) was a French mathematician who worked on problems of mathematical physics, especially the theory of heat conduction.

e.g., Sneddon, 1951; Arfken, 1970; Lee, 1965; Tolstov, 1976). The class of complete orthonormalized systems includes the Legendre's system of spherical functions $Y_{lm}(\vartheta, \varphi)$, $l = 0, 1, \ldots, m = -l, -l+1, \ldots, l-1, l$ considered above. Any bounded function "on the surface of a sphere" that is dependent on angles ϑ and φ may be expanded in such functions. When there is no dependence on φ, complete systems on a sphere are formed by Legendre polynomials $P_l(\cos\vartheta)$.

One of the most widely used and complete systems of functions, orthonormalized over the interval $[-\pi, +\pi]$, is the trigonometric system:

$$\frac{1}{\sqrt{2\pi}}, \quad \frac{\cos n\tau}{\sqrt{\pi}}, \quad \frac{\sin n\tau}{\sqrt{\pi}}, \qquad n = 1, 2, \ldots \tag{1.243}$$

The orthonormality of this system of functions may be easily verified directly. The expansion of a certain function in a series over trigonometric functions forms its *Fourier series*. However, sometimes, a general expansion (1.237) over any complete orthonormalized system of functions is also called a Fourier series (in a wider sense).

Because the trigonometric functions (1.243) are periodic, a function being expanded will be represented by the Fourier series, whatever the values of τ, only if it is periodic and has the same period 2π, that is, $f(\tau) = f(\tau + 2n\pi)$, $n = \pm 1, \pm 2, \ldots$, or if it is specified within the finite segment $b - a = 2L > 0$. In the latter case, in (1.243), the variable τ to $\pi x/L$ must be replaced and the reference point of the coordinate x shifted to the center of the interval $[a, b]$, that is, $x' = x - a - L, -L \le x' \le +L$ is introduced. The function in question, if a Fourier series is set for it, will be expanded, in this case periodically, to the whole proper Ox axis. A nonperiodic function specified over an infinite interval will be correctly represented by a Fourier series only at the final segment $2L$. For it to be represented over the whole Ox axis, the Fourier integral (see below) must be used.

If a Fourier series represents a function that has jump discontinuities (finite jumps), it will, at the point of a jump $x = x_0$, converge on the half sum of the values of the function located on both sides of the jump:

$$\sum_{k=0}^{\infty} c_n \varphi_n(x_0) = \frac{1}{2}[f(x_0 - 0) + f(x_0 + 0)] \,. \tag{1.244}$$

Example 1.22

Write down the Fourier expansion over interval $[-L, +L]$, selecting, as a complete system of functions,[36] exponents with imaginary index $\exp(in\pi x/L)$, $n = 0, \pm 1, \ldots$

36) The completeness of the system follows from that previously used. The functions $\sin n\tau$ and $\cos n\tau$ are linearly expressed through $\exp(in\tau)$. This is why the notation of the exponents signifies another form of trigonometric series.

Solution. Make sure that the components in question are mutually orthogonal over the interval $[-L, +L]$:

$$\int_{-L}^{L} \exp\left\{\frac{i(m-n)\pi x}{L}\right\} \mathrm{d}x = 2L\delta_{mn} .$$

Write down the required expansion as

$$f(x) = \sum_{n=-\infty}^{\infty} F_n \exp\left\{\frac{i n \pi x}{L}\right\} . \tag{1.245}$$

In order to find the expansion coefficients F_n, multiply both members of (1.245) by $\exp(i m\pi x/L)$ and integrate over the interval in question. Owing to the orthogonality of the exponents, after integration in the sum over n, only one member with $n = m$ is left. This will allow you to find the coefficients of the Fourier series:

$$F_m = \frac{1}{2L} \int_{-L}^{L} f(x) \exp\left\{\frac{i m \pi x}{L}\right\} \mathrm{d}x . \tag{1.246}$$

As follows from (1.246), if $f(x)$ is a real function, then the Fourier coefficients (1.246), being, in the general case, complex quantities, satisfy the condition $F_{-n} = F_n^*$. This condition ensures the reality of the sum of the series (1.245). □

The Fourier expansion, obviously, may be generalized to include the case of functions that depend on several variables.

Problems

1.129. Expand the periodic function specified within the interval $[-\pi, +\pi]$ in the Fourier series under the conditions $f(x) = x$ for $0 \le x \le \pi$ and $f(-x) = f(x)$.

1.130. Do the same for the function under the conditions $f(x) = a$ at $0 \le x \le \pi$ and $f(-x) = -f(x)$.

1.131. Expand the periodic function specified within the interval $[-L, +L]$ in the Fourier series under the conditions $f(x) = a$ when $0 \le x < L/2$ and $f(x) = 0$ when $L/2 < x \le L$ and $f(-x) = f(x)$.

1.3.8
Fourier Integral

We will now consider a system of functions dependent on the real parameter λ, which takes a continuous series of values:

$$\varphi(x, \lambda) = \frac{1}{\sqrt{2\pi}} e^{i\lambda x} , \quad -\infty < \lambda < \infty . \tag{1.247}$$

These functions are determinate and bounded whatever the real values of the coordinate x may be, that is, within the infinite interval $-\infty < x < \infty$. Using the representation (1.219) of the delta function, we compute the integral

$$\int_{-\infty}^{\infty} \varphi(x,\lambda)\varphi^*(x',\lambda)\mathrm{d}\lambda = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\lambda\, e^{i\lambda(x-x')} = \delta(x-x') \,.$$

The resulting relation coincides with (1.242) (when there are no discrete values of λ) and evidences the completeness of the system of functions $\varphi(x,\lambda)$. This is why any function of a rather large class, defined over the whole proper Ox axis, may be expanded in functions $\varphi(x,\lambda)$:

$$f(x) = \int_{-\infty}^{\infty} F(\lambda)\varphi(x,\lambda)\mathrm{d}\lambda = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\lambda)e^{i\lambda x}\mathrm{d}\lambda \,. \tag{1.248}$$

The function $F(\lambda)$ is called the *Fourier image* of the original function $f(x)$ or its *Fourier amplitude*. It may be found in the same way as the Fourier series coefficients were found in Example 1.22: by multiplying both members of equality (1.248) by $\varphi^*(x,\mu)$ and integrating over the coordinate x. Changing the order of integration, we have

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)\varphi^*(x,\mu)\mathrm{d}x &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\lambda\, F(\lambda)\int_{-\infty}^{\infty} e^{ix(\lambda-\mu)}\mathrm{d}x \\ &= \int_{-\infty}^{\infty} \mathrm{d}\lambda\, F(\lambda)\delta(\lambda-\mu) = F(\mu) \,. \end{aligned} \tag{1.249}$$

This is the equality that allows us to find the Fourier amplitude of the specified function $f(x)$.

The direct and inverted Fourier transforms are often written more easily in their asymmetric form:

$$f(x) = \int_{-\infty}^{\infty} F(\lambda)e^{i\lambda x}\frac{\mathrm{d}\lambda}{2\pi} \,, \quad F(\lambda) = \int_{-\infty}^{\infty} f(x)e^{-i\lambda x}\mathrm{d}x \,. \tag{1.250}$$

The reality of the Fourier integral is ensured by the relation

$$F(-\lambda) = F^*(\lambda) \tag{1.251}$$

when λ and $f(x)$ are real.

Expansion in the Fourier integral is easily generalized for the case of several dimensions. For instance, in three-dimensional space, the Fourier transform may be written as

$$f(\boldsymbol{r}) = \int F(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\,, \quad F(\boldsymbol{k}) = \int f(\boldsymbol{r}) e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} \mathrm{d}^3\boldsymbol{r}\,. \tag{1.252}$$

In both integrals, integration is done over the whole space.

Example 1.23

Obtain the expansion in the Fourier integral for an infinite interval $-\infty < x < \infty$ by the way of passage to the limit $L \to \infty$ in formulas (1.245) and (1.246).

Solution. When $L \to \infty$, the adjacent members summed as per (1.245) are almost equal. This is why summation may be replaced by integration over $\mathrm{d}n = (L/\pi)\mathrm{d}\lambda$ within $-\infty, +\infty$. By labeling $\lim_{L\to\infty} 2LF_n$ through $F(\lambda)$, we get, from (1.245) and (1.246), relation (1.250). □

Besides the already mentioned sources, for more information about expansion in systems of functions, series, and integrals, see Arfken (1970), Sneddon (1951), Madelung (1957), and Tolstov (1976).

Problems

1.132. Express the Fourier image of the derivative $f'(x)$ through the Fourier image $F(\lambda)$ of the function $f(x)$. It is presumed that the integral $\int_{-\infty}^{\infty} |f(x)|\mathrm{d}x$ is convergent.

1.133. Do the same for the function $f(ax)\exp(ibx)$.

1.134. Find the Fourier image of the function $f(x) = (1 + x^2)^{-1}$.

Hint: Regarding x as a complex variable, close the path of integration with an arc of an infinite radius and use the residue theorem.

1.135. Find the Fourier image of the function $\exp(-\alpha^2 x^2)$.

1.136. Find the three-dimensional Fourier image of the function $f(r) = \exp(-\alpha^2 r^2)$.

1.137*. Find the three-dimensional Fourier image of the function $G(r) = r^{-1}$.

1.138*. Expand the plane wave $\exp(ikr\cos\theta)$ in series over Legendre polynomials $P_l(\cos\theta)$. Find the expansion coefficients, using the orthogonality of Legendre polynomials.

1.139. Assume that the directions of the vectors $\boldsymbol{k}$ and $\boldsymbol{r}$ are specified in a spherical coordinate system by the angles (θ, ϕ) and ϑ, φ, respectively. Expand the plane wave $\exp(i\boldsymbol{k}\cdot\boldsymbol{r})$ in a series over spherical Legendre functions.

Hint: Use the summation theorem for spherical functions.

1.140*. Prove the identity

$$\frac{1}{\sqrt{r_\perp^2 + z^2}} = \int_0^\infty e^{-k|z|} J_0(k r_\perp) \mathrm{d}k \,,$$

where $r_\perp$ and z are cylindrical coordinates.

1.4 Answers and Solutions

1.1 As follows from (1.6), which is the result of definition (1.3), $|\widehat{a}|^2 = 1$ whatever the angles of rotation are, that is, $|\widehat{a}| = \pm 1$. However, when the angle of rotation equals zero (identical transformation), $|\widehat{a}| = 1$. Since the elements of the rotation matrix are continuous functions of angles, the last value is preserved for all values of the rotation angles. When the axes are inverted, $|\widehat{g}| = -1$. The product of the matrices $\widehat{a}\widehat{g} = \widehat{g}\widehat{a}$, for which $|\widehat{a}\widehat{g}| = |\widehat{a}||\widehat{g}| = -1$, corresponds to the rotation accompanied by the reflection of the axes. Transformations with determinant $+1$ are called proper and transformations with determinant -1 are called nonproper.

1.3

$$P'_{\alpha\beta\cdots\kappa} = |\widehat{a}| a_{\alpha\mu} a_{\beta\nu} \cdots a_{\kappa\sigma} P_{\mu\nu\cdots\sigma} \,. \tag{1.253}$$

Here $|\widehat{a}|$ is the determinant of the transformation matrix. When the three axes are inverted, the transformation matrix $a_{\alpha\beta} = -\delta_{\alpha\beta}$, and that is why $|\widehat{a}| = -1$ and $P'_{\alpha\beta\cdots\kappa} = (-1)^{s+1} P_{\alpha\beta\cdots\kappa}$, in keeping with the definition of a pseudotensor of rank s. Formula (1.5) correctly describes transformations of a polar tensor during rotations and reflections but does not describe reflections of a pseudotensor (even though it correctly describes its rotations).

The rule of the transformation of the asymmetric tensor of rank 3 $e_{\alpha\beta\gamma}$ that describes rotations and reflections must also contain the determinant $|\widehat{a}|$. In the absence of the determinant, the components of the tensor would change their sign during reflections.

1.4

$$T_{\alpha\beta} = \frac{1}{2}(T_{\alpha\beta} + T_{\beta\alpha}) + \frac{1}{2}(T_{\alpha\beta} - T_{\beta\alpha}) \,. \tag{1.254}$$

1.5

$$T_{\alpha\beta} = T^h_{\alpha\beta} + T^{ah}_{\alpha\beta} \,, \quad \text{where} \quad T^h_{\alpha\beta} = \frac{1}{2}(T_{\alpha\beta} + T^*_{\beta\alpha}) \,,$$
$$T^{ah}_{\alpha\beta} = \frac{1}{2}(T_{\alpha\beta} - T^*_{\beta\alpha}) \,. \tag{1.255}$$

1.9 $T_{\alpha\beta}$ form a polar tensor of rank 2.

1.10

$$C_\alpha = \frac{1}{2} e_{\alpha\beta\gamma} A_{\beta\gamma} \,, \tag{1.256}$$

that is, $C_1 = A_{23} = -A_{32}$, $C_2 = A_{31} = -A_{13}$, and $C_3 = A_{12} = -A_{21}$.

1.11 $[\boldsymbol{A} \times \boldsymbol{B}]$ is a pseudovector or a polar antisymmetric tensor of rank 2 dual to it: $A_\beta B_\gamma - A_\gamma B_\beta$.$[\boldsymbol{A} \times \boldsymbol{B}] \times \boldsymbol{C}$ is a polar vector and $[\boldsymbol{A} \times \boldsymbol{B}] \cdot \boldsymbol{C}$ is a pseudoscalar.

1.13

$$\mathrm{d}S_\alpha = e_{\alpha\beta\gamma} \mathrm{d}x_\beta \mathrm{d}x'_\gamma = \frac{1}{2} e_{\alpha\beta\gamma} \mathrm{d}S_{\beta\gamma} \,, \tag{1.257}$$

where $\mathrm{d}S_{\beta\gamma} = \mathrm{d}x_\beta \mathrm{d}x'_\gamma - \mathrm{d}x_\gamma \mathrm{d}x'_\beta$ is the projection of the area of the parallelogram onto the coordinate plane $x_\beta x_\gamma$.

1.14

$$\mathrm{d}V = [\mathrm{d}\boldsymbol{r} \times \mathrm{d}\boldsymbol{r}'] \cdot \mathrm{d}\boldsymbol{r}'' = e_{\alpha\beta\gamma} \mathrm{d}x_\alpha \mathrm{d}x'_\beta \mathrm{d}x''_\gamma \,. \tag{1.258}$$

The element of volume is a pseudoscalar. When $\mathrm{d}\boldsymbol{r} = \boldsymbol{e}_1 \mathrm{d}x_1$, $\mathrm{d}\boldsymbol{r}' = \boldsymbol{e}_2 \mathrm{d}x_2$, and $\mathrm{d}\boldsymbol{r}'' = \boldsymbol{e}_3 \mathrm{d}x_3$, we get the usual expression for an element of volume in Cartesian coordinates: $\mathrm{d}V = \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3$.

1.16

$$\cos\theta = \cos\vartheta \cos\vartheta' + \sin\vartheta \sin\vartheta' \cos(\alpha - \alpha') \,. \tag{1.259}$$

1.17 $(\boldsymbol{A} \times \boldsymbol{B})_0 = i(A_{-1}B_{+1} - A_{+1}B_{-1})$, $(\boldsymbol{A} \times \boldsymbol{B})_{\pm 1} = \pm(A_0 B_{\pm 1} - A_{\pm 1} B_0)$, $\boldsymbol{A} \cdot \boldsymbol{B} = \sum_{\mu=-1}^{+1} (-1)^\mu A_{-\mu} B_\mu$, $r_\mu = r(4\pi/3)^{1/2} (-1)^\mu Y_{1\mu}(\vartheta, \alpha)$.

1.18

$$\widehat{g} = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \,. \tag{1.260}$$

1.19 When changing from Cartesian unit vectors to spherical ones (see Figure 1.12a), we have $e'_\mu = a_{\mu\beta} e_\beta$, where e_β ($\beta = 1, 2, 3$) are Cartesian and e'_μ ($\mu = r, \vartheta, \alpha$) are spherical unit vectors.

$$\widehat{a} = \begin{pmatrix} \sin\vartheta\cos\alpha & \sin\vartheta\sin\alpha & \cos\vartheta \\ \cos\vartheta\sin\alpha & \cos\vartheta\cos\alpha & -\sin\vartheta \\ -\sin\alpha & \cos\alpha & 0 \end{pmatrix} ,$$

$$\widehat{a}^{-1} = \begin{pmatrix} \sin\vartheta\cos\alpha & \cos\vartheta\cos\alpha & -\sin\alpha \\ \sin\vartheta\sin\alpha & \cos\vartheta\sin\alpha & \cos\alpha \\ \cos\vartheta & -\sin\vartheta & 0 \end{pmatrix} .$$

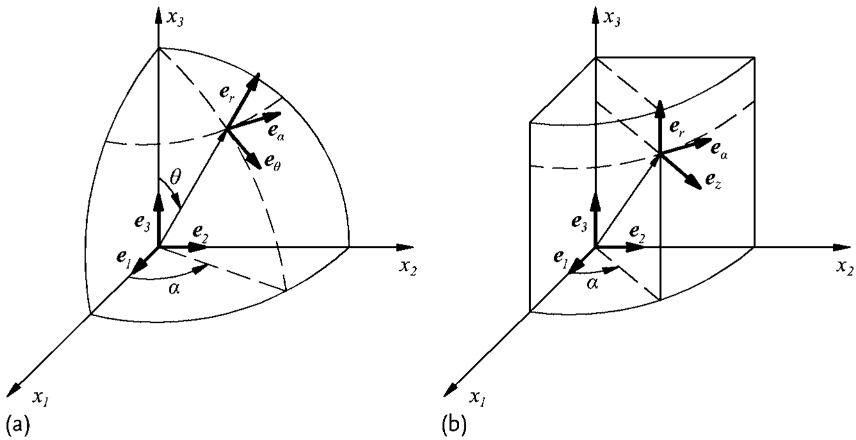

Figure 1.12 Changing from Cartesian to spherical orts (a), and changing from Cartesian to cylindrical orts (b).

When changing from Cartesian unit vectors to cylindrical ones e_r, e_α, e_z (see Figure 1.12b), we have

$$\widehat{a} = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \widehat{a}^{-1} = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

1.20 Using the results obtained in the previous problem, we get

$$\widehat{g}(\alpha_1\theta\,\alpha_2) = \widehat{g}(\alpha_2)\widehat{g}(\theta)\widehat{g}(\alpha_1) =$$
$$\begin{pmatrix} \cos\alpha_1\cos\alpha_2 - \cos\theta\sin\alpha_1\sin\alpha_2; & \sin\alpha_1\cos\alpha_2 + \cos\theta\cos\alpha_1\sin\alpha_2; & \sin\theta\sin\alpha_2 \\ -\cos\alpha_1\sin\alpha_2 - \cos\theta\sin\alpha_1\cos\alpha_2; & -\sin\alpha_1\sin\alpha_2 + \cos\theta\cos\alpha_1\cos\alpha_2; & \sin\theta\cos\alpha_2 \\ \sin\alpha_1\sin\theta & -\sin\theta\cos\alpha_1 & \cos\theta \end{pmatrix}. \tag{1.261}$$

1.21

$$\widehat{D}(\alpha_1\theta\,\alpha_2) =$$
$$\begin{pmatrix} (1/2)(1+\cos\theta)e^{i(\alpha_1+\alpha_2)}; & -(i/\sqrt{2})\sin\theta\, e^{i\alpha_2}; & -(1/2)(1-\cos\theta)e^{i(\alpha_2-\alpha_1)} \\ -(i/\sqrt{2})\sin\theta\, e^{i\alpha_1}; & \cos\theta; & -(i/\sqrt{2})\sin\theta\, e^{-i\alpha_1} \\ -(1/2)(1-\cos\theta)e^{i(\alpha_1-\alpha_2)}; & -(i/\sqrt{2})\sin\theta\, e^{-i\alpha_2}; & (1/2)(1+\cos\theta)e^{-i(\alpha_1+\alpha_2)} \end{pmatrix}.$$

1.22 The zero angle rotation matrix equals 1 (identical transformation) and when rotation is by a small angle, $|\varepsilon_{\alpha\beta}| \ll 1$. To prove the antisymmetry of $\widehat{\varepsilon}$, we will use the invariance of $r^2 = \delta_{\alpha\beta}x_\alpha x_\beta$ relative to the rotation. Since $x'_\alpha = x_\alpha + \varepsilon_{\alpha\beta}x_\beta$, we have $r'^2 = r^2 + 2\varepsilon_{\alpha\beta}x_\alpha x_\beta$ to small quantities of the first order. As follows from the invariance of r^2, $\varepsilon_{\alpha\beta}x_\alpha x_\beta = 0$ when x_α are arbitrary, which is possible only when $\varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha}$.

Now, we introduce a vector with components $\delta\varphi_\alpha = (1/2)e_{\alpha\beta\gamma}\varepsilon_{\beta\gamma}$. Then, $\boldsymbol{r}' = \boldsymbol{r} + \delta\boldsymbol{\varphi} \times \boldsymbol{r}$, which shows that $\delta\boldsymbol{\varphi}$ is the vector of an infinitely small rotation whose direction indicates the axis of rotation and its size indicates the angle of rotation.

1.23 If rotations are specified by small vectors $\delta\boldsymbol{\varphi}_1$ and $\delta\boldsymbol{\varphi}_2$, then, after the second rotation,

$$\boldsymbol{r}'' = \boldsymbol{r}' + \delta\boldsymbol{\varphi}_2 \times \boldsymbol{r}' = \boldsymbol{r} + (\delta\boldsymbol{\varphi}_1 + \delta\boldsymbol{\varphi}_2) \times \boldsymbol{r} + \delta\boldsymbol{\varphi}_2 \times (\delta\boldsymbol{\varphi}_1 \times \boldsymbol{r}) \,.$$

The vector of the resulting rotation $\delta\boldsymbol{\varphi} = \delta\boldsymbol{\varphi}_1 + \delta\boldsymbol{\varphi}_2$ may be introduced only if the last member of the second order is disregarded.

In the general case, the noncommutative character of the rotation matrices is shown by expression (1.261): the completion of rotation in the sequence $\alpha_2, \theta, \alpha_1$, inverse with respect to the matrix it is written for (1.261) corresponds to the substitution $\alpha_1 \to \alpha_2$. In this event, the form of the matrix will change if $\theta \neq 0$. The case of $\theta = 0$ is in keeping with rotation by angles α_1 and α_2 around the same Ox_3 axis and such rotations are commutative.

1.24 Any tensor of rank 2 may be written as $T_{\alpha\beta} = S_{\alpha\beta} + A_{\alpha\beta}$, whereas any Hermitian tensor may be written as $T^h_{\alpha\beta} = S_{\alpha\beta} + iA_{\alpha\beta}$, where $S_{\alpha\beta}$ and $A_{\alpha\beta}$ are symmetric and antisymmetric real tensors. The antisymmetric tensor $A_{\alpha\beta}$ is equivalent to a vector (see Problem 1.10) which will not be reduced to zero by any rotations. This is why only the real symmetric part of the any tensor of rank 2 may be diagonalized.

1.25

$$S_{\alpha\beta} = S^{(1)} n^{(1)}_\alpha n^{(1)}_\beta + S^{(2)} n^{(2)}_\alpha n^{(2)}_\beta + S^{(3)} n^{(3)}_\alpha n^{(3)}_\beta \,. \tag{1.262}$$

1.26 Computing a determinant (1.27) while keeping in mind that the principal values of tensor $S^{(i)}$ may be invariant only if such are the coefficients of an algebraic cubic equation, we find three invariants:

$$I_1 = S_{11} + S_{22} + S_{33} = S^{(1)} + S^{(2)} + S^{(3)}, \tag{1.263}$$

$$I_2 = D_{11} + D_{22} + D_{33} = S^{(1)}S^{(2)} + S^{(1)}S^{(3)} + S^{(2)}S^{(3)}, \tag{1.264}$$

$$I_3 = D = S^{(1)}S^{(2)}S^{(3)} \,, \tag{1.265}$$

where $D = |\widehat{S}|$ is the determinant of the tensor and $D_{\alpha\beta}$ are the algebraic cofactors of the determinant. Expressions in the second members of the equalities follow from Viète's theorem[37] about the connection between the coefficients of a cubic equation with its roots. The result is valid for any tensor of rank 2.

37) François Viète (1540–1603) was an French mathematician, and a lawyer by trade.

1.27 The row and column expansions of the determinant $D = |\widehat{T}|$ are written, respectively, as

$$T_{\alpha\beta} D_{\gamma\beta} = D\delta_{\alpha\gamma} , \quad D_{\gamma\alpha} T_{\gamma\beta} = D\delta_{\alpha\beta} ,$$

where $D_{\gamma\alpha} = (-1)^{\alpha+\gamma} \Delta_{\gamma\alpha}$ is an algebraic cofactor, and $\Delta_{\gamma\alpha}$ is the minor determinant D, that is, the determinant remaining after the elimination in it of the γ row and the α column. In accordance with the results obtained in the previous problem, since D is invariant and since $\delta_{\alpha\beta}$ is a tensor, then the algebraic cofactors $D_{\gamma\alpha}$ also form a tensor. Relations

$$T^{-1}_{\alpha\beta} = \frac{D_{\beta\alpha}}{D} \tag{1.266}$$

form a tensor inverse to $\widehat{T}$. To make the inverse tensor possible, it is necessary and sufficient that $D = |\widehat{T}| \neq 0$.

1.29

1. $A^2(\boldsymbol{B} \cdot \boldsymbol{C}) + (\boldsymbol{A} \cdot \boldsymbol{B})(\boldsymbol{A} \cdot \boldsymbol{C})$.
2. $[(\boldsymbol{A} \times \boldsymbol{B}) \times \boldsymbol{C}] \cdot [(\boldsymbol{A}' \times \boldsymbol{B}') \times \boldsymbol{C}']$.

1.31

$$\begin{aligned}
&(\boldsymbol{A} \cdot \boldsymbol{A}')(\boldsymbol{B} \cdot \boldsymbol{B}')(\boldsymbol{C} \cdot \boldsymbol{C}') + (\boldsymbol{A} \cdot \boldsymbol{B}')(\boldsymbol{B} \cdot \boldsymbol{C}')(\boldsymbol{C} \cdot \boldsymbol{A}') \\
&+ (\boldsymbol{B} \cdot \boldsymbol{A}')(\boldsymbol{C} \cdot \boldsymbol{B}')(\boldsymbol{A} \cdot \boldsymbol{C}') - (\boldsymbol{A} \cdot \boldsymbol{C}')(\boldsymbol{C} \cdot \boldsymbol{A}')(\boldsymbol{B} \cdot \boldsymbol{B}') \\
&- (\boldsymbol{A} \cdot \boldsymbol{B}')(\boldsymbol{B} \cdot \boldsymbol{A}')(\boldsymbol{C} \cdot \boldsymbol{C}') - (\boldsymbol{B} \cdot \boldsymbol{C}')(\boldsymbol{C} \cdot \boldsymbol{B}')(\boldsymbol{A} \cdot \boldsymbol{A}') .
\end{aligned}$$

1.32 Now, we will present our proof for a vector and tensor of rank 2.

1. In accordance with the situation in the problem, at any rotation $A'_\alpha = A_\alpha$, that is, $A'_x = A_x$, $A'_y = A_y$, and $A'_z = A_z$. Rotating the coordinate system around the Oz axis by angle π, we get $A'_x = -A_x$, $A'_y = -A_y$, and $A'_z = A_z$. These equalities are compatible with the previous ones only if $A_x = A_y = 0$. Performing rotation around the Ox axis by angle π, we will similarly prove that $A_z = 0$, that is, vector $\boldsymbol{A} = 0$.
2. Any tensor of rank 2 may be represented as the sum of a symmetric tensor and an antisymmetric tensor: $T_{\alpha\beta} = S_{\alpha\beta} + A_{\alpha\beta}$. An antisymmetric tensor is equivalent to a certain pseudovector and, in accordance with what was proven above, its components do not depend on the reference frame only if they are equal to zero. So we will consider a symmetric tensor $S_{\alpha\beta}$.

We will select a coordinate system where the symmetric tensor has a diagonal form $S^{(\alpha)}\delta_{\alpha\beta}$. If $S^{(\alpha)}$ are not equal to each another, then the components of the tensor depend on the selection of the axis, that is, what digit (1, 2, or 3) denotes the selected one. Only when $S^{(1} = S^{(2)} = S^{(3)} = S$ do the components of the tensor $S\delta_{\alpha\beta}$ not depend on the selection of the axis.

1.33

$$\overline{n_\alpha} = 0\,, \overline{n_\alpha n_\beta} = \frac{1}{3}\delta_{\alpha\beta}, \overline{n_\alpha n_\beta n_\gamma} = 0\,,$$
$$\overline{n_\alpha n_\beta n_\gamma n_\mu} = \frac{1}{15}(\delta_{\alpha\beta}\delta_{\gamma\nu} + \delta_{\alpha\gamma}\delta_{\beta\nu} + \delta_{\alpha\nu}\delta_{\beta\gamma})\,.$$

1.34

$$\frac{a^2}{3}, \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{3}, \frac{\boldsymbol{a}}{3}, \frac{2a^2}{3}, \frac{2\boldsymbol{a}\cdot\boldsymbol{b}}{3}, \frac{(\boldsymbol{a}\cdot\boldsymbol{b})(\boldsymbol{c}\cdot\boldsymbol{d}) + (\boldsymbol{a}\cdot\boldsymbol{c})(\boldsymbol{b}\cdot\boldsymbol{d}) + (\boldsymbol{a}\cdot\boldsymbol{d})(\boldsymbol{b}\cdot\boldsymbol{c})}{15}\,.$$

1.35 $n^2, n'^2, l^2, \boldsymbol{n}\cdot\boldsymbol{n}', (\boldsymbol{n}\times\boldsymbol{n}')\cdot\boldsymbol{l}, (\boldsymbol{n}\cdot\boldsymbol{l})^2, (\boldsymbol{n}'\cdot\boldsymbol{l})^2, (\boldsymbol{n}\cdot\boldsymbol{l})(\boldsymbol{n}'\cdot\boldsymbol{l})\,.$

1.36 $\boldsymbol{n}\cdot\boldsymbol{l}, \boldsymbol{n}'\cdot\boldsymbol{l}, \boldsymbol{n}_1\cdot(\boldsymbol{n}_2\times\boldsymbol{n}_3)\,.$

1.37

1. $a_\alpha{}^\beta = \boldsymbol{e}'_\alpha\cdot\boldsymbol{e}^\beta$.
2. If $\boldsymbol{e}_\beta = (\widehat{a}^{-1})_\beta{}^\alpha$, then $(\widehat{a}^{-1})_\beta{}^\alpha = \boldsymbol{e}'^\alpha\cdot\boldsymbol{e}_\beta = a^\alpha{}_\beta \neq a_\alpha{}^\beta$; in accordance with the definition of the inverse matrix $a_\alpha{}^\beta a^\gamma{}_\beta = \delta^\gamma_\alpha$, $a_\beta{}^\alpha a^\beta{}_\gamma = \delta^\alpha_\gamma = \delta^\gamma_\alpha$.
3. $\boldsymbol{e}'^\alpha = a^\alpha{}_\beta\boldsymbol{e}^\beta\,, \quad \boldsymbol{e}^\alpha = a_\beta{}^\alpha\boldsymbol{e}'^\beta$.
4. $A'_\beta = a_\beta{}^\alpha A_\alpha\,, \quad A'^\beta = a^\beta{}_\alpha A^\alpha\,; \quad A_\alpha = a^\beta{}_\alpha A'_\beta\,, \quad A^\alpha = a_\beta{}^\alpha A'^\beta\,.$
5. $g'_{\alpha\beta} = a_\alpha{}^\gamma a_\beta{}^\mu g_{\gamma\mu}\,, \quad g'^{\alpha\beta} = a^\alpha{}_\gamma a^\beta{}_\mu g^{\gamma\mu}\,, \quad g_{\alpha\beta} = a^\gamma{}_\alpha a^\mu{}_\beta g'_{\gamma\mu}\,,$
   $g^{\alpha\beta} = a_\gamma{}^\alpha a_\mu{}^\beta g'^{\gamma\mu}\,.$

The formulas in answer 4 are generalized directly for the cases of the transformation of covariant, contravariant, and mixed components of tensors of any rank.

1.38 When the systems of coordinates are inverted, the components of the vectors of both bases change their signs: $\boldsymbol{e}'_\alpha = -\boldsymbol{e}_\alpha$, $\boldsymbol{e}'^\alpha = -\boldsymbol{e}^\alpha$, $\alpha = 1, 2, 3$.

1.39

$$\boldsymbol{A}\cdot\boldsymbol{B} = g^{\alpha\beta}A_\alpha B_\beta = g_{\alpha\beta}A^\alpha B^\beta = A^\alpha B_\alpha = A_\alpha B^\alpha = \text{inv}\,. \tag{1.267}$$

$$\mathrm{d}l^2 = \mathrm{d}\boldsymbol{r}\cdot\mathrm{d}\boldsymbol{r} = g^{\alpha\beta}\mathrm{d}x_\alpha\mathrm{d}x_\beta = g_{\alpha\beta}\mathrm{d}x^\alpha\mathrm{d}x^\beta = \mathrm{d}x^\alpha\mathrm{d}x_\alpha = \text{inv}\,. \tag{1.268}$$

Note: In all cases when covariant and contravariant components do not coincide, the operation of tensor contraction must be done as a summation assumed over one upper and one lower index. Any tensor summed over two upper and two lower symbols is not a tensor of any rank.

1.40

$$C_\alpha = \sqrt{g}(A^\beta B^\gamma - A^\gamma B^\beta)\,, \quad C^\alpha = (1/\sqrt{g})(A_\beta B_\gamma - A_\gamma B_\beta)\,, \tag{1.269}$$

where $g = |\hat{g}|$ and numbers α, β, γ form a circular permutation 1, 2, 3.

The formulas shown may be regarded as the generalization of expression (1.23) for the case of an oblique basis. Having written (1.269) in the form of

$$C_\alpha = E_{\alpha\beta\gamma} A^\beta B^\gamma\,, \quad C^\alpha = E^{\alpha\beta\gamma} A_\beta B_\gamma\,, \tag{1.270}$$

we find a representation for an antisymmetric tensor of rank 3 in the oblique basis:

$$E_{\alpha\beta\gamma} = \sqrt{g} e_{\alpha\beta\gamma}\,, \quad E^{\alpha\beta\gamma} = \frac{1}{\sqrt{g}} e^{\alpha\beta\gamma}\,, \tag{1.271}$$

where $e_{\alpha\beta\gamma}$ and $e^{\alpha\beta\gamma}$, related to the orthogonal basis, are similar and determined by conditions (1.21). It is easy to verify that $E^{\alpha\beta\gamma}$ is produced from $E_{\alpha\beta\gamma}$ (and vice versa) in accordance with the rule of upping and lowering indices as per (1.35).

1.41 $\cos\theta = \frac{A_\alpha B^\alpha}{(A_\beta A^\beta B_\gamma B^\gamma)^{1/2}}$.

1.42 $g_{\mu\nu} = \left(e^\alpha_{(D)}\right)_\mu \left(e^{(D)}_\alpha\right)_\nu$, $g^{\mu\nu} = \left(e^\alpha_{(D)}\right)^\mu \left(e^{(D)}_\alpha\right)^\nu$, $g^\mu_\nu = \left(e^\alpha_{(D)}\right)^\mu \left(e^{(D)}_\alpha\right)_\nu = \delta^\mu_\nu$.

1.44

$$A^2 = g_{\alpha\beta} A^\alpha A^\beta = A^\alpha A_\alpha\,; \quad \cos\theta = \frac{A_\alpha B^\alpha}{\left(A^2 B^2\right)^{1/2}}\,,$$

in complete analogy with the result obtained in Problem 1.41 for an affine system.

1.45 In accordance with the common rules (1.51), we have

$$E^{\mu\nu\lambda} = \frac{\partial q^\mu}{\partial x^\alpha} \frac{\partial q^\nu}{\partial x^\beta} \frac{\partial q^\lambda}{\partial x^\gamma} e^{\alpha\beta\gamma}\,,$$

wherefrom the antisymmetry of $E^{\mu\nu\lambda}$ over any pair of symbols follows. This allows us to write $E^{\alpha\beta\gamma} = S e^{\alpha\beta\gamma}$, where S is a certain scalar. To define it, we consider a special case and get

$$E^{123} = \frac{\partial q^1}{\partial x^\alpha} \frac{\partial q^2}{\partial x^\beta} \frac{\partial q^3}{\partial x^\gamma} e^{\alpha\beta\gamma} = \left|\frac{\partial q^\lambda}{\partial x^\alpha}\right| = J^{-1} = g^{-1/2}\,,$$

where (1.39) and (1.53) are used. Therefore, $S = g^{-1/2}$, and so we have obtained, in another way, the second formula (1.271).

1.46

1. $\mathrm{d}l_{(1)} = \sqrt{g_{11}}\mathrm{d}q^1$, $\mathrm{d}l_{(2)} = \sqrt{g_{22}}\mathrm{d}q^2$, $\mathrm{d}l_{(3)} = \sqrt{g_{33}}\mathrm{d}q^3$.

2. The covariant basis (1.46) is such a vector.
3. Using the result obtained in Problem 1.44, we find

$$\cos\vartheta_{12} = \frac{g_{12}}{\sqrt{g_{11}g_{22}}}\,, \quad \cos\vartheta_{13} = \frac{g_{13}}{\sqrt{g_{11}g_{33}}}\,, \quad \cos\vartheta_{23} = \frac{g_{23}}{\sqrt{g_{22}g_{33}}}\,.$$

4. For the curvilinear coordinate system to be orthogonal, it is necessary and sufficient that the equalities $g_{12} = g_{23} = g_{13} = 0$ be valid in every point of space.

1.47 For a spherical system $g_{rr} = 1$, $g_{\vartheta\vartheta} = r^2$, $g_{\alpha\alpha} = r^2\sin^2\vartheta$, $g_{r\vartheta} = g_{r\alpha} = g_{\vartheta\alpha} = 0$; $g^{rr} = 1$, $g^{\vartheta\vartheta} = r^{-2}$, $g^{\alpha\alpha} = r^{-2}\sin^{-2}\vartheta$, $g^{r\vartheta} = g^{r\alpha} = g^{\vartheta\alpha} = 0$; $\boldsymbol{e}_r = \boldsymbol{e}^r = \boldsymbol{e}_{r*}$, $\boldsymbol{e}_\vartheta = r^2\boldsymbol{e}^\vartheta = r\boldsymbol{e}_r$, $\boldsymbol{e}_\alpha = r^2\sin^2\vartheta\,\boldsymbol{e}^\alpha = r\sin\vartheta\,\boldsymbol{e}_{\alpha*}$.

For a cylindrical system $g_{rr} = g_{zz} = 1$, $g_{\alpha\alpha} = r^2$, $g_{r\alpha} = g_{rz} = g_{\alpha z} = 0$; $g^{rr} = g^{zz} = 1$, $g^{\alpha\alpha} = r^{-2}$, $g^{r\alpha} = g^{rz} = g^{\alpha z} = 0$; $\boldsymbol{e}_r = \boldsymbol{e}^r = \boldsymbol{e}_{r*}$, $\boldsymbol{e}_\alpha = r^2\boldsymbol{e}^\alpha = r\boldsymbol{e}_{\alpha*}$, $\boldsymbol{e}_z = \boldsymbol{e}^z = \boldsymbol{e}_{z*}$.

The asterisks mark basic unit orts introduced in Problem 1.18. Covariant and contravariant basic vectors have different dimensions and are different in length. Their lengths are, generally speaking, not unity.

1.50 $R = \frac{a^2}{2} + ab - \frac{b^3}{3}$.

1.52 $\boldsymbol{l}, \boldsymbol{l}$.

1.54

$$\frac{\mathrm{d}r}{A_r} = \frac{r\,\mathrm{d}\alpha}{A_\alpha} = \frac{\mathrm{d}z}{A_z}\,, \quad \frac{\mathrm{d}r}{A_r} = \frac{r\mathrm{d}\vartheta}{A_\vartheta} = \frac{r\sin\vartheta\,\mathrm{d}\alpha}{A_\alpha}\,.$$

1.55 Because the gradient contains the first derivatives over the coordinates, the following notation may be used:

$$\mathrm{grad}\,\frac{(\boldsymbol{p}\cdot\boldsymbol{r})}{r^3} = \frac{\mathrm{grad}\,(\boldsymbol{p}\cdot\boldsymbol{r})}{r^3} + (\boldsymbol{p}\cdot\boldsymbol{r})\mathrm{grad}\,\frac{1}{r^3}\,.$$

Using the results obtained in Problems (1.52) and (1.53), we finally get the following:

$$\mathrm{grad}\,\frac{(\boldsymbol{p}\cdot\boldsymbol{r})}{r^3} = -\frac{3(\boldsymbol{p}\cdot\boldsymbol{r})\boldsymbol{r}}{r^5} + \frac{\boldsymbol{p}}{r^3}\,.$$

1.56 Direct the polar axis along the vector $\boldsymbol{p}$ and project the vector $\boldsymbol{E}$ onto the orts of spherical coordinates:

$$E_r = \frac{2p\cos\vartheta}{r^3}\,, \quad E_\vartheta = \frac{p\sin\vartheta}{r^3}\,, \quad E_\varphi = 0\,.$$

In spherical coordinates, vector lines are found with the following system of equations:

$$\frac{\mathrm{d}r}{E_r} = \frac{r\mathrm{d}\vartheta}{E_\vartheta} = \frac{r\sin\vartheta\,\mathrm{d}\varphi}{E_\varphi}\,.$$

The reduction of the E_φ component to zero means that the differential $d\varphi = 0$, that is, $\varphi =$ const must also become zero and, therefore, all the vector lines lie within the planes passing through the vector $\boldsymbol{p}$. Inserting the nonzero projections of $\boldsymbol{H}$ into the only remaining equation and eliminating common factors, we get a first-order differential equation with separable variables: $dr/r = 2\cot\vartheta\, d\vartheta$. The termwise integration of the first and second members gives $\ln r - \ln r_0 = 2\ln\sin\vartheta$ or $r(\vartheta) = r_0 \sin^2\vartheta$, where r_0 is the constant of integration that has the meaning of the distance between the vector line and the origin in the plane perpendicular to the vector $\boldsymbol{p}$.

1.57

$$\nabla_{\pm 1} = \mp\frac{1}{\sqrt{2}} e^{\pm i\alpha}\left(\sin\vartheta\frac{\partial}{\partial r} + \frac{\cos\vartheta}{r}\frac{\partial}{\partial\vartheta} \pm \frac{i}{r\sin\vartheta}\frac{\partial}{\partial\alpha}\right),$$

$$\nabla_0 = \cos\vartheta\frac{\partial}{\partial r} - \frac{\sin\vartheta}{r}\frac{\partial}{\partial\vartheta}\,.$$

1.59 3, 0, 0, $2\boldsymbol{\omega}$.

1.60

$$\boldsymbol{H} = \frac{3(\boldsymbol{m}\cdot\boldsymbol{r})\boldsymbol{r}}{r^5} - \frac{\boldsymbol{m}}{r^3}\,.$$

1.63 $\frac{\varphi' \boldsymbol{r}}{r}$, $3\varphi + r\varphi'$, 0, $\boldsymbol{l}\varphi + \boldsymbol{r}(\boldsymbol{l}\cdot\boldsymbol{r})\frac{\varphi'}{r}$.

1.64 $\varphi(r) = \frac{\text{const}}{r^3}$.

1.65 $(\boldsymbol{a}\cdot\boldsymbol{b})$, $\boldsymbol{a}\times\boldsymbol{b}$; $4(\boldsymbol{a}\cdot\boldsymbol{r})$, $\boldsymbol{a}\times\boldsymbol{r}$; 0, $(2\varphi + r\varphi')\boldsymbol{a} - \boldsymbol{r}(\boldsymbol{a}\cdot\boldsymbol{r})\frac{\varphi'}{r}$; $-2(\boldsymbol{a}\cdot\boldsymbol{r})$, $3(\boldsymbol{r}\times\boldsymbol{a})$.

1.66 $\boldsymbol{A} + (\boldsymbol{r}\cdot\boldsymbol{A}')\frac{\boldsymbol{r}}{r}$, $(\boldsymbol{A}'\cdot\boldsymbol{B} + \boldsymbol{A}\cdot\boldsymbol{B}')\frac{\boldsymbol{r}}{r}$, $\frac{\varphi'}{r}(\boldsymbol{r}\cdot\boldsymbol{A}) + \frac{\varphi}{r}(\boldsymbol{r}\cdot\boldsymbol{A}')$, $\frac{\varphi'}{r}(\boldsymbol{r}\times\boldsymbol{A}) + \frac{\varphi}{r}(\boldsymbol{r}\times\boldsymbol{A}')$, $\frac{\boldsymbol{l}\cdot\boldsymbol{r}}{r}(\varphi'\boldsymbol{A} + \varphi\boldsymbol{A}')$.

1.68

$$\int_V (\operatorname{grad}\varphi\cdot\operatorname{curl}\boldsymbol{A})\,dV = \oint_S (\boldsymbol{A}\times\operatorname{grad}\varphi)\cdot d\boldsymbol{S} = \oint_S \varphi\operatorname{curl}\boldsymbol{A}\cdot d\boldsymbol{S}\,.$$

1.69 $\boldsymbol{a}V$, $\boldsymbol{a}V$.

1.70* Use the method of the scalar multiplication of a constant vector by each of the integrals in question to find the following relations:

$$\oint_S \boldsymbol{n}\varphi\, dS = \int_V \operatorname{grad}\varphi\, dV\,; \tag{1.272}$$

$$\oint_S (\boldsymbol{n} \times \boldsymbol{A}) \mathrm{d}S = \int_V \operatorname{curl} \boldsymbol{A} \, \mathrm{d}V \; ; \tag{1.273}$$

$$\oint_S (\boldsymbol{n} \cdot \boldsymbol{b}) \boldsymbol{A} \mathrm{d}S = \int_V (\boldsymbol{b} \cdot \nabla) \boldsymbol{A} \, \mathrm{d}V \; ; \tag{1.274}$$

$$\oint_S T_{\alpha\beta} \boldsymbol{n}_\beta \mathrm{d}S = \int_V \frac{\partial T_{\alpha\beta}}{\partial x_\beta} \mathrm{d}V \; . \tag{1.275}$$

All these relations may be regarded as generalizations of the Gauss–Ostrogradskii theorem:

$$\oint_S \boldsymbol{n}(\ldots) \mathrm{d}S = \int_V \nabla(\ldots) \mathrm{d}V \; , \tag{1.276}$$

where the $(\ldots)$ symbol denotes a tensor of any rank.

1.76 $\int_S (\nabla u \times \nabla f) \cdot \mathrm{d}\boldsymbol{S}$.

1.81* As per the common rule (1.105), covariant divergence is expressed as follows:

$$\Gamma^\mu_{\mu\alpha} = \frac{1}{2} g^{\mu\lambda} \left(\frac{\partial g_{\mu\lambda}}{\partial q^\alpha} + \frac{\partial g_{\alpha\lambda}}{\partial q^\mu} - \frac{\partial g_{\alpha\mu}}{\partial q^\lambda} \right) = \frac{1}{2} g^{\mu\lambda} \frac{\partial g_{\mu\lambda}}{\partial q^\alpha} \; ,$$

that is,

$$A^\mu_{;\mu} = \frac{\partial A^\mu}{\partial q^\mu} + \frac{1}{2} g^{\mu\lambda} \frac{\partial g_{\mu\lambda}}{\partial q^\alpha} A^\alpha \; . \tag{1}$$

Now consider the determinant $g = |g_{\mu\nu}|$. Its differential equals the sum of the differentials of all its elements multiplied by the respective algebraic cofactors: $\mathrm{d}g = D^{\mu\nu} \mathrm{d}g_{\mu\nu}$, where $D^{\mu\nu} = (-1)^{\mu+\nu} \Delta^{\mu\nu}$, $\Delta^{\mu\nu}$ is the minor. On the other hand, the algebraic cofactors are expressed through the components of the inverse tensor, that is, $g^{\mu\nu}$: $D^{\mu\nu} = g g^{\mu\nu}$ (see Problem 1.27). In the end, we have

$$\mathrm{d}g = g g^{\mu\nu} \mathrm{d}g_{\mu\nu} \; , \quad g^{\mu\nu} \frac{\partial g_{\mu\nu}}{\partial q^\lambda} = \frac{1}{g} \frac{\partial g}{\partial q^\lambda} \; .$$

Having inserted the latter quantity in (1), we find the expression specified in the condition for the problem.

1.82

$$\Delta S = \frac{1}{\sqrt{g}} \frac{\partial}{\partial q^\mu} \left(\sqrt{g} g^{\mu\nu} \frac{\partial S}{\partial q^\nu} \right) \; . \tag{1.277}$$

1.83

$$T^{\mu\nu}{}_{;\mu} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial q^\mu} \left(\sqrt{g} T^{\mu\nu}\right) + \Gamma^\nu_{\mu\lambda} T^{\mu\lambda} \,.$$

1.84

$$A^{\mu\nu}{}_{;\nu} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial q^\nu} \left(\sqrt{g} A^{\mu\nu}\right) \,.$$

1.86

$$g_{\mu\nu;\lambda} = g^{\mu\nu}{}_{;\lambda} = 0 \,.$$

1.90*

$$\begin{aligned}
(\Delta \boldsymbol{A})_r &= \Delta A_r - \frac{A_r}{r^2} - \frac{2}{r^2} \frac{\partial A_\alpha}{\partial \alpha}, \\
(\Delta \boldsymbol{A})_\alpha &= \Delta A_\alpha - \frac{A_\alpha}{r^2} + \frac{2}{r^2} \frac{\partial A_r}{\partial \alpha}, \\
(\Delta \boldsymbol{A})_z &= \Delta A_z \,.
\end{aligned} \tag{1.278}$$

1.91*

$$\begin{aligned}
(\Delta \boldsymbol{A})_r &= \Delta A_r - \frac{2}{r^2} A_r - \frac{2}{r^2 \sin\vartheta} \frac{\partial}{\partial\vartheta}(A_\vartheta \sin\vartheta) - \frac{2}{r^2 \sin\vartheta} \frac{\partial A_\alpha}{\partial \alpha}, \\
(\Delta \boldsymbol{A})_\vartheta &= \Delta A_\vartheta - \frac{A_\vartheta}{r^2 \sin^2\vartheta} + \frac{2}{r^2} \frac{\partial A_r}{\partial\vartheta} - \frac{2\cos\vartheta}{r^2 \sin^2\vartheta} \frac{\partial A_\alpha}{\partial\alpha}, \\
(\Delta \boldsymbol{A})_\alpha &= \Delta A_\alpha - \frac{A_\alpha}{r^2 \sin^2\vartheta} + \frac{2}{r^2 \sin\vartheta} \frac{\partial A_r}{\partial\alpha} + \frac{2\cos\vartheta}{r^2 \sin^2\vartheta} \frac{\partial A_\vartheta}{\partial\alpha} \,.
\end{aligned} \tag{1.279}$$

1.92 (i) $A + B \ln r$; (ii) $A + B\alpha$; (iii) $A + Bz$.

1.93 (i) $A + \frac{B}{r}$; (ii) $A + B \ln\tan(\frac{\vartheta}{2})$; (iii) $A + B\alpha$.

1.94*

$$x = \pm \left[\frac{(\xi + a^2)(\eta + a^2)(\zeta + a^2)}{(b^2 - a^2)(c^2 - a^2)} \right]^{1/2},$$

$$y = \pm \left[\frac{(\xi + b^2)(\eta + b^2)(\zeta + b^2)}{(c^2 - b^2)(a^2 - b^2)} \right]^{1/2},$$

$$z = \pm \left[\frac{(\xi + c^2)(\eta + c^2)(\zeta + c^2)}{(a^2 - c^2)(b^2 - c^2)} \right]^{1/2};$$

$$h_1 = \frac{\sqrt{(\xi - \eta)(\xi - \zeta)}}{2R_\xi}, \quad h_2 = \frac{\sqrt{(\eta - \zeta)(\eta - \xi)}}{2R_\eta},$$

$$h_3 = \frac{\sqrt{(\zeta - \xi)(\zeta - \eta)}}{2R_\zeta};$$

$$\Delta = \frac{4}{(\xi - \eta)(\xi - \zeta)(\eta - \zeta)} \left[(\eta - \zeta) R_\xi \frac{\partial}{\partial \xi} \left(R_\xi \frac{\partial}{\partial \xi} \right) \right.$$

$$\left. + (\zeta - \xi) R_\eta \frac{\partial}{\partial \eta} \left(R_\eta \frac{\partial}{\partial \eta} \right) + (\xi - \eta) R_\zeta \frac{\partial}{\partial \zeta} \left(R_\zeta \frac{\partial}{\partial \zeta} \right) \right],$$

where $R_u = \sqrt{(u + a^2)(u + b^2)(u + c^2)}$. The formulas for x, y, and z show that there are eight threes: x, y, z for every three values of ξ, η, ζ. One may make sure that the ellipsoidal system of coordinates is orthogonal by finding gradients $\nabla\xi, \nabla\eta, \nabla\zeta$ and then the scalar products $\nabla\xi \cdot \nabla\eta$ and so on, all of which turn out to be equal to zero. The gradients may be found directly from the equations determining ellipsoidal coordinates (see the condition for the problem), resulting in a gradient from both members of each of these equations.

1.95*

$$z = \pm \left[\frac{(\xi + c^2)(\eta + c^2)}{c^2 - a^2} \right]^{1/2}, \quad r = \left[\frac{(\xi + a^2)(\eta + a^2)}{a^2 - c^2} \right]^{1/2};$$

$$h_1 = \frac{\sqrt{\xi - \eta}}{2R_\xi}, \quad h_2 = \frac{\sqrt{\xi - \eta}}{2R_\eta}, \quad h_3 = r,$$

where

$$R_\xi = \sqrt{(\xi + a^2)(\xi + c^2)}, \quad R_\eta = \sqrt{(\eta + a^2)(-\eta - c^2)};$$

$$\Delta = \frac{4}{\xi - \eta} \left[R_\xi \frac{\partial}{\partial \xi} \left(R_\xi \frac{\partial}{\partial \xi} \right) + R_\eta \frac{\partial}{\partial \eta} \left(R_\eta \frac{\partial}{\partial \eta} \right) \right] + \frac{1}{r^2} \frac{\partial^2}{\partial \alpha^2}.$$

1.96*

$$x = \pm \left[\frac{(\xi + a^2)(\zeta + a^2)}{a^2 - b^2} \right]^{1/2}, \quad r = \left[\frac{(\xi + b^2)(\zeta + b^2)}{b^2 - a^2} \right]^{1/2};$$

$$h_1 = \frac{\sqrt{\xi - \zeta}}{2R_\xi}, \quad h_2 = r, \quad h_3 = \frac{\sqrt{\xi - \zeta}}{2R_\zeta},$$

where

$$R_\xi = \sqrt{(\xi + a^2)(\xi + b^2)}, \quad R_\zeta = \sqrt{(\zeta + a^2)(-\zeta - b^2)};$$

$$\Delta = \frac{4}{\xi - \zeta} \left[R_\xi \frac{\partial}{\partial \xi} \left(R_\xi \frac{\partial}{\partial \xi} \right) + R_\zeta \frac{\partial}{\partial \zeta} \left(R_\zeta \frac{\partial}{\partial \zeta} \right) \right] + \frac{1}{r^2} \frac{\partial^2}{\partial \alpha^2}.$$

1.97*

$$h_\xi = h_\eta = \frac{a}{\cosh\xi - \cos\eta}\,, \quad h_\alpha = \frac{a\sin\eta}{\cosh\xi - \cos\eta}\,;$$

$$\Delta = \frac{(\cosh\xi - \cos\eta)^3}{a^2}\left[\frac{\partial}{\partial\xi}\left(\frac{1}{\cosh\xi - \cos\eta}\frac{\partial}{\partial\xi}\right)\right.$$
$$+\frac{1}{\sin\eta}\frac{\partial}{\partial\eta}\left(\frac{\sin\eta}{\cosh\xi - \cos\eta}\frac{\partial}{\partial\eta}\right)$$
$$\left.+\frac{1}{\sin^2\eta(\cosh\xi - \cos\eta)}\frac{\partial^2}{\partial\alpha^2}\right]\,.$$

1.98* The surfaces $\rho = \text{const}$ are toroids:

$$(\sqrt{x^2+y^2} - a\coth\rho)^2 + z^2 = \left(\frac{a}{\sinh\rho}\right)^2\,;$$

the surfaces $\xi = \text{const}$ are spherical segments:

$$(z - \arctan\xi)^2 + x^2 + y^2 = \left(\frac{a}{\sin\xi}\right)^2;$$

$$h_\rho = h_\xi = \frac{a}{\cosh\rho - \cos\xi}\,, \quad h_\alpha = \frac{a\sinh\rho}{\cosh\rho - \cos\xi}\,.$$

1.99 $x^\nu Z_\nu(x) + C\,, \quad -x^{-\nu} Z_\nu(x) + C.$

1.100 $1, \frac{1}{2}, \frac{1}{2^n n!}$.

1.107*

1. $xu'' + u' + x(a^2 - \frac{n^2}{x^2})u = 0$.
2. The integral is computed with the use of the equations for the functions $u(x)$ and $v(x) = J_n(bx)$.
3. The first equality (1.175) directly follows from (1.174); the second one results from passage to the limit.

1.111

$$P_0 = 1\,, \quad P_1 = x\,, \quad P_2 = \frac{1}{2}(3x^2 - 1)\,, \quad P_3 = \frac{1}{2}(5x^3 - 3x)\,,$$
$$P_4 = \frac{1}{8}(35x^4 - 30x^2 + 3)\,, \quad P_5 = \frac{1}{8}(63x^5 - 70x^3 + 15x)\,.$$

1.112* The integral that should be computed contains the product of the derivatives $\int_{-1}^{1}[(x^2-1)^l]^{(l)}[(x^2-1)^{l'}]^{(l')}\mathrm{d}x$. To make things clear, $l' \le l$. Integrating by parts, l times, we find $(-1)^l \int_{-1}^{1}(x^2-1)^l[(x^2-1)^{l'}]^{(l+l')}\mathrm{d}x$. The second factor under the integral is other than zero only when $l' = l$. In this case, beginning by integrating over the angle ϑ, we get

$$\int_{-1}^{1}(x^2-1)^l[(x^2-1)^l]^{(2l)}\mathrm{d}x = (-1)^l(2l)!2\int_0^{\pi/2}(\sin\vartheta)^{2l+1}\mathrm{d}\vartheta$$
$$= (-1)^l(2l)!B\left(l+1,\frac{1}{2}\right) .$$

The last integral here is expressed through a beta function (see the definition in Abramovitz and Stegun, 1965; Gradshtein and Ryzhik, 2007):

$$B(z,w) \equiv \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)} = 2\int_0^{\pi/2}(\sin\vartheta)^{2z-1}(\cos\vartheta)^{2w-1}\mathrm{d}\vartheta .$$

Carefully eliminating factorials and gamma functions, we get the result specified in the condition for the problem.

1.114 $P_2 = \frac{1+3\cos 2\vartheta}{4}$.

1.115

$$\frac{1}{\sin\vartheta}\frac{\mathrm{d}}{\mathrm{d}\vartheta}\sin\vartheta\frac{\mathrm{d}P_l^m}{\mathrm{d}\vartheta} + \left[l(l+1) - \frac{m^2}{\sin^2\vartheta}\right]P_l^m = 0 .$$

1.116* Use the Leibniz formula to find

$$(1-x^2)^{m/2}[(x+1)^l(x-1)^l]^{(l+m)}$$
$$= (-1)^{m/2}\sum_{k=0}^{l+m}\frac{(l+m)!l!l!}{k!(l+m-k)!(k-m)!(l-k)!}(x+1)^{k-m/2}(x-1)^{l-k+m/2} ;$$

and

$$(1-x^2)^{-m/2}[(x+1)^l(x-1)^l]^{(l-m)}$$
$$= (-1)^{-m/2}\sum_{s=0}^{l-m}\frac{(l-m)!l!l!}{s!(l-m-s)!(s+m)!(l-s)!}(x+1)^{s+m/2}(x-1)^{l-s-m/2} .$$

In both sums, the limits of summation are, actually, determined by the presence in the denominators of the factorials of negative integers. In the second sum, replace the summation index $s \to k-m$. As a result, in both sums, factors that depend on k become the same. Comparing them with each other, we find the formula specified in the condition for the problem.

1.117

$$P_l^0 = P_l,$$

$$P_1^1 = -2P_1^{-1} = (1-x^2)^{1/2} = \sin\vartheta,$$

$$P_2^1 = -6P_2^{-1} = 3x(1-x^2)^{1/2} = 3\cos\vartheta\sin\vartheta,$$

$$P_2^2 = 24P_2^{-2} = 3(1-x^2) = 3\sin^2\vartheta,$$

$$P_3^1 = -12P_3^{-1} = \frac{3}{2}(5x^2-1)(1-x^2)^{1/2} = \frac{3}{2}(5\cos^2\vartheta-1)\sin\vartheta,$$

$$P_3^2 = 120P_3^{-2} = 15x(1-x^2) = 15\cos\vartheta\sin^2\vartheta,$$

$$P_3^3 = -720P_3^{-3} = 15(1-x^2)^{3/2} = 15\sin^3\vartheta\ .$$

In the general case, adjoint Legendre polynomials contain radicals $(1-x^2)^{1/2}$ and, therefore, strictly speaking, are not polynomials.

1.118* When computing the normalization integral, use formula (1.198) and the method used to solve Problem 1.112*:

$$\int_{-1}^{1}[P_l^m(x)]^2 \mathrm{d}x = (-1)^m \frac{(l+m)!}{(l-m)!}\int_{-1}^{1} P_l^m(x)P_l^{-m}(x)\mathrm{d}x$$
$$= \frac{(2l)!(l+m)!}{2^{2l}l!l!(l-m)!}B(l+1,1/2)\ ,$$

$$C_{lm} = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\ .$$

Expressing the beta function through gamma functions and reducing fractions, we find the normalization factor (down to the phase factor over the module equaling 1, which remains arbitrary).) In the end, we get the normalized spherical Legendre function:

$$Y_{lm}(\vartheta,\varphi)$$
$$= \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}(1-\cos^2\vartheta)^{m/2}\left(\frac{\mathrm{d}}{\mathrm{d}\cos\vartheta}\right)^{l+m}(\cos^2\vartheta-1)^l e^{im\varphi}\ .$$

1.119

$$\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\sin\vartheta\frac{\partial Y_{lm}}{\partial\vartheta} + \frac{1}{\sin^2\vartheta}\frac{\partial^2 Y_{lm}}{\partial\varphi^2} + l(l+1)Y_{lm} = 0\ .$$

1.120 $-\frac{5}{3}$, $\quad -2$, $\quad 0$, $\quad \exp(\alpha)+\exp(-2\alpha)$.

1.121 $0\,, \quad f(a)\delta(x-a)\,, \quad 82\delta(x-4)+2\delta(x+1)\,.$

1.123

$$\delta(\boldsymbol{r}) = \frac{1}{2\pi r_\perp}\delta(r_\perp)\delta(z)\,,$$
$$\delta(\boldsymbol{r}-\boldsymbol{a}) = \frac{1}{a_\perp}\delta(r_\perp - a_\perp)\delta(\alpha-\alpha_0)\delta(z-a_z)\,.$$

To achieve passage to the limit $\boldsymbol{a} \to 0$, one must not only make values $a_\perp$ and a_z tend to zero, but must also average the second member over the azimuthal angle α_0, since a zero vector has no direction.

1.124

$$\delta(\boldsymbol{r}) = \frac{1}{4\pi r^2}\delta(r)\,, \quad \delta(\boldsymbol{r}-\boldsymbol{a}) = \frac{1}{a^2}\delta(r-a)\delta(\cos\vartheta - \cos\vartheta_0)\delta(\alpha-\alpha_0)\,.$$

1.125

$$f'(x) = g(x) + 2\delta(x-1)\,, \quad \text{where} \quad g(x) = \begin{cases} 3x^2\,, & \text{if } x < 1\,, \\ 2x\,, & \text{if } x > 1\,. \end{cases}$$

1.126

$$f'(x) = \frac{\mathrm{d}f}{\mathrm{d}x} + \sum_{k=1}^{n} \Delta f_k \delta(x-a_k)\,,$$

where $\Delta f_k = f(a_k+0) - f(a_k-0), \mathrm{d}f/\mathrm{d}x$ is a proper ("classical") derivative in the areas of the smooth variation of the function.

1.127

$$\frac{(-1)^m m!}{(n-m)!} f^{(m-n)}(0) \quad \text{at} \quad m \geq n\,, \quad 0 \quad \text{at} \quad m < n\,.$$

1.128 When $\boldsymbol{r} \neq \boldsymbol{r}'$, G is a bounded differentiable function and the equation is satisfied, since $\Delta G = 0$. When $\boldsymbol{r} \to \boldsymbol{r}'$, the function has a singularity. To find out the nature of that singularity ΔG when $\boldsymbol{r} \to \boldsymbol{r}'$, integrate (1.225) over the volume of a small sphere of radius $R \to 0$ with its center at the point $\boldsymbol{r} = \boldsymbol{r}'$. Using the Gauss–Ostrogradskii theorem, we get

$$\int_V \Delta G \,\mathrm{d}V = \int_V \operatorname{div}\operatorname{grad}\left(\frac{1}{r}\right)\mathrm{d}V = \oint_S \left(\nabla\frac{1}{r}\right)\cdot \mathrm{d}\boldsymbol{S}$$
$$= -\int \frac{1}{R^2}R^2 \mathrm{d}\Omega = -4\pi\,.$$

The same value is obtained by the integral over the volume in the second member of the equation, which is how it is satisfied.

1.129

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\cos(2k+1)x}{(2k+1)^2} \,.$$

1.130

$$f(x) = \frac{4a}{\pi} \sum_{k=0}^{\infty} \frac{\sin(2k+1)x}{2k+1} \,.$$

1.131

$$f(x) = \frac{a}{2} + \frac{2a}{\pi} \sum_{k=0}^{\infty} (-1)^k \frac{\cos((2k+1)\pi x/L)}{2k+1} \,.$$

The Fourier series of $f(-x) = f(x)$, which is a function that is even within the interval $[-L, +L]$, contains only cosines. The odd function $f(-x) = -f(x)$ is expandable in sines. A function that has no clear-cut parity contains both sines and cosines in its Fourier expansion.

1.132 $i\lambda F(\lambda)$.

1.133

$$\frac{1}{a} F\left(\frac{\lambda - b}{a}\right) \,.$$

1.134 $\pi \exp(-|\lambda|)$.

1.135

$$\frac{\sqrt{\pi}}{\alpha} \exp\left(-\frac{\lambda^2}{4\alpha^2}\right) \,.$$

1.136

$$\frac{\pi^{3/2}}{\alpha^3} \exp\left(-\frac{k^2}{4\alpha^2}\right) \,,$$

where $\boldsymbol{k}$ is a radius vector of the three-dimensional space of Fourier variables (see (1.252)).

1.137* $\frac{4\pi}{k^2+\kappa^2}$.

1.138* To compute Fourier integral (1.252), use spherical coordinates and select the Oz axis along the vector $\boldsymbol{k}$. Firstly, doing integration over angles and then over r, we find

$$F(k) = \lim_{R\to\infty} \frac{4\pi}{k^2}[1 - \cos(kR)] \,.$$

Formally, the function in the second member has no limit. However, it is easy to understand that the limit of the cosine may be regarded as effectively equal to zero since, when the inverted Fourier transformation is being done, the member with infinitely oscillating cosine will make zero contribution. As a result, we have $F(k) = 4\pi/k^2$.

1.139 Write down the expansion as

$$\exp(ikr\cos\theta) = \sum_{l=0}^{\infty} u_l(kr) P_l(\cos\theta)$$

and, using the orthogonality of Legendre polynomials (see Problem 1.112*), find an integral representation of the functions u_l sought:

$$u_l(kr) = \frac{2l+1}{2} \int_{-1}^{1} e^{ikrx} P_l(x) \mathrm{d}x \,.$$

Use the Rodrigues formula and integrate l times by parts to get

$$u_l(kr) = \frac{(2l+1)(-ikr)^l}{2^{l+1} l!} \int_{-1}^{1} e^{ikrx} (x^2-1)^l \mathrm{d}x \,.$$

Further, expand the exponent in a power series and integrate this absolutely convergent series termwise. Only terms with even powers of x are left:

$$u_l(kr) = \frac{(2l+1)(-ikr)^l}{2^l l!} \sum_{m=0}^{\infty} \frac{(ikr)^{2m}}{(2m)!} \int_0^1 x^{2m} (x^2-1)^l \mathrm{d}x \,.$$

Finally, the transition to a new integration variable $t = x^2$, $\mathrm{d}x = \mathrm{d}t/2\sqrt{t}$ allows us to express the integral in the latter equality through a beta function:

$$\int_0^1 t^{k-1/2}(1-t)^l dt = B(k+1/2, l+1) = \frac{\Gamma(k+1/2)\Gamma(l+1)}{\Gamma(l+k+3/2)} \,.$$

In the end, bundle all the factors to get the following series:

$$\begin{aligned} u_l(kr) &= i^l(2l+1)\frac{(kr)^l}{1\cdot 3\ldots(2l+1)} \\ &\quad\times \left\{1 - \frac{k^2r^2/2}{1!(2l+3)} + \frac{(k^2r^2/2)^2}{2!(2l+3)(2l+5)} - \ldots\right\} \\ &= i^l(2l+1)\, j_l(kr)\,, \end{aligned}$$

where $j_l(kr)$ is a spherical Bessel function; see formulas (10.1.2) in Abramovitz and Stegun (1965).

1.140*

$$\exp(i\boldsymbol{k}\cdot\boldsymbol{r}) = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} i^l j_l(kr)\, Y^*_{lm}(\theta, \phi)\, Y_{lm}(\vartheta, \varphi) \,.$$

2
Basic Concepts of Electrodynamics: The Maxwell Equations

2.1
Electrostatics

2.1.1
The Coulomb Law

Electrodynamics deals with interactions between various microscopic and macroscopic objects or bodies caused by a certain fundamental property whose quantitative measure is called *electric charge*. The sign of the charge of a body may be either positive or negative. Bodies whose charges are of the same sign repel each other, whereas bodies whose charges are of opposite signs are mutually attracted. The law governing the interaction in a vacuum between two small bodies at rest was experimentally discovered and formulated by the French physicist and military engineer Charles-Augustin de Coulomb (1736–1806), as early as in the eighteenth century:

$$\boldsymbol{\mathcal{F}}_{12} = k\frac{q_1 q_2}{r_{12}^2}\frac{\boldsymbol{r}_{12}}{r_{12}}\,, \tag{2.1}$$

where q_1 and q_2 are the charges of two bodies separated by a distance determined by radius vector $\boldsymbol{r}_{12}$ (Figure 2.1), and $\boldsymbol{\mathcal{F}}_{12}$ is the force, whose source is body 1, affecting body 2. We have $\boldsymbol{\mathcal{F}}_{12} = -\boldsymbol{\mathcal{F}}_{21}$, that is, Coulomb's force conforms to Newton's[1] third law. The coefficient k is determined by the choice of the units of measurement of electric charge and other physical quantities. If the force $\mathcal{F}_{12} = 1$ dyn and $r_{12} = 1$ cm (the CGS system), then $k = 1$, and the charge $q = |q_1| = |q_2|$ equals 1 statC (statcoulomb; CGS electrostatic unit), whose dimensionality is $(\text{length})^{3/2} \times (\text{mass})^{1/2}/(\text{time})$ $(\text{cm}^{3/2}\,\text{g}^{1/2}/\text{s})$. If $\mathcal{F}_{12} = 1\,\text{N}$ and $r_{12} = 1\,\text{m}$ (the International System of Units, SI), then $q = 3 \times 10^9\,\text{cm}^{3/2}\,\text{g}^{1/2}/\text{s} = 1\,\text{C}$. Then, coefficient k does not equal 1 and does not have a clear physical meaning. The last unit, coulomb, was named after the discoverer of the law of electrostatic interaction. It is

1) Isaac Newton (1643–1727) was an outstanding English physicist and mathematician. He created the theoretical bases of mechanics and astronomy, discovered the universal gravitation law, and developed (in parallel with Leibniz) differential and integral calculus.

Foundations of Classical and Quantum Electrodynamics, First Edition. Igor N. Toptygin.

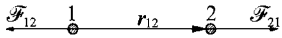

Figure 2.1 Coulomb interaction between two small bodies.

chosen from purely practical reasons. In this book, we will mostly use the CGS or physical system of units, also called the absolute Gaussian system of units. It is the most convenient one to use when dealing with the fundamental laws of physics.

Later, in the twentieth century, Coulomb's law was confirmed with very high precision by measuring the energies of atomic levels. It was also found that electric charge, just as many other physical quantities, may be quantized, that is, its size is always divisible by a certain minimal portion called *the elementary charge*, $e_0 \approx 4.8 \times 10^{-10}$ statC. The elementary charge is that of an electron, positron, proton, antiproton, and many other elementary and composite particles. The more "elementary" particles, quarks, carry fractional charges $e_0/3$ and $2e_0/3$ of either sign. Yet quarks, possibly, exist only within more complex particles and may never be isolated.

The elementary charge is a very small portion. The charges of macroscopic bodies often include a tremendous number of elementary charges, such as about 10^{22}. This is the number of free electrons in 1 cm^3 of a metal body. In this case, the spatial distribution of an electric charge may be expressed through its volume density:

$$\rho(\boldsymbol{r}) = \frac{\Delta q}{\Delta V} , \tag{2.2}$$

where Δq is a charge occupying a macroscopically small volume ΔV, that is, a volume containing a large number ($N \gg 1$) of elementary charges. The linear dimensions of such a volume must be small as compared with the other dimensions involved in a problem. A volume like that may be considered as, approximately, point-like and its volume density may be treated as a point function.

As shown in Section 1.3, point charges may also be considered as having volume density if the Dirac delta function is used: the distribution of charges $e_1, e_2, \ldots$ at positions $\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots$ may be described by the function

$$\rho(\boldsymbol{r}) = \sum_{a=1}^{N} e_a \delta(\boldsymbol{r} - \boldsymbol{r}_a) . \tag{2.3}$$

The fundamental property of a charge is its *exact conservation* whatever natural phenomena it may be involved in, such as chemical or nuclear reactions, transmutation of elementary particles, and so on. The total electric charge of all the particles involved is always the same before and after such an event.

2.1.2
Electric Field

From the viewpoint of physics, the existence of force (2.1) may be interpreted in two ways. One may believe that a charged body directly affects another charged body, without any involvement of an intervening medium (the long-range action). However, an alternative view is that a charged body changes the properties of the surrounding space, creating an *electric field* in it. The second body, through its electric charge, perceives the influence of the field, which creates the force of interaction. When we consider fixed charges, that is, in electrostatics, both of these views are equivalent. All the efficiency and inevitability of accepting the concept of a field become evident when considering numerous and various phenomena related to the movement of charged bodies and the evolution of their electrical and magnetic interactions. This is why, from the very start, we will accept the field paradigm and introduce its quantitative vector characteristic, its *strength* $\boldsymbol{E}(\boldsymbol{r})$.

Field $\boldsymbol{E}(\boldsymbol{r})$ determines the force $\mathcal{F}$ affecting a *resting* small body carrying charge q located at point $\boldsymbol{r}$:

$$\mathcal{F} = q\boldsymbol{E}(\boldsymbol{r}) . \tag{2.4}$$

This definition of the strength of an electric field is also true for alternating electric fields $\boldsymbol{E}(\boldsymbol{r}, t)$. According to Coulomb's law and the definition of the electric field (2.4), the field created by a point charge (2.1) will be expressed as follows:

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{e}{r^2}\frac{\boldsymbol{r}}{r} \tag{2.5}$$

(the physical system of units, e is the electric charge).

Now we have the formulation of another experimental law, *the principle of the superposition of fields in vacuum*: the field created by several point charges is equal to the geometric sum of fields $\boldsymbol{E}_i$ created by each of these charges individually, regardless of there being other sources of the field:

$$\boldsymbol{E} = \boldsymbol{E}_1 + \boldsymbol{E}_2 + \cdots \tag{2.6}$$

This is a very general law that governs both the electric and the magnetic fields of resting and arbitrary moving bodies.

Electrostatic equations We will calculate the electric field $\boldsymbol{E}$ created by a spatially limited system of charges described by a volume density ρ. We use the principle of the superposition of fields. We isolate the small element of the volume $\mathrm{d}V'$ (Figure 2.2) and enter, on the basis of Coulomb's law, in the form (2.5), the field created by the charges of this element at the point with radius vector $\boldsymbol{r}$:

$$\mathrm{d}\boldsymbol{E}(\boldsymbol{r}) = \frac{(\boldsymbol{r} - \boldsymbol{r}')\rho(\boldsymbol{r}')\mathrm{d}V'}{|\boldsymbol{r} - \boldsymbol{r}'|^3} .$$

Then, integrating over the whole distribution, we find

$$\boldsymbol{E}(\boldsymbol{r}) = \int \frac{(\boldsymbol{r} - \boldsymbol{r}')\rho(\boldsymbol{r}')\mathrm{d}V'}{|\boldsymbol{r} - \boldsymbol{r}'|^3} . \tag{2.7}$$

This expression describes the electric field of any limited system of charges at an arbitrary point. At that point, the volume density $\rho(\boldsymbol{r}')$ may have delta-like singularities. Here, the volume integral will converge at all points where point charges are not present and the point $\boldsymbol{r}' = \boldsymbol{r}$ is not singular for it.

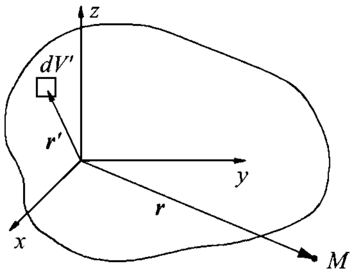

Figure 2.2 Finding an electric field using the superposition principle.

Expression (2.7) may be simplified with the use of the following identity:

$$\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3} = -\nabla \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} , \quad (\boldsymbol{r} \neq \boldsymbol{r}') ,$$

where ∇ is Hamilton's operator. Moving it to outside the integral sign, since differentiation and integration use different variables, $\boldsymbol{r}$ and $\boldsymbol{r}'$, we have

$$\boldsymbol{E}(\boldsymbol{r}) = -\nabla\varphi(\boldsymbol{r}) , \tag{2.8}$$

where the scalar function of the point

$$\varphi(\boldsymbol{r}) = \int \frac{\rho(\boldsymbol{r}')\mathrm{d}V'}{|\boldsymbol{r}-\boldsymbol{r}'|} + C \tag{2.9}$$

is called the *electrostatic potential*. This quantity is defined up to a constant. That is, in electrostatics, field strength $\boldsymbol{E}$ is a potential (vortexless) vector (see Section 1.2):

$$\operatorname{curl} \boldsymbol{E} = 0 . \tag{2.10}$$

The value of div $\boldsymbol{E}$ and, at the same time, the equation satisfied by the electrostatic potential $\varphi(\boldsymbol{r})$ may be obtained by applying the div operation to both parts of equality (2.8):

$$\operatorname{div} \boldsymbol{E} = -\nabla^2\varphi = -\Delta\varphi . \tag{2.11}$$

We insert Laplace's operator Δ under the integral (2.9) and use identity (1.225):

$$\Delta \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = -4\pi\delta(\boldsymbol{r}-\boldsymbol{r}') .$$

This gives us

$$\operatorname{div} \boldsymbol{E}(\boldsymbol{r}) = 4\pi\rho(\boldsymbol{r}) \tag{2.12}$$

and

$$\Delta\varphi(\boldsymbol{r}) = -4\pi\rho(\boldsymbol{r}) . \tag{2.13}$$

The latter equation is called the *Poisson equation*[2]. It allows us to find the electrostatic potential by integrating a differential equation in partial derivatives, rather than by integrating (2.9) directly. Equations (2.10) and (2.12) are differential equations used for determining vector $\boldsymbol{E}$ in electrostatics. Their *integral form* is obtained by applying the Stokes and Gauss–Ostrogradskii theorems to (2.10) and (2.12), respectively:

$$\oint_l \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = \int_S \operatorname{curl} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = 0, \tag{2.14}$$

$$\oint_S \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = 4\pi \int_V \rho \mathrm{d}V = 4\pi q\,. \tag{2.15}$$

Equations (2.10) and (2.12)–(2.15) follow from Coulomb's law. However, they are of a more general nature than the specific expressions (2.7) and (2.9) related to field strength and electrostatic potential.

What follows from relation (2.14) is that the work along any closed circuit of an electrostatic field on charge e equals zero: $R = \oint_l \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = 0$. Accordingly, the action required to move a charge from point A to point B does not depend on the particular way of travel:

$$R_{AB} = \int_l e\boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = -e \int_A^B \nabla\varphi \cdot \mathrm{d}\boldsymbol{l} = e(\varphi_A - \varphi_\mathrm{B})\,. \tag{2.16}$$

What follows from (2.15) is that the flux of the vector of the electric field strength passing through any closed surface is equal to the electric charge located inside that surface multiplied by 4π (*electrostatic Gauss theorem*).

Boundary conditions Consider surfaces where the value ρ has singularities (i.e., jumps or becomes infinite). Such surfaces may be a material (made of dielectrics, conductors, etc.). Differential equations, such as (2.10) and (2.12), may not be applied to singular surfaces and must be replaced by certain conditions, such as the "matching" of the components of the field strength derived from the integral form of the said equations.

Example 2.1

Use (2.14) to show that the components of the electric field strength tangential to any surface are continuous.

Solution. Make up a circuit (Figure 2.3) in the shape of a small rectangle with dimensions $a \times h$ whose surface is perpendicular to the surface in question S.

2) Siméon Denis Poisson (1781–1840) was a French mathematician, mechanical engineer, physicist, and astronomer.

Vectors $\boldsymbol{\tau}$, $\boldsymbol{\nu}$, and n make up three mutually perpendicular orts. Calculating the circulation of vector $\boldsymbol{E}$ along the circuit and using (2.14), we find that

$$\oint\limits_{l} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} \approx (E_{2\tau} - E_{1\tau})a + C_h = 0 \,,$$

where C_h is an integral over the side portions of the circuit. When $h \to 0$, we have $C_h \to 0$ if the field is limited on the surface in question (which we will assume). Letting $a \to 0$, we arrive at the precise relation

$$\boldsymbol{E}_{2\tau} = \boldsymbol{E}_{1\tau} \,. \tag{2.17}$$

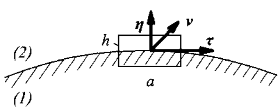

Figure 2.3 Obtaining a boundary condition for the tangential components of an electric field.

□

This relation is valid for two-dimensional vectors, the projections of the field $\boldsymbol{E}$ onto the surface at a certain point.

Example 2.2

Using (2.15), show that the component E_n, normal to any surface, meets the condition

$$E_{2n} - E_{1n} = 4\pi\sigma \,. \tag{2.18}$$

Here, the value σ, which is *the surface density of the charge*, is different from zero only if the volume density of the charge ρ has a delta-like singularity on the surface in question:

$$\rho(\boldsymbol{r}) = \tilde{\rho}(\boldsymbol{r}) + \sigma(u, v)\delta(z) \,,$$

where $\tilde{\rho}(\boldsymbol{r})$ is the finite component of volume density, the coordinate z is counted along the normal to the surface, and u and v are the coordinates of a point on the surface. Such a singularity appears if a finite charge is distributed in an extremely thin surface layer, that is, if expressed in the language of mathematics, $\sigma = \lim_{h\to 0}(h\rho)$.

Solution. Create an auxiliary volume in the shape of a small cylinder with base ΔS and height h (Figure 2.4). The normal $\boldsymbol{n}$ is directed from volume 1 into volume 2.

Applying equality (2.15) to the selected volume, we get, in turn,

$$\oint \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} \approx E_{n2}\Delta S + E_{n1}\Delta S + \Phi_h = (E_{2n} - E_{1n})\Delta S + \Phi_h$$
$$\approx 4\pi(\tilde{\rho} h \Delta S + \sigma \Delta S) .$$

Here, Φ_h is the flux directed through the side surface. Having $h \to 0$ so that the bases of the cylinder coincide with the surface in question, we arrive at the precise boundary condition (2.18).

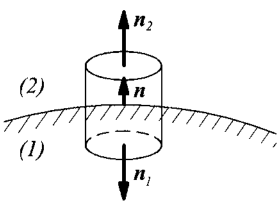

Figure 2.4 Obtaining a boundary condition for the normal components of an electric field.

□

As applied to electrostatic potential, using (2.8), we get the following boundary conditions:

$$\frac{\partial \varphi_1}{\partial \tau} = \frac{\partial \varphi_2}{\partial \tau} , \quad \frac{\partial \varphi_1}{\partial n} - \frac{\partial \varphi_2}{\partial n} = 4\pi\sigma . \tag{2.19}$$

Because the potential φ has been determined to a constant, the first condition (2.18) may be replaced with the condition of the continuity of the potential on the surface[3] S:

$$\varphi_1 = \varphi_2 . \tag{2.20}$$

Problems

2.1. A flat plate of large width, its thickness a, is evenly charged throughout its volume, the density of the charge being $\rho = \text{const}$. Disregarding edge effects, find the potential φ and strength $\boldsymbol{E}$ of the electric field. Consider the extreme case of an infinite thin plate, express the potential and strength of the field through the surface density of the charge, and verify the feasibility of the boundary conditions.

2.2. A charge is distributed in space according to the periodic law $\rho(x, y, z) = \rho_0 \cos \alpha x \cos \beta y \cos \gamma z$, producing an infinite spatial periodic lattice. Find the potential φ of the electric field.

3) Note, however, the case of a double layer (Example 2.6) when condition (2.20) is not satisfied.

2.3*. The electric charge distribution in Problem 2.2 is limited by surfaces $z = \pm z_0$ and $z_0 = \pi/2\gamma$ in the z direction and forms a flat $2z_0$ thick plate. Find the electrostatic potential throughout space.

Using the condition ($z_0\rho_0 = \text{const}$), pass to the limit $z_0 \to 0$ and introduce the surface charge density $\sigma(x, y)$.

2.4. A circular cylinder of infinite length, its radius R, is uniformly charged throughout its volume or surface so that the charge density is κ per unit length. Find the potential φ and the strength E of the electric field.

2.5. Find the potential φ and strength E of the electric field for a uniformly charged infinitely long straight filament. Its charge density per unit length is κ.

2.6*. The filament considered in the previous problem is charged nonuniformly: $\kappa(z) = \kappa_0 \cos \gamma z$. Find the electrostatic potential. In what region of space will both filaments have approximately the same potential? Examine extreme cases.

2.7. Find the potential φ and strength E of the electric field for a uniformly charged line segment of length $2a$, lying between $-a$ and a along the z axis, when its total charge is q.

2.8. Find the shapes of the equipotential surfaces of the uniformly charged segment considered in the previous problem.

2.9. Find the potential φ and strength E of the electric field of a sphere of radius R and full charge q uniformly distributed throughout the volume.

2.10. Do the same for when the charge is uniformly distributed throughout the surface of the sphere.

2.11. Inside a sphere uniformly charged throughout its volume with density ρ, there is an uncharged spherical cavity of radius R_1 whose center is located at a ($R > R_1 + a$) from the center of the sphere. Find the electric field $\boldsymbol{E}$ inside the cavity.

2.12. The space between two concentric spheres of radii R_1 and R_2 ($R_1 < R_2$) is charged with volume density $\rho = \alpha/r^2$. Find the full charge q, potential φ, and strength $\boldsymbol{E}$ of the electric field. Consider the extreme case of $R_2 \to R_1$, assuming $q = \text{const}$.

2.13. The distribution of a charge is spherically symmetric: $\rho = \rho(r)$. Write φ and $\boldsymbol{E}$ as single integrals over r, dividing the distribution of the charges into spherical layers.

2.14. Use the result obtained in the previous problem to solve Problems 2.9 and 2.12.

2.15. In the ground state of a hydrogen atom, the charge of its electron is distributed with density

$$\rho(r) = -\frac{e_0}{\pi a^2} \exp\left(-\frac{2r}{a}\right) ,$$

where $a = 0.529 \times 10^{-8}$ is the Bohr[4] radius and e_0 is the elementary charge. Find the potential φ_e and strength E_{er} of the electric field of the charge carried by the electron and the full potential φ and strength E of the field existing in the atom, considering the charge of the proton as lumped at the origin. Use a computer to plot the values of φ and E.

Hint: It is also good to integrate over spherical layers.

2.16. Regarding the atomic nucleus as a uniformly charged sphere, find the maximum value of the strength of its electric field E_{max}. The radius of the nucleus is $R = 1.5 \times 10^{-13} A^{1/3}$ cm and its is charge Ze_0 (A is the atomic number, Z is the charge number , and e_0 is the elementary charge). Compare E_{max} with the E value at the distance of the Bohr radius a/Z from the nucleus.

2.17. In Problem 2.10, write the expression for the volume density of a charge through the delta function and find the potential and electric field by integrating over spherical layers.

2.18. Find the potential φ and strength $\boldsymbol{E}$ of the electric field existing at the axis of a uniformly charged thin round disk of radius R. The charge of the disk is q. Make sure that on the surface of the disk the normal component $\boldsymbol{E}$ jumps by $4\pi\sigma$. Consider the field at large distances from the disk.

2.19. Express the potential φ of a uniformly charged thin circular ring of charge q and radius R. Do this through the full elliptical integral of the first kind:

$$K(k) = \int_0^{\pi/2} \frac{d\beta}{\sqrt{1 - k^2 \sin^2 \beta}} .$$

Hint: While integrating over the azimuthal angle, use the replacement $\alpha' = \pi - 2\beta$.

2.20. From the overall formula used in the previous problem, obtain the potential φ of the electric field (i) at the axis of the ring, (ii) at large distances from the ring, and (iii) close to the filament of the ring (find the asymptotic values of the elliptical integral in the reference literature)

Example 2.3

Use Green's identity (1.97) to prove by the rule of contraries the uniqueness theorem: the solution of the electrostatic problem inside the declared bounded volume V is unique if inside the volume V, the distribution of charge $\rho(\boldsymbol{r})$ is specified and either potential φ or its normal derivative $\partial\varphi/\partial n$ on the surface S is specified as well.

4) Niels Henrik David Bohr (1885–1962) was an outstanding Danish physicist, and one of the founders of quantum physics.

Solution. Presume that there are two different solutions of the electrostatic problem, φ_1 and φ_2 both satisfying the Poisson equation

$$\Delta\varphi_{1,2} = -4\pi\rho \quad \text{inside the volume } V \tag{1}$$

and the boundary conditions on S

$$\begin{aligned} \varphi_{1,2} &= f && \text{(Dirichlet's conditions)} \quad \text{or} \\ \frac{\partial\varphi_{1,2}}{\partial n} &= F && \text{(Neumann's conditions)}\,,^{5)} \end{aligned} \tag{2}$$

where f and F are the specified functions of coordinates. Assume (1.97) $\varphi = \psi = \varphi_1 - \varphi_2$:

$$\int\limits_V (\varphi\Delta\varphi + |\nabla\varphi|^2)\mathrm{d}V = \oint\limits_S \varphi\frac{\partial\varphi}{\partial n}\mathrm{d}S\,. \tag{3}$$

It follows from (1) that $\Delta\varphi = 0$. The integral on the right side of (3) is, consequently, also reduced to zero (2). As the result, we get

$$\int\limits_V |\nabla\varphi|^2\mathrm{d}V = 0\,, \tag{4}$$

which is possible only if $\nabla\varphi = 0$ and $\varphi = \text{const}$. In the case of Dirichlet's conditions, const $= 0$, that is, $\varphi_1 = \varphi_2$. In the case of Neumann's conditions const $\neq 0$, yet the two electrical potentials differing by the constant are physically equivalent. The solution is unique. □

Example 2.4

Use Green's identity (1.98) to express the electrostatic potential $\varphi(\boldsymbol{r})$ inside volume V through the known density $\rho(\boldsymbol{r})$ existing inside the volume and the values φ and $\partial\varphi/\partial n$ existing on the surface S bounding the volume.

Solution. In Green's identity (1.98) we identify function φ with the potential being sought and $\psi = 1/R$ with the reverse distance $R = |\boldsymbol{r} - \boldsymbol{r}'|$ from the point in question to the $\mathrm{d}V'$ element of integration. Use the Poisson equation $\Delta\varphi(\boldsymbol{r}) = -4\pi\rho(\boldsymbol{r})$ and the equation $\Delta'\psi(\boldsymbol{r}-\boldsymbol{r}') = -4\pi\delta(\boldsymbol{r}-\boldsymbol{r}')$, which is satisfied by function ψ (see formula (1.225)). Then, (1.225) gives us

$$\varphi(\boldsymbol{r}) = \int\limits_V \frac{\rho(\boldsymbol{r}')}{R}\mathrm{d}V' + \frac{1}{4\pi}\oint\limits_S \left[\frac{1}{R}\frac{\partial\varphi}{\partial n'} - \varphi\frac{\partial}{\partial n'}\frac{1}{R}\right]\mathrm{d}S'\,.$$

□

5) Peter Gustav Lejeune Dirichlet (1805–1859) and Karl Gottfried Neumann (1832–1925) were German mathematicians.

Example 2.5

A system of charges occupies a limited volume of size l in a space. Using the representation of the electrostatic potential as a volumetric integral (2.9), find the value of the potential existing at a distance $r \gg l$ from the system to within terms of order $(l/r)^2$. What values describing the system of charges do you need to do that?

Solution. Expanding the integrand in (2.9) for a small ratio, $r'/r \le l/r$, we get

$$\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = \frac{1}{r}\left(1 - 2\frac{\boldsymbol{r}\cdot\boldsymbol{r}'}{r^2} + \frac{r'^2}{r^2}\right)^{-1/2}$$
$$\approx \frac{1}{r} + \frac{\boldsymbol{r}\cdot\boldsymbol{r}'}{r^3} + \frac{1}{2r^5}x_\alpha x_\beta\left(3x'_\alpha x'_\beta - r'^2\delta_{\alpha\beta}\right) .$$

Here, the last term is written in the tensorial form, summating by repeated indices. Inserting the result in integral (2.9), we find the following:

$$\varphi(\boldsymbol{r}) = \frac{q}{r} + \frac{\boldsymbol{p}\cdot\boldsymbol{r}}{r^3} + \frac{Q_{\alpha\beta}x_\alpha x_\beta}{2r^5} , \tag{2.21}$$

where the notation below is used:

$$q = \int \rho(\boldsymbol{r}')\mathrm{d}V', \tag{2.22}$$

which is the full charge of the system;

$$\boldsymbol{p} = \int \rho(\boldsymbol{r}')\boldsymbol{r}'\mathrm{d}V' \tag{2.23}$$

which is the *dipole moment* of the system of charges;

$$Q_{\alpha\beta} = \int \rho(\boldsymbol{r}')\left(3x'_\alpha x'_\beta - r'^2\delta_{\alpha\beta}\right)\mathrm{d}V' \quad (Q_{\alpha\alpha} = 0) , \tag{2.24}$$

which is the *quadrupole moment tensor* of the system of charges. The expansion of the potential (2.21) may be continued. It is called *expansion by multipole moments* (multipoles). □

Example 2.6

A double electric layer includes two surfaces located at a small distance s from one another. The surface of one of them is charged positively with density σ, and the surface of the other one is charged negatively with density $-\sigma$, so the layer is overall electrically neutral. Write the expression for the electrostatic potential created by the double layer. Show that the values of the potential on each side of the double layer are connected by the following relations:

$$\varphi_2 - \varphi_1 = 4\pi\kappa , \quad \frac{\partial\varphi_2}{\partial n} = \frac{\partial\varphi_1}{\partial n} , \tag{2.25}$$

where $\kappa = \sigma s$ is the dipole moment per unit surface area.

Solution. Introduce the dipole moment density vector $\boldsymbol{\kappa} = \kappa s \boldsymbol{n}$ where the ort of the normal $\boldsymbol{n}$ must be directed from the positive surface to the negative one (see Figure 2.5). Regard the double layer as one surface and use the expression of the potential of a dipole, found in the $\mathrm{d}S$ surface element as per (2.21): $\mathrm{d}\varphi_M = \boldsymbol{\kappa} \cdot \boldsymbol{r}\mathrm{d}S/r^3$ where $\boldsymbol{r}$ connects element $\mathrm{d}S$ with the point of observation M. It is easy to see that the value $\boldsymbol{n} \cdot \boldsymbol{r}\mathrm{d}S/r^3 = \pm\mathrm{d}\Omega$ is an element of a spatial viewing angle at which the $\mathrm{d}S$ area may be seen from the point of observation (the sign is positive if the positive side of the surface is visible, and negative if the negative side is visible). According to the superposition principle, we have the following:

$$\varphi_M = \int\limits_{\Omega} (\pm\kappa)\mathrm{d}\Omega \ . \tag{2.26}$$

If the orientation of the surface is such as to make the signs of all its elements the same, then $\varphi_M = \pm\kappa\Omega$. If the observation point is located on one of the sides of the double layer surface, then, obviously, $\varphi_M = \pm 2\pi\kappa$. This results in a jump of the potential (2.25) at the time the double layer is crossed. The continuity of $E_n = -\partial\varphi/\partial n$ follows from the full surface density of the charge of the double layer being zero.

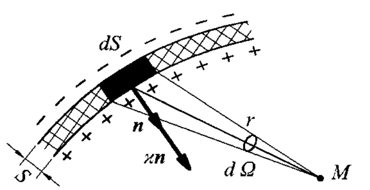

Figure 2.5 Finding the electrostatic potential of a double electric layer.

□

Problems

2.21. In spherical coordinates, write the expressions of the potentials of an electric dipole with moment $\boldsymbol{p}$ and electric quadrupole for the case of the axially symmetric distribution of charges. Also find the $\boldsymbol{E}_d$ and $\boldsymbol{E}_q$ field strengths for these systems.

2.22*. Expand the electric potential outside a bounded system of charges in multipoles and find the expressions for the multipole moments in spherical coordinates. Use course-of-value function (1.182) for the Legendre polynomials and the addition theorem (1.195) for spherical functions.

2.23*. Generalize the result obtained in the previous problem so as to make the expansion of the electrostatic potential in Legendre's spherical functions applicable to observation points located within the system of charges.

2.24. A thin circular ring of radius R consists of two uniformly and oppositely charged half-rings, their charges being q and $-q$. Find the potential φ and strength $\boldsymbol{E}$ of the electric field at the axis of the ring and at a small distance from it. What is the field at large distances from the ring?

2.25. The surface of a sphere of radius R is charged as per the $\sigma = \sigma_0 \cos\theta$ law. Find the potential φ of the electric field using multipole expansion in spherical coordinates.

2.26*. The sources of an electric field are positioned axially and symmetrically. There are no field sources close to the symmetry axis of the system. Express the potential φ and strength $\boldsymbol{E}$ of the electric field that exists close to the symmetry axis through the values of the potential and its derivatives at the said axis.

2.27. Find the potential φ of the electric field of a uniformly charged thin circular ring using multipole expansion in spherical coordinates. The charge of the ring is Q and its radius is R.

2.28. Find the potential φ of an electric field at large distances from the following systems of charges: (i) charges q, $-2q$, and q positioned along the Oz axis at a distance a from each other (linear quadrupole); (ii) charges $\pm q$ positioned at the apexes of a square, its side equaling a, so that the adjacent charges have different signs – charge $+q$ is positioned at the beginning of the coordinates and the sides of the square are parallel to the Ox and Oy axes (flat quadrupole).

2.29*. Find the potential φ of an electric field at large distances from the following systems of charges: (i) linear octupole (Figure 2.6a); (ii) spatial octupole (Figure 2.6b).

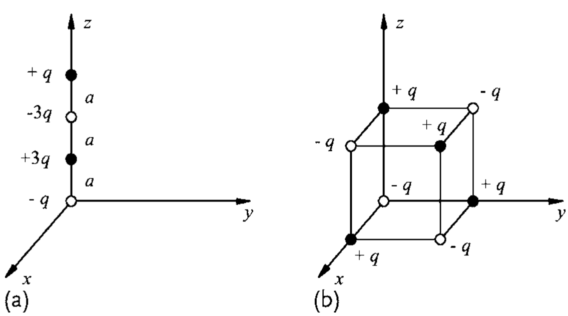

Figure 2.6 Linear octupole (a), and spatial octupole (b).

2.30. A point charge q is positioned at a point whose spherical coordinates are r_0, θ_0, α_0. Expand the potential φ of this charge in multipoles.

2.31. An ellipsoid with semiaxes a, b, and c is uniformly charged through its volume. Its full charge is q. Find the potential φ at large distances from the ellipsoid to a quadrupole member. Consider particular cases of an ellipsoid of rotation with semiaxes.[6] Find the potential, to a quadrupole member, at large distances from the ellipsoid. Consider particular cases of an ellipsoid of rotation with semiaxes $a = b$, c and a sphere ($a = b = c$).

Hint: When integrating over the volume of the ellipsoid, use generalized spherical coordinates $x = ar \sin\theta \cos\alpha$, $y = br \sin\theta \sin\alpha$, and $z = cr \cos\theta$.

2.32. Two coaxial uniformly charged thin circular rings of radii a and b ($a > b$) and charges q and $-q$, respectively, are positioned in the same plane. Find the potential φ at large distances from this system of charges. Compare it with the potential of a linear quadrupole (see Problem 2.28).

2.33*. Show that the distribution of charge $\rho = -(\boldsymbol{p}' \cdot \nabla)\delta(\boldsymbol{r})$ circumscribes an elementary dipole of moment $\boldsymbol{p}'$ placed at the origin. Explain the result using the rendition of the delta function.

Hint: Proceed from the multipole expansion in Cartesian coordinates.

2.34. Prove that the distribution of charges

$$\rho = q \prod_{i=1}^{n} (\boldsymbol{a}_i \cdot \nabla)\delta(\boldsymbol{r})$$

creates the potential

$$\varphi(\boldsymbol{r}) = q \prod_{i=1}^{n} (\boldsymbol{a}_i \cdot \nabla)\frac{1}{r} .$$

2.35. Using the results obtained in Problem 2.28 and considering that the quadrupole moment is a second-rank tensor, find the potential φ of an electric field at large distances from a linear quadrupole, the orientation of whose axis is determined by polar angles γ and β. What is the other way of solving this problem?

2.36*. A spatial octupole (Figure 2.6b) is turned around its Oz axis at an angle β. Find the field φ at large distances from it by transforming the component of the octupole moment. Compare this with other ways of solving the problem.

2.37. Find the potential φ of an electric field at large distances from a flat quadrupole positioned on a plane passing through the Oz axis (Figure 2.7). Obtain the components of the quadrupole moment both directly and by turning the flat quadrupole as in the first task in Problem 2.28.

2.38. A sphere of radius R is uniformly polarized, and the dipole moment per unit volume is $\boldsymbol{P}$. Find the potential φ of the electric field.

6) Atomic nuclei that have a quadrupole moment may, somewhat approximately, be regarded as ellipsoids of rotation.

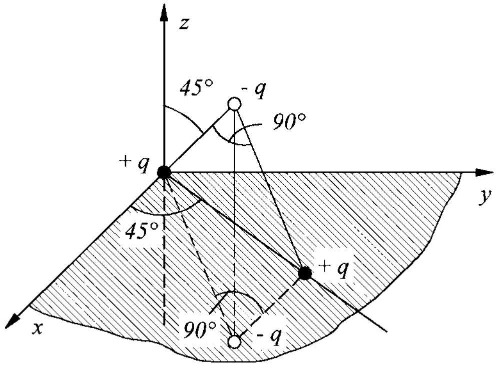

Figure 2.7 Finding the potential of a flat quadrupole.

2.39. A two-dimensional charge distribution is characterized by charge density $\rho(\boldsymbol{r})$ that is independent of the z coordinate. If $\rho \neq 0$ in a finite region S of the xy plane, then the potential φ may be expanded in two-dimensional multipoles. Find this expansion.

Hint: Break a system of charges into charged filaments and use the superposition principle and the expansion

$$\ln(1 + u^2 - 2u\cos\alpha) = -2\sum_{k=1}^{\infty} \frac{\cos k\alpha}{k} u^k \,, \quad |u| < 1 \,.$$

2.40. Expand the potential φ of a line whose charge is κ per unit length in two-dimensional multipoles. The charged line is parallel to the Oz axis and passes through the point $(r_0, \ \alpha_0)$ of the xy plane.

2.41. Find the potential φ of the electric field at a large distance from two close parallel linear charges κ and $-\kappa$ positioned at distance a from one another (linear dipole).

2.42. On a disk of radius R, there is a double electric layer, the density of its moment $\tau = \text{const}$. Find the potential φ and strength $\boldsymbol{E}$ of the electric field at the symmetry axis perpendicular to the plane of the disk.

2.43. Find the strength $\boldsymbol{E}$ of the electric field of a double electric layer, with moment $\tau = \text{const}$, occupying the semiplane $y = 0, x > 0$. Compare it with the magnetic field of an infinite rectilinear current directed along the Oz axis. Solve the problem in two ways: (i) the direct summation of strengths created by the small elements of the double layer; (ii) by finding first the electrostatic potential φ.

2.44. Find the equations of the force lines of the system of two point charges: $+q$ at point $z = a$ and $\pm q$ at point $z = -a$. Draw the force lines. Are there points of equilibrium in the field?

Hint: Owing to symmetry, force lines are located on planes $\alpha = \text{const}$, and E_z and E_R are independent of α (cylindrical coordinates). Variables in the differential equation of the force lines are divided after substitution

$$u = \frac{z+a}{r}, \quad v = \frac{z-a}{r}.$$

2.45. Using the result obtained in the previous problem, find the equations of the force lines of a point dipole positioned at the origin.

2.46. Find the equation for the force lines of the linear quadrupole considered in Problem 2.28. Draw the force lines with a computer.

2.47. Prove that the strength of the flux of the electric field of a point charge q that passes through a certain unclosed surface S is $q\Omega$. Here, Ω is a spatial viewing angle the circuit bounding the surface S is seen at from the point where the charge q is positioned ($\Omega > 0$ if the negative side of the surface is seen from that point).

2.48. A charge q_1 lies at the axis of symmetry of a circular disk of radius a, at a distance a from the plane of the disk. Find the charge q_2 that must be placed at a point located symmetrically in relation to the disk so that the flux of the electric field penetrating the disk in the direction of the charge q_1 equals Φ.

2.49*. Without integrating the differential equations for the force lines, find the equation for those lines of a system of n collinear charges $q_1, q_2, \ldots, q_n$ placed at the points $z_1, z_2, \ldots, z_n$ on the z axis. Apply the theorem proven in Problem 2.47 to a force tube generated by rotating a force line about its axis of symmetry.

2.50. Use the result obtained in the previous problem to find the equation for the force lines of a system of two point charges (compare this with Problem 2.44) and a linear quadrupole (compare this with Problem 2.46).

2.51. Uniformly charged filaments carrying charges κ_1 and $-\kappa_2$ per unit length are parallel and lie at a distance h from one another. Find the relation between κ_1 and κ_2 for which the surfaces of this system having equal potentials will include circular cylinders of finite radii and determine the radii and the positions of the axes of the cylinders.

2.52. Point charges q_1 and $-q_2$ lie at a distance h from each other. Show that the equipotential surfaces of this system include a sphere of finite radius. Find the coordinates of the center of that sphere and its radius. Find the potential φ on the surface of the sphere if $\varphi(\infty) = 0$.

2.53. Find the distribution of charges creating, in spherical coordinates, a potential of the form $\varphi(\mathbf{r}) = (q/r)\exp(-\alpha r)$, where α and q are constants.

2.54. Find the distribution of charges creating, in spherical coordinates, a potential of the form

$$\varphi(r) = \frac{e_0}{a}\left(\frac{a}{r} + 1\right)\exp\left(-\frac{2r}{a}\right),$$

where e_0 and a are constants.

2.1.3 Energy and Forces in Electrostatic Fields

Find the interaction energy of a system of electric point charges. One charge e_1 positioned at a point with radius vector $\boldsymbol{r}_1$ creates, in space, a potential $\varphi(\boldsymbol{r}) = e_1/|\boldsymbol{r}-\boldsymbol{r}_1|$. When another charge e_2 is moved from infinity to the point with radius vector $\boldsymbol{r}_2$, the following work must be performed: $R = -e_2\varphi_{12} = -e_1\varphi_{21} = -(1/2)(e_1\varphi_{21} + e_2\varphi_{12})$, where $\varphi_{ab} = e_b/|\boldsymbol{r}_a-\boldsymbol{r}_b|$ is the potential created by the charge e_b at the point $\boldsymbol{r}_a$ (for charges of the same sign, work performed by an external source is negative). That work becomes the potential energy of interaction of the charges: $W = -R$. Generalizing this consideration for the case of N charges, we find their interaction energy (*potential energy* in the terms of classical mechanics):

$$W = \frac{1}{2}\sum_{a\neq b=1}^{N} \frac{e_a e_b}{|\boldsymbol{r}_a - \boldsymbol{r}_b|} = \frac{1}{2}\sum_{a=1}^{N} e_a\varphi(\boldsymbol{r}_a)\,, \tag{2.27}$$

where $\varphi(\boldsymbol{r}_a)$ is the potential created at the location of charge e_a by all the other charges. Terms in which $a = b$ must be excluded.

Now we generalize the expressions obtained for the case of the continuous distribution of charges in space described by the volume density $\rho(\boldsymbol{r})$. Because the element of volume $\mathrm{d}V$ contains the charge $\mathrm{d}e = \rho(\boldsymbol{r})\mathrm{d}V$, the latter amount in (2.27) must be replaced with an integral:

$$W = \frac{1}{2}\int \rho(\boldsymbol{r})\varphi(\boldsymbol{r})\mathrm{d}V\,, \tag{2.28}$$

where integration is performed over the whole volume occupied by this system of charges.

Example 2.7

Two systems of charges, their volumetric densities being ρ_1 and ρ_2, create, at a specified point in space, potentials $\varphi_1(\boldsymbol{r})$ and $\varphi_2(\boldsymbol{r})$. Use (2.28) to find expressions for the self-energies of the systems W_{11} and W_{22} and their mutual energy W_{12}. How may the forces of interaction between the systems be found?

Solution. Inserting $\rho = \rho_1 + \rho_2, \varphi = \varphi_1 + \varphi_2$ into (2.28) and using (2.9), we find $W = W_{11} + W_{22} + W_{12}$, where individual terms may be written in various ways:

$$W_{ii} = \frac{1}{2}\int \rho_i\varphi_i\mathrm{d}V = \frac{1}{2}\int \frac{\rho_i(\boldsymbol{r})\rho_i(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}V\,\mathrm{d}V', \tag{2.29}$$

$$\begin{aligned} W_{ik} &= \frac{1}{2}\int (\rho_i\varphi_k + \rho_k\varphi_i)\mathrm{d}V = \int \rho_i\varphi_k\mathrm{d}V = \int \rho_k\varphi_i\mathrm{d}V \\ &= \int \frac{\rho_i(\boldsymbol{r})\rho_k(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}V\,\mathrm{d}V'\,. \end{aligned} \tag{2.30}$$

The interaction energy depends on the generalized coordinates $q_\alpha^{(i)}$ and $q_\beta^{(k)}$ of both systems determining the locations and orientation of the charged bodies. According to the general rules of mechanics, the derivative of the potential energy of interaction at its coordinate, used with the opposite sign, determines the force that tends to increase that coordinate:

$$\mathcal{F}_\alpha^{(i)} = -\frac{\partial W_{ik}}{\partial q_\alpha^{(i)}}\,, \quad \mathcal{F}_\beta^{(k)} = -\frac{\partial W_{ik}}{\partial q_\beta^{(k)}}\,. \tag{2.31}$$

□

To express the energy of a system of charges through the strength of its electric field $\boldsymbol{E}$, we expand the integration in (2.28) to include all the infinite space, since areas with $\rho = 0$ make zero contribution, and transform the integrand:

$$\rho\varphi = \frac{1}{4\pi}(\operatorname{div}(\varphi\boldsymbol{E}) - \boldsymbol{E}\cdot\nabla\varphi) = \frac{1}{4\pi}(\operatorname{div}(\varphi\boldsymbol{E}) + E^2)\,. \tag{2.32}$$

We transform the integral $\operatorname{div}(\varphi\boldsymbol{E})$ according to the Gauss–Ostrogradskii theorem:

$$\int \operatorname{div}(\varphi\boldsymbol{E})\mathrm{d}V = \oint\limits_{S\to S_\infty} \varphi\boldsymbol{E}\cdot\mathrm{d}\boldsymbol{S} \to 0\,.$$

According to (2.21), a potential located on an infinitely far surface decreases no more slowly than $1/r$, whereas the field E decreases no more slowly than $1/r^2$. This is why the surface integral is reduced to zero. Having inserted (2.32) into (2.28), we have

$$W = \frac{1}{8\pi}\int E^2\mathrm{d}V\,, \tag{2.33}$$

where integration is performed over the whole space.

We must pay attention to the possibility of different interpretations of expressions (2.28) and (2.33). The first integral is contributed to only by areas with charges ($\rho \neq 0$) and, therefore, we should consider energy as inherent to electric charges. The second formula allows us to interpret energy as a property of the electric field. Energy is present everywhere in space where $E \neq 0$ with volumetric density

$$w = \frac{1}{8\pi}E^2\,. \tag{2.34}$$

As the following chapters of the book will show, only the second interpretation correctly explains nonstationary electromagnetic phenomena.

It is also necessary to note that expressions (2.27), (2.28), and (2.33) are not equivalent, which will become obvious once we have inserted the density

$$\rho(\boldsymbol{r}) = \sum_{a=1}^{N} e_a\delta(\boldsymbol{r} - \boldsymbol{r}_a)$$

that describes a system of point charges into (2.28) and (2.9). Using these two formulas, we get the following:

$$W = \frac{1}{2} \sum_{a \neq b=1}^{N} \frac{e_a e_b}{|\boldsymbol{r}_a - \boldsymbol{r}_b|} + \frac{1}{2} \sum_{a=1}^{N} \frac{e_a^2}{|\boldsymbol{r}_a - \boldsymbol{r}_a|} .$$

Here, only the first sum with $a \neq b$ corresponds to the expression (2.27) that describes the interaction energy of a system of charges. The second sum contains self-energies of the point charges. Each of its term diverges owing to the divergence of the Coulomb potential when $r \to 0$. The obvious lack of sense in this result means that classical electrodynamics becomes inapplicable when distances are small. The problem with the divergence of the self-energy of a point object is fundamental and occurs not just in classical electrodynamics but also in the contemporary theory of elementary particles based on quantum mechanics and the theory of relativity. At the modern stage of development of science, we may find the self-energies of elementary particles only by measurement. The energy of their electrical interaction is found with the use of formula (2.27). When charges are distributed in a volume or on a surface with finite density, there is no problem in finding energies. They may be found according to (2.28) and (2.33).

Example 2.8

A system consisting of N point charges is placed in an external field whose sources are located far from the system in question (Figure 2.8). This is why the potential of the external field, within the size l of the limited system, changes slowly. The objective is to determine the energy of the interaction of the system with the external field to a quadrupole member.

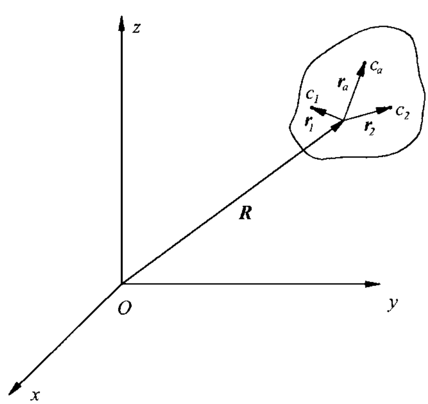

Figure 2.8 Finding the interaction energy of a system of charges with an external field.

Solution. Write the interaction energy through the potential of the external field as in (see (2.30))

$$W = \sum_{a=1}^{N} e_a \varphi(\boldsymbol{R} + \boldsymbol{r}_a) .$$

Using the smoothness of potential condition, expand it in a power series:

$$\varphi(\boldsymbol{R} + \boldsymbol{r}_a) \approx \varphi(\boldsymbol{R}) + x_\alpha^a \frac{\partial \varphi}{\partial x_\alpha} + \frac{1}{2} x_\alpha^a x_\beta^a \frac{\partial^2 \varphi}{\partial x_\alpha \partial x_\beta} + \ldots$$

If there are no sources of the external field within the system in question, then the potential φ in this area satisfies Laplace's equation:

$$\Delta \varphi = \frac{\partial^2 \varphi}{\partial x_\alpha \partial x_\alpha} = 0 .$$

This allows us to write the last member of the expansion of the potential as

$$\left(x_\alpha^a x_\beta^a - \frac{1}{3} r_a^2 \delta_{\alpha\beta} \right) \frac{\partial^2 \varphi}{\partial x_\alpha \partial x_\beta} .$$

Inserting these expansions into the original expression for energy, we get the following:

$$W = q\varphi - \boldsymbol{p} \cdot \boldsymbol{E} + \frac{Q_{\alpha\beta}}{6} \frac{\partial^2 \varphi}{\partial x_\alpha \partial x_\beta} + \ldots \tag{2.35}$$

Here, φ and $\boldsymbol{E}$ are taken with the argument R, whereas q, $\boldsymbol{p}$, and $Q_{\alpha\beta}$ are the multipole moments of the system of point charges:

$$q = \sum_{a=1}^{N} e_a , \quad \boldsymbol{p} = \sum_{a=1}^{N} e_a \boldsymbol{r}_a , \quad Q_{\alpha\beta} = \sum_{a=1}^{N} e_a \left(3 x_\alpha^a x_\beta^a - r_a^2 \delta_{\alpha\beta} \right) . \tag{2.36}$$

What is expanded here is the ratio between the l/L size of the system and the scale L measure of the inhomogeneity of the external field. □

Problems

2.55. Find the force $\mathcal{F}$ and the torque strength $\boldsymbol{N}$ applied to a dipole with moment $\boldsymbol{p}$ in the external field $\boldsymbol{E}$.

2.56. Find the electrostatic energy of a sphere of radius R with charge q uniformly distributed (i) throughout the volume, (ii) over the surface, and (iii) according to the law described in Problem 2.12.

2.57. The parallel planes of two coaxial uniformly charged rings of the same radii R are located at a distance a from one another. The work that must be performed to move point charge q from infinity to the center of each ring equals A_1 and A_2, respectively. Find the charges q_1 and q_2 of the rings.

2.58. Find the energy of interaction U between the electron cloud and the nucleus of a hydrogen atom. The distribution of charges in an atom is described in Problem 2.15.

2.59. At a certain approximation, we may assume that the electron clouds created by the two electrons in an atom of helium are identical and may be described by the following volumetric charge density:

$$\rho = -\frac{8e_0}{\pi a^3} \exp\left(-\frac{4r}{a}\right) ,$$

where a is the radius of the atom, according to Bohr, and e_0 is an elementary charge. Find in this approximation, the energy of interaction U between electrons in an atom of helium (the zero-order approximation in perturbation theory).

2.60*. Prove that the equilibrium of a system of stationary point charges interacting only by means of electrical forces, in the absence of any connections, is unstable (the Earnshaw theorem).

Hint: To do this, use the first Lyapunov[7] theorem (Izerman, 1974, p. 222): if the potential energy $U(q_\alpha)$ of a balanced conservative system has no minimum, and this is found by considering second-degree terms in the expansion $U(q_\alpha)$ in a series by degrees q_α, such an equilibrium is unstable.

2.61. The centers of two spheres carrying charges q_1 and q_2 are at a distance a from one another ($a > R_1 + R_2$, where R_1 and R_2 are the radii of the spheres). The charges are distributed in a spherically symmetric way. Find the energy of interaction U and the interaction force $\mathcal{F}$ between the spheres.

2.62. A soap bubble hanging on the end of an open tube contracts owing to its surface tension (the coefficient of surface tension is α). Assuming that the dielectric strength of air, that is, the field strength at which break occurs, equals E_0, find if the contraction may be prevented by placing a large charge on the bubble. What is the minimum equilibrium radius R of the bubble?

2.63*. Two thin parallel coaxial rings of radii a and b carry uniformly distributed charges q_1 and q_2. The distance between the planes of the rings is c. Find the energy of interaction U of the rings and the force $\mathcal{F}$ acting between them.

2.64. Find the force $\boldsymbol{\mathcal{F}}$ and torque $\boldsymbol{N}$ applied to an electric dipole with moment $\boldsymbol{p}$ by the field of a point charge q.

7) Alexandr Mikhailovich Lyapunov (1857–1918) was a Russian mathematician, and a founder of the theory of stability and equilibrium in mechanical systems.

2.65. A dipole with moment $\boldsymbol{p}_1$ is placed at the origin, and another dipole with moment $\boldsymbol{p}_2$ is at the point whose radius vector is $\boldsymbol{r}$. Find the energy of interaction U and the force $\mathcal{F}$ between the two dipoles. What orientation of the dipoles will make the force maximal?

2.66. A system of charges that may be described as having a volume density $\rho(\boldsymbol{r})$ occupies a finite region in the neighborhood of a certain point O. The system is placed in an external electric field that, near this point, may be represented as

$$\varphi_1(\boldsymbol{r}) = \sum_{l,m} \sqrt{\frac{4\pi}{2l+1}} a_{lm} r^l Y_{lm}(\theta, \alpha) .$$

Find the energy U of interaction between the system and the external field φ_1 by expressing it in terms of a_{lm} and the multipole moments Q_{lm} of the system. (compare this with Example 2.8).

Recommended literature:

Tamm (1976); Frenkel (1926); Landau and Lifshitz (1975); Jackson (1999); Bredov *et al.* (2003); Smythe (1950); Stratton (1948); Medvedev (1977); Panofsky and Phillips (1963); Feynman *et al.* (1963); Sommerfeld (1952)

2.2 Magnetostatics

2.2.1 Current Density and the Magnetic Field. Biot–Savart Law

Charged particles interact at rest by electrostatic Coulomb forces. As experiment shows, the movement of charged particles produces an added interaction called magnetic interaction. The first experiments in the study of magnetic interaction involved only macroscopic bodies (permanent magnets, conductors with current in them, etc.) The action of an electric current on a magnetic needle was discovered by the Danish physicist Hans Christian Oersted (1777–1851) in 1820. However, in the twentieth century, magnetic interactions between microscopic particles, such as electrons, ions, and so on, were studied. Below, we will formulate the laws of magnetic interactions, first for microscopic particles and then we will derive, therefrom, laws for macroscopic objects. All the formulas will be written in the absolute Gaussian system of units.

In the study of magnetic phenomena, the concept of magnetic field plays a fundamental role. A small particle moving at a constant speed $v \ll c$, where $c \approx 3 \times 10^{10}$ cm/s is the speed of light in a vacuum, creates *the strength of the magnetic field*:

$$\boldsymbol{h} = \frac{e}{c} \frac{\boldsymbol{v} \times \boldsymbol{R}}{R^3} , \tag{2.37}$$

where e is the charge carried by the particle, and $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}'$ is the radius vector drawn from the point where the particle is located to the point of observation. If a large number of particles move at an identical speed, then, according to the superposition principle, particles contained in $\mathrm{d}V$, which is a macroscopically small volume, create a field

$$\mathrm{d}\boldsymbol{H} = \sum_{(\mathrm{d}V)} \frac{e}{c}\frac{\boldsymbol{v}\times\boldsymbol{R}}{R^3} = \frac{\rho\mathrm{d}V}{c}\frac{\boldsymbol{v}\times\boldsymbol{R}}{R^3} = \frac{\boldsymbol{j}\times\boldsymbol{R}}{cR^3}\mathrm{d}V\,. \tag{2.38}$$

Here, ρ is the density of electric charge, and $\boldsymbol{j}(\boldsymbol{r}) = \rho(\boldsymbol{r})\boldsymbol{v}$ is *the density of electric current*, that is, the charge transferred by the charged particles in one unit of time across a unit area oriented perpendicular to the velocities of the charges $\boldsymbol{v}$. However, if the particles move at varying speeds like, for instance, those taking part in chaotic heat motion, then $\boldsymbol{v}$ should be understood as the average (drifting) speed. The volumetric density of electric current may also be created by single point particles. Using expression (2.3) for the charge density of point particles, we get

$$\boldsymbol{j}(\boldsymbol{r}) = \sum_a e_a\boldsymbol{v}_a\delta(\boldsymbol{r}-\boldsymbol{r}_a)\,. \tag{2.39}$$

If current flows in a quasilinear conductor whose cross-section is S and whose diameter is small in comparison with its length, we may then introduce the current strength $J = jS$ and perform a substitution:

$$\boldsymbol{j}\mathrm{d}V \rightarrow J\mathrm{d}\boldsymbol{l}\,, \tag{2.40}$$

where $\mathrm{d}\boldsymbol{l}$ is the directional element of the medial of the conductor. The strength of the magnetic field created by the element of the conductor with current will be as follows:

$$\mathrm{d}\boldsymbol{H} = \frac{J}{c}\frac{\mathrm{d}\boldsymbol{l}\times\boldsymbol{R}}{R^3}\,. \tag{2.41}$$

This is exactly the expression regarded as the empirical Biot–Savart law.[8] In the case of steady current, a direct experiment with a partial conductor is impossible. The conductor must form a close circuit. Integrating over the whole distribution of currents, we get

$$\boldsymbol{H}(\boldsymbol{r}) = \frac{1}{c}\int\frac{\boldsymbol{j}(\boldsymbol{r}')\times\boldsymbol{R}}{R^3}\mathrm{d}V'\,,\quad \boldsymbol{H}(\boldsymbol{r}) = \frac{J}{c}\oint\frac{\mathrm{d}\boldsymbol{l}\times\boldsymbol{R}}{R^3}\,. \tag{2.42}$$

The first of these expressions is applicable to steady-volume current and the second is applicable to a closed quasi-linear conductor. Note that the strength of the magnetic field is expressed through the vectorial product of true (polar; see Section 1.1) vectors and is an *axial vector* (pseudovector).

The unit of current strength in the SI is the ampere (A). When the current strength is 1 A, a charge of 1 C (coulomb) is carried through the cross-section of the conductor every second. In the absolute Gaussian system of units, the dimensionality of current strength is $(\text{length})^{3/2}\times(\text{mass})^{1/2}/(\text{time})^2$ $(\text{cm}^{3/2}\text{g}^{1/2}/\text{s}^2)$; $1\,\text{A} = 3\times10^9\text{cm}^{3/2}\text{g}^{1/2}/\text{s}^2$. The dimensionality of current density is $(\text{length})^{3/2}\times(\text{mass})^{1/2}/(\text{time})^2$ divided by length squared in the appropriate system of units.

8) Jean-Baptiste Biot (1774–1862) and Felix Savart (1791–1841) were French physicists who empirically discovered the law named after them in 1820.

2.2.2 Lorentz Force and Ampère's Formula

We have not yet indicated the way in which the strength of the magnetic field is measured. For that purpose, we will use a fundamental empirical fact, the expression for the force acting on a moving charged particle in the presence of electric and magnetic fields *(Lorentz force)*[9]:

$$\boldsymbol{\mathcal{F}}_{\mathrm{L}} = e\boldsymbol{E} + \frac{e}{c}\boldsymbol{v} \times \boldsymbol{H} \,. \tag{2.43}$$

Here, the first term on the right-hand side is the already known Coulomb force acting on a stationary particle in an electric field. The second term is the added force acting on a moving particle in the presence of a magnetic field. Expression (2.43) for the Lorentz force is one of the most elegant formulas in classical physics. Its simplicity and universality produces a profound impression: in an electromagnetic field, any charged particle moving at whatever the speed will experience force (2.43). This allows us to reduce the measurements of strengths $\boldsymbol{E}$ and $\boldsymbol{H}$ to the measurements of mechanical values $\boldsymbol{\mathcal{F}}$ and $\boldsymbol{v}$. Strength $\boldsymbol{E}$ is determined by the force acting on a test particle at rest ($\boldsymbol{v} = 0$); strength $\boldsymbol{H}$ is determined by the added force acting on a moving particle.

Having summed up (2.43), the effects on all the particles in the volume $\mathrm{d}V$, when $\boldsymbol{E} = 0$, as was done above, we get the magnetic force acting on an element of the volume of any medium where there is an electric current of density $\boldsymbol{j}$:

$$\mathrm{d}\boldsymbol{\mathcal{F}} = \sum_{(\mathrm{d}V)} \frac{e}{c}\boldsymbol{v} \times \boldsymbol{H} = \frac{1}{c}\boldsymbol{j} \times \boldsymbol{H}\mathrm{d}V \,. \tag{2.44}$$

We get the force acting on an element of quasi-linear current by substitution (2.40):

$$\mathrm{d}\boldsymbol{\mathcal{F}} = \frac{J}{c}\mathrm{d}\boldsymbol{l} \times \boldsymbol{H} \tag{2.45}$$

(Ampère formula).[10] In either case, one gets the value of the full force by integrating over the whole area where currents are flowing.

2.2.3 Conservation of Electric Charge and the Continuity Equation

As follows from the previous consideration, magnetic phenomena, that is, the creation of a magnetic field and its effect on particles and currents, are, basically, electrokinetic, that is, they are observed when charged particles are moving. At the

9) Hendrik Antoon Lorentz (1853–1928), a Nobel Prize recipient, was an outstanding Dutch physicist, the creator of the classical electronic theory of matter. His work laid the basis for special relativity

10) André-Marie Ampère (1775–1836) was a French physicist, and one of the founders of electrodynamics.

same time, as already noted in Section 2.1, electric charge is conserved in all the various natural phenomena that have been studied to date. In connection with this, we must find out what implications related to the relation of two values, the density of electric charge and the density of electric current, follow from the law of charge conservation. Consider this issue for a general case when these two densities depend on both coordinates and time. Having specified a finite volume V bounded by surface S, we may write, as an implication of charge conservation

$$\oint_S \boldsymbol{j} \cdot \mathrm{d}\boldsymbol{S} = -\frac{\mathrm{d}q}{\mathrm{d}t} ,$$

where $q(t)$ is an electric charge in volume V. It may change only if particles leave the volume. The integral in the left-hand side of the equation represents exactly a charge leaving the volume in a unit of time. Having expressed the charge q through the density, we get the integral formula of the law of the conservation of charge:

$$-\frac{\mathrm{d}}{\mathrm{d}t} \int_V \rho(\boldsymbol{r}, t) \mathrm{d}V = \oint_S \boldsymbol{j}(\boldsymbol{r}, t) \cdot \mathrm{d}\boldsymbol{S} . \tag{2.46}$$

The use of the Gauss–Ostrogradskii theorem leads to the following:

$$\int_V \left(\frac{\partial \rho}{\partial t} + \operatorname{div} \boldsymbol{j} \right) \mathrm{d}V = 0 .$$

This is true when the volume for integrating is selected arbitrarily, which means that the *continuity equation*

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \boldsymbol{j} = 0 \tag{2.47}$$

is valid at any point in space.[11)]

In physics, equations of this kind have enormous importance because they express, in differential form, the law of the conservation of any conserved substance.

In a static (resting) system of charges $\rho = \text{const}$, $\boldsymbol{j} = 0$, there is no magnetic field. The constant (in time) magnetic field is a stationary case where the distribution of charges in space does not change ($\rho = \text{const}$) but there is a stationary current $\boldsymbol{j}(\boldsymbol{r}) \neq 0$ flowing. The condition of stationarity that follows from (2.47) must be met:

$$\operatorname{div} \boldsymbol{j} = 0 . \tag{2.48}$$

Stationary (constant) currents either flow within closed circuits or extend to infinity.

11) Except, probably, for a finite number of points.

2.2.4
Equations of Magnetostatics. Vector Potential

To obtain differential equations satisfied by the strength of a steady magnetic field, we will use the Biot–Savart law in the form of (2.42) and the continuity equation for a stationary case (2.48). First, we note the following implication of (2.42):

$$\boldsymbol{H}(\boldsymbol{r}) = \operatorname{curl} \boldsymbol{A}(\boldsymbol{r}) \,, \tag{2.49}$$

where

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{1}{c}\int \frac{\boldsymbol{j}(\boldsymbol{r}')}{R}\mathrm{d}V' \tag{2.50}$$

is *the vector potential of the magnetic field.* If the current flows in a quasi-linear conductor, the vector potential can be found by using the following formula:

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{J}{c}\oint \frac{\mathrm{d}\boldsymbol{l}'}{R}. \tag{2.50$'$}$$

Having applied the operation div $= \nabla\cdot$ to equality (2.49), we get

$$\operatorname{div} \boldsymbol{H}(\boldsymbol{r}) = 0 \tag{2.51}$$

whatever the distribution of currents in space. Vector $\boldsymbol{H}$ is, therefore, solenoidal. The physical meaning of this condition is the absence, in nature, of magnetic charges analogous to electric ones.[12)]

Now, we apply the operation curl $= \nabla\times$ to both parts of the parity (2.49). On the right-hand side of the parity, we will see $\operatorname{div}\boldsymbol{A}$ and $\Delta\boldsymbol{A}$, since $\operatorname{curl}\operatorname{curl}\boldsymbol{A} = \operatorname{grad}\operatorname{div}\boldsymbol{A} - \Delta\boldsymbol{A}$.

Example 2.9

Prove that if the distribution of currents is restricted in space, the vector potential (2.50) satisfies the condition

$$\operatorname{div} \boldsymbol{A}(\boldsymbol{r}) = 0\,. \tag{2.52}$$

Solution. Insert operator ∇ under the sign of integration, changing the order of differentiating over $\boldsymbol{r}$ and integrating over $\mathrm{d}V'$, and use the identity $\nabla R = -\nabla' R$, where $R = |\boldsymbol{r} - \boldsymbol{r}'|$. With use of (2.48), we get

$$\nabla\cdot\frac{\boldsymbol{j}(\boldsymbol{r}')}{R} = \frac{\nabla'\cdot\boldsymbol{j}(\boldsymbol{r}')}{R} - \nabla'\cdot\frac{\boldsymbol{j}(\boldsymbol{r}')}{R} = -\nabla'\cdot\frac{\boldsymbol{j}(\boldsymbol{r}')}{R}\,.$$

12) Certain elementary particle theories assume the existence of magnetic monopoles, that is, particles carrying magnetic charges. However, all efforts to find them experimentally have been unsuccessful so far.

This results in

$$\operatorname{div} \boldsymbol{A} = \frac{1}{c} \int \nabla' \cdot \frac{\boldsymbol{j}(\boldsymbol{r}')}{R} \mathrm{d}V = \frac{1}{c} \oint\limits_{S_\infty} \frac{\boldsymbol{j} \cdot \mathrm{d}\boldsymbol{S}}{R} \to 0 \, .$$

The reduction of the integral to zero on the infinitely remote surface requires that the current density, at large distances, decreases faster than r^{-1}. □

Example 2.10

Obtain a heterogeneous differential equation for the vector potential $\boldsymbol{A}$ by directly applying the Laplace operator to integral (2.50).

Solution. Insert the Laplace operator, affecting coordinates $\boldsymbol{r}$, under the sign of integration and use identity (1.225). We get the Poisson equation:

$$\Delta \boldsymbol{A}(\boldsymbol{r}) = -\frac{4\pi}{c} \boldsymbol{j}(\boldsymbol{r}) \, . \tag{2.53}$$

Equation (2.49) gives, with the help of (2.53) and (2.52),

$$\operatorname{curl} \boldsymbol{H}(\boldsymbol{r}) = \frac{4\pi}{c} \boldsymbol{j}(\boldsymbol{r}) \, . \tag{2.54}$$

□

Magnetostatic equations (2.51) and (2.54) and the vector potential (2.53) are of more general nature than the specific integral (2.42) and (2.50) from which they were derived. Differential equations are good for solving a wider scope of problems and allow one to produce solutions in forms different from (2.42) and (2.50). In certain cases, the integral form of equations is more convenient for field $\boldsymbol{H}$:

$$\oint\limits_S \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{S} = 0 \, , \quad \oint\limits_l \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} = \frac{4\pi}{c} J \, . \tag{2.55}$$

These equations are derived in exactly the same way as the respective equations for the electrostatic field. The first equation, S, deals with an arbitrary selected bounded surface and the second deals with an arbitrary selected closed circuit. Current J is the full current flowing through the arbitrary surface restricted by the closed circuit l. This is why the second equation in (2.55) is sometimes called the full current theorem. The first equation shows that the magnetic flux crossing any bounded surface equals zero.

The vector potential $\boldsymbol{A}$ is an auxiliary value through which the observed physical value, the strength of the magnetic field, is expressed according to (2.49). That is why the vector potential is not synonymous.

The specified field $\boldsymbol{H}$ contains a whole family of vector potentials differing by a gradient of an arbitrary differentiable scalar function:

$$\boldsymbol{A}'(\boldsymbol{r}) = \boldsymbol{A}(\boldsymbol{r}) + \nabla\chi(\boldsymbol{r}) . \tag{2.56}$$

Since $\nabla \times \nabla\chi = 0$, the vector potentials $\boldsymbol{A}$ and $\boldsymbol{A}'$ represent the same strength of magnetic field $\boldsymbol{H}$. Vector potential $\boldsymbol{A}'$ satisfies condition (2.52) if χ is a solution of Laplace's equation $\Delta\chi = 0$.

Boundary conditions On a surface where the current density jumps or extends to infinity, certain boundary conditions must be set for normal and tangential field components. They are derived in the same way as boundary conditions for electrostatic fields. However, since magnetostatic equations are different from electrostatic ones, the boundary conditions are also different:

$$H_{2n} - H_{1n} = 0 , \quad H_{2\tau} - H_{1\tau} = \frac{4\pi}{c} i_\nu . \tag{2.57}$$

The directions of the orts $\boldsymbol{n}$, $\boldsymbol{\nu}$, and $\boldsymbol{\tau}$ are shown in Figure 2.3. The value $i_\nu = \lim_{h\to 0} j_\nu h$ is the density of the surface current. It is not equal to zero since a current of finite strength flows in the thin layer. A layer like this may be regarded as an infinitely thin surface on which conditions (2.57) must be satisfied. Both of these conditions may be also written in the vector form:

$$\boldsymbol{n}\cdot(\boldsymbol{H}_2 - \boldsymbol{H}_1) = 0 , \quad \boldsymbol{n}\times(\boldsymbol{H}_2 - \boldsymbol{H}_1) = \frac{4\pi}{c}\boldsymbol{i} . \tag{2.58}$$

Problems

2.67. May an electric current with volume density $\boldsymbol{j}(\boldsymbol{r}) = \boldsymbol{j}_0 \cos(\boldsymbol{k}\cdot\boldsymbol{r})$, where $\boldsymbol{j}_0$ and $\boldsymbol{k}$ are constant vectors, ensure the stationary (independent of time) distribution of charges in space?

2.68. Find the distribution in space of an electric current creating a magnetic field:

$$H_r = H_0\left(\frac{1}{3} - \frac{r^2}{5a^2}\right)\cos\vartheta , \quad H_\vartheta = H_0\left(\frac{2r^2}{5a^2} - \frac{1}{3}\right)\sin\vartheta ,$$

$$H_\alpha = 0 \quad \text{if} \quad r \le a ;$$

$$H_r = \frac{2H_0a^3}{15r^3}\cos\vartheta , \quad H_\vartheta = \frac{H_0a^3}{15r^3}\sin\vartheta , \quad H_\alpha = 0 \quad \text{if} \quad r \ge a .$$

2.69. Show that the homogeneous magnetostatic field $\boldsymbol{H}$ may be obtained from the vector potential $\boldsymbol{A} = \boldsymbol{H}\times\boldsymbol{r}/2$. Does it satisfy the condition div $\boldsymbol{A} = 0$?

2.70. A wire conductor of radius a is coaxial with a thin conductive cylindrical casing of radius b. Steady currents of the same magnitude J flow through both of these conductors in opposite directions. Find the magnetic field $\boldsymbol{H}$ created by this system at all the points of space. Solve this problem in two ways: by integrating the magnetostatic differential equations and by using the integral forms of these equations.

2.71. Find the strength of the magnetic field $\boldsymbol{H}$ created by a steady current J flowing through an infinite cylindrical conductor with circular cross-section of radius a. Solve the problem in the simplest way, with the use of a magnetostatic equation in its integral form (2.55), and also by introducing the vector potential $\boldsymbol{A}$.

2.72. Solve the preceding problem for a hollow cylindrical conductor of internal radius a and external radius b.

2.73. A rectilinear infinitely long band has width a. A current J uniformly distributed through the width of the band flows along it. Find the magnetic field $\boldsymbol{H}$. Verify the result by considering the extreme case of a field at large distances.

2.74. Oppositely directed currents of the same magnitude J flow within two thin infinitely long plates coinciding with two edges of an infinite prism of rectangular cross-section. The width of the plates is a and the distance between them is b. Find the force of interaction per unit length f.

2.75. Find the vector potential $\boldsymbol{A}$ and the magnetic field $\boldsymbol{H}$ created by two rectilinear parallel currents J flowing in opposite directions. The distance between the currents is $2a$.

2.76. Find the magnetic field $\boldsymbol{H}$ created by two parallel planes with the same surface currents $i = \text{const}$. Consider two cases: (i) the currents flow in opposite directions and (ii) the currents are directed the same way.

2.77. Find the magnetic field $\boldsymbol{H}$ inside a cylindrical cavity hollowed out in an infinitely long cylindrical conductor. The radii of the cavity and conductor are a and b, respectively, and the distance between their parallel axes is $d (b > a + d)$. Current J is uniformly distributed throughout the cross-section.

2.78*. Find the vector potential $\boldsymbol{A}$ and the magnetic field $\boldsymbol{H}$ created at an arbitrary point by a thin ring of radius a with current J. Express the result through elliptical integrals.

Hint: Use the method used in solving Problem 2.19.

2.79*. Show that if a magnetic field has axial symmetry and is circumscribed, in cylindrical coordinates, by a vector potential with components $A_\alpha(r, z)$, $A_r = A_z = 0$, then the equation of the magnetic force lines has the form of $rA_\alpha(r, z) = \text{const}$.

Hint: Consider the magnetic field flux inside the pipe formed by rotating one of the force lines around its axis of symmetry (compare this with the solution of Problem 2.49*).

2.80. Express the strength $\boldsymbol{H}$ and vector potential $\boldsymbol{A}$ of an axially symmetric magnetic field outside its sources through the strength of the magnetic field $H(z)$ at the axis of symmetry.

2.81. On the basis of the Biot–Savart law (2.41), show that the strength at a certain point of the magnetic field of a closed circuit with current J is expressed by the formula $\boldsymbol{H} = -(J/c)\text{grad}\,\Omega$, where Ω is a spatial angle at which the circuit is visible from that point (compare this with Problem 2.47).

2.82*. Prove the theorem of the uniqueness of the solutions of magnetostatic problems: magnetostatic equations (2.51) and (2.54) and boundary conditions (2.57) uniquely determine the strength of the magnetic field of a restricted-in-space system of steady currents.

2.83. Show that the magnetic field of an infinitely long densely wound cylindrical solenoid (n coils per unit length, current J) is described by the formulas

$$\boldsymbol{H} = \frac{4\pi}{c} n J \boldsymbol{e}_z \quad \text{inside the solenoid and} \quad \boldsymbol{H} = 0 \quad \text{outside it,}$$

where the Oz axis is directed along the solenoid.

Hint: Use the theorem of the uniqueness of the solutions of magnetostatic problems.

2.84. Find the magnetic field $\boldsymbol{H}$ at the axis of a finite densely wound cylindrical solenoid. The height of the cylinder is h, its radius a, the number of coils per unit length is n, and the current strength is J.

2.85*. A sphere of radius a carries charge e distributed over its surface. It rotates around one of its diameters with angular velocity ω. Find the magnetic field inside and outside the sphere.

Example 2.11

Following from the integral representation of vector potential (2.50), find its approximate value at large distances from a bounded-in-space system of steady currents occupying a region of size l. Show that the required value, to terms of l/r order, is

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\boldsymbol{m} \times \boldsymbol{r}}{r^3}\,, \quad \text{where} \quad \boldsymbol{m} = \frac{1}{2c}\int \boldsymbol{r} \times \boldsymbol{j}(\boldsymbol{r})\mathrm{d}V\,. \tag{2.59}$$

Vector $\boldsymbol{m}$ is called *the magnetic dipole moment* of a system of currents.

Solution. Inserting the expansion

$$\frac{1}{R} \approx \frac{1}{r} + \frac{\boldsymbol{r}\cdot\boldsymbol{r}'}{r^3}$$

into (2.50), we get

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{1}{cr}\int \boldsymbol{j}(\boldsymbol{r}')\mathrm{d}V' + \frac{1}{cr^3}\int (\boldsymbol{r}\cdot\boldsymbol{r}')\boldsymbol{j}(\boldsymbol{r}')\mathrm{d}V'\,. \tag{1}$$

The first integral on the right-hand side is reduced to zero, which follows from the identity

$$\boldsymbol{a} \cdot \boldsymbol{j}(\boldsymbol{r}') = \mathrm{div}'[(\boldsymbol{a} \cdot \boldsymbol{r}')\boldsymbol{j}(\boldsymbol{r}')] ,$$

where $\boldsymbol{a}$ is an arbitrary constant vector, div' is found at the coordinates $\boldsymbol{r}'$, and the condition of the stationarity of currents is used (2.48). Integrating both parts of the equality by volume and moving on to integrating over an infinitely remote surface, we get $\int \boldsymbol{j}(\boldsymbol{r}')\mathrm{d}V' = 0$. This means that magnetostatics lacks a term analogous to the Coulomb potential.

To transform the second integral into (1), we will use the identity

$$[\boldsymbol{r}' \times \boldsymbol{j}(\boldsymbol{r}')] \times \boldsymbol{r} = \boldsymbol{j}(\boldsymbol{r}')(\boldsymbol{r} \cdot \boldsymbol{r}') - \boldsymbol{r}'\mathrm{div}'[\boldsymbol{j}(\boldsymbol{r}')(\boldsymbol{r} \cdot \boldsymbol{r}')] .$$

We multiply, in a scalar way, both parts of this equality by an arbitrary vector $\boldsymbol{a}$ and integrate over all space:

$$\boldsymbol{a} \cdot \int [\boldsymbol{r}' \times \boldsymbol{j}(\boldsymbol{r}')]\mathrm{d}V' \times \boldsymbol{r} = \boldsymbol{a} \cdot \int \boldsymbol{j}(\boldsymbol{r}')(\boldsymbol{r} \cdot \boldsymbol{r}')\mathrm{d}V' - \int (\boldsymbol{a} \cdot \boldsymbol{r}')\mathrm{div}'[\boldsymbol{j}(\boldsymbol{r}')(\boldsymbol{r} \cdot \boldsymbol{r}')]\mathrm{d}V' . \tag{2}$$

We transform the latter integral with the identity

$$(\boldsymbol{a} \cdot \boldsymbol{r}')\mathrm{div}'[\boldsymbol{j}(\boldsymbol{r}')(\boldsymbol{r} \cdot \boldsymbol{r}')] - \boldsymbol{a} \cdot \boldsymbol{j}(\boldsymbol{r}')(\boldsymbol{r} \cdot \boldsymbol{r}') .$$

Having integrated over all space and used the Gauss–Ostrogradskii theorem, we find that the term on the right-hand side containing div' has been reduced to zero. Finally, owing to the arbitrariness of vector $\boldsymbol{a}$, it follows from equality (2) that

$$\int [\boldsymbol{r}' \times \boldsymbol{j}(\boldsymbol{r}')]\mathrm{d}V' \times \boldsymbol{r} = 2 \int \boldsymbol{j}(\boldsymbol{r}')(\boldsymbol{r} \cdot \boldsymbol{r}')\mathrm{d}V' . \tag{3}$$

Having used this identity in (1), we get a representation of the vector potential as it is given in the condition in the example. □

Problems

2.86. Find the magnetic moment $\boldsymbol{m}$ of the rotating sphere considered in Problem 2.85*. Do the same for the case when the charge is uniformly distributed throughout the volume. Find the magnetic field at a large distance from the sphere and compare this with the precise solution of the said problem.

2.87. Show that the magnetic moment of a flat circuit may be found with the following formula:

$$\boldsymbol{m} = \frac{J}{c} S \boldsymbol{n} , \tag{2.60}$$

where J is the current in the circuit, S is the area bounded by the circuit with the current, and $\boldsymbol{n}$ is the ort of the normal to the plane of the circuit.

2.88. Let a system of charged particles with the same ratio e/m of charge to mass perform a finite (bounded-in-space) movement in a certain external field. Show that the magnetic moment of such a system of particles is proportional to its moment of impulse:

$$\boldsymbol{m} = \eta \boldsymbol{L} , \tag{2.61}$$

where $\eta = e/2mc$ is the gyromagnetic ratio, and $\boldsymbol{L} = \sum_a \boldsymbol{r}_a \times \boldsymbol{p}_a$ is the moment of impulse of the system of particles.

2.89. A system consists of two particles with different e/m ratios. Express its magnetic moment through its full mechanical moment $\boldsymbol{L}$ in the system of the center of masses. Consider the case of a particle and an antiparticle ($e_2 = -e_1$, $m_2 = m_1$).

2.90. The electron of an excited hydrogen atom creates an orbital current

$$j_r = j_\vartheta = 0 , \quad j_\alpha = \frac{e\hbar r^3}{3^8 \pi m_e a^7} \exp\left(-\frac{2r}{3a}\right) \sin\theta \cos^2\theta$$

expressed through the same notation as in Problem 2.15; $\hbar \approx 1.054 \times 10^{-27}$ erg s – Planck's constant[13] divided by 2π. Find the magnetic moment of the orbital current.

2.91*. The density of the current created by the spin magnetic moment[14] of the electron in a hydrogen atom in the ground state is described by the function $\boldsymbol{j} = (c/e)\mathrm{curl}\,[\rho(r)\boldsymbol{\mu}_s]$, where $\boldsymbol{\mu}_s$ is the constant vector and $\rho(r)$ is the volume density of the charge distribution, independent of angles and exponentially decreasing when the values of r are large. Show that the magnetic field at the origin equals $-(8/3e)\pi\rho(0)\boldsymbol{\mu}_s$. What magnetic moment does the specified current create?

2.92. Find the magnetic field at the origin and the magnetic moment created by the current

$$j_r = j_\vartheta = 0 , \quad j_\alpha = \frac{e\hbar r^3}{2 \cdot 3^8 \pi m_e a^7} \exp\left(-\frac{2r}{3a}\right) \sin^3\vartheta$$

in one of the excited states of a hydrogen atom (in spherical coordinates).

2.93. A uniformly charged ellipsoid of rotation, its full charge equaling e, with semiaxes a, a, and b rotates around its axis of symmetry with angular velocity $\boldsymbol{\omega}$. Find its magnetic moment and magnetic field at large distances.

2.94. A thin disk of radius a with a uniformly distributed charge e rotates around its axis with angular velocity $\boldsymbol{\omega}$. Find the magnetic moment and field strength everywhere along the axis of the disk.

13) Max Planck (1858–1947) was an outstanding German physicist, and one of the founders of quantum physics. He actively developed the special theory of relativity, and was a Nobel Prize recipient.

14) For spin moments, see the solution of Problem 2.88.

Example 2.12

Investigate the possibility of introducing the pseudoscalar potential $\boldsymbol{H}(\boldsymbol{r}) = -\nabla\psi(\boldsymbol{r})$. Use the example of a quasi-linear closed conductor with current to investigate the role of the nonsimple connectedness of the applicable domain ψ, and obtain the equation and boundary conditions for the potential.

Solution. The representation $\boldsymbol{H} = -\nabla\psi$ leads to the equality $\operatorname{curl}\boldsymbol{H} = 0$ and, therefore, according to (2.54), is possible only in regions where current density $\boldsymbol{j} = 0$. In those regions, the Laplace equation for the potential

$$\Delta\psi = 0 \tag{2.62}$$

follows from (2.51). However, the existence of regions where $\boldsymbol{j} \neq 0$ and the potential does not exist makes space non-simple connected and causes the ambiguity of ψ. Let us say, a field is created by a quasi-linear conductor with current. Select an arbitrary closed circuit encompassing the conductor with current (Figure 2.9), and find the value of the potential at point A after path tracing:

$$\tilde{\psi}_A = \psi_A + \oint_l \mathrm{d}\psi\ .$$

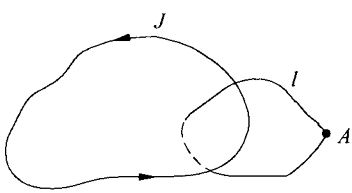

Figure 2.9 Increment of the pseudoscalar potential while tracing a closed circuit encompassing a conductor with current.

If $\psi(\boldsymbol{r})$ were a nonambiguous continuous function, then $\oint \mathrm{d}\psi = 0$. However, actually,

$$\oint_l \mathrm{d}\psi = \oint \nabla\psi \cdot \mathrm{d}\boldsymbol{l} = -\oint \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} = -\frac{4\pi}{c} J\ ,$$

which means that

$$\psi_A - \tilde{\psi}_A = \frac{4\pi}{c} J\ . \tag{2.63}$$

The latter relation shows that $\psi(\boldsymbol{r})$ is an ambiguous function of the coordinates and, after any closed circuit that encompasses a circuit with current has been traced, it is incremented by $4\pi J/c$, where J is the full current flowing through an arbitrary surface bounded by the circuit of integration.

Condition (2.63) is similar to the boundary condition (2.25) on the surface of the double electric layer with strength $\kappa_m = J/c$. That is why the circuit with current may be enveloped by an arbitrary surface where, we believe, condition (2.63) must be satisfied. The second boundary condition, the continuity of a potential on a double layer, is also satisfied because the normal component of a magnetic field is continuous on any surface:

$$\frac{\partial \psi_2}{\partial n} = \frac{\partial \psi_1}{\partial n} \,. \tag{2.64}$$

□

Problems

2.95. Find the pseudoscalar potential ψ of the magnetic field created by an infinitely long rectilinear conductor with current J. Find the components of the magnetic field.

2.96*. Find the pseudoscalar potential of the magnetic field of a closed linear circuit with current. Solve the problem (i) by integrating the Laplace equation for the potential and (ii) by using the known expression (2.50′) for the vector potential.

Hint: When solving the problem in the first way, use the concept of solving the Laplace equation as an integral for a closed surface; see Example 2.4.

2.2.5
Energy and Forces in Magnetostatic Fields

To find work to be performed in order to move circuit l with current J in external magnetic field $\boldsymbol{H}$, let every element of the counter $\mathrm{d}\boldsymbol{l}$ shift by the small $\delta \boldsymbol{s}$. The element of current is affected by the Ampère force (2.45), which means that the work δA needed to move the whole circuit is written as

$$\delta A = \frac{J}{c} \oint_l \delta \boldsymbol{s} \cdot [\mathrm{d}\boldsymbol{l} \times \boldsymbol{H}] = \frac{J}{c} \int_{\delta S} \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{S} = \frac{J}{c} \delta \Phi \,,$$

where $\mathrm{d}\boldsymbol{S} = \delta \boldsymbol{s} \times \mathrm{d}\boldsymbol{l}$, δS is the full surface to be traced by the full circuit with current during its small displacement (and, possibly, deformation). The integral in the previous chain of equalities represents the increment of the magnetic flux crossing the circuit with current when it is moved.

$$\Phi = \int_S \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{S} = \int_S \operatorname{curl} \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{S} = \oint_l \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{l} \tag{2.65}$$

This increment $\delta \Phi$ is exactly equal to the flux crossing the added surface traced by the circuit (Figure 2.10).

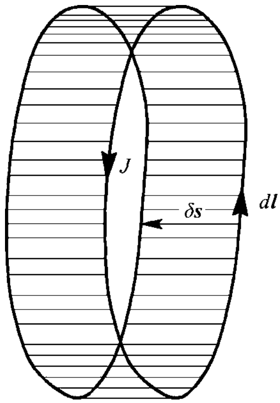

Figure 2.10 A surface circumscribed by a closed circuit with current when it is displaced.

The work found is, obviously, the increment of the energy of the circuit in the external field:

$$\delta A = \delta W \ . \tag{2.66}$$

The elementary work allows us to find the generalized forces $\mathcal{F}_\alpha$ acting on the circuit with current because it is related to the increment of the generalized coordinates δq_α known in mechanics by the relation

$$\delta A = \sum_\alpha \mathcal{F}_\alpha \delta q_\alpha \ .$$

However, the analogy with mechanics may be deepened by considering the *potential function* of the circuit with current in the external field:

$$U = -\frac{J}{c}\Phi \ . \tag{2.67}$$

The work of magnetic forces is done because the potential function decreases, $\delta A = -\delta U$, and the generalized forces are found with the use of the usual formulas of mechanics:

$$\mathcal{F}_\alpha = -\frac{\partial U}{\partial q_\alpha} \ . \tag{2.68}$$

One should bear in mind that the potential function plays the role of potential energy only when the displacement of the circuit is infinitely slow. When speeds are finite, there is the phenomenon of magnetic induction (see the following section), which changes the magnitude of the current in the circuit and results in the appearance of electromotive forces alongside magnetic ones.

Now consider interaction between two circuits, *a* and *b*, and there are no other circuits. The field of the second circuit plays the role of the external field, so the potential function of their interaction will be

$$U_{ab} = U_{ba} = -\frac{J_a}{c}\Phi_{ab} = -\frac{J_b}{c}\Phi_{ba} = -\frac{J_a J_b}{c^2}\oint_{l_a}\oint_{l_b}\frac{\mathrm{d}\boldsymbol{l}_a \cdot \mathrm{d}\boldsymbol{l}_b}{R} \ .$$

Here,

$$\Phi_{ba} = \oint\limits_{l_b} \boldsymbol{A}_a \cdot \mathrm{d}\boldsymbol{l}_b$$

is the magnetic flux of the field of circuit a crossing the circuit b, and formula (2.50′) is used for the vector potential. R is the distance between $\mathrm{d}\boldsymbol{l}_a$ and $\mathrm{d}\boldsymbol{l}_b$. It is convenient to write the potential function in the symmetric form:

$$U_{\text{int}} = \frac{1}{2}(U_{ab} + U_{ba}) = -\frac{1}{2c^2} \sum_{a \neq b = 1}^{2} L_{ab} J_a J_b \,, \tag{2.69}$$

where

$$L_{ab} = \oint\limits_{l_a} \oint\limits_{l_b} \frac{\mathrm{d}\boldsymbol{l}_a \cdot \mathrm{d}\boldsymbol{l}_b}{R} \quad (L_{ba} = L_{ab}) \,. \tag{2.70}$$

This is the *coefficient of mutual induction* of the two circuits. It is determined (outside magnetizable material mediums) only by the shapes and locations of circuits with currents in them as well as by the directions in which they are traced (the directions of the currents). The forces of the interaction of the circuits are found through U_{int}, according to (2.68). The magnetic flux created by circuit a crossing circuit b is also expressed through the coefficient of mutual induction:

$$\Phi_{ba} = \frac{1}{c} L_{ba} J_a. \tag{2.70′}$$

If the thicknesses of the conductors are not small in comparison with their lengths, then, in the above formulas, $J\mathrm{d}\boldsymbol{l} \to \boldsymbol{j}\mathrm{d}V$ must be replaced and integration must be done over the volume of each conductor. This will result in the following expression:

$$U_{\text{int}} = -\frac{1}{2^2} \sum_{a \neq b = 1}^{2} \int\limits_{V_a} \int\limits_{V_b} \frac{\boldsymbol{j}_a(\boldsymbol{r}_a) \cdot \boldsymbol{j}(\boldsymbol{r}_a)}{R} \mathrm{d}V_a \mathrm{d}V_b = -\frac{1}{2c^2} \sum_{a \neq b = 1}^{2} L_{ab} J_a J_b \,, \tag{2.71}$$

where the coefficient of mutual induction is

$$L_{ab} = \frac{1}{J_a J_b} \int\limits_{V_a} \int\limits_{V_b} \frac{\boldsymbol{j}_a(\boldsymbol{r}_a) \cdot \boldsymbol{j}_b(\boldsymbol{r}_b)}{R} \mathrm{d}V_a \mathrm{d}V_b \,, \tag{2.72}$$

and so on. All these expressions are also good for an arbitrary number of currents if the summation of all currents is done.

Formulas (2.71) and (2.72) allow us to build the potential function of interaction between individual elements of the same current:

$$U = -\frac{1}{2c} \int\limits_V \boldsymbol{j}(\boldsymbol{r}) \cdot \boldsymbol{A}(\boldsymbol{r}) \mathrm{d}V = -\frac{1}{2c^2} \int\limits_V \frac{\boldsymbol{j}(\boldsymbol{r}) \cdot \boldsymbol{j}(\boldsymbol{r}')}{R} \mathrm{d}V \mathrm{d}V' = -\frac{1}{2c^2} L J^2 \,. \tag{2.73}$$

These expressions may be considered as the result of interaction between individual current threads. Any current of finite thickness may be divided into such threads. The value

$$L = \frac{1}{J^2} \int_V \frac{\boldsymbol{j}(\boldsymbol{r}) \cdot \boldsymbol{j}(\boldsymbol{r}')}{R} \mathrm{d}V \mathrm{d}V' \tag{2.74}$$

is called the *coefficient of self-induction*. When it is being found, one should account for the finiteness of the cross-section of the conductor, or else the applied integral will diverge.

The potential function differs only by the sign from the energy that must be used to create the relevant current distributions in space. This is why the full magnetic energy W of continuous currents may be written as follows:

$$W = -U = \frac{1}{2c} \int_V \boldsymbol{j}(\boldsymbol{r}) \cdot \boldsymbol{A}(\boldsymbol{r}) \mathrm{d}V = \frac{1}{2c^2} \int_V \frac{\boldsymbol{j}(\boldsymbol{r}) \cdot \boldsymbol{j}(\boldsymbol{r}')}{R} \mathrm{d}V \mathrm{d}V' . \tag{2.75}$$

These expressions are good for the steady currents in the most general case. They include both the self-energies of any number of conductors with current and their interaction energies.

Magnetic energy allows the field interpretation after transformation of expressions (2.75) in a way similar to how it is done in electrostatics. Assuming that $\boldsymbol{j} = (c/4\pi)\mathrm{curl}\, \boldsymbol{H}$ as per (2.54) and using the identity

$$\boldsymbol{A} \cdot \mathrm{curl}\, \boldsymbol{H} = \mathrm{div}[\boldsymbol{H} \times \boldsymbol{A}] + \boldsymbol{H} \cdot \mathrm{curl}\, \boldsymbol{A} ,$$

we have

$$W = \frac{1}{8\pi} \int_V \boldsymbol{H} \cdot \mathrm{curl}\, \boldsymbol{A} \mathrm{d}V + \frac{1}{8\pi} \oint_S [\boldsymbol{H} \times \boldsymbol{A}] \cdot \mathrm{d}\boldsymbol{S} ,$$

where S is the surface bounding the volume of integration V. In the case of a system of currents of finite size, one may integrate over the whole infinite space. The integral of an infinitely remote surface will be reduced to zero, and magnetic energy will be expressed only through the strength of the magnetic field $\boldsymbol{H}$:

$$W = \int w \mathrm{d}V , \quad \text{where} \quad w = \frac{1}{8\pi} H^2 , \tag{2.76}$$

that is, the volume density of magnetic energy. As per the latter expression, magnetic energy, unlike in (2.75), is concentrated not just in regions where currents flow, but is poured over the whole space with volume density w. The analogy with the electrostatic formula (2.33) is obvious.

Example 2.13

Within a system of continuous currents with density $\boldsymbol{j}(\boldsymbol{r})$, restricted in space, an external magnetic field is weakly inhomogeneous. Express the potential function of the system through its magnetic moment.

Solution. Using formulas (2.65) and (2.67), write the required potential function:

$$U = -\frac{J}{c}\oint \boldsymbol{A}\cdot \mathrm{d}\boldsymbol{l} \to -\frac{1}{c}\int \boldsymbol{j}(\boldsymbol{r})\cdot \boldsymbol{A}(\boldsymbol{R}+\boldsymbol{r})\mathrm{d}V\ , \tag{1}$$

where $\boldsymbol{r}$ is counted from a certain point within the system of currents (see Figure 2.7). Represent the vector potential of the external field as

$$\boldsymbol{A}(\boldsymbol{R}+\boldsymbol{r}) \approx \boldsymbol{A}(\boldsymbol{R}) + (\boldsymbol{r}\cdot\nabla)\boldsymbol{A}(\boldsymbol{R})\ ,$$

where operator ∇ acts on $\boldsymbol{R}$. The substitution of this expression into (1) reduces the first integral to zero, which was the result of solving Example 2.11. Transform the second integral using identity (3) from the solution of the same example and substitute, in the said identity, vector ∇ for vector $\boldsymbol{r}$. This will produce the following:

$$\frac{1}{c}\int \boldsymbol{j}(\boldsymbol{r})(\boldsymbol{r}\cdot\nabla)\mathrm{d}V = \frac{1}{2c}\int [\boldsymbol{r}\times \boldsymbol{j}(\boldsymbol{r})]\mathrm{d}V\times\nabla = \boldsymbol{m}\times\nabla\ .$$

Substituting this result into (1), we find the potential function:

$$U = -[\boldsymbol{m}\times\nabla]\cdot \boldsymbol{A}(\boldsymbol{R}) = -\boldsymbol{m}\cdot\boldsymbol{H}\ , \tag{2.77}$$

where $\boldsymbol{H}(\boldsymbol{R}) = \nabla\times\boldsymbol{A}(\boldsymbol{R})$ is the strength of the external magnetic field and $\boldsymbol{m}$ is the magnetic moment. □

Problems

2.97. Find the force $\boldsymbol{F}$ and torque moment $\boldsymbol{N}$ acting on a closed thin conductor with current in a homogeneous magnetic field $\boldsymbol{H}$. The shape of the circuit formed by the conductor does not matter. Solve this problem in two ways: (i) by directly summating forces and moments of forces applied to current elements and (ii) by using the potential function. Express the result through the magnetic moment $\boldsymbol{m}$.

2.98. Find the potential function U of two small currents whose magnetic moments are $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$. Find the interaction force $\boldsymbol{F}$ of these currents and torque moments $\boldsymbol{N}$ applied to them. Consider the particular case of $\boldsymbol{m}_1 \parallel \boldsymbol{m}_2$.

2.99. Show that the forces acting between small currents tend to set the magnetic moments of those currents in parallel to each other and to the line connecting the centers.

2.100. Find the potential function u_{12} of two parallel infinitely long rectilinear currents $\mathcal{J}_1$ and $\mathcal{J}_2$ and the force f of their interaction per unit length.

2.101. A square frame with current $\mathcal{J}_2$ is placed so that two of its sides are parallel to a long rectilinear conductor with current $\mathcal{J}_1$ (Figure 2.11). The side of the square equals a. Find the force $\boldsymbol{F}$ acting on the frame and the torque moment $\boldsymbol{N}$ in relation to the OO' axis.

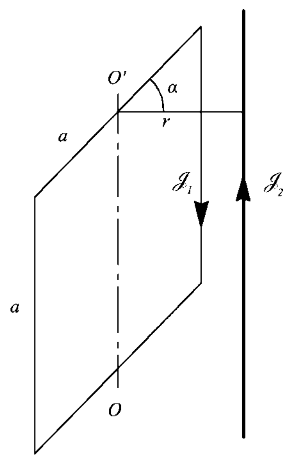

Figure 2.11 A rectangular frame with current in the neighborhood of a straight conductor.

2.102. A frame with current $\mathcal{J}_2$ consists of a circular arc with angle $2(\pi - \alpha)$ and a chord connecting its ends (Figure 2.12). The radius of the arc is a. Along the normal of the frame's plane is a long rectilinear conductor with current $\mathcal{J}_1$. Find the moment of forces applied to the frame.

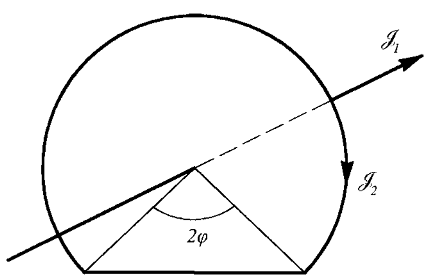

Figure 2.12 Interaction between a frame with current and a straight conductor.

2.103. Within a thin conductive cylindrical casing of radius b runs a coaxial conductor of radius a. Find the self-induction $\mathcal{L}$ per unit length.

2.104. A line consists of two thin cylindrical coaxial cases of radii a and b $(a < b)$. Find the self-induction $\mathcal{L}$ per unit length.

2.105. A thin rectilinear conductor and a ring of radius a lie in the same plane. The distance between the center of the ring and the conductor is b Find the coefficient of mutual induction L_{12} and the force of interaction F if the magnitude of the current in the conductor is $\mathcal{J}_1$ and that in the ring is $\mathcal{J}_2$.

2.106*. Two thin rings of radii a and b are placed so that their planes are perpendicular to the segment of a rectilinear line that connects the centers of the rings. Find the coefficient of mutual induction L_{12}. Express the result through elliptical integrals. Consider, for instance, the extreme case of $l \gg a,\ b$.

2.107. Find the interaction force F between the two circular currents considered in the previous problem.

2.108. Find the self-induction $\mathcal{L}$ of a unit length of an infinitely long densely wound cylindrical solenoid of arbitrary, but not necessarily circular, cross-section. The area of the cross-section is S and the number of coils per unit length is n.

2.109. Find the self-induction L of a solenoid of finite length h and of radius a ($h \gg a$) to terms a/h. Replace the current flowing in the winding with an equivalent surface current.

2.110. Find the self-induction L of a toroidal solenoid. The radius of a torus is b, the number of coils is N, and the cross-section of the torus is a circle of radius a. Find the self-induction per unit length of the solenoid in the extreme case of $b \to \infty,\ N/b = \text{const}$. Solve the same problem for a toroidal solenoid whose cross-section is a rectangle with sides a and h.

2.111. Find the self-induction $\mathcal{L}$ of a unit length of a dual conductor line consisting of two rectilinear parallel conductors of radii a and b, with the distance between axial lines h. The conductors carry equal-in-magnitude but oppositely directed currents $\mathcal{J}$.

2.112*. Show that the self-induction of a thin closed conductor of circular cross-section may be approximately found using the formula[15] $L = l/2 + L'$, where l is the length of the conductor and L' is the coefficient of the mutual induction of the two linear circuits. One of the circuits may coincide with the axial line of the quasi-linear conductor in question, and the other coincides with the section line between the surface of the conductor and a arbitrary nonclosed surface bearing on its axial line (Figure 2.13).

2.113. Find the self-induction L of a thin wire ring of radius b. The radius of the wire is $a \ll b$.

15) The first and second members of the expression L may be seen as the inner and outer self-induction, respectively, because they determine the magnetic energy stored both inside and outside the conductor.

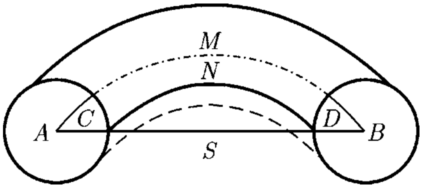

Figure 2.13 Finding the self-induction of a thin closed conductor.

Hint: Use the formula from the previous problem.

2.114. Find the coefficient of mutual induction L_{12} of two parallel line segments of length a placed at a distance l from one another and coinciding with two sides of a rectangle.[16)]

2.115. Find the coefficient of mutual induction L_{12} of two equal squares, each of their sides being a, located at a distance l from one another and coinciding with two opposite sides of a rectangular parallelepiped. Find the interaction force F between them.

2.116. Find the self-induction L of a wire square with side b. The radius of the wire is $a \ll b$.

Hint: The results obtained in Problems 2.112* and 2.114 may be used to solve this one.

Recommended literature:
Tamm (1976); Landau and Lifshitz (1975); Medvedev (1977); Frenkel (1926); Jackson (1999); Bredov *et al.* (2003); Feynman *et al.* (1963); Smythe (1950); Panofsky and Phillips (1963); Stratton (1948); Sommerfeld (1952).

2.3 Maxwell's Equations. Free Electromagnetic Field

2.3.1 The Law of Electromagnetic Induction

The previous sections dealt with the laws concerning electric charges at rest and steady currents (with the exception of the law of the conservation of electric charge, formulated for a general case). In this section, the overall picture of nonstationary electromagnetic phenomena and the equations describing those phenomena will be considered. Their basic singularity consists of the fact that electric and magnetic fields changing in time unlike static fields, may not exist independently. A connec-

16) Here, the coefficient of mutual induction sought does not have an immediate physical meaning because the currents flowing in the segments may not be closed. However, the inductance of closed circuits having parallel rectilinear portions may be easily expressed through it (see the following two problems).

tion appears between them, causing the formation of a single electromagnetic field described by two strengths, $\boldsymbol{E}(\boldsymbol{r}, t)$ and $\boldsymbol{H}(\boldsymbol{r}, t)$. In the most obvious form, this connection was shown in the experiments of the English physicist Michael Faraday[17] in 1831. If a closed conductor is placed in a time-dependent magnetic field, an electric current proportionate to the alternations of the magnetic flux crossing the circuit formed by the conductor begins to flow in that conductor. An electric current starts to flow also when a conductor moves in a magnetostatic but inhomogeneous magnetic field and when a closed conductor rotates in such a field. Whatever the case, the magnetic flux $\Phi = \int \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{S}$ crossing the circuit must alternate. Here $\mathrm{d}\boldsymbol{S}$ is a vector element of restricted surface S bearing on the circuit.

The appearance of an electric current means that an electric field that causes and sustains that current is induced in the conductor. The circulation of the strength of the electric field in the closed conductor, called *the electromotive force of induction*, may serve as the measure of this effect:

$$\mathcal{E}_{\text{ind}} = \oint_l \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} \, . \tag{2.78}$$

On the basis of the objective character of the laws of nature, we must assume that when the magnetic field in a given region of space alternates, an induced electric field appears there whether or not there is a conductor wherein that field may produce a current. The quantitative connection between magnetic and electric fields discovered by Faraday may be represented in the following way:

$$\mathcal{E}_{\text{ind}} = -\frac{1}{c}\frac{\mathrm{d}\Phi}{\mathrm{d}t} \, . \tag{2.79}$$

This is called *the law of electromagnetic induction*. From this relation, we can produce an integral connection between field strengths $\boldsymbol{E}$ and $\boldsymbol{H}$. If we assuming that an arbitrary circuit l is at rest and bounds surface S bearing on it, we get the following from (2.78) and (2.79):

$$\oint_l \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = -\frac{1}{c}\frac{\mathrm{d}}{\mathrm{d}t}\int_S \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{S} = -\frac{1}{c}\int_S \frac{\partial \boldsymbol{H}}{\partial t} \cdot \mathrm{d}\boldsymbol{S} \, . \tag{2.80}$$

If the integral on the left-hand side is transformed into the surface integral as per Stokes's theorem, we get

$$\int_S \left(\operatorname{curl} \boldsymbol{E} + \frac{1}{c}\frac{\partial \boldsymbol{H}}{\partial t}\right) \cdot \mathrm{d}\boldsymbol{S} = 0 \, .$$

17) Michael Faraday (1791–1867) was an outstanding English experimental physicist. He introduced the concept of the field, studied many electric and magnetic phenomena, and established a connection between them. Faraday experimentally substantiated the law of the conservation of electric charge.

Finally, considering the free selection of the surface S, we get the following differential equation:

$$\operatorname{curl} \boldsymbol{E} + \frac{1}{c}\frac{\partial \boldsymbol{H}}{\partial t} = 0 . \tag{2.81}$$

Relations (2.80) and (2.81) represent the integral and differential forms of Faraday's law of electromagnetic induction.

The differential form of the law of electromagnetic induction (2.81) may be regarded as a generalization of the electrostatic equation (2.10), $\operatorname{curl} \boldsymbol{E} = 0$, that allows us to describe nonstationary phenomena. It is easy to make sure that the magnetostatic equation (2.54), $\operatorname{curl} \boldsymbol{H} = 4\pi \boldsymbol{j}/c$, is also inapplicable for describing phenomena changing with time. Applying the div operation to both parts of this equation, we get $\operatorname{div} \boldsymbol{j} = 0$, which is not in keeping with the continuity equation (2.47) in the general case.

During the 1860s, outstanding Scottish physicist James Clerk Maxwell[18] generalized the equations of the electromagnetic field, thus allowing him to describe a rather wide spectrum of nonstationary electromagnetic phenomena:

$$\operatorname{curl} \boldsymbol{E}(\boldsymbol{r}, t) = -\frac{1}{c}\frac{\partial \boldsymbol{H}(\boldsymbol{r}, t)}{\partial t} , \tag{2.82}$$

$$\operatorname{curl} \boldsymbol{H}(\boldsymbol{r}, t) = \frac{1}{c}\frac{\partial \boldsymbol{E}(\boldsymbol{r}, t)}{\partial t} + \frac{4\pi}{c}\boldsymbol{j}(\boldsymbol{r}, t) , \tag{2.83}$$

$$\operatorname{div} \boldsymbol{E}(\boldsymbol{r}, t) = 4\pi\rho(\boldsymbol{r}, t) , \tag{2.84}$$

$$\operatorname{div} \boldsymbol{H}(\boldsymbol{r}, t) = 0 . \tag{2.85}$$

The appearance of the equations for divergences (2.12) and (2.51) was maintained when describing nonstationary phenomena. Maxwell added a term with a derivative from the strength of the electric field to (2.54):

$$\frac{1}{4\pi}\frac{\partial \boldsymbol{E}}{\partial t} . \tag{2.86}$$

It is called *displacement current*. It is added to the right-hand side of (2.83) summed up with the current $\boldsymbol{j}$ created by charged particles (*conduction current*), and, along with it, takes part in the creation of the magnetic field. It is exactly the presence of displacement current that allows us to ensure the conservation of electric charge: by applying the div operation to both parts of (2.83) and using (2.84), we get the continuity equation (2.47). If the sources of the field ρ and $\boldsymbol{j}$ are specified, Maxwell's equations are linear, which leads to *the principle of superposition of fields* in a vacuum. Electric and magnetic fields may exist separately only if they are static. These cases were considered in Sections 2.1 and 2.2.

18) James Clerk Maxwell (1831–1879) was an outstanding Scottish scientist. He was the author of the theory of electromagnetic phenomena and also made outstanding contributions to statistical physics (such as the Maxwell distribution and Maxwell's "demon") and many other areas.

Example 2.14

Show that Maxwell's equations (2.84) and (2.85) may be considered as universal (the same for all problems) initial condition for (2.83) and (2.82), respectively.

Solution. We apply the operator div $= \nabla\cdot$ to both sides of (2.83) and use the continuity equation (2.47). The result is

$$\frac{\partial}{\partial t}(\operatorname{div}\boldsymbol{E} - 4\pi\rho) = 0\,.$$

Hence it follows,that the difference div $\boldsymbol{E}(\boldsymbol{r}, t) - 4\pi\rho(\boldsymbol{r}, t)$ is not time dependent. If it equal zero at $t = t_0$, we have div $\boldsymbol{E} = 4\pi\rho$ at any moment in time. We obtain an analogous conclusion for div $\boldsymbol{H} = 0$. □

On surfaces where ρ and $\boldsymbol{j}$ have singularities, field strengths must satisfy the boundary conditions following from the integral forms of Maxwell's equations. Because the derivatives $\partial\boldsymbol{E}/\partial t$ and $\partial\boldsymbol{H}/\partial t$ are limited (the speed of any value alternating in time is finite), the boundary conditions, in the general case, are not different from the conditions we had previously, (2.17), (2.18), and (2.57), and have the following forms:

$$\boldsymbol{n}\times(\boldsymbol{E}_2 - \boldsymbol{E}_1) = 0\,, \tag{2.87}$$

$$\boldsymbol{n}\cdot(E_2 - E_1) = 4\pi\sigma\,, \tag{2.88}$$

$$\boldsymbol{n}\times(\boldsymbol{H}_2 - \boldsymbol{H}_1) = \frac{4\pi}{c}\boldsymbol{i}\,, \tag{2.89}$$

$$\boldsymbol{n}\cdot(\boldsymbol{H}_2 - \boldsymbol{H}_1) = 0\,. \tag{2.90}$$

Here, σ and $\boldsymbol{i}$ represent the surface densities of the electric charge and current, respectively.

Electrodynamic problems may be posed in various ways. The simplest is the case when the sources ρ and $\boldsymbol{j}$ of a field may be considered to be specified, that is, the distributions of charges and currents in space are known in advance. In a case like this, the solution of the problem comes down to integrating the system of linear equations (2.82)–(2.85) with boundary conditions (2.87)–(2.90). Yet far from every problem may be posed in this way because, very often, the movement of charged particles is not known in advance.

In cases like that, Maxwell's equations must be supplemented with equations for particle movement, all solved simultaneously. Because, in the general case, the coordinates and speeds of particles depend, in complicated ways, on field strengths, the systems of equations become nonlinear. The principle of the superposition of fields does not hold. This way of posing problems is more precise and far more complicated.

2.3.2
The Systems of Measurement Units of Electric and Magnetic Values

So far, we have used the absolute Gaussian system of units. It is the best suited for representing fundamental laws because Maxwell's equations obviously include one of the world's most important constants, the speed of light in a vacuum c, which is also the terminal velocity of any material bodies and the terminal speed of the propagation of signals and interactions (see Chapters 3 and 4). This facilitates the establishment of a connection between electrodynamics and the special and general theories of relativity.

Besides the Gaussian system, the International System of Units (SI) is also frequently used in applied electrodynamics. In that system, the electromagnetic field in a vacuum is characterized by four vectors, $\boldsymbol{E}$, $\boldsymbol{D}$, $\boldsymbol{B}$, and $\boldsymbol{H}$. In this system, Maxwell's equations acquire the following forms:

$$\operatorname{curl} \boldsymbol{E}(\boldsymbol{r}, t) = -\frac{\partial \boldsymbol{B}(\boldsymbol{r}, t)}{\partial t} , \tag{2.91}$$

$$\operatorname{curl} \boldsymbol{H}(\boldsymbol{r}, t) = \frac{\partial \boldsymbol{D}(\boldsymbol{r}, t)}{\partial t} + \boldsymbol{j}(\boldsymbol{r}, t) , \tag{2.92}$$

$$\operatorname{div} \boldsymbol{D}(\boldsymbol{r}, t) = \rho(\boldsymbol{r}, t) , \tag{2.93}$$

$$\operatorname{div} \boldsymbol{B}(\boldsymbol{r}, t) = 0 . \tag{2.94}$$

Vectors $\boldsymbol{E}$ and $\boldsymbol{D}$ and vectors $\boldsymbol{B}$ and $\boldsymbol{H}$ are interlinked, in pairs, by the following relations:

$$\boldsymbol{D} = \epsilon_0 \boldsymbol{E} , \quad \boldsymbol{B} = \mu_0 \boldsymbol{H} , \tag{2.95}$$

where

$$\epsilon_0 = \frac{10^7}{4\pi c^2}\ \mathrm{F/m} \quad \text{and} \quad \mu_0 = 4\pi \times 10^{-7}\ \mathrm{H/m} \tag{2.96}$$

are the electric and magnetic constants. This values individually have no reasonable physical sense. But the combination $(\epsilon_0 \mu_0)^{-1/2} = c$ is the velocity of light in a vacuum.

The simplicity of the notation used for Maxwell's equations in the SI is deceptive. Besides the already noted faults (four vectors instead of two, the use of the "electrical and magnetic permeability of a vacuum," which is a relict of the times of "luminiferous ether"), there are others, such as the different dimensions of all the four vectors characterizing the same object, the electromagnetic field; the introduction, besides the three basic values (length, time, mass) present in the Gaussian system, of the fourth value with its independent dimension (electric current strength measured in amperes). The value of the latter unit is purely practical.

On the basis of the above reasoning, we will continue using the Gaussian system of units. It is very easy to write the basic formulas of electrodynamics and transfer units from one system to the other with the use of the tables presented in Appendix A and also partially borrowed from the book by Sivukhin (1977, & 85).

2.3.3
An Analysis of the System of Maxwell's Equations

Below, in examples and problems, we will look at the basic properties of the system of Maxwell's equations and the conclusions that follow from them concerning the pecularities of electromagnetic phenomena.

Example 2.15

Use Maxwell's equations (2.82)–(2.85) to generalize the expressions produced earlier, (2.34) and (2.76), for the densities of the energies of electric and magnetic fields for the case of an alternating electromagnetic field. For that purpose, use the principle of the conservation of the full energy of any isolated physical system, keeping in mind that the value $\rho \boldsymbol{v} \cdot \boldsymbol{\mathcal{F}}_{\mathrm{L}} = \boldsymbol{j} \cdot \boldsymbol{\mathcal{F}}_{\mathrm{L}} = \boldsymbol{j} \cdot \boldsymbol{E}$, where $\boldsymbol{\mathcal{F}}_{\mathrm{L}}$ is the Lorentz force (2.43), represents the work an electromagnetic field does on charged particles moving with speed $\boldsymbol{v}$ (within a unit volume per unit time). Find the expression for the density of the energy flux of an electromagnetic field and write the energy balance of the field in the form of an equation of continuity with the source.

Solution. Using (2.82) and (2.83), we produce the bilinear expression

$$\boldsymbol{H} \cdot \operatorname{curl} \boldsymbol{E} - \boldsymbol{E} \cdot \operatorname{curl} \boldsymbol{H} = -\frac{1}{c}\left(\boldsymbol{H} \cdot \frac{\partial \boldsymbol{H}}{\partial t} + \boldsymbol{E} \cdot \frac{\partial \boldsymbol{E}}{\partial t}\right) - \frac{4\pi}{c} \boldsymbol{j} \cdot \boldsymbol{E} .$$

Using the identity (1.87) and dividing both parts of the latter equality by $4\pi/c$, we write it as an equation of continuity with the source:

$$\frac{\partial w}{\partial t} + \operatorname{div} \boldsymbol{\gamma} = -\boldsymbol{j} \cdot \boldsymbol{E} , \tag{2.97}$$

where

$$w = \frac{1}{8\pi}(E^2 + H^2) , \quad \boldsymbol{\gamma} = \frac{c}{4\pi} \boldsymbol{E} \times \boldsymbol{H} . \tag{2.98}$$

For the purpose of the physical interpretation of this relation, we write it in its integral form, integrating both parts over the volume V restricted by the surface S:

$$-\frac{\mathrm{d}}{\mathrm{d}t} \int_V w \mathrm{d}V = \oint_S \boldsymbol{\gamma} \cdot \mathrm{d}\boldsymbol{S} + \int_V \boldsymbol{j} \cdot \boldsymbol{E} \mathrm{d}V . \tag{2.99}$$

The last integral on the right-hand side represents the work done by the electromagnetic field on particles (a magnetic vector does no work) within the volume in question. According to the energy conservation law, while this work is being done, the energy of the field decreases. Therefore, the value w determined by expression (2.98) represents the energy

$$W = \int_V \frac{1}{8\pi}(E^2 + H^2) \mathrm{d}V \tag{2.100}$$

of the field within the volume V decreases. Therefore, the value w determined by expression (2.98) represents *the density of the energy of an electromagnetic field* in the most general case. When $\boldsymbol{j} = 0$, the loss of energy within the volume may only be due to energy leaving the volume through its surface. That is why the vector $\boldsymbol{\gamma}$ (*Poynting vector*)[19] should be identified as[20] *the density of electromagnetic energy flux.* □

Example 2.16

Prove the theorem of the uniqueness of the solutions of Maxwell's equations. Let the sources of the field (the functions $\rho(\boldsymbol{r}, t)$ and $\boldsymbol{j}(\boldsymbol{r}, t)$) be specified. Let the initial conditions, that is, the values $\boldsymbol{E}(\boldsymbol{r}, 0)$ and $\boldsymbol{H}(\boldsymbol{r}, 0)$, within a certain volume V also be specified, and, on its surface, S, the boundary conditions, let the components of one of the vectors ($\boldsymbol{E}_\tau$ or $\boldsymbol{H}_\tau$) be specified for all moments of time. In these conditions, the solutions of Maxwell's equations within the said volume are unique.

Solution. Our proof will be by the rule of contraries with the presumption of the existence of two different solutions. $\boldsymbol{E}_1, \boldsymbol{H}_1$ and $\boldsymbol{E}_2, \boldsymbol{H}_2$. We formulate the differences $\boldsymbol{E} = \boldsymbol{E}_1 - \boldsymbol{E}_2$ and $\boldsymbol{H} = \boldsymbol{H}_1 - \boldsymbol{H}_2$. They satisfy Maxwell's equations (2.82)–(2.85) wherein $\rho = \boldsymbol{j} = 0$. They also satisfy the initial conditions $\boldsymbol{E} = \boldsymbol{H} = 0$ when $t = 0$ within the volume V and satisfy the boundary conditions $\boldsymbol{E}_\tau = 0$ or $\boldsymbol{H}_\tau = 0$ on the surface S. We use (2.99), wherein we state that $\boldsymbol{J} = 0$:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_V \frac{1}{8\pi}(E^2 + H^2)\mathrm{d}V = -\frac{c}{4\pi} \oint_S [\boldsymbol{E} \times \boldsymbol{H}]_n \mathrm{d}S \,.$$

The normal component of the vectorial product $\boldsymbol{E} \times \boldsymbol{H}$ is expressed through the tangent components $\boldsymbol{E}_\tau$ and $\boldsymbol{H}_\tau$ on the surface S and it is reduced to zero owing to the boundary conditions. So, from the previous equality, we get

$$\int_V (E^2 + H^2)\mathrm{d}V = \text{const} \,.$$

However, since $E = H = 0$ when $t = 0$, the constant on the right-hand side of the equality also equals zero. This is possible only if $\boldsymbol{E} = \boldsymbol{H} = 0$ is identical throughout the volume. The theorem has been proven. □

19) The English physicist John Henry Poynting (1852–1914) introduced the concept of electromagnetic energy flux in 1884. In physics of continuous media, the first ideas regrading energy movement and energy flux were developed by the Russian physicist Nickolay Alekseevich Umov (1846–1915) in 1884. That is why the Russian language literature sometimes refers to the vector $\boldsymbol{\gamma}$ as the Umov–Poynting vector

20) The question of the nonambiguity of the definition of the density of energy flux is discussed in Section 4.3. See also Problem 2.117.

Example 2.17

Generalize formulas (2.8) and (2.49) linking field strengths with scalar and vectorial potentials φ and $\boldsymbol{A}$ for the general case of an electromagnetic field depending on time. Indicate the *family* of potentials that describe the specified electromagnetic field $\boldsymbol{E}$, $\boldsymbol{H}$.

Solution. As follows from Maxwell's equation (2.85), vector $\boldsymbol{H}$ is solenoidal in the most general case and, therefore, its representation (2.49) through the vectorial potential also holds for a nonstationary field:

$$\boldsymbol{H}(\boldsymbol{r}, t) = \nabla \times \boldsymbol{A}(\boldsymbol{r}, t) \ . \tag{2.101}$$

However, formula (2.8) for an alternating field is no longer good because curl $\boldsymbol{E} \neq 0$ in the general case. We get generalization (2.8) if we use equality (2.101) in (2.82). Changing the sequence of differential operations, we have curl$(\boldsymbol{E} + \partial \boldsymbol{A}/\partial t) = 0$, wherefrom it follows that the nonrotational vector within the brackets in the latter expression, may be expressed as the gradient of scalar potential $-\nabla\varphi(\boldsymbol{r}, t)$. This produces the relation between the electric field $\boldsymbol{E}$ and electromagnetic potentials

$$\boldsymbol{E}(\boldsymbol{r}, t) = -\nabla\varphi(\boldsymbol{r}, t) - \frac{1}{c}\frac{\partial \boldsymbol{A}(\boldsymbol{r}, t)}{\partial t} \ . \tag{2.102}$$

A large family of possible potentials corresponds to the specified field strengths. Indicating

$$\boldsymbol{A}'(\boldsymbol{r}, t) = \boldsymbol{A}(\boldsymbol{r}, t) + \nabla\chi(\boldsymbol{r}, t) \ , \tag{2.103}$$

where $\chi(\boldsymbol{r}, t)$ is any differentiable scalar function, we get $\boldsymbol{H}' = \nabla \times \boldsymbol{A}' = \nabla \times \boldsymbol{A} = \boldsymbol{H}$. This means that the transformation of vector potential (2.103) does not change the magnetic vector. For the electrical vector to also remain unchanged, it is necessary and sufficient that the scalar potential be transformed simultaneously:

$$\varphi'(\boldsymbol{r}, t) = \varphi(\boldsymbol{r}, t) - \frac{1}{c}\frac{\partial\chi(\boldsymbol{r}, t)}{\partial t} \ . \tag{2.104}$$

The transformation of potentials (2.103) and (2.104) is called *gauge* (gradient) transformation. Therefore, many electromagnetic potentials $\boldsymbol{A}, \varphi$ interlinked by a gauge transformation that may be obtained with various functions $\chi(\boldsymbol{r}, t)$ correspond to the specified electromagnetic field $\boldsymbol{E}$, $\boldsymbol{H}$. □

Because the selection of electromagnetic potentials $\boldsymbol{A}$, φ is nonsynonymous, additional conditions may be imposed on them. The following gauge conditions of potentials are the most widespread:

$$\operatorname{div} \boldsymbol{A} + \frac{1}{c}\frac{\partial\varphi}{\partial t} = 0 \quad \text{(Lorentzian gauge)} \ ; \tag{2.105}$$

$$\operatorname{div} \boldsymbol{A} = 0 \qquad \text{(Coulomb gauge)} \ . \tag{2.106}$$

Example 2.18

Obtain equations for electromagnetic potentials for the cases of Lorentzian and Coulomb gauges from Maxwell's equations. To what extent do conditions (2.105) and (2.106) limit the possibility of gauge transformation of potentials?

Solution. Having inserted field strengths (2.101) and (2.102), expressed through potentials, into (2.83) and (2.84), we get the following system of equations:

$$\Delta \boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} = -\frac{4\pi}{c}\boldsymbol{j} - \nabla\left(\nabla \cdot \boldsymbol{A} + \frac{1}{c}\frac{\partial \varphi}{\partial t}\right),$$

$$\Delta \varphi = -4\pi\rho - \frac{1}{c}\frac{\partial}{\partial t}\nabla \cdot \boldsymbol{A}\,.$$

Using the Lorentzian gauge (2.105), we find the following:

$$\Delta \boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} = -\frac{4\pi}{c}\boldsymbol{j}\,, \tag{2.107}$$

$$\Delta \varphi - \frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} = -4\pi\rho\,. \tag{2.108}$$

Both potentials satisfy the *inhomogeneous d'Alembert*[21] *equations*. If the Coulomb gauge is used,

$$\Delta \boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} = -\frac{4\pi}{c}\boldsymbol{j} - \frac{1}{c}\frac{\partial}{\partial t}\nabla\varphi\,, \tag{2.109}$$

$$\Delta \varphi = -4\pi\rho\,. \tag{2.110}$$

Potentials $\boldsymbol{A}'$ and φ' will satisfy the Lorentzian condition (2.105) as will $\boldsymbol{A}$ and φ if the function χ is the solution of homogeneous d'Alembert's equation:

$$\Delta \chi - \frac{1}{c^2}\frac{\partial^2 \chi}{\partial t^2} = 0\,. \tag{2.111}$$

In the case of the Coulomb calibration, the function χ must satisfy Laplace's equation. Therefore, the gauge conditions limit only the class of functions taking part in the transformation. In static fields, the conditions of both Lorentz and Coulomb coincide. □

21) Jean le Rond d'Alembert (1717–1783) was a French philosopher, mathematician, and physicist, and one of the founders of mathematical physics.

Problems

2.117. May or may not the density of the energy flux of the electromagnetic field from (2.97) be found in a unique way? Indicate other expressions, different from (2.98) but compatible with (2.97) and (2.99).

2.118. A sphere of radius a carrying a charge e uniformly distributed through its volume rotates, with alternating angular velocity $\boldsymbol{\Omega}(t)$ around one of its diameters. Write densities $\rho(\boldsymbol{r}, t)$ and $\boldsymbol{j}(\boldsymbol{r}, t)$ and see if the continuity equation (2.47) is satisfied.

2.119. Do the same for a sphere uniformly charged over its surface.

2.120. An electromagnetic field is expanded in harmonic components (i.e., to Fourier integral (1.250), by time):

$$\boldsymbol{E}(\boldsymbol{r}, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \boldsymbol{E}_\omega(\boldsymbol{r}) e^{-i\omega t} \mathrm{d}\omega\,, \quad \boldsymbol{E}_\omega(\boldsymbol{r}) = \int_{-\infty}^{\infty} \boldsymbol{E}(\boldsymbol{r}, t) e^{i\omega t} \mathrm{d}t \tag{2.112}$$

and, similarly, so is the vector $\boldsymbol{H}$. Write down Maxwell's equations for Fourier harmonics. Show the connection between Fourier harmonics $\boldsymbol{E}_\omega$ and $\boldsymbol{E}_{-\omega}$ as well as between the harmonics of the field strengths and electromagnetic potentials.

2.121. An electromagnetic disturbance of a finite duration crosses a unit area. Express the spectral density of the energy flux of the electromagnetic field $\boldsymbol{\Gamma}_\omega$ through Fourier harmonics $\boldsymbol{E}_\omega$ and $\boldsymbol{H}_\omega$. The value $\boldsymbol{\Gamma}_\omega$ is normalized by the condition

$$\int_0^{\infty} \boldsymbol{\Gamma}_\omega \mathrm{d}\omega = \boldsymbol{\Gamma} = \frac{c}{4\pi}\int_{-\infty}^{\infty} \boldsymbol{E}(t) \times \boldsymbol{H}(t) \mathrm{d}t\,,$$

where $\boldsymbol{\Gamma}$ is the full density of the energy flux crossing the unit area over the whole duration of the electromagnetic disturbance.

Hint: When integrating over time, use expression (1.219) for the delta function.

2.122. An electromagnetic field is expanded in plane waves, that is, in Fourier integral (1.252) by three coordinates:

$$\boldsymbol{E}(\boldsymbol{r}, t) = \frac{1}{(2\pi)^3}\int \boldsymbol{E}_{\boldsymbol{k}}(t) e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \mathrm{d}^3 k\,, \quad \boldsymbol{E}_{\boldsymbol{k}}(t) = \int \boldsymbol{E}(\boldsymbol{r}, t) e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} \mathrm{d}^3 r\,. \tag{2.113}$$

Here, integration is done over the whole infinite coordinate space and over the whole wave vector space $\boldsymbol{k}$. Write Maxwell's equations for Fourier spatial harmonics.

2.123. Show that the Fourier components of the expansion of an nonrotational vector in plane waves are parallel $\boldsymbol{k}$ (longitudinal), whereas the Fourier components of a solenoidal vector are perpendicular $\boldsymbol{k}$ (transverse).

2.124. Expand the potential $\varphi(r)$ and field strength $\boldsymbol{E}(\boldsymbol{r})$ of a point charge e at rest in plane waves.

2.125. An electromagnetic field is expanded in plane monochromatic waves, that is, in the Fourier integral by three coordinates and by time:

$$\boldsymbol{E}(\boldsymbol{r}, t) = \frac{1}{(2\pi)^4} \int \boldsymbol{E}_{k\omega} e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} \mathrm{d}^3 k \mathrm{d}\omega \, ,$$

$$\boldsymbol{E}_{k\omega} = \int \boldsymbol{E}(\boldsymbol{r}, t) e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} \mathrm{d}^3 r \mathrm{d}t \, . \tag{2.114}$$

Write down the Maxwell's equations for Fourier harmonics.

2.126. Write down the d'Alembert equations and the Lorentzian condition for the Fourier components of the potentials $\varphi(\boldsymbol{r}, t)$ and $\boldsymbol{A}(\boldsymbol{r}, t)$. Consider all three versions of Fourier expansions (2.112), (2.113), and (2.114).

2.3.4 Free Electromagnetic Field

The most important property of Maxwell's equations is the possibility of the existence of an electromagnetic field with no sources, that is, when $\rho = \boldsymbol{j} = 0$. In such a case, the system of equations is as follows:

$$\operatorname{curl} \boldsymbol{E}(\boldsymbol{r}, t) = -\frac{1}{c}\frac{\partial \boldsymbol{H}(\boldsymbol{r}, t)}{\partial t} \, , \tag{2.115}$$

$$\operatorname{curl} \boldsymbol{H}(\boldsymbol{r}, t) = \frac{1}{c}\frac{\partial \boldsymbol{E}(\boldsymbol{r}, t)}{\partial t} \, , \tag{2.116}$$

$$\operatorname{div} \boldsymbol{E}(\boldsymbol{r}, t) = 0 \, , \tag{2.117}$$

$$\operatorname{div} \boldsymbol{H}(\boldsymbol{r}, t) = 0 \, . \tag{2.118}$$

It has various nonzero solutions of the wave type, some of which are considered hereafter. This exact property of his equations allowed Maxwell to predict a new important physical effect, the possibility of the existence, in free space, and propagation at the speed of light of electromagnetic waves. This led ultimately to the establishment of the wave nature of light and eventually to the discovery of the connection between electromagnetic and optical phenomena. The theoretical discoveries made by Maxwell and his followers stimulated the experimental investigation of electromagnetic waves. The most important investigations in this regard were the works of the German physicist Heinrich Hertz,[22] who, in 1888, experimentally proved the existence of electromagnetic waves predicted by Maxwell's theory, measured their speed, and studied various wave-related phenomena (interference, diffraction, and polarization). On the basis of the works of Hertz, the

22) Heinrich Rudolf Hertz (1857–1894) was an outstanding German physicist, experimenter, and theorist.

Russian physicist Alexander Popov[23] and the Italian engineer and entrepreneur Guglielmo Marconi[24] achieved the first radio transmission in the 1890s, laying the ground for contemporary radio, television, mobile communications, and so on. In 1909, Marconi received the Nobel Prize in physics for his discoveries.

Example 2.19

From systems (2.115)–(2.118) derive second-order equations satisfied by the vectors $\boldsymbol{E}$ and $\boldsymbol{H}$ separately.

Solution. We apply the curl operation to (2.115) and use (2.116), identity (1.83), and (2.117). This gives us a *wave equation*, or the *homogeneous d'Alembert equation*:

$$\Delta \boldsymbol{E} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} = 0\,. \tag{2.119}$$

In a similar way, we get the same kind of equation for magnetic fields:

$$\Delta \boldsymbol{H} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{H}}{\partial t^2} = 0\,. \tag{2.120}$$

□

Problems

2.127. By changing over to variables $\xi = x - ct$ and $\eta = x + ct$, show that the wave equation (2.119) has a one-dimensional solution of the form

$$\boldsymbol{E}(x,t) = \boldsymbol{E}_1 F(x - ct) + \boldsymbol{E}_2 \Phi(x + ct)\,,$$

where $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$ are constant vectors and F and Φ are arbitrary twice-differentiable functions. What is the physical meaning of this solution? How may it be written so that the surface of constant phase (the plane whereon $\boldsymbol{E}$ preserves its constant meaning) is perpendicular to the specified unit vector $\boldsymbol{n}$?

2.128. Using Maxwell's equations (2.115)–(2.118), show that the plane waves considered in the previous problem satisfy the *transversality condition:*

$$\boldsymbol{n}\cdot\boldsymbol{E} = 0\,, \quad \boldsymbol{n}\cdot\boldsymbol{H} = 0\,. \tag{2.121}$$

Also show that in plane waves propagating *in the same direction*, the electrical and magnetic vectors are mutually perpendicular and interconnected by the following relations:

$$\boldsymbol{H} = \boldsymbol{n}\times\boldsymbol{E}\,, \quad \boldsymbol{E} = \boldsymbol{H}\times\boldsymbol{n}\,, \quad E = H\,. \tag{2.122}$$

23) Alexander Stepanovich Popov (1859–1906) was a Russian physicist and electrical engineer, and the inventor of the radio. He created the first ever radio receiver ("thunderstorm register") and, in 1896, sent the first radiogram of just two words: "Heinrich Hertz."

24) Guglielmo Marchese Marconi (1874–1937) was an Italian physicist, engineer, and entrepreneur who patented his discovery in 1901 and sent and received radio messages across the Atlantic.

Find a connection between the density of electromagnetic energy and the density of energy flux (Poynting vector) in such waves.

2.129*. One may conveniently describe a plane monochromatic wave[25] using the complex functions

$$\boldsymbol{E} = \boldsymbol{E}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}\,, \quad \boldsymbol{H} = \boldsymbol{H}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} \tag{2.123}$$

while understanding the physical meanings of fields as real parts of these functions. Here $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$ are constant vectors (*complex amplitudes*) of value ω (*circular frequency*) and $\boldsymbol{k}$ (*wave vector*) are real constants (when waves are propagating in a vacuum).

1. Show that the frequency and wave vector of a plane monochromatic wave propagating in a vacuum are connected by the dispersion dependence[26] $\omega^2 = c^2 k^2$ and the amplitudes satisfy the conditions $\boldsymbol{H}_0 = \boldsymbol{n} \times \boldsymbol{E}_0$ and $\boldsymbol{E}_0 = -\boldsymbol{n} \times \boldsymbol{H}_0$ where $\boldsymbol{n} = \boldsymbol{k}/k$ is the unit vector determining the direction of motion of the plane on which *phase* $\varphi(\boldsymbol{r}, t) = \boldsymbol{k}\cdot\boldsymbol{r} - \omega t$ has a steady value: $\varphi(\boldsymbol{r}, t) = \text{const}$.
2. Show that the distance between the adjacent maximums of electric or magnetic fields, in direction $\boldsymbol{n}$, equals the wavelength $\lambda = 2\pi/k$.
3. Show that the average energy values over the oscillation period $T = 2\pi/\omega$, quadratic by monochromatic components of fields of the form $A = A_0(\boldsymbol{r})e^{-i\omega t}$ and $B = B_0(\boldsymbol{r})e^{-i\omega t}$, may be found with the use of the following formulas:

$$\overline{A^2} = \overline{[\mathrm{Re}(A)]^2} = \frac{1}{2}|A|^2\,, \quad \overline{AB} = \overline{\mathrm{Re}(A)\mathrm{Re}(B)} = \frac{1}{2}\mathrm{Re}(AB^*)\,. \tag{2.124}$$

4. Show that the energy density averaged over the field's periods of a plane wave and its Poynting vector are connected by the relation $\boldsymbol{\gamma} = cw\boldsymbol{n}$, where $w = |E_0|^2/8\pi$.

2.130*. In the general case, the complex amplitude of a plane monochromatic wave has the form $\boldsymbol{E}_0 = \boldsymbol{E}_{01} + i\boldsymbol{E}_{02}$, where $\boldsymbol{E}_{01}$ and $\boldsymbol{E}_{02}$ are arbitrary real vectors. Show that the complex amplitude can be formed as

$$\boldsymbol{E}_0 = (\boldsymbol{\mathcal{E}}_1 + i\boldsymbol{\mathcal{E}}_2)e^{i\alpha}\,, \quad \boldsymbol{\mathcal{E}}_1\cdot\boldsymbol{\mathcal{E}}_2 = 0\,, \tag{2.125}$$

where $\boldsymbol{\mathcal{E}}_1$ and $\boldsymbol{\mathcal{E}}_2$ are two mutually perpendicular real vectors and α ($-\pi < \alpha \le \pi$) is a certain initial phase.

1. Express the initial phase α and amplitudes $\boldsymbol{\mathcal{E}}_1$ and $\boldsymbol{\mathcal{E}}_2$ through the initially specified vectors $\boldsymbol{E}_{01}$ and $\boldsymbol{E}_{02}$.
2. Show that the observed field (the real part of the complex vector $\boldsymbol{E}$) may be written in the following form:

$$\boldsymbol{E} = \boldsymbol{\mathcal{E}}_1\cos(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t + \alpha) - \boldsymbol{\mathcal{E}}_2\sin(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t + \alpha)\,. \tag{2.126}$$

25) "One-color" wave oscillating with just one frequency.
26) In the general case, such a connection between the frequencies and the wave vectors of Fourier harmonics (2.114) does not exist

3. Show that the end of vector $\boldsymbol{E}$ circumscribes, at the given point in space, either an ellipse (*elliptical polarization*) or a circle (*circular polarization*) or oscillates along a certain rectilinear line (*linear polarization*). In the case of elliptical or circular polarization, two opposed directions of rotation are possible, and in the case of linear polarization,the end of the vector may oscillate in either of the two mutually perpendicular directions. This is why when the direction of wave propagation is specified, there are two different independent types of polarization.
4. How do you determine the direction of the rotation of vector $\boldsymbol{E}$ in relation to the direction of wave propagation?

2.131. Two plane monochromatic linearly polarized waves of the same frequency propagate along the Oz axis. The first wave is polarized along Ox, and its amplitude is a. The second wave is polarized along Oy, and its amplitude is b, and it leads the first wave by phase χ. Consider the polarization of the resulting wave as it depends on a/b.

2.132. In the previous problem, investigate the dependence of polarization on phase shift χ for the case of $a = b$.

2.133. Circularly polarized waves propagate along the Oz axis. Write the complex orts $\boldsymbol{e}^{(\sigma)}$, $\sigma = 1, 2$, corresponding to waves with right and left helicities, normalized by the condition $\boldsymbol{e}^{(\sigma)} \cdot \boldsymbol{e}^{(\sigma')*} = \delta_{\sigma\sigma'}$.

2.134. Two circularly polarized waves $E_{1,2} = E_0 \boldsymbol{e}^{(1,2)} \exp[i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t \pm \alpha)]$ have different initial phases. The orts of polarization $\boldsymbol{e}^{(1,2)}$ were determined in the previous problem. Find the amplitude of the resulting wave and determine its polarization.

2.3.5
The Partial Polarization of Waves

If waves propagating in the specified direction $\boldsymbol{n}$ are generated by many independent sources, then their initial phases are, as a rule, random and, even if the degree of their monochromaticity is high (a narrow range of frequencies $\Delta\omega$), the resulting field $\boldsymbol{E}(t)$ will be a random function of time. Such a field will be just partially polarized. Customarily, it is characterized by the *tensor of polarization*

$$J_{\alpha\beta} = \overline{E_\alpha(t) E^*_\beta(t)}\,, \quad \alpha, \beta = 1, 2\,, \tag{2.127}$$

where the values of the components of the field are taken at one point and averaged out as a result of a rather long observation time. If the sources of the field are stationary, the so averaged products of its components will not depend on time. We will call the sum of the diagonal components of the polarization tensor

$$J_{\alpha\alpha} = I \tag{2.128}$$

the intensity of the field because this value characterizes the density of its energy. In accordance with definition (2.127), $J_{\alpha\beta} = J^*_{\beta\alpha}$, that is, the polarization tensor is a Hermitian one (see Section 1.1).

Example 2.20

Use considerations of symmetry, and write the polarization tensor of fully nonpolarized waves. Also write all the possible polarization tensors for fully polarized monochromatic waves.

Solution. Although there is no polarization, all the directions of the electrical and magnetic vectors in the plane perpendicular to the direction of propagation are equally probable. In a case like this, there is the isotropic tensor

$$J_{\alpha\beta} = \frac{I}{2}\delta_{\alpha\beta}\,, \quad \alpha, \beta = 1, 2\,. \tag{2.129}$$

The following tensor corresponds to a fully polarized monochromatic wave with amplitude (2.125):

$$J_{\alpha\beta} = \frac{1}{2}\begin{pmatrix} \mathcal{E}_1^2 & \pm i\mathcal{E}_1\mathcal{E}_2 \\ \mp i\mathcal{E}_1\mathcal{E}_2 & \mathcal{E}_2^2 \end{pmatrix}\,. \tag{2.130}$$

Its axes are directed along the real vectors $\boldsymbol{\mathcal{E}}_1$ and $\boldsymbol{\mathcal{E}}_2$. A tensor like this, when $\mathcal{E}_1$ and $\mathcal{E}_2$ are arbitrary and there are nondiagonal components of two signs, describes elliptically polarized waves with two possible directions of rotation. Circularly polarized waves are described by tensors

$$J_{\alpha\beta} = \frac{1}{2}I\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}\,, \quad J_{\alpha\beta} = \frac{1}{2}I\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}\,, \tag{2.131}$$

which describe waves of different helicities. Finally, the two directions of linear polarization correspond to tensors

$$J_{\alpha\beta} = \frac{1}{2}I\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\,, \quad J_{\alpha\beta} = \frac{1}{2}I\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\,. \tag{2.132}$$

When waves are fully polarized, their determinants $|J_{\alpha\beta}| = J_{11}J_{22} - |J_{12}|^2$ are reduced to zero. However, when polarization is partial, $|J_{\alpha\beta}| > 0$ and $|J_{\alpha\beta}|$ attains its maximum value of $I^2/4$ when waves are fully nonpolarized. □

Example 2.21

Represent $J_{\alpha\beta}$ of partially polarized waves as the sum of two tensors,

$$J_{\alpha\beta} = J_{\alpha\beta}^{(n)} + J_{\alpha\beta}^{(p)}\,, \tag{1}$$

of which

$$J_{\alpha\beta}^{(n)} = \frac{I_n}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2}$$

describes a fully nonpolarized wave and

$$J_{\alpha\beta}^{(p)} = \frac{1}{2}\begin{pmatrix} B & D \\ D^* & C \end{pmatrix} \tag{3}$$

describes a fully polarized wave. From the conditions

$$B \geq 0\,, \quad C \geq 0\,, \quad BC - |D|^2 = 0 \tag{4}$$

find the intensities I_n and I_p of a fully nonpolarized wave and a fully polarized wave and also the degree of polarization of a wave

$$P = \frac{I_p}{I_p + I_n} \tag{2.133}$$

and the degree of its depolarization

$$\rho = 1 - P\,. \tag{2.134}$$

Solution. From (1), (2), and (3), we find

$$B = 2J_{11} - I_n\,, \quad C = 2J_{22} - I_n\,, \quad D = 2J_{12}\,, \quad D^* = 2J_{21}\,.$$

Because the determinant of tensor (3) equals zero, we get

$$I_n = J_{11} + J_{22} \pm \sqrt{(J_{11} - J_{22})^2 + 4|J_{12}|^2}\,. \tag{2.135}$$

The solution with a minus sign before the square root gives us

$$B = J_{11} - J_{22} + \sqrt{(J_{11} - J_{22})^2 + 4|J_{12}|^2} \geq 0\,,$$

$$C = J_{22} - J_{11} + \sqrt{(J_{11} - J_{22})^2 + 4|J_{12}|^2} \geq 0\,,$$

whereas the solution with a plus sign gives $B \leq 0$ and $C \leq 0$ and must be discarded. The intensity of the fully polarized wave is

$$I_p = \frac{1}{2}(B + C) = \sqrt{(J_{11} - J_{22})^2 + 4|J_{12}|^2} = \sqrt{(J_{11} + J_{22})^2 - 4|J_{\alpha\beta}|}\,. \tag{2.136}$$

The degree of polarization is

$$P = \frac{I_p}{I_p + I_n} = \sqrt{1 - \frac{4|J_{\alpha\beta}|}{(J_{11} + J_{22})^2}}\,. \tag{2.137}$$

The trace $J_{\alpha\alpha} = J_{11} + J_{22}$ and determinant $|J_{\alpha\beta}|$ are values invariant in relation to rotations (see (1.263) and (1.265)), so the division of full strength into I_n and I_p does not depend on the selection of the axes in the plane which is perpendicular to the direction of wave propagation. □

Problems

2.135*. Represent a tensor of polarization as an expansion in its own vectors (see Example 1.1). In what way is this expansion connected with the representation of this tensor considered in Example 2.21?

2.136. A plane monochromatic wave with intensity I propagates along the Oz axis and is elliptically polarized with semiaxes a and b. The major semiaxis a lies at an angle θ to the Ox axis. Write the tensor of polarization and consider the possible particular cases.

2.137. An electromagnetic wave is the superposition of two incoherent "quasi-monochromatic" waves of equal intensities I with approximately equal frequencies and wave vectors. Both of these waves are linearly polarized, the directions of their polarization being specified in the plane perpendicular to their wave vector by orts $e^{(1)}(1, 0)$ and $e^{(2)}(\cos\vartheta, \sin\vartheta)$. Write the tensor of polarization I_{ik} to the resultant of the wave and find its degree of depolarization.

2.138. Solve the previous problem for the case of the waves having different strengths ($I_1 \neq I_2$) and the directions of polarization lying at an angle of $\pi/4$.

2.139*. The tensor of polarization of an electromagnetic wave, which is Hermitian, may be represented as

$$I_{ik} = \frac{1}{2} I \left(\delta_{ik} + \sum_{l=1}^{3} \xi_l \hat{\tau}_{ik}^{(l)} \right) = \frac{1}{2} I \begin{pmatrix} 1 + \xi_3 & \xi_1 - i\xi_2 \\ \xi_1 + i\xi_2 & 1 - \xi_3 \end{pmatrix} ,$$

where I is the full intensity of the wave, ξ_i are the real-valued parameters satisfying the condition $\xi^2 = \xi_1^2 + \xi_2^2 + \xi_3^2 \leq 1$ (Stokes' s parameters), and $\hat{\tau}^{(l)}$, $l = 1, 2, 3$, are matrices

$$\hat{\tau}^{(1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \hat{\tau}^{(2)} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \hat{\tau}^{(3)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

Find the physical meaning of the parameters ξ_i. For that purpose, express the degree of depolarization ρ of the wave through ξ_i and find the polarizations of the two principal waves that the partially polarized wave break into in the following three cases: (i) $\xi_1 \neq 0$, $\xi_2 = \xi_3 = 0$; (ii) $\xi_2 \neq 0$, $\xi_1 = \xi_3 = 0$; (iii) $\xi_3 \neq 0$, $\xi_1 = \xi_2 = 0$.

2.140. We will call the superposition of plane monochromatic waves in free space a wave packet:

$$\Psi(\boldsymbol{r}, t) = \int \psi(\boldsymbol{k}) \exp[i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega(k)t)] \mathrm{d}^3 k ,$$

where $\Psi(\boldsymbol{r}, t)$ is any Cartesian component of the vector $\boldsymbol{E}$ or $\boldsymbol{H}$. (i) Find the condition making Ψ the solution of the homogeneous d'Alembert wave equation regardless of the form of the amplitude function $\psi(\boldsymbol{k})$. (ii) Create a one-dimensional wave packet for time $t = 0$. Use Gauss' s distribution $\psi(k) = a_0 \exp[-(k - k_0)^2/\Delta k^2]$, $k = k_x$, where a_0, k_0, and Δk are constants, as the amplitude function. Find the connection between the width Δx of the wave packet and Δk, and also for its width in the space of wave numbers.

2.141. At moment $t = 0$, a one-dimensional wave packet (see the previous problem) $\Psi(x)$ has the form

$$\text{(i)}\quad \Psi(x) = A_0 e^{-\alpha|x|/2}\,; \quad \text{(ii)}\quad \Psi(x) = \begin{cases} A_0 & \text{when } |x| < a\,, \\ 0 & \text{when } |x| > a\,, \end{cases}$$

where A_0 and a are constants. Find the product $\overline{\Delta x^2} \cdot \overline{\Delta k^2}$, where $\overline{\Delta x^2} = \bar{x^2} - \bar{x}^2$, and similarly for $k = k_x$. Averaging should be done by the distribution of the wave intensity in the x- and k-spaces, that is, according to the formulas

$$\bar{x} = \int_{-\infty}^{\infty} x|\Psi(x)|^2 \mathrm{d}x \left[\int_{-\infty}^{\infty} |\Psi(x)|^2 \mathrm{d}x\right]^{-1},$$

$$\bar{k} = \int_{-\infty}^{\infty} k|\psi(k)|^2 \mathrm{d}k \left[\int_{-\infty}^{\infty} |\psi(k)|^2 \mathrm{d}k\right]^{-1},$$

and so on.

2.142. The Ψ wave packet is formed by the superposition of plane monochromatic waves with different frequencies in free space. The amplitude function has the form of the Gaussian distribution $\psi(\omega) = a_0 \exp[-(\omega - \omega_0)^2/\Delta\omega^2]$, where a_0, ω_0, and $\Delta\omega$ are constants. Find the dependency of the amplitude of the packet on time at the point $x = 0$. Find the connection between the duration of the wave impulse Δt and the interval of frequencies $\Delta\omega$.

2.143. The $\Psi(x, t)$ wave packet is formed by the superposition of plane monochromatic waves with different frequencies in free space. The form of the packet, when $x = 0$ and $\Psi(0, t) = u(t)$, is known. Find the amplitude function $\psi(k)$.

2.144. A certain object illuminated by light with wavelength λ is observed through a microscope. Find $\Delta x_{\min}$, the minimum possible size of that object allowed by the condition $\Delta x \Delta k \geq 1$.

2.145. The position of a certain object is being determined by radar. What is the ultimate accuracy of that determination, the distance to the object being l and the wavelength being λ?

2.3.6
Analytical Signal

The Fourier expansion of the type (2.112) of a *real function* $U'(t)$ may be written as an integral only for *positive* frequencies:

$$U'(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(\omega)e^{-i\omega t}\mathrm{d}\omega = \frac{1}{2\pi}\int_{0}^{\infty} a(\omega)\cos[\varphi(\omega) - \omega t]\mathrm{d}\omega \;, \qquad (2.138)$$

where $u(\omega) = a(\omega)\exp[i\varphi(\omega)]$, whereas, owing to the condition $u(-\omega) = u^*(\omega)$, $a(\omega)$ and $\varphi(\omega)$ are the real even functions of the frequency. Generalize the latter integral in (2.138) and consider the complex function

$$U(t) = \frac{1}{2\pi}\int_{0}^{\infty} a(\omega)\exp[i(\varphi(\omega) - \omega t)]\mathrm{d}\omega = U'(t) + i\,U''(t)\;. \qquad (2.139)$$

The imaginary part

$$U''(t) = \frac{1}{2\pi}\int_{0}^{\infty} a(\omega)\sin[i(\varphi(\omega) - i\omega t)]\mathrm{d}\omega \qquad (2.140)$$

is determined by the real part $U'(t)$ synonymously because it is derived from the latter by substituting $\varphi(\omega) - \pi/2$ for phase $\varphi(\omega)$ of every Fourier harmonic.

The value $U(t)$ is called the*analytical signal*. It is often used in the theory of wave fields, the theory of oscillations, radio engineering, and so on. The characteristic property of the function $U(t)$ is that it contains Fourier harmonics only with positive frequencies. This is why if $U(t)$ is restricted and all t are real, that is, the integral on ω converges, it will remain limited also when complex values $t = t' + it''$ lie in the lower semiplane ($t'' < 0$). This is because when $t' + it''$ is substituted for t into (2.139), an added factor $\exp(\omega t'')$ appears under the integral. If $t'' < 0$, this factor will even increase the convergence of the integral by ω. That is, the function $U(t)$, considered when t are complex, *is analytical (has no singularity points)* in the lower semiplane.

Problems

2.146*. Using the analyticity of the function $U(t)$ in the lower semiplane of complex t, find an integral connection[27] between its real and imaginary parts:

$$U'(t) = -\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{U''(t')}{t'-t}\mathrm{d}t'\;, \quad U''(t) = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{U'(t')}{t'-t}\mathrm{d}t'\;. \qquad (2.141)$$

27) In mathematics, formulas (2.141) are named Hilbert transforms. They play an important role in physics and are mentioned in the physics literature as dispersion relations. David Hilbert (1862–1943) was an outstanding German mathematician who believed in the unlimited potential of the human mind and in the unity of mathematics and natural history.

Here, the symbol $\mathcal{P}$ refers to the principal value of the integral (without this indication, the calculation of the integrals is impossible because of there being poles at the point $t' = t$).

2.147. Express the energy Γ of a plane nonmonochromatic wave that has crossed a unit area over its finite lifetime through an analytical signal, equating $E(t)$ with $U'(t)$. Write the spectral density of the energy Γ_ω ($\Gamma = \int_0^\infty \Gamma_\omega \mathrm{d}\omega$).

2.148. The quasi-monochromatic signal $U'(t) = A(t)\cos[\Phi(t) - \omega_0 t]$ is a cosine wave whose amplitude $A(t)$ and an added phase $\Phi(t)$ are alternating slowly. Express A and Φ through an analytical signal $U(t)$ whose real part is $U'(t)$.

2.149. An attenuating source of radiation emits a signal $U'(t) = A_0 \Theta(t) e^{-\gamma t/2} \sin \omega_0 t$, where $\Theta(t)$ is a staircase function, γ is the attenuation constant, and $A_0 = \text{const}$. What conditions will make the signal quasi-monochromatic? Find the energy distribution over frequencies using the concept of the analytical signal. Evaluate the frequency range of this signal and the product $\Delta\omega\Delta t$.

2.150. Show that an electromagnetic field in free space may be described by just one vector potential $\boldsymbol{A}(\boldsymbol{r}, t)$ under the condition that $\varphi = 0$.

2.3.7
The Hamiltonian Form of Equations for a Free Electromagnetic Field

A free electromagnetic field is a linear oscillating system with an infinite number of degrees of freedom. Oscillations occur at every point of space, and oscillations occurring at adjacent points are interconnected through a wave equation that includes derivatives by coordinates. In mechanics, a linear oscillating system may be described with the use of normal (or principal) coordinates, wherein it becomes equivalent to a set of independent harmonic oscillators. This kind of representation, which is possible for an electromagnetic field, is rather convenient to use while solving certain classical electrodynamic problems (Ginzburg, 1987) and is absolutely necessary when regarding the field as a quantum system.

We will assume that $\varphi = 0$ (see Problem 2.150) and describe a field with the vector potential $\boldsymbol{A}(\boldsymbol{r}, t)$ that satisfies the equations

$$\Delta \boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} = 0\,, \quad \operatorname{div} \boldsymbol{A} = 0\,. \tag{2.142}$$

Then we will impose periodicity conditions on $\boldsymbol{A}$ as its boundary conditions:

$$\boldsymbol{A}(x, y, x, t) = \boldsymbol{A}(x + L, y, z, t) = \boldsymbol{A}(x, y + L, z, t) = \boldsymbol{A}(x, y, z + L, t)\,. \tag{2.143}$$

The whole space has become divided into regions of the size $V = L^3$ where the field behaves in this way. Physical considerations make it clear that if a volume is large, the physical processes inside it weakly depend on the conditions existing at its bounds.

We expand the vector potential into plane waves with a certain wave vector $\boldsymbol{k}$ and polarization $\sigma = 1, 2$:

$$A(\boldsymbol{r}, t) = \sum_{k\sigma} q_{k\sigma}(t) A_{k\sigma}(\boldsymbol{r}) , \quad \text{where} \quad A_{k\sigma}(\boldsymbol{r}) = \boldsymbol{e}_{k\sigma} A^0 e^{i\boldsymbol{k}\cdot\boldsymbol{r}} , \tag{2.144}$$

where A^0 is the normalization factor that will be selected later and $\boldsymbol{e}_{k\sigma}$ are the unit vectors of the polarization of plane waves, satisfying transverse $\boldsymbol{k} \cdot \boldsymbol{e}_{k\sigma} = 0$ and mutual orthogonality conditions $\boldsymbol{e}^*_{k\sigma'} \cdot \boldsymbol{e}_{k\sigma} = \delta_{\sigma\sigma'}$. From periodicity conditions (2.143), we get $\exp(ik_x L) = \exp(ik_y L) = \exp(ik_z L) = 1$, that is,

$$k_x = \frac{2\pi}{L} n_1 , \quad k_y = \frac{2\pi}{L} n_2 , \quad k_z = \frac{2\pi}{L} n_3 , \tag{2.145}$$

where $n_1, n_2, n_3 = 0, \pm 1, \pm 2, \ldots$ are independent and take on only numeric values. That is, the amount in (2.144) must be extended to all possible values of n_1, n_2, and n_3, which form an infinite but countable multitude. The basic functions $A_{k\sigma}(\boldsymbol{r})$ form a complete orthogonal system of functions that satisfy the normality condition

$$\int_V A^*_{k'\sigma'}(\boldsymbol{r}) \cdot A_{k\sigma}(\boldsymbol{r}) \mathrm{d}V = (A^0)^2 V \delta_{kk'} \delta_{\sigma\sigma'} . \tag{2.146}$$

The latter equality is easily verified with the use of the explicit form (2.144) of basic functions. From the orthogonality of functions (2.146) and the wave equation (2.142), we get a system of equations for Fourier amplitudes:

$$\ddot{q}_{k\sigma} + \omega_k^2 q_{k\sigma} = 0 , \quad \text{where} \quad \omega_k = ck \geq 0 . \tag{2.147}$$

They satisfy the equations of *independent* harmonic oscillators and are *principal* (but complex) coordinates of an electromagnetic field. The common solution of (2.147) has the following form:

$$q_{k\sigma}(t) = b_{k\sigma} e^{-i\omega_k t} + c_{k\sigma} e^{i\omega_k t} , \tag{2.148}$$

where b and c are complex constants. From the reality of $A(\boldsymbol{r}, t)$, that is, from the equality $A^*(\boldsymbol{r}, t) = A(\boldsymbol{r}, t)$, by equating coefficients of exponents with the same attributes, we obtain the following:

$$\boldsymbol{e}_{k\sigma} b_{k\sigma} = \boldsymbol{e}^*_{-k\sigma} c^*_{-k\sigma} .$$

If the orts of polarization are selected so that the following conditions are satisfied

$$\boldsymbol{e}_{k\sigma} = \boldsymbol{e}^*_{-k\sigma} , \tag{2.149}$$

then we also get $b_{k\sigma} = c^*_{-k\sigma}$ and $c_{k\sigma} = b^*_{-k\sigma}$ and for the complex Fourier amplitude

$$q_{k\sigma}(t) = b_{k\sigma} e^{-i\omega_k t} + b^*_{-k\sigma} e^{i\omega_k t} . \tag{2.150}$$

These relations allow us to write (2.144) as

$$A(\boldsymbol{r},t) = \sum_{k\sigma} \left[b_{k\sigma}(t) A_{k\sigma}(\boldsymbol{r}) + b^*_{k\sigma}(t) A^*_{k\sigma}(\boldsymbol{r}) \right], \quad \text{where} \quad b_{k\sigma}(t) = b_{k\sigma} e^{-i\omega_k t} \tag{2.151}$$

satisfies the equation of the motion of the oscillator (2.147). Note that here the form of the expansion in plane waves is different from a similar expansion in Problem 2.122 and has certain methodological advantages.

Example 2.22

Formulate the Hamiltonian function of a free electromagnetic field in principal coordinates and write the field equations in the Hamiltonian form.

Solution. We find the field energy

$$W = \frac{1}{8\pi} \int_V \left(E^2 + H^2 \right) \mathrm{d}V \tag{2.152}$$

in the principal area V. For that purpose, we find field strengths using (2.144):

$$\boldsymbol{E} = -\frac{1}{c}\dot{\boldsymbol{A}} = -\frac{1}{c}\sum_{k\sigma} \dot{q}_{k\sigma} \boldsymbol{A}_{k\sigma} = -\frac{1}{c}\sum_{k\sigma} \dot{q}^*_{k\sigma} \boldsymbol{A}^*_{k\sigma} , \tag{2.153}$$

$$\boldsymbol{H} = \operatorname{curl} \boldsymbol{A} = i \sum_{k\sigma} q_{k\sigma} [\boldsymbol{k} \times \boldsymbol{A}_{k\sigma}] = -i \sum_{k\sigma} q^*_{k\sigma} [\boldsymbol{k} \times \boldsymbol{A}^*_{k\sigma}] . \tag{2.154}$$

Using the orthogonality condition (2.146), we get the following from (2.152):

$$W = \frac{(A^0)^2 V}{8\pi c^2} \sum_{k\sigma} \left(\dot{q}^*_{k\sigma}(t) \dot{q}_{k\sigma}(t) + c^2 k^2 q^*_{k\sigma}(t) q_{k\sigma}(t) \right) . \tag{2.155}$$

If, we insert $q_{k\sigma}(t) = b_{k\sigma}(t) + b^*_{-k\sigma}(t)$, based on (2.150), we get

$$W = \frac{(A^0)^2 V}{2\pi c^2} \sum_{k\sigma} \omega_k^2 b^*_{k\sigma}(t) b_{k\sigma}(t) . \tag{2.156}$$

The full energy of the field in volume V has been expressed as the sum of the energies of individual eigenmodes expressed through complex principal coordinates. If we insert real variables $Q_{k\sigma}$ and $P_{k\sigma}$,

$$b_{k\sigma} = \frac{1}{2}\left(Q_{k\sigma} + \frac{i P_{k\sigma}}{\omega_k} \right) . \tag{2.157}$$

As follows from their definition (2.157), these variables also satisfy the equation of motion of a harmonic oscillator (2.147). The energy of the system expressed through the generalized coordinates and momenta is called the Hamiltonian function of the system. If we denote the Hamiltonian function through $\mathcal{H}$ and insert complex coordinates expressed through real variables (2.157) into (2.156), we obtain

$$\mathcal{H} = \frac{(A^0)^2 V}{8\pi} \sum_{k\sigma} \left(P_{k\sigma}^2 + \omega_k^2 Q_{k\sigma}^2\right) . \tag{2.158}$$

At this point, it is convenient to select the normalization constant A^0 so that the effective masses of the field oscillators become unity:

$$A^0 = \sqrt{\frac{4\pi c^2}{V}} . \tag{2.159}$$

Finally, the Hamiltonian function assumes its canonical form:

$$\mathcal{H} = \sum_{k\sigma} \mathcal{H}_{k\sigma} , \quad \text{where} \quad \mathcal{H}_{k\sigma} = \frac{1}{2}\left(P_{k\sigma}^2 + \omega_k^2 Q_{k\sigma}^2\right) . \tag{2.160}$$

Canonical equations or Hamiltonian equations are derived from the Hamiltonian function (2.160) in the usual way,

$$\dot{Q}_{k\sigma} = \frac{\partial \mathcal{H}}{\partial P_{k\sigma}} = P_{k\sigma} , \quad \dot{P}_{k\sigma} = -\frac{\partial \mathcal{H}}{\partial Q_{k\sigma}} = -\omega_k^2 Q_{k\sigma} , \tag{2.161}$$

and lead to the correct equation of motion:

$$\ddot{Q}_{k\sigma} + \omega_k^2 Q_{k\sigma} = 0 . \tag{2.162}$$

This proves that Q and P are canonical variables for field oscillators. □

Depending on the normalization selected, coordinate basic functions assume the form

$$\boldsymbol{A}_{k\sigma}(\boldsymbol{r}) = \boldsymbol{e}_{k\sigma} \sqrt{\frac{4\pi c^2}{V}} e^{i\boldsymbol{k}\cdot\boldsymbol{r}} . \tag{2.163}$$

Problems

2.151•. If the size L of the principal region of a field is large compared with the wavelengths of the oscillations in question, then the wave vector and frequency alternate in a quasi-continuous way and a large number of field oscillators fall in a small ($\Delta\omega \ll \omega$) frequency interval. Having in mind that two independent polarizations occur for every value $\boldsymbol{k}$, show that the number of eigenmodes of the field falling in a small interval of frequencies $\mathrm{d}\omega$ or wave vectors $\mathrm{d}k$ is written as

$$\mathrm{d}N = \frac{2V}{(2\pi)^3} k^2 \mathrm{d}k \mathrm{d}\Omega_k = \frac{2V}{(2\pi c)^3} \omega^2 \mathrm{d}\omega \mathrm{d}\Omega_k , \tag{2.164}$$

where $\mathrm{d}\Omega_k$ is the spatial angle at which the wave vector is oriented inside.

2.152•. Two unit vectors of polarization, complex in the general case, satisfy the conditions of orthogonality $\boldsymbol{e}^{(s')*}\cdot\boldsymbol{e}^{(s)} = \delta_{ss'}$ and transverse $\boldsymbol{n}\cdot\boldsymbol{e}^{(s)} = 0$, where $s, s' = 1, 2$, $\boldsymbol{n}$ is a unit vector in the direction of wave propagation. Prove the relations

$$e_\alpha^{(s)} e_\beta^{(s)*} = \delta_{\alpha\beta} - n_\alpha n_\beta \ , \quad (\boldsymbol{a}\cdot\boldsymbol{e}^{(s)})(\boldsymbol{b}\cdot\boldsymbol{e}^{(s)*}) = [\boldsymbol{a}\times\boldsymbol{n}]\cdot[\boldsymbol{b}\times\boldsymbol{n}] \ , \tag{2.165}$$

where summing is done by the reiterating index s.

Suggested literature:
Tamm (1976); Landau and Lifshitz (1975); Sivukhin (1980); Bredov *et al.* (2003); Medvedev (1977); Jackson (1999); Ginzburg (1987); Sommerfeld (1954); Frenkel (1926); Panofsky and Phillips (1963).

2.4
Answers and Solutions

2.1

$$\varphi_1 = -2\pi\rho z^2 \ , \quad \boldsymbol{E}_1 = 4\pi\rho z \boldsymbol{e}_z \quad (|z| < \frac{a}{2}) \ ,$$
$$\varphi_2 = -\pi\rho a(2|z| - \frac{a}{2}) \ , \quad \boldsymbol{E}_2 = 2\pi\rho a z \boldsymbol{e}_z/|z| \quad (|z| > \frac{a}{2}) \ .$$

The Oz axis is directed along the normal to the surface of a plate. When we changed to the charged surface, $a \to 0$, the product $\rho a = \sigma$ remains fixed. The inner region disappears, and the boundary condition (2.18) is satisfied on the plane $z = 0$

2.2

$$\varphi(x, y, z) = \frac{4\pi\rho_0}{\alpha^2 + \beta^2 + \gamma^2} \cos\alpha x \cos\beta y \cos\gamma z \ .$$

2.3*

$$\begin{aligned}\varphi_1 &= \frac{4\pi\rho_0}{k^2}\left[\cos\gamma z + \frac{\gamma\cosh\lambda z}{\lambda(\cosh(\pi\lambda/2\gamma) + \sinh(\pi\lambda/2\gamma))}\right]\cos\alpha x\cos\beta y \ , \\ &\quad |z| < z_0 \ ; \\ \varphi_2 &= \frac{4\pi\rho_0\gamma}{k^2\lambda}\frac{\exp[\lambda(\pi/1\gamma - |z|)]}{1 + \tanh(\pi\lambda/2\gamma)}\cos\alpha x\cos\beta y \ , \quad |z| > z_0 \ .\end{aligned} \tag{2.166}$$

Here $k^2 = \alpha^2 + \beta^2 + \gamma^2$, $\lambda = \sqrt{\alpha^2 + \beta^2}$. During the limiting transition to the charged plane, we have $\varphi = (2\pi\sigma(x, y)/\lambda)\exp(-\lambda|z|)$ throughout the space where $\sigma(x, y) = \sigma_0\cos\alpha x\cos\beta y$ and $\sigma_0 = \lim_{z_0\to 0}(4z_0\rho_0/\pi)$. The exponential decrease of the charge along the Oz axis is due to the plane having oppositely charged areas.

2.4 The simplest method of solving the problem is with the use of the Gaussian electrostatic theorem. If one solves it by the method of integrating Poisson's equation, one should write Laplace's operator in cylindrical coordinates and use the fact that, owing to the symmetry of the system, the potential depends only on r.

In the case of volumetric charge distribution,

$$\varphi_1 = \kappa\left(1 - \frac{r^2}{R^2}\right) , \quad E_1 = \frac{2\kappa r}{R^2} \quad (r \le R) ;$$
$$\varphi_2 = -2\kappa \ln \frac{r}{R} , \quad E_2 = \frac{2\kappa}{r} \quad (r \ge R) .$$

In the case of surface charge distribution, $\varphi_1 = 0$, $\varphi_2 = -2\kappa \ln(r/R)$.

2.5 $\varphi = -2\kappa \ln(r/R)$ and $E = 2\kappa/r$, where R is an arbitrary constant.

2.6* $\varphi(r, z) = 2\kappa_0 K_0(\gamma r) \cos \gamma z$, where $K_0(\gamma r)$ is the modified Bessel function (1.163). The arbitrary constant is selected so that $\varphi|_{r\to\infty} \to 0$. When $\gamma r \ll 1$, $\varphi(r, z) \approx -2\kappa(z) \ln \gamma r$, that is, the potential found changes to the potential of a charged filament with the local value of linear charge density (see Problem 2.5).

2.7 $\varphi(x, y, z) = -\frac{q}{2a} \ln \left| \frac{z-a+\sqrt{(z-a^2)+x^2+y^2}}{z+a+\sqrt{(z+a)^2+x^2+y^2}} \right| .$

2.8 We introduce the following notation:

$$z_1 = z + a , \quad z_2 = z - a , \quad r_{1,2} = \sqrt{x^2 + y^2 + z_{1,2}^2} , \quad C = \frac{z_1 + r_1}{z_2 + r_2} .$$

As follows from the result in the previous problem,

$$r_1 + r_2 = 2a\frac{C+1}{C-1} = \text{const} . \tag{1}$$

(one should keep in mind that $z_1 - z_2 = 2a$).

Equality (1) shows that equipotential surfaces are ellipsoids of rotation whose focuses coincide with the ends of the fragment.

2.9

$$\varphi_1(r) = \frac{q}{R}\left(\frac{3}{2} - \frac{r^2}{2R^2}\right) , \quad \boldsymbol{E}_1 = \frac{q\boldsymbol{r}}{R^3} , \quad (r \le R) ;$$
$$\varphi_2(r) = \frac{q}{r} , \quad \boldsymbol{E}_2 = \frac{q\boldsymbol{r}}{r^3} , \quad (r \ge R) .$$

2.10

$$\varphi_1(r) = \frac{q}{R} , \quad \boldsymbol{E}_1 = 0 , \quad (r < R) ; \quad \varphi_2(r) = \frac{q}{r} , \quad \boldsymbol{E}_2 = \frac{q\boldsymbol{r}}{r^3} , \quad (r > R) .$$

2.11 The electric field in the cavity is homogeneous:

$$\boldsymbol{E} = \frac{4}{3}\pi\rho\boldsymbol{r} - \frac{4}{3}\pi\rho(\boldsymbol{r} - \boldsymbol{a}) = \frac{4}{3}\pi\rho\boldsymbol{a} \,.$$

2.12 $q = 4\pi\alpha(R_2 - R_1)$;

$$E_1 = 0\,, \quad \varphi_1 = \frac{q}{R_2 - R_1}\ln\frac{R_2}{R_1} \quad \text{when} \quad r \le R_1\,;$$

$$E_2 = \frac{q(r - R_1)}{(R_2 - R_1)r^2}\,,$$
$$\varphi_2 = \frac{q}{R_2 - R_1}\left(1 - \ln\frac{r}{R_2} - \frac{R_1}{r}\right) \quad \text{when} \quad R_1 \le r \le R_2\,;$$

$$E_3 = \frac{q}{r^2}\,; \quad \varphi_3 = \frac{q}{r} \quad \text{when} \quad r \ge R_2\,.$$

When $R_2 \to R_1 \equiv R$ and the value of the charge q is fixed, we get the field of a sphere uniformly charged over its surface.

2.13

$$\varphi(r) = \frac{4\pi}{r}\int_0^r \rho(r')r'^2\mathrm{d}r' + 4\pi\int_r^\infty \rho(r')r'\mathrm{d}r'\,;$$

$$\boldsymbol{E}(r) = \frac{4\pi\boldsymbol{r}}{r^3}\int_0^r \rho(r')r'^2\mathrm{d}r'\,.$$

2.15 The field of the electron cloud in an atom:

$$\varphi_\mathrm{e}(r) = -\frac{e_0}{r}\left(1 - e^{-2r/a}\right) + \frac{e_0}{a}e^{-2r/a};$$

$$E_{er} = -\frac{e_0}{r^2}\left[1 - \left(\frac{2r}{a} + 1\right)e^{-2r/a}\right] + \frac{2e_0}{a^2}e^{-2r/a}\,.$$

The potential of the complete electric field in the atom: $\varphi(r) = \varphi_\mathrm{e}(r) + e_0/r$.

2.16 The field strength is maximum on the surface of the nucleus:

$$E_\mathrm{max} = \frac{Ze_0}{R^2} = 6.4 \times 10^{18}\frac{Z}{A^{2/3}}\,\mathrm{V/cm}\,,$$
$$\frac{E_\mathrm{B}}{E_\mathrm{max}} = \left(\frac{RZ}{a}\right)^2 \approx 3 \times 10^{-10}A^{2/3}Z^2\,.$$

2.18

$$\varphi = \frac{2q}{R^2}\left(\sqrt{R^2+z^2} - |z|\right) ;$$

$$E_x = E_y = 0 , \quad E_z = \frac{2q}{R^2}\left(\frac{z}{|z|} - \frac{z}{\sqrt{R^2+z^2}}\right) ,$$

where z is the coordinate of the observation point, counted from the surface of the disk.

2.19 Owing to the symmetry of the system, the potential φ will not depend on the azimuthal angle α, which allows us to draw the surface xz through the point of observation without violating commonality. Then (Figure 2.14)

$$r_{12} = \sqrt{r^2 + R^2 - 2rR\sin\vartheta\cos\alpha'}$$

and

$$\varphi(r,\vartheta) = 2\kappa R \int_0^{\pi} \frac{\mathrm{d}\alpha'}{\sqrt{r^2+R^2-2rR\sin\vartheta\cos\alpha'}} ,$$

where $\kappa = q/2\pi R$.

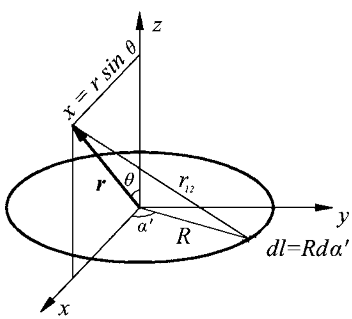

Figure 2.14 Finding the potential of a thin charged ring.

Having substituted $\alpha' = \pi - 2\beta$ and introduced the notation

$$k^2 = \frac{4rR\sin\vartheta}{\sqrt{r^2+R^2+2rR\sin\vartheta}} ,$$

we get

$$\varphi(r,\vartheta) = \frac{4\kappa R}{\sqrt{r^2+R^2+2rR\sin\vartheta}} \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\beta}{\sqrt{1-k^2\sin^2\beta}}$$

$$= \frac{2k\kappa}{\sqrt{rR\sin\vartheta}} K(k) .$$

2.20 (i) $\varphi = q/\sqrt{R^2 + z^2}$, where z is the distance from the plane of the ring to the point of observation; (ii) $\varphi = q/r$; (iii) having expressed the distance from the observation point to the filament of the ring through r', we get

$$1 - k^2 \approx \frac{r'^2}{4R^2}\,, \quad K(k) = \ln(8R/r')\,, \quad \text{and} \quad \varphi(\boldsymbol{r}) = -2\kappa \ln r' + \text{const}\,,$$

where $r' \ll R$, as it should be in the case of a linear charge.

2.21

$$\varphi_d = \frac{p\cos\theta}{r^2}\,, \quad E_{dr} = \frac{2p\cos\theta}{r^3}\,, \quad E_{d\vartheta} = \frac{p\sin\theta}{r^3}\,, \quad E_{d\alpha} = 0\,;$$

$$\varphi_q = \frac{Q}{4r^3}(3\cos^2\theta - 1) = \frac{QP_2(\cos\theta)}{2r^3}\,,$$

$$E_{qr} = \frac{3QP_2(\cos\theta)}{2r^4}\,, \quad E_{q\vartheta} = \frac{3Q\cos\theta\sin\theta}{2r^4}\,, \quad E_{q\alpha} = 0\,.$$

Here, $P_2(\cos\theta)$ is Legendre's polynomial.

2.22*

$$\varphi(\boldsymbol{r}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\sqrt{\frac{4\pi}{2l+1}}\,\frac{Q_{lm}Y_{lm}(\theta,\alpha)}{r^{l+1}}\,,$$

where Q_{lm} is a multipole moment of order l, m:

$$Q_{lm} = \sqrt{\frac{4\pi}{2l+1}}\int \rho(\boldsymbol{r}')r'^l\,Y^*_{lm}(\theta',\alpha')\mathrm{d}V'\,.$$

Here, integration is done over the whole volume occupied by the system of charges.

2.23* Use a spherical surface of radius r to divide the area of integration by r' in (2.9) into the inner and outer regions. In the inner region, use expansion (1.182), and substitute r' for a to get the same sum as in the previous problem, except that the multipole moments $Q_{lm}(r)$ become functions of r. In the outer region, in expansion (1.182), change $r \to a$, $r \to r'$. In the end, we get

$$\varphi(\boldsymbol{r}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\sqrt{\frac{4\pi}{2l+1}}\left(\frac{Q_{lm}(r)}{r^{l+1}} + r^l Q'_{lm}(r)\right)Y_{lm}(\theta,\alpha)\,,$$

where

$$Q'_{lm}(r) = \sqrt{\frac{4\pi}{2l+1}}\int \frac{\rho(\boldsymbol{r}')}{r'^{l+1}}\,Y^*_{lm}(\theta',\alpha')\mathrm{d}V'\,.$$

In the latter interval, integration over r' is done within the range from r to ∞.

2.24 If a positively charged semiring occupies region $x > 0$ in the xy plane, then, when $x,\ y \ll (R^2 + z^2)/R$, by expanding the integrand in the integral $\int(\kappa/r_{12})\mathrm{d}l$ in a series, we get

$$\varphi = \frac{4qRx}{\pi(R^2 + z^2)^{3/2}} ,$$

which gives

$$E_x = -\frac{4qR}{\pi(R^2 + z^2)^{3/2}} , \quad E_y = 0 , \quad E_z = \frac{12qRxz}{\pi(R^2 + z^2)^{5/2}} .$$

When $z \gg R$, we get the field of an electric dipole whose moment is directed along the Ox axis and is equal to $4qR/\pi$.

2.25

$$\varphi_1 = \frac{4\pi}{3}\sigma_0 r \cos\vartheta \quad (r \le R) ,$$

$$\varphi_2 = \frac{4\pi}{3}\frac{\sigma_0 R^3}{r^2}\cos\vartheta \quad (r \ge R) .$$

Inside the sphere there is a homogeneous electric field whose strength equals $E_{1z} = -(4\pi\sigma_0)/3$. Outside the sphere, there is the field of a dipole with moment $(4\pi\sigma_0 R^3)/3$.

2.26* Owing to the axial symmetry of the field, the Laplace equation written in cylindrical coordinates (the polar axis is directed along the axis of symmetry of the system) assumes the following form:

$$\frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{\partial^2\varphi}{\partial z^2} = 0 . \tag{1}$$

We find the solution of equation (1) in the form of a power series of r:

$$\varphi(r, z) = \sum_{n=0}^{\infty} a_n(z) r^n , \quad a_0(z) = \varphi(0, z) \equiv \Phi(z) , \tag{2}$$

where $\Phi(z)$ is the potential on the axis of symmetry of the system.

Having inserted (2) into (1), regrouped the terms, and equated the coefficients of the resulting series to zero, we find the recurrent relations for finding the coefficients $a_n(z)$, wherefrom it follows that

$$\varphi(r, z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2}\Phi^{(2n)}(z)\left(\frac{r}{2}\right)^{2n} = \Phi(z) - \frac{r^2}{4}\Phi''(z) + \dots ,$$

$$E_r = -\frac{\partial\varphi}{\partial r} = \frac{r}{2}\Phi''(z) + \dots , \quad E_\alpha = 0 ,$$

$$E_z = -\frac{\partial\varphi}{\partial z} = -\Phi'(z) + \dots$$

2.27 Multipole moments need to be found:

$$Q_{lm} = \sqrt{\frac{4\pi}{2l+1}} \int \kappa R^l Y_{lm}^* \left(\frac{\pi}{2}, \alpha\right) R d\alpha \,,$$

$$Q'_{lm} = \sqrt{\frac{4\pi}{2l+1}} \int \frac{\kappa}{R^{l+1}} Y_{lm}^* \left(\frac{\pi}{2}, \alpha\right) R d\alpha \,,$$

where $\kappa = q/2\pi R$. Using formulas (1.84), (1.89), and (1.92), we find

$$\varphi(r, \vartheta) = \frac{q}{r} \sum_{n=0}^{\infty} (-1)^n \frac{(2n-1)!!}{(2n)!!} \left(\frac{R}{r}\right)^{2n} P_{2n}(\cos\vartheta) \quad \text{when } r > R \,,$$

$$\varphi(r, \vartheta) = \frac{q}{R} \sum_{n=0}^{\infty} (-1)^n \frac{(2n-1)!!}{(2n)!!} \left(\frac{r}{R}\right)^{2n} P_{2n}(\cos\vartheta) \quad \text{when } r < R \,.$$

Both formulas are also valid when $r = R(\vartheta \neq \pi/2)$.

2.28

1. $\varphi \approx qa^2 \frac{3z^2 - r^2}{r^5} = 2qa^2 \frac{P_2(\cos\vartheta)}{r^3}$.
2. $\varphi \approx \frac{3qa^2 \sin^2\vartheta \cos\alpha \sin\alpha}{r^3}$.

2.29*

1. $\varphi \approx \frac{6qa^3 P_3(\cos\vartheta)}{r^4} = qa^3 \frac{15\cos^3\vartheta - 9\cos\vartheta}{r^4}$.
2. $\varphi \approx \frac{15qabcxyz}{r^7} = \frac{15qabc \sin^2\vartheta \cos\vartheta \sin\alpha \cos\alpha}{r^4}$.

2.30 $\varphi(r, \vartheta, \alpha) = q \sum_{l,m} \frac{4\pi}{2l+1} \frac{r^l}{r_0^{l+1}} Y_{lm}^*(\vartheta_0, \alpha_0) Y_{lm}(\vartheta, \alpha) \quad \text{when } r < r_0$;

$\varphi(r, \vartheta, \alpha) = q \sum_{l,m} \frac{4\pi}{2l+1} \frac{r_0^l}{r_0^{l+1}} Y_{lm}^*(\vartheta_0, \alpha_0) Y_{lm}(\vartheta, \alpha) \quad \text{when } r > r_0$.

2.31 $\varphi(x, y, z) \approx \frac{q}{r} + q \frac{a^2(3x^2 - r^2) + b^2(3y^2 - r^2) + c^2(3z^2 - r^2)}{10r^5}$.

In the case of an ellipsoid of rotation ($a = b$),

$$\varphi(r, \vartheta) = \frac{q}{r} + q \frac{c^2 - a^2}{5} \frac{P_2(\cos\vartheta)}{r^3} \,.$$

In the case of a sphere ($a = b = c$),

$$\varphi = \frac{q}{r} \,.$$

2.32 In spherical coordinates with the polar axis lying along the axis of symmetry of the system and the pole located in the center of the rings,

$$\varphi(r,\vartheta) = -\frac{q^2(a^2-b^2)}{2}\frac{P_2(\cos\vartheta)}{r^3} .$$

This is the potential of a linear quadrupole where the charges $-q$ are located at a distance of $\sqrt{a^2-b^2}/2$ from the central charge $2q$.

2.33* Find the multipole moments

$$q = -\int (\boldsymbol{p}'\cdot\nabla)\delta(\boldsymbol{r})\mathrm{d}V = -\oint(\boldsymbol{p}'\cdot\boldsymbol{n})\delta(\boldsymbol{r})\mathrm{d}S = 0 ,$$

because $\delta(\boldsymbol{r}) = 0$ everywhere, except $\boldsymbol{r} = 0$;

$$\begin{aligned} p_\alpha &= -\int x_\alpha(\boldsymbol{p}'\cdot\nabla)\delta(\boldsymbol{r})\mathrm{d}V \\ &= -\int x_\alpha p'_n\frac{\partial\delta(\boldsymbol{r})}{\partial x_n}\mathrm{d}V = \int p'_n\frac{\partial x_\alpha}{\partial x_n}\delta(\boldsymbol{r})\mathrm{d}V . \end{aligned}$$

The latter transform involves integrating by parts. The recurring index n involves summation. The surface integral that results from it is reduced to zero because $\delta(\boldsymbol{r}) = 0$ if $\boldsymbol{r} \neq 0$. $\delta(\boldsymbol{r}) = 0$ when $\boldsymbol{r} \neq 0$. Then we get

$$p_\alpha = p'_n\frac{\partial x_\alpha}{\partial x_n} = p'_n\delta_{\alpha n} = p'_\alpha .$$

All the multipole moments of a higher order are proportional to the components $\boldsymbol{r}$ in the presence of $\boldsymbol{r} = 0$, which is why they are reduced to zero. Consider, for instance, the components of the quadrupole moment. Indeed,

$$\begin{aligned} Q_{\alpha\beta} &= -\int x_\alpha x_\beta p'_n\frac{\partial\delta(\boldsymbol{r})}{\partial x_n}\,\mathrm{d}V \\ &= \int\delta(\boldsymbol{r})p'_n\frac{\partial x_\alpha x_\beta}{\partial x_n}\,\mathrm{d}V = p'_\alpha x_\beta + p'_\beta x_\alpha\,|_{r=0} = 0 \ . \end{aligned}$$

2.34 After integrating by parts n times, we get

$$\varphi(\boldsymbol{r}) = q(-1)^n\int\delta(\boldsymbol{r}')\prod_i(\boldsymbol{a}_i\cdot\nabla')\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}V = q\prod_i(\boldsymbol{a}_i\cdot\nabla)\frac{1}{r} .$$

2.35 The easiest way is to use the formula $\varphi = \frac{qa^2(3z'^2-r^2)}{r^5}$ (see the solution of Problem 2.28) to express z' in it through x, y, and z (Figure 2.15). We get

$$\varphi = \frac{qa^2}{r^3}[3(\cos\vartheta\cos\gamma + \sin\vartheta\sin\gamma\cos(\alpha-\beta))^2 - 1] .$$

The same result may be obtained with the use of the fact that the aggregate of the components of a quadrupole moment is a tensor of the second rank. In the system of axes x', y', and z', the components of the quadrupole moment are

$$Q'_{xx} = Q'_{yy} = Q'_{xy} = Q'_{xz} = Q'_{yz} = 0\,, \quad Q'_{zz} = 2qa^2\,.$$

The matrix of the transformation coefficients has the following form:

$$\widehat{a} = \begin{pmatrix} \cos\gamma\cos\beta & -\sin\beta & \sin\gamma\cos\beta \\ \cos\gamma\sin\beta & \cos\beta & \sin\gamma\sin\beta \\ -\sin\gamma & 0 & \cos\gamma \end{pmatrix}.$$

Using this matrix, we find the components $Q_{\alpha\beta}$ in the system xyz by the formula

$$Q_{\alpha\beta} = \sum_{\gamma,\delta} a_{\alpha\gamma} a_{\beta\delta} Q'_{\gamma\delta}\,,$$

and then use formula (2.21).

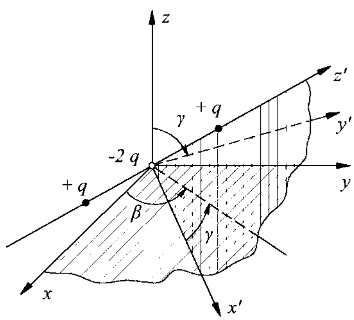

Figure 2.15 Finding the potential of a linear quadrupole.

2.36* $\varphi = \frac{15qabcz}{2r^7}[(y^2-x^2)\sin 2\beta + 2xy\cos 2\beta] = \frac{15qabc}{2r^4}\sin^2\vartheta\cos\vartheta\sin 2(\alpha-\beta)$.

2.37 $\varphi = \frac{qa^2}{4r^3}(3\sin^2\vartheta\sin 2\alpha - 3\cos 2\vartheta - 1)$.

2.38 According to the superposition principle, we may write

$$\varphi(\boldsymbol{r}) = \int_V \frac{\boldsymbol{P}\cdot(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\mathrm{d}V' = \int \boldsymbol{P}\cdot\nabla'\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}V'\,.$$

Transforming this expression with the use of the Gauss–Ostrogradskii theorem, we get $\varphi(\boldsymbol{r}) = \int_S \frac{P_n}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}S$, where S is the inside surface of the polarized sphere and $P_n = P\cos\vartheta$. Using the results obtained in Problem 2.25, we find

$$\varphi_1 = \frac{4\pi P r}{3}\cos\vartheta \quad (r \le R)\,, \quad \varphi_2 = \frac{4\pi P R^3}{3r^2}\cos\vartheta \quad (r \ge R)\,.$$

2.39

$$\varphi(\boldsymbol{r}) = -2\kappa \ln r + 2\sum_{n=1}^{\infty} \frac{A_n \cos n\alpha + B_n \sin n\alpha}{n r^n} ,$$

where $\kappa = \int \rho(\boldsymbol{r}')\mathrm{d}S'$ is the full charge per unit length of the distribution and $A_n = \int \rho(\boldsymbol{r}')r'^n \cos n\alpha' \mathrm{d}S'$ and $B_n = \int \rho(\boldsymbol{r}')r'^n \sin n\alpha' \mathrm{d}S'$ are two-dimensional multipole moments of order n.

What follows from these formulas, for instance, is that the potential of a dipole, in the two-dimensional case, has the form $\varphi = (2\boldsymbol{p}\cdot\boldsymbol{r})/(r^2)$, where $\boldsymbol{p} = \int \rho(\boldsymbol{r}')\boldsymbol{r}'\mathrm{d}S'$ is the dipole moment of the distribution per unit length and $\boldsymbol{r}$ is the radius vector in the xy plane.

2.40 $\varphi(r, \alpha) = -2\kappa \ln r + \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{r_0}{r}\right)^n \cos n(\alpha - \alpha_0)$ when $(r > r_0)$,
$\varphi(r, \alpha) = -2\kappa \ln r + \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{r}{r_0}\right)^n \cos n(\alpha - \alpha_0)$ when $(r < r_0)$.

2.41 $\varphi(\boldsymbol{r}) \approx (2\kappa a)/r \cos\alpha = (2\boldsymbol{p}\cdot\boldsymbol{r})/(r^2)$, where $\boldsymbol{p}$ is the dipole moment per unit length, $\boldsymbol{r}$ is the radius vector in the xy plane $(r \gg a)$, and the z axis is directed along one of the linear charges.

2.42 At the axis of symmetry of the disk (the z axis is directed from the negative side of the disk to its positive side),

$$\varphi(z) = \tau\Omega = 2\pi\tau\left(1 - \frac{|z|}{\sqrt{R^2 + z^2}}\right)\frac{z}{|z|} ;$$

$$E_x = E_y = 0 , \quad E_z = \frac{2\pi a^2 \tau z}{|z|(a^2 + z^2)^{3/2}} .$$

2.43

1. In cylindrical coordinates:

$$E_\alpha = \frac{2\tau}{r} , \quad E_r = E_z = 0 ;$$

2. $\varphi = 2\tau(\pi - \alpha)$, $\quad E_\alpha = -\frac{1}{r}\frac{\varphi}{\alpha} = \frac{2\tau}{r}$; $\quad E_r = E_z = 0$.

Field $\boldsymbol{E}$ coincides with the magnetic field of the rectilinear current $\mathcal{I} = \tau c$.

2.44 The equation of force lines

$$(z + a)[(z + a)^2 + r^2]^{-1/2} \pm (z - a)[(z - a)^2 + r^2]^{-1/2} = C ,$$

where C is a constant. Figure 2.16a shows the force lines for the case of opposite charges. In the case of like charges, there is a neutral point $r = 0$, $z = 0$ in the field (Figure 2.16b).

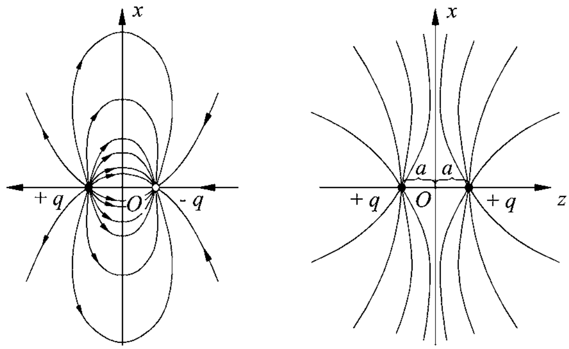

Figure 2.16 The force lines of fields created by opposite and like charges.

2.45 It is expedient to change to spherical coordinates. Letting a tend to zero, expanding in a series, and discarding terms of order a^2 and order, we get $r = C\sin^2\vartheta$.

2.46

$$r = C\sqrt{\sin^2\vartheta|\cos\vartheta|}\,, \quad C = \text{const}\,.$$

One should remember that in the case of a quadrupole of finite dimensions the formula obtained is only valid for large distances (Figure 2.17).

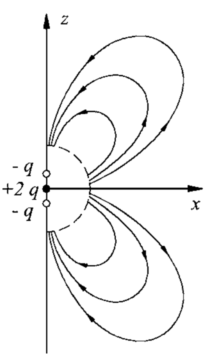

Figure 2.17 The force lines of a linear quadrupole.

2.48 $q_2 = \frac{\Phi+\sqrt{2}(\sqrt{2}-1)\pi q}{\sqrt{2}(\sqrt{2}-1)\pi}$.

2.49* Consider a force tube formed by the rotation of a certain force line around the z axis. Having applied the electrostatic Gaussian theorem to the volume limited by the side surface of the tube and by two planes $z = \text{const}$, containing no charges, we find that the flux passing through any cross-section of the tube normal to its axis $\Phi(z) = \sum_i q_i \Omega_i(z)$ (see Problem 2.47) does not depend on z (when z changes within the range from z_k to z_{k+1}) Here, $\Omega_i(z) = 2\pi(\pm 1 - \cos\alpha_i)$ is the spatial angle at which the negative side of such a cross-section may be viewed from the point z_i where the charge q_i is located; α_i is the angle between the direction of the z axis and the radius vector of the point of the circuit of the normal cross-section with coordinates (r, z). We must have a plus sign when $z > z_i$, and a minus sign when $z < z_i$. If, when z changes, the normal cross-section of the tube crosses the charge q_k, then $\Phi(z)$ will abruptly change to $\pm 4\pi q_k$. However, when this happens, $\sum_i q_i \cos\alpha_i$ will not change. Having expressed $\cos\alpha_i$ through z, z_i, and r, we get the equation for the family of force lines sought:

$$\sum_i \frac{q_i(z - z_i)}{\sqrt{r^2 + (z - z_i)^2}} = C\,, \quad C = \text{const}\,.$$

2.51 Select a cylindrical system of coordinates whose z axis coincides with the axis of the cylinder (Figure 2.18). Instead of the condition $\varphi|_S = \text{const}$ on the surface S of the cylinder, it is more convenient to use the condition that follows from it: $\partial\varphi/\partial\alpha|_S = 0$. Having completed differentiating, we get

$$\frac{\kappa_1 x_1}{R^2 + x_1^2 - 2Rx_1\cos\alpha} = \frac{\kappa_2 x_2}{R^2 + x_2^2 - 2Rx_2\cos\alpha}\,.$$

Discard the denominators and, separately, equate the terms with $\cos\alpha$ and without it. Now we have that when $\kappa_1 = \kappa_2$, any cylindrical surface whose axis is parallel to the charged filaments and lies in the same plane as them, while the radius satisfies the condition $R^2 = x_1 x_2$, will be equipotential. When $x_1 = 0$, there is a solution $\kappa_2 = 0$. This is the case of cylindrical equipotential surfaces in the field of a single filament.

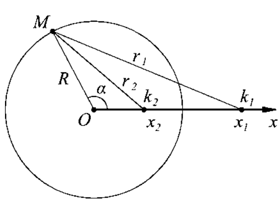

Figure 2.18 The equipotential cylindrical surface of two oppositely charged parallel filaments.

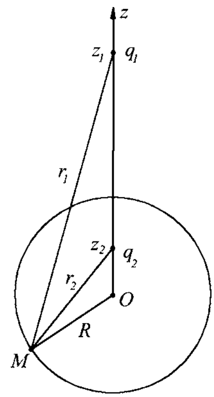

Figure 2.19 The equipotential spherical surface of two opposite point charges.

2.52 Use Figure 2.19. The radius R of the sphere sought and the position of its center are determined by the equations

$$R^2 = z_1 z_2 \,, \qquad \frac{z_1}{z_2} = \frac{q_1^2}{q_2^2} \,.$$

The potential on the surface of that sphere equals zero.

2.53

$$\Delta\varphi = q\Delta\frac{e^{-\alpha r}}{r} = q\Delta\frac{1}{r} + q\Delta\frac{e^{-\alpha r}-1}{r}$$
$$= -4\pi q\delta(\boldsymbol{r}) + \frac{q}{r}\frac{\partial^2}{\partial r^2}\left(r\frac{e^{-\alpha r}-1}{r}\right) = -4\pi q\delta(\boldsymbol{r}) + \frac{q\alpha^2 e^{-\alpha r}}{r} \,.$$

So there is a point charge q at the origin and a spherically distributed volume charge with density $\rho = -\frac{q\alpha^2 e^{-\alpha r}}{4\pi r}$, $\int \rho \mathrm{d}V = -q$.

2.54 A point charge e_0 at the origin is surrounded by a volume charge with density $\rho(r) = -(e_0/\pi a^3)e^{-2r/a}$. Such is the charge distribution in a hydrogen atom (compare this with Problem 2.15)

2.55

$$\mathcal{F} = (\boldsymbol{p}\cdot\nabla)\boldsymbol{E} \,, \quad \boldsymbol{N} = \boldsymbol{p}\times\boldsymbol{E} \,.$$

2.56

$$\frac{3q^2}{5R} \,, \quad \frac{q^2}{2R} \,, \quad \frac{q^2}{R_2 - R_1}\left(1 - \frac{R_1}{R_2 - R_1}\ln\frac{R_2}{R_1}\right) \,.$$

2.57

$$q_{1,2} = \frac{R\sqrt{R^2 + a^2}}{qa^2} \left(\sqrt{R^2 + a^2} A_{1,2} - R A_{1,2}\right) .$$

2.58

$$U = \int \frac{e_0}{r} \rho(r) \mathrm{d}V = -\frac{e_0^2}{a} .$$

2.59 $U = \frac{5e_0^2}{4a}$.

2.60* The necessary condition of the minimum of potential energy is the reduction to zero of all the first derivatives and the positiveness of all the second derivatives of potential energy at generalized coordinates when the coordinates have certain values (at the point of equilibrium of the system):

$$\frac{\partial U}{\partial q_\alpha} = 0 , \quad \frac{\partial^2 U}{\partial q_\alpha^2} > 0 , \quad \alpha = 1, 2, \ldots, 3N ,$$

where $3N$ is the number of degrees of freedom N of point particles. q_α may be understood as their Cartesian coordinates. Select a particle and assign the coordinates q_1, q_2, and q_3 to it. The electrodynamic potential of all other particles at this point $\varphi(q_1, q_2, q_3)$ satisfies Laplace's equation:

$$\Delta\varphi = \frac{\partial^2\varphi}{\partial q_1^2} + \frac{\partial^2\varphi}{\partial q_2^2} + \frac{\partial^2\varphi}{\partial q_3^2} = 0 .$$

Therefore, the potential energy $U = e\varphi(q_1, q_2, q_3)$ may not have second derivatives of the same sign at the point where this charge is located. This reasoning is valid for any charge. Thus, the potential energy has no minimum, and the equilibrium of this system of point charges is unstable.

2.61

$$U = \frac{q_1 q_2}{a} , \quad F = \frac{q_1 q_2}{a^2} .$$

2.62

$$R = \frac{32\pi\alpha}{E_0^2} .$$

2.63*

$$U = \oint\limits_l \oint\limits_2 \frac{\kappa_1 \kappa_2 \mathrm{d}l_1 \mathrm{d}l_2}{r_{12}}$$

$$= \frac{q_1 q_2}{4\pi^2 ab} \int\limits_0^{2\pi} \int\limits_0^{2\pi} \frac{ab \mathrm{d}l_1 \mathrm{d}l_2}{\sqrt{c^2 + a^2 + b^2 - 2ab\cos(\alpha_1 - \alpha_2)}} ,$$

where integration is done over all the elements of both rings dl_1 and dl_2, and α_1 and α_2 are the angles indicating the locations of the elements. Integrating on $d\alpha_2$ and replacing $\alpha_1 = \pi - 2\alpha$, we get

$$U = \frac{q_1 q_2 k}{\pi\sqrt{ab}} K(k) ,$$

where

$$k = \frac{2\sqrt{ab}}{\sqrt{c^2 + (a+b)^2}} \quad \text{and} \quad K(k) = \int_0^{\pi/2} \frac{d\alpha}{\sqrt{1 - k^2 \sin^2\alpha}}$$

is the full elliptical integral of the first kind.

When finding the force $F = -\frac{\partial U}{\partial \nabla c} = -\frac{U}{k}\frac{\partial k}{\partial c}$, one should use the formula

$$2k^2 \frac{dK(k)}{d(k^2)} = \frac{E(k)}{1-k^2} - K(k)$$

(see Gradshtein and Ryzhik, 2007), where $E(k) = \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2\sin^2\alpha}\, d\alpha$ is the full elliptical integral of the second kind.

Finally,

$$F = \frac{q_1 q_2 c k^3}{4\pi(ab)^{3/2}} \frac{E(k)}{1-k^2} .$$

2.64 $F = -\frac{3q\boldsymbol{r}(\boldsymbol{p}\cdot\boldsymbol{r})}{r^5} + \frac{q\boldsymbol{p}}{r^3}$, $\quad \boldsymbol{N} = \frac{q\boldsymbol{p}\times\boldsymbol{r}}{r^3}$.

2.65

$$U = p_1 p_2 \frac{\sin\vartheta_1 \sin\vartheta_2 \cos\varphi - 2\cos\vartheta_1\cos\vartheta_2}{r^3} ,$$

where $\vartheta_1 = \angle(\boldsymbol{r}, \boldsymbol{p}_1)$, $\vartheta_2 = \angle(\boldsymbol{r}, \boldsymbol{p}_2)$, and φ is the angle between planes $(\boldsymbol{r}, \boldsymbol{p}_1)$ and $(\boldsymbol{r}, \boldsymbol{p}_2)$,

$$F = 3p_1 p_2 \frac{\sin\vartheta_1 \sin\vartheta_2 \cos\varphi - 2\cos\vartheta_1\cos\vartheta_2}{r^4} .$$

The force is at its maximum when $\vartheta_1 = \vartheta_2 = \vartheta = 0$, that is, when the dipoles are parallel.

2.66

$$U_{21} = \int \rho(\boldsymbol{r}')\varphi_1(\boldsymbol{r}')dV'$$
$$= \sum_{l,m} \sqrt{\frac{4\pi}{2l+1}} a_{lm} \int r'^l Y_{lm}(\vartheta', \alpha')\rho(\boldsymbol{r}')dV' = \sum_{l,m} a_{lm} Q^*_{lm} .$$

Here, $Q_{lm} = \int r^l Y^*_{lm}(\vartheta, \alpha)\rho(\boldsymbol{r})dV$ are the multipole moments of the charge distribution in question.

2.67 Yes, under the condition $\boldsymbol{j}_0 \cdot \boldsymbol{k} = 0$. Otherwise $\partial\rho/\partial t \neq 0$.

2.68

$$j_r = j_\vartheta = 0\,, \quad j_\alpha = \frac{H_0 c r}{4\pi a^2}\sin\theta \quad \text{when } r < a\,; \quad \boldsymbol{j} = 0 \quad \text{when } r > a\,.$$

2.70

$$H_r = H_z = 0\,, \quad H_\alpha = \begin{cases} \frac{2Jr}{ca^2} & \text{when } r < a\,, \\ \frac{2J}{cr} & \text{when } a \le r \le b\,, \\ 0 & \text{when } r > b\,, \end{cases}$$

2.71 Consider solving the problem by the method of the vector potential. If the z axis is directed along the axis of the cylinder, then the rectangular components of $\boldsymbol{A}$ will satisfy the equations

$$\Delta A_x = 0\,, \quad \Delta A_y = 0\,, \quad \Delta A_z = -\frac{4\pi}{c} j_z\,, \tag{1}$$

where $j_z = 0$ when $r > a$ and $j_z = J/\pi a^2$ when $r \le a$.

Because the equations for A_x and A_y do not include any specified current J, these components may be regarded as equal to zero; A_z will depend only on the distance r to the z axis. Integrating the equation for A_z and using the continuity condition A_z and H_α at the bound $r = a$ and the restriction condition for H when $r = 0$, we get when $r < a$,

$$A_z = C - \frac{J}{c}\left(\frac{r}{a}\right)^2\,, \quad B_\alpha = \frac{2J}{ca^2} r\,, \quad H_\alpha = \frac{2J}{ca^2} r\,, \tag{2}$$

and when $r > a$,

$$A_z = C - \frac{J}{c}\left(1 + 2\ln\frac{r}{a}\right)\,, \quad H_\alpha = \frac{2cI}{cr}\,, \quad H_\alpha = \frac{2J}{cr}\,. \tag{2'}$$

The constant C is arbitrary.

2.72 When $r < a$,

$$A_z = C_1\,, \quad \boldsymbol{H} = 0\,;$$

when $a \le r \le b$,

$$A_z = \frac{2Ja^2}{c(b^2 - a^2)}\left(\ln\frac{r}{a} - \frac{r^2}{2a^2}\right) + C_2\,, \quad H_\alpha = \frac{2J}{c(b^2 - a^2)}\left(r - \frac{a^2}{r}\right)\,;$$

when $r > b$,

$$A_z = \frac{2J}{c}\ln\frac{b}{r} + C_3\,, \quad H_\alpha = \frac{2J}{cr}\,.$$

The remaining components $\boldsymbol{A}$ and $\boldsymbol{H}$ are equal to zero. Any two constants included in A_z may be expressed through a third one using the conditions of the continuity of the vector potential at the bounds.

2.73

$$H_x = -\frac{2J}{ca}\left(\arctan\frac{a+2x}{2y} + \arctan\frac{a-2x}{2y}\right),$$

$$H_y = -\frac{J}{ca}\ln\frac{\left(x-\frac{a}{2}\right)^2+y^2}{\left(x+\frac{a}{2}\right)^2+y^2}, \quad H_z = 0.$$

The y axis is perpendicular to the stripe and crosses its center.

2.74 The plates repel one another with force

$$f = \frac{4J^2}{c^2a^2}\left(a\arctan\frac{a}{b} - \frac{1}{2}b\ln\frac{a^2+b^2}{b^2}\right).$$

2.75

$$A_z = \frac{2J}{c}\ln\frac{r_2}{r_1} = \frac{J}{c}\ln\frac{(a+x)^2+y^2}{(a-x)^2+y^2},$$

$$H_x = \frac{\partial\partial A_z}{\partial y} = -\frac{8J}{c}\frac{axy}{r_1^2r_2^2},$$

$$H_y = -\frac{\partial A_z}{\partial x} = -\frac{2J}{c}\left(\frac{a-x}{r_1^2}+\frac{a+x}{r_2^2}\right).$$

In the plane that is perpendicular to conductors with current, the coordinates of such conductors are $(a,0)$ for current $+J$ and $(-a,0)$ for current $-J$; r_1, and r_2 are distances from the points $(a,0)$ and $(-a,0)$ to the observation point.

2.76 (i) Between the planes $H = \frac{4\pi}{c}i$, in the rest of space $H = 0$; (ii) between the planes $H = 0$, in the rest of space $H = \frac{4\pi}{c}i$. In both cases, the magnetic field is perpendicular to the current and parallel to the planes.

2.77 $H_y = \frac{2Jd}{c(b^2-a^2)}$, $H_x = H_z = 0$; The y axis is normal to the plane drawn through the axes of the cylinders.

2.78* In a cylindrical system of coordinates whose z axis is perpendicular to the plane of the ring and crosses its center,

$$A_\alpha = \frac{2J}{c}\left(\frac{q}{r}\right)^{\frac{1}{2}}\left[\left(\frac{2}{k}-k\right)K(k) - \frac{2}{k}E(k)\right], \quad A_z = A_r = 0,$$

where $K(k)$ and $E(k)$ are full elliptical integrals, $k^2 = \frac{4ar}{(a+r)^2+z^2}$.

The components of the magnetic field are as follows:

$$H_r = \frac{2J}{c}\cdot\frac{z}{r\sqrt{(a+r)^2+z^2}}\left[-K(k) + \frac{a^2+r^2+z^2}{(a-r)^2+z^2}E(k)\right],$$

$$H_z = \frac{2J}{c} \cdot \frac{z}{\sqrt{(a+r)^2 + z^2}} \left[K(k) + \frac{a^2 - r^2 - z^2}{(a-r)^2 + z^2} E(k) \right] , \quad H_\alpha = 0 \, .$$

At the axis of the coil ($r = 0$), these expressions become

$$H_r = 0 \, , \quad H_z = \frac{2\pi a^2 J}{c(a^2 + z^2)^{3/2}} \, .$$

2.79* The magnetic flux passing through any cross-section of such a tube will be the same. This is why the equation for the surface of the tube is as follows:

$$N = \int_S \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{S} = f(r, z) = \text{const} \, ,$$

where the surface of integration S is a circle of radius r in a plane perpendicular to the axis of symmetry (the center of the circle lies on the axis of symmetry). Since A_α is independent of α, the use of the Stokes theorem gives us

$$\int \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{S} = \oint \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{l} = 2\pi r A_\alpha(r, z) = \text{const} \, .$$

The intersection lines of these surfaces with planes $\alpha = \text{const}$ give us the lines of magnetic induction sought.

2.80 The components of the magnetic field are

$$H_z = -\frac{\partial \psi}{\partial z} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} H^{(2n)}(z) \left(\frac{r}{2}\right)^{2n} = H_z - \frac{r^2}{4} H''(z) + \ldots ,$$

$$H_r = -\frac{\partial \psi}{\partial r} = \sum_{n=1}^{\infty} \frac{(-1)^n}{(n-1)!n!} H^{(2n-1)}(z) \left(\frac{r}{2}\right)^{2n-1} = -\frac{r}{2} H'(z) + \ldots ,$$

$$H_\alpha = 0 \, .$$

The vector potential is expressed through the strength of the magnetic field with the use of the Stokes theorem and the relation $\boldsymbol{H} = \operatorname{curl} \boldsymbol{A}$:

$$A_\alpha(r, z) = \frac{1}{r} \int_0^r H_z r \mathrm{d}r = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1)!} H^{(2n)}(z) \left(\frac{r}{2}\right)^{2n+1}$$
$$= \frac{r}{2} H(r) - \ldots$$

2.82* Assume the existence of two different fields $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$ satisfying (2.51), (2.54), and (2.57) when the volume and surface currents are those specified. The difference of strengths $\boldsymbol{H} = \boldsymbol{H}_1 - \boldsymbol{H}_2$ satisfies the equations

$$\operatorname{curl} \boldsymbol{H} = 0 \, , \quad \operatorname{div} \boldsymbol{H} = 0 \, ,$$

and the vector $\boldsymbol{H}$ is continuous everywhere. Assuming that $\boldsymbol{H} = \text{curl}\,\boldsymbol{A}$, we get

$$H^2 = \boldsymbol{H}\cdot\text{curl}\,\boldsymbol{A} = \text{div}\,[\boldsymbol{A}\times\boldsymbol{H}] + \boldsymbol{A}\cdot\text{curl}\,\boldsymbol{H} = \text{div}\,[\boldsymbol{A}\times\boldsymbol{H}]\,.$$

Now, integrate the two parts of the latter equality for the whole space,

$$\int H^2 \mathrm{d}V = \oint [\boldsymbol{A}\times\boldsymbol{H}]\cdot \mathrm{d}\boldsymbol{S} \to 0\,,$$

if at large distances the product AH decreases faster than r^{-2}. In a bounded system of currents, the latter condition is satisfied a fortiori (see Example 2.11). Therefore, $\boldsymbol{H} = 0$ throughout space and the solution of the problem is unique.

2.84

$$H_z = \frac{2\pi n J}{c}(\cos\theta_1 + \cos\theta_2)\,,$$

where

$$\cos\theta_1 = \frac{h - z}{\sqrt{a^2 + (h-z)^2}}\,, \quad \cos\theta_2 = \frac{z}{\sqrt{a^2+z^2}}$$

(see Figure 2.20).

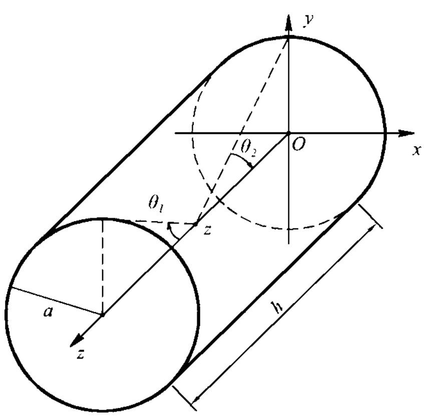

Figure 2.20 Finding the magnetic field of a solenoid.

2.85* Solve the problem with the use of a vector potential. The density of the surface current results from the rotation of a sphere in a spherical system of coordinates. Its polar axis directed along $\boldsymbol{\omega}$ has the form

$$\boldsymbol{i} = \boldsymbol{e}_\alpha \frac{e\omega}{4\pi a}\sin\theta\,. \tag{1}$$

The vector potential satisfies Laplace's equation and, on the surface of the sphere, the boundary condition that follows from (2.57). Since the current is symmetric, the vector potential may be selected so that only the component A_α independent of α should be other than zero. We write the equation for it using formulas (1.279):

$$\Delta A_\alpha - \frac{1}{r^2 \sin^2\theta} A_\alpha = 0 \,. \tag{2}$$

We find the solution in the form

$$A_\alpha = F(r)\sin\theta \,, \tag{3}$$

which satisfies the condition $\operatorname{div} \boldsymbol{A} = 0$. Seeking $F(r)$ with the use of equation (2) and the boundary condition, we find that A_α and $\boldsymbol{H} = \operatorname{curl} \boldsymbol{A}$:

$$\boldsymbol{H} = \frac{2e\boldsymbol{\omega}}{3ca} \quad \text{at } r < a \,; \qquad \boldsymbol{H} = \frac{3\boldsymbol{r}(\boldsymbol{m}\cdot\boldsymbol{r})}{r^5} - \frac{\boldsymbol{m}}{r^3} \quad \text{at } r > a \,.$$

Here,

$$\boldsymbol{m} = \frac{ea^2\boldsymbol{\omega}}{3c}$$

is the magnetic moment of the system (see Problem 2.86).

2.86 Using the delta function, we write the current of a rotating sphere (see formula (1) in the solution of the previous problem), expressing it through its volume density:

$$\boldsymbol{j} = \boldsymbol{e}_\alpha \frac{e\omega}{4\pi a^3} \sin\theta\, \delta(r-a) \,.$$

Using formula (2.59), we find the magnetic moment $\boldsymbol{m} = ea^2\boldsymbol{\omega}/3c$ obtained in a different way in the previous problem. Because the charge is uniformly distributed throughout the volume, we get $\boldsymbol{m} = ea^2\boldsymbol{\omega}/5c$. The magnetic field found according to the approximate formula (2.59) in the outer region $r > a$ coincides with the exact solution.

2.88 Having inserted (2.39) (the expression for current density created by point particles) into (2.59), we get

$$\boldsymbol{m} = \sum_a \frac{e_a}{2c}[\boldsymbol{r}_a \times \boldsymbol{v}_a] = \sum_a \frac{e_a}{2m_a c}[\boldsymbol{r}_a \times \boldsymbol{p}_a] \,,$$

where $\boldsymbol{p}_a = m_a \boldsymbol{v}_a$ is the momentum of an individual particle. When $e_a/m_a = e/m$, we get formula (2.61), which determines the magnetic moment of a system of particles conditioned by their movement in space.

One should keep in mind that the magnetic moment is among the most important physical characteristics, being inherent to many macroscopic bodies, such as permanent magnets, the Earth, the Sun, and the stars, or to almost all microscopic particles, either charged ones such as electrons, protons, and atomic nuclei or electrically neutral ones such as neutrons and atoms. The inner moments of microscopic particles are called their spin moments. They are not related to their movement as a whole and are conditioned by the singularities of their inner structures, which are mostly unknown. There is a relation of the type

$$\boldsymbol{m}_s = \eta_s \boldsymbol{L}_s$$

(2.61) between the magnetic $\boldsymbol{m}_s$ and mechanical $\boldsymbol{L}_s$ moments, yet the proportionality coefficient η_s does not coincide with $\eta = e/2mc$ and is different for different particles. Spin magnetic moment of elementary particles is a quantum phenomenon that, in classical electrodynamics, remains unexplained.

2.89

$$\boldsymbol{m} = \frac{1}{2c}\left(\frac{e_1}{m_1^2} + \frac{e_2}{m_2^2}\right)\frac{m_1 m_2}{m_1 + m_2}\boldsymbol{L} .$$

2.90

$$\boldsymbol{m} = \mu_\mathrm{B} \boldsymbol{e}_z , \quad \text{where } \mu_\mathrm{B} = \frac{e\hbar}{2m_\mathrm{e}c} \approx 0.9 \times 10^{-20} \mathrm{erg/G}$$

which is the *Bohr magneton.*

2.91* $\boldsymbol{m} = \boldsymbol{\mu}_s$.

2.92

$$\boldsymbol{H}(0) = \frac{4\mu_\mathrm{B}}{405a^3}\boldsymbol{e}_z , \quad \boldsymbol{m} = 6\mu_\mathrm{B}\boldsymbol{e}_z .$$

2.93

$$\boldsymbol{m} = \frac{ea^2}{5c}\boldsymbol{\omega} , \quad \boldsymbol{H} = \frac{3\boldsymbol{r}(\boldsymbol{m}\cdot\boldsymbol{r})}{r^5} - \frac{\boldsymbol{m}}{r^3} .$$

2.94

$$\boldsymbol{m} = \frac{ea^2}{4c}\boldsymbol{\omega} , \quad \boldsymbol{H} = \left[\left(1 + \frac{a^2}{z^2}\right)^{1/2} + \left(1 + \frac{a^2}{z^2}\right)^{-1/2} - 2\right]\frac{2e|z|}{a^2c}\boldsymbol{\omega} .$$

2.95 At points where $j = 0$, one may assume that $\boldsymbol{H} = -\operatorname{grad}\psi$. Then the equation $\operatorname{curl}\boldsymbol{H} = 0$ is valid whatever the values of ψ, and what follows from the equation $\operatorname{div}\boldsymbol{H} = 0$ is

$$\Delta\psi = 0 .$$

The latter equation must be solved under an additional condition:

$$\int_l \boldsymbol{H}\cdot \mathrm{d}\boldsymbol{l} = \frac{4\pi}{c}J ,$$

where l is any circuit enveloping current J. We introduce cylindrical coordinates r, α, and z and look for a solution in the form $\psi = \psi(\alpha)$.

Finally, we get

$$\psi = -\frac{2J}{c}\alpha\,, \quad H_\alpha = \frac{2J}{cr}\,, \quad H_r = H_z = 0\,.$$

2.96*

1. To ensure the synonymous (simple) scalar potential ψ of the magnetic field, select a certain surface S (Figure 2.21) bearing on a circuit with current, and assume that when crossing this surface, ψ experiences a jump:

$$\psi(2) - \psi(1) = \frac{4\pi}{c}J\,. \tag{1}$$

Points 1 and 2 are infinitely close to one another on the different sides of the surface, and the direction from 1 to 2 makes a right-handed system with the direction of the current.

The solution of Laplace's equation may be written as (see Example 2.4):

$$\psi = \frac{1}{4\pi}\oint\left[\frac{1}{r}\cdot\frac{\partial\psi}{\partial n} - \psi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right]\mathrm{d}S\,. \tag{2}$$

In expression (2) integration must be done over the infinitely remote closed surface S' as well as over all closed surfaces Σ_i, which are located at finite distances from the origin, wherein ψ or $\partial\psi/\partial n$ experiences breaks. In the case in question, the integral of an infinitely remote surface equals zero, the source of the field (the circuit with current) having finite dimensions. Surfaces where the normal derivative $\partial\psi/\partial n = -H_n$ has a tear are absent because H_n is a continuous value. This is why in (2) integration must be done over one surface Σ enveloping S.

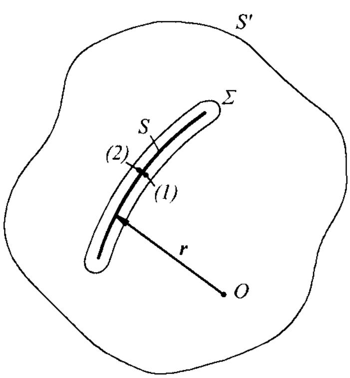

Figure 2.21 Finding the pseudoscalar potential of a circuit with current.

We retract Σ until it coincides with S. Because the values $\frac{1}{r}$, $\partial\psi/\partial n$, and $(\partial/\partial n)(1/r)$ are continuous on the surface S, formula (2) will assume the form

$$\psi = -\frac{1}{4\pi}\int[\psi(1) - \psi(2)]\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\mathrm{d}S\ , \tag{3}$$

where integration is now done over the unclosed surface S.
Using equality (1), we get

$$\psi = \frac{J}{c}\int\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\mathrm{d}S = -\frac{J}{c}\int\frac{\boldsymbol{r}\cdot d\boldsymbol{S}}{r^3}\ . \tag{4}$$

The integral $\int\frac{\boldsymbol{r}\cdot\mathrm{d}\boldsymbol{S}}{r^3}$ is the spatial angle the circuit with current is seen at from the observation point, which is why formula (4) may be written as

$$\psi = -\frac{J}{c}\Omega\ .$$

The sign of Ω is positive if the radius vector $\boldsymbol{r}$ drawn from the observation point to a certain point on the surface S and the direction of the current in the circuit make up a right-handed system.

2. Transform the integral (2.50′) of the circuit into an integral of the surface bearing on the circuit. Using formula (1.99), we

$$\boldsymbol{A} = \frac{J}{c}\int\mathrm{d}\boldsymbol{S}\times\nabla\left(\frac{1}{r}\right) = \frac{J}{c}\int\nabla_M\left(\frac{1}{r}\right)\times\mathrm{d}\boldsymbol{S}\ ,$$

where ∇_M means differentiating over the coordinates of the observation point M. Calculating $\boldsymbol{H} = \mathrm{curl}\,\boldsymbol{A}$, we find the following:

$$\boldsymbol{H} = \frac{J}{c}\int(\mathrm{d}\boldsymbol{S}\cdot\nabla)\left(\frac{1}{r}\right) = \frac{J}{c}\nabla_M\int\mathrm{d}\boldsymbol{S}\cdot\nabla_M\left(\frac{1}{r}\right)\ . \tag{5}$$

(In the transform, the equality $\Delta\left(\frac{1}{r}\right) = 0$ was used and it is assumed that point $r = 0$ is not located on the surface of integration.) Comparing (5) with the formula $\boldsymbol{H} = -\mathrm{grad}\,\psi$, we get

$$\psi = -\frac{J}{c}\int\mathrm{d}\boldsymbol{S}\cdot\nabla_M\left(\frac{1}{r}\right) = -\frac{J}{c}\int\frac{\boldsymbol{r}\cdot\mathrm{d}\boldsymbol{S}}{r^3} = -\frac{J}{c}\Omega\ .$$

2.97 $\boldsymbol{F} = 0$ and $\boldsymbol{N} = \boldsymbol{m}\times\boldsymbol{H}$, where $\boldsymbol{m} = \frac{\mathcal{I}}{c}\int\boldsymbol{n}\cdot\mathrm{d}\boldsymbol{S}$ is the magnetic moment of the circuit with current.

2.98 $U = \frac{\boldsymbol{m}_1\cdot\boldsymbol{m}_2}{r^3} - \frac{3(\boldsymbol{m}_1\cdot\boldsymbol{r})(\boldsymbol{\gamma}_2\cdot\boldsymbol{r})}{r^5}$;

$$\boldsymbol{F}_2 = -\boldsymbol{F}_1 = \frac{3}{r^5}[(\boldsymbol{m}_1\cdot\boldsymbol{r})\boldsymbol{m}_2 + (\boldsymbol{m}_2\cdot\boldsymbol{r})\boldsymbol{m}_1 + (\boldsymbol{m}_1\cdot\boldsymbol{m}_2)\boldsymbol{r}] - \frac{15}{r^7}(\boldsymbol{m}_1\cdot\boldsymbol{r})(\boldsymbol{m}_2\cdot\boldsymbol{r})\boldsymbol{r}\ ,$$

where $\boldsymbol{r}$ is the radius vector drawn from the first current to the second one, and $\boldsymbol{F}_1$ and $\nu\,\boldsymbol{F}_2$ are the forces acting on the first and second currents.

$$\boldsymbol{N}_1 = \frac{3(\boldsymbol{m}_2 \cdot \boldsymbol{r})(\boldsymbol{m}_1 \times \boldsymbol{r})}{r^5} + \frac{\boldsymbol{m}_2 \times \boldsymbol{m}_1}{r^3},$$

$$\boldsymbol{N}_2 = \frac{3(\boldsymbol{m}_1 \cdot \boldsymbol{r})(\boldsymbol{m}_2 \times \boldsymbol{r})}{r^5} + \frac{\boldsymbol{m}_1 \times \boldsymbol{m}_2}{r^3},$$

where $\boldsymbol{N}_1$ and $\boldsymbol{N}_2$ are the rotary moments applied to the first and second currents, respectively. One should note that $\boldsymbol{N}_1 \neq -\boldsymbol{N}_2$ but

$$\boldsymbol{N}_1 + \nu\,\boldsymbol{N}_2 + (\nu\,\boldsymbol{r} \times \boldsymbol{F}_2) = 0\,.$$

If the magnetic moments are parallel ($\boldsymbol{m}_1 = m_1\boldsymbol{n}$, and $\boldsymbol{m}_2 = m_2\boldsymbol{n}$ and $\boldsymbol{r} = r\boldsymbol{r}_0$, $\boldsymbol{n}$ and $\boldsymbol{r}_0$ are unit vectors), then

$$\boldsymbol{F}_2 = \frac{3m_1 m_2[2\boldsymbol{n}\cos\vartheta - \boldsymbol{r}_0(5\cos^2\vartheta - 1)]}{r^4},$$

where ϑ is the angle between $\boldsymbol{n}$ and $\boldsymbol{r}_0$.

2.100 The potential function of the current J_2 in the field of the current J_1 is

$$u_{21} = \frac{2J_1J_2}{c^2}\ln a + \text{const}\,,$$

where a is the distance between the currents.

The force acting on a unit length of the second current is

$$f = -\frac{\partial u_{21}}{\partial a} = -\frac{2J_1J_2}{c^2 a}\,.$$

When the currents are parallel ($\mathcal{J}_1$ and $\mathcal{J}_2$ being of the same sign), there is an attraction.

2.101 The force $\boldsymbol{F}$ and the rotary moment $\boldsymbol{N}$ are found by differentiating the potential function:

$$U(r,\alpha) = -\frac{J_1J_2a}{c^2}\ln\frac{4r^2 + a^2 + 4ar\cos\alpha}{4r^2 + a^2 - 4ar\cos\alpha}\,.$$

2.102 $N = \frac{4JJ'a}{c}(\sin\varphi - \varphi\cos\varphi)w$.

2.103 $\mathcal{L} = \frac{1}{2} + 2\ln\frac{b}{a}$.

2.104 $\mathcal{L} = 2\ln\frac{b}{a}$.

2.105 $L_{12} = 4\pi\left(b - \sqrt{b^2 - a^2}\right)$;

$$F = \frac{\mathcal{I}_1\mathcal{I}_2}{c^2}\cdot\frac{\partial L_{21}}{\partial b} = \frac{4\pi\mathcal{I}_1\mathcal{I}_2}{c^2}\left(1 - \frac{b}{\sqrt{b^2 - a^2}}\right)\,.$$

2.106* In this problem, it is convenient to use formula (2.70). After integrating, as in Problem 2.19, we get

$$L_{12} = 4\pi\sqrt{ab}\left[\left(\frac{2}{k} - k\right) K(k) - \frac{2}{k} E(k)\right] ,$$

where

$$K(k) = \int_0^{\pi/2} \frac{d\psi}{\sqrt{1 - k^2 \sin^2 \psi}} ,$$

$$E(k) = \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2 \psi} d\psi , \quad k^2 = \frac{4ab}{(a+b)^2 + l^2} .$$

When $l \gg a, b$, the parameter k is small:

$$k^2 \approx \frac{4ab}{l^2} , \quad k \approx \frac{2\sqrt{ab}}{l} \ll 1 ,$$

This is why approximate formulas may be used for E and K (see Gradshtein and Ryzhik, 2007):

$$K(k) = \frac{\pi}{2}\left(1 + \frac{1}{4}k^2 + \frac{9}{64}k^4\right) , \quad E(k) = \frac{\pi}{2}\left(1 - \frac{1}{4}k^2 - \frac{3}{64}k^4\right) .$$

Leaving only terms proportional to k^3 in the expression for L_{12}, we get the following in the first nonvanishing approximation:

$$L_{12} = \frac{2\pi^2 a^2 b^2}{l^3} .$$

The latter result is also easy to get from the equality $L_{12} = c\Phi_{12}/\mathcal{I}_1$ regarding the rings with current as magnetic dipoles.

2.107 In the notation of the previous problem

$$F = \frac{4\pi\mathcal{I}_1\mathcal{I}_2}{c^2} \frac{l}{\sqrt{(a+b)^2 + l^2}} \left[-K(k) + \frac{a^2 + b^2 + l^2}{(a+b)^2 + l^2} E(k)\right] .$$

2.108 $\mathcal{L} = 4\pi n^2 S$. For a solenoid of a large but finite length h, disregarding the tip effect, we get the full induction

$$L = 4\pi n^2 S h .$$

2.109 Find the magnetic energy using the formula

$$W = \frac{1}{2c^2} \int \frac{\boldsymbol{i}_1 \cdot \boldsymbol{i}_2}{R} dS_1 dS_2 .$$

Here, $\mathrm{d}S_1$ and $\mathrm{d}S_2$ are the elements of the surface of the solenoid, R is the distance between them, the density of the surface current that replaces the current flowing in the solenoid winding is expressed through $\boldsymbol{i}$ ($i_1 = i_2 = i = n\mathcal{I}$), and n is the number of coils per unit length.

It is convenient to solve the integral in cylindrical coordinates:

$$W = \frac{\pi n^2 a^2 \mathcal{I}^2}{c^2} \int_0^h \mathrm{d}z_1 \int_0^h \mathrm{d}z_2 \oint \frac{\cos\alpha\,\mathrm{d}\alpha}{\sqrt{(z_1 - z_2)^2 + 4a^2 \sin^2\frac{\alpha}{2}}}$$
$$= \frac{2\pi^2 a^2 n^2 \mathcal{I}^2 h \left(1 - \frac{8a}{3\pi h}\right)}{c^2} ,$$

where all the terms of order $\left(\frac{a}{h}\right)^2$ and higher are discarded. Therefore,

$$L = 4\pi^2 a^2 n^2 h \left(1 - \frac{8a}{3\pi h}\right) .$$

If the a/h term is disregarded, compared with 1, we get the result obtained in the previous problem:

$$L = 4\pi^2 a^2 n^2 h = 4\pi n^2 S h .$$

2.110 For a circular cross-section

$$L = 4\pi N^2 \left(b - \sqrt{b^2 - a^2}\right) .$$

For the infinite solenoid, we find the self-induction per unit length $\mathcal{L} = \frac{L}{2\pi b}$ by passing to the limit $b \to \infty$ with a specified number of coils per unit length $n = \frac{N}{2\pi b}$:

$$\mathcal{L} = 4\pi^2 n^2 a^2 = 4\pi n^2 S$$

(compare this with Problem 2.108).

For a rectangular cross-section

$$L = 2N^2 h \ln\frac{2b + a}{2b - a} .$$

When $b \gg a$, we again get $\mathcal{L} = 4\pi n^2 S$.

2.111 Using formula (2.75), we find the magnetic energy of a unit length of a line. The vector potential of a rectilinear conductor with current was found in Problem 2.71. For conductor 1 (Figure 2.22), we write it as

$$A_{1z} = C - \frac{\mathcal{J} r_1^2}{c a^2} \qquad \text{when} \quad r_1 < a ,$$
$$A_{1z} = C - \frac{\mathcal{J}}{c}\left(1 + 2\ln\frac{r_1}{a}\right) \qquad \text{when} \quad r_1 > a . \tag{1}$$

We find the vector potential created by conductor 2 by substituting into (1) $-\mathcal{J}$ for $\mathcal{J}$, b for a, and r_2 for r_1.

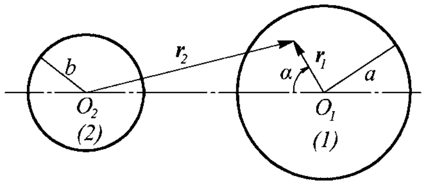

Figure 2.22 Finding the self-induction of a long dual conductor line.

We find the magnetic energy:

$$W = \frac{\mathcal{J}}{2\pi c a^2} \int\limits_1 (A_{1z} + A_{2z}) \mathrm{d}S_1 - \frac{\mathcal{J}}{2\pi c b^2} \int\limits_2 (A_{1z} + A_{2z}) \mathrm{d}S_2 \,. \tag{2}$$

The integrals in (2) may be solved with the use of a mathematical reference book. Then, considering the connection between the coefficient of induction and the magnetic energy of the system, we finally, get

$$\mathcal{L} = 1 + 2 \ln \frac{h^2}{ab} \,.$$

2.112* The full magnetic energy of a current flowing in a conductor consists of two parts:

$$W = W_1 + W_2 \,, \tag{1}$$

where

$$W_1 = \frac{1}{8\pi} \int H_1^2 \mathrm{d}V$$

is the energy is stored inside the conductor (so integration is done over the volume of that conductor) and

$$W_2 = \frac{1}{8\pi} \int H_2^2 \mathrm{d}V$$

is the energy stored in the rest of the space.

Presume that we may introduce the parameter r_0 that has the dimensionality of length and satisfies the condition

$$a \ll r_0 \ll R \,, \tag{2}$$

where a is the radius of the conductor and R is the radius of curvature of the axis line of the conductor (which, in the general case, changes from point to point). Then, at distances less than r_0, the magnetic field may be regarded as coinciding with the field of an infinite rectilinear conductor. For instance, inside the conductor in question,

$$H_1 = \frac{2\mathcal{J} r}{c a^2}$$

(see Problem 2.71). This allows us to find the "inner" energy W_1:

$$W_1 = \frac{l \mathcal{J}^2}{4c^2} \, . \tag{3}$$

To find the "external" energy W_2, we make up an auxiliary surface S bearing on an arbitrary circuit lying on the surface of the conductor and introduce the scalar potential ψ. The scalar potential will experience a jump at S

$$\psi_+ - \psi_- = \frac{4\pi}{c} \mathcal{J} \, . \tag{4}$$

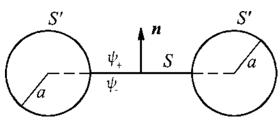

Figure 2.23 Finding the magnetic energy stored in the space outside a conductor.

The integral that W_2 is expressed through may be transformed in the following way:

$$\int (\boldsymbol{H} \cdot \boldsymbol{H}) \mathrm{d}V = -\int \boldsymbol{H} \cdot \nabla \psi \mathrm{d}V = -\int \div (\psi \boldsymbol{H}) \mathrm{d}V = -\oint \psi H_n \mathrm{d}S$$

(here, index 2 is omitted and the equation $\div \boldsymbol{H} = 0$ is used). In the latter integral, integration must be done over the two sides of the auxiliary surface S and over the surface of the conductor S' (see Figure 2.23, showing the cross-section between the conductor and a certain plane). The integral for an infinitely remote surface is reduced to zero owing to the finite dimensions of the conductor with current. Therefore,

$$W_2 = -\frac{1}{8\pi} \int_{S'} \psi H_n \mathrm{d}S + \frac{1}{8\pi} \int_S \psi_+ H_n \mathrm{d}S - \frac{1}{8\pi} \int_S \psi_- H_n \mathrm{d}S \, . \tag{5}$$

The first of these integrals is reduced to zero because, as per condition (2), the magnetic field on the surface S' coincides with the field of the rectilinear conductor and, therefore, has only a tangential constituent. To transform the other two integrals, we use equality (4) and the continuity condition of the component H_n. Now we get

$$W_2 = \frac{\mathcal{J}}{2c} \int_S H_n \mathrm{d}S \, . \tag{6}$$

At large distances from the conductor ($r > r_0$), the magnetic field does not depend on the distribution of current over the cross-section of the conductor, so the current may be regarded as flowing along the axis. At small distances ($a \le r < r_0$), this field coincides with the magnetic field of an infinite circular cylinder, and the current may also be regarded as flowing along the axis. Therefore, the integral in formula (6) is the magnetic flux created by the current flowing along the axis of the conductor through the surface bearing on a closed circuit that lies on the surface of the conductor. Using the expression of flux through the coefficient of magnetic induction, we get the following:

$$W_2 = \frac{\mathcal{J}^2}{2c^2} L' . \tag{7}$$

By formulas (1), (3), and (7) using the connection between the coefficient of self-induction and the magnetic energy of the system, we get the required formula for the coefficient of self-induction:

$$L = \frac{l}{2} + L' . \tag{8}$$

2.113 Using the result obtained in the previous problem, we get

$$L' = 4\pi b \left(\ln \frac{8b}{a} - 2 \right) .$$

And for full self-induction, we have

$$L = 4\pi b \left(\ln \frac{8b}{a} - \frac{7}{4} \right) .$$

2.114 $L_{12} = 2l - 2\sqrt{a^2 + l^2} + 2a \ln \frac{a+\sqrt{a^2+l^2}}{l}$.

2.115 Using the result obtained in Problem 2.114, we get

$$L_{12} = 8 \left[l - 2\sqrt{a^2 + l^2} + 2\sqrt{2a^2 + l^2} \right.$$
$$\left. + a \ln \frac{a + \sqrt{a^2 + l^2}}{l} - a \ln \frac{a + \sqrt{2a^2 + l^2}}{\sqrt{a^2 + l^2}} \right] ,$$
$$F = \frac{8\mathcal{I}_1\mathcal{I}_2}{c^2} \left[\frac{a^2 + 2l^2}{l\sqrt{a^2 + l^2}} - \frac{l\sqrt{2a^2 + l^2}}{a^2 + l^2} - 1 \right] .$$

2.116 $L = 2b + 8b \left[\ln \frac{2b}{a(1+\sqrt{2})} + \sqrt{2} - 2 \right]$.

2.117

$$\gamma' = \frac{c}{4\pi} \boldsymbol{E} \times \boldsymbol{H} + \nabla \times \boldsymbol{K} ,$$

where $\boldsymbol{K}$ is any vector that depends on the strength of an electromagnetic field.

2.118

$$\rho = \frac{3e}{4\pi a^3}\Theta(a - r)\,; \quad \boldsymbol{j} = \rho\boldsymbol{\Omega} \times \boldsymbol{r}\,,$$

where Θ is a step function (1.212).

2.119

$$\rho = \frac{e}{4\pi a^2}\delta(a - r)\,; \quad \boldsymbol{j} = \rho\boldsymbol{\Omega} \times \boldsymbol{r}\,.$$

2.120

$$\begin{aligned}
\operatorname{curl} \boldsymbol{E}_\omega &= \frac{i\omega}{c}\boldsymbol{H}_\omega\,,\\
\operatorname{curl} \boldsymbol{H}_\omega &= -\frac{i\omega}{c}\boldsymbol{E}_\omega + \frac{4\pi}{c}\boldsymbol{j}_\omega\,,\\
\operatorname{div} \boldsymbol{E}_\omega &= 4\pi\rho_\omega\,,\\
\operatorname{div} \boldsymbol{H}_\omega &= 0\,.
\end{aligned}$$

$$\boldsymbol{E}_\omega = -\nabla\varphi_\omega + \frac{i\omega}{c}\boldsymbol{A}_\omega\,, \quad \boldsymbol{H}_\omega = \nabla \times \boldsymbol{A}_\omega\,.$$

In integrals (2.112) the change of the sign of frequency is equivalent to the change of the sign of the imaginary unit i. This is why $\boldsymbol{E}_{-\omega} = \boldsymbol{E}^*_\omega$. This relation between Fourier harmonics is valid for any real function.

2.121

$$\boldsymbol{\Gamma}_\omega = \frac{c}{4\pi^2}\mathrm{Re}[\boldsymbol{E}_\omega \times \boldsymbol{H}^*_\omega]\,.$$

2.122

$$\begin{aligned}
i\boldsymbol{k} \times \boldsymbol{E}_k &= -\frac{1}{c}\dot{\boldsymbol{H}}_k\,,\\
i\boldsymbol{k} \times \boldsymbol{H}_k &= \frac{1}{c}\dot{\boldsymbol{E}}_k + \frac{4\pi}{c}\boldsymbol{j}_k\,,\\
i\boldsymbol{k} \cdot \boldsymbol{E}_k &= 4\pi\rho_k\,,\\
\boldsymbol{k} \cdot \boldsymbol{H}_k &= 0\,.
\end{aligned}$$

2.124 The Fourier transform of a point charge was found in Problem 1.137*:

$$\varphi_k = \frac{4\pi e}{k^2}\,, \quad \boldsymbol{E}_k = -i\boldsymbol{k}\varphi_k\,.$$

Another way to solve the problem is to use Fourier integral

$$\varphi(\boldsymbol{r}) = \int \varphi_k \exp(i\boldsymbol{k} \cdot \boldsymbol{r})\mathrm{d}^3k\,.$$

Having used the Laplace operator under the integral, we find $\Delta\varphi(\boldsymbol{r}) = \int(-k^2\varphi_{\boldsymbol{k}}) \exp(i\boldsymbol{k}\cdot\boldsymbol{r})\mathrm{d}^3k$, wherefrom it follows that $(\Delta\varphi)_{\boldsymbol{k}} = -k^2\varphi_{\boldsymbol{k}}$. On the other hand, applying the Fourier transform to both parts of the Poisson equation $\Delta\varphi(\boldsymbol{r}) = -4\pi e\delta(\boldsymbol{r})$, we get $(\Delta\varphi)_{\boldsymbol{k}} = -4\pi e$. Finally, what we now get is the solution above.

2.125

$$\begin{aligned}
\boldsymbol{k}\times\boldsymbol{E}_{k\omega} &= \frac{\omega}{c}\boldsymbol{H}_{k\omega}\,,\\
\boldsymbol{k}\times\boldsymbol{H}_{k\omega} &= -\frac{\omega}{c}\boldsymbol{E}_{k\omega} - i\frac{4\pi}{c}\boldsymbol{j}_{k\omega}\,,\\
i\boldsymbol{k}\cdot\boldsymbol{E}_{k\omega} &= 4\pi\rho_{k\omega}\,,\\
\boldsymbol{k}\cdot\boldsymbol{H}_{k\omega} &= 0\,.
\end{aligned}$$

2.126 Expansion in harmonic constituents:

$$\Delta\boldsymbol{A}_\omega + \frac{\omega^2}{c^2}\boldsymbol{A}_\omega = -\frac{4\pi}{c}\boldsymbol{j}_\omega\,,\quad \Delta\varphi_\omega + \frac{\omega^2}{c^2}\varphi_\omega = -4\pi\rho_\omega\,,$$
$$\operatorname{div}\boldsymbol{A}_\omega - \frac{i\omega}{c}\varphi_\omega = 0\,.$$

Expansion in plane waves:

$$\begin{aligned}
\ddot{\boldsymbol{A}}_k + k^2c^2\boldsymbol{A}_k &= 4\pi c\boldsymbol{j}_k\,,\\
\ddot{\varphi}_k + k^2c^2\varphi_k &= 4\pi c^2\rho_k\,,\\
ic\boldsymbol{k}\cdot\boldsymbol{A}_k + \dot{\varphi}_k &= 0\,.
\end{aligned}$$

Expansion in plane monochromatic waves:

$$\begin{aligned}
\left(k^2 - \frac{\omega^2}{c^2}\right)\boldsymbol{A}_{k\omega} &= \frac{4\pi}{c}\boldsymbol{j}_{k\omega}\,,\\
\left(k^2 - \frac{\omega^2}{c^2}\right)\varphi_{k\omega} &= 4\pi\rho_{k\omega}\,,\\
\boldsymbol{k}\cdot\boldsymbol{A}_{k\omega} - \frac{\omega}{c}\varphi_{k\omega} &= 0\,.
\end{aligned}$$

2.127 In the variables ξ, η a one-dimensional wave equation becomes $\partial^2\boldsymbol{E}/\partial\xi\partial\eta = 0$, wherefrom we obtain the required solution. It describes two *plane waves* of arbitrary form propagating in opposite directions along the Ox axis with speed c. The solution describing the propagation of plane waves in the direction $\boldsymbol{n}$ is as follows:

$$\boldsymbol{E}(\boldsymbol{r},t) = \boldsymbol{E}_1F(\boldsymbol{n}\cdot\boldsymbol{r} - ct) + \boldsymbol{E}_2\Phi(\boldsymbol{n}\cdot\boldsymbol{r} + ct)\,.$$

2.128 $\boldsymbol{\gamma} = cw\boldsymbol{n}\,,\quad w = (E^2 + H^2)/8\pi = E^2/4\pi$. Energy is carried through a vacuum at speed c.

2.130*

1.

$$\tan 2\alpha = \frac{2\boldsymbol{E}_{01} \cdot \boldsymbol{E}_{02}}{E_{01}^2 - E_{02}^2} ,$$
$$\boldsymbol{\mathcal{E}}_1 = \boldsymbol{E}_{01} \cos\alpha + \boldsymbol{E}_{02} \sin\alpha , \quad \boldsymbol{\mathcal{E}}_2 = -\boldsymbol{E}_{01} \sin\alpha + \boldsymbol{E}_{02} \cos\alpha . \tag{1}$$

To determine the type of polarization and the directions of rotation, we write (2.126) as a projection on the coordinate axes by selecting the usually used *right-hand* system of coordinates with the Ox axis directed along $\boldsymbol{\mathcal{E}}_1$ and the Oz axis running in the direction of the propagation of the wave $\boldsymbol{k}$.

2. Then we find in the general case the equation of the ellipse:

$$\frac{E_x^2}{\mathcal{E}_1^2} + \frac{E_y^2}{\mathcal{E}_2^2} = 1 . \tag{2}$$

This means elliptic polarization if $\mathcal{E}_1 \neq \mathcal{E}_2 \neq 0$, circular polarization if $\mathcal{E}_1 = \mathcal{E}_2$, and linear polarization if $\mathcal{E}_1 \neq 0$ and $\mathcal{E}_2 = 0$ or $\mathcal{E}_1 = 0$ and $\mathcal{E}_2 \neq 0$.

3. Write the argument of the trigonometric functions so that it increases as t increases. We get

$$E_x = \mathcal{E}_1 \cos(\omega t - \boldsymbol{k} \cdot \boldsymbol{r} - \alpha),$$

$$E_y = \pm\mathcal{E}_2 \sin(\omega t - \boldsymbol{k} \cdot \boldsymbol{r} - \alpha) ,$$

where $\mathcal{E}_1 \geq 0$ and $\mathcal{E}_2 \geq 0$, the plus sign in the second formula corresponds to the three right-hand-side vectors $\boldsymbol{\mathcal{E}}_1$, $\boldsymbol{\mathcal{E}}_2$, and $\boldsymbol{k}$, and the minus sign corresponds to the three left-hand-side vectors. When we have a plus sign, the wave has right-hand helicity, that is, the direction of rotation of the vector $\boldsymbol{E}$ and the direction of propagation make a right-handed screw. When we have a minus sign, the helicity is left-handed (a left-handed screw). Historically, the opposite terminology is used in optics: the right-handed rotation of the vector $\boldsymbol{E}$ is called left-handed and the rotation in the opposite direction is called right-handed.

2.131 Following the method used in the previous problem, we write

$$\boldsymbol{E}_0 = \boldsymbol{E}_1 + \boldsymbol{E}_2 = a\boldsymbol{e}_x + be^{i\chi}\boldsymbol{e}_y = (\boldsymbol{\mathcal{E}}_1 + i\boldsymbol{\mathcal{E}}_2)e^{i\alpha}$$

and find

$$\boldsymbol{\mathcal{E}}_1 = a\cos\alpha\,\boldsymbol{e}_x + b\cos(\chi - \alpha)\boldsymbol{e}_y,$$

$$\boldsymbol{\mathcal{E}}_2 = -a\sin\alpha\,\boldsymbol{e}_x + b\sin(\chi - \alpha)\boldsymbol{e}_y,$$

$$\tan 2\alpha = \frac{b^2 \sin 2\chi}{a^2 + b^2 \cos 2\chi} , \quad -\pi < \alpha \leq \pi .$$

(the latter equality follows from the condition $\boldsymbol{\mathcal{E}}_1 \cdot \boldsymbol{\mathcal{E}}_2 = 0$). The helicity of the resulting wave is determined by the sign of the product $\boldsymbol{\mathcal{E}}_2 \cdot \boldsymbol{e}_{y'}$, where $\boldsymbol{e}_{y'} = \boldsymbol{n} \times \boldsymbol{\mathcal{E}}_1/\mathcal{E}_1$ is the third ort, one of the three right-handed ones including $\boldsymbol{e}_{x'} = \boldsymbol{\mathcal{E}}_1/\mathcal{E}_1$ and $\boldsymbol{n}$. Using the expressions obtained, we find $\boldsymbol{\mathcal{E}}_2 \cdot \boldsymbol{e}_{y'} = ab \sin\chi/\mathcal{E}_1$. According to the results obtained in the previous problem, when $ab > 0$ and $\sin\chi > 0$, $0 < \chi < \pi$, the helicity is right-handed and when $\sin\chi < 0$, $-\pi < \chi < 0$, it is left-handed. When $\chi = 0$ and when $\chi = \pm\pi$, the polarizations are linear in the two mutually perpendicular planes.

2.132 When $\chi = 0$, the polarization is linear, and the plane of polarization passes through the bisector of the angle between the Ox and Oy axes. When $\chi = \pi$, the polarization is also linear, and the plane of polarization passes through the bisector of the angle between the Ox and $-Oy$ axes. When $\chi = \pi/2$, the polarization is circular with right-handed helicity (Figure 2.24a). When $\chi = -\pi/2$, the polarization is circular with left-handed helicity (Figure 2.24b). In all other cases, the polarization is elliptical; when $0 < \chi < \pi$, the helicity is right-handed and when $-\pi < \chi < 0$, it is left-handed.

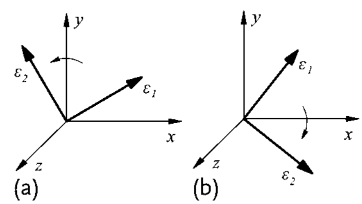

Figure 2.24 Circular polarization with right-hand helicity (a), and circular polarization with left-hand helicity (b).

2.133 $\boldsymbol{e}^{(1)} = \boldsymbol{e}^{(2)*} = (\boldsymbol{e}_x + i\boldsymbol{e}_y)/\sqrt{2}$.

2.134 $\boldsymbol{\mathcal{E}} = \sqrt{2}E_0\boldsymbol{e}'$. The polarization is linear, and ort $\boldsymbol{e}'$ makes an angle α with the Ox axis (Figure 2.25).

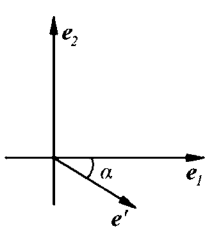

Figure 2.25 Linear polarization.

2.135*

$$J_{\alpha\beta} = I^{(1)} e_\alpha^{(1)} e_\beta^{(1)*} + I^{(2)} e_\alpha^{(2)} e_\beta^{(2)*} ,$$

where $I^{(1)}$ and $I^{(2)}$ are the principal values of the Hermitian tensor (real values), and $\boldsymbol{e}^{(1)}$ and $\boldsymbol{e}^{(2)}$ are its own vectors, which are, in the general case, complex and describe elliptical polarization. They satisfy the conditions $\boldsymbol{e}^{(\sigma)} \cdot \boldsymbol{e}^{(\sigma')*} = \delta_{\sigma\sigma'}$, $\boldsymbol{e}_\alpha^{(\sigma)} e_\beta^{(\sigma)*} = \delta_{\alpha\beta}$ (sum over repeated indices). The values introduced in Example 2.21 are expressed through $I^{(1)} \geq I^{(2)}$:

$$I_n = 2I^{(2)} , \quad I_p = I^{(1)} - I^{(2)} , \quad P = \frac{I^{(1)} - I^{(2)}}{I^{(1)} + I^{(2)}} , \quad \rho = \frac{2I^{(2)}}{I^{(1)} + I^{(2)}} .$$

2.136 Introduce rectangular axes $x' \parallel a$ and $y' \parallel b$. In these axes, the complex amplitude will be follows:

$$\boldsymbol{E}_0 = a\boldsymbol{e}_{x'} \pm ib\boldsymbol{e}_{y'} .$$

where the plus sign corresponds to the left-handed elliptical polarization and the minus sign corresponds to right-handed. elliptical polarization The intensity $I = a^2 + b^2$. The phase is selected as equal to zero for x' components of the field. Now, expressing the orts $\boldsymbol{e}_{x'}$ and $\boldsymbol{e}_{y'}$ through $\boldsymbol{e}_x$ and $\boldsymbol{e}_y$, we get the following for the components I_{ik}:

$$I_{11} = a^2 \cos^2\vartheta + b^2 \sin^2\vartheta,$$

$$I_{22} = a^2 \sin^2\vartheta + b^2 \cos^2\vartheta,$$

$$I_{12} = (b^2 - a^2) \sin\vartheta \cos\vartheta \mp iab = I_{21}^* .$$

The upper sign corresponds to left-handed elliptical polarization and the lower sign corresponds to right-handed elliptical polarization. When $b = 0$, the polarization is linear and tensor I_{ik} is as follows:

$$I_{ik} = I \begin{pmatrix} \cos^2\vartheta & \sin\vartheta\cos\vartheta \\ \sin\vartheta\cos\vartheta & \sin^2\vartheta \end{pmatrix} .$$

When $a = b = \sqrt{I/2}$, the polarization is circular and

$$I_{ik} = \frac{I}{2} \begin{pmatrix} 1 & \mp i \\ \pm i & 1 \end{pmatrix} .$$

2.137 The amplitude of the total wave

$$\boldsymbol{E} = \boldsymbol{E}_1 + \boldsymbol{E}_2 = E(\boldsymbol{e}^{(1)} + \boldsymbol{e}^{(2)} e^{i\alpha}) ,$$

where α is the randomly changing phase shift, $|\boldsymbol{E}|^2 = I$. The components of the tensor of polarization are equal by definition (see (2.127)):

$$I_{ik} = \overline{E_i E_k^*} = \overline{I(\boldsymbol{e}^{(1)} + \boldsymbol{e}^{(2)} e^{i\alpha})_i (\boldsymbol{e}^{(1)} + \boldsymbol{e}^{(2)} e^{-i\alpha})_k} .$$

When averaging by time, we get $\overline{e^{\pm i\alpha}} = 0$, which is why the tensor of polarization is as follows:

$$I_{ik} = I \begin{pmatrix} 1 + \cos^2\vartheta & \sin\vartheta\cos\vartheta \\ \sin\vartheta\cos\vartheta & 1 - \cos^2\vartheta \end{pmatrix} .$$

Therefore, using the result from Example 2.21, we get

$$P = |\cos\vartheta| .$$

The same result may be obtained by diagonalization of tensor I_{ik}. Its principal values are $I_1 = 1 + |\cos\vartheta|$ and $I_2 = 1 - |\cos\vartheta|$. Therefore, again, $P = (I_1 - I_2)/(I_1 + I_2) = |\cos\vartheta|$. The basic vectors are $\boldsymbol{e}_1 = \left(\cos\frac{\vartheta}{2}, \sin\frac{\vartheta}{2}\right)$ and $\boldsymbol{e}_2 = \left(-\sin\frac{\vartheta}{2}, \cos\frac{\vartheta}{2}\right)$. In the case in question, they are real.

The resulting wave includes its nonpolarized portion with intensity $I(1-|\cos\vartheta|)$ and the portion linearly polarized along the direction $\boldsymbol{e}_1 = \left(\cos\frac{\vartheta}{2}, \sin\frac{\vartheta}{2}\right)$ with intensity $I|\cos\vartheta|$:

$$(I_{ik}) = I(1 - |\cos\vartheta|)(\delta_{ik}) + I|\cos\vartheta| \begin{pmatrix} \cos^2\frac{\vartheta}{2} & \sin\frac{\vartheta}{2}\cos\frac{\vartheta}{2} \\ \sin\frac{\vartheta}{2}\cos\frac{\vartheta}{2} & \sin^2\frac{\vartheta}{2} \end{pmatrix} .$$

The resulting wave is fully polarized (although not monochromatic) when $\vartheta = 0$. When $\vartheta = \pi/2$, it is fully depolarized.

2.138 Polarization tensor

$$I_{ik} = \begin{pmatrix} 1_1 + \frac{1}{2}I_2 & \frac{1}{2}I_2 \\ \frac{1}{2}I_2 & \frac{1}{2}I_2 \end{pmatrix}$$

(the x_1 axis coincides with the direction of polarization of the first wave).

The degree of polarization

$$P = \frac{\sqrt{I_1^2 + I_2^2}}{I_1 + I_2} .$$

The resulting wave includes a nonpolarized wave with intensity $(I_1 + I_2)(1 - P)$ and a linearly polarized wave. The direction of linear polarization makes an angle

$$\vartheta = \arctan\frac{\sqrt{I_1^2 + I_2^2} - I_1}{I_2}$$

with the direction of polarization of the first wave.

2.139* $\rho = (1 - \xi)/(1 + \xi)$. When $\xi = 0$, the wave is nonpolarized and when $\xi = 1$, it is fully polarized. That is why the ξ is called the degree of polarization.

Assume that $\xi_i = \xi \eta_i$, where $\eta_1^2 + \eta_2^2 + \eta_3^2 = 1$. Then

$$I_{ik} = \frac{I}{2}(1-\xi)\delta_{ik} + \frac{I\xi}{2}\left(1 + \sum_{l=1}^{3} \eta_l \tau_{ik}^{(l)}\right) .$$

The first member of this expression corresponds to the fully nonpolarized condition and the second one corresponds to the fully polarized condition. In case (i), $\eta_3 = 1$ and $\eta_1 = \eta_2 = 0$.

Comparing

$$I''_{ik} = I\xi \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

with the expression $I_{ik} = I n_i n_k^*$, we see that in this case, $n_1 = 1$ and $n_2 = 0$, that is, tensor I''_{ik} describes a wave that is linearly polarized in the direction of the x axis (the wave propagates in the z direction).

Similarly, it is easy to see that in case (ii), $\eta_1 = 1$ and $\eta_2 = \eta_3 = 0$, the wave is linearly polarized in the direction that makes an angle of 45° with the x axis, and in case (iii), $\eta_2 = 1$ and $\eta_1 = \eta_3 = 0$, the wave is circularly polarized.

2.140 The wave packet satisfies the homogeneous d'Alembert equation under the condition $\omega^2 = c^2 k^2$. The packet

$$\Psi(x,0) = A(x,0)\exp(ik_0 x) , \quad \text{where}$$

$$A(x,0) = a_0\sqrt{\pi}\Delta k \exp\left(-\frac{x^2 \Delta k^2}{4}\right) ,$$

corresponds to the Gaussian amplitude function.

The amplitude of the wave packet $A(x,0)$ is shaped as the Gaussian curve. It becomes vanishingly small when $|x\Delta k| \gg 1$. Therefore, the width of the packet Δx is connected through the relation $\Delta x \cdot \Delta k \approx 1$ with the width Δk in the space of wave vectors. That relation has a universal character and is valid for electromagnetic or any other waves. In quantum mechanics, it plays a special role for the waves of probability, leading to the uncertainty relation for the coordinates and impulse of a microscopic particle.

2.141 (i) $\overline{\Delta x^2} \cdot \overline{\Delta k^2} = \frac{1}{2}$, (ii) $\overline{\Delta x^2} \cdot \overline{\Delta k^2} \to \infty$.

2.142

$$\Psi(0,t) = A(0,t)\exp(-i\omega_0 t) , \quad \text{where}$$

$$A(0,t) = a_0\sqrt{\pi}\Delta\omega \exp\left[-\frac{t^2\Delta\omega^2}{4}\right] , \quad \Delta t \cdot \Delta\omega \approx 1 .$$

2.143

$$\psi(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(t)e^{ikct}\mathrm{d}t .$$

2.144

$$\Delta x_{\min} = \frac{\lambda}{2\pi \sin\theta} ,$$

where θ is half of the angle of the conical beam spread of the light directed from the microscope lens to the object under observation.

2.145 The width Δx of the wave impulse cast by the radar is connected to the transverse variation of the wave vectors $k_\perp$ by the relation $\Delta x \cdot k_\perp \geq 1$. On the other hand, it is obvious that $\Delta x / l \approx k_\perp / k$. From these two estimations, find the uncertainty in the determination of the position of the object:

$$\Delta x \geq \sqrt{l\lambda} .$$

2.146* Consider integral (1) over the closed circuit C (Figure 2.26), where point $t' = t$ is circumvented from below along the smaller semicircle. Owing to the lack of special points of the integrand inside the circuit of integration, this integral is reduced to zero:

$$\oint_C \frac{U(t')}{t' - t} \mathrm{d}t' = 0 . \tag{1}$$

The integral along the great-circle arc is also reduced to zero because of the fast decrease of $U(t')$ when $|t'| \to \infty$ in the lower semiplane. The integral along the minor semicircle c is calculated directly and gives the half-residue of the integrand:

$$\int_c \frac{U(t')}{t' - t} \mathrm{d}t' = i\pi U(t) . \tag{2}$$

The remaining integral along the real axis with the excepted point must be calculated in terms of the principal value, that is, the fragment $(t - \rho, t + \rho)$ of the real axis is excepted from integration when $\rho \to 0$. Finally, we get the following from (1) and (2):

$$i\pi U(t) + \mathcal{P} \int_{-\infty}^{\infty} \frac{U(t')}{t' - t} \mathrm{d}t' = 0 , \tag{3}$$

After separating the real part from the imaginary one, we get relations (2.141).

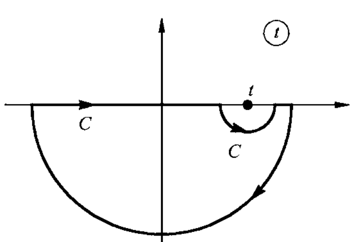

Figure 2.26 The path of integration leading to Hilbert transforms.

2.147 $\Gamma_\omega = \frac{ca^2(\omega)}{4\pi^2}$.

2.148

$$A(t) = |U(t)| \,, \quad \Phi(t) = \omega_0 t + \arctan\left(i \frac{U^* - U}{U^* + U}\right) .$$

2.149

$$|a(\omega)|^2 = \frac{A_0^2}{(\omega - \omega_0)^2 + \gamma^2/4} .$$

The signal will be narrow band (quasi monochromatic) when $\gamma \ll \omega_0$. If we define Δt as the time needed for the intensity of the signal $I \propto |U(t)|^2$ to decrease e times and $\Delta\omega$ as the detuning causing the spectral power to also decrease e times, then $\Delta t \cdot \Delta\omega = \sqrt{e-1}/2$. The shape of the spectrum obtained in this problem is called *the Lorentz contour*.

2.150 Select Lorentzian calibration of potentials (2.105). The function χ (see Example 2.19) satisfies the homogeneous d'Alembert equation (2.111). φ and $\partial\chi/\partial t$ also satisfy that equation. This is why when using the calibrating transform, we may select χ, which will reduce the scalar potential to zero: $\varphi' = \varphi - (1/c)(\partial\chi/\partial t) = 0$.

2.151• The formulas stated in the problem are derived from the obvious expression, $\Delta N = 2\Delta n_1 \Delta n_2 \Delta n_3$, for the number of the eigenmodes of the field.

2.152• Expand the real orts $\boldsymbol{\epsilon}_\alpha, \alpha = 1, 2, 3$ of a certain Cartesian system of coordinates in three mutually orthogonal orts $\boldsymbol{e}^{(1)}$, $\boldsymbol{e}^{(2)}$, and $\boldsymbol{n}$: $\boldsymbol{\epsilon}_\alpha = e^{(1)}\alpha \boldsymbol{e}^{(1)} + e_\alpha^{(2)} \boldsymbol{e}^{(2)} + n_\alpha \boldsymbol{n}$. Having obtained the scalar products, we get $\delta_{\alpha\beta} = \boldsymbol{\epsilon}_\alpha \cdot \boldsymbol{\epsilon}_\beta = e_\alpha^{(1)} e_\beta^{(1)*} + e_\alpha^{(2)} e_\beta^{(2)*} + n_\alpha n_\beta$.

3
The Special Theory of Relativity and Relativistic Kinematics

3.1
The Principle of Relativity and Lorentz Transformations

3.1.1
Properties of Space–Time and Intervals

The special theory of relativity is the science concerned with the study of the commonest properties of space, time, and motion. These properties established by investigations of mechanical and especially electromagnetic phenomena were formulated in the early twentieth century in the works of Albert Einstein,[1)] Hendrik Lorentz, Henri Poincaré,[2)] Max Planck, Hermann Minkowski, and other outstanding physicists and mathematicians. The main postulates and inferences of the special theory of relativity were confirmed in the course of further development of experimental and theoretical physics and modern technologies (quantum mechanics and quantum field theory, physics and technics of acceleration of the charged particles, nuclear power engineering, etc.). They underlie the most fundamental concepts of modern physical science that provide the basis for the development of the most up-to-date theories.

The special theory of relativity itself appeared on the basis of the notions about the characteristics of space, time, and motion, elaborated by classical mechanics. But, these notions were deepened, critically analyzed, and changed, and some important points were added by Einstein, taking into account new experimental data obtained during research into electromagnetic phenomena, especially in moving media. We enumerate the basic features of space and time which are well known from classical mechanics.

1) Albert Einstein (1870–1955), an ingenious German physicist, was the founder of the special and general theories of relativity. His works in the fields of electrodynamics, the theory of gravitation, quantum mechanics, statistical physics, and physical kinetics underlie modern physics and cosmology. He was a Nobel Prize Winner.

2) Jules Henri Poincaré (1854–1912) was an outstanding French mathematician, physicist, astronomer, and philosopher.

Foundations of Classical and Quantum Electrodynamics, First Edition. Igor N. Toptygin.

1. There are *inertial frames of reference* (IFR) with respect to which a free particle ("material point" of classical mechanics) moves uniformly and rectilinearly (the rest is a particular case of such motion).

   Note: In what follows, it will be assumed that a system of three rectangular axes of Cartesian coordinates is chosen in each IFR. The coordinates of an arbitrary point x, y, z (they form radius vector $\boldsymbol{r} = (x, y, z)$) can be measured with a scale ruler; they are actually the shortest distances (perpendicular lines) between the chosen point and the coordinate planes. This means that the *geometry of three-dimensional space is Euclidean*. Moreover, each IFR has a clock to count time intervals Δt. Any periodic physical process may serve as the clock.

2. In any IFR, the space free of matter and physical fields is *uniform and isotropic* and time is *homogeneous*.

   Note: Space uniformity means the equivalence of all parts of space, that is, any mechanical system placed in an arbitrary region of free space behaves exactly as it does in any other region. It is impossible to distinguish one part of uniform space from another on the basis of the behavioral characteristics of mechanical phenomena. Space isotropy means the equivalence of all directions of space, that is, neither the properties nor the behavior of mechanical objects depend on their orientation in an isotropic space. Finally, time homogeneity implies the equivalence of all moments of time: any mechanical system travels similarly regardless of the moment of time at which the movement was triggered, that is, when the initial conditions of the motion were fixed. Certainly, the initial conditions should be identical in different cases.

3. Any mechanical phenomenon behaves similarly under similar initial conditions in all IFR (*the principle of relativity of classical mechanics* or IFR equivalence).
4. Interactions between bodies and information-carrying signals can propagate with infinite velocity.

   Note: The possibility of infinite interaction propagation velocities is implied in the concept of forces dependent on the coordinates of the interacting bodies: $\boldsymbol{F}(\boldsymbol{r}_1(t) - \boldsymbol{r}_2(t))$. The equations of motion in classical mechanics contain radius vectors of the bodies, $\boldsymbol{r}_1(t)$ and $\boldsymbol{r}_2(t)$, at one and the same time t. A change in the position of one body at such a moment causes a change in the force acting on the second body regardless of the distance between them. The interaction propagates instantaneously.

5. Coordinates and time in two IFRs moving at relative speed V along the Ox axis (Figure 3.1) are related by the expressions

$$x' = x - Vt\,, \quad y' = y\,, \quad z' = z\,, \quad t' = t\,, \tag{3.1}$$

which are called *Galilean transformations.*[3)]

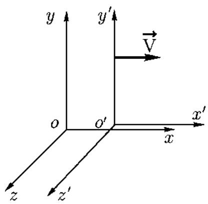

Figure 3.1 Two inertial reference frames.

To recall, time is untransformable: it is absolute and the same in all frames of reference. This property of time in classical mechanics has a direct bearing on the possibility of an infinite propagation velocity of signals and interactions. Let a device placed in system S isotropically emit in all directions a signal propagating with infinite speed. Detectors situated at any points in space and in any reference frame (not necessarily inertial ones) moving with finite velocities with respect to S record this signal at the moment at which it is emitted. In other words, such signals may serve as "time markers," with the help of which the same, that is, absolute, time $t' = t'' = \ldots = t$ is measured in all systems.

Simultaneity is defined in a like manner: if two given signals are simultaneously emitted at two different points of system S, they are synchronously detected in system S'. Events that are simultaneous in one frame of reference are simultaneous in another. In classical physics, simultaneity is absolute.

In the case of an arbitrary direction of the relative velocity, $\boldsymbol{r}' = \boldsymbol{r} - \boldsymbol{V}t$ and $t' = t$ instead of (3.1). Differentiation of the former equality with respect to absolute time yields the velocity addition law of classical mechanics:

$$\boldsymbol{v}' = \boldsymbol{v} - \boldsymbol{V} \,. \tag{3.2}$$

The velocities are summed up in accordance with the vector addition law in three-dimensional space (parallelogram rule); their absolute values are not limited.

Let us turn now to the formulation of the basic postulates of the special theory of relativity and their comparison with the principles of classical mechanics.

1. The postulate of IFRs and their definition remain valid in the special theory of relatively.
2. The limits of applicability of the assertion of space–time symmetry properties (uniformity and isotropy of free space, homogeneity of time) are extended to cover not only mechanical but also all physical phenomena. This implies, for example, as a consequence of space uniformity, any physical phenomenon (pendulum oscillations, emission of electromagnetic waves from an antenna, decay of elementary particles, etc.) should behave in a similar manner in any place

3) Galileo Galilei (1564–1642) was an outstanding Italian physicist and astronomer, and one of the founders of exact natural science.

of the free space. The notions of space isotropy and time homogeneity are generalized by analogy. Although physicists deal with physical phenomena par excellence, none of them seem to doubt that the notions being discussed can be extended to any, not only physical, natural phenomena.

3. The principle of relativity of classical mechanics is generalized to all physical phenomena without any exception as one of the most universal laws of nature. In this general sense, it can be formulated as follows: *any natural phenomenon behaves similarly in all IFR given their initial state is the same.*
4. Interactions between bodies and signals transmitting information cannot propagate with an arbitrarily high velocity. There is a maximum (limiting) propagation velocity of signals and interactions that coincides with the velocity of light in a vacuum:

$$c = 299\,792\,458\,\mathrm{m/s} \approx 3 \times 10^{10}\,\mathrm{cm/s}\,. \tag{3.3}$$

Another, equivalent formulation of this proposition is the velocity of light in a vacuum is identical in all reference frames; it depends neither on the source of motion nor on the light detector *(principle of the constancy of the velocity of light).*[4] *The unity of points 3 and 4 represents the principle of relativity formulated by Einstein* (Einstein, 1905).

One can readily see that these postulates do not agree with the laws of classical mechanics. Given the speed of light in system S is c, it becomes equal to $\boldsymbol{c}' = \boldsymbol{c} - \boldsymbol{V}$ in another IFR, S', that moves relative to the initial system S with velocity $\boldsymbol{V}$. Speed c' in absolute magnitude may be either higher or lower than c depending on the angle between $\boldsymbol{c}$ and $\boldsymbol{V}$; this confirms the incompatibility of Einstein's principle of relativity with classical notions.

Each event is considered in a certain IFR. Let us characterize it by the spatial coordinates x, y, z and the moment of time t at which it occurred. Collectively, these four quantities (ct, x, y, z) determine the coordinates of the event in four-dimensional Minkowski space–time.[5] The time coordinate ct is chosen in such a way that its dimension coincides with that of the spatial coordinates.

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2 \tag{3.4}$$

is called the squared interval between two closely occurring events (or close points in the four-dimensional space–time). In another IFR, the interval $\mathrm{d}s'^2$ will be expressed in a similar way through differentials of the primed space–time coordinates.

4) For the first time the constancy of the light velocity was demonstrated in the experiment conducted by Michelson and Morley. Albert Abraham Michelson (1852–1931) – was an American physicist, the author of precision optic experiments, and Nobel Prize Winner. Modern experiments on check STR and their results are discussed in the article by Aleshkevich (2012).

5) Hermann Minkowski (1864–1909) was a German physicist and a mathematician. He introduced the idea of unifying the three space dimensions and time into one joint four-dimensional space–time. He was one of the founders of the special theory of relativity.

Example 3.1

Using the postulates 1–4 formulated above, prove the equality

$$\mathrm{d}s^2 = \mathrm{d}s'^2 = \mathrm{inv} \tag{3.5}$$

(interval invariance) for the case of transition from one IFR, S, to another, S'.

Solution. On transition from one IFR to another, differentials of four-dimensional coordinates must be linearly related. The coefficients in the formulas of constraints depend only on the relative speed of the two systems and not on their coordinates and time; otherwise, the property of space–time uniformity would be violated. Let the origins of the coordinates of systems S and S' coincide at $t = t' = 0$, and let a flash of light at the common origin of these coordinate occur at this moment. After time $\mathrm{d}t$, the spherical front (three-dimensional space isotropy) will be described in system S by the equation of a spherical surface,

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2 = 0\,,$$

and in system S' by a similar equation in primed coordinates (equality of systems!). The absolute value of the speed of light is the same in both systems:

$$\mathrm{d}s'^2 = c^2\mathrm{d}t'^2 - \mathrm{d}x'^2 - \mathrm{d}y'^2 - \mathrm{d}z'^2 = 0\,.$$

It follows from the two last equalities that equating $\mathrm{d}s^2$ to zero yields the equality $\mathrm{d}s'^2 = 0$, and vice versa. The values $\mathrm{d}s^2$ and $\mathrm{d}s'^2$ have a similar order of smallness only in the case of their linear relationship, $\mathrm{d}s^2 = k\mathrm{d}s'^2$, which does not exclude the linear relation between differentials of the four-dimensional coordinates of the two systems. Because both quantities are scalars in the three-dimensional space (see Section 1.1), coefficient k may depend only on the squared three-dimensional vector V^2 (i.e., the relative speed of the two IFR) but not its direction. It follows from the equal status of the two systems that $\mathrm{d}s'^2 = k\mathrm{d}s^2$, or $\mathrm{d}s^2 = k^2\mathrm{d}s^2$, that is, $k = \pm 1$. However, the value of $k = -1$ cannot ensure the equality of three IFR: at $\mathrm{d}s^2 = -\mathrm{d}s'^2$ and $\mathrm{d}s'^2 = -\mathrm{d}s''^2$ one has $\mathrm{d}s^2 = -\mathrm{d}s'^2 = \mathrm{d}s''^2$, which distinguishes system S'. Therefore, the sole possibility is to choose $k = 1$, which leads to invariance of the small interval in the case of its transformation into any initial frame of reference. Integration allows this property to be extended to the finite intervals between arbitrary events 1 and 2: $s_{12} = s'_{12}$, that is,

$$s_{12} = [c^2(t_1 - t_2)^2 - (x_1 - x_2)^2 - (y_1 - y_2)^2 - (z_1 - z_2)^2]^{1/2} = \mathrm{inv}\,. \tag{3.6}$$

□

Example 3.2

Analyze Einstein's train thought experiment: let observer A be placed in the center of a long "train" moving with a relativistic[6] speed and let observer B stand on the ground as the train moves past. The devices at the front and back of the train equidistant from observer A give off two short flashes of light. These flashes reach both observers simultaneously at the moment they align. How do the two observers answer the following question: Which flash was emitted earlier than the other?

Solution. Observer A says: "The signals originated from the points equidistant from me and simultaneously reached me at the same time. Therefore, they were emitted simultaneously."

Observer B answers: "The signals reached me simultaneously but I was closer to the front of the train than to the back. Consequently, the signal from the back covered a longer distance, that is, it was emitted earlier than the signal from the front of the train." □

3.1.2
Lorentz Transformations

The last example shows that time, space, and the notion of simultaneity lose their absolute character when the finite speed of signal propagation is taken into account, that is, the period of time Δt between two events for a stationary observer differs from the period between the same events $\Delta t'$ for a moving observer. Time passes at different rates in different frames of reference, in conflict with the Galilean transformations (3.1). Their generalization taking account of the finite speed c was found[7] by Lorentz[8] in 1904:

$$x = \frac{x' + Vt'}{\sqrt{1 - V^2/c^2}} , \quad y = y' , \quad z = z' , \quad t = \frac{t' + Vx'/c^2}{\sqrt{1 - V^2/c^2}} . \tag{3.7}$$

This transformation corresponds to the case when the respective axes of the coordinate systems S and S' are parallel to each other, the relative velocity $\boldsymbol{V}$ is directed along the Ox axis (Figure 3.1), and the origins of the coordinate systems coincide at $t = t' = 0$. More general transformations of such type will be considered in Section 4.1.

In this and the next chapters we shall frequently use the notations

$$\beta = \frac{V}{c} , \quad \gamma = \frac{1}{\sqrt{1 - \beta^2}} . \tag{3.8}$$

6) That is, with speed comparable with the limiting speed c.
7) See the solution of Problem 3.1* and Section 4.1 for the derivation of the Lorentz transformations.
8) Hendrick Anton Lorentz (1853–1928) was an outstanding Dutch physicist, and the originator of the classical electronic theory of matter. Lorentz transformations underlie the special theory of relativity.

The inverses of the (3.7) transforms can be obtained by changing the sign of the relative velocity; they have the following form in notations (3.8):

$$x' = \gamma(x - \beta ct)\,, \quad y' = y\,, \quad z' = z\,, \quad t' = \gamma(t - \beta\frac{x}{c})\,. \tag{3.9}$$

The Lorentz transforms satisfy the correspondence principle: they can be converted into Galilean transformations (3.1) at $V \ll c$, $\gamma \approx 1$.

Example 3.3

Let the space–time coordinates x, ct of the reference frame S be plotted on the mutually perpendicular Cartesian axes. Plot the coordinate axes x' and ct' on this graph. Indicate the position of the x and ct axes when x' and ct' are plotted on the rectangular axes.

Solution. The time axis of system S' is a straight line $x' = 0$. In accordance with (3.9), this straight line in the axes system ct, x is given by $ct = x/\beta$ and makes angle $\alpha = \arctan(V/c)$ with the ct axis (Figure 3.2). The x' axis in system S' is specified by $ct' = 0$, which assumes the form $ct = \beta/x$ in system S and is a straight line making the same α angle with the x axis. The new coordinate axes became skew angular! (affine transformation altering the angle between the axes – see Section 1.1.). By plotting ct' and x' on the rectangular axes, one arrives at the skew-angular ct and x axes deflected through the α angle in different directions (Figure 3.3).

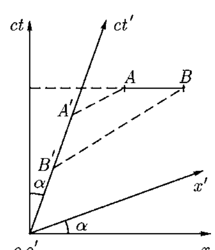

Figure 3.2 Connection between axes of two inertial frames of reference.

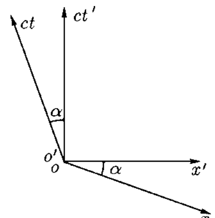

Figure 3.3 Connection between axes for inverse Lorentz transformation.

Figure 3.2 clearly illustrates the relativity of simultaneity. Events A and B are projected on the time axis ct at a single point AB, which suggests their simultaneity in system S, whereas their projections onto the time axis ct' of system S' are different. This means that these events are not simultaneous in system S' and that event B occurs early than event A. However, an attempt to determine the time that elapsed between the two events, $\Delta t' = A'B'/c$, from Figure 3.2 using the usual trigonometric relations fails and it is hardly possible to obtain the correct result ensuing from the Lorentz transformation (3.9):

$$\Delta t' = \gamma \beta c^{-1}(x_B - x_A) . \tag{3.10}$$

Not only rotation of the axes but also a change of the scale on transition to another system needs to be taken into consideration if such a result is to be obtained. □

Example 3.4

The time measured with a clock that is motionless with respect to a certain object is called the *proper time* of this object. Relate a small period of intrinsic time, $d\tau$, to coordinate time, dt, in the system with respect to which the object moves. Demonstrate that proper time is an invariant of the Lorentz transformation.

Solution. Let the object be at rest in system S'. In this system, $dt' = d\tau$ and $dx' = dy' = dz' = 0$. In system S the quantities dx, dy, and dz are the distances passed by the objects in time dt. It follows from the invariance of interval (3.5) that

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 = c^2(1 - \frac{v^2}{c^2})dt^2 ,$$

whence

$$d\tau = \frac{ds}{c} = \sqrt{1 - \frac{v^2}{c^2}}dt = \text{inv} . \tag{3.11}$$

Here, $\boldsymbol{v}(t)$ denotes the velocity of the inertial system instantaneously accompanying the object of interest and therefore coincident with the velocity of the object itself. In such form, the formula for the intrinsic time interval is suitable for non-uniformly moving objects too. □

Example 3.5

A solid rod has length l_0 in its eigenframe. What length would an observer measure if the rod moved with respect to the observer at velocity $\boldsymbol{v}$? What is the volume of a certain body measured by the observer if the volume of this body in its rest frame is $\mathcal{V}_0$?

Solution. The length depends on the mutual orientation of the rod and the relative velocity vector $\boldsymbol{v}$. Given both the rod and the velocity vector are aligned along the x axis, the distance between the ends of the rod in its rest frame $x'_B - x'_A = l_0$. To measure the length of the rod, an observer in system S must simultaneously (by his watch) make notches against the ends of the x axis in his own frame of reference: $x_B - x_A = l$, $t_B - t_A = 0$. Using the Lorentz transformations (3.9), we obtain the *Lorentz contraction of the length of a moving object*:

$$l = l_0\sqrt{1 - \frac{v^2}{c^2}}\ . \tag{3.12}$$

If the rod is oriented across the velocity vector, then $l = l_0$, because, according to (3.9), transverse scales do not change. The volume $\mathcal{V}$ of a moving body shrinks only if its longitudinal size decreases:

$$\mathcal{V} = \mathcal{V}_0\sqrt{1 - \frac{v^2}{c^2}}\ . \tag{3.13}$$

□

Example 3.6

How does the four-dimensional volume $\Delta\Omega = c\Delta t\mathcal{V}$ transform when passing into a different IFR?

Solution. Using (3.13) and (3.11), we find

$$\Delta\Omega = c\Delta t\mathcal{V} = c\Delta\tau\mathcal{V}_0 = \mathrm{inv}\ . \tag{3.14}$$

The four-dimensional volume is an invariant of the Lorentz transform. □

Example 3.7

The velocity of a particle in system S' is $\boldsymbol{v}' = \mathrm{d}\boldsymbol{r}'/\mathrm{d}t'$. Express its velocity $\boldsymbol{v} = \mathrm{d}\boldsymbol{r}/\mathrm{d}t$ in system S through $\boldsymbol{v}'$ and the relative speed of the system, $\boldsymbol{V}$. Can the absolute value of velocity $\boldsymbol{v}$ exceed the speed of light c?

Solution. Write down formulas (3.7) for differentials,

$$\mathrm{d}x = \frac{\mathrm{d}x' + V\mathrm{d}t'}{\sqrt{1 - V^2/c^2}} = \frac{\mathrm{d}t'(v'_x + V)}{\sqrt{1 - V^2/c^2}}\ , \quad \mathrm{d}y = \mathrm{d}y'\ , \quad \mathrm{d}z = \mathrm{d}z'\ ,$$
$$\mathrm{d}t = \frac{\mathrm{d}t' + V\mathrm{d}x'/c^2}{\sqrt{1 - V^2/c^2}} = \frac{\mathrm{d}t'(1 + v'_x V/c^2)}{\sqrt{1 - V^2/c^2}}\ ,$$

and divide the first three equalities term by term by the fourth one. The result is the *sum rule for relativistic velocities*:

$$v_x = \frac{v'_x + V}{1 + V v'_x/c^2} , \quad v_y = \frac{v'_y \sqrt{1 - V^2/c^2}}{1 + V v'_x/c^2} , \quad v_z = \frac{v'_z \sqrt{1 - V^2/c^2}}{1 + V v'_x/c^2} . \tag{3.15}$$

At $V \ll c$ and $|v'_x| \ll c$ these formulas turn into the relativistic velocity addition law (3.2):

$$v_x = v'_x + V , \quad v_y = v'_y , \quad v_z = v'_z .$$

When $v' \to c$, they account for the limiting character of the speed of light. Writing $v'_x = c \cos \vartheta$ and $v'^2_y + v'^2_z = c^2 \sin^2 \vartheta$ for the last case and calculating v^2 from (3.15), we find $v^2 = c^2$. In other words, the addition of the relative motion speed $\boldsymbol{V}$ in accordance with the relativistic law does not change the absolute value of $v' = c$. □

3.1.3
Pseudo-Euclidean Geometry

Let us familiarize ourselves in more detail with the geometry of four-dimensional Minkowski space–time. Individual points in the four-dimensional space–time denote the coordinates and time of a certain "event." The sequence of kinematic states of any body (i.e., its coordinates at different moments of time) is depicted by the *world line*.

In a given inertial system, the world lines are inherent not only in moving bodies but also in motionless bodies. By way of example, the world line of a body resting at a (spatial) point x_0 of the x axis is the straight line AB parallel to the ct axis and passing through x_0 (Figure 3.4); the world line of a body moving with constant speed $v = \text{const}$ (and passing through the origin of the coordinates at $t = 0$) is the straight line CD ($\tan \alpha = v/c$); the world line of a body moving with varying speed $v(t)$ is the curve MN; the world line of a light beam emitted at moment $t = 0$ from the origin of the coordinates in the direction of the x axis is the bisectrix of coordinate angle OF.

We have already found expressions (3.4) and (3.6) for the interval, that is, a quantity playing the role of the invariant distance between the points of the four-dimensional space. It follows from these expressions that the Minkowski space has non-Euclidean geometry, in which the Pythagorean[9] theorem, in particular, does not hold. Such geometry is referred to as *Pseudo-Euclidean*, where, unlike the Euclidean geometry of a three-dimensional space, equating the interval s_{12} to zero

9) Pythagoras (ca 570 to ca 500 BC) was an Ancient Greek philosopher and mathematician.

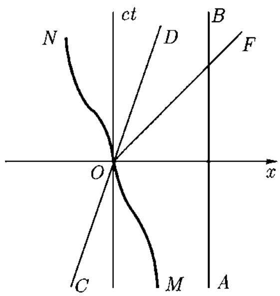

Figure 3.4 Examples of world lines.

does not mean that points 1 and 2 necessarily coincide. Depending on the relationship between $c^2(t_1 - t_2)^2$ and $(\boldsymbol{r}_1 - \boldsymbol{r}_2)^2$, intervals can be either real or imaginary. The intervals for which $s_{12}^2 > 0$ are called *time-like intervals* and those for which $s_{12}^2 < 0$ are *space-like intervals*; these properties occur in all IFR owing to interval invariance. Zero intervals are termed *light-like or null intervals*.

The character of an interval is closely related to the notion of causality; it determines the possibility of an causal relationship between events occurring at space–time points 1 and 2. If $s_{12}^2 > 0$, event 2 can be casually related to event 1 (and vice versa): it is possible to send a signal with velocity $v = r_{12}/(t_2 - t_1) < c$, from point 1 that will trigger event 2. The same is possible in the case of a light-like interval, only the signal should propagate with the limiting speed c. Events separated by a space-like interval cannot be casually conditioned because the signals cannot propagate with velocity $v = r_{12}/(t_2 - t_1) > c$.

Example 3.8

Let coordinates x, ct be plotted on the rectangular Cartesian axes in a certain plane. Designate on this plane regions 1, 2, and 3 possessing the following properties.

Region 1 – absolutely future; all events denoted by points in this region occur in any IFR after the event corresponding to the origin of the coordinates.

Region 2 – absolutely past; events in this region regardless of the IFR precede the event at the origin of the coordinates.

Region 3 – absolutely remote; there are IFR in which events in this region occur before, after, or simultaneously with the event (0,0), but there are no IFR in which events in this region and (0,0) would occur at one point of the three-dimensional space (such a variant is feasible for regions 1 and 2).

Solution. The regions sought are depicted in Figure 3.5.

They are divided by bisectrices of coordinate angles. In region 1, $t > 0$ and for all possible IFR $t' > 0$, because the ct' axes of these systems can be oriented only between the two lines corresponding to light-like intervals (i.e., between the lines of intersection of the plane and the light cone). If the ct' axis passes through a given point and the origin of the coordinates, one can find from $V = c \tan \alpha$ (see Example 3.3) the speed of the IFR in which the two events being considered occur at a single point of the three-dimensional space. Region 2 has analogous properties

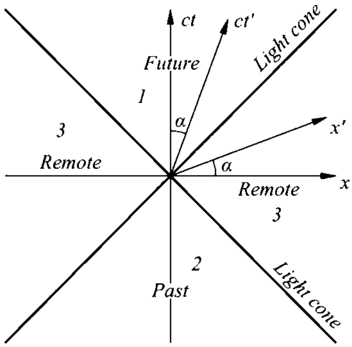

Figure 3.5 Kinematic plane of the special theory of relativity.

In region 3, the above formula defines IFR in which the event of interest occurs simultaneously with the event at the origin of the coordinates. □

In the pseudo-Euclidean space, two kinds of coordinates (contravariant and covariant) need to be introduced along with two types of tensor indices, as in the consideration of affine transformations in the three-dimensional space (see Section 1.1). The squared interval (3.4) between the adjacent points can be written in tensor notation:

$$ds^2 = g_{ik} dx^i dx^k \,, \tag{3.16}$$

where

$$dx^0 = c dt \,, \quad dx^1 = dx \,, \quad dx^2 = dy \,, \quad dx^3 = dz \tag{3.17}$$

are the differentials of contravariant four-dimensional coordinates, and

$$g_{ik} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{3.18}$$

is the metric tensor. Here and hereafter, summation must be over four values $i, k = 0, 1, 2, 3$ of the coincident Latin indices.

The contravariant metric tensor must be determined, as in the three-dimensional space (see formula (1.36) and point (3) in Problem 1.37), from the relations

$$g_{ik} g^{kl} = g_i^l = \delta_i^l \,, \tag{3.19}$$

where δ_i^l is the four-dimensional Kronecker symbol. It is readily seen from (3.18) that the contravariant components g^{ik} coincide with the respective covariant ones, that is, $g^{ik} = g_{ik}$.

Any four quantities, A^0, A^1, A^2, A^3, defined in all IFR and undergoing transformation on transition from one system to another as coordinates and time, that is, according to the rule

$$A^0 = \gamma(A'^0 + \beta A'^1)\,, \quad A^1 = \gamma(A'^1 + \beta A'^0)\,, \quad A^2 = A'^2\,, \quad A^3 = A'^3\,, \tag{3.20}$$

give rise to the contravariant components of the four-dimensional vector (4-vector) A^i, $i = 0, 1, 2, 3$. The three-dimensional vector $\boldsymbol{A} = (A^1, A^2, A^3)$ is called the space part and quantity A^0 is called the time part of 4-vector A^i. The covariant constituents of this vector A_i are defined in accordance with the lowering of index rule (1.35):

$$A_i = g_{ik} A^k = (A^0, -\boldsymbol{A})\,. \tag{3.21}$$

The scalar product of two 4-vectors and the square of a 4-vector are invariants of the Lorentz transforms. They are defined as the natural extension of formulas (1.267):

$$A_i B^i = A^i B_i = g_{ik} A^i B^k = \text{inv}\,, \quad A_i A^i = g_{ik} A^i A^k = \text{inv}\,. \tag{3.22}$$

Similarly to an interval, an arbitrary 4-vector can be null or isotropic ($A_i A^i = 0$), time like ($A_i A^i > 0$), or space like ($A_i A^i < 0$). Examples of four-dimensional vectors, besides the 4-radius vector in the Minkowski space, are provided by the 4-velocity u^i of a particle and its 4-acceleration w^i:

$$u^i = \frac{\mathrm{d}x^i}{\mathrm{d}\tau} = \left(\frac{c}{\sqrt{1 - v^2/c^2}}\,, \quad \frac{\boldsymbol{v}}{\sqrt{1 - v^2/c^2}}\right)\,, \quad w^i = \frac{\mathrm{d}u^i}{\mathrm{d}\tau} = \frac{\mathrm{d}^2 x^i}{\mathrm{d}\tau^2}\,, \tag{3.23}$$

where $\boldsymbol{v} = \mathrm{d}\boldsymbol{r}/\mathrm{d}t$ is the three-dimensional velocity of the particle and $\mathrm{d}\tau$ is the differential of proper time. Evidently, w^i is a 4-vector because $\mathrm{d}x^i/\mathrm{d}\tau$ is the ratio of 4-vector $\mathrm{d}x^i$ to scalar (invariant) $\mathrm{d}\tau$. For the same reason, w^i is a 4-vector too.

The phase of a monochromatic plane wave considered in Section 2.3, $\varphi = \boldsymbol{k} \cdot \boldsymbol{r} - \omega t$, is a relativistic invariant (scalar). It follows from the fact that a change of the phase $\Delta\varphi/2\pi$ determines the number of zero values of the field between two space–time points independent of the reference frame chosen. However, since $(ct, \boldsymbol{r})$ make up the four-dimensional radius vector, the frequency ω and the wave vector $\boldsymbol{k}$ of the plane wave also give rise to the wave 4-vector

$$k^i = \left(\frac{\omega}{c}, \boldsymbol{k}\right)\,, \quad k_i = \left(\frac{\omega}{c}, -\boldsymbol{k}\right)\,, \tag{3.24}$$

and the phase can be written as the scalar product of two 4-vectors:

$$\varphi = -k_i x^i\,. \tag{3.25}$$

Problems

3.1*. Derive the Lorentz transform (3.7) based on the special theory of relativity postulates 1–4 and interval invariance proved above (see formula (3.6)).

3.2. Let system S' move with respect to system S with speed V along the x axis. The clock in system S' at point (x'_0, y'_0, z'_0) passes at moment t'_0 point (x_0, y_0, z_0) in system S containing a clock that shows time t_0 at the same moment. Write the formulas of the Lorentz transformation for this case.

3.3. System S' moves with respect to system S with velocity V. All clocks in each of IFR are mutually synchronized. Prove that of the clocks present in systems S and S', the clock in one of them whose readings are sequentially compared with the readings of the two clocks in the other reference frame always runs more slowly. Express one period of time through the other. (The readings of the moving clocks are compared when they are aligned.)

3.4. The length of a rod moving along its axis in a certain frame of reference can be found in the following way: by measuring the period of time during which the rod passes a fixed point in this system and multiplying it by the rod's speed. Show that this method yields the ordinary Lorentz contraction.

3.5. System S' moves with respect to system S with speed V. At the moment when the origins of the coordinates coincided, the clocks of both systems localized at the origins showed the same time $t = t' = 0$. What coordinates will the world point have henceforth in either system if the clocks at this point in systems S and S' show the same time $t = t'$? Find the law governing the motion of this point.

3.6. Let time be measured making use of the periodic process of alternate reflection of a light spot from two mirrors attached to the ends of a rod of length l. One period is the time for the light spot to travel from one mirror to the other and back. The light clock is motionless in system S' and oriented parallel to the direction of motion. Show, on the basis of the postulate of the constancy of the speed of light, that the proper time interval $\mathrm{d}\tau$ is expressed through the period of time $\mathrm{d}t$ in system S by formula (3.11).

3.7. Solve the preceding problem for the case of the light clock oriented perpendicular to the direction of the relative velocity.

3.8. A "train" AB has length $l_0 = 8.64 \times 10^8$ km in the system in which it is at rest. The train travels past the first observer, who is at rest on the "platform," with velocity $V = 240\,000$ km/s. Find the time interval Δt in which the train will pass by this observer. What interval of the train passing by the first observer will be registered by a motionless second observer in the train?

3.9. What time Δt by the Earth clock would it take a rocket traveling at constant speed $V = \sqrt{0.9999}c$ to reach the star Proxima Centauri (lying four light years[10] away) and return? Calculate the duration of the space flight mission to ensure adequate provision of victuals and equipment for the crew members. What is the kinetic energy stored in such rocket if its mass is 10 000 kg?

3.10. Two scales of rest length l_0 are moving uniformly in opposite directions parallel to the common Ox axis. An observer connected with one of them notices that

10) A light year is the distance that light travels in a vacuum in 1 year.

the left and right ends of the scales coincide after a period of time Δt. What is the relative velocity v of the scales? What is the order in which their ends coincide for the observers connected with each scale and for the observer relative to which both scales move in opposite directions with equal speed?

3.11. Derive formulas of the Lorentz transformation from system S' to system S for the radius vector $\boldsymbol{r}$ and time t without assuming that the velocity $\boldsymbol{V}$ of system S' with respect to S is parallel to the Ox axis. Represent the result in the vector form.

Hint: Expand $\boldsymbol{r}$ into longitudinal and transverse components with respect to $\boldsymbol{V}$ and make use of Lorentz transformations (3.7).

3.12. Write formulas for the Lorentz transformations for an arbitrary 4-vector $A^i = (A^0, \boldsymbol{A})$ without assuming the speed $\boldsymbol{V}$ of system S' with respect to S is parallel to the Ox axis.

3.13. Derive formulas for velocity addition for the case when velocity $\boldsymbol{V}$ of system S' with respect to S has an arbitrary direction. Represent the formulas in vector form.

3.14. Suppose three frames of reference, S, S', and S'' are given. System S'' moves with respect to system S' at speed V' parallel to the Ox' axis and system S' moves with respect to S at speed V parallel to the Ox axis. The respective axes of all systems are parallel. Write the Lorentz transformations from S'' to S and derive from them the parallel velocity addition formula.

3.15*. Prove formula

$$\sqrt{1-\frac{v^2}{c^2}} = \frac{\sqrt{1-v'^2/c^2}\cdot\sqrt{1-V^2/c^2}}{1+\boldsymbol{v}'\cdot\boldsymbol{V}/c^2}\,,$$

where $\boldsymbol{v}$ and $\boldsymbol{v}'$ are the particle velocities in systems S and S', and $\boldsymbol{V}$ is the velocity of S' with respect to S.

3.16. Prove the relation

$$v = \frac{\sqrt{(\boldsymbol{v}'+\boldsymbol{V})^2-(\boldsymbol{v}'\times\boldsymbol{V})^2/c^2}}{1+\boldsymbol{v}'\cdot\boldsymbol{V}/c^2}\,,$$

where $\boldsymbol{v}$ and $\boldsymbol{v}'$ are the particle velocities in systems S and S', $\boldsymbol{V}$ and is the velocity of S' with respect to S.

3.17*. Three consecutive transformations of the reference frame occur: (i) transition from system S to system S' moving with respect to S at speed $\boldsymbol{V}$ parallel to the Ox axis; (ii) transition from system S' to system S'' moving with respect to S' with velocity $\boldsymbol{v}$ parallel to the Oy' axis; (iii) transition from system S'' to system S''' moving with respect to S'' with a velocity equal to the relativistic sum of velocities $-\boldsymbol{v}$ and $-\boldsymbol{V}$.[11] Prove that system S''' is at rest with respect to system S,

11) Note that the resulting velocity depends on the order in which the velocities are summated.

as expected, and $t''' = t$; however, S''' is turned with respect to system S through a certain angle in the xy *plane (Thomas precession)*. Calculate angle φ of Thomas precession.

Hint: Use general formulas for the Lorentz transformation (see Problem 3.10) and velocity addition (see Problem 3.12) and write them in the projections onto Cartesian axes.

3.18. Two scales of length l_0 are moving in opposite directions with equal velocities v with respect to a certain reference frame. What is the length l of each of the scales measured in the frame of reference connected with the other scale?

3.19. Two electron beams are directed oppositely with velocities $v = 0.9c$ with respect to a laboratory system of coordinates. What is the relative speed V of the electrons (i) from the standpoint of an observer in the laboratory and (ii) from the standpoint of an observer traveling together with one of the electron beams?

3.20. The effects resulting from the collision of two elementary particles do not depend on their uniform motion as a whole; these effects are determined only by the relative velocity of the particles. One and the same velocity can be imparted to the colliding particles in two ways (for simplicity, the particles are assumed to have equal masses m): (i) one accelerator speeds up the particles to energy E; then, the fast particles hit a motionless target composed of identical particles; (ii) two similar accelerators are positioned so that the particle beams they generate are directed oppositely; each accelerator must speed up the particles to energy $E_0 < E$. Compare the values of E and E_0. Consider, in particular, an ultrarelativistic case.

3.21. Find the formulas for the transformation of acceleration $\dot{\boldsymbol{v}}$ for the case when system S' moves relative to system S with an arbitrarily directed speed $\boldsymbol{V}$. Represent these transformation formulas in vector form.

3.22. Express the components of four-dimensional acceleration w_i through the usual acceleration $\dot{\boldsymbol{v}}$ and particle velocity $\boldsymbol{v}$. Find the square of 4-acceleration. Is the four-dimensional acceleration space like or time like?

3.23. Express particle acceleration $\dot{\boldsymbol{v}}'$ in the instantaneously accompanying IFR through its acceleration $\dot{\boldsymbol{v}}$ in the laboratory reference frame. Consider cases when only the value or only the direction of particle velocity $\boldsymbol{v}$ changes.

3.24. A relativistic particle performs a "uniformly accelerated" one-dimensional motion (acceleration $\dot{v} \equiv w$ is constant in the rest reference frame of the particle). Find the dependence of particle velocity $v(t)$ and coordinate $x(t)$ on time t in a laboratory reference frame if the starting velocity is v_0 and the initial coordinate is x_0. Consider, in particular, nonrelativistic and ultrarelativistic limits.

Hint: Use the result obtained in the preceding problem.

3.25. The rocket considered in Problem 3.9 initially at rest accelerates to $v = \sqrt{0.9999}c$. Its acceleration is $|v| = 20\,\mathrm{m/s^2}$ in the system instantaneously accompanying it. How much time will it take for the rocket to accelerate by the clock in a motionless frame of reference and by the clock on board the rocket?

Hint: Disregard the influence of inertial forces on the clock rate aboard the rocket.[12)]

3.26*. A particle travels with velocity $\boldsymbol{v}$ and acceleration $\dot{\boldsymbol{v}}$ so that its speed in a laboratory frame of reference S changes by $\delta \boldsymbol{v} = \dot{\boldsymbol{v}} \delta t$ for a short span of time δt. Let S' be the IFR instantaneously accompanying the particle at moment t, and let S'' be a similar system accompanying it at moment $t + \delta t$. Using the Lorentz transformations, show with an accuracy of up to terms linear in $\delta \boldsymbol{v}$ that the coordinates and time in these systems are related by the formulas

$$\boldsymbol{r}'' = \boldsymbol{r}' + \Delta\boldsymbol{\varphi} \times \boldsymbol{r}' - t' \Delta \boldsymbol{v}\,, \quad t'' = t' - \frac{\boldsymbol{r}' \cdot \Delta \boldsymbol{v}}{c^2}\,, \tag{1}$$

where

$$\Delta \boldsymbol{v} = \gamma \left[\delta \boldsymbol{v} + (\gamma - 1) \frac{\boldsymbol{v} \cdot \delta \boldsymbol{v}}{\boldsymbol{v}^2} \boldsymbol{v} \right], \quad \delta \varphi = (\gamma - 1) \frac{\delta \boldsymbol{v} \times \boldsymbol{v}}{\boldsymbol{v}^2}\,. \tag{2}$$

What is the geometric sense of transformations (1)? What forms do expressions (2) assume in the case of $v \ll c$ in the first nonvanishing approximation?

Hint: It is convenient to consider a sequence of transformations $S'' \to S \to S'$ using the formulas from the answer to Problem 3.11.

3.27. A particle has velocities $\boldsymbol{v}$ and $\boldsymbol{v}'$ in IFR S and S', respectively. Find the relationship between angles ϑ and ϑ' which these velocities make with the similarly directed Ox and Ox' axes. The relative speed of the systems is V.

3.28. System S' moves relative to system S with speed $\boldsymbol{V}$ while two bodies move with respect to S at velocities $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, respectively. What is the α angle between the velocities of these bodies in system S and system S'?

Hint: It may be helpful to use the results obtained in Problems 3.11 and 3.13.

3.29. What happens to the angle between the velocities of the two bodies considered in the previous problem when the velocity of system S' with respect to system S tends toward c?

3.30. A beam of light propagates in system S' at an angle ϑ' to the x' axis. What angle ϑ does it make with the x axis in system S?

3.31. At a certain moment, the direction of a light beam coming from a star makes angle ϑ with the Earth's orbital velocity $\boldsymbol{v}$ (in a system connected with the Sun). Find how the direction from the Earth to the star changes during half a year (light aberration), making no assumptions related to the smallness of v/c.

3.32. Find the shape of a visible curve circumscribed by a star in the sky by virtue of annual aberration. The star's polar coordinates in a system connected with the Sun

12) This means that it is necessary to calculate the sum of proper times $\mathrm{d}\tau = \mathrm{d}t\sqrt{1 - v^2/c^2}$ in the sequence of IFR accompanying the rocket expressed as integral $\int \mathrm{d}\tau$. See Fock (1955), Weinberg (1972), and Misner *et al.* (1973) for details.

are ϑ, α (the polar axis being directed perpendicular to the Earth's orbital plane). The Earth's orbital velocity $v \ll c$.

3.33. A beam of light in a certain frame of reference makes a solid angle $d\Omega$. How will this angle change on transition to a different IFR?

3.34•. A certain source emits light isotropically in all directions in its rest system so that fraction $dN = d\Omega_0/4\pi$ of the total emission propagates within the solid angle $d\Omega_0$. Find the function for the distribution of the emitted light over angles $f(\vartheta) = dN/d\Omega$ in a system with respect to which the source moves at velocity $\boldsymbol{v}$. Draw polar emission diagrams for different values of $\gamma: \gamma = (1 - v^2/c^2)^{-1/2}$. Study thoroughly the relativistic case $\gamma \gg 1$. What is the characteristic angular width of emission from a moving source in this case?

3.35•. A monochromatic source generates in its reference frame an electromagnetic wave with frequency ω_0. Show, using the transformation law (3.24) of the wave 4-vector, that an observer will record the wave frequency

$$\omega = \omega_0 \frac{\sqrt{1 - V^2/c^2}}{1 - (V/c)\cos\theta} ,$$

where V is the relative speed of the source and the observer, and θ is the angle between the direction of the beam and the relative speed in the observer's system (*Doppler effect*)[13]. Also express wave vector $\boldsymbol{k}$ in the observer's system through quantities ω_0 and $\boldsymbol{k}_0$ in the source frame.

3.36. Find the frequency ω of the light wave observed in the transverse Doppler effect (the direction of propagation of light is perpendicular to the direction of the movement of the source in the system connected with the light detector). What is the direction of propagation of the wave considered in the system connected to the source?

3.37. The wavelength of light emitted from a certain source in the system where the source is at rest is λ_0. What wavelength λ will be recorded by (i) an observer approaching the source at speed V and (ii) an observer departing from the source at the same speed.

3.38. A source emitting light of frequency ω_0 isotropically in all directions in its frame of reference moves uniformly and rectilinearly with respect to the observer at speed V and passes by with an impact parameter d at the moment of maximum proximity. The number of photons emitted per unit time per unit solid angle (photon flux intensity) is J_0 in the source's rest frame. Find the dependence of frequency ω_0 and photon flux intensity J recorded by the observer from the angle between the direction of the beam and speed $\boldsymbol{V}$. At which angles $\theta = \theta_0$ do the recorded frequency and photon flux intensity coincide with ω_0 and J_0? What fraction of the photons is recorded by the observer in intervals $0 \le \theta \le \theta_0$ and $\theta_0 \le \theta \le \pi$?

13) Christian Doppler (1803–1853) was an Austrian physicist, mathematician and astronomer. He was the organizer of the first-over Physical institute at the Viennese university.

Draw the graphs of the dependencies $\omega(\theta)$ and $J(\theta)$ for $V/c = 1/3$ and $V/c = 4/5$. What character does these dependencies have at $V/c \to 1$?

3.39. Find the angular distribution of the luminous power I (the light energy emitted per unit time and per unit solid angle) and the total light flux from the source considered in the preceding problem.

Hint: Each photon has energy $\hbar\omega$, where $\hbar$ is the reduced Planck constant.

3.40. A mirror moves normally to its own plane at speed $\boldsymbol{V}$. Find the law of reflection of a flat monochromatic wave from the mirror (substituting the principle of equality between the angles of incidence and reflection at $V = 0$) and the law of frequency transformation in such a reflection. Consider, in particular, the case of $V \to c$.

3.41. Solve the previous problem for the case when the mirror performs a translational motion along its own plane.

3.42. A nontransparent cube with edges of length l_0 moves in the rest proper frame with respect to an observer at speed V (Figure 3.6). The observer takes a photograph at the moment when beams of light emitted from the surface of the cube arrive at the camera lens at a right angle to the direction of motion (in the camera's system). The cube is seen at a small solid angle; therefore, the beams originating at different points on the cube's surface can be regarded as parallel.

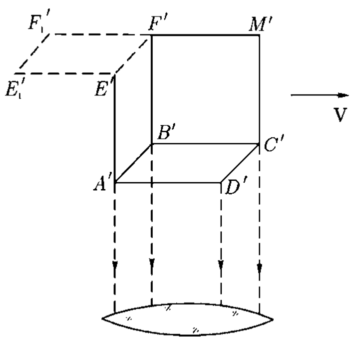

Figure 3.6 Photographing a cube at a right angle.

What will the image at the photoplate look like? Draw a diagram of the image and plot the vertices and edges of the cube to be photographed. Calculate their relative lengths. Identify a motionless object to which the photograph thus obtained is equivalent. What would the image of the moving cube look like if the Galilean transformations held?

3.43. A thin rod $M'N'$ of length l_0 is at rest in system S' and oriented as shown in Figure 3.7. System S' moves with speed $\boldsymbol{V} \parallel Ox$ with respect to a photoplate AB resting in system S. As the rod moves past the photoplate a short light flash occurs during which the beams are incident normally on the photoplate's xy plane.

(i) What is the image length l on the photoplate? Can it become equal to or bigger than l_0? (ii) At what tilt angle α' will only an end of the rod be photographed? (iii) What is the tilt angle α between the rod and the Ox axis?

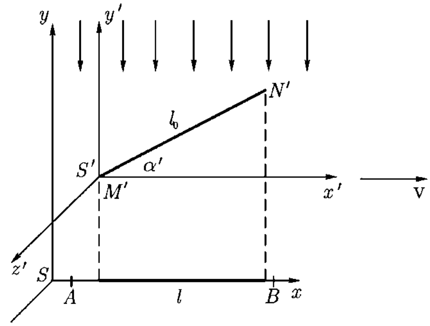

Figure 3.7 Photographing an inclined thin core.

3.44. A sphere moving with speed V is photographed by an observer at rest at a small solid angle. A beam of parallel light rays from the sphere is incident on the camera lens making a right angle with the direction of speed $\boldsymbol{V}$. What will be the shape of the image on the photoplate? What part of the surface of the sphere will be photographed?

Hint: Represent the sphere as a set of thin disks moving parallel to their planes and draw an image of each disk.

3.45. A motionless observer takes a photograph of a moving nontransparent cube at the moment when rays of light coming from its surface make an arbitrary α angle with the cube's speed $\boldsymbol{V}$ (in the observer's frame). The cube is seen at a small solid angle; therefore, beams of parallel rays are incident on the photoplate normal to its surface (Figure 3.8). Show that the photograph must be identical with that of a motionless cube rotated through a certain angle. Find the image rotation angle at different values of V and fixed α. At what V value will a single cube face $A'B'$ (and a single face $B'C'$) be photographed?

3.46*. A spacecraft moves uniformly along a straight line connecting it with an observer. At specified distances l_1 and $l_2 < l_1$ from the observer, the spacecraft gives off two short light flashes, which are recorded by the observer's watch at moments t_1 and t_2. What is the speed V with which the spacecraft approaches the observer and at what moment t_* will it arrive? What is the "visible" speed of the spacecraft related to its real speed in the observer's frame?

Hint: The "visible" or "apparent" speed is the ratio of the distance covered $l_1 - l_2$ to the time interval $\Delta t = t_2 - t_1$ recorded by the observer. It is in this way that the speed would have been defined in classical mechanics, where $v \ll c$.

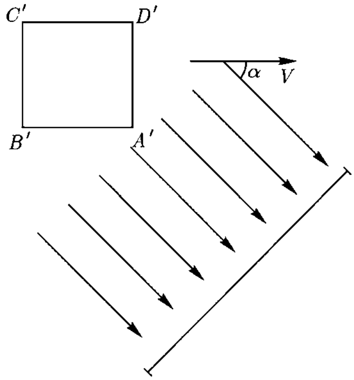

Figure 3.8 Photographing an inclined cube.

3.47*. A spacecraft approaches an observer with known speed V. To determine the length of the spacecraft, the observer sends two short light pulses that are reflected from the mirrors at the front and the back of the spacecraft and returned to the observer (simultaneously recorded by his watch). How can one find the lengths of the spacecraft on the basis of such a thought experiment: (i) the "apparent" length a_* defined as the distance between the mirrors from which the reflected pulses simultaneously reach the observer; (ii) length a in the observer's frame; (iii) length a_0 in the spacecraft's proper frame in which it is at rest? What will be the lengths if the spacecraft moves away from the observer?

3.48*. A spacecraft moves at speed $\boldsymbol{V} =$ const with respect to an observer standing very far away from its path (Figure 3.9). Calculate the "apparent" speed of the spacecraft (in the sense this notion is used in Problem 3.46*) using light signals. Find projections of this speed onto a beam directed toward the observer and onto the plane perpendicular to the beam. Under what conditions are the "apparent" speed superluminal? Consider, in particular, the case of a high relativistic factor of the spacecraft, $\gamma \gg 1$, and a small α angle.

3.49. Introduce a wave 4-vector describing the propagation of a plane monochromatic wave in a medium which has refractive index n and moves with speed $\boldsymbol{V}$. The phase velocity of the wave in a motionless dielectric medium $v' = c/n$. Find the formulas for transformation of the frequency, phase velocity, and angle between the wave vector and the relative speed.

3.50. A plane wave propagates with speed V in the direction of the movement of the medium. The wavelength in a vacuum is λ. Find the wave velocity v with respect to a laboratory frame of reference *(Fizeau experiment)*.[14] The refractive index n is determined in system S' connected with the medium and depends on the wavelength λ' in this system. Perform calculations to first-order accuracy in V/c.

14) Armand Hippolyte Louis Fizeau (1819–1896) was a French optical physicist who conducted many fundamental optical experiments.

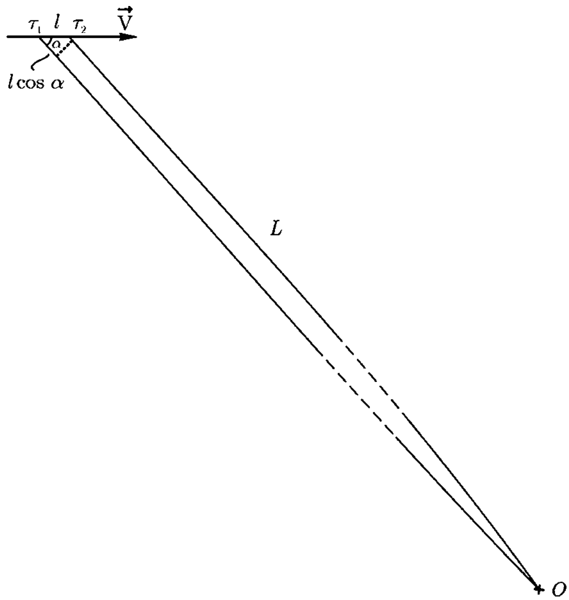

Figure 3.9 Calculation of "apparent" speed of the spacecraft.

Recommended literature:
Einstein (1905, 1953); Landau and Lifshitz (1975); Mandelstam (1950); Fock (1955); Ginzburg (1979a); Misner *et al.* (1973); Ugarov (1997); Bredov *et al.* (2003); Tolman (1969); Papapetrou (1955); Minkowski (1909); Einstein *et al.* (1952); Burke (1980); Weisskopf (1960); Born (1962); Bolotovskii (1985, 1990)

3.2
Kinematics of Relativistic Particles

3.2.1
Energy and Momentum

As known from nonrelativistic classical mechanics, the most compact and simplest notation system for recording the laws of motion (in the absence of macroscopic dissipative forces) is based on the variation principle and has the form

$$\delta S = 0 \,. \tag{3.26}$$

Here

$$S = \int_{(1)}^{(2)} L \mathrm{d}t \tag{3.27}$$

is the *action* of the mechanical system being considered and L is its *Lagrangian*.[15] Particle coordinates and instants of time are given in states 1 and 2. As a mechanical system travels along a physical (true) trajectory, the action takes a stationary value as manifested in the equating of its first variation (3.26) to zero. For a free nonrelativistic particle, the Lagrangian function coincides with its kinetic energy expressed through velocity,

$$L = \frac{mv^2}{2} , \tag{3.28}$$

where v is the velocity and $m > 0$ is a positive constant, the inert mass of a particle. In nonrelativistic classical mechanics, it is defined as the coefficient of proportionality between force $\boldsymbol{F}$ and acceleration $\dot{\boldsymbol{v}}$ that the particle acquires under the action of this force:

$$\dot{\boldsymbol{v}} = \frac{1}{m}\boldsymbol{F} .$$

In relativistic mechanics, the variation principle facilitates the formulation of relativistic invariant equations allowing the motion in any IFR to be described. Moreover, the variation principle makes it possible to describe electromagnetic as well as other classical and quantum fields. Therefore, we shall use it as a basis to expound the fundamentals of relativistic mechanics and the classical field theory.

We use two general physical principles to define the kind of action for a free relativistic particle: (i) relativistic invariance of the action ensuring the identity of the laws of mechanics in all IFR, that is, the relativity principle; (ii) the correspondence principle implying that the relativistic action at velocities $v \ll c$ leads to the nonrelativistic formulas (3.27) and (3.28).

The principle of relativistic invariance in conjunction with the properties of uniformity and isotropy of 4-space in IFR allows the action of a free particle to be expressed through the sole invariant, the length of the particle's world line between points 1 and 2 of the four-dimensional space:

$$S = \alpha \int_{(1)}^{(2)} \mathrm{d}s . \tag{3.29}$$

Both the value and the sign of invariant constant a are found from the correspondence principle. Using formula (3.11), we write (3.29) in the form of a time integral in the laboratory system:

$$S = \alpha c \int_{(1)}^{(2)} \sqrt{1 - \frac{v^2}{c^2}}\mathrm{d}t . \tag{3.30}$$

15) Joseph Louis Lagrange (1736–1813) was an outstanding French mathematician. He developed the variational approach currently used in many fields of physics.

This gives the Lagrangian function. In the nonrelativistic limit $v \ll c$,

$$L = \alpha c \left(1 - \frac{v^2}{2c^2}\right) = -mc^2 + \frac{mv^2}{2} \,. \tag{3.31}$$

The last equality is obtained by substituting $a = -mc$ and ensures equivalence of expressions (3.28) and (3.31) for the Lagrangian in the nonrelativistic limit because the system energy in classical mechanics is determined with an accuracy of up to a constant and the Lagrangian is determined even more arbitrarily.

According to (3.30), the Lagrangian for a free relativistic particle has the form

$$L = -mc^2 \sqrt{1 - \frac{v^2}{c^2}} \,. \tag{3.32}$$

It should be emphasized that mass m entering all preceding formulas is an invariant quantity similar in all IFR. Otherwise, relativistic invariance of the theory breaks down.

Example 3.9

Using Lagrangian (3.32), calculate the energy $\mathcal{E}$ and momentum $\boldsymbol{p}$ of a relativistic particle as a function of its velocity.

Solution. Using formulas

$$\boldsymbol{p} = \frac{\partial L}{\partial \boldsymbol{v}} \,, \quad \mathcal{E} = \boldsymbol{p} \cdot \boldsymbol{v} - L \,,$$

which are known from mechanics, we find

$$\boldsymbol{p} = \frac{m\boldsymbol{v}}{\sqrt{1 - v^2/c^2}} \,, \quad \mathcal{E} = \frac{mc^2}{\sqrt{1 - v^2/c^2}} \,. \tag{3.33}$$

□

The momentum of a relativistic particle at $v \ll c$ transforms into the well-known expression $\boldsymbol{p} = m\boldsymbol{v}$ of Newtonian mechanics. The energy in this limit

$$\mathcal{E} = mc^2 + \frac{mv^2}{2} \,, \tag{3.34}$$

contains, besides the nonrelativistic term (kinetic energy $mv^2/2$), a constant

$$\mathcal{E}_0 = mc^2 \,, \tag{3.35}$$

called *the rest energy* of a particle. This constant, unlike an arbitrary term in the potential energy of a nonrelativistic system, is observable and can transform partly or totally into other forms of energy. Specifically, the energy released during the combustion of a chemical or nuclear fuel is actually drawn from its rest energy.

Because energy (3.33) contains the rest energy, it is referred to as *the total energy*. In this sense, the total energy is different from the sum of the kinetic and potential energies $E = T + U$ in nonrelativistic mechanics which is also known as the total energy. *The kinetic energy* of a particle, T, differs from its total energy by the amount of rest energy:

$$T = \mathcal{E} - mc^2 \,. \tag{3.36}$$

Expressions (3.33) establish a meaningful relationship between the total energy, momentum, and velocity:

$$\mathcal{E}\boldsymbol{v} = c^2 \boldsymbol{p} \,. \tag{3.37}$$

Example 3.10

Derive the equation of motion of a relativistic particle by varying the action (3.29) (where $a = -mc$) and preserving the covariant form at all stages of the calculations, that is, by expressing all quantities through 4-vectors and 4-invariants. After varying 4-coordinates, pass to integration over an invariant parameter conveniently chosen as proper time τ.

Solution. Writing (3.29) in the form

$$S = -mc \int_{(1)}^{(2)} \sqrt{\mathrm{d}x^k \mathrm{d}x_k} \,,$$

we have

$$\delta S = -mc \int_{(1)}^{(2)} \frac{\delta(\mathrm{d}x^k \mathrm{d}x_k)}{2\mathrm{d}s} = -mc \int_{(1)}^{(2)} \frac{\mathrm{d}x^k}{\mathrm{d}s} \delta \mathrm{d}x_k = -m \int_{(1)}^{(2)} u^k \mathrm{d}\delta x_k \,,$$

because $c\mathrm{d}x^k/\mathrm{d}s = u^k$ is a 4-velocity and differentiation and variation are permutation operations. Integration of the last expression by parts leads to

$$\delta S = -mu^k \delta x_k \big|_{(1)}^{(2)} + m \int_{(1)}^{(2)} \delta x_k \mathrm{d}u^k \,.$$

Because the 4-coordinates of points 1 and 2 are fixed, $\delta x_k = 0$ at these points. Setting $\mathrm{d}u^k = (\mathrm{d}u^k/\mathrm{d}\tau)\mathrm{d}\tau$ yields

$$\delta S = m \int_{(1)}^{(2)} \frac{\mathrm{d}u^k}{\mathrm{d}\tau} \delta x_k \mathrm{d}\tau = 0 \,.$$

Values δx_k being arbitrary and independent. The equation of motion is

$$m\frac{\mathrm{d}u^k}{\mathrm{d}\tau} = 0 \,. \tag{3.38}$$

□

This equation implies the equating to zero of 4-acceleration of a free particle, that is, a quite natural and even trivial result. For all that, this notation allows a very useful relativistic interpretation of the particle's energy and momentum. The 4-vector

$$p^k = mu^k = \left(\frac{mc}{\sqrt{1-v^2/c^2}}\,, \quad \frac{m\boldsymbol{v}}{\sqrt{1-v^2/c^2}}\right) \tag{3.39}$$

is called the *four-dimensional momentum* of a particle. Its spatial part coincides with the three-dimensional momentum (3.33), whereas the temporal component equals $\mathcal{E}/c$:

$$p^k = \left(\frac{\mathcal{E}}{c}, \boldsymbol{p}\right) \tag{3.40}$$

Because the square of the 4-vector is invariant,

$$p^k p_k = \left(\frac{\mathcal{E}}{c}\right)^2 - p^2 = \left(\frac{\mathcal{E}'}{c}\right)^2 - p'^2 \,.$$

Let a particle rest in a primed system, that is, $p' = 0$, $\mathcal{E}' = mc^2$. The preceding equality implies the relationship between energy and momentum:

$$p^k p_k = \left(\frac{\mathcal{E}}{c}\right)^2 - p^2 = (mc)^2 \,, \quad \text{or} \quad \mathcal{E} = \sqrt{c^2p^2 + m^2c^4} \,. \tag{3.41}$$

Here, the plus sign is chosen in front of the square root in order to obtain the correct relativistic limit. Integration of the energy and momentum into one 4-vector allows them to be easily transformed into any IFR. The transformation is realized in accordance with (3.33).

The formulas above admit the particles' motion with a limiting velocity[16)] $v = c$ but only in the case when the mass of the particles $m = 0$. From relations (3.37) and (3.41), for such particles we have

$$\mathcal{E} = cp \,. \tag{3.42}$$

16) The hypothesis about the existence of particles (tachyons), which move with velocities $v > c$ has been discussed in the scientific literature. Such particles have not been found in experiments. One can become acquainted with their hypothetical properties from the reviews by Barashenkov (1974) and Feinberg (1967).

A similar relationship between energy and momentum is roughly valid for any *ultrarelativistic* particle having energy $\mathcal{E} \gg mc^2$. The ratio $\gamma = \mathcal{E}/mc^2 = (1 - v^2/c^2)^{-1/2}$ is called the *relativistic factor* (Lorentz factor).

Zero mass is inherent in the quanta of the electromagnetic field or photons (high-energy photons are called gamma quanta) and possibly some other hypothetical (yet unobserved experimentally) particles. Photon energy and momentum in a vacuum are related to the frequency ω and wave vector $\boldsymbol{k}$ by quantum formulas (see Chapter 6):

$$\mathcal{E} = \hbar\omega , \quad p = \frac{\hbar\omega}{c} = \hbar k , \tag{3.43}$$

where $\hbar \approx 1.05 \times 10^{-34}$ J s $\approx 1.05 \times 10^{-27}$ erg s is the *reduced Planck constant*.

3.2.2
Kinematic Problems

Diverse and meaningful information about such processes as particle decays and collisions can be obtained only on the basis of the laws of energy and momentum conservation without regard for the explicit form of interaction between the particles. The energy and momentum of a closed (not interacting with external fields or bodies) system of particles are integrals of motion. The relationship between this fundamental fact and space–time symmetry will be demonstrated in Section 4.2.

The general kinematic problem can be formulated as follows. Suppose there are a few particles $a, b, \ldots$ with 4-momenta p_a^i, $p_b^i, \ldots$ localized away from one another and therefore unable to interact. When moving, the particles come closer to each other and interact. The character of interaction is inessential as far as the solution of kinematic problems is concerned; what is really needed is the interaction must cease after the subsequent dispersion of particles. Interactions may lead to either elastic collisions between the particles in which both they and their internal states remain unaltered or decays and inelastic collisions resulting in a change of the particles' masses and internal states and even in the appearance of new particles. The process of interaction can be represented in the form

$$a + b + \cdots \to m + n + \cdots$$

If particles $m, n, \ldots$ in the finite state are far from one another, have 4-momenta $p_m^i, p_n^i, \ldots$, and do not interact, the law of 4-momentum conservation for the entire system can be written in the form of the equality

$$p_a^i + p_b^i + \cdots = p_m^i + p_n^i + \cdots , \tag{3.44}$$

which is satisfied in any IFR. This equality provides a basis for all kinematic calculations. Also, it should always be borne in mind that the square of any 4-momentum is invariant and therefore has the same value in all IFR. The laboratory frame of reference S (L-system) and the center of inertia frame S' (C-system) appear to be especially convenient for kinematic calculations. The latter is defined as a system in which the full three-dimensional momentum of the particles $\boldsymbol{p} = \boldsymbol{p}_a + \boldsymbol{p}_b + \ldots$ is equal to zero.

As follows from (3.44), both the total energy and the full three-dimensional momentum are conserved unlike the total mass understood as the sum of the masses of individual particles. The sums of masses before and after inelastic collision are as a rule different:

$$\Delta M = m_a + m_b + \cdots - (m_n + m_k + \cdots) \neq 0 . \tag{3.45}$$

The quantity $Q = c^2 \Delta M$ is referred to as *the energy efficiency of a reaction* and ΔM is referred to as *the mass defect*, which is compatible with the total mass of the particles involved in the reaction. An example of such inelastic collisions is provided by annihilation of an electron–positron or muon pair resulting in the emission of gamma quanta:

$$e^+ + e^- \to \gamma_1 + \gamma_2 , \quad \mu^+ + \mu^- \to \gamma_1 + \gamma_2 .$$

Because gamma quanta have no mass, $\Delta M = 2m_\mathrm{e}$ and $\Delta M = 2m_\mu$. In norelativistic processes (exemplified by chemical reactions), the mass is strictly speaking not conserved either. However, the mass defect accounts only for a small fraction of the total mass of the substances involved in the reaction (see Problem 3.85 for estimates). For this reason, the law of conservation of mass in chemical reactions is usually considered to be an exact law of nature.

It should be borne in mind that the notion of mass defect has a strict sense only in the nonrelativistic limit when the quantity ΔM is small compared with $m_a + m_b + \ldots$ In the relativistic case, the mass of a system of interacting particles has the same sense as the mass of a single particle and must be defined as an invariant quantity,

$$M = \left[\frac{\mathcal{E}^2}{c^4} - \frac{P^2}{c^2}\right]^{1/2} = \left[\left(\sum_a \frac{\mathcal{E}_a}{c^2}\right)^2 - \left(\sum_a \frac{\boldsymbol{p}_a}{c}\right)^2\right]^{1/2} , \tag{3.46}$$

that is, through the total energy $\mathcal{E}$ and the full three-dimensional momentum $\boldsymbol{P}$ of the system, which are additive quantities expressed as the sums of energies and momenta of all particles. The mass such defined is the same before and after any particle interaction process because $\mathcal{E}$ and $\boldsymbol{P}$ taken separately are conserved. However, such mass is nonadditive and depends on the angles between the particles' momenta. It becomes additive again and is expressed through the total energies of the particles, $M = \sum_a E_a/c^2$, only in the C-system in which $\boldsymbol{P} = 0$. In this IFR where a system of particles as a whole is motionless ($P = 0$), the relationship between the mass and the total energy assumes the form (3.35) analogous to the rest energy of a single particle:[17)]

$$\mathcal{E} = Mc^2 . \tag{3.47}$$

17) The author advises to readers to get acquainted with L.B. Okun's articles (1989; 2000, and especially 2008) in which concepts of mass and energy of rest in the relativistic physics are discussed very interestingly and clear.

Reactions such as

$$a + b \to c + d\,, \tag{3.48}$$

that is, reactions in which two particles turn into two other particles, are called two-particle reactions (a particular case of a two-particle reaction is elastic scattering of two particles). The kinematics of two-particle reactions is convenient to describe with the use of *invariant variables* s, t, and u expressed through 4-momenta of the particles involved in the reactions:

$$\begin{aligned} &s = (p_a + p_b)_i (p_a + p_b)^i\,, \quad t = (p_a - p_c)_i (p_a - p_c)^i\,, \\ &u = (p_a - p_d)_i (p_a - p_d)^i\,. \end{aligned} \tag{3.49}$$

Any of these quantities (s, t, u) can be expressed through two others with the help of the relation[18)]

$$s + t + u = \left(m_a^2 + m_b^2 + m_c^2 + m_d^2\right) c^2\,. \tag{3.50}$$

A kinematic plane onto which the values of variables s, t, and u are plotted provides insight into the kinematics of two-particle reactions. The laws of conservation of energy and momentum confine the parameter values on the kinematic plane to a region of values feasible (physical) for a given reaction.

Many formulas of relativistic kinematics assume a simpler form in a system of units where the velocity of light is $c = 1$. In this case, mass, energy, and momentum are measured in identical units, for example, in megaelectronvolts ($1\,\text{MeV} = 10^6\,\text{eV} = 10^{-3}\,\text{GeV} = 1.602 \times 10^6$ erg). Such a system of units is used in certain problems in this section. In some cases, the masses of elementary particles are measured in electron mass units, m_e (i.e., using the system in which $m_\text{e} = 1$).

The masses of some elementary particles and their generally accepted notation are presented for reference in Table 3.1. The bars denote antiparticles, which may differ from the respective particle not only in the sign of the electric charge but also in other quantum numbers. However, the masses of particles and antiparticles are identical. The electroneutral particles listed in the table include the photon γ, the Z boson, all types of neutrino ν, mesons π^0 and K^0, the neutron n, and the hyperon Λ, whereas the remaining particles have electric charges with absolute values equal to the elementary charge. Charged leptons bear the names electron (positron), muon, and taon. Baryons p and n are called nucleons (nuclear particles).

In 2012, experimental evidence was presented for the existence of the Higgs boson H, which had been predicted for a long time on the basis of theoretical work. It has mass $m_H \approx 126$ GeV, zero charge, and apparently, zero spin (Rubakov, 2012). Brief data on the Higgs theory are available in Problem 4.132*.

Table 3.2 shows the values of the binding energy B of some atomic nuclei. The superscript indicates the number of nucleons (protons and neutrons) in the nucleus and the subscript indicates the number of protons and the nuclear charge in

18) Any two variables, for example, s and t, can be chosen as independent quantities. All others (particle energies and scattering angles in the laboratory frame of reference and in the center of mass frame) are expressed through them (see Problems 3.104–3.107).

Table 3.1 Masses of some elementary particles

Particle	Mass (m_e units)	Mass (MeV)	Particle	Mass (m_e units)	Mass (MeV)
Photon γ	0	0	*Hadrons*		
Intermediate bosons			*Mesons*		
$W^{\pm}$	1.59×10^5	8.10×10^4	$\pi^{\pm}$	273	139.6
Z	1.81×10^5	9.24×10^4	π^0	264	135.0
Leptons			$K^{\pm}$	965.9	493.6
$e^{\pm}$	1	0.511	$K^0, \bar{K}^0$	974.0	497.7
$\nu_e, \bar{\nu}_e$	0	0	Baryons		
$\mu^{\pm}$	207	105.7	$p, \bar{p}$	1836	938.2
$\nu_\mu, \bar{\nu}_\mu$	< 0.5	< 0.25	$n, \bar{n}$	1839	939.5
$\tau^{\pm}$	3491.4	1784.1	$\Lambda, \bar{\Lambda}$	2183.2	1115.6
$\nu_\tau, \bar{\nu}_\tau$	< 68.5	< 35	$\Omega^{\pm}$	3272.8	1672.4

Table 3.2 Binding energies of atomic nuclei

Isotopes	$^2H_1$	$^4He_2$	$^7Li_3$
B (MeV)	2.23	28.11	38.96

elementary charge units. The binding energy

$$B = \Delta M c^2 = \sum \mathcal{E}_{0n} - \mathcal{E}_0 \,, \tag{3.51}$$

where $\mathcal{E}_{0n}$ is the rest energy of a free nucleon, and E_0 is the rest energy of the nucleus. Summation is performed over all nucleons of a given nucleus.

Problems

3.51. Derive the equation of motion for a free relativistic particle by varying action (3.30) with the Lagrangian (3.32) and the noninvariant variable of integration, that is, coordinate time t. Confirm the equivalence of the equation thus obtained and (3.38).

3.52. Express the momentum p of a relativistic particle through its kinetic energy T.

3.53. Express the velocity $\boldsymbol{v}$ of a relativistic particle through its momentum $\boldsymbol{p}$.

3.54. A particle with mass[19] m has energy $\mathcal{E}$. Find the velocity v of the particle. Consider, in particular, nonrelativistic and ultrarelativistic limits.

19) Subsequently we will use everywhere the term "mass" to always mean invariant mass, determined by formulas (3.42) and (3.47). The often used term "rest mass" is superfluous.

3.55. Find the approximate expressions for the kinetic energy T of a particle with mass m (i) from its velocity v and (ii) from its momentum p with an accuracy of up to v^4/c^4 and p^4/m^4c^4, respectively, at $v \ll c$.

3.56. Find the velocity v of a particle with mass m and charge e that passed through potential difference V (the starting velocity equals zero). Simplify the general formula for nonrelativistic and ultrarelativistic cases (taking account of two expansion terms for each case).

3.57. Find the velocity v of the particles in the following cases: (i) electrons in a vacuum tube ($\mathcal{E} = 300\,\text{eV}$); (ii) electrons in a 300 MeV synchrotron; (iii) protons in a 680 MeV synchrocyclotron; (iv) protons in a 10 GeV synchrotron-based particle accelerator; (v) protons in a 7 TeV collider.

3.58. A beam of charged particles at the exit from an accelerator has kinetic energy T and current strength J. Find the pressure force F exerted by the beam on the absorbing target and the power W released in the target. The particle has mass m and charge e.

3.59. A body moves with relativistic velocity v through a gas, a unit volume of which contains N slowly propagating particles with mass m. Find the pressure p that the gas exerts on a surface element normal to its velocity. if the particles are elastically reflected from the surface of the body.

3.60. A particle is accelerated in the gap between hollow cylindrical electrodes, "drift tubes," of a linear accelerator, with the particle's path being aligned along their common axis. The acceleration is due to the action of a high-frequency ($\nu =$ const) electric field. Only those particles that pass through all interspaces between the tubes subjected to the accelerating field are accelerated. What should the length of the drift tubes be to ensure that a particle with charge e and mass m flies through the accelerating gaps at the moments when a maximum voltage V_e is applied to them? Also estimate the total length of an accelerator having N drift tubes.

3.61. A vertically incident flux of monochromatic muons is born in the upper layers of the atmosphere.[20] Find the ratio of the muon flux intensities at height h above sea level (I_h) and at sea level (I_0) assuming that the air layer of thickness h only weakens the flux owing to the natural decay of the particles. The muon energy $\mathcal{E} = 4.2\times10^8$ eV, $h = 3$ km, and mean lifetime of a resting muon $\tau_0 = 2.2\times10^{-6}$ s.

3.62. A reference frame S' moves with respect to system S at speed $\boldsymbol{V}$. A particle of mass m having energy $\mathcal{E}'$ and velocity $\boldsymbol{v}'$ in the system S' moves at an angle ϑ' to direction $\boldsymbol{V}$. Find the angle ϑ between the particle's momentum $\boldsymbol{p}$ and direction $\boldsymbol{V}$ in system S. Express the particle's energy and momentum in system S through ϑ' and $\mathcal{E}'$ or through ϑ' and v'. Consider, in particular, the ultrarelativistic case of $\mathcal{E}' \gg mc^2$ and $V \approx c$. Show that in this case the approximate formula $\vartheta' \approx (1/\gamma)tg(\vartheta/2)$ can be used in a certain (what?) range of angles.

20) A simplified formulation of the problem is presented.

3.63. System S' moves with respect to system S with speed $\boldsymbol{V}$. The angular distribution of the particles having similar energy $\mathcal{E}'$ in system S' is described by the function $\mathrm{d}W/\mathrm{d}\Omega' = F'(\vartheta', \alpha')$, where $\mathrm{d}W$ is a fraction of the particles moving in system S' within the solid angle $\mathrm{d}\Omega'$. It is usually normalized so that

$$\int \mathrm{d}W = \int F'(\vartheta', \alpha')\mathrm{d}\Omega' = 1 .$$

Angle ϑ' is counted from direction $\boldsymbol{V}$. Find the angular distribution of such particles in system S. Specifically, consider an ultrarelativistic case.

3.64•. Show that a volume element in the space of three-dimensional momenta $\mathrm{d}^3 p$ transforms on transition to a different IFR in the following way:

$$\frac{\mathrm{d}^3 p}{\mathcal{E}} = \frac{\mathrm{d}^3 p'}{\mathcal{E}'} ,$$

where $\mathcal{E}$ and $\mathcal{E}'$ are the corresponding energies.

3.65•. The momentum distribution function $f(\boldsymbol{p})$ of the particles is normalized so that the total number $\mathrm{d}N$ of those particles whose momenta lie in volume $\mathrm{d}^3 p$ is given by $\mathrm{d}N = f(\boldsymbol{p})\mathrm{d}^3 p$. Find the law of transformation of the distribution function on transition to a different IFR.

3.66•. The number of particles $\mathrm{d}N$ in a volume element $\mathrm{d}V$ having momentum components within limits $p_x, p_x+\mathrm{d}p_x$; p_y, $p_y+\mathrm{d}p_y$; and $p_z, p_z+\mathrm{d}p_z$ is expressed in the form

$$\mathrm{d}N = f(\boldsymbol{r}, \boldsymbol{p}, t)\mathrm{d}V\mathrm{d}^3 p ,$$

where $\mathrm{d}^3 p = \mathrm{d}p_x\mathrm{d}p_y\mathrm{d}p_z$ is the volume element in the momentum space and $f(\boldsymbol{r}, \boldsymbol{p}, t)$ is the coordinate and momentum distribution function, that is, the particle number density in the phase space. Find the relativistic transformation law $f(\boldsymbol{r}, \boldsymbol{p}, t)$.

3.67. Particles of type 1 having velocity $\boldsymbol{v}_1$ in system S are scattered by particles at rest of type 2. How is the scattering cross-section $\mathrm{d}\sigma_{12}$ transformed on transition to the reference frame S' in which particles of types 1 and 2 have the velocities $\boldsymbol{v}'_1$ and $\boldsymbol{v}'_2$, respectively? Consider, in particular, the case where velocities $\boldsymbol{v}'_1$ and $\boldsymbol{v}'_2$ are parallel.

Hint: The scattering cross-section $\mathrm{d}\sigma_{12}$ is the ratio of the number of particles scattered per unit time into the solid angle $\mathrm{d}\Omega$ by a single scattering center to the flux density of scattered particles $J_{12} = n_1 v_0$, where n_1 is the number of particles scattered per unit volume and $v_0 = |\boldsymbol{v}_1 - \boldsymbol{v}_2|$ is the relative velocity of particles of types 1 and 2 (cf. Problem 3.18).

3.68. Pion π^0 propagates with velocity v and decays in flight into two gamma quanta. Find the angular distribution $\mathrm{d}W/\mathrm{d}\Omega$ of gamma quanta originating from decay in the laboratory frame of reference taking into account that it is spherically symmetric in the pion rest system.

3.69. Express the energy of the pion considered in the preceding problem through the ratio f of the number of decay gamma quanta emitted into the forward semisphere to the number of gamma quanta emitted into the backward semisphere.

3.70. A neutral pion decays in flight into two gamma quanta. Show that the minimal spreading angle of gamma quanta, $\vartheta_{\min}$, is determined by the condition $\cos(\vartheta_{\min}/2) = v/c$ in the reference frame in which the velocity of the pion is v.

3.71. Find the dependence of the energy of a gamma quantum resulting from the decay of a pion (see Problem 3.68) on the angle ϑ between the directions of quantum propagation and pion movement. Calculate the energy spectrum of decay gamma quanta in the laboratory frame of reference.

Hint: It follows from the laws of conservation of energy and momentum that the energy of a gamma quantum in the pion rest system is $\mathcal{E}' = mc^2/2$ (m is the mass of a pion).

3.72. Show that whatever the shape of the energy spectrum of pions, the energy spectrum of decay gamma quanta in the laboratory frame of reference exhibits a maximum at $\mathcal{E} = \mathcal{E}'$, with $\mathcal{E}' = mc^2/2$, where m is the mass of a pion. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be the arbitrary values of the energy of decay gamma quanta on either side of the maximum corresponding to identical values of the distribution function . Express the mass m of a pion through $\mathcal{E}_1$ and $\mathcal{E}_2$.

Hint: Make use of the energy spectrum of gamma quanta found in Problem 3.71.

3.73. Determine the mass m of a certain particle knowing that it decays into two particles with masses m_1 and m_2. The values of momenta p_1 and p_2 resulting from the decay and the angle ϑ between their directions are known from experience. Calculate the mass of a charged pion decaying as $\pi \to \mu + \nu$ if it is known from experiment that the pion was at rest prior to the decay and the muon acquired momentum $p_\mu = 29.8\,\mathrm{MeV}/c$ after the decay. The mass of a muon is given in Table 3.1. The mass of a neutrino is consider to be zero.

3.74. Determine the mass m_1 of a certain particle knowing that it is one of the two particles resulting from the decay of a particle with mass m and momentum p. Momentum p_2, mass m_2, and the exit angle ϑ_2 of the second particle originating from the decay are also known.

3.75. A particle with mass m_1 and velocity v hits a particle at rest with mass m_2 that captures it. Find the mass m and speed V of the resultant particle.

3.76. A body at rest having mass m_0 breaks into two pieces with masses m_1 and m_2 respectively. Calculate the kinetic energies T_1 and T_2 of the disintegration products. Find the decay energy distribution in the rest system of the disintegrating particle between (i) the alpha particle and the daughter nucleus produced in the alpha-particle decay of U^{238}, (ii) a μ meson and a neutrino (ν) resulting from pion decay ($\pi \to \mu + \nu$), and (iii) a gamma quantum and the recoil nucleus associated with the emission of the gamma quantum.

3.77. A resting particle a decays as $a \to b + d$. Express the decay energy $Q_a = m_a - m_b - m_d$ ($c = 1$) through the kinetic energy T_b of one of the decay particles and masses m_b and m_d. Calculate the decay energy and mass M_Σ of the Σ^+ particle decaying as $\Sigma^+ \to n + \pi^+$ with the use of the experimentally found value of $T_{\pi^+} = 91.7$ MeV and the neutron and pion masses presented in Table 3.1. Do the same for the Σ^+ decay following a different scheme, $\Sigma^+ \to p + \pi^0$, if $T_p = 18.8$ MeV.

3.78. A free excited nucleus at rest (excitation energy $\Delta\mathcal{E}$) emits a gamma quantum. Find its frequency ω. The mass of the excited nucleus is m. Why is $\omega \neq \Delta\mathcal{E}/\hbar$? How will the result change if the nucleus is rigidly fixed in the crystal lattice (Mössbauer effect).[21)]

3.79*. A resting particle a with mass m decays as $a \to a_1 + a_2 + a_3$ into three particles with masses m_1, m_2, and m_3 and kinetic energies T_1, T_2, and T_3, respectively. Study the kinematics of such a decay using the Dalitz diagram.[22)] To this aim, introduce variables $x = (T_2 - T_3)/\sqrt{3}$, $y = T_1$ and consider the xy plane. Each concrete decay has a definite point in this plane.

1. Prove that the law of conservation of energy defines a limited area on the xy plane in the form of an equiangular triangle. Confirm that the lengths of perpendicular lines dropped from the point denoting a given decay to the sides of the triangle are equal to the kinetic energies of the particles being formed.
2. Make certain that introduction of two quantities, x and y, is sufficient to determine the values of the momenta of the newly formed particles and the angles between these momenta in the rest system of the disintegrating particle.
3. The law of conservation of three-dimensional momentum accounts for the fact that not all points inside the triangle correspond to true decays. Find the area on the xy plane in which decays are kinematically possible for the particular case of $m_2 = m_3 = 0$ and $m_1 \neq 0$.

3.80. Draw a Dalitz diagram (see the condition in the preceding problem) for the decays of μ and K mesons:

$$\text{(i)} \quad \mu^\pm \to e^\pm + 2\nu\,, \quad \text{(ii)} \quad K^\pm \to \pi^0 + e^\pm + \nu\,.$$

In the latter process, an ultrarelativistic electron is usually born and its rest mass can be disregarded. Determine the maximum energies of the particles.

3.81. Draw a Dalitz diagram (see Problem 3.79*) for the decay of a resting K^+ meson according to the following scheme:

$$K^+ \to \pi^- + \pi^+ + \pi^+\,.$$

21) Rudolf Mössbauer (1929–2011) was a German physicist. He was awarded the Nobel prize for his research into the resonance absorption of γ-rays and the discovery of the scattering effect without recoil in a crystalline lattice.

22) Richard Henry Dalitz (1925–2006) was an Australian-born British theoretical physicist.

The energy of decay $Q = m_K - 3m_\pi \approx 75\,\text{MeV} < m_\pi$ $(c = 1)$,; therefore, the newly emerging pions may be arbitrarily regarded as nonrelativistic. What is the maximum energy of each particle?

3.82. Draw a Dalitz diagram (see the condition for Problem 3.79*) for the decay of an ω meson according to the following scheme:

$$\omega \to \pi^+ + \pi^- + \pi^0 .$$

Assume the masses of the three mesons are identical, and the decay energy $Q = m_\omega - 3m_\pi \approx 360\,\text{MeV} > m_\pi$, $m_\omega \approx 780\,\text{MeV}$ $(c = 1)$. What is the maximum energy of each meson?

3.83*. The condition for Problem 3.79* contains the rules for drawing a Dalitz diagram of the decay of three particles. The probability of the $\mathrm{d}W$ decay has the form

$$\mathrm{d}W = \rho \mathrm{d}\Gamma .$$

Here, ρ is a quantity depending on the interaction forces responsible for the decay and on the particles' momenta, and $\mathrm{d}\Gamma$ is an element of the phase volume Γ specified by integral

$$\Gamma = \int \frac{\mathrm{d}^3 p_1}{\mathcal{E}_1} \frac{\mathrm{d}^3 p_2}{\mathcal{E}_2} \frac{\mathrm{d}^3 p_3}{\mathcal{E}_3} \delta\left(p^i - p_1^i - p_2^i - p_3^i\right) ,$$

where p^i is the 4-momentum of a particle ($p^i = (m, 0)$ decaying from the rest state), $p_\alpha^i = (\mathcal{E}_\alpha, \boldsymbol{p}_\alpha)$, $\alpha = 1, 2, 3$, are the 4-momenta of the particles being formed, and $\mathrm{d}^3 p_\alpha$ is a volume element of the momentum space. The four-dimensional delta function expresses the law of conservation of 4-momentum in the decay and indicates that integration is performed only over those values of momenta $\boldsymbol{p}_1$, $\boldsymbol{p}_2$, and $\boldsymbol{p}_3$ that are compatible with the laws of conservation of energy and momentum.

Express $\mathrm{d}\Gamma$ through $\mathrm{d}x$ and $\mathrm{d}y$ and show that the phase volume Γ is expressed (on the appropriate scale) through the area of the permitted region in the Dalitz diagram. Furnish proofs for the general case of $m_1 \neq m_2 \neq m_3 \neq 0$.

3.84. A particle with mass m hits a motionless particle with mass m_1. The resulting reaction gives rise to a few particles with total mass M. There is no reaction if $m + m_1 < M$ and the kinetic energy of the incoming particle is low; it is forbidden by the law of conservation of energy. Find the minimum value of the kinetic energy of the projectile particle (the energy threshold T_0 of the reaction) starting from which the reaction becomes energetically feasible.

3.85. An intermediate W^+ boson is born in the reaction $\nu_\mu + p \to \mu^- + W^+ + p$. Calculate the threshold energy T_0 of neutrino ν_μ.

3.86. Find energy thresholds T_0 of the following reactions: (i) the birth of a pion in nucleon–nucleon collision ($N + N \to N + N + \pi$); (ii) pion photoproduction off a nucleon ($N + \gamma \to N + \pi$); (iii) the birth of a K meson and a Λ hyperon in the collision between a pion and a nucleon ($\pi + N \to \Lambda + K$); (iv) the birth of a proton–antiproton pair in the collision between a proton with mass m_p and a nucleus of mass m. Specifically, consider the collision with a proton. Estimate the antiproton-producing threshold on a nucleus with mass number A assuming that $m \approx m_p A$.

3.87. Derive an approximate expression for the energy threshold T_0 of reactions in which a change of the mass ΔM of the colliding particles is a small part of their total mass M ("reaction between nonrelativistic particles"). Apply the formula thus obtained to the determination of the energy threshold T_0 of the following reactions: (i) deuterium photofission (reaction $\gamma + \mathrm{H}_1^2 \to p + n$); (ii) reaction $\mathrm{He}_2^4 + \mathrm{He}_2^4 \to \mathrm{Li}_3^7 + p$. Compare the approximate values thus obtained with the exact ones (see Problem 3.84).

3.88. Estimate the mass defect ΔM and the relative change of mass $\Delta M/M$ in the following processes:

1. The reaction of hydrogen burning in the Sun and stars (occurring through a few parallel channels):

$$2e^- + 4p \to {}^4\mathrm{He} + 2\nu + Q\,, \quad Q = 26.2\,\mathrm{MeV}\,.$$

   The energy yield of the reaction does not include neutrino ν because this energy is carried away from the system and does not heat surrounding matter.
2. The reaction of deuterium and tritium in controlled thermonuclear reactors (or a hydrogen bomb):

$${}^2\mathrm{H} + {}^3\mathrm{H} \to {}^4\mathrm{He} + n + Q\,, \quad Q = 17.6\,\mathrm{MeV}\,.$$

3. Fission of uranium by a slow neutron:

$$n + {}^{235}\mathrm{U} \to \mathrm{A} + \mathrm{B} + (2 \div 3)n + Q\,, \quad Q = 200\,\mathrm{MeV}\,.$$

4. Combustion of methane in a gas burner:

$$\mathrm{CH_4} + 2\mathrm{O_2} \to \mathrm{CO_2} + 2\mathrm{H_2O} + Q\,, \quad Q = 0.67\,\mathrm{eV}\,.$$

5. Ice melting at 0 °C: the energy is absorbed, $Q = -0.6 \times 10^{-2}$ eV/molecule that passes from a piece of ice to water.

3.89*. An electron and a positron having three-dimensional momenta $\boldsymbol{p}_-$ and $\boldsymbol{p}_+$ in the L-system annihilate in flight and form two gamma quanta. Calculate the invariant mass of the system. What additional information is needed for the calculation of quantum energies $\mathcal{E}_1$ and $\mathcal{E}_2$?

3.90. Prove that the birth of an electron–positron pair by a single gamma quantum is possible only if the reaction involves a particle with rest mass $m_1 \neq 0$ (this particle undergoes no changes; its role is to absorb part of the energy and momentum, thereby facilitating the fulfillment of the laws of conservation). Find the threshold T_0 for the reaction in which the pair are born.

3.91. Prove that the law of energy–momentum conservation forbids annihilation of an electron–positron pair accompanied by the emission of one gamma quantum but allows annihilation resulting in the emission of two photons.

3.92. A particle with energy $\mathcal{E}$ and mass m_1 hits a particle of mass m_2 at rest. Find the center-of-mass velocity v with respect to the laboratory frame of reference in such a collision.

3.93*. A particle with mass m_1 and energy $\mathcal{E}_0$ undergoes elastic collision with a motionless particle having mass m_2. Express scattering angles ϑ_1 and ϑ_2 of the particles in the laboratory frame of reference through their energies $\mathcal{E}_1$ and $\mathcal{E}_2$ after the collision.

3.94. On the basis of the solution of the preceding problem, express the energy of the particles undergoing elastic scattering through the scattering angles in the laboratory frame of reference.

3.95. An ultrarelativistic particle having mass m and energy $\mathcal{E}_0$ is scattered elastically from a nucleus at rest with mass $M \gg m$. Find the dependence of the particle's finite energy $\mathcal{E}$ on its scattering angle ϑ.

3.96. Solve the previous problem for the case of inelastic scattering of a particle from the nucleus. The excitation energy of the nucleus ΔE in its rest system satisfies the inequality $mc^2 \ll \Delta E \ll Mc^2$.

3.97. A particle with mass m undergoes elastic collision with a motionless particle of the same mass. Express the kinetic energy T_1 of the scattered particle through the kinetic energy T_0 of the projectile particle and the scattering angle ϑ_1.

3.98. Using the results obtained in Problem 3.95, find, for the nonrelativistic case, the dependence of the kinetic energies T_1 and T_2 of the particles undergoing elastic collision on the initial kinetic energy T_0 of the first particle and scattering angles ϑ_1 and ϑ_2 in the laboratory frame of reference (assuming the second particle to be at rest prior to the collision).

3.99. Particles with masses m_1 and m_2 undergo elastic collision. The velocities of the particles in the C-system are v_1' and v_2', respectively, the scattering angle is ϑ', and the speed of the C-system with respect to the laboratory frame of reference is V. Determine the scattering angle χ of the particles in the laboratory system. Consider, in particular, the case of $m_1 = m_2$.

3.100. A single quantum of light with frequency ω_0 is scattered from a uniformly moving free electron. The initial momentum of the electron, $\boldsymbol{p}_0$, makes angle ϑ_0 with the direction of quantum movement. Find the dependence of frequency ω of the scattered photon on the direction of its propagation *(Compton[23] effect)*. Consider, in particular, the case of an electron at rest prior to the collision.

23) Arthur Holly Compton (1892–1962) was an American physicist and Nobel Prize recipient.

3.101. A photon with energy $\hbar\omega_0$ is scattered by an ultrarelativistic electron having mass m and energy $\mathcal{E}_0 \gg \hbar\omega_0$. Find the maximum energy $\hbar\omega$ of the scattered photon.

3.102. Find the change in the energy of the electron after its collision with a photon. The initial energies of the electron and photon are $\mathcal{E}_0$ and $\hbar\omega_0$, respectively, and the angle between their momenta is ϑ. Consider the result. Under what conditions would electrons be accelerated by the impacting photons?

3.103. Express invariant variables s, t, and u (3.49) for the case of elastic scattering of identical particles through mass m, the absolute value of momentum q, and scattering angle ϑ in the C-system.

3.104. Suppose a particle b is at rest in the laboratory frame of reference. Express the energy $\mathcal{E}_a$ of particle a in the laboratory frame and energies $\mathcal{E}'_a$ and $\mathcal{E}'_b$ of the particles in the C-system through the invariant variable s (see (3.49)). Do the same for the absolute values of three-dimensional momenta p_a and p' $(p'_a = p'_b = p')$ using a system of units where the speed of light $c = 1$.

3.105. Express energies $\mathcal{E}_c$ and $\mathcal{E}_d$ of the particles produced in a two-particle reaction through the invariant variables (3.49). The energies $\mathcal{E}_c$ and $\mathcal{E}_d$ belong to the laboratory frame of reference.

3.106. Express the angle Θ between three-dimensional momenta $\boldsymbol{p}_a$ and $\boldsymbol{p}_c$ in the laboratory frame of reference in the case of a two-particle reaction through the invariant variables s, t, and u (3.49). Express the angle Θ' between momenta $\boldsymbol{p}'_a$ and $\boldsymbol{p}'_c$ in the C-system through the same variables.

3.107. Construct the region of acceptable s and t values (see (3.49) for the reaction $\gamma + p \to \pi^0 + p$ (photoproduction of a π^0 meson off a proton). What point of this region corresponds to the reaction threshold? What is the threshold value T_0 of the gamma-quantum energy in the laboratory frame of reference? What kinetic energy T_π does the π^0 meson have in the laboratory system at the threshold energy of the gamma quantum?

3.108. Two gamma quanta are transformed into an electron–positron pair. The energy of one of them is specified as equaling $\mathcal{E}_1$. At what $\mathcal{E}_2$ values of the second quantum and angle ϑ between the momenta is this reaction feasible? Depict these values on the plane of variables $\mathcal{E}_2$, $\cos\vartheta$. Also, find the region of acceptable s and t values (3.49). Write the energy in units of mc^2, where m is the electron mass.

3.109*. Draw on the kinematic plane of variables s, t (3.49) the physical regions corresponding to the following processes:

1. $\pi^+ + p \to \pi^+ + p$ – elastic scattering.
2. $\pi^- + \overline{p} \to \pi^- + \overline{p}$ – elastic scattering of antiparticles.
3. $\pi^+ + \pi^- \to p + \overline{p}$ – the birth of a proton–antiproton pair.

All mesons and all nucleons have identical masses (m and M, respectively).

3.110. Prove that emission and absorption of a photon by a free electron in a vacuum is impossible (based on the law of conservation of energy and momentum).

3.111. Prove that the uniform motion of a charged free particle in a medium with refractive index $n(\omega)$ (particle's mass m, charge e, velocity $\boldsymbol{v}$) can be associated with the emission of electromagnetic waves[24] *(Vavilov–Cherenkov effect)*.[25] Express the angle ϑ between the direction of wave propagation and the direction of the particle's velocity through v, ω, and $n(\omega)$.

Hint: In a resting medium having coefficient of refraction $n(\omega)$, a photon has energy $\mathcal{E} = \hbar\omega$ and momentum $p = \hbar\omega n(\omega)/c$.

3.112. Prove that a free electron traveling in a medium with velocity v may absorb electromagnetic waves the frequencies ω of which satisfy $v > c/n(\omega)$, where $n(\omega)$ is the refractive index of the medium.

3.113. An excited particle having, generally speaking, an intricate structure and containing electric charges (e.g., an atom) moves uniformly with velocity $\boldsymbol{v}$ in a medium with refractive index $n(\omega)$. When the particle passes from the excited state to the normal state, it emits a quantum with frequency ω_0 (in the rest system). This quantum is observed in the laboratory frame of reference at an angle ϑ to the direction of the particle's movement. What frequency ω is observed in the laboratory system? *(Doppler effect in a refractive medium.)* Consider, specifically, the case of $\omega_0 \to 0$.

Hint: Disregard the second-order terms in $\hbar$ and assume that $\hbar\omega_0 \ll mc^2$, where m is the particle's mass.

3.114. The particle considered in Problem 3.113 moves uniformly through a medium when in the normal state (the remaining conditions for Problem 3.113 are unaltered). Prove that emission occurring in this case is accompanied by excitation of the particle. Clarify what conditions are needed for such emission to occur. Find the frequency ω of this emission *(superlight Doppler effect)*.

3.115. It follows from the laws of conservation of energy and momentum that Cherenkov emission of a single quantum with frequency ω is impossible if the medium has refractive index $n(\omega) \leq 1$ (see Problem 3.112). Specifically, single-quantum Cherenkov emission of sufficiently hard photons becomes impossible because at high frequencies $n(\omega) < 1$. Show that a uniform movement of a fast charged particle with energy $\mathcal{E}_0$ through a medium can be associated with the emission of two photons at a time, one of which (with frequency ω_2) may be hard (i.e., $n(\omega_2) \to 1$). Make clear what conditions the frequency ω_1 of other photon and the velocity v_0 of the particle ($\hbar\omega_1 \ll cp_0$) must satisfy in order to make

24) A similar effect may occur during the passage of a neutral particle with an electric or magnetic moment through matter.

25) Sergei Ivanovich Vavilov (1891–1951) was an outstanding Soviet physicist. His main works are related to physical optics and the history of physics. Pavel Alekseyevich Cherenkov (1904–1990) was a Soviet physicist and Nobel Prize recipient.

such a process possible (hard Vavilov–Cherenkov radiation). What is the maximum energy of a hard quantum?

3.116. Consider the kinematics of hard Vavilov–Cherenkov radiation (see the preceding problem) on the assumption that the electron is ultrarelativistic, $\mathcal{E}_0 \gg mc^2$, and the exit angle ϑ_2 of a hard quantum is small. Determine the maximum value $(\hbar\omega_2)_{\max}$ of the hard quantum energy that can be reached in this case; consider characteristic specific cases.

3.117. The crystal lattice can receive momentum only in discrete portions $\boldsymbol{q} = 2\pi\hbar\boldsymbol{g}$, where $\boldsymbol{q}$ is the reciprocal lattice vector. In the case of a crystal lattice whose elementary cell is a rectangular parallelepiped with edges a_1, a_2, and a_3, $\boldsymbol{g} = (\frac{n_1}{a_1}, \frac{n_2}{a_2}, \frac{n_3}{a_3})$, where n_1, n_2, and n_3 are any integer numbers. Clarify the angular distribution patterns of the particles scattered from a single crystal on the assumption that a crystal having a very large mass cannot absorb the particle's energy.

3.118. Taking account of the relationship $p_0 = 2\pi\hbar/\lambda_0$ between the particle's momentum p_0 and the corresponding wavelength λ_0, derive the Bragg–Wulff[26] condition, $2a\sin(\vartheta/2) = n\lambda_0$, where a is the distance between the crystal planes, ϑ is the scattering angle of the particle, and n is an integer.

3.119. Clarify the character of the energy spectrum of bremsstrahlung quanta originating from the scattering of charged particles by a single crystal (see Problem 3.117). The angle between the direction of propagation of a bremsstrahlung quantum and the starting momentum of a particle is constant and small, $\vartheta \ll 1$. The particle is ultrarelativistic, $\mathcal{E}_0 \gg mc^2$.

3.120•. In the classical model, an electron can be represented as a charged sphere of certain radius R. Determine R on the assumption that the electron mass is of electrostatic origin, that is, its rest energy equals the electrostatic energy of the charged sphere.

Suggested literature:
Landau and Lifshitz (1976); Barashenkov (1974); Okun (1989, 2000); Medvedev (1977); Feinberg (1967); Byckling and Kajantie (1973); Smorodinskii (1972); Lightman *et al.* (1975) and materials specified in Section 3.1.

26) William Henry Bragg (1862–1942) and William Lawrence Bragg (1890–1971) were English physicists, father and son. They were Nobel Prize recipients, and the founders of research into crystalline lattices using X-rays. Georgi (Yuri) Viktorovich Wulff (1863–1925) was a Russian crystallographer and crystallophysicist.

3.3 Answers and Solutions

3.1*

1. As follows from space and time uniformity, transformations must be linear:

$$x' = \alpha x + \alpha' y + \beta' z + \beta t + p, \text{etc.}\,, \tag{1}$$

where $\alpha, \alpha', \beta \ldots$ are constant coefficients that may depend on the relative speed V. If these quantities depended on coordinates and time, it would mean that transformation law (1) is not equally valid for different points in space and different moments of time, in conflict with the postulate of space–time uniformity.

2. Let three spatial axes of two reference frames, S and S', coincide at $t = t'$ (Figure 1.1). Under these conditions, the free terms in equalities (1) (constant p, etc.) tend to vanish. Moreover, the xy plane coincides with the $x'y'$ plane. This means that at $z = 0$, $z' = 0$ and both equalities must be fulfilled at any x', y', z' and x, y, z values, respectively. This is possible only if the relationship between z and z' has the form $z' = kz, k = \text{const}$. The directions of the y and z axes (space isotropy!) being arbitrary, there must be a similar relationship with the same coefficient k between y and y': $y' = ky$.
Let us represent transformations x' and t' in the forms

$$x' = \alpha x + \beta t + \alpha' y + \beta' z\,, \quad t' = \sigma x + \delta t + \sigma' y + \delta' z\,. \tag{2}$$

In the plane $x' = 0$, we have $x = Vt$ at any z and y because system S' moves with speed V with respect to system S. Substituting these values of x and x' into the first equality in (2) yields $\alpha' = \beta' = 0$ and $\beta = -\alpha V$. Finally, let us turn to the transformation formula for t'. We set the clock in system S' so that $t' = 0$ at $x = 0$ and $t = 0$. This is possible only at $\sigma' = \delta' = 0$. As a result, we have the following transformation formulas:

$$x' = \alpha(V)(x - Vt)\,, \quad y' = k(V)y\,, \quad z' = k(V)z\,, \quad t' = \sigma(V)x + \delta(V)t\,, \tag{3}$$

where the coefficients depend explicitly on the relative speed.

3. Now, let us use the condition of equality of systems S and S'. This means the formulas for transition from S' to S must be derived from the transition formulas (3) by substitution of $-V$ for V:

$$\begin{aligned} x &= \alpha(-V)(x' + Vt')\,, \qquad & y &= k(-V)y'\,, \\ z &= k(-V)z'\,, \qquad & t &= \sigma(-V)x' + \delta(-V)t'\,. \end{aligned} \tag{4}$$

Let us first turn to the formulas for y and z. The cases of (3) and (4) differ only in the direction of the relative velocity, which is perpendicular to yz plane

in either case. But the two directions are absolutely equivalent (space isotropy, equivalence between right and left!); therefore, $k(-V) = k(V)$. The transformation from y to y' and back from y' to y gives $y = k^2 y$, that is, $k^2 = 1$, $k = \pm 1$. The value of $k = -1$ conforms to the opposite orientation of the y and y' axes; for this reason, the agreement with Figure 1.1 is possible only at $k = 1$.
We substitute the values of x' and t' from (3) for x into formula (4) to obtain

$$x = [\alpha(-V)\alpha(V) + V\sigma(V)\alpha(-V)]x + \alpha(-V)V[\delta(V) - \alpha(V)]t . \tag{5}$$

This equality holds at all x and t. The transformation coefficients must satisfy the relation

$$\delta(V) = \alpha(V) . \tag{6}$$

We do not need the second relation following from (5). We use instead the light velocity invariance postulate.

4. Let a short light signal be emitted from the coincident reference points at the moment of coincidence of systems S and S' ($t = t' = 0$). The point of intersection between the wave and the x axis moves in system S at speed $x/t = c$. Because the light signal propagates with equal speed in all reference frames, the wavefront velocity in system S' is the same: $x'/t' = c$. Bearing in mind these conditions and dividing term by term the equation for x' by equation for t', we obtain from (3)

$$c = \frac{\alpha(c - V)}{\sigma c + \alpha} .$$

Hence,

$$\sigma(V) = -\frac{V}{c^2}\alpha(V) . \tag{7}$$

To determine coefficient $a(V)$, we consider the spherical wavefront equations for systems S and S':

$$x^2 + y^2 + z^2 = (ct)^2 , \quad x'^2 + y'^2 + z'^2 = (ct')^2 . \tag{8}$$

In these equations the property of light velocity invariance is used again; therefore, c is the same. Since $y' = y$ and $z' = z$, then $(ct')^2 - x'^2 = (ct)^2 - x^2$. Using (3), (6), and (7), we obtain from the last equation

$$\alpha^2(1 - \frac{v^2}{C^2})(c^2t^2 - x^2) = (ct)^2 - x^2 ,$$

whence

$$\alpha(V) = \pm(1 - \frac{V^2}{c^2})^{-1/2} . \tag{9}$$

Here again, only the plus sign must be taken because the minus sign corresponds to the opposite direction of the x and x' axes. Bringing together the results (3)–(9) leads to the Lorentz transformations (3.7).

3.2

$$x - x_0 = \frac{x' - x'_0 + V(t' - t'_0)}{\sqrt{1-\beta^2}} , \quad y - y_0 = y' - y'_0 ,$$

$$z - z_0 = z' - z'_0 , \quad t - t_0 = \frac{t' - t'_0 + \frac{V}{c^2}(x' - x'_0)}{\sqrt{1-\beta^2}} .$$

3.5 The coordinates of the clocks showing the same time $t = t'$ in systems S and S' are

$$x = \frac{c^2}{V}\left(1 - \frac{1}{\gamma}\right) t , \quad x' = -\frac{c^2}{V}\left(1 - \frac{1}{\gamma}\right) t .$$

The point at which $t = t'$ moves uniformly in each of the systems, S and S', follows from these formulas. If a reference frame with respect to which this point is motionless is introduced, systems S and S' move in opposite directions with equal speeds $V_0 = (c^2)/V(1 - 1/\gamma)$ (V_0 being the relativistic "half" of speed V in the sense that relativistic addition of two V_0 speeds yields V).

3.6 In system S', the duration of one period $T' = 2l/c$; in system S, time T_1 of the movement of the light spot along the rod in the direction of the relative speed V is calculated from the equation

$$T_1 = \frac{1}{c}\left(l\sqrt{1 - \frac{V^2}{c^2}} + V T_1\right) ,$$

and the movement in the backward direction T_2 is determined by substituting $-V$ for V. For the ratio of T' to $T = T_1 + T_2$, we find

$$\frac{T'}{T} = \sqrt{1 - \frac{V^2}{c^2}} ,$$

hence, (3.11).

3.8 $\Delta t = l_0/V\gamma = 36\,\text{min}$; $\Delta t' = l_0/V = 60\,\text{min}$.

3.9 By the Earth clock, $\Delta t = 8$ years. When calculating the store of victuals and equipment, one should be base the calculation on the period of time $\Delta t_0 = 0.01\Delta t \approx 1$ month measured by the clock in the rocket;

$$T = mc^2(\gamma - 1) \equiv 2.5 \times 10^{16}\,\text{kWh} .$$

This amount of energy is a few orders of magnitude greater than the current global annual production of electrical energy.

3.10 $v = \frac{2l_0\Delta t}{(\Delta t)^2 + l_0^2/c^2}$. An observer connected with the first scale (Figure 3.10a) starts seeing the left ends coincide, then he sees the right ends coming together.

An observer connected with the second scale sees the same events in the reverse order (Figure 3.10b). From the standpoint of an observer relative to whom the scales are moving with a similar speed, the ends coincide simultaneously.

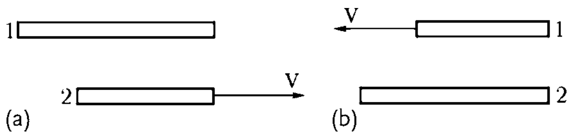

Figure 3.10 Observation of moving scales for an observer connected with the first scale (a), and for an observer connected with the second scale (b).

3.11 Let us introduce the transverse and longitudinal components of the radius vector $\boldsymbol{r}$:

$$\boldsymbol{r}_\parallel = \boldsymbol{V}\frac{\boldsymbol{r}\cdot\boldsymbol{V}}{V^2}\,, \quad \boldsymbol{r}'_\parallel = \boldsymbol{V}\frac{\boldsymbol{r}'\cdot\boldsymbol{V}}{V^2}\,; \quad \boldsymbol{r}_\perp = \boldsymbol{r} - \boldsymbol{r}_\parallel\,, \quad \boldsymbol{r}'_\perp = \boldsymbol{r}' - \boldsymbol{r}_\parallel\,.$$

Applying the Lorentz transformations (3.7) to $\boldsymbol{r}_\parallel$ and $\boldsymbol{r}_\perp$, we arrive at

$$\boldsymbol{r}_\parallel = \gamma(\boldsymbol{r}'_\parallel + \boldsymbol{V}t')\,, \quad \boldsymbol{r}_\perp = \boldsymbol{r}'_\perp\,.$$

Finally,

$$\boldsymbol{r} = \gamma(\boldsymbol{r}' + \boldsymbol{V}t') + (\gamma - 1)\frac{(\boldsymbol{r}'\times\boldsymbol{V})\times\boldsymbol{V}}{V^2}\,, \quad t = \gamma\left(t' + \frac{\boldsymbol{r}'\cdot\boldsymbol{V}}{c^2}\right)\,.$$

3.12

$$\boldsymbol{A} = \gamma\left(\boldsymbol{A}' + \frac{\boldsymbol{V}}{c}A'_0\right) + (\gamma - 1)\frac{(\boldsymbol{A}'\times\boldsymbol{V})\times\boldsymbol{V}}{V^2}\,, \quad A_0 = \gamma\left(A'_0 + \frac{\boldsymbol{A}'\cdot\boldsymbol{V}}{c}\right)\,.$$

3.13

$$\boldsymbol{v} = \boldsymbol{v}_\parallel + \boldsymbol{v}_\perp = \frac{\boldsymbol{v}' + \boldsymbol{V} + \boldsymbol{V}[(\boldsymbol{v}'\cdot\boldsymbol{V}) + V^2](\gamma - 1)/V^2}{\gamma\,(1 + \boldsymbol{v}'\cdot\boldsymbol{V}/c^2)}\,,$$

where $\boldsymbol{v}$ and $\boldsymbol{v}'$ are the velocities in systems S and S'. Also, it is possible to simply differentiate with respect to time the radius vector $\boldsymbol{r}$ expressed through $\boldsymbol{r}'$ and t' using the formula obtained in Problem 3.10.

3.17* The Thomas rotation angle is defined by the relation

$$\varphi = -\arccos\frac{v^2\sqrt{1 - V^2/c^2} + V^2\sqrt{1 - v^2/c^2}}{V^2 + v^2 - V^2v^2/c^2}\,.$$

At $v, V \ll c$, $\varphi \approx 0$. At $v \to c$, $\varphi \to -\arccos\sqrt{1 - \frac{V^2}{c^2}}$. At $V \to c$, $\varphi \to \frac{\pi}{2}$.

3.18

$$l = l_0 \frac{1 - v^2/c^2}{1 + v^2/c^2} \ .$$

3.19 (i) $V = 2 \times 0.9c = 1.8c$; (ii) $V = 0.994c$.

3.20 The relative velocity of two particles in a system connected with one of them is $V = 2v/(1 + v^2/c^2)$. Hence,

$$\mathcal{E} = \frac{mc^2}{\sqrt{1 - V^2/c^2}} = mc^2 \left[2 \left(\frac{\mathcal{E}_0}{mc^2} \right)^2 - 1 \right] \ .$$

In the ultrarelativstic case of $\mathcal{E}_0 \gg mc^2$, $\mathcal{E} = 2\mathcal{E}_0^2/mc^2$. In the case when electrons are accelerated ($mc^2 = 0.5$ MeV), at $\mathcal{E}_0 = 50$ MeV, the power gain of the accelerator can be as high as 200-fold, that is, $\mathcal{E} = 10\,000$ MeV. For the proton beams specified in the condition for the problem, the relativistic factors $\gamma = \mathcal{E}/mc^2 \approx 10^5$ and $\gamma_0 \approx \gamma^{1/2}/\sqrt{2} \approx 230$. The energy in the beam must be $\mathcal{E}_0 \approx 230$ GeV.

3.21 This problem like Problem 3.12 can be solved in two ways. The result is

$$\dot{\boldsymbol{v}} = \frac{1}{\gamma^2 s^2} \dot{\boldsymbol{v}}' - \frac{(\gamma - 1)(\dot{\boldsymbol{v}}' \cdot \boldsymbol{V})\boldsymbol{V}}{\gamma^3 s^3 V^2} - \frac{(\dot{\boldsymbol{v}}' \cdot \boldsymbol{V})\boldsymbol{v}'}{\gamma^2 s^3 c^2} \ ,$$

where $s = 1 + \boldsymbol{v}' \cdot \boldsymbol{V}/c^2$. It follows from these formulas that if a particle moves with constant acceleration $\dot{\boldsymbol{v}}'$ in one frame of reference, the acceleration $\dot{\boldsymbol{v}}$ in another frame is, generally speaking, time dependent (because the transformation formulas contain the variable velocity $\boldsymbol{v}'$ of the particle).

3.22 $w_i w^i = -\gamma^6 \left[\dot{\boldsymbol{v}}^2 - \frac{(\dot{\boldsymbol{v}} \times \boldsymbol{v})^2}{c^2} \right] = -\gamma^4 \left[\dot{\boldsymbol{v}}^2 + \gamma^2 \frac{(\boldsymbol{v} \cdot \dot{\boldsymbol{v}})^2}{c^2} \right] < 0$, that is, the four-dimensional acceleration – the space-like vector.

3.23 Let S' be a system instantaneously accompanying the particle. According to the answer to Problem 3.21,

$$\dot{\boldsymbol{v}}' = \gamma^2 \left[\dot{\boldsymbol{v}} + \frac{\gamma - 1}{v^2} (\dot{\boldsymbol{v}} \cdot \boldsymbol{v}) \boldsymbol{v} \right] \ . \tag{1}$$

Hence, the square of acceleration

$$\dot{\boldsymbol{v}}'^2 = \gamma^4 \left[\dot{\boldsymbol{v}}^2 + \frac{\gamma^2 (\dot{\boldsymbol{v}} \cdot \boldsymbol{v})^2}{c^2} \right] = \gamma^6 \left[\dot{\boldsymbol{v}}^2 - \left(\dot{\boldsymbol{v}} \times \frac{\boldsymbol{v}}{c} \right)^2 \right] \ . \tag{2}$$

Given only the value of the particle's velocity varies, $\dot{\boldsymbol{v}} \parallel \boldsymbol{v}$ and

$$\dot{\boldsymbol{v}}' = \gamma^3 \dot{\boldsymbol{v}} \ . \tag{3}$$

If only the particle's direction changes, $\boldsymbol{v} \perp \dot{\boldsymbol{v}}$ and $\boldsymbol{v} \cdot \dot{\boldsymbol{v}} = 0$; therefore, $\boldsymbol{v} \perp \dot{\boldsymbol{v}}$ and $\boldsymbol{v} \cdot \dot{\boldsymbol{v}} = 0$,

$$\dot{\boldsymbol{v}}' = \gamma^2 \dot{\boldsymbol{v}} . \tag{4}$$

Result (2) can be obtained in a different (simpler) way by making use of the expression for the square of four-dimensional acceleration found in the previous problem. The square of $w_i w^i$ is a 4-invariant. This means that its calculation in both system S and system S' must yield the same result. Bearing in mind that the particle's velocity $\boldsymbol{v}' = 0$, we arrive at formula (2).

3.24

$$v(t) = \frac{wt + v_0 \left(1 - \beta_0^2\right)^{-1/2}}{\sqrt{1 + c^{-2} \left(wt + v_0 \left(1 - \beta_0^2\right)^{-1/2}\right)^2}},$$

$$x(t) = \frac{c^2}{w} \left\{ \sqrt{1 + c^{-2} \left(wt + v_0 \left(1 - \beta_0^2\right)^{-1/2}\right)^2} - \left(1 - \beta_0^2\right)^{-1/2} \right\} + x_0 .$$

In the ultrarelativistic limit,

$$v(t) \approx c , \quad x(t) \approx ct + x_0 + \frac{c v_0}{w \sqrt{1 - \beta_0^2}} .$$

In the nonrelativistic limit,

$$v(t) = v_0 + wt , \quad x(t) = x_0 + v_0 t + \frac{1}{2} w t^2 .$$

3.25 The acceleration time measured by the clock in a motionless system is

$$T = \frac{1}{|\dot{v}|} \int_0^v \frac{\mathrm{d}v}{(1 - v^2/c^2)^{3/2}} = \frac{v}{|\dot{v}| \sqrt{1 - v^2/c^2}} = 47.5 \text{ years} .$$

The acceleration time measured by the clock connected with the rocket is

$$\tau = \frac{c}{2|\dot{v}|} \ln \left| \frac{1 + v/c}{1 - v/c} \right| = 2.5 \text{ years} .$$

3.26* Formulas (1) describe the Lorentz transformation with a low relative velocity $\Delta \boldsymbol{v}$ and the rotation through angle $\Delta \varphi = |\Delta \boldsymbol{\varphi}|$, with the axis of rotation passing through the origin of the coordinates and parallel to vector $\Delta \boldsymbol{\varphi}$. Owing to the smallness of $\Delta \boldsymbol{v}$ and $\Delta \boldsymbol{\varphi}$, these transformations can be performed in any order. Therefore, the instantaneously accompanying system is a rotating one. This rotation is a purely kinematic relativistic effect called Thomas precession (see Problem 3.16).

At $v \ll c$, formulas (2) assume the form

$$\Delta \boldsymbol{v} \approx \delta \boldsymbol{v}\ , \quad \Delta \boldsymbol{\varphi} \approx \frac{1}{2c^2} \delta \boldsymbol{v} \times \boldsymbol{v}\ .$$

In this limit, the quantity

$$\omega_T = \frac{\delta \boldsymbol{\varphi}}{\delta t} \approx \frac{1}{2c^2} \dot{\boldsymbol{v}} \times \boldsymbol{v}$$

can be regarded as the angular velocity of Thomas precession of the instantaneously accompanying system with respect to the laboratory system S.

3.27

$$\tan \vartheta = \frac{v' \sqrt{1 - V^2/c^2} \sin \vartheta'}{v' \cos \vartheta' + V}\ .$$

3.28 In system S, $\cos \alpha = \boldsymbol{v}_1 \cdot \boldsymbol{v}_2 / |\boldsymbol{v}_1||\boldsymbol{v}_2|$. In system S',

$$\cos \alpha' = \frac{(\boldsymbol{v}_1 - \boldsymbol{V}) \cdot (\boldsymbol{v}_2 - \boldsymbol{V}) - \frac{1}{c^2}(\boldsymbol{v}_1 \times \boldsymbol{V}) \cdot (\boldsymbol{v}_2 \times \boldsymbol{V})}{\sqrt{(\boldsymbol{v}_1 - \boldsymbol{V})^2 - \frac{1}{c^2}(\boldsymbol{v}_1 \times \boldsymbol{V})^2} \sqrt{(\boldsymbol{v}_2 - \boldsymbol{V})^2 - \frac{1}{c^2}(\boldsymbol{v}_2 \times \boldsymbol{V})^2}}\ .$$

3.29 The angle in system S' tends to zero. To be sure, we assume that $\boldsymbol{V} = \boldsymbol{V}_0 c$, where $|\boldsymbol{V}_0 = 1$. We calculate $\cos \alpha'$ using the formula obtained in the preceding problem. We then use the identity

$$(\boldsymbol{a} \times \boldsymbol{b}) \cdot (\boldsymbol{a}_1 \times \boldsymbol{b}_1) = (\boldsymbol{a} \cdot \boldsymbol{a}_1)(\boldsymbol{b} \cdot \boldsymbol{b}_1) - (\boldsymbol{a} \cdot \boldsymbol{b}_1)(\boldsymbol{a}_1 \cdot \boldsymbol{b})\ ,$$

and obtain

$$\cos \alpha' = \frac{c^2 - \boldsymbol{v}_1 \cdot \boldsymbol{V} - \boldsymbol{v}_2 \cdot \boldsymbol{V} + (\boldsymbol{v}_1 \cdot \boldsymbol{V})(\boldsymbol{v}_2 \cdot \boldsymbol{V})/c^2}{\sqrt{(c - \boldsymbol{v}_1 \cdot \boldsymbol{V}/c)^2} \cdot \sqrt{(c - \boldsymbol{v}_2 \cdot \boldsymbol{V}/c)^2}} = 1\ ,$$

whence $\alpha' = 0$. Narrowing of the angular distribution is a characteristic relativistic effect that manifests itself in many phenomena.

3.30

$$\cos \vartheta = \frac{\cos \vartheta' + \beta}{1 + \beta \cos \vartheta'}\ .$$

3.31 Determination of the aberration angle reduces to the calculation of two angles (Figure 3.11): the angle α_1 between the direction of the AC ray and the direction of the velocity of the Earth $\boldsymbol{v}$ in its first position and the angle α_2 between the direction of the BC ray and the direction of the Earth's velocity $\boldsymbol{v}'$ in its second position (in half a year). The aberration angle δ can be defined as $\delta = (\pi - \alpha_2) - \alpha_1 = \pi - \alpha_1 - \alpha_2$. Angles α_2 and α_2 are calculated from the formulas in Problem 3.26* and are expressed through angle ϑ observed in the frame of reference connected with the Sun between the ray of light OC and the Earth's velocity vector:

$$\tan(\pi - \alpha_1) = \frac{\sin\vartheta}{\gamma(\cos\vartheta - \beta)} , \quad \tan(\pi - \alpha_2) = -\frac{\sin\vartheta}{\gamma(\cos\vartheta + \beta)} ,$$

where $\beta = v/c$, $\gamma = 1/\sqrt{1-\beta^2}$. Hence,

$$\tan\frac{\delta}{2} = \sqrt{\frac{1-\cos\delta}{1+\cos\delta}} = \beta\gamma\sin\vartheta .$$

Note that all three angles between the velocities depicted in Figure 3.11 belong to different frames of reference and that the figure itself is conventional (e.g., it shows sections $AC = CO = CB = c$).

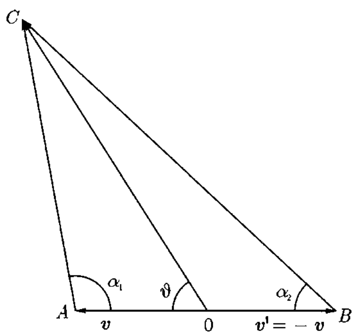

Figure 3.11 Diagram for alculation of an angle of an aberration.

The results obtained suggest, in particular, that the aberration angle δ depends only on the relative velocity v of the Earth and the Sun and does not depend on the velocity of the Solar system with respect to a star.

3.32 If the Earth's position on its orbit is determined by the azimuthal angle φ and $\boldsymbol{a} = (0, a_\vartheta, a_\alpha)$ is the vector drawn between a point (ϑ, α) on the celestial sphere and the point corresponding to the apparent position of the star on this sphere, then

$$a_\vartheta = -\beta\cos\vartheta\sin(\alpha - \chi) , \quad a_\alpha = -\beta\cos(\alpha - \varphi) .$$

This means that the visible position of the star on the celestial sphere circumscribes during a year an ellipse with semiaxes $\beta\cos\vartheta$ and β.

3.33 Let us consider a beam inside the solid angle $\mathrm{d}\Omega = \sin\vartheta\mathrm{d}\vartheta\mathrm{d}\alpha$ in system S. In system S', this beam is observed inside the angle $\mathrm{d}\Omega' = \sin\vartheta'\mathrm{d}\vartheta'\mathrm{d}\alpha'$. The angle $\alpha = \alpha'$, and $\cos\vartheta' = (\cos\vartheta - \beta)/(1 - \beta\cos\vartheta)$. Hence,

$$\mathrm{d}\Omega' = \sin\vartheta'\mathrm{d}\vartheta'\mathrm{d}\alpha' = \frac{1-\beta^2}{(1-\beta\cos\vartheta)^2}\mathrm{d}\Omega .$$

Then, obviously, $\int\mathrm{d}\Omega' = \int\mathrm{d}\Omega = 4\pi$.

3.34• In system S, the same fraction $\mathrm{d}N$ of the total radiation occurs in another solid angle $\mathrm{d}\Omega$. Therefore, using the formula from the answer to the preceding problem, we obtain

$$f(\vartheta) = \frac{1-\beta^2}{(1-\beta\cos\vartheta)^2} = \frac{1}{\gamma^2[1-(1-\gamma^{-2})^{1/2}\cos\vartheta]^2} ,$$

where ϑ is the angle between the direction of observation and the velocity of the source. The light angular distribution becomes anisotropic with increasing $\beta = v/c$ (Figure 3.12). At $\gamma \gg 1$, the distribution becomes sharply anisotropic and the largest part of the light is emitted in the direction of the relative velocity: $f(0)/f(\pi) \approx 4\gamma^2 \gg 1$. The distribution function at angles $\vartheta \ll 1$ can be simplified by expanding $\cos V$ into a series:

$$f(\vartheta) = \frac{4}{\gamma^2(\gamma^{-2}+\vartheta^2)^2} .$$

It follows from the last formula that any light source moving with a relativistic velocity emits light in the forward direction within the angular opening of the cone,

$$\Theta_0 \approx \gamma^{-1} \ll 1 ,$$

if its emission in a rest system is not substantially different from an isotropic one.

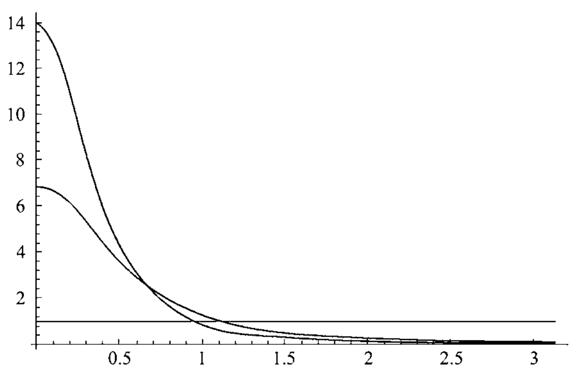

Figure 3.12 Contraction of the angle distribution with increase of relativistic factor.

3.35•

$$\boldsymbol{k} = \gamma\left(\boldsymbol{k}_0 + \frac{V\omega_0}{c^2}\right) + \frac{\gamma-1}{V^2}(\boldsymbol{k}_0 \times \boldsymbol{V}) \times \boldsymbol{V} .$$

3.36 If ω_0 is the frequency in a system where the source is at rest and V is its speed with respect to the light detector, the device detects a lower frequency $\omega =$

$\omega_0\sqrt{1-\frac{V^2}{c^2}}$ (redshift). The angle α between the beam and the direction of detector movement in its rest system is given by

$$\cos\alpha = -\frac{V}{c}\,.$$

The angle α is close to 90° only at $V \ll c$. If $V \to c$, $\alpha \to \pi$.

3.37 (i) $\lambda = \lambda_0\sqrt{(1-V/c)/(1+V/c)}$; (ii) $\lambda = \lambda_0\sqrt{(1+V/c)/(1-V/c)}$.

3.38

$$\omega = \omega_0\frac{\sqrt{1-\beta^2}}{1-\beta\cos\theta}\,, \quad J = J_0\frac{(1-\beta^2)^{3/2}}{(1-\beta\cos\theta)^2}\,.$$

The frequencies coincide, $\omega = \omega_0$, at $\theta = \theta_0$, where $\cos\theta_0 = (1-\sqrt{1-\beta^2})/\beta$; in this case, $J = J_0\sqrt{1-\beta^2}$. The intensities become equal, $J = J_0$, at $\theta = \theta_1 < \theta_0$, $\cos\theta_1 = [1-(1-\beta^2)^{3/4}]/\beta$. When a source of light situated far from the observer begins to approach him so that $\theta < \theta_0$, the $\omega > \omega_0$ owing to the Doppler effect (blueshift). If, in addition, $\theta < \theta_1$, the intensity J also exceeds J_0 and the moving source looks brighter than the motionless one. The maximum intensity occurs at $\theta = 0$: $J_{\max} = J_0(1+\beta)^{3/2}/\sqrt{1-\beta}$. At $\theta > \theta_0$, the $\omega < \omega_0$, and the observer sees the "red" shift, the light intensity being lower than that from the motionless source. These effects are especially apparent at $V \approx c$, when

$$\omega_{\max} = \omega_0\sqrt{\frac{1+\beta}{1-\beta}} \gg \omega \quad \text{and} \quad J_{\max} = J_0\frac{(1+\beta)^{3/2}}{(1-\beta)^{1/2}} \gg J_0\,,$$

and angle

$$\theta_0 \approx \sqrt{2}(1-\beta)^{1/4} \ll 1\,,$$

so the light begins to "redden" when the source is still far from the observer but approaches him. It occurs starting from distances $l \approx d/\theta_0$.

The number of photons emitted per unit laboratory time in the angle range $0 < \theta < \theta_0$ is

$$N_1 = J_0(1-\beta^2)\int_0^{\theta_0}\frac{2\pi\sin\theta\,d\theta}{(1-\beta\cos\theta)^2}$$
$$= 2\pi J_0\sqrt{1-\beta^2}\frac{1+\beta-\sqrt{1-\beta^2}}{\beta} = 2\pi J_0\sqrt{1-\beta^2}(1+\cos\theta_0)\,,$$

and in the range $\theta_0 < \theta < \pi$ is

$$N_2 = 2\pi J_0\sqrt{1-\beta^2}\frac{\sqrt{1-\beta^2}-1+\beta}{\beta} = 2\pi J_0\sqrt{1-\beta^2}(1-\cos\theta_0)\,.$$

Evidently, $N_1 + N_2 = 4\pi J_0\sqrt{1-\beta^2}$ corresponds to the total number of photons emitted per unit time in all directions. N_1 is equal to N_2 at $\beta \ll 1$, when $\cos\theta_0 \approx 0$. If β approaches unity, N_1 becomes much greater than N_2. Thus, in this ultrarelativistic case, the largest part of light is emitted within a narrow cone $\theta < \theta_0$ and undergoes a violet shift.

3.39 On the basis of the solution of the preceding problem, we obtain

$$I = J\hbar\omega = I_0\frac{(1-\beta^2)^2}{(1-\beta\cos\theta)^3} ,$$

where $I_0 = J_0\hbar\omega_0$ is the isotropically distributed light intensity in the rest system of the source. The overall luminous flux

$$\Phi = \int\limits_{(4\pi)} I\,\mathrm{d}\Omega = 2\pi I_0(1-\beta^2)^2\int\limits_0^{\pi}\frac{\sin\theta\,\mathrm{d}\theta}{(1-\beta^2\cos\theta)^3} = 4\pi I_0 = \Phi_0$$

is identical in the rest system of the source and in the laboratory system.

3.40 Let us introduce system S' connected with a mirror (S being the laboratory system) and denote by α_1' and α_2' the angles made by the wave vectors $\boldsymbol{k}_1'$ and $\boldsymbol{k}_2'$ of the incident and reflected waves with the direction of the mirror speed $\boldsymbol{V}$ (Figure 3.13).

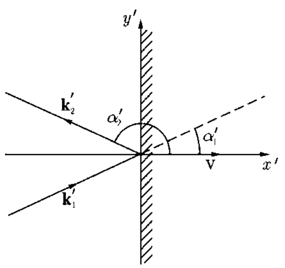

Figure 3.13 Reflection of light from a moving mirror.

The frequency before and after reflection is denoted by ω_1' and ω_2', respectively. The analogous quantities in system S are denoted by the same letters without primes. We proceed from the known laws of reflection in system S', viz., $\omega_1' = \omega_2' = \omega'$ and $\alpha_2' = \pi - \alpha_1'$, whence $\cos\alpha_2' = -\cos\alpha_1'$.

Expressing ω' through ω and $\cos\alpha'$ through $\cos\alpha$ with the help of the Lorentz transformation formulas and solving the resultant equation with respect to ω_2 and $\cos\alpha_2$, we obtain

$$\cos\alpha_2 = -\frac{(1+\beta^2)\cos\alpha_1 - 2\beta}{1-2\beta\cos\alpha_1+\beta^2} , \quad \omega_2 = \omega_1\frac{1-2\beta\cos\alpha_1+\beta^2}{1-\beta^2} .$$

If $\beta \to 1$, the normal incidence onto the approaching mirror corresponds to $\omega_2 \to 0$, and the normal incidence onto the receding mirror corresponds to $\omega_2 \to \infty$.

3.41 $\omega_1 = \omega_2$. The angle of incidence equals the angle of reflection.

3.42 An image is produced by light quanta that simultaneously strike the photoplate even though they are not simultaneously emitted from different points of the moving body because the points lie at different distances from the photoplate and because the events concurrent in one system are not such in another. For this reason, the image of a moving object is dissimilar to that of a motionless one.

The quanta emitted simultaneously from different points at edge $A'B'$ in system S' (a cube) reach the photoplate at the same time. The length of the AB image is the same as in the case of a motionless cube; it is determined only by the contraction depending on the distance to the object and the focal length of the camera lens. Assume the image length to be 1.

The image of edge $E'F'$ of a motionless cube is fused with the $A'B'$ image (in the limiting case of an arbitrarily small solid angle when all the rays are parallel). The quanta from edge $E'F'$ of a moving cube will reach the photoplate at the same time as the quanta from edge $A'B'$ if the former were emitted earlier ($\Delta t = l_0/c$) than the latter (in system S). At this time, edge $E'F'$ was in position $E_1'F_1'$ and had already covered the distance of Vl_0/c before the light was emitted from edge $A'B'$. As a result, edge AB does not obstruct edge $E'F'$ and the images of edges $A'E'$ and $B'F'$ have length $V/c = \beta$ rather than zero as in the motionless cube; therefore, the photograph of the entire $A'B'F'E'$ facet looks like a rectangular $ABFE$ with an aspect ratio of $1 : \beta$.

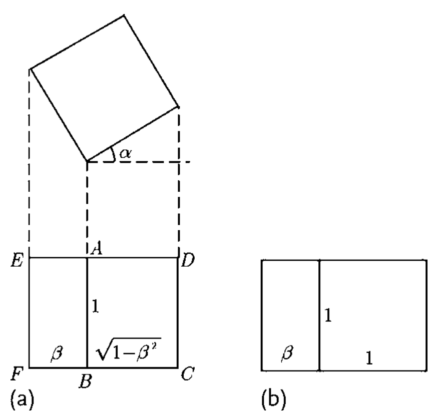

Figure 3.14 Photographing a cube at a right angle: cube image (a), and image with Galilean transformations (b).

In system S, the quanta creating images of edges $A'B'$ and $C'D'$ are simultaneously emitted by the cube. It follows from the Lorentz transformations that in system S' the quanta from edge $C'D'$ must be emitted earlier than those from edge $A'B'$ ($\Delta t' = \gamma Vl/c^2$, where l is the length of edges $B'C'$ and $A'D'$ in system S). It may be assumed that two events occurred in system S' (one $\Delta t'$ later than the other) at the points spaced $\Delta x' = l_0$ apart. The distance between them in system S can be found from (3.1):

$$l \equiv \Delta x = \gamma(\Delta x' - V\Delta t') .$$

Substituting $\Delta x'$ and $\Delta t'$ yields $l = l_0\sqrt{1-\beta^2}$ (the length of edges BC and AD in system S). These edges underwent the usual Lorentz contraction. Their images (taking into account the contraction inside the camera) have lengths $\sqrt{1-\beta^2}$.

Figure 3.14a shows a cube image. Curiously, a similar image is produced by a motionless cube rotated with respect to $\boldsymbol{V}$ through the angle $\alpha = \arcsin(V/c)$. In this case, the visible shape of the object does not undergo deformation due to the Lorentz contraction; the object just "turned" through angle α. It appears that this result is applicable (see Weisskopf, 1960, and subsequent problems) to any object and any angle between the velocity and the viewing direction provided the object is seen at a small solid angle.

Were the Galilean transformations true, the edges $A'D'$ and $B'C'$ would not undergo Lorentz contraction and the image would have the shape shown in Figure 3.14b. The rear (relative to the direction of motion) face of the cube would be photographed as before, but with different edge ratios. In other words, the visible shape of the moving object would be distorted.

3.43

1. $l = l_0|\sqrt{1-\beta^2}\cos\alpha' - \beta\sin\alpha'|$, $\beta = V/c$. The value of α'_{max} at which function $|\sqrt{1-\beta^2}\cos\alpha' - \beta\sin\alpha'|$ has a maximum is given by the condition $\tan\alpha'_{max} = -\beta/\sqrt{1-\beta^2}$. In this case, $l = l_0$, that is, the longest length l is equal to l_0, and the image is equivalent to the image of a motionless rod oriented parallel to the photoplate. The rod "turns" through the angle $\pi - \alpha'_{max}$.
2. $\alpha' = \arctan(\sqrt{1-\beta^2}/\beta)$; in this case, the image resembles a motionless rod oriented normally to the photoplate.
3. If two observers motionless in system S simultaneously make notches at points M and N on the xy plane past which the ends of the rod are moving at a given moment, the resulting section MN and the Ox axis make angle

$$\alpha = \arctan\left(\frac{\tan\alpha'}{\sqrt{1-\beta^2}}\right) .$$

3.44 The image will have the circular shape, and the semisphere hatched in Figure 3.15 will be photographed. It is confined by plane $A'B'$ making angle

$$\alpha' = \arctan\frac{\beta}{\sqrt{1-\beta^2}}$$

with the direction of speed V (in the system of the sphere). Despite the natural intuitive expectations, the observer does not see a moving sphere as an oblate ellipsoid in the direction of motion. The Lorentz contraction becomes invisible, but this does not mean that it is nonexistent.

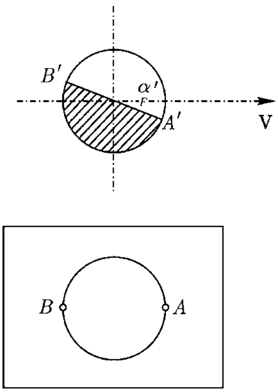

Figure 3.15 Photographing a sphere.

3.45 The visible positions of the cube are schematically depicted in Figure 3.16. At $V/c < \cos\alpha$, the front face $A'D'$ and the lower face $A'B'$ can be seen. If the dimensions of the object are not contracted in the camera optical system,

$$AB = l_0\sqrt{1-\beta^2}\frac{\sin\alpha}{1-\beta\cos\alpha}\,, \quad AD = l_0\frac{\cos\alpha-\beta}{1-\beta\cos\alpha}\,.$$

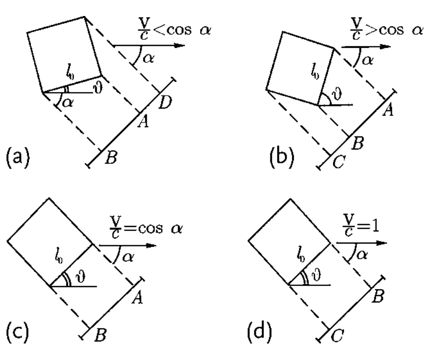

Figure 3.16 Photographing an inclined cube. Explanations to four images see in the text and the formulas resulted at the solution of the problem.

These formula can be used to find angle ϑ through which the cube is rotated:

$$\vartheta = \frac{\pi}{2} - \alpha - \theta\,, \quad \tan\theta = \frac{\cos\alpha-\beta}{\sin\alpha\sqrt{1-\beta^2}}\,.$$

At $V/c = \cos\alpha$, we have $\vartheta = \pi/2 - \alpha$, and only the lower face $A'B'$ is seen. At $V/c > \cos\alpha$, both the lower face and the rear face can be seen,

$$\vartheta = \frac{\pi}{2} - \alpha + \arctan\frac{\beta-\cos\alpha}{\sqrt{1-\beta^2}\sin\alpha}\,.$$

Finally, at $V/c \to 1$, the observer sees the rear face alone, and the lower face is Lorentz contracted to 0, $\vartheta = \pi - \alpha$.

3.46* The speed V of the spacecraft equals the ratio of distance $l_1 - l_2$ to time $\Delta\tau = \tau_2 - \tau_1$ between the moments at which the light flashes occur. The velocity of light being independent of the velocities of the source and the detector, we have $\tau_{1,2} = t_{1,2} - l_{1,2}/c$. Therefore,

$$V = \frac{l_1 - l_2}{\Delta t + (l_1 - l_2)/c}\,, \tag{1}$$

where $\Delta t = t_2 - t_1$ – is the period of time between the flashes recorded by the observer. By writing (1) in the form

$$l_1 - l_2 = \frac{V}{1 - V/c}\Delta t = V_* \Delta t\,, \tag{2}$$

we find that the proportionality factor between distance $l_1 - l_2$ and time Δt (the "apparent" speed $V_* = V/(1 - V/c)$) exceeds the velocity of light at $V > c/2$. The "apparent" speed and the "apparent" position of an object arise from the fact that the observer actually sees the object's position at the moment $t - l/c$ preceding the observation owing to the finite value of the speed of light.

A similar effect of the finiteness of the speed of sound is very apparent in when observing a supersonic aircraft in flight: the sound propagates more slowly than the plane and comes to the observer from the points of the trajectory behind the plane.

The moment when the spacecraft arrives is $t_* = \tau_1 + l_1/V = \tau_2 + l_2/V$. Here, all times are given in one and the same reference frame (the observer's system).

3.47* Three quantities are used to determine the lengths sought, two velocities c and V and the time interval Δt between the moments at which the signals are emitted. We denote times of signal reflection from the back and front mirrors by τ_1 and τ_2 (in the observer's system). The positions of the spacecraft at moments τ_1 and τ_2 are shown in Figure 3.17. The signals are emitted at times

$$t_1 = \tau_1 - \frac{l + a}{c}\,, \quad t_2 = \tau_2 - \frac{l - V\Delta\tau}{c}\,, \quad \Delta\tau = \tau_2 - \tau_1\,. \tag{1}$$

The signals return to the observer at times

$$t_1' = \tau_1 + \frac{l + a}{c}\,, \quad t_2' = \tau_2 + \frac{l - V\Delta\tau}{c}\,. \tag{2}$$

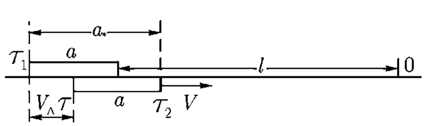

Figure 3.17 Diagram for the calculation of the lengths of the moving spacecraft.

It follows from the condition $t_1' = t_2'$ that $a = (c - V)\Delta\tau$. Using formulas (1), we find the time that elapsed between the moments at which the observer sends signals as $\Delta t = t_2 - t_1 = 2\Delta\tau$, that is, $\Delta\tau = \Delta t/2$. These data and Figure 3.17 lead to

$$a_* = a + V\Delta\tau = \frac{a}{1 - V/c} = \frac{1}{2}c\Delta t\,, \quad a = \frac{1}{2}\left(1 - \frac{V}{c}\right)c\Delta t\,. \tag{3}$$

Owing to the finiteness of the speed of light, the "apparent" length of the spacecraft, a_*, is greater than its "true" length a in the observer's system.

The length of the spacecraft in the proper frame, a_0, is related to its length in the observer's frame, a, by the Lorentzian formula (3.12). Using (3) and (3.12), we find

$$a_0 = \frac{1}{2}\sqrt{\frac{1 - V/c}{1 + V/c}}c\Delta t = \sqrt{\frac{1 - V/c}{1 + V/c}}a_*\,.$$

Despite the intuitive notion of the Lorentz scale contraction, the proper length of the spacecraft a_0 is smaller than the "apparent' length,' that is, the "visible" (with the help of instruments) length a_*. This "paradox" is attributable to the method used to measure the lengths using light signals and assuming the finiteness of the speed of light. All three aforementioned lengths coincide at $c \to \infty$.

The case of a spacecraft moving away from the observer stems from the previous one after the substitution of $-V$ for V. In this case, the apparent length a_* becomes smaller than other two lengths, but the relationship between a and a_0 remains unaltered.

3.48* Let us consider the movement of the spacecraft over distance $l \ll L$ and disregard its dimensions. Under these conditions, the rays coming from the beginning and the end of section l to the observer may be regarded as parallel. The light emitted from the beginning of section l (moment τ_1) is recorded by the observer at the moment $t_1 = \tau_1 + (L + l\cos\alpha)/c$. The light emitted from the end of section l (moment τ_2) reaches the observer at the moment $t_2 = \tau_2 + L/c$ (see Figure 3.9). Because $l = V\Delta\tau$, $\Delta t = t_2 - t_1 = \Delta\tau(1 - V\cos\alpha/c)$. The "apparent" speed

$$V_* = \frac{l}{\Delta t} = \frac{V}{1 - \beta\cos\alpha}\,, \quad \beta = \frac{V}{c}\,. \tag{1}$$

The projection of vector $\boldsymbol{V}_*$ directed along $\boldsymbol{V}$ onto the beam direction and on the plane perpendicular to the beam gives

$$V_{*\parallel} = \frac{V\cos\alpha}{1 - \beta\cos\alpha}\,, \quad V_{*\perp} = \frac{V\sin\alpha}{1 - \beta\cos\alpha}\,. \tag{2}$$

At $\gamma = (1 - \beta^2)^{-1/2} \gg 1$ and at $\alpha \ll 1$

$$V_{*\parallel} = \frac{2c}{\gamma^{-2} + \alpha^2}\,, \quad V_{*\perp} = \frac{2c\alpha}{\gamma^{-2} + \alpha^2}\,. \tag{3}$$

Therefore, $V_{*\|\max} \approx 2c\gamma^2 \gg c$ and $V_{*\perp\max} \approx c\gamma \gg c$. Astronomers frequently observe superlight velocities of macroscopic objects (clouds of relativistic plasma). Their origin can probably be explained on the basis of formulas (1)–(3).

3.49 Let a plane wave with frequency ω' and wave vector $\boldsymbol{k}' = (k' \cos\alpha', k' \sin\alpha', 0)$, $\boldsymbol{k}' \perp Oz$ propagate in the frame of reference S' connected with a medium. The phase velocity of the wave, $v' = c/n = \omega'/k'$, in this system does not depend on the angle α' determining the direction of wave propagation. The field components are proportional to $e^{-ik'_i x'^i}$, where $k'_i = (\omega'/c, -\boldsymbol{k}')$. Phase $k'_i = (\omega'/c, -\boldsymbol{k}')$ being an invariant with respect to the Lorentz transformation, k_i is the 4-vector (the wave 4-vector). Using formulas (3.13) and (3.17), we can find the components of k_i in the frame of reference S relative to which the medium moves with speed $\boldsymbol{V} \parallel Ox$; hence,

$$\omega = \gamma\omega'(1 + \beta n \cos\alpha')\,, \quad \tan\alpha = \frac{\sin\alpha'}{\gamma(\cos\alpha' + \beta/n)}\,, \tag{1}$$

$$v = \frac{\omega}{k} = c\frac{1 + \beta n \cos\alpha'}{\sqrt{n^2 + 2\beta n \cos\alpha' + \beta^2(1 - n^2 \sin^2\alpha')}}\,. \tag{2}$$

It follows from (2) that the phase velocity in the moving medium depends on the direction of propagation. It gives rise to a peculiar kind of anisotropy associated with the motion of the medium.

3.50 The velocity being sought can be found from formula (2) in the preceding problem:

$$v = c\frac{1 + \beta n(\lambda')}{n(\lambda') + \beta} \approx \frac{c}{n(\lambda')} + V\left(1 - \frac{1}{n^2(\lambda')}\right)\,.$$

Here, $\lambda' = 2\pi c/\omega'$, ω' is the frequency observed in system S' relative to which the medium is at rest. Using formula (1) in the preceding problem, we find to first-order accuracy in V/c

$$\frac{\lambda'}{\lambda} = \frac{\omega}{\omega'} = 1 + \frac{nV}{c}\,,$$

whence

$$\frac{c}{n(\lambda')} = \frac{c}{n(\lambda)} - \frac{c}{n^2} \cdot \frac{\mathrm{d}n}{\mathrm{d}\lambda}\lambda\frac{nV}{c}$$

and finally

$$v = \frac{c}{n(\lambda)} + V\left(1 - \frac{1}{n^2(\lambda)} - \frac{\lambda}{n(\lambda)}\frac{\mathrm{d}n(\lambda)}{\mathrm{d}\lambda}\right)\,.$$

3.51

$$p = \frac{1}{c}\sqrt{T(T + 2mc^2)}$$

3.52

$$\frac{d\boldsymbol{p}}{dt} = 0\,, \quad \text{where} \quad \boldsymbol{p} = \frac{m\boldsymbol{v}}{\sqrt{1 - v^2/c^2}}\,.$$

3.53 $v = \frac{cp}{\sqrt{p^2+m^2c^2}}$.

3.54 $\beta = \frac{v}{c} = \sqrt{1 - (\frac{\mathcal{E}_0}{\mathcal{E}})^2}\,, \quad \mathcal{E}_0 = mc^2\,.$

In the nonrelativistic case, $\beta \approx \sqrt{2T/\mathcal{E}_0}$, and in the ultrarelativistic case, $\beta = 1 - \mathcal{E}_0^2/2\mathcal{E}^2$.

3.55

1. $T = \frac{1}{2}mv^2 + \frac{3mv^2}{8c^2} + \ldots,$
2. $T = \frac{p^2}{2m} - \frac{p^4}{8m^3c^2} + \ldots.$

3.56

$$v = \sqrt{\frac{2eV}{m} \cdot \frac{1 + eV/2mc^2}{(1 + eV/mc^2)^2}}\,.$$

Specifically, at $eV \ll mc^2$,

$$v = \sqrt{\frac{2eV}{m}}\left(1 - \frac{3}{4}\frac{eV}{mc^2}\right) \ll c\,,$$

and at $eV \gg mc^2$,

$$v = c\left[1 - \frac{1}{2}\left(\frac{mc^2}{eV}\right)^2\right] \approx c\,.$$

3.57 (i) $v = 3.42 \times 10^{-2}$ s; (ii) $v = 0.999\,998\,5$ s; (ii) 0.81 s; (iv) 0.9956 s.

3.58 $F = \frac{J}{ce}\sqrt{T(T + 2mc^2)}$, $W = \frac{J}{e}T$.

3.59 $p = \frac{2mv^2N}{1-v^2/c^2}$.

The pressure is the same in the system connected with the body and in the system connected with the gas. This inference can be proved either by direct calculation of the pressure in each of these reference frames or by the Lorentz transformation for the four-dimensional force (see Section 4.2).

3.60 The length of the nth tube is

$$L_n = \frac{v_n}{2\nu} = \frac{c}{2\nu}\sqrt{1 - \left(\frac{mc^2}{nV_e e + mc^2}\right)^2}\,,$$

where v_n is the velocity of the particle in the nth tube. At the onset of acceleration, $mc^2 \gg neV_e$ and $L_n \approx (1/2\nu)\sqrt{2eV_e/m} \cdot \sqrt{n}$. In the ultrarelativistic limit, $T_n \gg mc^2$, $v \approx c$, and $L_n \approx c/2\nu$.

The estimation of the accelerator length is

$$L = \sum_n L_n \approx \frac{c}{2\nu} \int_0^N \sqrt{1 - \left(\frac{mc^2}{nV_e e + mc^2}\right)^2} \mathrm{d}n =$$

$$= \frac{c}{2\nu e V_e} \left[\sqrt{(NeV_e + mc^2)^2 - m^2c^4} - mc^2 \arccos \frac{mc^2}{NeV_e + mc^2} \right] .$$

3.61 The ratio of intensities is

$$\frac{I_h}{I_0} = \exp \frac{h}{v\tau} \approx \exp\left[\frac{h}{\tau_0 c} \cdot \frac{m_\mu c^2}{\mathcal{E}}\right] \approx 2.5$$

where $\tau = \tau_0/\sqrt{1 - v^2/c^2}$ is the half-life of a muon propagating with velocity v. In the absence of the relativistic time transformation, we would have for the intensity ratio (assuming the velocity of muons equals c)

$$\frac{I'_h}{I'_0} \approx \exp \frac{h}{\tau_0 c} \approx 94.4 .$$

Observations agree with the first result ($I_h/I_0 \approx 2.5$); in other words, they provide direct experimental evidence for the existence of a relativistic effect (slowing of time in a moving clock).

3.62

$$\tan\vartheta = \frac{1}{\gamma} \frac{p' \sin\vartheta'}{p' \cos\vartheta' + V\mathcal{E}'/c^2} = \frac{1}{\gamma} \frac{\sin\vartheta'}{\cos\vartheta' + V/v'} ,$$

where

$$\gamma = \frac{1}{\sqrt{1 - V^2/c^2}} , \quad \mathcal{E} = \gamma(\mathcal{E}' + p'V\cos\vartheta') ,$$

where p and p' are the particle's momenta in systems S and S', respectively.

The approximate formula presented in the condition for the ultrarelativistic case may be used if

$$\cos\left(\frac{\vartheta'}{2}\right) \gg \sqrt{\left|1 - \frac{V}{v'}\right|} ,$$

where $v' = p'c^2/\mathcal{E}'$ is the particle's velocity in system S'. The energy in the relativistic case assumes the form

$$\mathcal{E} \approx pc \approx 2\gamma\mathcal{E}' \cos^2(\vartheta'/2) .$$

3.63 Let us consider $\mathrm{d}N$ particles propagating in system S' inside a solid angle $\mathrm{d}\Omega'$. In system S, the same particles would propagate inside a solid angle $\mathrm{d}\Omega = \sin\vartheta\,\mathrm{d}\vartheta\,\mathrm{d}\alpha$ made by the velocity vectors of these particles in system S. The angular distribution of the particles in system S is described by function $F(\vartheta, \alpha)$ found from the equality

$$F(\vartheta, \alpha)\mathrm{d}\Omega = F'(\vartheta', \alpha')\mathrm{d}\Omega' = \mathrm{d}W = \frac{\mathrm{d}N}{N} . \tag{1}$$

The angle ϑ' should be expressed through ϑ with the help of the formula

$$\cos^2\vartheta = \frac{1}{1+\tan^2\vartheta} = \frac{(\cos\vartheta' + (V/v'))^2}{(\cos\vartheta' + (V/v'))^2 + (1/\gamma^2)\sin^2\vartheta'} ,$$

which ensues from the solution of Problem 3.62 ($v' = p'c^2/\mathcal{E}'$ is the velocity of particles in system S'). Bearing in mind that $\alpha = \alpha'$, we eventually arrive at

$$F(\vartheta, \alpha) = F'[\vartheta'(\vartheta), \alpha]\frac{\gamma^2\left[(\cos\vartheta' + (V/v'))^2 + (1/\gamma^2)\sin^2\vartheta'\right]^{\frac{3}{2}}}{1 + (V/v')\cos\vartheta'} . \tag{2}$$

In the case of ultrarelativistic particles, $v' = c$ and the angular distribution in system S becomes simplified (see Problem 3.33):

$$F(\vartheta, \alpha) = F'[\vartheta'(\vartheta), \alpha]\frac{(1 + (V/c)\cos\vartheta)^2}{1 - V^2/c^2} . \tag{3}$$

We note that the particles propagating in system S at different angles ϑ have different energies despite the equivalence of their energies in system S'.

3.65•

$$f'(\boldsymbol{p}') = \gamma\left(1 + \frac{\boldsymbol{p}'\cdot\boldsymbol{V}}{\mathcal{E}'}\right) f\left(\gamma\left(p'_x + V\mathcal{E}'/c^2\right), p'_y, p'_z\right) ,$$
$$\gamma = \left(1 - \frac{V^2}{c^2}\right)^{-1/2} .$$

3.66• The distribution function f is an invariant quantity. This means that in the transition to the other frame of reference, S',

$$f'(\boldsymbol{r}', \boldsymbol{p}', t') = f(\boldsymbol{r}, \boldsymbol{p}, t) ,$$

where $\boldsymbol{r}$, $\boldsymbol{p}$, and t on the right-hand side of the equality should be expressed through primed quantities with the help of the Lorentz transformations (3.20).

3.67 Let us denote the number of scattered and scattering particles per unit volume in system S by n_1 and n_2, respectively. The total number of particles, $\mathrm{d}N$, scattered into the solid angle interval $\mathrm{d}\Omega$ in time t by the scattering particles enclosed in volume V is expressed, in accordance with the definition of the cross-section, by $\mathrm{d}N = \mathrm{d}\sigma_{12} J_{12} n_2 V t$, where $J_{12} = n_1 v_1$. In system S', an analogous expression holds for the number $\mathrm{d}N$: $\mathrm{d}N = \mathrm{d}\sigma_{12} J_{12} n_2 V t$, where $J_{12} = n_1 v_1$ (in this system, $\mathrm{d}N$ is the number of particles scattered into the solid angle $\mathrm{d}\Omega'$ corresponding to $\mathrm{d}\Omega$). Thus,

$$\mathrm{d}N = \mathrm{d}\sigma_{12} n_1 n_2 v_1 V t = \mathrm{d}\sigma'_{12} n'_1 n'_2 |\boldsymbol{v}'_1 - \boldsymbol{v}'_2| V' t' \,. \tag{1}$$

Similarly to the four-dimensional density enc, $en\boldsymbol{v}$ of the electric current, the totality of four quantities $(n_i c, n_i \boldsymbol{v}_i)$ is a 4-vector. It follows that

$$n_1 n_2 = n'_1 n'_2 \left(1 - \frac{\boldsymbol{v}'_1 \cdot \boldsymbol{v}'_2}{c^2}\right) , \tag{2}$$

because the scalar product of two 4-vectors is invariant. Taking account of formula (2) and invariance of the 4-volume, $Vt = V't'$, we eventually have

$$\mathrm{d}\sigma'_{12} = \mathrm{d}\sigma_{12} \frac{v_1 \left(1 - (\boldsymbol{v}'_1 \cdot \boldsymbol{v}'_2)/(c^2)\right)}{|\boldsymbol{v}'_1 - \boldsymbol{v}'_2|} \,. \tag{3}$$

In the specific case when $\boldsymbol{v}'_1 \parallel \boldsymbol{v}'_2$, we have

$$\boldsymbol{v}_1 = \frac{\boldsymbol{v}'_1 - \boldsymbol{v}'_2}{1 - (\boldsymbol{v}'_1 \cdot \boldsymbol{v}'_2)/(c^2)}$$

(see Problem 3.13) and it follows from (3) that the cross-section is invariant:

$$\mathrm{d}\sigma_{12} = \mathrm{d}\sigma'_{12} \,. \tag{4}$$

Such a case occurs, for example, in the transformation of the laboratory frame of reference into the C-system. It should be noted that if the flux is described by the formula $J_{12} = n_1 \tilde{v}$, where $\tilde{v} = v_1(1 - (\boldsymbol{v}'_1 \cdot \boldsymbol{v}'_2)/(c^2))$, then the cross-section is invariant in the case of the arbitrary Lorentz transformation.

3.68 $\mathrm{d}W = \frac{\mathrm{d}\Omega}{4\pi\gamma^2(1-\beta\cos\vartheta)^2}$, $\int_{4\pi} \mathrm{d}W = 1$, where $\beta = v/c$.

3.69 $f = (1+\beta)/(1-\beta)$, from which it follows that $\mathcal{E} = mc^2(f+1)/(2\sqrt{f})$, where m is the mass of a pion.

3.71 The photon momentum being $p = \mathcal{E}/c$, we have (see Problem 3.62)

$$\mathcal{E} = \frac{\mathcal{E}'}{\gamma(1-\beta\cos\vartheta)} , \quad \mathcal{E}' = \frac{mc^2}{2} , \quad \beta = \frac{v}{c} \,.$$

Comparison of the ensuing expression $\mathrm{d}\mathcal{E} = -(\mathcal{E}' d(1-\beta\cos\vartheta))/(\gamma(1-\beta\cos\vartheta)^2)$ with the angular distribution of decay gamma quanta found in the solution of Problem 3.68 yields the probability distribution for the decay photon energies:

$$\mathrm{d}W(\mathcal{E}) = \frac{|\mathrm{d}\mathcal{E}|}{\mathcal{E}_{\max} - \mathcal{E}_{\min}} ,$$

where

$$\mathcal{E}_{\min} = \mathcal{E}' \sqrt{\frac{1-\beta}{1+\beta}}$$

is the minimum value of the energy of the decay gamma quantum (at $\vartheta = \pi$) and

$$\mathcal{E}_{\max} = \mathcal{E}' \sqrt{\frac{1+\beta}{1-\beta}}$$

is the maximum value of the energy of the decay gamma quantum (at $\vartheta = 0$). This means that the spectrum of decay gamma quanta has a rectangular shape in the laboratory frame of reference, which suggests an equal probability of any energy values within the interval between $\mathcal{E}_{\min}$ and $\mathcal{E}_{\max}$.

3.72 $m = \frac{2\sqrt{\mathcal{E}_1\mathcal{E}_2}}{c^2}$.

3.73 $m^2 = m_1^2 + m_2^2 + 2\left[\sqrt{(p_1^2+m_1^2)(p_2^2+m_2^2)} - p_1 p_2 \cos\vartheta\right], \quad c = 1;$ $m_\pi = 139.58\,\mathrm{MeV}$.

3.74 $m_1^2 = m^2 + m_2^2 - 2\left[\sqrt{(p^2+m^2)(p_2^2+m_2^2)} - p\,p_2 \cos\vartheta_2\right], \quad c = 1.$

3.75 $m^2 = \mathcal{E}^2 - p^2 = m_1^2 + m_2^2 + \frac{2m_1m_2}{\sqrt{1-v^2}}, \; V = \frac{p}{\mathcal{E}} = \frac{m_1 v}{m_1 + m_2\sqrt{1-v^2}}, \quad c = 1.$

3.76 $T_1 = \frac{(m_0-m_1)^2-m_2^2}{2m_0}; \; T_2 = \frac{(m_0-m_2)^2-m_1^2}{2m_0}, \quad c = 1;$
(i) $T_\alpha/T_n = 58{,}5$; (ii) $T_\nu/T_\mu = 7{,}27$; (iii) $T_\gamma/T_n \approx (2mc^2)/(\Delta\mathcal{E})$, where m is the mass of the initial nucleus and $\Delta\mathcal{E}$ is the energy of its excitation, with $mc^2 \gg \Delta\mathcal{E}$.

It follows from the common formulas for T_1 and T_2, and from the foregoing examples that a lighter particle has higher energy.

3.77 $Q_a = T_b\left[1 + \frac{T_b+2m_b}{m_d+\sqrt{T_b^2+2T_bm_b+m_d^2}}\right];$

$$Q_{\Sigma^+} = 109.6\,\mathrm{MeV}; \quad M_{\Sigma^+} = 1188.7\,\mathrm{MeV} \quad (\Sigma^+ \to n + \pi^+);$$

$$Q_{\Sigma^+} = 116.1\,\mathrm{MeV}; \quad M_{\Sigma^+} = 1189.3\,\mathrm{MeV} \quad (\Sigma^+ \to n + \pi^0).$$

The two M_{Σ^+} values are in excellent agreement with each other.

3.78

$$\omega = \frac{\Delta\mathcal{E}}{\hbar}\left(1 - \frac{\Delta\mathcal{E}}{2mc^2}\right).$$

The energy $\hbar\omega$ carried away by the quantum is smaller than $\Delta\mathcal{E}$ by energy $(\Delta\mathcal{E})^2/(2mc^2)$ carried away by the recoil nucleus. When the nucleus is rigidly fixed in the crystal lattice, the latter receives no energy (because its mass $M \gg m$ is too large), and the quantum carries away the entire energy, $\hbar\omega = \Delta\mathcal{E}$.

3.79*

1. The law of energy conservation limits the equiangular triangle ABC (Figure 3.18a), the height of which BO is equal to the decay energy $Q = m - m_1 - m_2 - m_3$ ($c = 1$). The distance between point D and base AC obviously equals T_1. The distances from D to AB and BC are easy to calculate; they are equal to T_2 and T_3, respectively.

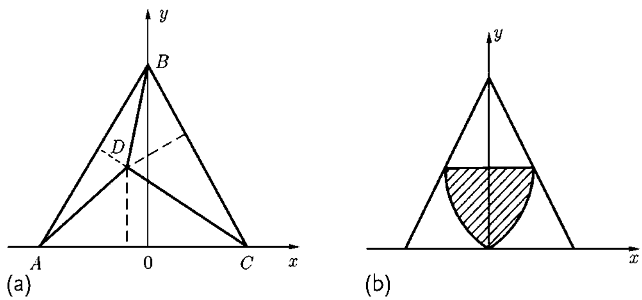

Figure 3.18 The Dalitz diagram for the decay of a resting particle into three particles. (a) The image of concrete decay (point D). (b) The image of the resolved area which all decays of the given type keep within.

2. If the masses of all particles are given, their momenta are defined by specifying two energies, for example, T_1 and T_2 (since $T_3 = Q - T_1 - T_2$), or their two linear combinations, x and y. The momenta of decay particles are the sides of a triangle ($\boldsymbol{p}_1 + \boldsymbol{p}_2 + \boldsymbol{p}_3 = 0$) in the rest system of a decaying particle. The angles of the triangle characterize the relative directions of outgoing particles and can be found from the known p_1, p_2, and p_3.
3. The boundaries of the allowed region are defined by the conditions

$$p_1 + p_2 \geq p_3 \,, \quad -p_3 \leq p_1 - p_2 \leq p_3 \,.$$

These conditions lead to the hatched region in Figure 3.18b bounded by the straight line $y = (m - m_1)^2/2m$ from above and by the hyperbola $x = \pm\sqrt{\frac{y^2+2m_1 y}{3}}$ from below.

3.80 The Dalitz diagram has the shape shown in Figure 3.18b.

1. $T_{1\max} \approx T_{2\max} \approx T_{3\max} \approx 69.8\,\text{MeV}$.
2. $T_{1\max} = \frac{(m-m_1)^2}{2m} \approx 127\,\text{MeV}$, $T_{2\max} = T_{2\max} = \frac{m^2-m_1^2}{2m} \approx 228\,\text{MeV}$.

The maximum momenta of all three particles are identical.

3.81 The Dalitz diagram in the approximation $Q \ll m_\pi$ is shown in Figure 3.19.

$$OB = Q \,, \quad R = Q/3 \,, \quad T_{\max} = 2Q/3 \approx 50\,\text{MeV} \,.$$

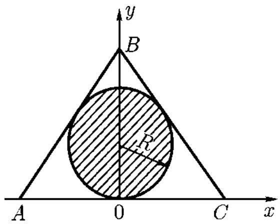

Figure 3.19 The Dalitz diagram for the decay of a kaon into three pions.

3.82 The Dalitz diagram is shown in Figure 3.20. $OB = Q$, $T_{\max} \approx 210$ MeV. The inner closed curve is given by the equation

$$x = \pm\sqrt{\frac{(2m_\pi y + y^2)[(m_\omega - m_\pi)^2 - 4m_\pi^2 - 2m_\omega y]}{3[(m_\omega - m_\pi)^2 - 2m_\omega y]}} .$$

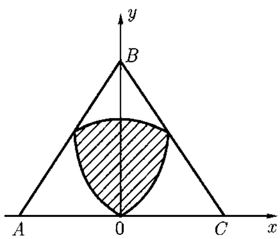

Figure 3.20 The Dalitz diagram for the decay of an ω meson decay into three pions.

3.83* The delta function of the 4-vector should be understood as the product of four delta functions of its components:

$$\delta(p_i - p_{i1} - p_{i2} - p_{i3}) = \delta(\boldsymbol{p} - \boldsymbol{p}_1 - \boldsymbol{p}_2 - \boldsymbol{p}_3)\delta(\mathcal{E} - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3) . \qquad (1)$$

Integration over $d^3 p_3$ using (1) leads to

$$\Gamma = \int \frac{d^3 p_1 d^3 p_2}{\mathcal{E}_1 \mathcal{E}_2 \mathcal{E}_3} \delta\left(\sqrt{p_1^2 + p_2^2 + m_3^2 + 2p_1 p_2 \cos\vartheta} - \mathcal{E}_3\right) , \qquad (2)$$

where $\mathcal{E}_3 = m - \mathcal{E}_1 - \mathcal{E}_2$, and ϑ is the angle between $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$.

Let us represent $d^3 p_2$ in the form $d^3 p_2 = p_2^2 dp_2 d\Omega_2$, where $d\Omega_2$ is an element of the solid angle. If the direction of $\boldsymbol{p}_1$ is assumed to be the polar axis, then $d\Omega_2 = 2\pi \sin\vartheta d\vartheta$. Moreover, $p_2 dp_2 = \mathcal{E}_2 d\mathcal{E}_2$ as follows from (3.41). Let us transform the delta function into (2) using formula (1.209):

$$\delta\left(\sqrt{p_1^2 + p_2^2 + m_3^2 + 2p_1 p_2 \cos\vartheta} - \mathcal{E}_3\right) = 2\mathcal{E}_3 \delta\left(2p_1 p_2 \cos\vartheta + p_1^2 + p_2^2 + m_3^2 - \mathcal{E}_3^2\right) . \qquad (3)$$

Because $-1 \leq \cos\vartheta \leq 1$, integral (2) is nonzero only if the following inequalities are fulfilled:

$$p_1 + p_2 \geq p_3\,, \quad p_1 - p_2 \leq p_3\,, \quad p_1 - p_2 \geq -p_3\,;$$

it is these inequalities that define the boundaries of the permitted region in the Dalitz diagram.

Integration over $\mathrm{d}\vartheta$ with the use of (3) and (1.207) gives

$$\Gamma = \pi \int \frac{\mathrm{d}^3 p_1 \mathrm{d}\mathcal{E}_2}{\mathcal{E}_1 p_1} = 4\pi^2 \int \mathrm{d}\mathcal{E}_1 \mathcal{E}_2\,.$$

Let us turn to integration over variables

$$x = \frac{T_2 - T_3}{\sqrt{3}} = \frac{\mathcal{E}_1 + 2\mathcal{E}_2 + m_3 - m_2 - m}{\sqrt{3}}\,, \quad y = T_1 = \mathcal{E}_1 - m_1\,,$$

which are used to construct the Dalitz diagram. The transformation of element $\mathrm{d}\mathcal{E}_1\mathrm{d}\mathcal{E}_2$ yields

$$\Gamma = 2\sqrt{3}\pi^2 \int \mathrm{d}x\mathrm{d}y\,,$$

where the integration domain is bounded by the inner curve in the diagram (see Figures 3.18b–3.20).

The latter formula indicates that the element of the phase volume $\mathrm{d}\Gamma = 2\sqrt{3}\pi^2\mathrm{d}x\mathrm{d}y$ is proportional to the area element in the Dalitz diagram. The energies T_1, T_2, and T_3 of the particles resulting from decay can be measured in an experiment and the respective points can be on the Dalitz diagram. The density of such points is proportional to ρ (see the condition for the problem), which can be thus found from the experimental data.

3.84 Let us consider the energy–momentum 4-vector p^i of the system of particles. It is conserved, that is, its respective components are equal before and after the reaction. When the value of the kinetic energy T_0 corresponds to the reaction threshold, the newly formed particles are at rest in the C-system (it is noteworthy that the particles cannot rest at the threshold value of T_0 in the laboratory frame of reference because that would mean violation of the law of momentum conservation). The vector of the full 4-momentum of the system before the reaction has the following form in the laboratory system:

$$p^{(0)i} = \left(\frac{\mathcal{E}_0}{c} + m_1 c, \boldsymbol{p}_0\right)\,,$$

where $\mathcal{E}_0$ is the total energy and $\boldsymbol{p}_0$ is the full momentum at the reaction threshold.

The 4-momentum is $p'^i = (Mc, 0)$ after the reaction in the C-system. Taken together, the invariance of the square of the 4-vector and the law of 4-momentum conservation account for $p^{(0)i}p_i^{(0)} = p'^i p'_i$. By representing the last equality in the expanded form, we obtain

$$M^2c^2 = \frac{\mathcal{E}_0}{c^2} + 2m_1\mathcal{E}_0 + m_1^2c^2 - p_0^2 \ ;$$

therefore,

$$T_0 = \frac{c^2}{2m_1}(M - m_1 - m)(M + m_1 + m) \ .$$

3.85 $T_0 \approx 3.5 \times 10^6$ MeV $= 3.5$ TeV.

3.86 (i) $T_0 = 288$ MeV; (ii) $T_0 = 160$ MeV; (iii) $T_0 = 763$ MeV; (iv) $T_0 = \frac{2m_\mathrm{p}(m+2m_\mathrm{p})c^2}{m}$, where m_p is the proton mass. In the particular case of collision with a proton, $m = m_\mathrm{p}$, we have

$$T_0 = 6m_\mathrm{p}c^2 = 5.63\,\text{GeV} \ .$$

The approximate form for the threshold energy is

$$T_0 = \frac{2(A+2)}{A}m_\mathrm{p}c^2 \ .$$

At large A, $T_0 \approx 2m_\mathrm{p}c^2$.

3.87

$$T_0 = \left(1 + \frac{m}{m+1}\right)\Delta\mathcal{E} \ .$$

In case (i) we have from the above approximate form

$$\Delta\mathcal{E} = T_0 = 2.18\,\text{MeV} \quad (m = 0) \ .$$

Using the exact formula, we would have a $|Q|^2/2m_1c^2 \approx 0.0012$ MeV greater value, where $Q = -(M - m_1 - m)c^2$ is the thermal effect of the reaction.

In case (ii) the approximate formula gives $T_0 = 2|Q| = 7.96$ MeV. This value differs by 0.003 MeV from that obtained with the use of the exact formula.

3.88

	ΔM (g)	$\Delta M/M$
1	4.6×10^{-26}	0.65×10^{-2}
2	3.1×10^{-26}	0.35×10^{-2}
3	3.6×10^{-25}	0.85×10^{-3}
4	1.2×10^{-33}	1.4×10^{-11}
5	10^{-35}	3×10^{-13}

3.89*

$$M^2 = 2(m^2 + \mathcal{E}_-\mathcal{E}_+/c^4 - \boldsymbol{p}_- \cdot \boldsymbol{p}_+/c^2) > 0 ,$$

$$\mathcal{E}_1 = \frac{m^2c^4 + \mathcal{E}_-\mathcal{E}_+ - c^2\boldsymbol{p}_- \cdot \boldsymbol{p}_+}{(\mathcal{E}_- + \mathcal{E}_+)(1 - V\cos\theta/c)} , \quad \mathcal{E}_2 = \mathcal{E}_- + \mathcal{E}_+ - \mathcal{E}_1 ,$$

where $\boldsymbol{V} = (\boldsymbol{p}_- + \boldsymbol{p}_+)/M$, θ is the exit angle of the first quantum with respect to $\boldsymbol{V}$, and $\mathcal{E}_\pm$ are the electron and positron energies in the L-system.

3.90 The equation for the reaction has the form

$$\gamma + (\text{particle}) \to e^+ + e^- + (\text{particle}) .$$

The threshold can be found from the general formula (see Problem 3.84):

$$T_0 = \hbar\omega_0 = \frac{c^2}{2m_1}(m_1 + 2m - m_1)(m_1 + 2m + m_1) = 2mc^2\left(1 + \frac{m}{m_1}\right) ,$$

where m is the electron (or positron) mass. In the absence of a particle $m_1 \to 0$, the threshold energy $T_0 \to \infty$, which suggests that the reaction is impossible.

The last result can be obtained by demonstrating the impracticability of fulfillment of the equality $k_i = p_{+i} + p_{-i}$, where k_i, p_{+i}, and p_{-i} are the 4-momenta of the photon, positron, and electron, respectively. Taking the square of the two parts of the equality yields

$$k^i k_i = (\mathcal{E}_+ + \mathcal{E}_-)^2 - (\boldsymbol{p}_+ + \boldsymbol{p}_-)^2 .$$

However, $k_i k^i = 0$, but the invariant quantity on the right-hand side of the previous equality does not vanish at any $\boldsymbol{p}_+$ or $\boldsymbol{p}_-$. This becomes obvious on the transition to the frame of reference in which $\boldsymbol{p}_+ + \boldsymbol{p}_- = 0$.

3.92

$$v = \frac{c\sqrt{\mathcal{E}^2 - m_1^2c^4}}{\mathcal{E} + m_2c^2} .$$

3.93* According to the law of 4-momentum conservation,

$$p_{1i}^{(0)} + p_{2i}^{(0)} = p_{1i} + p_{2i} . \qquad (1)$$

To determine the scattering angle of the first particle, we transfer p_{1i} to the left and square both parts of the resulting equality:

$$p_{1i}^{(0)}p_1^{(0)i} + p_{2i}^{(0)}p_2^{(0)i} + p_{1i}p_1^i + 2p_{1i}^{(0)}p_2^{(0)i} - 2p_{1i}^{(0)}p_1^i - 2p_{2i}^{(0)}p_1^i = p_{2i}p_2^i . \qquad (2)$$

In accordance with (3.41), $p_{1i}^{(0)}p_1^{(0)i} = p_{1i}p_1^i = -m_1^2c^2$ and $p_{2i}^{(0)}p_2^{(0)i} = p_{2i}p_2^i = -m_2^2c^2$. The scalar products are transformed in the following way ($\boldsymbol{p}_2^{(0)} = 0$):

$$-p_{1i}^{(0)}p_2^{(0)i} = \boldsymbol{p}_1^{(0)} \cdot \boldsymbol{p}_2^{(0)} - \frac{1}{c^2}\mathcal{E}_1^{(0)}\mathcal{E}_2^{(0)} = -\mathcal{E}_0 m_2 , \quad p_{2i}^{(0)}p_1^i = -m_2\mathcal{E}_1 ,$$

$$p_{1i}^{(0)} p_1^i = \boldsymbol{p}_1^{(0)} \cdot \boldsymbol{p}_1 - \frac{1}{c^2} \mathcal{E}_1^{(0)} \mathcal{E}_1 = p_0 p_1 \cos \vartheta_1 - \frac{\mathcal{E}_0 \mathcal{E}_1}{c^2} ,$$

where $p_0 = (1/c)\sqrt{\mathcal{E}_0^2 - m_1^2 c^4}$. Substituting the expressions thus obtained into (2) yields

$$\cos \vartheta_1 = \frac{\mathcal{E}_1(\mathcal{E}_0 + m_2 c^2) - \mathcal{E}_0 m_2 c^2 - m_1^2 c^4}{c^2 p_0 p_1} .$$

Similarly,

$$\cos \vartheta_2 = \frac{(\mathcal{E}_0 + m_2 c^2)(\mathcal{E}_2 - m_2 c^2)}{c^2 p_0 p_2} .$$

3.94

$$\mathcal{E}_1 = m_1 c^2 \times \frac{(\gamma_0 + m_2/m_1)(1 + \gamma_0 m_2/m_1) \pm \cos \vartheta_1 (\gamma_0^2 - 1)\sqrt{m_2^2/m_1^2 - \sin^2 \vartheta_1}}{(\gamma_0 + m_2/m_1)^2 - (\gamma_0^2 - 1)\cos^2 \vartheta_1} , \tag{1}$$

$$\mathcal{E}_2 = \frac{(\gamma_0 + m_2/m_1)^2 + (\gamma_0^2 - 1)\cos^2 \vartheta_2}{(\gamma_0 + m_2/m_1)^2 - (\gamma_0^2 - 1)\cos^2 \vartheta_2} m_2 c^2 , \tag{2}$$

where

$$\gamma_0 = \frac{\mathcal{E}_0}{m_1 c^2} .$$

It follows from these formulas that scattering at $m_1 > m_2$ is possible only at angles ϑ_1 that do not exceed arcsin $\sqrt{m_2/m_1}$ (the expression under the radical in (1) must be positive). Two values of energy $\mathcal{E}_1$ correspond to each ϑ_1 value.

At $m_1 = m_2$, the scattering angle ϑ_1 does not exceed $\pi/2$ and each ϑ_1 value is matched with only one value of the energy corresponding to the choice of the plus sign in formula (1). The value of $\mathcal{E}_1 = m_1 c^2$ would correspond to the minus sign regardless of the scattering angle, but it would be contrary to the fact. For a similar reason, only the plus sign is left for $\mathcal{E}_2$ in the numerator of formula (2).

At $m_1 < m_2$, scattering at any angle is possible and each ϑ_1 value corresponds to a single value of $\mathcal{E}_1$. The plus sign should be chosen for formula (1) if $0 < \vartheta_1 < \pi/2$ and the minus sign should be chosen if $\pi/2 < \vartheta_1 < \pi$. With such choice of the signs, the scattering of the projectile particle at a larger angle corresponds to a greater energy loss, as could be expected.

3.95

$$\mathcal{E} \approx \frac{\mathcal{E}_0}{1 + \frac{\mathcal{E}_0}{Mc^2}(1 - \cos \vartheta)} .$$

3.96

$$\mathcal{E} \approx \frac{\mathcal{E}_0 - \Delta E}{1 + \frac{\mathcal{E}_0}{Mc^2}(1 - \cos\vartheta)} .$$

3.97

$$T_1 = \frac{T_0 \cos^2\vartheta_1}{1 + \frac{1}{2}\left(\frac{T_0}{mc^2}\right)\sin^2\vartheta_1} .$$

3.98

$$T_1 = T_0 \left(\frac{m_1}{m_1 + m_2}\right)^2$$

$$\times \left[1 + \left(\frac{m_2}{m_1}\right)^2 - 2\sin^2\vartheta_1 \pm 2\cos\vartheta_1 \sqrt{\left(\frac{m_2}{m_1}\right)^2 - \sin^2\vartheta_1}\,\right];$$

$$T_2 = \frac{4m_1 m_2}{(m_1 + m_2)^2} T_0 \cos^2\vartheta_2 .$$

The rule of signs is formulated in the solution of Problem 3.94.

3.99 The scattering angle of the particles, $\chi = \vartheta_1 + \vartheta_2$, is described by the formula (compare this with Problem 3.28)

$$\tan\chi = \frac{(v_1' + v_2')\sqrt{1 - V^2/c^2}\sin\vartheta'}{(V^2/c^2)v_1'\sin^2\vartheta' + (V - v_1')(1 - \cos\vartheta')} .$$

At $m_1 = m_2$, the velocities $v_1' = v_2' = V$ and

$$\tan\chi = \frac{2c^2\sqrt{1 - V^2/c^2}}{V^2 \sin\vartheta'} .$$

In this case, $\chi < 90°$. In the nonrelativistic limit, $\chi \to 90°$.

3.100 Acting as in the solution of Problem 4.42, we obtain

$$\omega = \frac{\omega_0\,(\mathcal{E}_0/c - p_0\cos\vartheta_0)}{\mathcal{E}_0/c - p_0\cos\vartheta_1 + (\hbar\omega_0/c)(1 - \cos\vartheta)} ,$$

where ϑ is the angle between the directions of motion of primary and scattered photons, and ϑ_1 is the angle between the directions of the initial electron motion and photon motion after scattering. If the electron was at rest prior to the collision, then

$$\omega = \frac{\omega_0}{1 + \frac{\hbar\omega_0}{mc^2}(1 - \cos\vartheta)} .$$

3.101 The energy of a scattered quantum is highest at $\vartheta_0 = \vartheta = \pi$, $\vartheta_1 = 0$, that is, in the case of a head-on collision resulting in the backward scattering of the quantum. Then,

$$\hbar\omega \approx \hbar\omega_0 \frac{2\mathcal{E}_0}{(mc^2)^2/2\mathcal{E}_0 + 2\hbar\omega_0} . \tag{1}$$

It follows from (1) that the quantum becomes much harder in the ultrarelativistic case, $\hbar\omega \gg \hbar\omega_0$. Two particular cases are worth mentioning. At $\hbar\omega_0 \ll mc^2(mc^2/\mathcal{E}_0)$, formula (1) gives $\mathcal{E}_0 \gg \hbar\omega = 4\hbar\omega_0(\mathcal{E}_0/mc^2)^2 \gg \hbar\omega_0$. If $\hbar\omega_0 \gg mc^2(mc^2\mathcal{E}_0)$, $\hbar\omega$ approaches $\mathcal{E}_0$.

3.102 $\mathcal{E} - \mathcal{E}_0 = \hbar\omega_0 \frac{p_0c(\cos\vartheta_0 - \cos\vartheta_1) + \hbar\omega_0(1-\cos\vartheta)}{\mathcal{E}_0 - p_0c\cos\vartheta_1 + \hbar\omega_0(1-\cos\vartheta)}$. The notation for the angles is the same as in Problem 3.100. The energy of an initially resting electron always increases after collision with a photon:

$$\mathcal{E} - mc^2 = \frac{(\hbar\omega_0)^2(1-\cos\vartheta)}{mc^2 + \hbar\omega(1-\cos\vartheta)} .$$

If the electron has momentum $p_0 \gg \hbar\omega/c$ before scattering, its energy increases after scattering if $\vartheta_0 < \vartheta_1$, and decreases in the opposite case. The maximum acceleration of the electron occurs at $\vartheta_0 = 0$, $\vartheta = \vartheta_2 = \pi$. In this case,

$$\mathcal{E} - \mathcal{E}_0 = 2\hbar\omega_0 \frac{p_0c + \hbar\omega_0}{\mathcal{E}_0 + p_0c + \hbar\omega_0} .$$

In the case of a nonrelativistic electron with $p_0c \gg \hbar\omega_0$, $\mathcal{E} - \mathcal{E}_0 = 2\hbar\omega_0(v_0/c) \ll \hbar\omega_0$. For a relativistic electron, $\mathcal{E} - \mathcal{E}_0 \approx \hbar\omega_0$, and the conditions for its acceleration are optimal.

3.103 $s = 4(m^2 + q^2)$, $t = -2q^2(1-\cos\vartheta)$, $u = -2q^2(1+\cos\vartheta)$.

3.104

$$\mathcal{E}_a = \frac{1}{2m_b}\left(s - m_a^2 - m_b^2\right) ,$$
$$p_a = \frac{1}{2m_b}\sqrt{\lambda\left(s, m_a^2, m_b^2\right)} ,$$
$$\mathcal{E}'_a = \frac{1}{2\sqrt{s}}\left(s + m_a^2 + m_b^2\right) ,$$
$$p' = \frac{1}{2\sqrt{s}}\sqrt{\lambda\left(s, m_a^2, m_b^2\right)} ,$$
$$\mathcal{E}'_b = \frac{1}{2\sqrt{s}}\left(s + m_b^2 - m_a^2\right) ,$$

where

$$\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz .$$

Because $\boldsymbol{p}_a = -\boldsymbol{p}_b$ in the C-system, the quantity s has the sense of the squared total energy in this reference frame:

$$s = \left(\mathcal{E}'_a + \mathcal{E}'_b\right)^2 = \left(\mathcal{E}'_c + \mathcal{E}'_d\right)^2 \; .$$

3.105 $\mathcal{E}_c = \frac{1}{2m_b}(m_b^2 + m_c^2 - u)\; \mathcal{E}_d = \frac{1}{2m_b}(m_b^2 + m_d^2 - t)\;; \quad c = 1\;.$

3.106

$$\cos\theta = \frac{(s - m_a^2 - m_b^2)(m_b^2 + m_c^2 - u) + 2m_b^2(t - m_a^2 - m_c^2)}{\sqrt{\lambda(s, m_a^2, m_b^2)}\sqrt{\lambda(u, m_b^2, m_c^2)}}\;;$$

$$\cos\theta' = \frac{s^2 + s(2t - m_a^2 - m_b^2 - m_c^2 - m_d^2) + (m_a^2 - m_b^2)(m_c^2 - m_d^2)}{\sqrt{\lambda(s, m_a^2, m_b^2)}\sqrt{\lambda(s, m_c^2, m_d^2)}}\;.$$

Here $c = 1$, and the quantity λ is determined in the answer to Problem 3.104.

3.107 The quantity $s = (\mathcal{E}'_\pi + \mathcal{E}'_{\mathrm{p}})^2$ has the sense of the square of the total energy of two particles in the C-system; therefore, it is always positive. The minimal value of $s_{\min} = (m + M)^2$ corresponds to the case of a pion (with mass m) and a proton (with mass M) both resting in the C-system. Thus, $s_{\min} = (m + M)^2$.

The cosine of the scattering angle θ' in the C-system is related to s and t by the formula

$$\cos\theta' = \frac{s^2 + s(2t - 2M^2 - m^2) + M^2(M^2 - m^2)}{(s - M^2)\sqrt{s^2 - 2s(M^2 + m^2) + (M^2 - m^2)^2}}\;. \tag{1}$$

Because $-1 \le \cos\theta' \le 1$, the substitution of $\cos\theta'$ from (1) into this two-sided inequality yields a t value acceptable at a given value of s.

The physical region is hatched in Figure 3.21. Point A corresponds to the reaction threshold with $s_A = (M + m)^2$ and $t_A = -m^2 M/(M + m)$,

$$T_0 = m + \frac{m^2}{2M}\;, \quad T_\pi = \frac{m^3}{2M(M + m)}\;.$$

3.108 The regions sought are shown in Figure 3.22.

3.109* The allowed regions for the first two processes are depicted in Figure 3.23a and those for the third process are depicted in n Figure 3.23b.

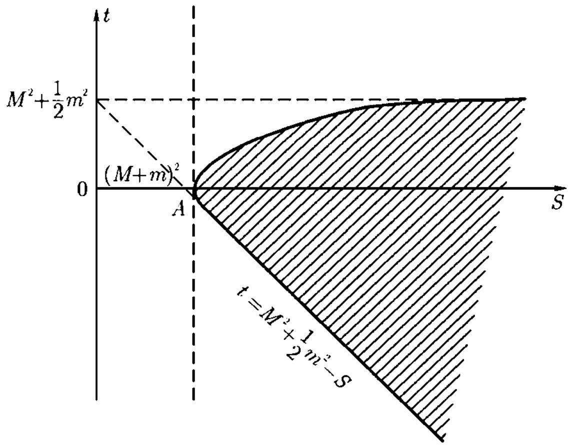

Figure 3.21 Photoproduction of a pion off a proton.

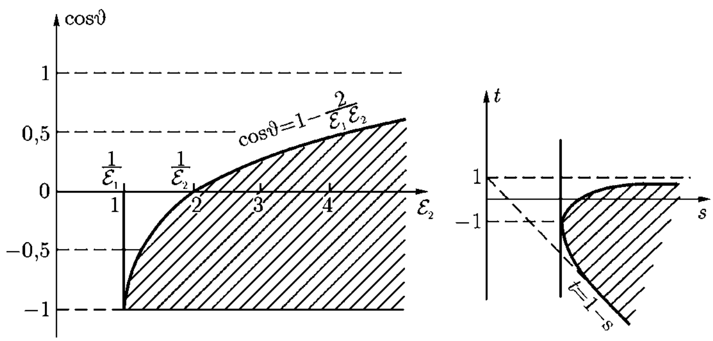

Figure 3.22 The birth of an electron–positron pair from two gamma quanta.

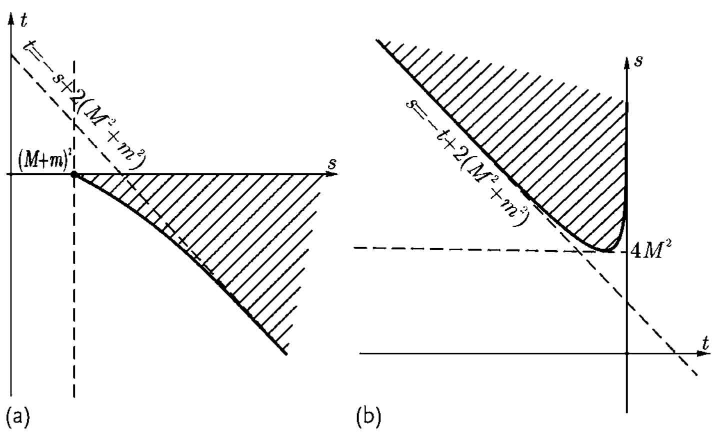

Figure 3.23 Elastic scattering of a pion by a proton (a), and elastic scattering of their antiparticles (b).

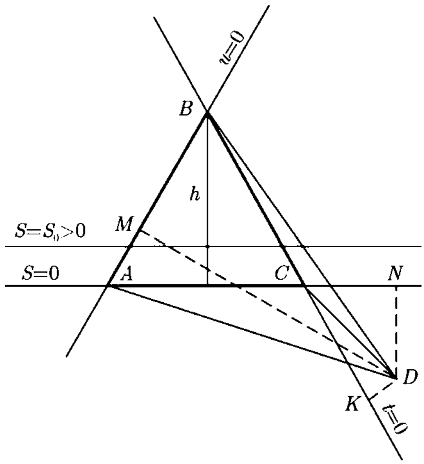

Figure 3.24 Construction of the common diagram for a process with four particles.

A common kinematic diagram can be drawn for all three processes regarding them as three possible channels of a single reaction involving two nucleons and two mesons. The initial and final states of the mesons and nucleons in such channels are characterized by different energies, momenta, and electric charges.[27)]

To construct the diagram (Figure 3.24), we draw three straight lines on which $s = 0$, $t = 0$, and $u = 0$, respectively, so that their intersections make an equiangular triangle with height $h = s + t + u = m_a^2 + m_b^2 + m_c^2 + m_d^2$ $(c = 1)$. The straight line parallel to the axis $s = 0$ and spaced $|s_0|$ apart corresponds to the values of $s = s_0 =$ const. This line should be drawn on the same side as the triangle if $s_0 > 0$ and on the opposite side if $s_0 < 0$. The lines $t =$ const and $u =$ const are drawn in a similar fashion.

As a result, an oblique coordinate system is built up on the plane and three quantities, s, t, and u (both positive and negative), are matched with any point in the plane. The sum of these three quantities satisfies the necessary condition (3.50). To be sure, we choose an arbitrary point D and drop from it perpendiculars on sides AB, BC, and AC or their extensions. Because ABC area $=$ ABD area $-$ $(BCD$ area $+$ ACD area),

$$DM - DN - DK = h = m_a^2 + m_b^2 + m_c^2 + m_d^2 \,. \tag{1}$$

But $-DN = s$, $-Dk = t$, and $DM = u$; hence, we have (3.50).

For our purpose, it is necessary to slightly change the definitions of s, t, and u in comparison with (3.49). Let

$$s = (p_{ai} + p_{bi})\left(p_a^i + p_b^i\right) \,, \quad t = (p_{ai} + p_{ci})\left(p_a^i + p_c^i\right) \,,$$
$$u = (p_{ai} + p_{di})\left(p_a^i + p_d^i\right) \,, \tag{2}$$

27) And a few more characteristics studied in quantum theory.

where $p^i = (-\mathcal{E}, -\boldsymbol{p})$ for the particles that disappear and $p^i = (\mathcal{E}, \boldsymbol{p})$ for those that are born in the course of the reaction. This law of signs corresponds to $\sum_a p_a^i = 0$, as in the case of decay. We label mesons by a and b and nucleons by c and d. Then, for process 3, $p_a^i = (-\mathcal{E}_a, -\boldsymbol{p}_a)$, $p_b^i = (-\mathcal{E}_b, -\boldsymbol{p}_b)$, $p_c^i = (\mathcal{E}_c, \boldsymbol{p}_c)$, and $p_d^i = (\mathcal{E}_d, \boldsymbol{p}_d)$; $s = \left(\mathcal{E}'_a + \mathcal{E}'_b\right)^2 = \left(\mathcal{E}'_c + \mathcal{E}'_d\right)^2 \geq 4M^2$; the acceptable values of t was obtained from the condition $|\cos\theta'| \leq 1$.

The boundary of the physical region is given by the equation

$$s = -t - \frac{(M^2 - m^2)^2}{t} + 2(M^2 + m^2) \geq 4M^2 \,. \tag{3}$$

It is a hyperbola with asymptotes $t = 0$ and $u = 0$ (Figure 3.25)

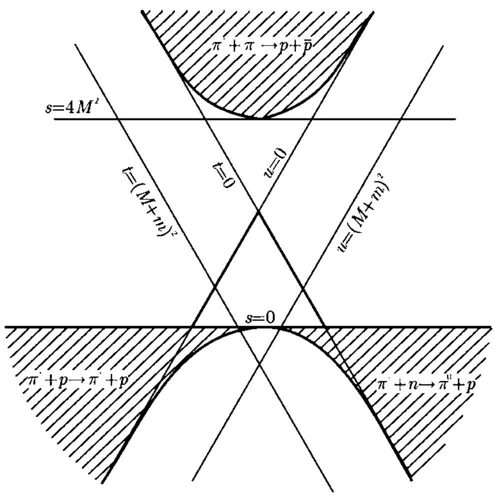

Figure 3.25 Acceptable regions for processes with four particles.

In the case of process 1, its assumed that $p_a^i = (-\mathcal{E}_a, -\boldsymbol{p}_a)$, $p_c^i = (-\mathcal{E}_c, -\boldsymbol{p}_c)$, $p_b^i = (\mathcal{E}_b, \boldsymbol{p}_b)$, and $p_d^i = (\mathcal{E}_d, \boldsymbol{p}_d)$. The physical region is bounded by the straight line $s = o$ and the hyperbola

$$s = -t - \frac{(M^2 - m^2)^2}{t} + 2(M^2 + m^2) \,, \quad t \geq (M + m)^2 \,, \tag{4}$$

which is the second branch of hyperbola (3).

The physical region for process 2 is constructed in an analogous manner. It follows from the diagram thus obtained being very similar to the Dalitz diagram for three-particle decay (see Problem 3.79*).

The resemblance is because the process in both cases involves four particles whose 4-momenta are related by the condition $p_{ai} + p_{bi} + p_{ci} + p_{di} = 0$ on the strength of the law of conservation. It is easy to see, taking into consideration the specified masses of all particles, $m_a^2 = p_{ai} p_a^i$, and so on, that only two independent invariants, for example, $s = (p_{ai} + p_{bi})(p_a^i + p_b^i)$ and $t = (p_{ai} + p_{ci})(p_a^i + p_c^i)$, can be composed from the 4-momenta of the particles. For this reason, a two-dimensional space (a kinematic plane) is needed to depict such processes.

3.111 If a particle propagating in a medium with 4-momentum p_0^i emits a photon with 4-momentum $k^i = (\hbar\omega/c, \hbar\omega\boldsymbol{n}/c)$, the laws of energy and momentum conservation may be expressed through the four-dimensional equality

$$p_0^i = p^i + k^i \,,$$

where p_i is the 4-momentum of the particle after emission of the photon. We transfer k^i to the left and square both parts of the resulting equality. The elementary transformations yield

$$\cos\vartheta = \frac{1}{n\beta}\left[1 + \frac{\pi\Lambda}{n\lambda}(n^2-1)\sqrt{1-\beta^2}\right] , \tag{1}$$

where $\Lambda = \hbar/mc$ is the Compton wavelength of the particle, $\lambda = 2\pi c/\omega n$ is the photon wavelength, and $\beta = v/c$. The second term is of the order of magnitude Λ/λ and is usually very small. Omitting this term expressing quantum corrections (Λ is proportional to $\hbar$) reduces expression (1) to the classical condition of Vavilov–Cherenkov radiation:

$$\cos\vartheta = \frac{1}{n\beta} \,.$$

3.113 We denote the 4-momenta of the particle before and after emission by p_{0i} and p_i. respectively. and the 4-momentum of the photon by k_i and write the law of conservation of energy and momentum in the form

$$p_{0i} - k_i = p_i \,.$$

Taking the square of both parts of this equality and omitting the term with $\hbar^2$ yields

$$(m^2 - m_0^2)c^2 - 2\boldsymbol{p}\cdot\boldsymbol{k} + \frac{2\mathcal{E}_0 k}{c} = 0 \,,$$

where m_0 is the mass of the excited particle and m is the mass of the particle in the normal state.

We represent the difference $c^2(m_0^2 - m^2)$ in the form $c^2(m_0 - m)(m_0 + m) \approx 2\hbar\omega_0 m$. Then,

$$n(\omega)\beta\cos\vartheta = 1 - \frac{\omega_0}{\omega}\sqrt{1-\beta^2} \,, \tag{1}$$

where $\beta = v/c$.

At $\omega_0 \to 0$, equality (1) transforms into the condition

$$n(\omega)\beta\cos\vartheta = 1$$

for the generation of Vavilov–Cherenkov radiation. This radiation is therefore unrelated to a change of the particle's internal state.

For $\omega_0 \neq 0$, we rewrite expression (1) in the form

$$\omega = \frac{\omega_0\sqrt{1-\beta^2}}{1-n(\omega)\beta\cos\vartheta} \,. \tag{2}$$

Formula (2) describes the Doppler effect in a refractive medium. It is applicable if $n(\omega)\beta\cos\vartheta < 1$ and differs from the corresponding formula describing the Doppler effect in a vacuum only by the presence of $n(\omega)$ in the denominator. No new qualitative events occur at $\beta \ll 1$, but the event becomes more complicated at $\beta \approx 1$ and in the presence of dispersion in the medium.

In the general case, formula (2) is a nonlinear equation with respect to ω (n is a function of ω!) and may have more than one solution. Moreover, a few lines (complex Doppler effect) may be observed in the laboratory frame of reference instead of a single line as in the case of an ordinary Doppler effect.

3.114 Acting as in the solution of Problem 3.115, we obtain the following results. The emission of frequency ω accompanied by excitation of the particle may occur if the velocity $v = \beta c$ of the particle's movement exceeds the threshold value $c/n(\omega)\cos\vartheta$ (ϑ is the angle between the directions of the particle's velocity and the photon's momentum). The necessary energy is drawn from the kinetic energy of the particle. Emission of such type is observed at a fixed value of ω only in a certain range of acute angles ϑ inside the Cherenkov cone, the surface of which is described by the equation $n\beta\cos\vartheta = 1$. The observed frequency ω is related to angle ϑ and quantities β and $n(\omega)$ by the formula

$$\omega = \frac{\omega_0\sqrt{1-\beta^2}}{n(\omega)\beta\cos\vartheta - 1} \quad [n(\omega)\beta\cos\vartheta > 1] \,,$$

which is actually an equation with respect to ω as in Problem 3.113. In the general case, this equation admits a few solutions (complex superlight Doppler effect).

3.115 We denote the angle between the initial momentum of the electron, $\boldsymbol{p}_0$, and the direction of soft quantum propagation by ϑ_1 and the angle between $\boldsymbol{p}_0$ and the direction of propagation of a hard quantum by ϑ_2. It follows from the law of conservation of 4-momentum (see Problem 3.113) on the assumption of $\hbar\omega_1 \ll \mathcal{E}_0$ and $\hbar\omega_0 \ll \mathcal{E}_0$ that

$$\cos\vartheta_2 = \frac{c}{v_0 n(\omega_1)} + \frac{\hbar\omega_2}{\hbar\omega_1} \cdot \frac{c - v_0\cos\vartheta_1}{v_0 n(\omega_1)} \,. \tag{1}$$

It can be seen that the hard Cherenkov quantum $\hbar\omega_2$ propagates inside the Cherenkov cone corresponding to the soft Cherenkov quantum with frequency $\hbar\omega_1$. The angular opening of this cone at a specified accuracy is determined by the condition $\cos\vartheta_1 = c/v_0 n(\omega_1)$. The inequality $v_0 > c/n(\omega_1)$ should be satisfied if hard Vavilov–Cherenkov radiation is to be generated, as in the case of ordinary Vavilov–Cherenkov radiation. This is possible only if $n(\omega_1) > 1$, which implies that one of the quanta must be soft enough. The solution of (1) with respect to $\hbar\omega_2$ gives

$$\hbar\omega_2 = \hbar\omega_1 \frac{n(\omega_1)v_0 \cos\vartheta_1 - c}{c - \cos\vartheta_2} \,. \tag{2}$$

The maximum value of $\hbar\omega_2$ is achieved at $\vartheta_1 = \vartheta_2 = 0$:

$$(\hbar\omega_2)_{\max} = \hbar\omega_1 \frac{n(\omega_1)v_0 - c}{c - v_0} \,. \tag{3}$$

3.116

$$\hbar\omega_2 = \frac{2\hbar\omega_1[n(\omega_1)\cos\vartheta_1 - 1]}{(mc^2/\mathcal{E}_0)^2 + 2(\hbar\omega_1/\mathcal{E}_0)[n(\omega_1)\cos\vartheta_1 - 1] + \vartheta_2^2} \,.$$

The maximum value of $\hbar\omega_2$ is achieved at $\vartheta_1 = \vartheta_2 = 0$.
At $\mathcal{E}_0 \ll (mc^2)^2/\hbar\omega_1$,

$$(\hbar\omega_2)_{\max} \approx 2\hbar\omega_1[n(\omega_1) - 1]\left(\frac{\mathcal{E}_0}{mc^2}\right)^2 \,,$$

and at $\mathcal{E}_0 \gg (mc^2)^2/\hbar\omega_1$,

$$(\hbar\omega_2)_{\max} \approx \mathcal{E}_0 \,.$$

It follows from the last expression that a hard Cherenkov quantum can carry away the largest fraction of the initial energy of an ultarelativistic electron.

3.117 The scattering angle assumes discrete values given by the equation

$$\sin\frac{\vartheta}{2} = \frac{\pi\hbar}{a p_0} \,,$$

where $\frac{1}{a} = \sqrt{\frac{n_1^2}{a_1^2} + \frac{n_2^2}{a_2^2} + \frac{n_3^2}{a_3^2}}$, and n_i are integers.

3.119 At $\hbar\omega \ll \mathcal{E}_0$,

$$\hbar\omega = \frac{(qc)^2/2\mathcal{E}_0}{(mc^2/\mathcal{E}_0)^2 + \vartheta^2 - 2(qc/\mathcal{E}_0)} \,.$$

The energy $\hbar\omega$ of a bremsstrahlung quantum takes on discrete values at fixed values of angle ϑ because the momentum transferred $\boldsymbol{q} = 2\pi\hbar\boldsymbol{g}$ is discrete.

3.120• According to the results for Problem 2.56, the electrostatic energy of a charged sphere is given by the expression $W = \alpha e^2/R$, where e is the net charge of the sphere and α is a numerical factor on the order of unity. The equality $W = m_e c^2$ leads to $R = \alpha e^2/m_e c^2$. The quantity

$$r_0 = \frac{e^2}{m_e c^2} = 2.8 \times 10^{-13}\,\text{cm}$$

is called *the classical electron radius*. This parameter emerges in many electrodynamic problems. However, one should not ascribe the literal sense of the radius of an elementary particle to this parameter because the classical theory that provides a basis for its estimation breaks down at rather large distances owing to quantum effects:

$$\Lambda = \frac{\hbar}{mc} .$$

For an electron, $\Lambda = 137 r_0 = 3.9 \times 10^{-11}$ cm. The quantity Λ is called the *Compton wavelength*.

4
Fundamentals of Relativistic Mechanics and Field Theory

4.1
Four-Dimensional Vectors and Tensors

4.1.1
Transformations of Tensors

On transition from one inertial frame of reference, S', to another, S, the contravariant components of the 4-vector undergo transformation in accordance with the following rule:

$$A^i = \Lambda^i{}_k A'^k \,, \tag{4.1}$$

where the Lorentz transformation matrix $\Lambda^i{}_k$ of a particular form (3.20) (a boost along the Ox axis) is given by

$$\Lambda^i{}_k = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{4.2}$$

The transformation matrix is frequently written by using parameter ψ ("rapidity") according to the formulas

$$\cosh\psi = \gamma \,, \quad \sinh\psi = \beta\gamma \,, \quad \cosh^2\psi - \sinh^2\psi = 1 \,. \tag{4.3}$$

In conformity with (3.21) and (3.18), the covariant components of the 4-vector are transformed with a different matrix:

$$A_i = \Lambda_i{}^k A'_k \,, \tag{4.4}$$

where

$$\Lambda_i{}^k = \begin{pmatrix} \cosh\psi & -\sinh\psi & 0 & 0 \\ -\sinh\psi & \cosh\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{4.5}$$

Foundations of Classical and Quantum Electrodynamics, First Edition. Igor N. Toptygin.

Definitions (4.1) and (4.4) are directly generalized to a tensor of any rank. For example, a mixed rank 2 tensor is a 16-component quantity $T_i{}^k$ $(i, k = 0, 1, 2, 3)$ undergoing transformation according to the rule

$$T_i{}^k = \Lambda_i{}^m \Lambda^k{}_n T'^n{}_m \,. \tag{4.6}$$

The following example is provided to consider more general Lorentz transformations than those matched with matrices (4.2) and (4.5).

Example 4.1

What is the form of the most general linear transformation of coordinates? What conditions are imposed on the transformation coefficients by the requirement of invariance of the interval between two events? How many independent parameters determine such a transformation?

Solution. The transformation has the form

$$x^k = \Lambda^k{}_n x'^n + a^k \,, \tag{4.7}$$

where $\Lambda^k{}_n$ and a^k are the transformation coefficients. The coordinates being real quantities, the transformation coefficients must be real too. An identical transformation is compatible with the values of

$$\Lambda^k{}_n = \delta^k_n \,, \quad a^k = 0 \,. \tag{4.8}$$

At $\Lambda^k{}_n = \delta^k_n$, one has

$$x^k = x'^k + a^k \,, \tag{4.9}$$

which means that a^k is the slip vector of the reference frame whose components are independent and arbitrary.

It follows from the condition of invariance of the small interval $\mathrm{d}s'^2 = \mathrm{d}s^2$ that

$$g_{ik}\mathrm{d}x^i\mathrm{d}x^k = g_{ik}\Lambda^i{}_m\Lambda^k{}_n\mathrm{d}x'^m\mathrm{d}x'^n = g_{mn}\mathrm{d}x'^m\mathrm{d}x'^n \,,$$

whence

$$g_{ik}\Lambda^i{}_m\Lambda^k{}_n = g_{mn} \,. \tag{4.10}$$

Multiplication of the last equality by g^{nr} yields

$$g_{ik}\Lambda^k{}_n g^{nr}\Lambda^i{}_m = \delta^r_m \,. \tag{4.11}$$

Bearing in mind the definition of the reciprocal matrix Λ^{-1},

$$\Lambda^n{}_i(\Lambda^{-1})^i{}_m = (\Lambda^{-1})^n{}_i\Lambda^i{}_m = \delta^n_m \,, \tag{4.12}$$

we find for it the following expression:

$$(\Lambda^{-1})^n{}_i = g_{ik}\Lambda^k{}_r g^{rn} = \Lambda_i{}^n \,. \tag{4.13}$$

We call the reader's attention to the reverse order of indices i and n on the right-hand and left-hand sides of the equation. This fact together with the explicit form of the metric tensor g_{ik} implies that the reciprocal matrix ensues from the initial one after the transposition and alteration of the sign of the elements in the first row and the first column (corresponding to index 0). The transformation opposite (4.7) is

$$x'^i = (\Lambda^{-1})^i{}_k(x^k - a^k) = \Lambda_k{}^i(x^k - a^k) \,. \tag{4.14}$$

Property (4.11) of the transformation matrix is known as pseudo-orthogonality. It means that 16 matrix elements $\Lambda^i{}_m \quad (i, m = 0, 1, 2, 3)$ are interrelated by 10 equations (rather than 16 because a change in the positions of indices m and r does not result in new equalities). In other words, the transformation matrix Λ is determined by six independent parameters, in agreement with the physical notions of the general form of the Lorentz transformation specified by six parameters (such as the value and direction of the relative velocity and the three angles determining the orientation of the spatial axes). These six parameters and four components of the slip vector a_k vary continuously and collectively make up a set of 10 independent quantities.

Transformation (4.7) is defined by two more discrete parameters, besides the 10 continuous ones; these parameters take only two values each. It follows from (4.11): because the determinant of a matrix product is equal to the product of determinants,

$$(\det \widehat{g})^2 (\det \widehat{\Lambda})^2 = 1 \,,$$

or

$$\det \widehat{\Lambda} = \pm 1 \,. \tag{4.15}$$

Assuming that $r = m = 0$ in (4.11), we have

$$(\Lambda^0{}_0)^2 - (\Lambda^1{}_0)^2 - (\Lambda^2{}_0)^2 - (\Lambda^3{}_0)^2 = 1 \,, \tag{4.16}$$

whence $(\Lambda^0{}_0)^2 \geq 1$ or

$$\frac{\Lambda^0{}_0}{|\Lambda^0{}_0|} = \pm 1 \,. \tag{4.17}$$

Thus, it is necessary to specify the signs of $\det \widehat{\Lambda}$ and $\Lambda^0{}_0$, besides the 10 continuous transformation parameters.

$\det \widehat{\Lambda} = +1$ and $\Lambda^0{}_0 = +1$ correspond to identical transformations, and the signs of these quantities cannot be altered by any change of continuous parameters.

The values of $\det \widehat{\Lambda} = -1$ and $\Lambda^0{}_0 = -1$ correspond to time reflection in the fixed coordinates (transformation $t' = -t$, $x'^\mu = x^\mu$). The values of $\Lambda^0{}_0 = -1$ and $\det \widehat{\Lambda} = +1$ correspond, for instance, to the inversion of all the four axes, $x'^k = -x^k$; finally, the values of $\Lambda^0{}_0 = +1$ and $\det \widehat{\Lambda} = -1$ correspond to the spatial inversion $x'^\mu = -x^\mu$, $t' = t$.

Transformation (4.7) makes up a group[1] called the Poincaré group. Transformations that do not contain a shift ($a^k = 0$) also constitute a group (the group of general Lorentz transformations). The transformations with $\Lambda^0{}_0 \geq 1$ and $\det \widehat{\Lambda} = 1$ build a six-parameter group of proper Lorentz transformations. □

Example 4.2

Derive the Lorentz transformation (3.5) from the general relations (4.7)–(4.16) obtained in the preceding example.

Solution. The transformations (3.7) correspond to the case shown in Figure 3.1, with $t = t' = 0$ at the moment when the three-dimensional points 0 and O' coincide. For this reason, it should be assumed that $a^i = 0$ in the general formulas (4.7). Let us turn now to the transformations of x^2 and x^3 coordinates. Because the planes x^1 and x^2 coincide, the value of $x^3 = 0$ must correspond to $x'^3 = 0$ regardless of the values of x^0, x^1, and x^2. This is possible only on the condition that in the linear relation

$$x'^3 = \Lambda^3{}_0 x^0 + \Lambda^3{}_1 x^1 + \Lambda^3{}_2 x^2 + \Lambda^3{}_3 x^3 ,$$

ensuing from (4.7), the coefficients $\Lambda^3{}_0 = \Lambda^3{}_1 = \Lambda^3{}_2 = 0$. Similar considerations suggest that $\Lambda^2{}_0 = \Lambda^2{}_1 = \Lambda^2{}_3 = 0$. Moreover, the equal status of x^2 and x^3 implies that $\Lambda^3{}_3 = \Lambda^2{}_2 = k(V)$, that is,

$$x'^2 = k(V) x^2 , \quad x'^3 = k(V) x^3 , \tag{4.18}$$

which contains the possible dependence of the transformation coefficient k on the relative velocity. Owing to the equal status of systems S and S', formulas (4.18) must yield inverse transformations on the replacement of $\boldsymbol{V}$ by $-\boldsymbol{V}$:

$$x^2 = k(-V) x'^2 , \quad x^3 = k(-V) x'^3 . \tag{4.19}$$

But both directions of $\boldsymbol{V}$ are perpendicular to the plane $x^2 x^3$ and absolutely tantamount (space isotropy!). Therefore, $k(-V) = k(V)$. Passing from x^2 to x'^2 and then back to x^2, we arrive at $x^2 = k^2(V) x^2$, that is, $k = \pm 1$. The value of $k = -1$ corresponds to the opposite orientation of the x^2 and x'^2 axes; therefore, only

$$\Lambda^3{}_3 = \Lambda^2{}_2 = k = +1 \tag{4.20}$$

1) See Bogolubov and Shirkov (1982) for an introduction to group theory.

corresponds to Figure 3.1.

Let us write equality (4.7) for $k = 0$ and $k = 1$:

$$x'^0 = \Lambda^0{}_0 x^0 + \Lambda^0{}_1 x^1 , \quad x'^1 = \Lambda^1{}_0 x^0 + \Lambda^1{}_1 x^1 . \tag{4.21}$$

We find the four unknown coefficients from the following relations:

1. The equation of plane $x'^1 = 0$ in system S has the form $x^1 = Vt = \beta x^0$. Using (4.21), we find

$$\Lambda^1{}_0 = -\beta \Lambda^1{}_1 . \tag{4.22}$$

2. It follows (4.11) at $r = 0$ and $m = 1$ that

$$\Lambda^0{}_0 \Lambda^0{}_1 - \Lambda^1{}_0 \Lambda^1{}_1 = 0 , \tag{4.23}$$

   whence we find (using (4.22)) that

$$\Lambda^0{}_1 = \frac{-\beta(\Lambda^1{}_1)^2}{\Lambda^0{}_0} . \tag{4.24}$$

3. We find from (4.11) at $r = m = 0$ and $r = m = 1$ two equations

$$(\Lambda^0{}_0)^2 - (\Lambda^1{}_0)^2 = 1, (\Lambda^1{}_1)^2 - (\Lambda^0{}_1)^2 = 1 . \tag{4.25}$$

   We calculate the transformation coefficients from the four algebraic equations (4.22)–(4.25), selecting the positive values of $\Lambda^0{}_0$ and $\Lambda^1{}_1$:

$$\Lambda^0{}_0 = \Lambda^1{}_1 = \gamma , \Lambda^0{}_1 = \Lambda^1{}_0 = -\beta\gamma \tag{4.26}$$

   (the proper Lorentz transformation).

□

4-tensors, like tensors in the three-dimensional space, can behave differently in the case of inversion of the spatial coordinate axes (see Section 1.1). The contravariant pseudotensor of the Nth rank is the totality of 4^N quantities $P^{ik\cdots l}$ that undergo transformation on reflections and turns in the four-dimensional Minkowski space in accordance with

$$P^{ik\cdots l} = |\widehat{\Lambda}| \Lambda^i{}_m \Lambda^k{}_n \cdots \Lambda^l{}_{\mathrm{p}} P'^{mn\cdots p} . \tag{4.27}$$

The determinant $|\widehat{\Lambda}| = -1$ if the transformation includes ab odd number (one or three) of coordinate axes. The pseudotensor is exemplified by a rank 4 antisymmetric unit tensor, e^{iklm}. It is defined by the following conditions:

1. The components of e^{iklm} change sign on permutation of any pair of indices (this condition, as in the three-dimensional space, makes all components having two or more coinciding indices become zero);

2. $e^{0123} = 1$;
3. Rule (4.27) for transition to a different coordinate system.

These conditions make e^{iklm} an invariant pseudotensor. In any system of coordinates

$$e'^{iklm} = e^{iklm} . \tag{4.28}$$

4.1.2 Dual Tensors

Tensor e^{iklm} can be used to build new tensors. For example, the arbitrary covariant 4-vector B_m can be matched with the rank 3 antisymmetric tensor

$$\tilde{B}^{ikl} = e^{iklm} B_m . \tag{4.29}$$

The antisymmetric covariant rank 2 tensor, $A_{lm} = -A_{ml}$, can be converted into another antisymmetric tensor

$$\tilde{A}^{ik} = (1/2) e^{iklm} A_{lm} . \tag{4.30}$$

Finally, the components of the rank 3 antisymmetric tensor can be used to create vector

$$\tilde{J}^i = (1/3!) e^{iklm} J_{klm} . \tag{4.31}$$

All three cases reduce to rewriting the components of the initial tensors. The pairs of tensors with and without the tilde are called dual to each other. The dual pairs comprise only antisymmetric tensors. If tensors with arbitrary symmetry are substituted into the right-hand sides of the last two equalities, summation of their symmetric parts makes a zero contribution and therefore does not influence the values of the left-hand sides.

Differentiation of the tensor with respect to 4-coordinates results in a change of its rank because the differential operator is a 4-vector.

Example 4.3

Show that operator of the 4-gradient $\partial/\partial x^k$ transforms as a true covariant 4-vector.

Solution. The use of (4.14) leads to

$$\frac{\partial}{\partial x^k} = \frac{\partial x'^m}{\partial x^k} \frac{\partial}{\partial x^m} = \Lambda_k{}^m \frac{\partial}{\partial x'^m} , \tag{4.32}$$

which coincides with the transformation rule (4.4) for the covariant 4-vector. For this reason, other indices are applied to denote the 4-vector, besides (4.32), explicitly using the covariant index (subscript), for example,

$$\frac{\partial \Phi}{\partial x^k} = \partial_k \Phi = \Phi_{,\,k} . \tag{4.33}$$

In the case of inversion of axes, the 4-gradient undergoes transformation as 4-coordinates. □

Suggested literature:
Landau and Lifshitz (1975); Medvedev (1977); Rashevskii (1959); Pauli (1921); Fock (1962); Burke (1980); Weinberg (1972); Ginzburg (1979a)

Problems

4.1. Show that the metric tensor (3.9) has the same form in all inertial frames of reference.

4.2. Prove the equalities

$$e^{iklm} A_i B_k C_l D_m = \begin{vmatrix} A_0 & A_1 & A_2 & A_3 \\ B_0 & B_1 & B_2 & B_3 \\ C_0 & C_1 & C_2 & C_3 \\ D_0 & D_1 & D_2 & D_3 \end{vmatrix} . \tag{4.34}$$

$$e_{i'k'l'm'} \Lambda_i{}^{i'} \Lambda_k{}^{k'} \Lambda_l{}^{l'} \Lambda_m{}^{m'} = \det|\widehat{\Lambda}| e_{iklm} . \tag{4.34'}$$

4.3. Prove equality (4.28) based on the definition of e^{iklm}.

4.4. A certain rank 2 contravariant tensor has the property of symmetry ($S^{ik} = S^{ki}$) or antisymmetry ($A^{ik} = -A^{ki}$). What do the corresponding relations for the covariant and mixed tensors look like?

4.5. Write the rank 2 tensor $T_i{}^k$ with arbitrary symmetry as the sum of a tensor proportional to the unit tensor δ_i^k and the tensor with zero trace.

4.6. Consider tensor T^{ikl} with arbitrary symmetry. Use its components to create tensors S^{ikl} and A^{ikl} symmetric and antisymmetric with respect to any pair of indices.

4.7. Show that

$$e_{iklm} = -e^{iklm} . \tag{4.35}$$

4.8*. Prove the identities

1. $e_{iklm}e^{iklm} = -24$.
2. $e_{iklm}e^{ikln} = -6\delta_m^n$.
3. $e_{iklm}e^{ikjr} = -2\left(\delta_l^j \delta_m^r - \delta_l^r \delta_m^j\right)$.
4.

$$e_{iklm}e^{iprs} = -\begin{vmatrix} \delta_k^p & \delta_k^r & \delta_k^s \\ \delta_l^p & \delta_l^r & \delta_l^s \\ \delta_m^p & \delta_m^r & \delta_m^s \end{vmatrix} . \tag{4.36}$$

5.

$$e_{iklm}e^{jprs} = -\begin{vmatrix} \delta_i^j & \delta_i^p & \delta_i^r & \delta_i^s \\ \delta_k^j & \delta_k^p & \delta_k^r & \delta_k^s \\ \delta_l^j & \delta_l^p & \delta_l^r & \delta_l^s \\ \delta_m^j & \delta_m^p & \delta_m^r & \delta_m^s \end{vmatrix} .$$

4.9. Convert equalities (4.29)–(4.31) and express vector B_m and tensors A_{lm} and J_{klm} through $\tilde{B}^{ikl}$, $\tilde{A}^{ik}$, and $\tilde{J}^i$ respectively. How do tensors with the tilde transform on reflection of coordinate axes if the initial tensors are polar?

4.10. Two nonparallel 4-vectors, A_i and B_k, having a common origin determine the two-dimensional hyperplane in a 4-space. Show that the tensor $\tilde{C}^{ik}$ dual to the antisymmetric tensor $A_i B_k - A_k B_i$, is orthogonal to any 4-vector lying in this hyperplane.

4.11*. Three noncoplanar (linearly independent) 4-vectors, A_i, B_j, and C_k, form the edges of a three-dimensional hyperparallelepiped in a four-dimensional pseudo-Euclidean space. Determine the volume of the parallelepiped. Show that the 4-vector $\tilde{V}_i$ dual to the rank 3 antisymmetric tensor representing the volume described above is orthogonal to any 4-vector belonging to the said three-dimensional hyperparallelepiped.

4.12. • Find the three-dimensional tensors into which the rank 2 4-tensor T^{ik} splits on spatial turns.

4.13. • Find the three-dimensional tensors into which the true rank 2 4-tensor A_{ik} splits on spatial turns and reflections.

Hint: Use the results obtained in Problem 1.10.

4.14. Prove the following identity for the antisymmetric tensor A_{ik} considered in the previous problem:

$$A_{ik}\tilde{A}^{kl} = \boldsymbol{p} \cdot \boldsymbol{a}\delta_i^l , \quad \tilde{A}_{ik}\tilde{A}^{kl} = A_{ik}A^{kl} + \frac{1}{2}A_{mn}A^{mn}\delta_i^l , \tag{4.37}$$

where $\tilde{A}_{ik}$ is the dual tensor, and $\boldsymbol{p}$ and $\boldsymbol{a}$ are the polar and axial 3-vectors, respectively, of which A_{ik} is comprised.

4.15. Two 4-vectors, A_i and B_i, are called parallel if

$$\frac{A_0}{B_0} = \frac{A_1}{B_1} = \frac{A_2}{B_2} = \frac{A_3}{B_3} .$$

Prove that the ratio of analogous components of the parallel 4-vectors is invariant with respect to the Lorentz transformation.

Hint: Make use of the equivalent ratio property.

4.16. Spatial rotations of the coordinate system make up a subgroup of proper Lorentz transformations. Write the Lorentz transformation matrix for the spatial turns choosing the Euler angles as transformation parameters (Figure 1.2).

4.17. The frame of reference S'' moves with respect to S' with velocity V' parallel to the x' axis and system S moves with velocity V parallel to the x axis. The similarly named axes of all three systems are parallel. The multiplication of the respective matrices yields the matrix of transformation from S'' to S. Derive the formula for addition of unidirectional velocities.

4.18. The system S' moves relative to S with velocity V, the direction of movement being specified in system S by angles Θ and Φ. The spatial axes of the two systems are parallel. Derive the Lorentz transformation matrix by multiplying the spatial rotation matrices and boost along one of the coordinate axes.

4.19*. Obtain the transformation matrix $\widehat{\Lambda}$ for the case considered in the previous problem making use of the general properties of this matrix mentioned in Example 4.1.

4.20. Write the infinitely small proper Lorentz transformation in the form

$$x^k = x'^k + \delta\Omega^k{}_l x'^l ,$$

where $\delta\Omega^k{}_l$ is the transformation parameters. What constraints does the requirement of interval invariance impose on matrix $\delta\Omega^k{}_l$ and how many independent transformation parameters are there? What is the geometric sense of quantity $\delta\Omega^k{}_l$?

4.21. Find the transformation rules for the derivatives of $A^i_{,i}$, $\partial_k A_i$, and $\partial_i T^i_k$, where A^i is a vector and T^i_k is a rank 2 tensor.

4.22. Show that the d'Alembert operator

$$\Box = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \tag{4.38}$$

is a relativistic invariant.

4.23. Define the 4-vector rotor A_i by analogy with the rotor of the three-dimensional vector curl $\boldsymbol{A}$. Is it possible to regard the 4-rotor as a 4-vector?

4.24*. A smooth closed contour l is given in the four-dimensional pseudo-Euclidean space. Prove Stokes's theorem for an arbitrary differentiable 4-vector $A_i(x) \equiv A_i(x^0, x^1, x^2, x^3)$:

$$\oint_l A_i \mathrm{d}l^i = \int_S \left(\frac{\partial A_k}{\partial x^i} - \frac{\partial A_i}{\partial x^k}\right) \mathrm{d}S^{ik} , \tag{4.39}$$

where S is an arbitrary open hypersurface resting on contour l, $\mathrm{d}S^{ik}$ is the directed element of this hypersurface, and $\mathrm{d}l^i$ is the directed element of contour l.

4.25*. Prove the Gauss–Ostrogradskii theorem in the four-dimensional pseudo-Euclidean space

$$\oint_{\Sigma} A^i \mathrm{d}S_i = \int_{\Omega} \frac{\partial A^i}{\partial x^i} \mathrm{d}^4 x \,, \tag{4.40}$$

where A^i is an arbitrary differentiable 4-vector, Σ is a closed three-dimensional hypersurface limiting the 4-volume Ω, $\mathrm{d}S_i$ is the directed element of the hypersurface, and $\mathrm{d}^4 x$ is an element of the 4-volume.

Hint: The elements of the three-dimensional hypersurface perpendicular to the coordinate axes building the 4-vector are given by the expressions (see Problem 4.11*)

$$\begin{aligned} \mathrm{d}S_0 &= \pm \mathrm{d}x^1 \mathrm{d}x^2 \mathrm{d}x^3 \,, \quad \mathrm{d}S_1 = \pm \mathrm{d}x^0 \mathrm{d}x^2 \mathrm{d}x^3 \,, \\ \mathrm{d}S_2 &= \pm \mathrm{d}x^0 \mathrm{d}x^1 \mathrm{d}x^3 \,, \quad \mathrm{d}S_3 = \pm \mathrm{d}x^0 \mathrm{d}x^1 \mathrm{d}x^2 \,. \end{aligned}$$

Note: The Stokes and Gauss–Ostrogradskii theorems hold for any 4-tensor of rank $s \neq 0$. All the indices, besides those over which summation is performed in equalities (4.39) and (4.40), remain free.

4.2 The Motion of Charged Particles in Electromagnetic Fields. Transformation of the Electric Field

4.2.1 Interaction of Charged Particles with the Electromagnetic Field

The action of the external electric field on a charged particle traveling at arbitrary speed, either constant or variable, is described by the *Lorentz force*

$$\boldsymbol{\mathcal{F}} = e\boldsymbol{E} + \frac{e}{c}\boldsymbol{v} \times \boldsymbol{H} \,. \tag{4.41}$$

We shall consider this inference as a fundamental experimental fact. Also, we shall use it together with some general physical principles to determine the form of the interaction between charged particles and the electromagnetic field, construct the Lagrangian and Hamiltonian functions, clarify the rules for the relativistic transformation of electromagnetic field strengths $\boldsymbol{E}$ and $\boldsymbol{H}$, and find the invariant form of the equations of motion of relativistic particles in the electromagnetic field.

To implement this program based on the variational approach, we need to determine the form of the action for a charged particle in the electromagnetic field . To this effect, the action (3.29) for a free particle should be supplemented with the term describing by the particle–field interaction:

$$S = -mc \int_{(1)}^{(2)} \mathrm{d}s + S_{\text{int}} \,. \tag{4.42}$$

It must be a relativistic invariant and contain the product of the particle's charge and the quantity characterizing the field; more importantly, it must lead to the experimentally confirmed corollaries including the Lorentz force (4.41).

The first of these conditions ensues from the relativity principle, whereas the second one can be considered as integration of the experimental data according to which the interaction between the field and the particles is described by the sole scalar characteristic of a particle, that is, its *electric charge* (see Chapter 2). The strengths $\boldsymbol{E}$ and $\boldsymbol{H}$ as the quantities related to the field are difficult to introduce in S_{int} because of the rather complicated procedure of their transformation into an arbitrary inertial system. For this reason, the field is described by another system of field functions, namely, the *four-dimensional vector potential* $A_i(x^0, x^1, x^2, x^3)$, where $i = 0, 1, 2, 3$, which was already used in Chapter 2 in three-dimensional notation. On the basis of these assumptions, the relativistically invariant expression for S_{int} is written in the maximally simple form

$$S_{\text{int}} = -\frac{e}{c} \int_{(1)}^{(2)} A_i(x^0, x^1, x^2, x^3)\mathrm{d}x^i \,, \tag{4.43}$$

where the integral is taken along the particle's world line, as in (3.29). Factor c^{-1} is specified by the absolute Gaussian system of units being used and the minus sign is specified by the requirement that the components of A_i coincide with the scalar φ and vector $\boldsymbol{A}$ potentials introduced in Chapter 2. The representation of the action in the form of (4.43) is sometimes referred to as the *hypothesis of minimal electromagnetic interaction*. All the numerous electromagnetic phenomena investigated thus far are in agreement with (4.43), and this expression can be regarded as a fairly well established fact for a wide group of physical events. It cannot be excluded, however, that these interactions will be reconsidered after the description of the electromagnetic interactions with certain elementary particles that have thus far not been observed experimentally, such as Dirac magnetic monopoles, the search for which is in progress.[2)]

The action for a few charged particles is written in the form of the sum of expressions (4.43)::

$$S_{\text{int}} = -\sum_a \frac{e_a}{c} \int_{(1)}^{(2)} A_i\left(x_a^0, x_a^1, x_a^2, x_a^3\right) \mathrm{d}x_a^i \,, \tag{4.44}$$

2) The classical formula (4.43) does not describe the interaction between the internal (spin) magnetic moment of an elementary particle and the electromagnetic field because the spin is a quantum characteristic of the particle. The approximate quasi-classical theory of an such interaction is expounded below in this section, and quantum theory is expounded in Chapters 6 and 7.

where each particle has its own initial (1) and final (2) points while index a at the coordinates denotes the particles' numbers. Here, A_i gives the value of the 4-potential created by external bodies and all the particles excepting the a-th one at the point where particle No a is localized. In this way, the action of a particle on itself is excluded from the interaction as mentioned in Chapter 2. It can be accounted for by the fact that modern electrodynamics (both classical and quantum) does not allow to calculate the self-action energy correctly.

Epression (4.44) means that the action on a given particle is mediated through the field created by other particles in its neighborhood (in accordance with the theory of relativity, the field must propagate with a finite velocity that does not exceed the limiting velocity c). The concept of particle interaction through an intermediate agent (field) is known as the *short-range interaction concept*. Its validity is confirmed in the modern quantum theory of elementary particles where the role of interaction carriers is played by the quanta of the respective fields, that is, photons, gluons, intermediate bosons, and so on. This picture is essentially different from the particle interaction concept in nonrelativistic mechanics based on the notion of potential energy $U(\boldsymbol{r}_1(t) - \boldsymbol{r}_2(t))$. The radius vectors of two particles enter the potential energy simultaneously and a change in the position of one particle immediately leads to a change of the force acting on the second particle. In this situation, the interaction propagation velocity is infinite, and no intermediate agent is needed for the interaction transfer. The notion of the field can be used in this case too, but only as a formal mathematical object rather than as an indispensable constituent of the physical process of particle interaction. Such an interaction picture is referred to as *long-range interaction*.

The long-range interaction concept is incompatible with the theory of relativity, which establishes the finite limiting velocity for the interaction propagation. As a result, the energy released and the momentum created by one of the interacting particles separated by a finite distance $|\boldsymbol{r}_1 - \boldsymbol{r}_2|$ must belong to a certain intermediate substance (field) during a finite time $|\boldsymbol{r}_1 - \boldsymbol{r}_2|/c$ until they are gained by the second particle. Thus, the notion of the field as physical reality is an essential attribute of the theory of relativity. Certainly, it ensues from experience because an electromagnetic field can exist as a self-consistent entity even in the absence of charged particles.

4.2.2
Equations of Motion of a Relativistic Particle

Let us now turn to the analysis of corollaries following from formulas (4.42) and (4.43) for the action describing a charged particle in the electromagnetic field:

$$S = -\int_{(1)}^{(2)} \left(mc\mathrm{d}s + \frac{e}{c} A_i \mathrm{d}x^i \right) . \tag{4.45}$$

Example 4.4

Determine the form of the Lagrangian function for a charge particle in the electromagnetic field making use of expression (4.45). Perform a transition to the nonrelativistic limit. Write the equation of motion in the Lagrangian form.

Solution. Pass to integration over coordinate time t in integral (4.45). Then, the integrand is the Lagrangian function by definition. Writing $\mathrm{d}s = c\sqrt{1 - v^2/c^2}\mathrm{d}t$ and $A_i \mathrm{d}x^i = A_0 c\mathrm{d}t - \boldsymbol{A}\mathrm{d}\boldsymbol{r} = c\varphi\mathrm{d}t - \boldsymbol{v}\cdot\boldsymbol{A}\mathrm{d}t$, (where $A_0 = \varphi$) with the help of (3.11), we find the *Lagrangian function* for a relativistic particle interacting with the electromagnetic field:

$$L = -mc^2\sqrt{1 - \frac{v^2}{c^2}} + \frac{e}{c}\boldsymbol{v}\cdot\boldsymbol{A} - e\varphi\ . \tag{4.46}$$

Here, the Cartesian coordinates of the particle $\boldsymbol{r} = (x, y, z)$ on which (as on time t) $\boldsymbol{A}$ and φ are dependent serve as the generalized coordinates.

The nonrelativistic approximation is

$$L = \frac{mv^2}{2} + \frac{e}{c}\boldsymbol{v}\cdot\boldsymbol{A} - e\varphi\ , \tag{4.47}$$

with constant mc^2 being omitted.

Generalization over a system of particles gives

$$L = -\sum_a m_a c^2\sqrt{1 - \frac{v_a^2}{c^2}} + \sum_a e_a\left(\frac{1}{c}\boldsymbol{v}_a\cdot\boldsymbol{A}_a - \varphi_a\right)\ . \tag{4.48}$$

The electromagnetic potentials at the point of particle localization are denoted by $\boldsymbol{A}_a$ and φ taking no account of the particle field. Time t is common for all the particles.

The equations of motion can be written in the Lagrangian form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0\ , \tag{4.49}$$

where q_i and $\dot{q}_i$ are the generalized coordinates and velocities (which are not necessarily 4-vectors). If the Cartesian coordinates of a charged particle are chosen as the generalized coordinates, (4.49) can be written in the vector form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \boldsymbol{v}} - \frac{\partial L}{\partial \boldsymbol{r}} = 0\ , \tag{4.50}$$

where $\boldsymbol{v} = \dot{\boldsymbol{r}}$. □

Example 4.5

Find the equation of motion for a charged particle in relativistic covariant form. Use action (4.45) and pass into invariant variable proper time τ under the integral.

Solution. The first action variation has the form

$$\delta S = \int_{(1)}^{(2)} \left(-mc\delta \mathrm{d}s - \frac{e}{c}\delta A_i \mathrm{d}x^i - \frac{e}{c} A_i \delta \mathrm{d}x^i \right) .$$

Further calculations lead to

$$\delta \mathrm{d}s = \delta \sqrt{\mathrm{d}x^i \mathrm{d}x_i} = \frac{\mathrm{d}x_i}{\mathrm{d}s}\delta \mathrm{d}x^i = \frac{1}{c} u_i \delta \mathrm{d}x^i ,$$

where $u_i = \mathrm{d}x_i/\mathrm{d}\tau$ is the particle's 4-velocity, and whence

$$\delta S = \int_{(1)}^{(2)} \left[\left(-mu_i - \frac{e}{c} A_i \right) \delta \mathrm{d}x^i - \frac{e}{c}\delta A_i \mathrm{d}x^i \right] .$$

Taking advantage of the permutation $\delta \mathrm{d}x^i = \mathrm{d}\delta x^i$ and integrating by parts, we obtain

$$\delta S = -\left(mu_i + \frac{e}{c} A_i \right) \delta x^i \bigg|_{(1)}^{(2)} + \int_{(1)}^{(2)} \left(m\mathrm{d}u_i \delta x^i + \frac{e}{c}\mathrm{d}A_i \delta x^i - \frac{e}{c}\delta A_i \mathrm{d}x^i \right) . \tag{4.51}$$

At the fixed 4-points 1 and 2, $\delta x^i = 0$; therefore, the term outside the integral vanishes. We perform transformations under the integral sign:

$$\mathrm{d}u_i = \frac{\mathrm{d}u_i}{\mathrm{d}\tau}\mathrm{d}\tau ; \quad \mathrm{d}A_i \delta x^i = \frac{\partial A_i}{\partial x^k}\frac{\mathrm{d}x^k}{\mathrm{d}\tau}\mathrm{d}\tau \delta x^i = \frac{\partial A_i}{\partial x^k} u^k \mathrm{d}\tau \delta x^i ;$$

$$\delta A_i \mathrm{d}x^i = \frac{\partial A_k}{\partial x^i}\mathrm{d}x^k \delta x^i = \frac{\partial A_k}{\partial x^i} u^k \mathrm{d}\tau \delta x^i .$$

We combine the last two items by introducing the antisymmetric tensor (4-potential rotor)

$$F_{ik} = \partial_i A_k - \partial_k A_i , \tag{4.52}$$

which is called the electromagnetic field tensor. As a result, there is action variation in the compact invariant representation:

$$\delta S = \int_{(1)}^{(2)} \left(m\frac{\mathrm{d}u_i}{\mathrm{d}\tau} - \frac{e}{c} F_{ik} u^k \right) \delta x^i \mathrm{d}\tau .$$

Equating this to zero and taking advantage of the independence of variations δx^i that are functions of the invariant parameter r, we find the equation of motion:

$$m\frac{\mathrm{d}u_i}{\mathrm{d}\tau} = \frac{e}{c}F_{ik}u^k\,, \quad i = 0, 1, 2, 3\,. \tag{4.53}$$

Evidently, there is a formal similarity between the relativistic equation (4.53) and the nonrelativistic Newton equation $m\dot{\boldsymbol{v}} = \boldsymbol{\mathcal{F}}$. The left-hand side of (4.53) is the product of mass and 4-acceleration. The right-hand side that describes the action of the electromagnetic field on a particle is called the *four-dimensional electromagnetic force*:

$$\mathcal{F}_i = \frac{e}{c}F_{ik}u^k\,. \tag{4.54}$$

The sense of the equation thus obtained becomes clearer after it is rewritten in the three-dimensional form. On the basis of the definition of 4-velocity, write down the left-hand side at $i = 1, 2, 3$ in the form $-\gamma\,\mathrm{d}\boldsymbol{p}/\mathrm{d}t$, where $\gamma = 1/\sqrt{(1 - v^2/c^2)}$ is the relativistic factor. Introduce the three-dimensional scalar $\varphi(\boldsymbol{r}, t)$ and the three-dimensional vector $\boldsymbol{A}(\boldsymbol{r}, t)$, $A_i = (\varphi, -\boldsymbol{A})$, $A^i = (\varphi, \boldsymbol{A})$ instead of A_i on the right-hand side. In these notations, we have the following nonzero values of the components of the electromagnetic field tensor:

$$\begin{aligned}
&F_{0x} = -F_{x0} = -\frac{\partial\varphi}{\partial x} - \frac{1}{c}\frac{\partial A_x}{\partial t}\,, \quad F_{0y} = -F_{y0} = -\frac{\partial\varphi}{\partial y} - \frac{1}{c}\frac{\partial A_y}{\partial t}\,,\\
&F_{0z} = -F_{z0} = -\frac{\partial\varphi}{\partial z} - \frac{1}{c}\frac{\partial A_z}{\partial t}\,,\\
&F_{xy} = -F_{yx} = -\mathrm{curl}_z\boldsymbol{A}\,, \quad F_{xz} = -F_{zx} = \mathrm{curl}_y\boldsymbol{A}\,,\\
&F_{yz} = -F_{zy} = -\mathrm{curl}_x\boldsymbol{A}\,.
\end{aligned} \tag{4.55}$$

This allows the spatial part of (4.55) to be written in the form

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = e\left(-\nabla\varphi - \frac{1}{c}\frac{\partial\boldsymbol{A}}{\partial t}\right) + \frac{e}{c}\boldsymbol{v}\times\mathrm{curl}\,\boldsymbol{A}\,. \tag{4.56}$$

But the derivative of momentum with respect to time is equal to the force acting on the particle. In the electromagnetic field, it must be the Lorentz force (4.41). Comparison of (4.56) with (4.41) makes it possible to express the fields $\boldsymbol{E}$ and $\boldsymbol{H}$ through the electromagnetic potentials:

$$\boldsymbol{E} = -\nabla\varphi - \frac{1}{c}\frac{\partial\boldsymbol{A}}{\partial t}\,, \quad \boldsymbol{H} = \nabla\times\boldsymbol{A}\,. \tag{4.57}$$

The equation of motion of a particle assumes the Newtonian form

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = e\boldsymbol{E} + \frac{e}{c}\boldsymbol{v}\times\boldsymbol{H}\,, \tag{4.58}$$

but with the relativistic velocity dependence of momentum (3.33) and the Lorentz force in the right-hand side. The time component of (4.53) describes a change of the particle's total energy:

$$\frac{d\mathcal{E}}{dt} = e\boldsymbol{E}\cdot\boldsymbol{v}\,. \tag{4.59}$$

This equation is not independent; it can be obtained by multiplying scalarwise both parts of (4.58) and $\boldsymbol{v}$. In this case, $\boldsymbol{v}\cdot d\boldsymbol{p} = d\mathcal{E}$. The magnetic field performs no work over a particle because the magnetic force is always perpendicular to velocity. □

Example 4.6

Let the equation of motion of a relativistic particle have the form (cf. (4.58))

$$\frac{d\boldsymbol{p}}{dt} = \boldsymbol{\mathcal{F}}\,,$$

where $\boldsymbol{\mathcal{F}}$ is a force of any nature (not necessarily electromagnetic). Write this equation in the relativistic covariant form introducing the 4-vector force for an arbitrary case.

Solution. Introduce the proper time differential $d\tau = \gamma^{-1}dt$ and write the equation of motion so that the left-hand side contains the 4-vector:

$$\frac{dp^\alpha}{d\tau} = \gamma\mathcal{F}^\alpha\,, \quad \frac{dp^0}{d\tau} = \frac{1}{c}\frac{d\mathcal{E}}{d\tau} = \frac{1}{c}\frac{d\mathcal{E}}{d\boldsymbol{p}}\cdot\frac{d\boldsymbol{p}}{d\tau} = \frac{\gamma}{c}\boldsymbol{\mathcal{F}}\cdot\boldsymbol{v}$$

(the second equation, like (4.59), is not independent). The totality of the quantities on the right-hand sides of these equalities makes up the 4-vector force:

$$\mathcal{F}^k = \left(\gamma\boldsymbol{\mathcal{F}}\cdot\frac{\boldsymbol{v}}{c}, \gamma\boldsymbol{\mathcal{F}}\right)\,, \tag{4.60}$$

where $\boldsymbol{\mathcal{F}}$ is the three-dimensional force acting on a relativistic particle. It can be expressed through the force applied to the particle in its rest system (see Problem 4.42). □

Example 4.7

Using relation (4.51), find the Hamiltonian function of a charged particle and write its equation of motion in the form of the Hamilton–Jacobi[3] equation.

Solution. We relate the action to a real physical trajectory, but regard it as being an ordinary function of the endpoint of the motion for 4-coordinates (point 2) using

3) Carl Gustav Jacob Jacobi (1804–1851) was a well-known German mathematician.

point 1 as the fixed point. This approach differs from that used in Example 4.5, where the action was considered for different arbitrarily chosen trajectories of the particle (and the one corresponding to the real motion was determined from the stationarity condition). At the physical path, the integral term on the right-hand side of (4.51) (action variation in the case of fixed endpoints) vanishes. We also have $\delta x^i|_{(1)} = 0$, because point 1 is fixed. Thus, $\delta S = -(mu_i + (e/c)A_i)\,\delta x^i$, whence $\partial_i S = -mu_i - (e/c)A_i$.

It is known from classical Hamiltonian mechanics that derivative actions with respect to spatial coordinates are equal to the components of the generalized momentum $\boldsymbol{P}$ and the time derivative taken with the opposite sign is the Hamiltonian function of the particle, $\mathcal{H}$:

$$\mathcal{H} = -\frac{\partial S}{\partial t}\,, \quad \boldsymbol{P} = \nabla S\,. \tag{4.61}$$

Let us name the quantity

$$P_i = -\frac{\partial S}{\partial x^i} = mu_i + \frac{e}{c}A_i \tag{4.62}$$

the covariant *generalized 4-momentum*. It is more convenient to use its contravariant components:

$$P^i = \left(\frac{\mathcal{E} + e\varphi}{c}\,, \quad \boldsymbol{p} + \frac{e}{c}\boldsymbol{A}\right)\,, \tag{4.63}$$

where $\boldsymbol{p}$ is the usual three-dimensional momentum (3.33), $\mathcal{E}$ is the total energy of the free particle, and $\mathcal{E} + e\varphi = \mathcal{H}$ is the Hamiltonian function. As known from mechanics, it must be expressed through generalized coordinates (in our case $\boldsymbol{r} = (x, y, z)$) and generalized momentum $\boldsymbol{P}$. We use (4.63) to find

$$\boldsymbol{p} = \boldsymbol{P} - \frac{e}{c}\boldsymbol{A} \tag{4.64}$$

and the second formula in (3.33) to find the *Hamiltonian function* of a relativistic particle:

$$\mathcal{H}(\boldsymbol{r}, \boldsymbol{P}, t) = \sqrt{m^2c^4 + c^2\left(\boldsymbol{P} - \frac{e}{c}\boldsymbol{A}(\boldsymbol{r}, t)\right)^2} + e\varphi(\boldsymbol{r}, t)\,. \tag{4.65}$$

In the nonrelativistic approximation, we find from (4.65)

$$\mathcal{H}(\boldsymbol{r}, \boldsymbol{P}, t) = \frac{1}{1m}\left(\boldsymbol{P} - \frac{e}{c}\boldsymbol{A}(\boldsymbol{r}, t)\right)^2 + e\varphi(\boldsymbol{r}, t) \tag{4.65$'$}$$

(constant mc^2 is omitted). Now, it is easy to write the Hamilton–Jacobi equation. To this effect, we use equalities (4.61) and substitute the Hamiltonian function (4.65) into the first of them after the replacement of $\boldsymbol{P}$ by ∇S:

$$\frac{\partial S}{\partial t} + \sqrt{m^2c^4 + c^2\left(\nabla S - \frac{e}{c}\boldsymbol{A}\right)^2} + e\varphi = 0\,. \tag{4.66}$$

A representation without the radical sign appears more convenient:

$$\left(\nabla S - \frac{e}{c}\boldsymbol{A}\right)^2 - \frac{1}{c^2}\left(\frac{\partial S}{\partial t} + e\varphi\right)^2 + m^2c^2 = 0\,. \tag{4.67}$$

The last two equalities are different forms of the representation of the *Hamilton–Jacobi equation* for a relativistic particle. In this method, the unknown quantity found from the above equations is the function $S(\boldsymbol{r}, t)$. □

4.2.3
Transformations of Electromagnetic Field Stress

Example 4.8

Draw tables relating components F^{ik} and F_{ik} to the intensities $\boldsymbol{E}$ and $\boldsymbol{H}$ on the basis of the definition of the electromagnetic field tensor (4.52) and relations (4.55) and (4.57). Write also the formulas for relativistic transformation of field vectors $\boldsymbol{E}$ and $\boldsymbol{H}$.

Solution.

$$F_{ik} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -H_z & H_y \\ -E_y & H_z & 0 & -H_x \\ -E_z & -H_y & H_x & 0 \end{pmatrix},$$

$$F^{ik} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -H_z & H_y \\ E_y & H_z & 0 & -H_x \\ E_z & -H_y & H_x & 0 \end{pmatrix}. \tag{4.68}$$

It is useful to compare these two tables with the general structure of the rank 2 antisymmetric tensor considered in Problem 4.11*. It follows from the comparison and from formulas (4.54) that $\boldsymbol{E}$ is a true vector and $\boldsymbol{H}$ is a pseudovector with respect to inversion of spatial axes.

Transformation into another frame of reference is performed in accordance with the general rule of type (4.6): $F_{ik} = \Lambda_i{}^l \Lambda_k{}^n F'_{ln}$, where matrix $\Lambda_i{}^l$ is given by equality (4.5). This leads to

$$F_{01} = \Lambda_0{}^0\Lambda_1{}^1 F'_{01} + \Lambda_0{}^1\Lambda_1{}^0 F'_{10} = (\cosh^2\psi - \sinh^2\psi)F'_{01} = F'_{01}\,,$$

$$F_{02} = \Lambda_0{}^0\Lambda_2{}^2 F'_{02} + \Lambda_0{}^1\Lambda_2{}^2 F'_{12} = \cosh\psi\, F'_{02} - \sinh\psi\, F'_{12}\,,$$

and so on. The use of tables (4.68) and (4.5) yields

$$\begin{aligned} E_x &= E'_x\,, \quad E_y = \gamma(E'_y + \beta H'_z), E_z = \gamma(E'_z - \beta H'_y)\,, \\ H_x &= H'_x\,, \quad H_y = \gamma(H'_y - \beta E'_z), H_z = \gamma(H'_z + \beta E'_y)\,. \end{aligned} \tag{4.69}$$

Here, the Ox axis is directed along the relative velocity $\boldsymbol{V}$ and the Oy and Oz axes are perpendicular to Ox. It is therefore easy to write (4.69) for an arbitrary direction of $\boldsymbol{V}$:

$$\begin{aligned} \boldsymbol{E}'_{\parallel} &= \boldsymbol{E}_{\parallel}\,, \quad \boldsymbol{E}'_{\perp} = \gamma(\boldsymbol{E}_{\perp} + \boldsymbol{V}\times\boldsymbol{H}/c)\,, \\ \boldsymbol{H}'_{\parallel} &= \boldsymbol{H}_{\parallel}\,, \quad \boldsymbol{H}'_{\perp} = \gamma(\boldsymbol{H}_{\perp} - \boldsymbol{V}\times\boldsymbol{E}/c)\,. \end{aligned} \tag{4.70}$$

The symbols $\parallel$ and $\perp$ label the constituents parallel and perpendicular to $\boldsymbol{V}$. At $V \ll c$, it should be assumed, with an accuracy of up to the linear terms, that $\gamma = 1$ in (4.69) and (4.70). □

Example 4.9

Construct all possible independent relativistic invariants from the components of the electromagnetic field tensor.

Solution. The spatial turns constitute part of the transformations referred to as the group of general Lorentz transformations; therefore, the number of Lorentz invariants cannot be greater than the number of invariants with respect to three-dimensional rotation. The two 3-vectors, $\boldsymbol{E}$ and $\boldsymbol{H}$, entering the electromagnetic field vector F_{ik}, give rise to only two true scalars, E^2 and H^2, and one pseudoscalar, $\boldsymbol{E}\cdot\boldsymbol{H}$. The latter is invariant with respect to the proper Lorentz transformations as follows from the tensor form of the representation (see (4.37)):

$$e^{iklm}F_{ik}F_{lm} = -8\boldsymbol{E}\cdot\boldsymbol{H}\,. \tag{4.71}$$

Under the axes inversion invariant is $(\boldsymbol{E}\cdot\boldsymbol{H})^2$.

$\boldsymbol{E}^2$ and $\boldsymbol{H}^2$ are not Lorentz invariants, but their difference is a true scalar:

$$2(\boldsymbol{H}^2 - \boldsymbol{E}^2) = F_{ik}F^{ik}\,. \tag{4.72}$$

This exhausts the list of independent invariants of the electromagnetic field tensor. □

Example 4.10

Consider transformation of the 4-potential consisting of its supplementation by a certain 4-vector f_i: $\tilde{A}_i = A_i + f_i$. What property must vector f_i have in order for the transformation not to change the components of the electromagnetic field tensor, that is, the strengths $\boldsymbol{E}$ and $\boldsymbol{H}$?

Solution. It follows from the conditions $\tilde{F}_{ik} = \partial_k\tilde{A}_i - \partial_i\tilde{A}_k = \partial_k A_i - \partial_i A_k = F_{ik}$ that $\partial_k f_i = \partial_i f_k$, where f is an arbitrary scalar function of 4-coordinates. The transformation of the 4-potential

$$\tilde{A}_i = A_i + \partial_i f \tag{4.73}$$

that leaves invariant the observed quantities (field strengths) is called the *gauge* (gradient) transformation. It is related to the simple mathematical fact of the zero value of the rotor of the scalar function gradient occurring in both three-dimensional and four-dimensional Euclidean spaces. In other words, the electromagnetic potentials are defined nonuniquely.

In the three-dimensional representation, the gauge transformation of potentials has the form that was already found in Chapter 2 (see (2.103), (2.104)):

$$\tilde{\varphi} = \varphi + \frac{1}{c}\frac{\partial f}{\partial t}\,, \quad \tilde{\boldsymbol{A}} = \boldsymbol{A} - \nabla f\,. \tag{4.74}$$

The observed and measured quantities in classical electrodynamics are the force (4.41) acting on a charged particle or changes of its energy (4.59) as well as the energy and the energy flux of the electromagnetic field itself. All these quantities are expressed through the strengths $\boldsymbol{E}$ and $\boldsymbol{H}$, which can be found in such a manner from experiment. The potentials $\boldsymbol{A}$ and φ play the role of an auxiliary function, from which the strengths are derived by differentiation with respect to formulas (4.57). The theory of the electromagnetic field permitting one to calculate tensor F_{ik} must be gauge invariant, that is, its corollaries must be independent of potential variations caused by transformation (4.73). This requirement has important implications not only for classical electrodynamics but also for the elaboration of modern theories of quantum fields and elementary particles. □

Recommended literature:
Landau and Lifshitz (1976); Goldstein (1950); Yzerman (1974); Landau and Lifshitz (1975); Bredov *et al*. (2003); Fock (1955); Pauli (1921); Frenkel (1926); Möller (1972)

Problems

4.26. Write in the form of tables tensors $\tilde{F}_{ik}$ and $\tilde{F}^{ik}$ dual to the electromagnetic field tensor through the strengths $\boldsymbol{E}$ and $\boldsymbol{H}$.

4.27. Write invariants of the electromagnetic field with the use of a tensor dual to tensor F_{ik}.

4.28*. System S contains a uniform electromagnetic field $\boldsymbol{E}$, $\boldsymbol{H}$. Find all possible inertial frames of reference in which the field has one of the following properties:

1. The strengths become either parallel, $\boldsymbol{E}' \parallel \boldsymbol{H}'$, or antiparallel.
2. One of the strengths vanishes, either $\boldsymbol{E}' = 0$ or $\boldsymbol{H}' = 0$.
3. The strengths become mutually perpendicular, $\boldsymbol{E}' \perp \boldsymbol{H}'$.
4. The absolute values of the strengths become equal, $E' = H'$.
5. Both strengths vanish, $E' = H' = 0$. At what values of the initial fields, $\boldsymbol{E}$ and $\boldsymbol{H}$, are these cases realized?

4.29. An infinitely long circular cylinder is uniformly charged with a linear density κ. A uniformly distributed current J flows along the cylinder axis. Find the reference system in which only the electric or only the magnetic field exists. Find the strength of these fields.

4.30•. Write the Maxwell equations (2.82)–(2.85) and the differential law of conservation of electric charge (2.47) in the covariant four-dimensional form through the electromagnetic field tensor (4.67).

4.31*. The system of differential equations for the magnetic lines of force of the form

$$\mathrm{d}\boldsymbol{r} \times \boldsymbol{H} = 0 \tag{1}$$

is not relativistically invariant and does not retain its form on transition to another inertial frame of reference.

1. Show that the system of equations for the field of a certain special form

$$\mathrm{d}\boldsymbol{r} \times \boldsymbol{H} + c\boldsymbol{E}\mathrm{d}t = 0\,, \quad \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{r} = 0 \tag{2}$$

   can be regarded as the relativistically invariant generalization of system (1).
2. Elucidate the structure of the fields for which such a generalization is possible by considering the consistency conditions for equations (2). How many independent equations does system (2) contain?
3. What is the form of the integrability condition for system (2)?
4. Make sure that the lines of force given by system (2) move in the transverse direction at speed $\boldsymbol{u} = c\boldsymbol{E} \times \boldsymbol{H}/H^2$, that is, remain in motion even in static fields.

4.32*. Show that the relativistically invariant system of equations for the electrical lines of force analogous to system (2) in the previous problem has the form

$$e_{iklm} F^{lm} \mathrm{d}x^k = 0\,. \tag{1}$$

What requirements are imposed on $\boldsymbol{E}$ and $\boldsymbol{H}$, the distribution of charges, and the distribution of currents by the compatibility and integrability conditions for system (1)? How do the lines of force determined by system (1) move?

4.33. Find the electromotive force of electromagnetic induction arising from the motion of a conductor in the magnetic field $\boldsymbol{H}$. Use the formulas for field strength transformation or for potential transformation.

4.34•. Find the fields φ, $\boldsymbol{A}$, $\boldsymbol{E}$, and $\boldsymbol{H}$ of the point charge e moving uniformly with velocity $\boldsymbol{V}$ by performing the Lorentz transformation from the reference frame in which the charge is at rest. Write the 4-potential in the explicit relativistically covariant form.

Hint: To write the 4-potential in the covariant form it should be expressed through the 4-velocity of the particle and the 4-radius vector relating the following two events: the observation of the field at the 3-point, $\boldsymbol{r}$, at moment t and its generation by the charge at the 3-point, $\boldsymbol{s}(t')$, at the preceding moment of time, t'.

4.35. Show that the electric field of a uniformly moving point charge "flattens" in the direction of motion. This results in the weakening of field E along the line of charge propagation compared with the Coulomb field. How does this weakening agree with the transformation formula $E_{\|} = E'_{\|}$?

4.36•. An electric dipole with moment $\boldsymbol{p}_0$ in the accompanying system moves with velocity $\boldsymbol{V}$. Find the electric field φ, $\boldsymbol{A}$, $\boldsymbol{E}$, $\boldsymbol{H}$ it creates.

4.37•. Charged particles perform a nonrelativistic periodic motion or remain at rest in a certain system of coordinates, thus creating dipole electric and magnetic moments $\boldsymbol{p}_0$ and $\boldsymbol{m}_0$. Derive the rule for the transformation of moments during transition to an arbitrary inertial frame of reference.

Hint: Represent the totality of dipole moments as the integral over the three-dimensional volume from a certain covariant "moment density" and make use of the Lorentz transformation.

4.38. An uncharged wire circuit shaped as a rectangle with sides a and b carries current J' and moves uniformly with velocity $\boldsymbol{V}$ parallel to its a side. The wire has a finite cross-section. Find the distribution of electric charges over the loop and its electric and magnetic moments observed in the laboratory frame of reference.

4.39•. Derive the equations of motion for a relativistic particle (4.58) using the Lagrangian function (4.46).

4.40•. Do the same using the Hamiltonian function (4.65).

4.41. Write the relativistic equation of motion of a particle under the action of force $\boldsymbol{\mathcal{F}}$ expressing the momentum explicitly through the particle's velocity $\boldsymbol{v}$. Consider, in particular, (i) the case in which only the magnitude of the velocity varies, (ii) the case in which only the direction of the velocity varies, and (iii) the case in which $v \ll c$.

4.42. Express through each other the 3-vector forces acting on a particle in the laboratory reference frame ($\boldsymbol{\mathcal{F}}$) and in the rest system ($\boldsymbol{\mathcal{F}}'$). The particle's velocity is $\boldsymbol{v}$. Verify the formulas thus obtained by applying them to the Lorentz force.

4.43. What force $\mathcal{F}$ acts from the observer's viewpoint in an instantaneously accompanying system on a body with mass m aboard a rocket and motionless relative to it if the rocket moves with relativistic velocity v over the circular orbit of radius R?

4.44. Two particles with charges e_1 and e_2 propagate parallel to the x axis at equal constant velocities v. Find the force of interaction between the particles in the laboratory frame of reference. Consider, in particular, the ultrarelativistic limit. Show that the force thus found can be calculated using the formula $\boldsymbol{\mathcal{F}} = -e_2 \nabla \psi$ from the so-called convective[4)] potential

$$\psi = e_1/\gamma^2 R\,, \quad \text{where}$$
$$R = \sqrt{(x_1 - x_2)^2 + (1 - \beta^2)\left[(y_1 - y_2)^2 + (z_1 - z_2)^2\right]}\,,$$

where $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are the radius vectors of the charges.

Hint: It is convenient to proceed from the Coulomb force in the accompanying system and transform it using the formulas from Problem 4.42.

4.45. Find the convective potential ψ of an infinitely long uniformly charged wire. The linear charge density in the frame of reference where the wire rests is κ. The wire moves translationally with velocity v at an angle α to its length (in the laboratory frame of reference). Consider, in particular, the cases $\alpha = 0$ and $\alpha = \pi/2$.

4.46. An infinitely long uniformly charged thread having linear charge density κ in the frame of reference where the thread rests moves uniformly along its length with velocity v. A point charge moves with the same velocity parallel to the straight line at distance r from the thread. Find the electromagnetic force $\mathcal{F}$ acting on the charge.

4.47. The distribution of electrons in a parallel beam exhibits axial symmetry and is characterized by the volume charge density ρ in the reference frame connected with electrons. The electrons are accelerated by the potential difference V. The total current in the beam is J. Find the electromagnetic force $\mathcal{F}$ applied to one of the electrons in the beam in the laboratory frame of reference.

Hint: It would be helpful to use the result from the previous problem.

4.48. Find the broadening Δa of the electron beam considered in the preceding problem along the path L due to mutual repulsion between the electrons. The circular cross-section of the beam has radius a. Assume the broadening to be small ($\Delta a \ll L$).

4.49*. A particle with mass m and charge e propagates at an arbitrary speed in a uniform constant electric field E. At starting moment $t = 0$, the particle was at the origin of the coordinates and had momentum $\boldsymbol{p}_0$. Determine the three-dimensional coordinates and time t as well as the energy of the particle in the laboratory frame of reference as a function of its proper time τ. Exclude τ and represent the three-dimensional coordinates depending on t. Consider, in particular, the nonrelativistic and relativistic limits.

4.50. Find in the analytical form the trajectory of the charged particle considered in the preceding problem. Draw trajectories for different initial conditions using a computer. Specifically, study the nonrelativistic and relativistic cases.

4) The convective potential of a system of charges moving as a whole is the coordinate function for which differentiation yields the components of the force acting in the laboratory frame of reference on a unit test charge traveling together with the charge system.

4.51. Find the path l of a relativistic charged particle with mass m, charge e, and initial energy $\mathcal{E}$ in a retarding uniform electric field $\boldsymbol{E}$ parallel to the particle's initial velocity.

4.52*. A relativistic particle with mass m and charge e propagates in a uniform constant magnetic field $\boldsymbol{H}$. At starting moment $t = 0$, the particle was at the point with radius vector $\boldsymbol{r}_0$ and had momentum $\boldsymbol{p}_0$. Calculate the momentum and energy as well as the coordinates of the particle as a function of its proper time and coordinate time.

4.53*. A nonrelativistic particle with mass m and charge e propagates in crossed uniform constant electric $\boldsymbol{E} = (0, E_y, E_z)$ and magnetic $\boldsymbol{H} = (0, 0, H)$ fields. At starting moment $t = 0$, the particle was at the origin of the coordinates and had velocity $\boldsymbol{v} = (v_{ox}, 0, v_{oz})$. Determine the time dependence of the velocity and coordinate components, and draw possible trajectories of the particle using a computer.

4.54. A relativistic particle propagates in parallel uniform constant electric $\boldsymbol{E}$ and magnetic $\boldsymbol{H}$ fields ($\boldsymbol{E} \parallel \boldsymbol{H} \parallel Oz$). At starting moment $t = 0$, the particle was at the origin of the coordinates and had momentum $\boldsymbol{p}_0 = (p_{ox}, 0, p_{oz})$. Determine the dependence of the momentum and energy components on the particle's proper time τ. Draw projections of the particle path on the coordinate planes using a computer.

4.55. Find the proper time dependence of the momentum and energy components for a relativistic particle propagating in mutually perpendicular uniform constant electric $\boldsymbol{E}$ and magnetic $\boldsymbol{H}$ fields ($H > E$). The initial momentum of the particle is $\boldsymbol{p}_0$, $x = y = z = 0$, at $t = 0$.

4.56. Solve the preceding problem for $E > H$. Study by means of the limiting transition $E \to H$ the particle's motion in mutually perpendicular electric and magnetic fields of equal absolute value.

4.57*. Find the law for the motion of a particle with charge e and mass m in the field of a plane electromagnetic wave having a four-dimensional potential of the form

$$A^i = \varepsilon^i f(s) \,, \quad \varepsilon^i = \text{const}$$

(a linearly polarized wave). Here, $f(s)$ is an arbitrary twice-differentiable function whose argument $s = n_l x^1$ is expressed through the zero 4-vector $n_1 = (1, -\boldsymbol{n})$ showing the direction of propagation of a plane wave ($n^l n_l = 0$, $\boldsymbol{n}^2 = 1$), and e^i is the four-dimensional space-like polarization vector normalized by the condition $e^i e_i = -1$ and orthogonal to the wave vector ($n^l e_l = 0$). The initial conditions of the general form are specified. Find also the changes of the particle's energy within a finite time. Consider, in particular, the cases of a periodic plane wave and a wave packet of finite length.

Hint: Show that the particle's proper time is proportional to argument s of the plane wave and use the result for the solution of the problem.

4.58*. Solve the preceding problem for a plane wave with arbitrary polarization. The four-dimensional potential $A^i(s)$ satisfies the condition $n_i A^i(s) = 0$. Do the same for the three-dimensional form expressing explicitly both the coordinates and the time through field intensities $\boldsymbol{E}(s)$ and $\boldsymbol{H} = \boldsymbol{n} \times \boldsymbol{E}(s)$.

4.59. A nonrelativistic charged particle with charge e and mass m passes through a two-dimensional electric field having the potential $\phi = k(x^2 - y^2)$, where $k =$ const > 0 (a strongly focusing lens). At time $t = 0$, the particle is at the point with coordinates (x_0, y_0, z_0) and the initial velocity v_0 is parallel to the z axis. Determine the particle's motion.

4.60. Write the differential equations of motion of a relativistic particle in an electromagnetic field proceeding from the Lagrangian function in cylindrical coordinates.

4.61*. A potential difference V is maintained between the plates of a cylindrical capacitor with radii a and b $(a < b)$. An axially symmetric magnetic field is enclosed in the space between the plates, the strength of which is parallel to the condenser axis. Electrons with zero initial velocity are emitted from the inner plate, which serves as the cathode. Find the critical value of the magnetic field flux Φ_{cr} between the plates at which the electrons do not reach the anode any longer owing to the bending of their paths in the magnetic field.

4.62. A long straight cylindrical cathode of radius a in which a uniformly distributed current $\mathcal{J}$ flows emits electrons with zero initial velocity. These electrons propagate under the action of the accelerating potential V toward the long coaxial anode of radius b. What must the minimum value of the potential difference V between the cathode and anode be in order for the electrons to reach the anode despite the bending of their trajectories by the magnetic field of current $\mathcal{J}$?

4.63. Current $\mathcal{J}$ flows in a long straight cylindrical wire of radius a. An electron breaks loose from the surface of the wire with initial velocity v_0 in the direction along the conductor. Find the maximum distance b that the electron can move away from the wire axis.

4.64. Solve Problem 4.62 using the Lorentz transformation in a frame of reference containing a single field ($\boldsymbol{E}$ or $\boldsymbol{H}$). Such a transformation was considered in Problem 4.29.

4.65•. Charged particles perform nonrelativistic motion in a bounded region of space.

1. Find the relationship between their kinetic and potential energies averaged over a long period of time.
2. Do the same assuming the presence of a uniform magnetic field $\boldsymbol{H}$ under the condition that $e_a/m_a = e/m$, where e_a and m_a are the charge and the mass of an individual particle, respectively.

4.66. Find the trajectory of the relative motion of nonrelativistic particles with charges e and e' and masses m_1 and m_2. Consider the solution in terms of the initial conditions given by the total energy of interacting nonrelativistic particles $\mathcal{E}$ and the angular momentum $\boldsymbol{l}$ of their relative motion.

4.67*. Find the differential scattering cross-section $\sigma(\theta)$ for nonrelativistic particles with charge e in the field of a motionless point charge e'. The velocity of the particles far from the motionless scattering center is v_0.

4.68. Find the differential scattering cross-section for charged particles scattered from identical particles. Write the differential cross-section through the scattering angle and energy determined in the center-of-mass system and in the laboratory frame of reference where the particles of the target are initially at rest.

4.69. A motionless particle having charge e' is bombarded by a limited stationary flux of nonrelativistic particles with charges e, masses m, and velocities v (Figure 4.1).

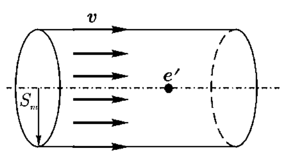

Figure 4.1 Interaction of a particle at rest with a stationary flux of particles.

The particle number density in the flux is n. Calculate the force acting on the motionless particle, disregarding the interactions between the projectile particles. Explain why this force becomes infinite at the beam radius $s_m \to \infty$. Does the force remain infinite if charge e' is one of the charges of a neutral system (neutral atom, plasma)?

4.70. A "test" particle having charge e and mass m propagates with velocity v in a gas composed of identical charged particles Their masses m', charges e', number density n', and velocity distribution are described by the function $f(\boldsymbol{v})$ ($\int f(\boldsymbol{v})\mathrm{d}^3v = n'$). Write the expression for the average force $\bar{\boldsymbol{\mathcal{F}}}(\boldsymbol{v})$ acting on the "test" particle.

Hint: Use the result obtained in the solution of the preceding problem. The velocity dependence of Coulomb logarithm can be disregarded.

4.71. A test particle having charge e, mass m, and initial velocity v_0 propagates in a medium composed of chaotically distributed motionless infinitely heavy identical particles with charges e' and number density n. How do the particle's energy and momentum vary with time under the action of the average-over-collisions force exerted by the medium?

4.72. Particles of a medium with charges e and masses m have equal absolute velocities v_0 characterized by a spherically symmetric distribution. Calculate the average force $\bar{\mathcal{F}}$ acting on a test particle with charge e' and mass m' that propagates with velocity $\boldsymbol{v}$.

Do the same for the case of particles propagating with velocities $\boldsymbol{v}_0$ of similar value and in a similar direction.

4.73*. Electrons perform chaotic thermal movements in the plasma and, besides, have an ordered velocity constituent arising under the action of a uniform electric field $\boldsymbol{E}$ created by an external source. Write the ordered estimation of the dependence of the average frictional force $\bar{\mathcal{F}}$ on the ordered speed $\boldsymbol{u}$ on the assumption that friction is due to collisions with motionless ions. Show that $\bar{\mathcal{F}}$ as a function of u has a maximum and estimate $\bar{\mathcal{F}}_{\text{max}}$. How will the electron gas behave under the effect of an electric field E if $E < \bar{\mathcal{F}}_{\text{max}}/e$ and $E > \bar{\mathcal{F}}_{\text{max}}/e$?

4.74*. A relativistic particle having charge $-e$ and mass m propagates in the field of a stationary point charge Ze. Find the equation for the particle trajectory. Consider all possible cases of the motion depending on the initial conditions given by the total energy of the particle, $\mathcal{E}$, and its angular momentum, $\boldsymbol{l}$.

4.75*. Study the process of incidence of a relativistic particle on the center of a Coulomb attraction field (see the previous problem, the case of $Ze^2 > lc$): (i) calculate the time Δt of the incidence from distance r; (ii) show that the particle's velocity tends toward c at $r \to 0$.

4.76*. A relativistic particle having charge e and mass m propagates in the field of an analogous stationary point charge Ze. Find the equation for the particle trajectory. Consider all possible cases of the motion depending on the initial conditions given by the total energy of the particle, $\mathcal{E}$, and its angular momentum, $\boldsymbol{l}$.

4.77. Calculate the deviation angle θ of a relativistic particle having charge $\pm e$, energy $\mathcal{E} > mc^2$, and angular momentum $l > Ze^2/c$ that propagates in the Coulomb field of a stationary charge Ze.

4.78. A relativistic particle having charge $\pm e$, mass m, and velocity at infinity v_0 undergoes small-angle scattering by the Coulomb field of a stationary charge Ze. Calculate the differential scattering cross-section $\mathrm{d}\sigma(\theta)/\mathrm{d}\Omega$.

4.79*. The magnetic field grows steadily as an electron accelerates in a betatron, giving rise to induction electromotive force that accelerates the electron but its orbit remains unaltered. Prove that the acceleration of the electron is possible on the condition that the total magnetic flux Φ across the orbit is twice the Φ_0 flux that would have been formed if the field enclosed by the orbit had been uniform and equal to the field at the orbit (the "2 : 1" betatron rule).

4.80*. Show that the energy of retarded interaction between two charged particles has, with an accuracy of up to the v^2/c^2 terms, the form[5)]

$$U(t) = \frac{e_1 e_2}{R}\left\{1 - \frac{1}{2c^2}[\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 + (\boldsymbol{v}_1 \cdot \boldsymbol{n})(\boldsymbol{v}_2 \cdot \boldsymbol{n})]\right\} , \tag{4.75}$$

where $\boldsymbol{R}$ is the radius vector with respect to the particles' position, $\boldsymbol{n} = \boldsymbol{R}/R$, and $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ and are the particles' velocities. All the quantities on the right-hand side of the equality are taken at moment t.

Hint: Use the expansions of Lienard–Wiechert[6)] potentials (see Chapter 5) as a power series in the retardation time taking account of only those terms that do not depend on accelerations and their derivatives. Perform gauge transformation of the potentials so that the scalar potential assumes the form of the Coulomb potential.

4.81. Find the approximate expression of the Lagrangian function for two interacting particles with charges e_1 and e_2 and masses m_1 and m_2 taking into consideration the retardation effect with an accuracy of up to correction terms on the order of v^2/c^2.

4.2.4
Dynamics of Orbital and Spin Magnetic Moments

As a charged particle propagates in a centrally symmetric electric field it retains its angular momentum with respect to the center of symmetry, $\boldsymbol{l} = r_0(t) \times \boldsymbol{p}(t)$, where $\boldsymbol{r}_0(t)$ is the radius vector of the particle. This accounts for the conservation of the magnetic moment $\boldsymbol{m}_l$ created by the orbital motion of the particle (*the orbital magnetic moment*):

$$\boldsymbol{m}_l = \frac{1}{2c}\int \boldsymbol{r} \times \boldsymbol{j}\,\mathrm{d}^3 r = \eta \boldsymbol{l} , \tag{4.76}$$

where the expression $\boldsymbol{j} = e\boldsymbol{v}\delta(\boldsymbol{r} - \boldsymbol{r}_0(t))$ for the current generated by a traveling point particle is substituted into the integral, and the coefficient of proportionality $\eta = ec/2\mathcal{E}$ is called the gyromagnetic ratio. For a nonrelativistic particle, $\eta = \eta_0 = e/2mc$ depends on the charge-to-mass ratio (see Problem 2.88).

Example 4.11

Derive the equation describing the motion of the orbital magnetic moment in the external magnetic field $\boldsymbol{H}$.

5) This expression is known as the Breit formula. An analogous expression is used in the approximate quantum description of retarded interaction. Gregory Breit (1899–1981) was an American physicist who worked in the fields of nuclear physics, quantum mechanics and electrodynamics, and high-energy physics.

6) Emil Wiechert (1861–1928) was a German physicist and geophysicist. Philip Lienard (1862–1947) was a German physicist and Nobel Prize recipient. He was an active antagonist of the theory of relativity.

Solution. A magnetic dipole in the external field undergoes the action of the moment of force $\boldsymbol{N} = \boldsymbol{m}_l \times \boldsymbol{H}$. The change in the angular momentum is given by the equation from mechanics

$$\frac{\mathrm{d}\boldsymbol{l}}{\mathrm{d}t} = \boldsymbol{N} = \boldsymbol{m}_l \times \boldsymbol{H} . \tag{4.77}$$

It follows from (4.76) that $\dot{\boldsymbol{m}}_l = \eta\dot{\boldsymbol{l}} - (\dot{\mathcal{E}}/\mathcal{E})\eta\boldsymbol{l}$. Owing to the conservation of the particle's energy in a constant magnetic field, $\dot{\mathcal{E}} = 0$, the equation of motion of the orbital magnetic moment assumes the form

$$\frac{\mathrm{d}\boldsymbol{m}_l}{\mathrm{d}t} = \eta\boldsymbol{m}_l \times \boldsymbol{H} . \tag{4.78}$$

□

Most elementary particles (and also compound microscopic particles and macroscopic bodies) possess, besides orbital mechanical and magnetic moments, intrinsic (spin) mechanical $\boldsymbol{s}$ and magnetic $\boldsymbol{m}$ moments, with the coefficient of proportionality between them being different for different particles:

$$\boldsymbol{m}_s = g\eta_0\boldsymbol{s} . \tag{4.79}$$

For an electron, $g_{\mathrm{e}} \approx 2(1 + \alpha/2\pi)$, where $\alpha = e^2/\hbar c \approx \frac{1}{137}$ is the fine structure constant; for a proton, $g_{\mathrm{p}} \approx 5.59$, and the value of $\eta_0 = e/2m_{\mathrm{p}}c$ is determined by the proton's charge and mass. It is roughly three orders of magnitude smaller than for the electron. A change in the spin mechanical moment with time[7] is described by an equation analogous to (4.77):

$$\frac{\mathrm{d}\boldsymbol{s}}{\mathrm{d}t} = \boldsymbol{m}_s \times \boldsymbol{H} . \tag{4.80}$$

The neutron has no electric charge but possesses a spin magnetic moment (($g_n\eta_0\boldsymbol{s}$, where $g_n \approx -3.83$). Spin quantization ($\boldsymbol{s} \cdot \boldsymbol{H}/H = \pm\hbar/2$, where $\hbar$ is the reduced Planck constant) orients the neutron's magnetic moment in the external magnetic field $\boldsymbol{H}(\boldsymbol{r})$ in only two ways, either along or against the field. The initial orientation is preserved if the condition of gradual alteration of the field or its adiabaticity is fulfilled; this condition consists of the low field rotation rate in the neutron rest system compared with the spin precession frequency, $\omega_{\mathrm{L}} = 2|\boldsymbol{m}_s|H/\hbar$, in the magnetic field. In this case, the motion of neutrons with a magnetic moment oriented either along or against the field can be regarded as the motion of a classical particle in a force field with potential energy

$$U(\boldsymbol{r}) = \mp|\boldsymbol{m}_s|H(\boldsymbol{r}) . \tag{4.81}$$

7) The spin moment has a quantum nature and its consistent description is achieved by the methods of quantum mechanics (see Chapters 6 and 7). In the formulas presented, $\boldsymbol{s}$ and $\boldsymbol{m}$ stand for the values of the spin moments of a microscopic particle averaged over the quantum mechanical state.

The energy U is usually very low; therefore, the magnetic field influences the motion of only very slow ("cold") neutrons.

Recommended literature:
Frenkel (1926); Berestetskii *et al.* (1982); Jackson (1999); Matyshev (2000)

Problems

4.82. Integrate the equation of motion of the magnetic moment in a magnetic field (4.78) and express the components $\boldsymbol{m}_l$ through it s starting value $\boldsymbol{m}_s$. What is the character of the motion of the magnetic moment?

4.83*. A particle with charge e and mass m having internal (spin) mechanical $\boldsymbol{s}$ and magnetic

$$\boldsymbol{m}_s = \frac{eg}{2mc}\boldsymbol{s}$$

moments performs nonrelativistic motion in the external electrostatic centrally symmetric field $\varphi(r)$. Calculate the interaction energy U between the spin and the external field in the first nonvanishing approximation in v/c taking into consideration the Thomas precession of the instantaneously accompanying system with angular velocity

$$\boldsymbol{\omega}_T = \frac{\dot{\boldsymbol{v}} \times \boldsymbol{v}}{2c^2} .$$

The origin of Thomas precession was clarified in Problem 3.26*.

Hint: The rates of alteration of the arbitrary vector $\boldsymbol{A}$ in the laboratory inertial and rotating systems of coordinates are related by the expression

$$\left(\frac{\mathrm{d}A}{\mathrm{d}t}\right)_{\text{lab}} = \left(\frac{\mathrm{d}A}{\mathrm{d}t}\right)_{\text{curl}} + \boldsymbol{\Omega} \times \boldsymbol{A} ,$$

where $\boldsymbol{\Omega}$ is the angular rotation rate (Landau and Lifshitz, 1976).

4.84*. Calculate the energy of interaction between spin $\boldsymbol{s}$ of a nonrelativistic nucleon and the nuclear force field commensurable with the potential energy $V(r)$. Disregard the action of weak electrostatic forces.

4.85**. Write the equation of motion for the spin moment of a particle (4.80) in the four-dimensional form holding for any invariant frame of reference.

Hint: Use the axial 4-vector S^k matching the three-dimensional vector of spin $\boldsymbol{s}$ in the particle's rest system. Use the particle's proper time as an independent variable.

4.86. Write the components of the 4-vector of spin S^k from the preceding problem in an arbitrary inertial frame of reference using the three-dimensional $\boldsymbol{s}$ determined in the particle's rest system.

4.87**.** Derive for a relativistic particle the equation of evolution of the three-dimensional vector $\boldsymbol{s}$ that characterizes the particle's spin in the accompanying reference frame. Consider spin motion in a constant uniform field for three principal cases: (i) the motion across the magnetic field, (ii) the motion along the magnetic field, and (iii) the arbitrary motion in an electric field.

Hint: Use the results from the preceding problem.

4.88. Make sure that the equation of spin motion (4.82) takes account of the Thomas precession considered in Problems 3.26* and 4.83*.

4.89. A neutron with magnetic moment $\boldsymbol{m}_n$ and kinetic energy T enters a magnetic field of strength $H = \text{const}$ having the plane boundary from a vacuum. Under what condition is the neutron reflected from the field?

4.90. Consider the possible trajectories of a cold neutron (mass M, magnetic moment $\boldsymbol{m}_n$) in the field of an infinitely long straight wire carrying current $\mathcal{J}$.

4.91. A flux of cold neutrons (velocity v_0, magnetic moment $\boldsymbol{m}_n$, mass M) is scattered from the magnetic field of an infinite straight wire with current $\mathcal{J}$. Calculate the differential transverse scattering length

$$l(\alpha) = \left|\frac{\mathrm{d}s}{\mathrm{d}\alpha}\right| ,$$

where $s(\alpha)$ is the impact parameter at which the neutron is scattered at angle α.

4.2.5
The Approximate Methods. Averaging over Rapid Movements

Most of the problems dealing with the motion of charged particles in nonuniform and alternating electric fields do not thus far have exact answers. Many approximate methods for the solution of such problems are based on the discrimination between "rapid" and "slow" movements and their sequential consideration. Let a particle be involved in a rapid quasi-periodic motion (e.g., rotation around the direction of a magnetic field or oscillations in a high-frequency wave) with a concomitant slower and regular alteration of its coordinates and energy associated with the slow alteration of the magnetic field or wave amplitude.

Then, it is possible to perform averaging over rapid movement that as a rule yields simpler equations approximately describing the smoothed motion of the particle at long enough times and distances. Such a method finds wide application in the studies of particle motion in weakly nonuniform electromagnetic fields slowly varying in time at $H > E$. In this case, the method is termed the *guiding center approximation* or the *drift approximation*.

Averaging over the Larmor rotation leads to the consideration of the Larmor circle (guiding center) rather than the particle itself (see the examples below and the next problems). The reader is referred to the following books and reviews: Bogolubov and Mitropolsky (1961); Sagdeev *et al.* (1988); Sivukhin (1965); Morozov and

Solovyov (1963); Alfven and Felthammar (1963); Chirkov (2001); Matyshev (2000); Toptygin (1985); Fleishman and Toptygin (2013).

Example 4.12

Average (over the Larmor rotation) the energy and momentum of a charged particle traveling in mutually perpendicular uniform and constant fields $\boldsymbol{E}$ and $\boldsymbol{H}(E < H)$. Find in this way the smoothed velocity of the motion of the particle's guiding center.

Solution. Perform the obviously invariant operation of averaging over proper time with the use of the formulas obtained in Problem 4.55; this operation does not affect the transformation law for the quantity being averaged. Denote the averaging by a bar and arrive at

$$\bar{p}_x = \gamma_E^2(\mathcal{E}_0 - v_E p_{0x})\frac{v_E}{c^2} , \qquad \bar{p}_y = 0 ,$$
$$\bar{p}_z = p_{0z} , \qquad \bar{\mathcal{E}} = \gamma_E^2(\mathcal{E}_0 - v_E p_{0x}) .$$

We determine the guiding center velocity $\boldsymbol{v}_c$ through averaged values:

$$\boldsymbol{v}_c = \frac{c^2\bar{\boldsymbol{p}}}{\bar{\mathcal{E}}} = \boldsymbol{v}_E + \boldsymbol{v}_\parallel , \quad \boldsymbol{v}_\parallel = \frac{c^2 p_{0z}}{\bar{\mathcal{E}}}\boldsymbol{h} , \tag{4.82}$$

where $\boldsymbol{h} = \boldsymbol{H}/H$ is the unit vector. The particle travels across magnetic and electric fields with the *electric drift velocity* $v_E < c$:

$$\boldsymbol{v}_E = c\frac{\boldsymbol{E}\times\boldsymbol{H}}{H^2} . \tag{4.83}$$

The electric drift velocity is unrelated to the magnitude and sign of the charge nor does it depend on the particle's mass and energy.

In the general case, the velocity along the magnetic force line has the form

$$v_\parallel = v_{0z}\frac{1 - E^2/H^2}{1 - v_{0x}E/cH} . \tag{4.84}$$

It reduces to the starting longitudinal velocity v_{0z}, if $E \ll H$. □

Example 4.13

A charged particle propagates in a constant but nonuniform magnetic field. Its strength $\boldsymbol{H}$ remains unaltered in terms of direction but changes slightly (at a distance on the order of the Larmor radius of the particle) in terms of absolute value. Calculate in the first nonvanishing approximation the transverse drift velocity of the particle related to the nonuniformity of the magnetic field.

Solution. Because the particle's energy is conserved, the transverse motion is described by the equation

$$\dot{\boldsymbol{v}}_\perp = \boldsymbol{\Omega} \times \boldsymbol{v}_\perp\,, \quad \boldsymbol{\Omega} = -\frac{ec}{\mathcal{E}}\boldsymbol{H}\,, \tag{1}$$

where $\boldsymbol{H} = H\boldsymbol{h}$ is the field at the point occupied by the particle. Let us represent its radius vector $\boldsymbol{R} + \boldsymbol{r}$ in the form of the sum of the guiding center $\boldsymbol{R}$ and the radius vector of the particle $\boldsymbol{r}$ with respect to the guiding center. As a result, $H(\boldsymbol{R} + \boldsymbol{r}) = H(\boldsymbol{R}) + (r_\perp \cdot \nabla)H(\boldsymbol{R})$ with an accuracy of up to the terms of the first order of smallness. Equation (1) assumes the form

$$\dot{\boldsymbol{v}}_\perp = \boldsymbol{\Omega} \times \boldsymbol{v}_\perp \left[1 + \frac{(\boldsymbol{r}_\perp \cdot \nabla)H}{H}\right]\,, \tag{2}$$

where $H(\boldsymbol{R})$ is now taken at the point of the guiding center and does not depend on the particle's coordinates.

Let us represent the velocity of the particle in the form $\boldsymbol{v}_\perp = \boldsymbol{v}_{0\perp} + \boldsymbol{v}'_\perp$, where $\boldsymbol{v}_{0\perp} = \dot{\boldsymbol{r}}_{0\perp}$ is the velocity in the uniform field $\boldsymbol{H}(\boldsymbol{R})$ and $\boldsymbol{v}'_\perp$ is the small addition stemming from the nonuniformity of the field. Let us further substitute $\boldsymbol{v}_\perp$ and $\boldsymbol{r}_\perp$ with their unperturbed values $\boldsymbol{v}_{0\perp}$ and $\boldsymbol{r}_{0\perp}$, taking $\dot{\boldsymbol{v}}_{0\perp} = \boldsymbol{\Omega} \times \boldsymbol{v}_{0\perp}$. Thus,

$$\dot{\boldsymbol{v}}'_\perp = \boldsymbol{\Omega} \times \left[\boldsymbol{v}'_\perp + \boldsymbol{v}_{0\perp}\frac{(\boldsymbol{r}_{0\perp} \cdot \nabla H)}{H}\right]\,. \tag{3}$$

Averaging (3) over the Larmor rotation, that is, the period $T = 2\pi/\Omega$, yields

$$\bar{\dot{\boldsymbol{v}}}_\perp = \frac{1}{T}\left[\boldsymbol{v}'_\perp(t + T) - \boldsymbol{v}'_\perp(t)\right] \approx 0$$

with an accuracy of up to the first-order terms, bearing in mind that additions to the first-order terms are of order 2 or higher. Hence, the transverse drift velocity

$$\boldsymbol{v}_\mathrm{g} = \bar{\boldsymbol{v}}_{0\perp} + \bar{\boldsymbol{v}}'_\perp = \bar{\boldsymbol{v}}'_\perp = \frac{-\overline{\boldsymbol{v}_{0\perp}(\boldsymbol{r}_{0\perp}} \cdot \nabla H)}{H}\,,$$

where $\boldsymbol{r}_{0\perp}(t) = R_\perp(\boldsymbol{e}_1 \sin \Omega t + \boldsymbol{e}_2 \cos \Omega t)$ is the radius vector of the particle in the uniform field and $R_\perp = cp_\perp/eH$ is the Larmor radius. Averaging over time yields the *gradient drift* velocity:

$$\boldsymbol{v}_\mathrm{g} = \frac{v_\perp R_\perp}{2H}\boldsymbol{h} \times \nabla H\,. \tag{4.85}$$

Unlike the velocity of electric drift, that of the gradient drift depends on the particle's energy and charge. The smallness parameter over which the expansion was performed is the dimensionless quantity

$$\frac{R_\perp|\nabla H|}{H} \ll 1\,. \tag{4}$$

□

Example 4.14

A nonrelativistic charged particle performs oscillations in a one-dimensional potential field $U(x)$ with period T_0. The rapidly changing electric field $E(x)\cos\omega t$, $\omega \gg 2\pi/T_0$, is switched on at a certain moment. The amplitude $E(x)$ depends slowly on the coordinate (amplitude modulation). Perform averaging over rapid oscillations of the electric field and derive the equation that describes the averaged motion of the particle in the resultant field. Estimate the validity limits of the description on the basis of the averaged equation.

Solution. The exact equation has the form

$$m\ddot{x} = \frac{\mathrm{d}U}{\mathrm{d}x} + eE(x)\cos\omega t\,. \tag{1}$$

Its averaging is based on distinguishing the small oscillating addition $q(t)$ to the particle's coordinates:

$$x(t) = X(t) + q(t)\,, \quad \bar{x}(t) = X(t)\,, \quad \bar{q}(t) = 0\,. \tag{2}$$

Bars denote averaging over fast oscillations with frequency ω. In such averaging, the slow coordinate $X(t)$ and the quantities expressed through this coordinate can be regarded as being constant at times on the order of period $T = 2\pi/\omega$ of the fast oscillations. The smallness q is due not to the smallness of the field E but is due to its rapid changes reducing the fluctuation amplitude of the particle.

After we have substituted (2) and taken account of the first-order terms in q, equation (1) assumes the form

$$\ddot{X} + \ddot{q} = -\frac{1}{m}\frac{\mathrm{d}U}{\mathrm{d}X} - \frac{1}{m}\frac{\mathrm{d}^2U}{\mathrm{d}X^2}q + \frac{e}{m}E(X)\cos\omega t + \frac{e}{m}\frac{\mathrm{d}E}{\mathrm{d}X}q\cos\omega t\,. \tag{3}$$

This expansion suggests the inequalities

$$\left|q\frac{\mathrm{d}^2U}{\mathrm{d}X^2}\right| \ll \left|\frac{\mathrm{d}U}{\mathrm{d}X}\right|\,, \quad \left|q\frac{\mathrm{d}E}{\mathrm{d}X}\right| \ll |E|\,,$$

and the next terms must be negligibly small. In other words, $|q|$ must be smaller than the smallest scale L of the changes of functions $E(X)$ and $\mathrm{d}U/\mathrm{d}X$.

Averaging (3) over period $2\pi/\omega$ yields the equation

$$\ddot{X} = -\frac{1}{m}\frac{\mathrm{d}U}{\mathrm{d}X} + \frac{e}{m}\frac{\mathrm{d}E}{\mathrm{d}X}\overline{q\cos\omega t}\,. \tag{4}$$

In this equation, q must be expressed through X and averaging must be performed for the last term. Subtracting (4) from (3) gives the equation for the oscillating addition:

$$\ddot{q} = -\frac{q}{m}\frac{\mathrm{d}^2U}{\mathrm{d}X^2} + \frac{e}{m}E(X)\cos\omega t + \frac{e}{m}\frac{\mathrm{d}E}{\mathrm{d}X}\left(q\cos\omega t - \overline{q\cos\omega t}\right)\,.$$

Because $q(t)$ oscillates with the driving force frequency, $\ddot{q} \sim \omega^2 q$, $qU''/m \sim \omega_0^2 q$ by the order of magnitude estimation and the last item has an extra small multiplier on the order of q/L compared with the next to last one. Disregarding the small terms results in the approximate equation

$$\ddot{q} = \frac{e}{m} E(X) \cos \omega t \,,$$

the solution of which dictated by the driving force has the form

$$q(t) = -\frac{e}{m\omega^2} E(X) \cos \omega t \,. \tag{5}$$

Turning back to equation (4) and averaging yields the equation being sought for the motion smoothed over rapid oscillations:

$$m\ddot{X} = -\frac{\mathrm{d}U_{\text{eff}}}{\mathrm{d}X} \,, \quad \text{where}$$

$$U_{\text{eff}}(X) = U(X) + \frac{e^2}{4m\omega^2} E^2(X) = U(X) + \frac{m\bar{\dot{q}}^2}{2} \tag{4.86}$$

that is, the effective potential energy taking account of the averaged action of the high-frequency field. □

Phase trajectories and phase portrait The character of particle motion in field $U_{\text{eff}}(X)$ depends on its concrete form and the initial conditions. The general picture of the character of such motion at different initial conditions emerges from knowledge of *singular points* on the phase plane (X, p), its *separatrix*, and the *phase portrait* of the system of interest, that is, characteristic types of phase trajectories.

The *singular points* are the points at which simultaneously $\dot{X} = 0$ and $\dot{p} = 0$. Because $\dot{X} = p/m$, all singular points lie at the OX axis (as in the simplest case under consideration) and coincide with the equilibrium points X_n, $U'_{\text{eff}}(X_n) = 0$. The equilibrium points are stable at $U''_{\text{eff}}(X_n) = k_n > 0$ and unstable at $k_n < 0$. At $k_n = 0$, the leading derivatives need to be considered.

It is easy to study the particle motion in the general form near the singular points. Expansion of the Hamiltonian function $H(p, X) = p^2/2m + U_{\text{eff}}(X)$ in small deviations p and $x = X - X_n$ from a singular point leads to the equation for the second-order curve at the phase plane:

$$p^2 \pm m k_n x^2 = 2m(E - E_n) \,,$$

where E is the total energy of the particle and $E_n = U_{\text{eff}}(X_n)$. The plus sign on the left-hand side of the equation corresponds to the stable point called the elliptical point or the center because phase trajectories remain in its vicinity and have an ellipsoid shape. In this case, only values of $E \geq E_n$ are possible.

The minus sign corresponds to hyperbolas, and the E values can be either higher or lower than E_n. At $E = E_n$, the phase trajectories degenerate to four straight lines originating from a singular point (two intersecting lines – asymptotes of a

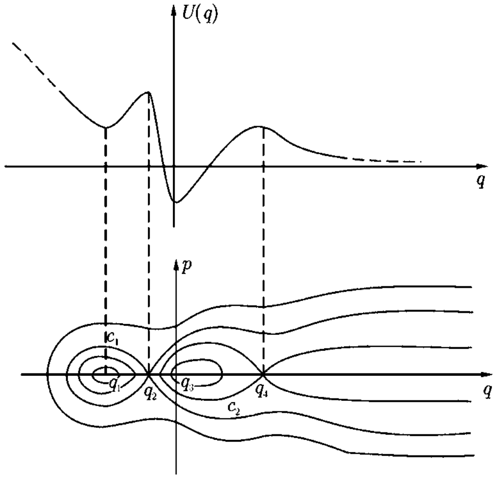

Figure 4.2 Example of a phase portrait and singular points.

hyperbola). The corresponding point is unstable (the particle moves away from it) and is referred to as the *hyperbolic point* or the *saddle*.

The phase trajectories passing through unstable singular points and separating regions of finite (at least on one side) and infinite motion are called separatrices. Examples of singular points and phase trajectories are presented in Figure 4.2.

It shows a one-dimensional potential field $U(q)$ and the phase portrait of a particle moving in this field. C_1 and C_2 are the seperatrices, points $(0, q_1)$ and $(0, q_3)$ at the phase plane are elliptical (centers), and points $(0, q_2)$ and $(0, q_4)$ are hyperbolic (saddles).

Adiabatic invariants Let us turn to the case opposite the one considered in Example 4.14. Let the external field $U(x, \lambda(t))$ in which a particle performs periodic motion with period T change slowly (adiabatically) with time so that parameter $\lambda(t)$ satisfies the condition

$$|T\dot{\lambda}| \ll \lambda \tag{4.87}$$

(certainly the motion will be nearly periodic rather than strictly periodic at variable λ). As known from mechanics (Landau and Lifshitz, 1976; Zaslavskii and Sagdeev, 1988), it is possible in such a case to construct the approximate integral of motion (i.e., *adiabatic invariant*):

$$I = \frac{1}{2\pi} \oint p \, \mathrm{d}q \,, \tag{4.88}$$

where integration is performed over the oscillation quasi-period, and p and q are the canonical variables. Although the adiabatic invariant (4.88) computed with exact p and q undergoes small oscillations with the system's oscillation period, its averaging over this period leads to a constant value that holds with a high (frequently exponential) degree of accuracy throughout many periods. In most cases, it is enough to calculate integral (4.88) in the zero approximation, that is, on the assumption that $\lambda = \text{const}$, and make the parameter λ dependent on t only after the explicit dependence $I(\lambda)$ has been found. Adiabatic invariants can be used to solve many problems related to the motion of a particle in slowly changing electromagnetic fields.

Example 4.15

Construct an adiabatic invariant related to the rotation of a charged particle around the direction of the magnetic field. The field undergoes slow changes in space and time.

Solution. Determine the algebraic invariant in system S' of the particle's guiding center. Take the canonical variables p' and q' in the zero approximation, that is, without regard for spatial and temporal nonuniformity of the field. Let q' be the azimuthal angle α' determining the particle's position in the Larmor orbit. The linear velocity of the particle has only the azimuthal constituent $v' = v'_\alpha = R'_\perp \dot{\alpha}'$, where $R'_\perp$ is the Larmor radius. The Lagrangian function (4.46) has the form

$$L = -mc^2\sqrt{1 - \frac{R'^2_\perp \dot{\alpha}'^2}{c^2}} + \frac{e}{c}A'_\alpha R'_\perp \dot{\alpha}' - e\varphi' ,$$

where $\boldsymbol{A}'$ and φ' are the electromagnetic potentials in the guiding center system.
Calculating the generalized momentum

$$p'_\alpha = \frac{\mathcal{E}' R'^2_\perp \dot{\alpha}'}{c^2} + \frac{e}{c}A'_\alpha R'_\perp$$

and substituting it into integral (4.88) yields two integrals, one of which is

$$\int_0^{2\pi} \mathcal{E}' R'^2_\perp \dot{\alpha}' \mathrm{d}\alpha' = \frac{2\pi e}{c} R'^2_\perp H' ,$$

because $\dot{\alpha}' = ceH'/\mathcal{E}'$ is the rotation rate. The second integral is substituted in the form

$$\int_0^{2\pi} A'_\alpha R'_\perp \mathrm{d}\alpha' = \oint \boldsymbol{A}' \cdot \mathrm{d}\boldsymbol{l}' = \int \operatorname{curl} \boldsymbol{A}' \cdot \boldsymbol{n} \mathrm{d}S = \pi R'^2_\perp H' ,$$

where Stokes's theorem is used. Combining the two last results, we arrive at

$$I' = \frac{3e}{2c} R'^2_\perp H' = \frac{3e}{2c}\frac{p'^2_\perp}{H'} .$$

On transition to the laboratory frame of reference moving along $\boldsymbol{H}'$ with the velocity of the particle $v_\parallel$, we have $H' = H$ and $p'_\perp = p_\perp$. Thus, the adiabatic invariant in the laboratory frame of reference

$$\frac{p_\perp^2}{H} = \text{const} . \tag{4.89}$$

Given the field is uniform but slowly changes with time, $p_\parallel = \text{const}$ and $p_\perp(t) = p_\perp(0)\sqrt{H(t)/H(0)}$. The particle's energy and transverse momentum vary in parallel. In a constant but spatially nonuniform field, the particle's energy $\mathcal{E} = \text{const}$, and angle ϑ between the direction of the momentum of the particle and the direction of the field changes as

$$\sin\vartheta = \sqrt{\frac{H(\boldsymbol{r})}{H(\boldsymbol{r}_0)}}\sin\vartheta_0 ,$$

where field H is taken at the point of the particle's guiding center. □

Problems

4.92. Provide a quantitative explanation for the origin of electric drift. To this effect, draw a few turns of the particle path in the mutually perpendicular fields $\boldsymbol{E}$ and $\boldsymbol{H}$ using a computer; specify the cause of its regular shifting in the direction of $\boldsymbol{E} \times \boldsymbol{H}$.

4.93. Do the same for the gradient drift.

4.94. A weak nonelectromagnetic force $\boldsymbol{\mathcal{F}}$ perpendicular to $\boldsymbol{H}$ is applied to a charged particle propagating in a uniform magnetic field $\boldsymbol{H}$. Calculate the velocity of the transverse drift caused by the applied force.

4.95. A magnetic field of constant absolute strength in the region of space being considered changes slightly (at distances on the order of the Larmor radius) its direction. Calculate the velocity of the transverse drift of a charged particle attributable to the force line curvature (such drift is termed *curvature*, or *centrifugal drift*). On the basis of the solutions of Examples 4.12 and 4.13, show that the velocity of motion of the particle's guiding center in the presence of a weakly nonuniform magnetic field and a low ($E \ll H$) electric field (taking account of the first-order transverse drifts) has the form

$$\dot{\boldsymbol{r}}_\mathrm{c} \equiv \boldsymbol{v}_\mathrm{c} = v_\parallel \boldsymbol{h} + \frac{c}{H}\boldsymbol{E} \times \boldsymbol{h} + \frac{1}{2H} v_\perp R_\perp \boldsymbol{h} \times \nabla H + v_\parallel R_\parallel \boldsymbol{h} \times (\boldsymbol{h} \cdot \nabla)\boldsymbol{h} , \tag{4.90}$$

where $\boldsymbol{h} = \boldsymbol{H}/H$ and $R_{\perp,\parallel} = c p_{\perp,\parallel}/eH$. What is the order of magnitude of the transverse drifts caused by slow time-dependent changes of the magnetic field and spatial nonuniformity of the electric field?

4.96. Prove by means of direct calculations the adiabatic invariance of quantity $I = p_\perp^2/H$ in the case of a slow time-dependent change of the magnetic field. To this end, calculate the change of the particle's transverse momentum for the Larmor period regarding its trajectory as a circumference in the plane perpendicular to $\boldsymbol{H}$, and relate it to the variation of the magnetic field.

4.97. Show that the rate of change in the energy of a charged particle in electric and magnetic fields ($E \ll H$) slowly changing in space and time in the guiding center approximation has the form

$$\dot{\mathcal{E}} = e\boldsymbol{E} \cdot \boldsymbol{v}_{\mathrm{c}} - \frac{e}{2} v_\perp R_\perp \boldsymbol{h} \cdot \operatorname{curl} \boldsymbol{E} \,. \tag{4.91}$$

This equation takes into account terms of up to the second order of smallness.

Note: Equations (4.89)–(4.91) form a complete system of equations describing the motion of a particle in a given slowly changing electric field ($E \ll H$). All the quantities on the right-hand sides of these equations should be taken at the point of the particle's guiding center. See Sivukhin (1965) for more details concerning the derivation and application of these equations.

4.98. On the basis of invariance of the quantity $I = p_\perp^2/H$ and the law of energy conservation, show that a magnetic field across the orbit of particle cyclotron rotation and the magnetic moment of a nonrelativistic particle created by its cyclotron rotation are preserved in the drift approximation. Under what additional conditions is the magnetic moment of a relativistic particle conserved?

4.99. A particle propagates in a weakly nonuniform constant magnetic field. Taking advantage of invariance of the quantity $I = p_\perp^2/H$ and the law of energy conservation show that the particle is subject to the action of force $\mathcal{F}$ directed along the magnetic force line and determine the magnitude of this force. Express it through the magnetic moment of the particle's cyclotron rotation.

4.100. A polarizing neutral system of charged particles having no dipole moment in the absence of the external field is placed in a weakly nonuniform static electric field $\boldsymbol{E}(\boldsymbol{r})$. Assuming the scalar polarizability β of the system to be known, write in the dipole approximation the nonrelativistic equation of motion of its center of mass. Find the potential energy and energy integral of the system.

4.101. Let the neutral system of the preceding problem be additionally subjected to the action of a weakly nonuniform magnetic field $\boldsymbol{H}(\boldsymbol{r})$. Write the equation of motion for the center of mass.

4.102*. A system of identical charged nonrelativistic particles is placed in an axially symmetric external potential field. Show that the weak external field $\boldsymbol{H}$ applied along the axis of symmetry of the system makes the system rotate as a whole with angular velocity

$$\boldsymbol{\Omega}_L = -\frac{e\boldsymbol{H}}{2mc}$$

(*Larmor precession*). Specify the criterion for the weakness of the magnetic field.

4.103. Region III, with an enhanced field ("magnetic plug"), lies between regions I and II, in which the static magnetic field is uniform and equals H. The maximum field strength is H_{m}; the lines of force are schematically represented in Figure 4.3.

A particle moving in region I has momentum $\boldsymbol{p}$ making angle ϑ with the force line direction at a certain instant. Find the relationship between ϑ, H, and H_{m} at which the particle is reflected from the region with the strong field on the assumption that the field changes slowly in space.

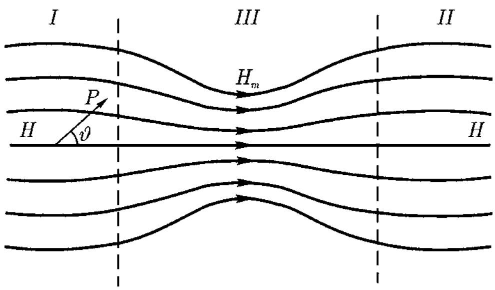

Figure 4.3 A magnetic plug.

4.104. The structure of the magnetic field in an adiabatic trap containing the axially symmetric field has the form schematically depicted in Figure 4.4. An amount of particles with isotropically distributed velocities is injected into the middle part of the trap, where the field strength is $\boldsymbol{H}$. What fraction of the particles, R, will be confined in the trap for a long time?

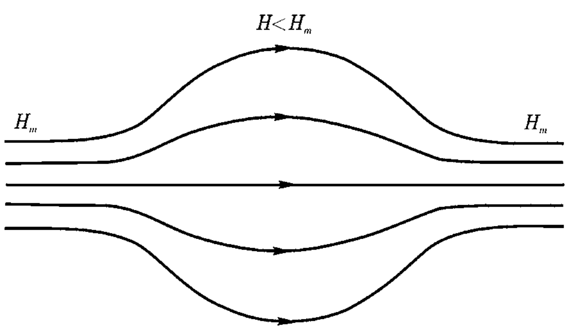

Figure 4.4 The structure of the magnetic field in an adiabatic trap.

4.105. A trap with the axially symmetric field depicted in Figure 4.4 keeps a fraction of particles that most of the time stay in its middle part, where the field is almost uniform. Let the field of the trap grow slowly with time so that the shape of the magnetic force lines does not change. Find how the distance between the guiding center of each particle and the trap axis changes.

4.106. A point charge q at rest is placed in a uniform magnetic field of strength H. A particle with charge e and mass m having the longitudinal constituent of velocity at infinity $v_{\|}$ is scattered from charge q. Find the line of force along which the particle's guiding center will move after scattering, assuming that the drift approximation is applicable and disregarding alteration of the longitudinal velocity during scattering. Before scattering, the guiding center moved along the force line for which the equation in cylindrical coordinates with the Oz axis passing through charge q and oriented along the field has the form $r = l, \varphi = 0$.

4.107. The Earth's magnetic field can be roughly represented as the field of a point dipole with magnetic moment $\mu \approx 8.1 \times 10^{25}$ G cm^3. A proton with energy $\mathcal{E} = 50$ MeV residing at a certain moment in the magnetic equatorial plane two Earth radii from the center of the globe moves across the magnetic lines of force. Find the law for the motion of the proton's guiding center in the drift approximation. What time, T, will it need to make a complete turn around the Earth? What is the Larmor radius of the proton? The Earth's radius $r_* = 6380$ km and its mass $M = 6\times 10^{27}$g.

4.108*. A proton resides in the geomagnetic equatorial plane (see the condition for the previous problem) at distance r from the center of the Earth; its momentum makes angle α with the direction of a magnetic line of force.

1. Disregard the gravitational field and show that the proton's guiding center undergoes azimuthal drift, besides performing motion along the magnetic force lines; find the drift angular velocity ω_{d} and express it through r and the geomagnetic width λ.
2. Specify the values of λ_{m} corresponding to the particle reflection points in the Earth's magnetic field.
3. Find the conditions under which the proton can reach the surface of the Earth.

4.109. The distance between two magnetic plugs in the adiabatic axially symmetric trap shown in Figure 4.4 slowly changes in accordance with the $l(t)$ law. Construct an adiabatic invariant associated with longitudinal fluctuation of a particle captured in the trap. Using this adiabatic invariant, express a change in the particle's energy through the distance between the plugs $l(t)$ assuming the field in the largest part of the trap to be quasi-uniform (reflection from "magnetic mirrors").

4.110**. The electric field $\boldsymbol{E}$ is uniform everywhere and directed along the Ox axis and the magnetic field is parallel to Oy and undergoes a jump of absolute value on plane $z = 0$ such that $H = H_1$ at $z < 0$ and $H = H_2 > H_1 \gg E$ at $z > 0$. A particle is initially in the region $z < 0$ and has momentum $\boldsymbol{p}_1 \perp \boldsymbol{H}_1$. Calculate the particle energy $\mathcal{E}_2$ averaged over the Larmor radius after its Larmor circle drifted to the region $z > 0$.

4.111***. A relativistic particle propagates in nonuniform electric and magnetic fields

$$\boldsymbol{E} = \boldsymbol{e}_x E_x(x) + \boldsymbol{e}_y E_y \ , \quad \boldsymbol{H} = \boldsymbol{e}_x H_x + \boldsymbol{e}_z H B_z(x) \ ,$$

where $E_y = \text{const} > 0$, $H_x = \text{const}$, $E_x(x)$, and $H_z(x)$ are the arbitrary (not necessarily slowly changing) functions x, taking the values $E_x = 0$ and $H_z = H_1$ at $x \to -\infty$ and $e_x = 0$ and $B_2 > b_1$ at $x \to +\infty$. Moreover, $E_y \ll H$ and $|H_x| \ll H_z$ everywhere.

Construct the adiabatic invariant of the particle associated with its motion in the direction of the Ox axis. Using this adiabatic invariant, find the change in the particle's energy during transition from the region with field H_1 to the region where field $H_z = H_2$. Analyze particular cases, including the case of a sharp H_z jump.

Hint: In this and the preceding problems, it is proposed within the framework of a simplified model to study the interaction between a charged particle and the shock wave front in the plasma which is in a magnetic field.

4.112. Generalize (4.87) derived in Example 4.14 to the case of one-dimensional motion of a relativistic particle in a rapidly oscillating field.

4.113*. A nonrelativistic charged particle propagates in a periodic constant potential field $U(x) = U_0[1 - \cos(2\pi x/L)]$ onto which the rapidly oscillating modulated electric field $E(x,t) = E_0 \sin(2\pi x/L) \cos \omega t$ is imposed.

1. Perform averaging over fast oscillations and find the averaged potential energy that could be used to describe the smoothed motion of the particle. Specify the validity limits of the method.
2. Find singular points at the phase plane of "slow" variables, p, X. Propose their classification and calculate the small oscillation frequencies near the stable equilibrium points.
3. Build up the phase portrait of the system.

4.114. Do the same on the assumption of a rapidly oscillating electric field:

$$E(x,t) = E_0 \cos\left(\frac{2\pi x}{L}\right) \cos \omega t \ .$$

4.115. Dirac's hypothesis suggests the existence of particles having magnetic charges (*"Dirac monopoles"*). According to this hypothesis, the value of the magnetic charge g is quantized:

$$\frac{e_0 g}{\hbar c} = \frac{1}{2} k \ , \quad k = 0, \pm 1, \pm 2 \dots \ ,$$

where e_0 is the elementary electric charge. Particles with magnetic charge in a magnetic field undergo the action of force $\mathcal{F} = g\boldsymbol{H}$. Calculate the minimal magnetic charge and energy increment of the magnetic monopole in the field $H = 1$ kE per unit path.

4.116*. Dirac proposed describing the vector potential of the magnetic charge g in the spherical system of coordinates (r, ϑ, α) by the expression $\boldsymbol{A} = (0, 0, (g/r) \tan(\vartheta/2))$. Calculate the strength of the magnetic field and its flux through the closed surface surrounding the origin of the coordinates. What difficulties regarding the formal and physical character are likely to be encountered when using it?

4.117*. A particle with electric charge e and mass m moves in the field of a magnetic monopole having a magnetic charge g. Find the integrals of its motion. Find the time dependence of the coordinates for a relativistic particle.

4.3
The Four-Dimensional Formulation of Electrodynamics. Introduction to Field Theory

4.3.1
Lagrangian and Hamiltonian Methods in Field Theory

Electrodynamics was the first field theory in the history of physics in which the methods of classical field theory were developed and verified (the principle of least action, Lagrangian and Hamiltonian description of the field, methods of the theory of transformations). Quantum electrodynamics was the first quantum field theory that enabled researchers to calculate various quantum dynamic effects with an accuracy far beyond that in other fields of physics.

At one time, the Lagrangian and Hamiltonian methods successfully applied in electrodynamics seemed to have no prospects for use in the quantum theory of elementary particles, especially for the description of strong interactions (see, e.g., Landau, 1960).[8] However, the important role of the methods of the classical field theory for the up-to-date theory of elementary particles is presently beyond any doubt in the light of the recently developed theory of electroweak interactions and the experimental discovery of intermediate bosons it predicted as well as the modern quantum theory of strong interactions (quantum chromodynamics) constructed largely by analogy with quantum electrodynamics. Therefore, this book includes sections dealing with the general principles of field theory. Whenever appropriate, we also use quantum notions that as a rule remain within the confines of the standard university course in quantum mechanics.

This section is focused on the Lagrangian form of field equations and the principle of least action, conservation laws, and the main notions of transformation theory. Further information on classical field theory and its links with modern quantum theories can be found in textbooks and monograph, such as Landau and Lifshitz (1975), Medvedev (1977), Gal'tsov *et al.* (1991), Linde (1990), Okun (1988, 1984), and Rubakov (2002)

The most natural way to introduce the variation principle into the theory of continuous systems, such as electromagnetic and other fields, is the limiting transition from the mechanics of discrete point masses to continuous medium mechanics (see Appendix B). This approach leads to the definition of the action as the integral over a certain fixed region Ω in the four-dimensional space–time from Lagrangian function density (or Lagrangian) $\mathcal{L}$:

$$S = \frac{1}{c}\int\limits_{\Omega} \mathcal{L}\left(q^A, q^B_{,i}, x\right) \mathrm{d}^4 x \,. \tag{4.92}$$

8) Lev Davidovich Landau (1908–1968) was an outstanding Soviet theoretical physicist and Nobel Prize recipient. He was the author (jointly with Evgeny Mikhailovich Lifshitz) of the well-known *Course of Theoretical Physics*, and was the founder of a highly efficient school of theoretical physics.

In the general case, the Lagrangian[9] depends on the functions[10] $q^A(x^k)$ describing the field of interest and their derivatives $q^A_{,k} \equiv \partial q^A/\partial x^k$, and on the four-dimensional coordinates x^k. The field functions q^A can be either real or complex depending on the field, and their number (the number of index A values) may differ. In what follows, the field functions q^A, like their variations, are considered to be independent.

The structure of the Lagrangian cannot be altogether arbitrary; it is subject to a number of constraints ensuing from the general physical principles and well-established symmetries inherent in physical phenomena and confirmed by experience. These constraints include

1. Relativistic invariance. Action S must be an invariant of the Lorentz transformation in order for its corollaries to be in agreement with the relativity principle.[11] Because d^4x is also a relativistic invariant, Lagrangian $\mathcal{L}$ must be invariant too.
2. Locality. The Lagrangian may depend only on the finite number of derivatives. All the quantities it contains are taken at a single spatiotemporal point. In electrodynamics, such structure of the theory naturally leads to the picture of particle interaction through the field at the point of localization of each particle.
3. Inclusion of derivatives of field functions not higher than first-order ones in the Lagrangian. This confines the field equations to those of second order in both coordinates and time.
4. The invariance of action with respect to certain transformations (besides Lorentz transformations) related to internal symmetries of the theory. Of special importance in the theory of elementary particles is gauge transformation generalizing the gauge (gradient) transformation of electromagnetic potentials, the sense and corollaries of which will be clarified below.
5. Reality of the Lagrangian accounting for the reality (the lack of an imaginary part) of the field energy and other physical quantities.

The following formulation of the variation principle for the field is proposed:

Field evolution in space and time occurs so that the action remains stationary with respect to relatively small changes at its fixed value at the boundary of the integration domain, that is,

$$\delta S = 0 \quad \text{at} \quad \delta q^A|_{\Sigma} = 0\,, \tag{4.93}$$

where Σ is the three-dimensional hypersurface limiting the four-dimensional integration volume Ω in ((4.92)).

9) Here, the factor $1/c$ is distinguished in order to have the usual relation between the action and the Lagrangian function in the nonrelativistic limit.

10) In this section, we use uppercase Latin indices to number components of the field functions ($A, B = 1, 2, \ldots, N$) and lowercase Latin letters to label four-dimensional coordinates ($i, k = 0, 1, 2, 3$). Lowercase Greek letters are used, as before, to denote the numbers of the spatial coordinate ($\alpha, \mu = 1, 2, 3$). In certain cases (when no misunderstanding is anticipated), the letters q and x without indices are used to denote the totality of field functions and 4-coordinates , respectively.

11) Certainly the action should not necessarily be a relativistic invariant in the formulation of nonrelativistic theory (see, e.g., Problems (4.114)–(4.120)).

Similar field equations can be obtained from Lagrangians $\mathcal{L}'$ differing in divergence of the arbitrary vector $F^m(q^A, x)$:

$$\mathcal{L}' = \mathcal{L} + \frac{\mathrm{d}F^m(q^A, x)}{\mathrm{d}x^m} . \tag{4.94}$$

This can be accounted for by the fact that the fourfold integral of divergence transforms to the three-dimensional integral over the hypersurface Σ, the variation of which is zero owing to the field functions $q^A(x^i)$ specified at this surface.

Example 4.16

Write the field equations in the Lagrangian form assuming the field Lagrangian to be known and using the variation principle for the field formulated above.

Solution. Impart small independent increments $\delta q^A(x^i)$ unrelated to the changes of 4-coordinates to field function components and calculate in the first order variation

$$\delta S = S[q^A + \delta q^A] - S[q^A] = \int\limits_{\Omega} \mathrm{d}^4 x \left(\frac{\partial \mathcal{L}}{\partial q^A} \delta q^A + \frac{\partial \mathcal{L}}{\partial q^A_{,i}} \delta q^A_{,i} \right) . \tag{1}$$

Here, the rule of summation of repeated symbols (both i and A) is applied. Using the equation

$$\delta q^A_{,i} \equiv \delta \frac{\partial q^A}{\partial x^i} = \frac{\partial}{\partial x^i} \delta q^A , \tag{2}$$

we take δq^A from under the sign by integrating by parts over (1):

$$\delta S = \int\limits_{\Omega} \mathrm{d}^4 x \left(\frac{\partial \mathcal{L}}{\partial q^A} - \frac{\mathrm{d}}{\mathrm{d}x^i} \frac{\partial \mathcal{L}}{\partial q^A_{,i}} \right) \delta q^A(x^k) + \int\limits_{\Sigma} \mathrm{d}^3 \Sigma_i \frac{\partial \mathcal{L}}{\partial q^A_{,i}} \delta q^A . \tag{3}$$

Here, $\mathrm{d}^3\Sigma_i$ is the projection of an element of the three-dimensional hypersurface Σ onto the i axis; since the field is specified at Σ, we have $\delta q^A|_\Sigma = 0$, and the last term on the right-hand side of (3) vanishes. Equating the first variation of action to zero, we have from (3) owing to the arbitrariness and independence of the function δq^A and the arbitrariness of the integration domain

$$\frac{\delta S}{\delta q^A} \equiv \frac{\partial \mathcal{L}}{\partial q^A} - \frac{\mathrm{d}}{\mathrm{d}x^i} \frac{\partial \mathcal{L}}{\partial q^A_{,i}} = 0 , \quad A = 1, 2, \ldots, N . \tag{4.95}$$

This is the field equation sought in the Lagrangian form. The identity on the left-hand side defines the variational derivative of the action functional. □

Example 4.17

Introduce the generalized field momentum density by analogy with mechanics of a system of material points

$$\pi_A = \frac{\partial \mathcal{L}}{\partial q^A_{,0}}, \quad A = 1, 2, \ldots, N\,, \tag{4.96}$$

construct the field Hamiltonian (Hamiltonian function density), and formulate the field evolution equation in the Hamiltonian form.

Solution. In analogy with classical mechanics of a system of material points, define the field Hamiltonian through its Lagrangian:

$$\mathcal{H}(q^A, \pi_B, q^C_{,\alpha}, x^k) = q^A_{,0}\pi_A - \mathcal{L}\,. \tag{4.97}$$

The right-hand side of this equality implies the summation over index A, and all "generalized velocities" $q^A_{,0}$ must be expressed through generalized momenta π_A with the help of equalities (4.96). As a result, the Hamiltonian becomes a function of q^A and π_A, spatial derivatives of $q^A_{,\alpha}$, and also coordinates x^k (in the presence of external fields or sources).

To obtain the field equations in the Hamiltonian form, it is convenient to use the variation principle again. However, it is impossible to directly translate the calculation from the preceding example to the case in question because there are twice as many Hamiltonian equations as Lagrangian equations, which requires twice as many independent variations to obtain them.

This difficulty can be overcome by introducing into the Lagrangian additional formally independent variables subject to the relevant coupling conditions. Let us first consider this approach as exemplified by the derivation of the Lagrange equations. Denote $s^A = q^A_{,0}$ and introduce quantities s^B into the right-hand side of equality (4.92):

$$S = \int\limits_{\Omega} \mathcal{L}\left(q^A, s^B, q^C_{,\alpha}, x^k\right) \mathrm{d}^4 x\,. \tag{1}$$

Evidently, the variational problem with the action (1) and coupling equations

$$s^A - q^A_{,0} = 0 \quad (A = 1, 2, \ldots, N) \tag{2}$$

is equivalent to the one considered in Example (4.16). Now, however, we may regard the variables q^A and s^A and their variations as being independent if we introduce the indefinite Lagrange multipliers $\lambda_A(x^k)$ in the problem:

$$S = \int\limits_{\Omega} \left[\mathcal{L}\left(q^A, s^B, q^C_{,\alpha}, x^k\right) - \lambda_A\left(s^A - q^A_{,0}\right)\right] \mathrm{d}^4 x\,. \tag{3}$$

Calculation of the first action variation as in Example (4.16) yields $2N$ equations containing *inter alia* the Lagrange multipliers:

$$\frac{\partial \mathcal{L}}{\partial q^A} - \frac{\mathrm{d}}{\mathrm{d}x^\mu}\frac{\partial \mathcal{L}}{\partial q^A_\mu} - \frac{\partial \lambda_A}{\partial x^0} = 0\,, \quad \frac{\partial \mathcal{L}}{\partial s^A} - \lambda_A = 0\,. \tag{4}$$

If the coupling equations (2) are taken into account, system (4) is equivalent to the Lagrange equations (4.95).

Let us take advantage of the fact that, according to (4), (2), and (4.96), the Lagrange multipliers coincide with the generalized momentum density:

$$\lambda_A = \frac{\partial \mathcal{L}}{\partial s^A} = \frac{\partial \mathcal{L}}{\partial q^A_{,0}} = \pi_A\,. \tag{5}$$

Introduce them into expression (3) and assume the system of q^A and π_A to be independent variables instead of quantities q^A and s^A:

$$S = \int\limits_\Omega \left[\pi_A q^A_{,0} - \mathcal{H}\left(q^A, \pi_B, q^C_{,\alpha}, x^k\right)\right]\mathrm{d}^4x\,. \tag{4.98}$$

Setting the first action variation equal to zero at independent δq^A and $\delta\pi_A$ yields the field equation in the Hamiltonian form:

$$\frac{\partial q^A}{\partial x^0} = \frac{\partial \mathcal{H}}{\partial \pi_A}\,, \quad \frac{\partial \pi_A}{\partial x^0} = -\frac{\partial \mathcal{H}}{\partial q^A} + \frac{\mathrm{d}}{\mathrm{d}x^\alpha}\frac{\partial \mathcal{H}}{\partial q^A_{,\alpha}}\,. \tag{4.99}$$

These equations are less symmetric than the Lagrange equations (4.95) because the coordinate x^0 is distinguished. □

Problems

4.118. The equation describing small-amplitude waves in an elastic isotropic medium has the following form in the three-dimensional notation:

$$\rho\frac{\partial^2 \boldsymbol{q}}{\partial t^2} = \mu\Delta\boldsymbol{q} + \left(K + \frac{\mu}{3}\right)\nabla(\nabla\cdot\boldsymbol{q})\,, \tag{1}$$

where the positive constants ρ, μ, and K are the medium density, shear modulus, and volume compression modulus respectively; the vector $\boldsymbol{q}(\boldsymbol{r}, t)$ describes the displacement of the small medium element under the effect of deformation caused by the propagating wave.

Write the Lagrangian and the Hamiltonian leading to equation (1).

4.119. A system of small macroscopic bodies ("material points") performs nonrelativistic motion when interacting with gravitational forces. Construct the action for a complete system consisting of substances and the gravitational field described by the Newtonian gravitational potential $\varphi(\boldsymbol{r}, t)$. Derive the equations of motion for the material points and the gravitational potential from the variation principle.

4.120. Do the same for a system of point charged particles, disregarding the magnetic interaction and gravity.

4.121. The distribution of stationary electric currents $\boldsymbol{j}(\boldsymbol{r})$ is specified in a space. Write down the action leading to magnetostatic equations.

4.122*. The complex scalar field $\psi(\boldsymbol{r}, t)$ in the nonrelativistic approximation has a Lagrangian

$$\mathcal{L} = \frac{i\hbar}{2}\left(\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}\right) - \frac{\hbar^2}{2m}|\nabla\psi|^2 - U|\psi|^2 , \tag{4.100}$$

where $m > 0$ is the particle mass, $\hbar$ is the reduced Planck constant, and $U(\boldsymbol{r}, t)$ is the potential field in which the particle propagates. Derive the equation of motion for $\psi(\boldsymbol{r}, t)$ from the Lagrangian and interpret it.

Hint: The complex field $\psi = q^1 + iq^2$ is equivalent to the two real functions $q^1(\boldsymbol{r}, t)$ and $q^2(\boldsymbol{r}, t)$. It is convenient to consider the linear combinations of variations of these functions, $\delta q^1 + i\delta q^2 = \delta\psi$ and $\delta q^1 - i\delta q^2 = \delta\psi^*$, as independent variations.

4.123*. Construct the Hamiltonian function density $\mathcal{H}$ using Lagrangian (4.100). Calculate the total field energy and propose the quantum interpretation of the result thus obtained.

4.124*. The Schrödinger equation for the wave function $\psi(\boldsymbol{r}, t)$ of a spin-free nonrelativistic charged particle has the form

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\left[\nabla - \frac{ie}{\hbar c}\boldsymbol{A}(\boldsymbol{r}, t)\right]^2\psi + e\varphi(\boldsymbol{r}, t)\psi , \tag{4.101}$$

where $\boldsymbol{A}(\boldsymbol{r}, t)$ and $\varphi(\boldsymbol{r}, t)$ are the electromagnetic potentials (specified real functions). Construct the Lagrangian leading to (4.101).

4.125*. The complex scalar field $\varphi(x^k)$ has a relativistically invariant Lagrangian

$$\mathcal{L} = \frac{1}{2}\left(\partial^k\varphi\partial_k\varphi^*\right) - \frac{m^2c^2}{2\hbar^2}|\varphi|^2 . \tag{4.102}$$

Find the equation of motion for the field $\varphi(x^k)$. What is the form of the dispersion relation between the wave vector and the frequency of the accompanying waves? Interpret it in terms of the de Broglie[12] quantum relations.

4.126*. Let the Lagrangian of the real scalar field φ have the form

$$\mathcal{L} = \frac{1}{2}\left(\partial^k\varphi\partial_k\varphi\right) - V(\varphi) , \tag{1}$$

12) Louis de Broglie (1892–1987) was an outstanding French physicist, a Nobel Prize recipient, and one of founders of quantum physics.

where the potential $V(\varphi)$ has the form

$$V(\varphi) = -\frac{\mu^2}{2}\varphi^2 + \frac{\lambda}{4}\varphi^4 \tag{2}$$

(Higgs[13] model). The quadratic term has changed sign compared with the potential in the preceding problem, and a positive fourth-order term is added.

Write down the differential equation for field φ. What character does the solution have at $\varphi^2 \ll \mu^2/\lambda$? Under what conditions for the given effective potential does the interpretation of φ as the wave function of field quanta continue to hold and what is the mass of these quanta?

4.3.2
The Action for an Electromagnetic Field

Let us move from the above examples and problems illustrating the effectiveness of the variation principle in the theory of various fields to the further consideration of an electromagnetic field, taking advantage of this extraordinarily powerful research tool. We shall start from the construction of the action suitable for the description of the classical theory in the most general case of a physical system composed of an electromagnetic field and charged relativistic particles, interactions between which are mediated through the field. This problem has already been considered for particular cases in Problems (4.120) and (4.121). Two constituent components of the action were constructed earlier (the action for a free particle (3.27) and the particle–field interaction term (4.43)). Generalizing these expressions to a system of particles yields

$$S_\mathrm{p} = -\sum_a m_a c \int \mathrm{d}s_a \tag{4.103}$$

and

$$S_\mathrm{int} = -\sum_a \frac{e_a}{c} \int A_i(x_a)\mathrm{d}x_a^i \,, \tag{4.104}$$

where index a denotes the particle number, and the summation is performed over all particles. For the comprehensive description of the system being considered, the action needs to be supplemented by S_em, corresponding to the electromagnetic field:

$$S = S_\mathrm{p} + S_\mathrm{int} + S_\mathrm{em} \,. \tag{4.105}$$

The last term must be expressed through the field functions, that is, the 4-potential A_i, and must comply with the invariance and symmetry principles formulated in the preceding paragraphs. One more important condition should be added

13) Peter Ware Higgs is an English physicist who in the 1960s predicted a new elementary particle, the Higgs boson.

for the electromagnetic field that imposes a limitation on the form of the action; namely, the action must be a relativistic and gauge invariant. All these requirements are satisfied by a single electromagnetic field invariant, $F^{ik}F_{ik}$[14], through which the action for the electromagnetic field is expressed:

$$S_{\text{em}} = -\frac{1}{16\pi c}\int F^{ik}F_{ik}\mathrm{d}^4x \,. \tag{4.106}$$

The numerical factor corresponds to the absolute Gaussian system of units.

The derivation of the equations for field motion from the action (4.105) implies variation of the electromagnetic potential at fixed values of the particle's 4-coordinates. It is convenient to write the potential-containing items uniformly, that is, in the form of integrals over particle world lines. The required writing is achieved by using the Dirac delta function, through which it is easy to express the charge and electric current density:

$$\rho(\boldsymbol{r}, t) = \sum_a e_a \delta\left(\boldsymbol{r} - \boldsymbol{r}_a(t)\right) \,, \quad \boldsymbol{j}(\boldsymbol{r}, t) = \sum_a e_a \boldsymbol{v}_a \delta\left(\boldsymbol{r} - \boldsymbol{r}_a(t)\right) \,. \tag{4.107}$$

Introduction of common time t for all the particles leads to $\mathrm{d}x^i_a/\mathrm{d}t = (c, \boldsymbol{v}_a)$ and

$$S_{\text{int}} = -\frac{1}{c^2}\int j^i(x)A_i(x)\mathrm{d}^4x \,, \tag{4.108}$$

where

$$j^i(x) = (c\rho(\boldsymbol{r}, t), \boldsymbol{j}(\boldsymbol{r}, t)) \tag{4.109}$$

is the *4-vector of current density*. Writing the current in such a form makes its vector nature apparent. Let us use the particle's velocity:

$$\begin{aligned} j^i(x) &= \sum_a e_a u^i_a \sqrt{1 - \frac{v_a^2}{c^2}}\,\delta\left(\boldsymbol{r} - \boldsymbol{r}_a(t)\right) \\ &= \sum_a c e_a \int u^i_a \delta^4\left(x - x_a(\tau_a)\right)\mathrm{d}\tau_a \,. \end{aligned} \tag{4.110}$$

In the last expression, integration is performed over invariant proper time of the particle a, $\mathrm{d}\tau_a = \sqrt{1 - v_a^2/c^2}\mathrm{d}t$, and

$$\delta^4(x - x_a) = \delta(\boldsymbol{r} - \boldsymbol{r}_a)\delta(x^0 - ct)$$

is a relativistic invariant too. This inference ensues from the invariance of d^4x and the integral

$$\int_\Omega \delta^4(x - x_a)\mathrm{d}^4x = 1 \,,$$

14) The second invariant of the Lorentz transformation, $e^{iklm}F_{ik}F_{lm}$, changes sign on inversion of the spatial axis, and its inclusion in the action leads to the equations having a different form in the right and left systems of coordinates.

which has a specified value for any four-dimensional region Ω if it includes the point x_a.

Example 4.18

Show that the action variation δS_{int} at a given current $j^i(x)$ is gauge invariant although the value of S_{int} depends on the calibration of 4-potential $A_i(x)$.

Solution. Substituting the potential $A_i = \tilde{A}_i - \partial_i f$ into (4.108) in accordance with (4.73) yields

$$S_{\text{int}} = \tilde{S}_{\text{int}} + \frac{1}{c^2}\int j^i(x)\partial_i f(x)\mathrm{d}^4 x\,.$$

Then, we transform the integrand

$$j^i(x)\partial_i f(x) = \partial_i(f j^i) - f\partial_i j^i\,.$$

The last term vanishes owing to the conservation of the electric charge (the continuity equation):

$$\partial_i j^i \equiv \frac{\partial\rho}{\partial t} + \operatorname{div}\boldsymbol{j} = 0\,. \tag{4.111}$$

Integration of the remaining term results in its transformation according to the Gauss–Ostrogradskii theorem into an integral over the three-dimensional hypersurface limiting the integration volume:

$$\int \partial_i(f j^i)\mathrm{d}^4 x = \oint f j^i \mathrm{d}^3\Sigma_i\,.$$

Because the field at this hypersurface is fixed, variation from the last integration equals zero. In other words, the variation δS_{int} is gauge invariant: $\delta S_{\text{int}} = \delta\tilde{S}_{\text{int}}$.
□

Example 4.19

Derive the Maxwell equations in four-dimensional form by varying action (4.105).

Solution. The equation $\delta S = 0$ takes the form

$$\delta\int\left(-\frac{1}{16\pi c}F^{ik}F_{ik} - \frac{1}{c^2}j^i A_i\right)\mathrm{d}^4 x = 0\,, \tag{1}$$

where the 4-potential undergoes variation and j^i should be regarded as a given field source. Hence,

$$\delta\left(F^{ik}F_{ik}\right) = 2F^{ik}\delta F_{ik} = 2F^{ik}\delta\left(\partial_i A_k - \partial_k A_i\right) = -4F^{ik}\partial_k\delta A_i\,.$$

Transformation of the last expression gives

$$F^{ik}\partial_k \delta A_i = \partial_k \left(F^{ik}\delta A_i\right) - \left(\partial_k F^{ik}\right)\delta A_i \,.$$

Integration of the first term on the right-hand side over the 4-volume using the Gauss–Ostrogradskii theorem yields

$$\int\limits_{\Omega} \partial_k \left(F^{ik}\delta A_i\right) \mathrm{d}^4 x = \oint\limits_{\Sigma} F^{ik}\delta A_i \mathrm{d}^3\Sigma_k = 0 \,,$$

because $\delta A_i = 0$ at the hypersurface Σ restricted the 4-volume. As a result, equation (1) assumes the form

$$\int\limits_{\Omega} \left(-\frac{1}{16\pi c}\partial_k F^{ik} - \frac{1}{c^2} j^i\right)\delta A_i \mathrm{d}^4 x = 0 \,,$$

from which the Maxwell equation in the 4-form follows owing to the arbitrariness of δA_i and the integration domain:

$$\partial_k F^{ik} = -\frac{4\pi}{c} j^i \,. \tag{4.112}$$

The second Maxwell equation is written through the dual field tensor; in accordance with (4.30), $\tilde{F}^{ik} = (1/2)e^{iklm}F_{lm} = e^{iklm}\partial_l A_m$, whence $\partial_i \tilde{F}^{ik} = e^{iklm}\partial_i\partial_l A_m = 0$. The last equality gives the second Maxwell equation:

$$\partial_i \tilde{F}^{ik} = 0 \,. \tag{4.113}$$

It can be written with the use of (4.36) through the initial field tensor:

$$\partial_l F_{ik} + \partial_i F_{kl} + \partial_k F_{li} = 0 \,. \tag{4.114}$$

□

Problem

4.127•. Derive the Maxwell equation in three-dimensional form from equalities (4.112) and (4.114).

4.3.3
Noether's Theorem and Integrals of Motion

The aforementioned symmetry properties of classical and quantum fields expressed as action invariants with respect to certain transformations of four-dimensional coordinates x^k and field functions $q^A(x)$ account for the conservation of some dynamic quantities that characterize the field. The search for motion integrals is as important in field theory as it is in classical mechanics. The relationship between the laws of conservation and field symmetry properties is established by Emmy Noether's[15] theorem considered in the next example.

Example 4.20

Let the action for a certain field be invariant with respect to an infinitely small transformation of the 4-coordinates and field functions of the form[16]

$$\begin{aligned} x'^k &= x^k + \delta x^k\,, & \delta x^k &= \Gamma^k{}_a \delta\lambda^a\,; \\ q'^A(x') &= q^A(x) + \delta^* q^A(x)\,, & \delta^* q^A(x) &= G^A_{\mathrm{B}a} q^B(x)\delta\lambda^a\,. \end{aligned} \tag{4.115}$$

Here $\lambda^a (a = 1, 2, \ldots, n)$ are the coordinate-independent and mutually independent transformation parameters; the values of $\lambda^a = 0$ correspond to the identical transformation; $\delta\lambda^a$ are their small increments; $\delta q^A(x) = \delta^* q^A(x) - q^A_{,l}\delta x^l$ is the variation of the function shape with an unaltered argument analogous to the one used in Example 4.16 and differing from the full variation $\delta^* q^A(x)$ (see above), which includes changes of the argument; all variations are linear in $\delta\lambda^a$.

Show that under these conditions it is possible to construct n 4-component quantities $J^k_a (a = 1, 2, \ldots, n)$, that is, generalized currents[17] or Noether currents, satisfying the continuity equations

$$\partial_k J^k_a = 0\,. \tag{4.116}$$

What conserved quantities are related to generalized currents?

Solution. A natural way to address this problem is to set to zero the action variation due to the transformation of interest (4.115) and extract the desired continuity equations (4.116) from the resulting equality, which must be linearly dependent on parameters $\delta\lambda^a$. Let us begin from the calculation of the action variation $\delta^* S$, labeling it with an asterisk because transformation (4.115) affects not only the form of the field functions q^a but also coordinate transformation:

$$\delta^* S = \int\limits_{\Omega'} \mathcal{L}\left(q'^A(x'), \frac{\partial q'^A(x')}{\partial x'^k}, x'^l\right) \mathrm{d}^4 x' - \int\limits_{\Omega} \mathcal{L}\left(q^A(x), q^A_{,k}, x^l\right) \mathrm{d}^4 x\,. \tag{4.117}$$

15) Amalie Emmy Noether (1882–1935) was a German mathematician and the author of fundamental works in theoretical physics establishing the relationship between the laws of conservation and symmetry properties of a physical system.

16) Transformations 4.115 are not necessarily infinitely small; they make up a Lie group (see, e.g., Bogolubov and Shirkov (1982)).

17) Index k is a vector symbol. The meaning of index a depends on the choice of transformation parameters λ^a. If λ^a is a scalar parameter, then J^k_a is a 4-vector. If λ^a is a vector symbol, then the Noether current is a rank 2 tensor, if λ^a includes two vector symbols, the Noether current is a rank 3 tensor, and so on.

Let us pass to variables x in the first integral and find with the help of (4.15)

$$d^4x' = \left|\frac{\partial x'^k}{\partial x^l}\right| = \left|\delta^k_l + \frac{\partial \delta x^k}{\partial x^l}\right| d^4x \approx \left(1 + \frac{\partial \delta x^l}{\partial x^l}\right) d^4x \ ,$$

where the Jacobian of the transformation from primed to unprimed coordinates is used and the last approximate equality is written with an accuracy of up to the first-order terms in δx. Furthermore,

$$\frac{\partial q'^A(x')}{\partial x'^k} = \frac{\partial x^l}{\partial x'^k}\frac{\partial}{\partial x^l}\left[q^A(x) + \delta q^A(x) + \frac{\partial q^A(x)}{\partial x^j}\delta x^j\right] .$$

Here again, we find with an accuracy up to the linear terms

$$\frac{\partial x^l}{\partial x'^k} \approx \frac{\partial}{\partial x'^k}\left(x'^l - \delta x^l(x')\right) \approx \delta^l_k - \frac{\partial \delta x^l(x)}{\partial x^k}$$

and eventually have

$$\frac{\partial q'^A(x')}{\partial x'^k} = q^A_{,k} + \delta q^A_{,k} + q^A_{,k,j}\delta x^j \ .$$

Using the formulas thus obtained, we reduce (4.117) to the form

$$\delta^* S = \int\limits_\Omega \Big\{\mathcal{L}\left(q^A + \delta q^A + q^A_{,j}\delta x^j, q^A_{,k} + \delta q^A_{,k} + q^A_{,kj}\delta x^j, x^l + \delta x^l\right) \\ \times \left(1 + \frac{\partial \delta x^n}{\partial x^n}\right) - \mathcal{L}\left(q^A, q^A_{,k}, x^l\right)\Big\} d^4x$$

and arrive, after elimination of the terms higher than the first order of smallness, at

$$\delta^* S = \int\limits_\Omega \left\{\mathcal{L}\frac{\partial \delta x^l}{\partial x^l} + \left(\frac{\partial \mathcal{L}}{\partial x^l} + \frac{\partial \mathcal{L}}{\partial q^A}q^A_{,l} + \frac{\partial \mathcal{L}}{\partial q^A_{,k}}q^A_{,kl}\right)\delta x^l\right\} d^4x \\ + \int\limits_\Omega \left(\frac{\partial \mathcal{L}}{\partial q^A} - \frac{d}{dx^k}\frac{\partial \mathcal{L}}{\partial q^A_{,k}}\right)\delta q^A d^4x + \int\limits_\Omega \frac{d}{dx^k}\left(\frac{\partial \mathcal{L}}{\partial q^A_{,k}}\delta q^A\right) d^4x \ .$$

Here, summation is performed over the repeated indices A, l, and k.

Convolution of the first integral yields a simple expression

$$\int\limits_\Omega \frac{d}{dx^k}\left(\mathcal{L}\delta x^k\right) d^4x \ ,$$

and the second is equal to zero owing to fulfillment of the Lagrangian equation (4.95). The quantities $\delta q^A = G^a_{Ba} q^B \delta\lambda^a - q^A_{,k}\delta x^k$ and $\delta x^k = \Gamma^k{}_a \delta\lambda^a$ are substituted into the remaining terms and the action variation is equated to zero:

$$\delta^* S = \delta\lambda^a \int\limits_\Omega d^4x \frac{d}{dx^k}\left\{\left(\mathcal{L}\delta^k_l - \frac{\partial \mathcal{L}}{\partial q^A_{,k}}q^A_{,l}\right)\Gamma^l{}_a + \frac{\partial \mathcal{L}}{\partial q^A_{,k}}G^A_{Ba}q^B\right\} = 0 \quad (4.118)$$

The transformation parameters $\delta\lambda^a$ do not depend on the coordinates and are therefore taken out of the integral.

Owing to the linear independence of quantities $\delta\lambda^a$ and the arbitrariness of the integration domain Ω, (4.118) gives n equalities (4.116), where

$$J_a^k = -\left(\mathcal{L}\delta_l^k - \frac{\partial\mathcal{L}}{\partial q_{,k}^A} q_{,l}^A\right) \Gamma^l{}_a - \frac{\partial\mathcal{L}}{\partial q_{,k}^A} G_{\mathrm{Ba}}^A q^B \tag{4.119}$$

are the *generalized currents (Noether currents)*.

The zero values of four-dimensional divergences express in the differential terms the laws of conservation of physical quantities related to generalized currents:

$$\frac{1}{c}\frac{\partial J_a^0}{\partial t} + \operatorname{div} \boldsymbol{J}_a = 0 . \tag{4.120}$$

Here J_a^0/c can be considered as the three-dimensional density of a certain quantity and $\boldsymbol{J}_a$ as its flux density. We use the three-dimensional Gauss–Ostrogradskii theorem regarding all quantities as diminishing fast enough in any space-like direction (i.e., with the distance from the origin of the coordinates along the spatial axes) and find in the usual way the integral laws of conservation:

$$Q_a(x^0) = \int J_a^0(x^0, x^1, x^2, x^3)\mathrm{d}^3x = \text{const} , \tag{4.121}$$

where integration is performed over the entire infinite three-dimensional space.

Noether currents are ambiguously defined by the method above because an arbitrary (in k index) vector j_a^k with zero divergence depending on the same quantities as the Lagrangian can be added into the braces of equality (4.118). This vector can be expressed through a rank 2 arbitrary antisymmetric tensor,

$$j_a^k = f_a^{lk}{}_{,l} , \quad f_a^{lk} = -f_a^{kl} , \tag{4.122}$$

since

$$j_{a,k}^k = \frac{\partial^2}{\partial x^k \partial x^l} f_a^{lk} \equiv 0 .$$

However, this ambiguity does not affect the values of integral quantities Q_a because the addition to the integral on the right-hand side of (4.121) has the form $\int f_a^{\mu 0}{}_{\mu} \mathrm{d}^3x$. This integral over the infinite three-dimensional volume of three-dimensional divergence is transformed according to the Gauss–Ostrogradskii theorem into the integral over the infinitely distant surface and vanishes because of the absence of the field at infinity. The distribution density of quantity Q_a in space and its three-dimensional current remain ambiguous; additional considerations are needed for the extension of the definition of these differential quantities (see the examples and problems below). □

Problems

4.128. Lagrangian (4.100) of a quantum mechanical particle is invariant with respect to a change in the phase of the wave function to a constant value, that is, transformation $\psi'(x,t) = \psi(x,t)\exp(i\alpha)$, $\alpha = \text{const}$ (neither coordinates nor time is transformed). Construct the generalized Noether current related to this transformation and clarify the quantity, conservation of which depends on invariance with respect to phase transformation.

4.129. Do the same for the relativistic Lagrangian (4.102). Is it possible to preserve for a generalized current in the relativistic case the same interpretation as in nonrelativistic quantum mechanics?

4.130. Do the same for the Lagrangian in Problem 4.124* (a nonrelativistic charged particle in the external electromagnetic field).

4.131*. What transformation of the wave function ψ in the previous problem is needed in order for gauge transformation of the potential

$$\boldsymbol{A} = \tilde{\boldsymbol{A}} - \nabla f(\boldsymbol{r},t)\,, \quad \varphi = \tilde{\varphi} + \frac{1}{c}\frac{\partial f(\boldsymbol{r},t)}{\partial t}$$

to alter neither the probability density and current nor the Lagrangian of a charged particle (as constructed in Problem 4.124*).

4.132*. The real vector field $V^k(x)$ interacts with the scalar complex field $\chi(x) = 2^{-1/2}(\chi_1(x)+i\chi_2(x))$. The Lagrangian describing the interacting fields has the form

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}(\partial_k V_i - \partial_i V_k)(\partial^k V^i - \partial^i V^k) \\ &+ \left[(\partial_k + igV_k)\chi^*\right]\left[(\partial^k - igV^k)\chi\right] + \mu^2\chi^*\chi - \lambda\left(\chi^*\chi\right)^2\,, \end{aligned} \tag{1}$$

where g is the interaction constant.

1. Show that the quanta of field V^k have zero mass in the absence of interaction ($g = 0$), and those of field χ have the mass found in Problem 4.126*.
2. In the presence of interaction, the quanta of vector field V^k acquire mass $m_v = g\mu/\sqrt{\lambda}$ (the Higgs mechanism of the origin of mass in certain particles).

Hint: To facilitate the calculations, first perform phase transformation of field $\chi \to 2^{-1/2}(\varphi_0 + \eta(x))\exp(i\zeta(x)/\varphi_0)$ and gauge transformation of field $V^k \to A^k(x) - (1/g\varphi_0)\partial^k\zeta(x)$, where $\zeta(x)$, φ_0, and $\eta(x)$ are real quantities.

4.133*. An infinitely small (by vector δa^k) shift of the frame of reference is described by transformation of the 4-coordinates and field functions

$$x'^k = x^k + \delta a^k\,, \quad q'^A(x') = q'^A(x+\delta a) = q^A(x)\,. \tag{4.123}$$

Show that the generalized Noether current is a rank 2 tensor (*canonical energy–momentum tensor*) and has the form

$$J_l^k = \frac{\partial\mathcal{L}}{\partial q^A_{,k}} q^A_{,l} - \mathcal{L}\delta^k_l\,. \tag{4.124}$$

What quantities are conserved given invariance of the Lagrangian with respect to the four-dimensional shift?

Example 4.21

Find the energy–momentum tensor of the electromagnetic field in the absence of charged particles based on Noether's theorem. The tensor being sought must have the following properties: (i) gauge invariance (i.e., must be expressed only through the components of the field F_{ik} tensor); (ii) symmetry with respect to permutation of tensor indices.[18)]

Solution. According to (4.92) and (4.106), the Lagrangian of a free electromagnetic field has the form

$$\mathcal{L}_{\text{em}} = -\frac{1}{16\pi} F^{ik} F_{ik} \ .$$

In this case, the field functions $q^A = A^i$. To calculate the partial derivative entering (4.124), we shall use the formula

$$\delta \mathcal{L}_{\text{em}} = \frac{\partial \mathcal{L}}{\partial A^i{}_{,k}} \delta A^i{}_{,k} = -\frac{1}{8\pi} F^{ik} \delta (A_{k,i} - A_{i,k}) = \frac{1}{4\pi} F^{ik} \delta A_{i,k} \ ,$$

whence

$$\frac{\partial \mathcal{L}_{\text{em}}}{\partial A^i{}_{,k}} = \frac{1}{4\pi} F_i{}^k \ .$$

Substituting the last quantity into (4.124), we find the *canonical momentum–energy tensor* of a free electromagnetic field:

$$J^{kl} = \frac{1}{4\pi} F^{ik} A_i{}^{,l} + \frac{1}{16\pi} (F^{mn} F_{mn})\, g^{kl} \ . \tag{4.125}$$

By virtue of Noether's theorem, this vector has zero divergence, $J^{kl}{}_{,k} = 0$, but it is asymmetric and noninvariant with respect to the gauge transformation of the 4-potential. To eliminate such important drawbacks, it should be supplemented by an appropriate tensor with zero divergence taken over the k index. As shown in Example 4.20, such a tensor must in itself represent divergence from a certain tensor (in the present case a rank 2 tensor) antisymmetric with respect to two symbols.

Bearing in mind the form of (4.125), we naturally choose $f^{ikl} = (a/4\pi) F^{ik} A^l$, $f^{kil} = -f^{ikl}$, with the so far undetermined multiplier a. Its divergence $f^{ikl}{}_{,i} = (a/4\pi) F^{ik} A^l{}_{,i}$, taking into account that $F^{ik}{}_{,i} = 0$ if $j^k = 0$ by virtue of the Maxwell

18) The second property is needed for the relationship between the momentum density and the moment of momentum density to have the same form as the relationship between the momentum and the moment in classical mechanics – see Problem 4.139*.

equation (4.112). It should be noted that $f^{ikl}{}_{,i,k} = (a/4\pi)F^{ik}A^{l}{}_{,i,k} = 0$. The addition of the rank 2 tensor $f^{ikl}{}_{,i}$ to the canonical tensor (4.125) gives

$$T^{kl} = J^{kl} + f^{ikl}{}_{,i} = \frac{1}{4\pi}F^{ik}\left(A_{i}{}^{,l} + aA^{l}{}_{,i}\right) + \frac{1}{16\pi}\left(F^{mn}F_{mn}\right)g^{kl} .$$

At $a = -1$, we obtain the *symmetric energy–momentum tensor* expressed only through the gauge-invariant components F^{ik}:

$$T^{kl} = \frac{1}{4\pi}\left(F^{ki}F_{i}{}^{l} + \frac{1}{4}g^{kl}F^{mn}F_{mn}\right) . \tag{4.126}$$

As follows from (4.126), the trace of the symmetric energy–momentum tensor of an electromagnetic field vanishes in all reference frames; $T^{k}{}_{k} = 0$.

Let us find the individual components of the energy–momentum tensor using tables (4.68). The quantity

$$T^{00} = \frac{1}{8\pi}\left(E^2 + H^2\right) \tag{4.127}$$

is the energy density of an electromagnetic field considered in Section 2.3. The components $T^{0\alpha}/c = T^{\alpha 0}/c = g^{\alpha}(\alpha = 1, 2, 3)$ make up the momentum density of the electromagnetic field,

$$\boldsymbol{g} = \frac{1}{4\pi c}\boldsymbol{E} \times \boldsymbol{H} , \tag{4.128}$$

which differs from the energy flux density γ (Pointing vector) in the factor $1/c^2$. The continuity equation for the components of tensor $T^{\alpha k}$ has the form $g^{\alpha}_{,t} + T^{\alpha\beta}_{\beta} = 0$. Writing it in the integral form with the help of the three-dimensional Gauss–Ostrogradskii theorem,

$$-\frac{\mathrm{d}G^{\alpha}}{\mathrm{d}t} = \oint_{S} T^{\alpha\beta}\mathrm{d}S_{\beta} , \tag{4.129}$$

where

$$\boldsymbol{G} = \int_{V} \boldsymbol{g}\mathrm{d}^3x \tag{4.130}$$

is the total field momentum in volume V, we can see that the spatial part of $T^{\alpha\beta}$ is the momentum flux density, whereas the integral in (4.129) gives the flux of the αth component of the momentum from the three-dimensional volume V through its surface S. The three-dimensional tensor

$$\sigma_{\alpha\beta} = \sigma^{\alpha\beta} = -T^{\alpha\beta} = \frac{1}{4\pi}\left(E_{\alpha}E_{\beta} + H_{\alpha}H_{\beta}\right) - \frac{1}{8\pi}\left(E^2 + H^2\right)\delta_{\alpha\beta} \tag{4.131}$$

is called the *Maxwell tension tensor*. □

Example 4.22

Construct the energy–momentum tensor of a system consisting of an electromagnetic field and charged relativistic particles interacting with it.

Solution. It should be expected that the divergence of the electromagnetic field tensor (4.126) (denoted here by T^{ik}_{em}) does not vanish in the presence of charged particles because both the energy and the momentum of the field may change during interaction with the particles. However, the total energy and momentum of a closed system composed of particles and an electromagnetic field must be conserved. Therefore, the quantity $T^{ik}_{\text{em},k}$ must transform to the divergence of a certain other tensor related to the particles:

$$T^{ik}_{\text{em},k} = -T^{ik}_{\text{part},k} \, . \tag{4.132}$$

The tensor $T^{ik} = T^{ik}_{\text{em}} + T^{ik}_{\text{part}}$ and can be regarded as the total energy–momentum tensor of a closed system.

Following this scheme, we calculate

$$T^{ik}_{\text{em},k} = \frac{1}{4\pi}\left(F^{il}F_l{}^k{}_{,k} + F_l{}^k F^{il}{}_{,k} + \frac{1}{2}F_{mn}F^{mn,i}\right) \, .$$

Next, we use Maxwell equations (4.112) and (4.114) to find

$$\frac{1}{4\pi}F^{il}F_l{}^k{}_{,k} = -\frac{1}{c}F^{il}j_l ;$$

$$F_l{}^k F^{il}{}_{,k} = -\frac{1}{2}F_{kl}F^{kl,i}$$

and obtain

$$T^{ik}_{\text{em},k} = -\frac{1}{c}F^{il}j_l \, .$$

Let us now use the expression for the 4-current in the form of (4.110) and the equation of particle motion (4.53):

$$\begin{aligned} T^{ik}_{\text{em},k} &= -\sum_a e_a \int u^a_k F^{ik}_a \delta^4(x - x_a)\mathrm{d}\tau_a \\ &= -\sum_a c m_a \int \frac{\mathrm{d}u^i_a}{\mathrm{d}\tau_a}\delta^4(x - x_a)\mathrm{d}\tau_a \\ &= -\sum_a c m_a u^i_a \delta(x - x_a)\big|^{+\infty}_{\tau_a=-\infty} \\ &\quad + \sum_a c m_a \int u^i_a \frac{\mathrm{d}}{\mathrm{d}\tau_a}\delta^4(x - x_a)\mathrm{d}\tau_a \, . \end{aligned}$$

The term outside the integral vanishes and the derivative of the delta function transforms as

$$\frac{\mathrm{d}}{\mathrm{d}\tau_a}\delta^4(x - x_a) = -\frac{\mathrm{d}x_a^k}{\mathrm{d}\tau_a}\frac{\partial}{\partial x^k}\delta^4(x - x_a) = -u_a^k\frac{\partial}{\partial x^k}\delta^4(x - x_a)\,.$$

We eventually arrive at

$$\frac{\partial T^{ik}_{\text{em}}}{\partial x^k} = -\frac{\partial}{\partial x^k}\sum_a cm_a\int u_a^i u_a^k \delta^4(x - x_a)\mathrm{d}\tau_a = -\frac{\partial T^{ik}_{\text{part}}}{\partial x^k}\,,$$

whence the symmetric energy–momentum tensor of the particles is found:

$$T^{ik}_{\text{part}} = \sum_a cm_a\int u_a^i u_a^k \delta^4(x - x_a)\mathrm{d}\tau_a\,. \tag{4.133}$$

The tensor thus obtained possesses all the necessary properties. The integral over the three-dimensional volume V gives

$$\begin{aligned}\int_V T^{00}_{\text{part}}\mathrm{d}^3x &= \sum_a cm_a\int u_a^0 u_a^0\sqrt{1 - \frac{v_a^2}{c^2}}\,\delta(\boldsymbol{r} - \boldsymbol{r}_a)\mathrm{d}^3x\,\delta(x^0 - ct)\mathrm{d}t\\ &= \sum_V m_a u_a^0 u_a^0\sqrt{1 - \frac{v_a^2}{c^2}} = \sum_V \mathcal{E}_a\end{aligned}$$

that is, the total energy of the particles in volume V, and integral

$$\frac{1}{c}\int_V T^{0\alpha}_{\text{part}}\mathrm{d}^3x = \frac{1}{c}\int_V T^{\alpha 0}_{\text{part}}\mathrm{d}^3x = \sum_a p_a^\alpha$$

gives the total particle momentum in the same volume. The three-dimensional tensor

$$T^{\alpha\beta}_{\text{part}} = \sum_a p_a^\alpha v_a^\beta\,\delta(\boldsymbol{r} - \boldsymbol{r}_a(t)) \tag{4.134}$$

is the momentum flux density of the particles. □

Problems

4.134. Reduce the energy–momentum tensor of an electromagnetic field to the diagonal form. Find all the frames of reference in which the tensor has the diagonal form. When is tensor diagonalization impossible?

4.135. Construct the energy–momentum tensor for the Lagrangian (4.100) corresponding to the nonrelativistic Schrödinger equation. Propose the quantum mechanical interpretation of the integrals over the three-dimensional volume from individual components of the energy–momentum tensor.

Hint: Include the particle rest energy into the Lagrangian to make possible the use of the general formula (4.124) in the present problem.

4.136. Let the energy–momentum tensor T^{ik} be known in a certain inertial frame of reference S. An observer travels relative to S with velocity u_i. What energy and momentum densities (per unit three-dimensional volume) will the observer measure in his own reference frame?

4.137. Show that the Maxwell strain tensor flux through the surface of a certain three-dimensional volume in a static case is equal to the total electromagnetic force applied to the particles contained in this volume (given they do not cross its boundaries). What quantity is added to this balance in an alternating field?

4.138. On the basis of the general results obtained in Example 4.20, express through the Lagrangian and its derivatives the generalized Noether current corresponding to the infinitely small proper Lorentz transformation (see Problem 4.20). Analyze a particular case when the transformation reduces only to rotation of the spatial axes. Distinguish in the Noether current the terms related to the momentum orbital moment and those related to the spin moment on the basis of the example of Lagrangian (4.100) for a spinless relativistic particle and the analogous Lagrangian for a relativistic particle with spin 1/2. Write the mean quantum mechanical values of these quantities in the form adopted in quantum mechanics.

4.139*. Use Noether's theorem to construct the 4-tensor of angular momentum density for an electromagnetic field in the absence of particles. To this effect, consider the infinitely small proper Lorentz transformation including both purely spatial turns and Lorentz pseudoturns (see Problem 4.20). By adding the momentum of a certain divergence to the tensor, reduce it to such a form in which the relationship between the momentum density and angular momentum density is similar to the relationship between the momentum and the moment of a particle system: $\boldsymbol{M} = \sum_a \boldsymbol{r}_a \times \boldsymbol{p}_a$.

4.140*. Construct the moment density tensor M_{kij} for a closed system consisting of a particle and an electromagnetic field. Clarify the meaning of the spatial $M_{\alpha\beta}$ and mixed $M_{0\alpha}$ components of the system's total angular momentum tensor.

4.141*. Use Noether's theorem to find motion integrals of a nonrelativistic charged particles in a uniform magnetic field $\boldsymbol{H}$. Show that the total angular momentum is a combination of the moments [of momentum] of a particle, $m[\boldsymbol{r} \times \boldsymbol{v}]$, and of the electromagnetic field, $\int \boldsymbol{r} \times \mathbf{g} d^3x$, respectively. The last moment includes the particle's Coulomb electric field and external magnetic field $\boldsymbol{H}$.

4.142. An electromagnetic field remains nonzero only inside a certain spatial volume V containing no charge. Show that the complete energy and momentum of the field are a 4-vector.

4.143. A system consists of particles and an electromagnetic field and occupies a finite volume. Find the expression for the flux density $\mathcal{R}$ of the field's angular momentum from the consideration of the balance of total angular momentum M_{ik} in this system. Use the results obtained in Problem 4.140*.

4.144•. Calculate the total momentum $\boldsymbol{P}$ of an electromagnetic field in the main region V, and express it through the canonical variables $Q_{\boldsymbol{k}\sigma}$ and $P_{\boldsymbol{k}\sigma}$ introduced by relation (2.157). Show that the momentum can be represented in the form

$$\boldsymbol{P} = \sum_{\boldsymbol{k}\sigma} \frac{\boldsymbol{k}\mathcal{H}_{\boldsymbol{k}\sigma}}{\omega_k}\,, \quad \text{where} \quad \mathcal{H}_{\boldsymbol{k}\sigma} = \frac{1}{2}\left(P_{\boldsymbol{k}\sigma}^2 + \omega_k^2 Q_{\boldsymbol{k}\sigma}^2\right) \tag{4.135}$$

is the Hamiltonian function of the $(\boldsymbol{k}, \sigma)$ mode (field oscillator).

4.4
Answers and Solutions

4.4 $S_{ik} = S_{ki}\,,\ A_{ik} = -A_{ki}\,,\ S_i{}^k = S^k{}_i \neq S_k{}^i\,,\ A_i{}^k = -A^k{}_i \neq -A_k{}^i\,.$

4.5 $T_i{}^k = \frac{1}{4} T_l{}^l \delta_i^k + \left(T_i{}^k - \frac{1}{4} T_l{}^l \delta_i^k\right).$

4.6 $S^{ikl} = T^{ikl} + T^{kli} + T^{lik} + T^{kil} + T^{ilk} + T^{lki}\,;\ A^{ikl} = T^{ikl} + T^{kli} + T^{lik} - T^{kil} - T^{ilk} - T^{lki}.$

4.9 $B_m = +1/6 e_{mikl} \tilde{B}^{ikl}$, $A_{lm} = -1/2 e_{lmik} \tilde{A}^{ik}$, $J_{klm} = e_{iklm} \tilde{J}^i$. Because e_{iklm} is a pseudotensor, $\tilde{B}_{ikl}$, $\tilde{A}^{ik}$, and $\tilde{J}^i$ are pseudotensors too if B_m, A_{lm}, and J_{klm} are true tensors.

4.10 Vector C_l, lying in plane (A_i, B_k), can be represented as a linear superposition of vectors A_i and B_k:

$$C_l = S A_l + P B_l\,,$$

where S and P are certain invariants. According to (4.30), the dual tensor has the form

$$\tilde{C}^{ik} = \frac{1}{2} e^{iklm} (A_l B_m - A_m B_l) = e^{iklm} A_l B_m\,.$$

Orthogonality of vector C_l and tensor $\tilde{C}^{ik}$ means conversion to zero of their scalar product $C_l \tilde{C}^{lk} = 0$, as is really the case owing to antisymmetry of pseudotensor e^{iklm} with regard to any pair of symbols.

4.11* The volume constructed on three vectors, $\mathrm{d}\boldsymbol{r}$, $\mathrm{d}\boldsymbol{r}'$, and $\mathrm{d}\boldsymbol{r}''$ in a three-dimensional space can be written in the form of a determinant (see Problem 1.14):

$$\mathrm{d}V = \begin{vmatrix} \mathrm{d}x_1 & \mathrm{d}x_2 & \mathrm{d}x_3 \\ \mathrm{d}x_1' & \mathrm{d}x_2' & \mathrm{d}x_3' \\ \mathrm{d}x_1'' & \mathrm{d}x_2'' & \mathrm{d}x_3'' \end{vmatrix} .$$

It has no direction (i.e., it is a pseudoscalar) but may have either a positive or a negative sign. By analogy, the three-dimensional volume in a 4-space can be expressed as a determinant in the form of a rank 3 antisymmetric 4-tensor:

$$V_{ikl} = \begin{vmatrix} A_i & A_k & A_l \\ B_i & B_k & B_l \\ C_i & C_k & C_l \end{vmatrix} .$$

There are four essentially different components of such a tensor (with indices 123, 120, 103, and 023) that can be interpreted as the components of the dual pseudovector:

$$\tilde{V}_i = \frac{1}{6} e^{iklm} V_{klm} = e^{iklm} A_k B_l C_m .$$

The fact that the three-dimensional hypervolume has a direction implies the possibility of different orientation of any element of the three-dimensional hypersurface in a four-dimensional space (to recall, an element of the two-dimensional surface in a three-dimensional space may have different orientation as well). Specifically, the component $\tilde{V}^0$ describes a usual three-dimensional volume constructed on the three-dimensional vectors $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ and taken with a particular sign (depending on which triad (right or left) is formed by vectors $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$). Depending on the sign, it can be directed either along the Ox^0 axis (i.e., into the future) or against it (i.e., into the past). Other components of $\tilde{V}^i$ have an analogous sense.

Any vector G_l belonging to a three-dimensional hypersurface can be expanded into 4-vectors on which it is built:

$$G_l = SA_l + PB_l + QC_l ,$$

where S, P, and Q are the invariants. Vectors G_l and $\tilde{V}^i$ are mutually perpendicular because $G_l \tilde{V}^l = 0$.

4.12 There are the rank 2 three-dimensional tensor $T^{\mu\nu}$, $\mu, \nu = 1, 2, 3$, two three-dimensional vectors $T^{0\mu}$, $T^{\mu 0}$, and the three-dimensional scalar T^{00}.

4.13 Any rank 2 antisymmetric 4-tensor includes the three-dimensional polar vector $\boldsymbol{p}$ and the three-dimensional axial vector $\boldsymbol{a}$:

$$A_{ik} = \begin{pmatrix} 0 & p_x & p_y & p_z \\ -p_x & 0 & -a_z & a_y \\ -p_y & a_z & 0 & -a_x \\ -p_z & -a_y & a_x & 0 \end{pmatrix} .$$

However, the inverse proposition is wrong because by no means do any set of three-dimensional vectors and a pseudovector form a rank 2 4-tensor.

4.16

$$\Lambda^i{}_k = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & \widehat{g} & \\ 0 & & & \end{pmatrix},$$

where the matrix of three-dimensional rotation $\widehat{g}(\alpha_1 \theta \alpha_2)$ is defined in the answer to Problem 1.20.

4.18 The matrix sought, $x^i = \Lambda^i{}_k x'^k$, can be represented in the form of the product

$$\widehat{\Lambda} = \widehat{\Lambda}(\Theta, \Phi)\widehat{\Lambda}(V)\widehat{\Lambda}^{-1}(\Theta, \Phi),$$

where $\widehat{\Lambda}(\Theta, \Phi)$ is the matrix of the spatial turn putting the Ox^3 axis into a new position Ox'^3 determined by the spherical angles Θ and Φ in the old system; $\Lambda(V)$ is the boost along the new Ox'^3 axis. The clockwise rotation through angle Φ around the Ox^3 axis and through angle Θ around the new Ox'^2 axis results in

$$\widehat{\Lambda}(\Theta, \Phi) = \begin{pmatrix} 1 & 0 & 0 & \\ 0 & \cos\Theta\cos\Phi & -\sin\Phi & \sin\Theta\cos\Phi \\ 0 & \cos\Theta\sin\Phi & \cos\Phi & \sin\Theta\sin\Phi \\ 0 & -\sin\Theta & 0 & \cos\Theta \end{pmatrix}.$$

The boost matrix along the Ox'^3 axis is obtained from (4.2) by permutation of rows and columns with numbers 1 and 3:

$$\widehat{\Lambda}(V) = \begin{pmatrix} \cosh\psi & 0 & 0 & \sinh\psi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\psi & 0 & 0 & \cosh\psi \end{pmatrix}.$$

Matrix $\widehat{\Lambda}^{-1}(\Theta, \Phi)$ is obtained by transposition $\widehat{\Lambda}(\Theta, \Phi)$. Multiplication of the matrices (to be performed with a degree of exactness) yields

$$\widehat{\Lambda}(V, \Theta, \Phi) = \begin{pmatrix} \cosh\psi & \sinh\psi\, n_x & \sinh\psi\, n_y & \sinh\psi\, n_z \\ \sinh\psi\, n_x & 1 + (\cosh\psi - 1)n_x^2 & (\cosh\psi - 1)n_x n_y & (\cosh\psi - 1)n_x n_z \\ \sinh\psi\, n_y & (\cosh\psi - 1)n_y n_x & 1 + (\cosh\psi - 1)n_y^2 & (\cosh\psi - 1)n_y n_z \\ \sinh\psi\, n_z & (\cosh\psi - 1)n_z n_x & (\cosh\psi - 1)n_z n_y & 1 + (\cosh\psi - 1)n_z^2 \end{pmatrix},$$

where $n_x = \sin\Theta\cos\Phi$, $n_y = \sin\Theta\sin\Phi$, and $n_z = \cos\Theta$ are the projections of the $\boldsymbol{V}/V$ unit vector onto the spatial axes of the initial system of coordinates.

4.19* Let us write the transformation (4.7) for differentials of the 4-coordinates:

$$\mathrm{d}x^0 = \Lambda^0{}_0\mathrm{d}x'^0 + \Lambda^0{}_\beta\mathrm{d}x'^\beta \,, \quad \mathrm{d}x^\alpha = \Lambda^\alpha{}_0\mathrm{d}x'^0 + \Lambda^\alpha{}_\beta\mathrm{d}x'^\beta \,. \tag{1}$$

We apply these transformations to the origin O' of the spatial coordinates in system S'. In this case, $\mathrm{d}x'^\beta = 0$, and $\mathrm{d}x^\alpha/\mathrm{d}x^0 = V^\alpha/c$ is the velocity of system S' with respect to S. It follows from (1) that

$$\Lambda^\alpha{}_0 = \Lambda^0{}_0 \left(\frac{V}{c}\right) n^\alpha \,, \tag{2}$$

where n^α is the projection of the unit vector onto the spatial axes. The equality (4.16) and the fact that $n^1n^1 + n^2n^2 + n^3n^3 = 1$ lead to

$$\Lambda^0{}_0 = \left(1 - \frac{V^2}{c^2}\right)^{-1/2} = \cosh\psi \quad \text{and} \quad \Lambda^\alpha{}_0 = n^\alpha \left(\frac{V}{c}\right)\cosh\psi \,. \tag{3}$$

Using (4.10) and assuming $m = 0$ and $l = \alpha = 1, 2, 3$, and taking account of (3), we have

$$\Lambda^0{}_\alpha = \frac{V}{c}\left(n^1\Lambda^1{}_\alpha + n^2\Lambda^2{}_\alpha + n^3\Lambda^3{}_\alpha\right) \,. \tag{4}$$

To find $\Lambda^\mu{}_\alpha$, we take into consideration that this matrix may depend only on the components of vector $\boldsymbol{V} = V\boldsymbol{n}$. Therefore, its general form is

$$\Lambda^\mu{}_\alpha = A\delta^\mu_\alpha + Bn^\mu n_\alpha \,, \tag{5}$$

where A and B depend only on the absolute value of V. It follows from (4) that

$$\Lambda^0{}_\alpha = \frac{V}{c}(A + B)n_\alpha \,. \tag{6}$$

Assuming that $m = \alpha$ and $l = \beta$ in (4.10) and substituting (5) and (6) into it, we obtain the equality

$$\left(\frac{V^2}{c^2}\right)(A + B)^2 n_\alpha n_\beta - A^2\delta_{\alpha\beta} - (2AB + B^2)n_\alpha n_\beta = -\delta_{\alpha\beta} \,. \tag{7}$$

This equality contains terms with tensor $\delta_{\alpha\beta}$ invariant with respect to the turns and terms with tensor $n_\alpha n_\beta$ that can be nullified by rotation of the coordinate system. The terms of both types must separately compensate each other if $A = 1$ and $B = \cosh\psi - 1$ are to be found (the signs are chosen so that the Lorentz transformation can be regarded as proper). It can be easily seen that the elements of the transformation matrix (3), $\Lambda^0{}_\alpha = n_\alpha(V/c)\cosh\psi$ and $\Lambda^\mu{}_\alpha = \delta^\mu_\alpha + (\cosh\psi - 1)n^\mu n_\alpha$, coincide with those obtained in the preceding problem.

4.20 The invariance of the interval leads to the condition $g_{il}\delta\Omega^l{}_k + g_{kl}\delta\Omega^l{}_i = 0$ or (using the rule of lowering indices) to

$$\delta\Omega_{ik} = -\delta\Omega_{ki} \tag{4.136}$$

(asymmetry). This property also holds in the case of raising of one of the indices: $\delta\Omega^i{}_k = -\delta\Omega_k{}^i$. In other words, there are a total of six transformation parameters corresponding to combinations of indices $i, k = 0,1; 0,2; 0,3;\ 1,2; 1,3; 2,3$.

The quantities $\delta\Omega^\alpha{}_0 = \delta\Omega^0{}_\alpha = V^\alpha/c$ are the small angles of pseudorotation in the $(0,\alpha)$ planes. The quantities $\delta\Omega^\alpha{}_\beta = \varphi_{\beta\to\alpha}$ are the small angles of usual turns in the (β,α) planes from the β axis to the α axis.

4.21 $A^i_{,i} = \partial_i A^i = \mathrm{inv}$ is the scalar, 4-divergence; $\partial_k A^i$ is the rank 2 covariant tensor, $\partial_i T^i{}_k$ is the covariant vector.

4.22 The invariance of the d'Alembert operator becomes obvious from writing it in tensor notation:

$$\square = -g^{ik}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^k}\ . \tag{4.137}$$

4.23 By analogy with the components of the 3-rotor, $\partial A_\alpha/\partial x_\beta - \partial A_\beta/\partial x_\alpha$ the four-dimensional rotor should be defined as the antisymmetric 4-tensor

$$F_{ik} = \frac{\partial A_k}{\partial x^i} - \frac{\partial A_i}{\partial x^k} \equiv A_{k,i} - A_{i,k}\ . \tag{4.138}$$

It has six essentially different components and cannot be reduced to the 4-vector. However, it can be matched, according to (4.39), with the rank 2 dual antisymmetric tensor

$$\tilde{F}_{ik} = \frac{1}{2}e^{iklm}F_{lm}\ . \tag{4.139}$$

4.24* Let us distinguish the small element of a two-dimensional hypersurface that can be regarded as flat in the form of a rectangle and introduce the local system of coordinates having the i and zk axes parallel to the sides of the rectangle (Figure 4.5). Consider the integral over the closed boundary of the rectangle:

$$\begin{aligned}
\oint_{(l_a)} A_j \mathrm{d}l^j &= \int_0^{\Delta x^k} \left[A_k(x^i + \Delta x^i, x^k + \eta) - A_k(x^i, x^k + \eta)\right]\mathrm{d}\eta \\
&\quad + \int_0^{\Delta x^i} \left[A_i(x^i + \xi, x^k) - A_i(x^i + \xi, x^k + \Delta x^k)\right]\mathrm{d}\xi \\
&\approx \frac{\partial A_k}{\partial x^i}\Delta x^i \Delta x^k - \frac{\partial A_i}{\partial x^k}\Delta x^i \Delta x^k \\
&= \left(\frac{\partial A_k}{\partial x^i} - \frac{\partial A_i}{\partial x^k}\right)\Delta S^{ik}_{(a)}\ .
\end{aligned}$$

We used the smallness of the rectangle and took into account only low-order terms.

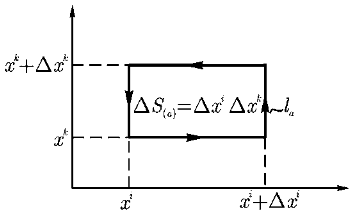

Figure 4.5 Diagram for the demonstration of the Stokes theorem.

The left-hand side of the equality is an invariant independent of the coordinate system; the right-hand side must be a similar invariant too; therefore, ΔS^{ik} is a rank 2 contravariant tensor describing the directed element of a two-dimensional hypersurface. Then, the entire hypersurface of interest should be divided into rectangles that are simultaneously elements of the tangential planes at the respective points of the hypersurface; the above relation can be written for each of them. Summation of the right-hand and left-hand sides of the approximate equality over the entire hypersurface enclosed by the contour l gives

$$\sum_a \oint_{(l_a)} A_j \mathrm{d}l^j \approx \sum_a \left(\frac{\partial A_k}{\partial x^i} - \frac{\partial A_i}{\partial x^k}\right) \Delta S^{ik}_{(a)} ,$$

where the areas of the rectangles tend to zero. The integral sum on the right-hand side of the equality is obtained in the limit depicting the integral over the entire hypersurface. The integrals over the internal sections shared by the adjacent rectangles enter the left-hand side of the equality with different signs and cancel each other. Only the integral over the outer boundaries of the rectangles remains. As a result, we have the equality given in the statement of the problem.

4.26

$$\tilde{F}_{ik} = \begin{pmatrix} 0 & H_x & H_y & H_z \\ -H_x & 0 & E_z & -E_y \\ -H_y & -E_z & 0 & E_x \\ -H_z & E_y & -E_x & 0 \end{pmatrix} ,$$

$$\tilde{F}^{ik} = \begin{pmatrix} 0 & -H_x & -H_y & -H_z \\ H_x & 0 & E_z & -E_y \\ H_y & -E_z & 0 & E_x \\ H_z & E_y & -E_x & 0 \end{pmatrix} .$$

4.27 $I_1 = F_{ik}F^{ik} = -\tilde{F}_{ik}\tilde{F}^{ik} = 2(H^2 - E^2)$, $I_2 = F_{ik}\tilde{F}^{ik} = -4\boldsymbol{E}\cdot\boldsymbol{H}$.

4.28*

1. If in the initial system $\boldsymbol{E} \times \boldsymbol{H} = 0$ (parallel intensities), this property will be preserved in any system moving along the common direction of $\boldsymbol{E}$ and $\boldsymbol{H}$ with an arbitrary velocity $V < c$.
   If in the initial system $\boldsymbol{E} \times \boldsymbol{H} \neq 0$, then the condition of parallelism $\boldsymbol{E}' \times \boldsymbol{H}' = 0$ can be used to find the system with the desired properties moving in the direction of $\boldsymbol{E} \times \boldsymbol{H}$:

$$\frac{\boldsymbol{V}}{c} = \frac{E^2 + H^2 - \sqrt{(E^2 - H^2)^2 + 4(\boldsymbol{E} \cdot \boldsymbol{H})^2}}{2(\boldsymbol{E} \times \boldsymbol{H})^2} \boldsymbol{E} \times \boldsymbol{H} , \quad \frac{V}{c} < 1 .$$

   In any other inertial frame moving with respect to this newly found system in the common direction $\boldsymbol{E}'$ and $\boldsymbol{H}'$, the intensities are also parallel and have the values

$$E'^2 = (1/2) \left[E^2 - H^2 + \sqrt{(E^2 - H^2)^2 + 4(\boldsymbol{E} \cdot \boldsymbol{H})^2} \right] ,$$

$$H'^2 = (1/2) \left[H^2 - E^2 + \sqrt{(E^2 - H^2)^2 + 4(\boldsymbol{E} \cdot \boldsymbol{H})^2} \right] .$$

2. At $\boldsymbol{E} \cdot \boldsymbol{H} = 0$ and $H^2 - E^2 > 0$ the electric field can be nullified and at $\boldsymbol{E} \cdot \boldsymbol{H} = 0$ and $E^2 - H^2 > 0$ the magnetic field can be turned to zero. The formulas from the first task are preserved, suggesting that

$$\boldsymbol{V} = c \frac{\boldsymbol{E} \times \boldsymbol{H}}{H^2} , \quad \boldsymbol{H}' = \frac{\boldsymbol{H}}{H} \sqrt{H^2 - E^2} \quad \text{at} \quad H > E ,$$

$$\boldsymbol{V} = c \frac{\boldsymbol{E} \times \boldsymbol{H}}{E^2} , \quad \boldsymbol{E}' = \frac{\boldsymbol{E}}{E} \sqrt{E^2 - H^2} \quad \text{at} \quad H < E .$$

   If $E = H$, the system being sought realizable in macroscopic bodies is absent because $V = c$.
3. Owing to invariance of $\boldsymbol{E} \cdot \boldsymbol{H}$, the property of orthogonality $\boldsymbol{E}' \cdot \boldsymbol{H}' = 0$ must be fulfilled in all inertial systems, including the initial one. If $\boldsymbol{E} \cdot \boldsymbol{H} \neq 0$, then $\boldsymbol{E}' \cdot \boldsymbol{H}' \neq 0$.
4. $E' = H'$ is possible only under the condition that $E = H$.

4.29 At $\kappa < J/c$ in a frame of reference moving with velocity $V = c^2\kappa/J$ parallel to the cylinder axis in the direction of the $\boldsymbol{E} \times \boldsymbol{H}$ vector, one has

$$E' = 0 , \quad H' = \frac{2J}{cr} \sqrt{1 - \frac{c^2\kappa^2}{J^2}} .$$

At $\kappa > J/c$ in a frame of reference moving with velocity $V = J/\kappa$ parallel to the cylinder axis in the direction of the $\boldsymbol{E} \times \boldsymbol{H}$ vector, one has

$$H' = 0 , \quad E' = \frac{2\kappa}{r} \sqrt{1 - \frac{J^2}{c^2\kappa^2}} .$$

At $\kappa = J/c$ there is no reference frame containing either an electric or a magnetic field alone. It follows from the above formulas that at $\kappa \to J/c$ the velocity of such a system would tend toward c and the strengths of both fields would tend toward zero.

4.30•

$$\partial_i F^{ik} = \frac{4\pi}{c} j^k\,, \quad \partial_k j^k = 0, \tag{1}$$

$$F_{ik,l} + F_{kl,i} + F_{li,k} = 0. \tag{2}$$

The second Maxwell equation can be written in a more compact form through the dual tensor $\tilde{F}^{ik} = e^{iklm}\partial_l A_m$:

$$\partial_i \tilde{F}^{ik} = 0\,. \tag{3}$$

It follows from equations (1) that the totality of quantities $j^k = (c\rho, \boldsymbol{j})$ forms the *four-dimensional current density vector.*

4.31*

1. At a fixed moment of time ($\mathrm{d}t = 0$), we obtain equations of the form $\boldsymbol{E}\cdot\mathrm{d}\boldsymbol{r} = 0$, $\mathrm{d}\boldsymbol{r} \times \boldsymbol{H} = 0$. It follows from the second equation that $\mathrm{d}\boldsymbol{r} \parallel \boldsymbol{H}$, that is, $\mathrm{d}\boldsymbol{r}$ is an element of the magnetic line of force. System (2) can be written in the form $F_{ik}\mathrm{d}x^k = 0$, which suggests its relativistic invariance.
2. The condition of compatibility for this system has the form $\boldsymbol{E}\cdot\boldsymbol{H} = 0$. It is relativistically invariant and indicates that the relativistically invariant magnetic lines can be introduced only for mutually perpendicular electric and magnetic fields.
3. The integration condition for the system has the form

$$\boldsymbol{H} \times \left(\mathrm{curl}\,\boldsymbol{E} + \frac{1}{c}\frac{\partial \boldsymbol{H}}{\partial t}\right) - \boldsymbol{E}\mathrm{div}\,\boldsymbol{H} = 0\,,$$

   or, in the covariant form, $F_{ik}e^{klmn}\partial_l F_{mn} = 0$; it is always fulfilled by reason of the Maxwell equation.
4. Writing equation (1) in the form ($\boldsymbol{E} \perp \boldsymbol{H}$),

$$\mathrm{d}\boldsymbol{r} = \frac{\boldsymbol{H}(\boldsymbol{H}\cdot\mathrm{d}\boldsymbol{r})}{H^2} + c\frac{\boldsymbol{E}\times\boldsymbol{H}}{H^2}\mathrm{d}t\,,$$

   we confirm the validity of the statement made in the condition for the problem.

4.32* In the three-dimensional system, the system specified in the statement of the problem takes the form

$$\mathrm{d}\boldsymbol{r} \times \boldsymbol{E} - c\boldsymbol{H}\mathrm{d}t = 0\,, \quad \boldsymbol{H}\cdot\mathrm{d}\boldsymbol{r} = 0\,,$$

which suggests the fulfillment of the parallelism condition $\mathrm{d}\boldsymbol{r} \times \boldsymbol{E} = 0$ for the increment of $\mathrm{d}\boldsymbol{r}$ and the electric vector $\boldsymbol{E}$ at any fixed moment of time ($\mathrm{d}t = 0$). The equations are compatible so far as $\boldsymbol{E} \cdot \boldsymbol{H} = 0$ and are integrable at

$$\boldsymbol{E} \times \left(\operatorname{curl} \boldsymbol{H} - \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t}\right) + \boldsymbol{H} \operatorname{div} \boldsymbol{E} = 0 .$$

The last equation imposes the following constraint on the charge and current distribution:

$$\boldsymbol{E} \times \boldsymbol{j} + c\rho \boldsymbol{H} = 0 .$$

Invariant force lines of the electric field cannot be introduced if the above conditions fail to be fulfilled. The lines of force move across their own direction at speed $\boldsymbol{u} = -c\boldsymbol{E} \times \boldsymbol{H}/E^2$.

4.34•

$$\varphi = \frac{e}{R^*} , \quad A = \frac{e\boldsymbol{V}}{cR^*} ,$$
$$\boldsymbol{E} = \frac{e\boldsymbol{R}}{\gamma^2 R^{*3}} = \frac{e\boldsymbol{R}\left(1 - V^2/c^2\right)}{R^3\left(1 - V^2/c^2 \sin^2\vartheta\right)^{3/2}} , \quad \boldsymbol{H} = \frac{\boldsymbol{V}}{c} \times \boldsymbol{E} ,$$

where $R^* = \sqrt{(x - Vt)^2 + (1 - \beta^2)(y^2 + z^2)}$, $(Vt, 0, 0)$ are the coordinates of a moving charge at time t, $\boldsymbol{R} = (x - Vt, y, z)$ is the radius vector connecting the charge and the observation point at moment t, and ϑ is the angle between $\boldsymbol{R}$ and $\boldsymbol{V}$.

To represent the 4-potential in the covariant form, we introduce the 4-velocity of the particle u^k and the 4-vector $R^k = (c(t - t'), \boldsymbol{R}(t'))$, where t' is the moment of field generation by the particle and $\boldsymbol{R}(t')$ is the 3-vector connecting the observation point with the charge position at moment t' (Figure 4.6). We assume $x = 0$ without restricting the generality.

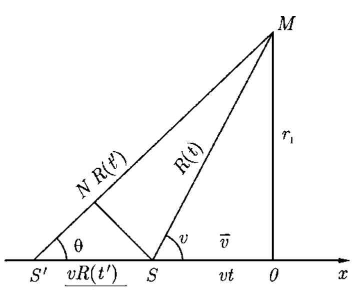

Figure 4.6 The explanation for the calculation of the field of a moving point charge.

R^k is a light-like vector, that is, $R^k R_k = 0$ and $c(t - t') = R(t')$. It follows from Figure 4.6 that the SS' segment equals the product of the particle's velocity and the delay time $vR(t')/c$; therefore, $S'N = \boldsymbol{v} \cdot \boldsymbol{R}(t')/c$, and $R(t') \sin\vartheta = r_\perp$. From the right-angled triangles MNS and MSO, we have $(MN)^2 = (MS)^2 - (NS)^2 = 3r_{\perp}^2| + (vt)^2 - (vR(t')/c)^2 \sin^2\vartheta = R^{*2}$.

On the other hand, $MN = S'M - S'N = R(t') - \boldsymbol{v}\cdot\boldsymbol{R}(t')/c = c(t-t') - \boldsymbol{v}\cdot\boldsymbol{R}(t')/c = \gamma^{-1}R_k u^k$, that is, $R^* = \gamma^{-1}R_k u^k$. Thus, the covariant form of the 4-potential is

$$A^k = \frac{eu^k}{u^l R_l} .$$

4.35 It follows from the formulas from the preceding problem that field E is $1 - V^2/c^2$ times weaker along the line of motion of the charge ($\vartheta = 0, \pi$) than the Coulomb field $E_C = e/R^2$. In the perpendicular direction ($\vartheta = \pi/2$), the field E is $(1 - V^2/c^2)^{-1/2}$ times stronger. At $V \approx c$, the field is high only in the narrow range of angles $\delta\vartheta \approx (1 - V^2/c^2)^{1/2}$ near the equatorial plane.

The condition $E_\parallel = E'_\parallel$ holds for the same points of the 4-space. But if a certain point lies on the x axis at a distance R from the charge in the charge rest system, it will lie $R\sqrt{1-\beta^2}$ apart in the laboratory reference frame. The comparison of $E_\parallel$ values at point $R\sqrt{1-\beta^2}$ and $E'_\parallel$ values at point R yields

$$E_\parallel = \frac{eR\sqrt{1-\beta^2}(1-\beta^2)}{(R\sqrt{1-\beta^2})^3} = \frac{e}{R^2} = E'_\parallel ,$$

as could be expected.

4.36•

$$\varphi = \frac{\boldsymbol{p}_0\cdot\boldsymbol{r}_*}{\gamma r_*^3} , \quad \boldsymbol{A} = \frac{\boldsymbol{V}}{c}\varphi , \quad \boldsymbol{E} = \frac{3\boldsymbol{R}(\boldsymbol{p}_0\cdot\boldsymbol{r}_*) - \boldsymbol{p}_0 r_*^2}{\gamma^2 r_*^5} , \quad \boldsymbol{H} = \frac{\boldsymbol{V}}{c}\times\boldsymbol{E} ,$$

where $\boldsymbol{R} = (x - Vt, y, z)$ and $\boldsymbol{r}_* = (x - Vt, y/\gamma, z/\gamma)$. The dipole moves along the x axis at moment t at a point with radius vector $\boldsymbol{V}t$.

4.37• By definition,

$$\boldsymbol{p} = \int \boldsymbol{r}\rho \mathrm{d}V , \quad \boldsymbol{m} = \frac{1}{2c}\int \boldsymbol{r}\times\boldsymbol{j}\mathrm{d}V .$$

The quantity $\mu^{\alpha\beta} = -\frac{1}{2c}(x^\alpha j^\beta - x^\beta j^\alpha)$ can be regarded as the spatial part of the rank 2 antisymmetric tensor and $\mu^{0\beta} = \frac{1}{c}j^0 x^\beta$ can be regarded as the elements of its first row (provided that $\mu^{00} = 0$). This follows from the fact that $(c\rho, \boldsymbol{j}) = j^k$ give rise to the 4-vector (see Problem 4.30•). Supplementing the first column by elements $\mu^{i0} = -\mu^{0i}$ yields the antisymmetric 4-tensor of "dipole moment density"[19] μ^{ik}.

19) The inverted commas are used here because the quantity introduced can be regarded as density only in the purely formal sense (the integral from it over volume gives the dipole moments of the system). Such an interpretation is much more justified for macroscopic bodies.

The ensemble of $\boldsymbol{p}$ and $\boldsymbol{m}$ is expressed in the form of the integral $\int \mu^{ik} \mathrm{d}V$:

$$p_\alpha = \int \mu^{0\alpha} \mathrm{d}V = -\int \mu^{\alpha 0} \mathrm{d}V\,, \quad m_\gamma = -\int \mu^{\alpha\beta} \mathrm{d}V$$

(in the last case, indices α, β, and γ should be cyclically repositioned). It follows from these formulas that the moments $(\boldsymbol{p}, \boldsymbol{m})$ are transformed as the product of a rank 2 antisymmetric 4-tensor and the volume and do not create by themselves a 4-tensor. Using rule (4.70) for the transformation of the antisymmetric tensor and rule (3.13) for volume transformation gives

$$\boldsymbol{p} = \boldsymbol{p}_0 - (\gamma - 1)\frac{(\boldsymbol{V} \cdot \boldsymbol{p}_0)\boldsymbol{V}}{\gamma V^2} + \frac{\boldsymbol{V}}{c} \times \boldsymbol{m}_0\,, \quad \boldsymbol{m} = \boldsymbol{m}_0 - (\gamma - 1)\frac{(\boldsymbol{V} \cdot \boldsymbol{m}_0)\boldsymbol{V}}{\gamma V^2} - \frac{\boldsymbol{V}}{c} \times \boldsymbol{p}_0\,.$$

4.38 Using the formulas for the transformation of four-dimensional current density, we find that sides 2 and 4 of the rectangle (Figure 4.7) are not charged, whereas sides 1 and 3 carry charges $q_1 = -q_3 = -V J' a/c^2$, where J' is the current in system S' connected to the loop. Hence (or from the results from the previous problem) the electric dipole moment of the loop present in S' equals $p = q_3 b = V m'/c$, where $m' = J' a b/c$ is the magnetic moment of the loop observed in system S'.

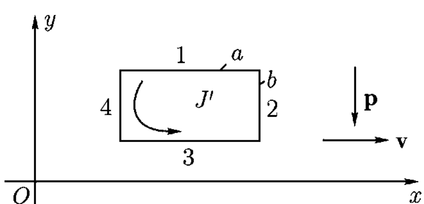

Figure 4.7 The explanation for a moving loop with electric current.

4.41

$$\frac{m\dot{\boldsymbol{v}}}{(1 - v^2/c^2)^{1/2}} + \frac{m\boldsymbol{v}(\boldsymbol{v} \cdot \dot{\boldsymbol{v}})}{c^2(1 - v^2/c^2)^{3/2}} = \boldsymbol{F}\,;$$

Special cases:

$$\frac{m\dot{\boldsymbol{v}}}{(1 - v^2/c^2)^{3/2}} = \mathcal{F} \quad \text{at} \quad \boldsymbol{v} \parallel \mathcal{F},$$

$$\frac{m\dot{\boldsymbol{v}}}{(1 - v^2/c^2)^{1/2}} = \mathcal{F} \quad \text{at} \quad \boldsymbol{v} \perp \mathcal{F},$$

$$m\dot{\boldsymbol{v}} = \mathcal{F} \quad \text{at} \quad v \ll c\,.$$

In the old scientific literature, the quantities $m(1 - v^2/c^2)^{-3/2}$ and $m(1 - v^2/c^2)^{-1/2}$ were referred to as longitudinal and transverse masses, respectively. However, it is impossible at an arbitrary angle between the force and velocity in a relativistic case to distinguish any factor as a proportionality coefficient between the force and acceleration to which the sense of mass could be attributed. Therefore, the terms "relativistic mass" and "velocity-dependent mass" cannot be defined correctly and their use appears impractical. The sole remaining mass, m, is a velocity-independent invariant quantity.

4.42

$$\mathcal{F} = \frac{1}{\gamma}\mathcal{F}' + \left(1 - \frac{1}{\gamma}\right)\frac{(\boldsymbol{v}\cdot\boldsymbol{F}')\boldsymbol{v}}{v^2}\,, \quad \mathcal{F}' = \gamma\mathcal{F} - (\gamma - 1)\frac{(\boldsymbol{v}\cdot\boldsymbol{F})\boldsymbol{v}}{v^2}\,,$$

where $\gamma = (1 - v^2/c^2)^{-1/2}$.

4.43 $\mathcal{F} = \gamma^2 \frac{mv^2}{R}$.

4.45

$$\psi(r, \alpha) = -\frac{2\kappa(1-\beta^2)}{\sqrt{(1-\beta^2)\cos^2\alpha + \sin^2\alpha}}\ln r\,,$$

where $\beta = v/c$ and r is the distance between the observation point and the wire.

4.46

$$\mathcal{F} = \frac{2e\kappa}{\gamma r}\,.$$

The problem can be solved in different ways:

1. By directly calculating the electromagnetic force acting on a moving point charge from the side of the linear charge and the current (take account of the Lorentz reduction!).
2. By determining the force in a reference frame that lacks a magnetic field and using the formulas for 4-force transformation.
3. By making use of the convection potential ψ obtained in Problem 4.45.

4.47

$$\mathcal{F} = e\left(1 - \frac{v^2}{c^2}\right)\frac{2J(r)}{vr}\,,$$

where r is the distance between an electron and the beam axis,

$$J(r) = \frac{2\pi v}{\sqrt{1-\beta^2}}\int\limits_0^r \rho(r) r\mathrm{d}r$$

is the current flowing through a circle of radius r, and

$$v = \left(1 + \frac{eV}{mc^2}\right)^{-1} \left(1 + \frac{eV}{2mc^2}\right) \sqrt{\frac{2eV}{m}}$$

is the velocity of electrons.

An electron at the beam surface is subject to force $\mathcal{F} = e\left(1 - v^2/c^2\right)(2J/va)$, where a is the beam radius.

4.48 An external electron is accelerated normally to the beam axis and the electron's velocity; therefore, in the laboratory frame of reference we have

$$\dot{v}_n = \frac{(1 - v^2/c^2)^{1/2}}{m}\mathcal{F} = \frac{2eJ(1 - v^2/c^2)^{3/2}}{mav}$$

(the solutions of Problems 4.41 and 4.47 can be used). Beam broadening

$$\Delta a = \frac{\dot{v}_n t^2}{2} = \frac{\dot{v}_n L^2}{2v^2} \ .$$

In accordance with the condition $\Delta a \ll L$, $\dot{v}_n L/v \ll v$ or $\dot{v}_n t \ll v < c$. Thus, the use of the nonrelativistic formula for the computation of Δa is justified.

The same value of Δa can be obtained by considering beam broadening in the reference frame moving together with beam electrons; in such a system, the electrons are subject to the electric force alone.

4.49* Integration of equations of motion written in the covariant form,

$$\frac{dp_x}{d\tau} = \frac{eE}{mc^2}\mathcal{E} \ , \quad \frac{dp_y}{d\tau} = 0, \quad \frac{dp_z}{d\tau} = 0 \ , \quad \frac{d\mathcal{E}}{d\tau} = \frac{eE}{m}p_x \ ,$$

yields the energy and momentum as the proper time function:

$$p_x = \left(\frac{\mathcal{E}_0}{c}\right) \sinh \kappa E\tau + p_{0x} \cosh \kappa E\tau \ , \quad p_y = p_{0y} \ , \quad p_z = 0 \ ,$$

$$\mathcal{E} = \mathcal{E}_0 \cosh \kappa E\tau + c p_{0x} \sinh \kappa E\tau \ , \quad \kappa = \frac{e}{mc} \ .$$

Repeated integration of equations $dx/d\tau = p_x/m$ and so on makes it possible to obtain the 4-coordinates as the proper time function. Specifically,

$$x^0(\tau) = ct = \frac{\mathcal{E}_0}{eE} \sinh \kappa E\tau + \frac{c p_{0x}}{eE} (\cosh \kappa E\tau - 1) \ .$$

It follows from the last equation that

$$\tau(t) = \frac{mc}{eE} \ln \frac{p_{0x} + eEt + \sqrt{(p_{0x} + eEt)^2 + m^2c^2 + p_{0y}^2}}{p_{0x} + \mathcal{E}_0/c} \ .$$

Note that $\tau > 0$ and that τ increases further with t regardless of the sign of the charge, at $e > 0$ and $e < 0$. Substituting τ expressed through t into the formulas for $x(\tau)$, $y(\tau)$, and $\mathcal{E}(\tau)$ yields

$$x(t) = \frac{c}{eE}\left[\sqrt{(p_{0x} + eEt)^2 + m^2c^2 + p_{0y}^2} - \frac{\mathcal{E}_0}{c}\right],$$

$$y(t) = \frac{cp_{0y}}{eE}\tau(t), \quad z(t) = 0,$$

$$\mathcal{E}(t) = \sqrt{\mathcal{E}_0^2 - c^2p_{0x}^2 + (cp_{0x} + eEct)^2}.$$

At $p_0 \ll mc$ and $t \ll mc/|e|E$ the motion is nonrelativistic and the expressions for x, y, and z transform into the usual nonrelativistic formulas of uniformly accelerated motion:

$$x(t) = \frac{p_{0x}}{m}t + \frac{eE}{2m}t^2, \quad y(t) = \frac{p_{0y}}{m}t.$$

After a rather long lapse of time since the onset of motion ($t \gg mc/|e|E$), the particle's velocity becomes close to c (even if it was initially low). Then,

$$x(t) = ct - \frac{mc^2}{eE}, \quad y(t) = \frac{cp_{0y}}{eE}\ln\frac{2|e|Et}{mc}$$

and the motion becomes uniform (with velocity c). The courses of $x(t)$ and $y(t)$ are shown in Figure 4.8. The motion observed at $p_{0y} = 0$ (see Figure 4.8a) is referred to as hyperbolic one.

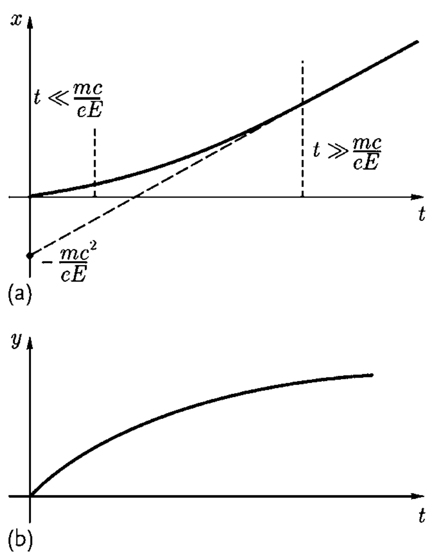

Figure 4.8 Coordinates of a particle moving in electric field as a function of time; $x(t)$ (a) and $y(t)$ (b).

4.50 The particle's trajectory is defined by the equation

$$x = \frac{\mathcal{E}_0}{eE}\left(\cosh\frac{Ee}{cp_{0y}}y - 1\right) + \frac{cp_{0x}}{eE}\sinh\frac{eE}{cp_{0y}}y\ .$$

In the nonrelativistic limit $\mathcal{E}_0 = mc^2$, $p_0 \ll mc$, and $|eEy| \ll |cp_{0y}$. The latter follows from the fact that $eE\tau$, that is, the momentum acquired by the particle, must be small compared with mc in the nonrelativistic case. Thus,

$$x = \frac{meEy^2}{2p_{0y}^2} + \frac{p_{0x}}{p_{0y}}y\ .$$

4.51

$$l = \frac{\mathcal{E} - mc^2}{eE}\ .$$

4.52* The four-dimensional equation of motion

$$\frac{\mathrm{d}p^k}{\mathrm{d}\tau} = \frac{e}{mc}F^{kl}p_l$$

gives the equations for the 4-momentum components:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\mathcal{E}}{c}\right) = 0\ , \quad \frac{\mathrm{d}p_x}{\mathrm{d}\tau} = -\omega_\mathrm{c}p_y\ , \quad \frac{\mathrm{d}p_y}{\mathrm{d}\tau} = \omega_\mathrm{c}p_z\ , \quad \frac{\mathrm{d}p_z}{\mathrm{d}\tau} = 0\ ,$$

where the *cyclotron frequency* $\omega_\mathrm{c} = eH/mc$ is either positive or negative depending on the sign of the charge. It follows from the equations and initial conditions that

$$\mathcal{E} = \mathcal{E}_0 = \sqrt{m^2c^4 + c^2p_0^2} = \text{const}\ , \quad p_z = p_{0z} = \text{const}\ ,$$

$$p_x = p_{0\perp}\cos(\omega_\mathrm{c}\tau + \alpha)\ , \quad p_y = p_{0\perp}\sin(\omega_\mathrm{c}\tau + \alpha)\ .$$

The use of equations $\mathrm{d}x/\mathrm{d}\tau = p_x/m$ and so on helps us to find the 4-coordinates as the proper time function:

$$x(\tau) = \frac{p_{0\perp}}{m\omega_\mathrm{c}}\sin(\omega_\mathrm{c}\tau + \alpha) + x_0 - \frac{p_{0y}}{m\omega_\mathrm{c}}\ , \quad t = \frac{\mathcal{E}_0}{mc^2}\tau\ ,$$

$$y(\tau) = -\frac{p_{0\perp}}{m\omega_\mathrm{c}}\cos(\omega_\mathrm{c}\tau + \alpha) + y_0 + \frac{p_{0x}}{m\omega_\mathrm{c}}\ , \quad z(\tau) = \frac{p_{0z}}{m\omega_\mathrm{c}}\tau + z_0\ .$$

The absolute value of the particle's momentum remains constant and it rotates around the direction of the magnetic field with angular velocity ω_c (as the proper time function) or with a lower angular velocity

$$\Omega = \frac{\omega_\mathrm{c}mc^2}{\mathcal{E}_0} = \omega_\mathrm{c}\gamma_0^{-1}$$

as a function of coordinate time. The particle itself moves along the helical line wound around a circular cylinder of radius

$$R_{\perp} = \frac{c p_{0\perp}}{|e| H} = \frac{c p_{\perp}}{|e| H}$$

(the *Larmor*[20] *radius* or *gyroradius* of the particle) with pitch $h = 2\pi |v_{0z}|/|\Omega|$. The cylinder axis coincides with a magnetic field force line having coordinates x_0, y_0 in the xOy plane.

4.53*

$$v_x = \frac{e E_y}{m\omega_c} + v_{0x} \cos \omega_c t \,, \quad x = \frac{e E_y}{m\omega_c} + \frac{v_{0x}}{\omega_c} \sin \omega_c t \,,$$

$$v_y = -v_{0x} \sin \omega_c t \,, \quad y = \frac{v_{0x}}{\omega_c} (\cos \omega_c t - 1) \,,$$

$$v_z = \frac{e E_z}{m} t + v_{0z} \,, \quad z = \frac{e E_z}{2m} t^2 + v_{0z} \,,$$

where $\omega_c = eH/mc$, $a\omega_c = v_{0x} - cE_y/H$.

The uniform motion occurs along the z axis under the action of the z component of the electric field. The motion in the xy plane is actually the circulation of a charge in a uniform magnetic field around a circle of radius $R = v_{0x}/\omega_c = cp_{0x}/eH$ with the center moving ("drifting") uniformly in the direction perpendicular to the plane $(\boldsymbol{E}, \boldsymbol{H})$. Drift velocity

$$v_E = \frac{e E_y}{H} \,.$$

The possible projections of the particle's path onto the plane are shown in Figure 4.9. The first, third, fifth, and seventh trajectories from top to bottom are trochoidal curves of the general form and the second and sixth trajectories from top to bottom are cycloids. The motion is nonrelativistic at all time moments when $v_0 \ll c$ and $E_y/H \ll 1$.

4.54

$$p_x = p_{0x} \cos \kappa H\tau - p_{0y} \sin \kappa H\tau \,, \quad p_y = p_{0y} \cos \kappa H\tau - p_{0x} \sin \kappa H\tau \,,$$

$$p_z = \left(\frac{p_{0z} E}{H}\right) \cosh \kappa E\tau + \left(\frac{\mathcal{E}_0 E}{cH}\right) \sinh \kappa E\tau \,,$$

$$\mathcal{E} = \mathcal{E}_0 \cosh \kappa E\tau + p_{0z} c \sinh \kappa E\tau \,, \quad \kappa = \frac{e}{mc} \,.$$

20) Joseph Larmor (1857–1942) was an English physicist-theorist and mathematician, developed electrodynamics of moving bodies, and the electronic theory of matter.

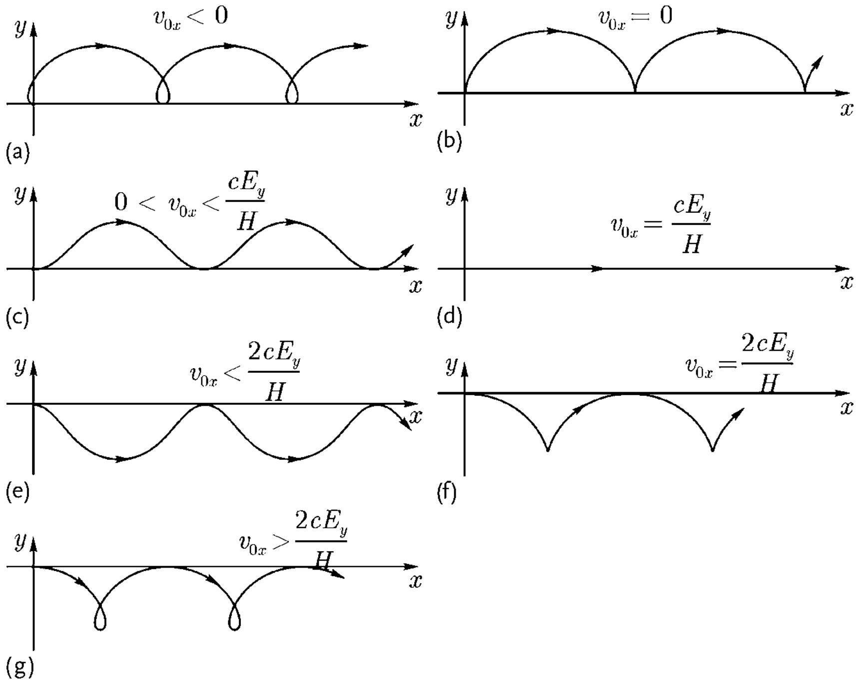

Figure 4.9 Different trajectories of a particle in crossed electric and magnetic fields.

4.55 There is another effective method to solve the problem, besides direct integration of the equation of motion in the frame of reference being considered. It is the Lorentz transformation from a system in which both the electromagnetic field and the particle motion look especially simple. Such a system S' is a system moving with respect to the initial one with velocity $\boldsymbol{v}_E = c\boldsymbol{E} \times \boldsymbol{H}/H^2$ along the Ox axis in which $\boldsymbol{E}' = 0$ and $\boldsymbol{H}' = \boldsymbol{H}\sqrt{H^2 - E^2}/H$ (see Problem 4.28*). Let us put the Oz axis along $\boldsymbol{H}$ and the axis Oy along $\boldsymbol{E}$ and write the solution of the task in system S':

$$p'_x = p'_{0x} \cos\omega'_c\tau - p'_{0y} \sin\omega'_c\tau\,, \quad p'_y = p'_{0x} \sin\omega'_c\tau + p'_{0y} \cos\omega'_c\tau\,,$$

$$p'_z = p'_{0z}\,, \quad \mathcal{E}' = \mathcal{E}'_0\,, \quad \omega'_c = \frac{eH'}{mc} = \frac{e\sqrt{H^2 - E^2}}{mc}\,.$$

Because both τ and ω'_c are invariants, only the momentum components and the energy should be transformed into the initial system.

As a result,

$$\mathcal{E} = \gamma_E^2(\mathcal{E}_0 - v_E p_{0x}) + \gamma_E^2\left(p_{0x} - \frac{v_E\mathcal{E}_0}{c^2}\right) v_E \cos\omega_c'\tau - \gamma_E p_{0y} v_E \sin\omega_c'\tau\ ,$$

$$p_x = \gamma_E^2(\mathcal{E}_0 - v_E p_{0x})\frac{v_E}{c^2} + \gamma_E^2\left(p_{0x} - \frac{v_E\mathcal{E}_0}{c^2}\right)\cos\omega_c'\tau - \gamma_E p_{0y}\sin\omega_c'\tau\ ,$$

$$p_y = p_{0y}\cos\omega_c'\tau + \gamma_E\left(p_{0x} - \frac{v_E\mathcal{E}_0}{c^2}\right)\sin\omega_c'\tau\ ,$$

$$p_z = p_{0z}\ ,\quad \gamma_E = \left(1 - \frac{v_E^2}{c^2}\right)^{-1/2} = \frac{H}{\sqrt{H^2 - E^2}}\ .$$

The proper time dependence of coordinates is calculated by means of single integration from the equations $\mathrm{d}x/\mathrm{d}\tau = p_x/m$ and so on.

4.56

$$\mathcal{E} = \frac{E(\mathcal{E}_0 E - c p_{0x} H)}{E^2 - H^2}\cosh\kappa\sqrt{E^2 - H^2}\tau + \frac{c p_{0y} E}{\sqrt{E^2 - H^2}}\sinh\kappa\sqrt{E^2 - H^2}\tau + \frac{H(c p_{0x} E - \mathcal{E}_0 H)}{E^2 - H^2}\ ,$$

$$p_x = \frac{H(\mathcal{E}_0 E - c p_{0x} H)}{c(E^2 - H^2)}\cosh\kappa\sqrt{E^2 - H^2}\tau + \frac{p_{0y} H}{\sqrt{E^2 - H^2}}\sinh\kappa\sqrt{E^2 - H^2}\tau + \frac{E(c p_{0x} E - \mathcal{E}_0 H)}{c(E^2 - H^2)}\ ,$$

$$p_y = p_{0y}\cosh\kappa\sqrt{E^2 - H^2}\tau + \frac{\mathcal{E}_0 E - c p_{0x} H}{c\sqrt{E^2 - H^2}}\sinh\kappa\sqrt{E^2 - H^2}\tau\ ,$$

$$p_z = p_{0z}\ ,\quad \kappa = \frac{e}{mc}\ .$$

Resolving uncertainties at $E \to H$ leads to

$$\mathcal{E} = \frac{(\mathcal{E}_0 - c p_{0x})\omega_c^2\tau^2}{2} + c p_{0y}\omega_c\tau + \mathcal{E}_0\ ,$$

$$p_x = \frac{(\mathcal{E}_0/c - p_{0x})\omega_c^2\tau^2}{2} + p_{oy}\omega_c\tau + p_{0x}\ ,$$

$$p_y = \left(\frac{\mathcal{E}_0}{c} - p_{0x}\right)\omega_c\tau + p_{0y}\ ,$$

$$p_z = p_{0z}\ ,\quad \omega_c = \frac{eH}{mc}\ .$$

The calculation of the particle's trajectory in the parametric form (with proper time being the parameter) reduces to single integration of the resultant expressions.

4.57* Let us create the tensor of the flat wave electromagnetic field:

$$F_{ik} = A_{k,i} - A_{i,k} = (n_i \varepsilon_k - n_k \varepsilon_i) f'(s) . \tag{1}$$

Because $n^i F_{ik} = 0$, the equation of motion (4.53)

$$m \frac{\mathrm{d}u_i}{\mathrm{d}\tau} = \frac{e}{c} F_{ik} u^k \tag{2}$$

gives $m\mathrm{d}(u_i n^i)/\mathrm{d}\tau = 0$. Hence,

$$u_i n^i = u_i(0) n^i = \text{const} , \tag{3}$$

where the initial value of the particle's 4-velocity is introduced, $u_i(0)$. Substituting $u_i = \mathrm{d}x_i/\mathrm{d}\tau$ into (3) and choosing the 4-coordinates such that $x_i(\tau) = 0$ at $\tau = 0$ yields

$$s = n_i x^i = n_i u^i(0)\tau . \tag{4}$$

Thus, the variables τ and s are different in the constant multiplier and one can pass to an independent variable s in (2):

$$m \frac{\mathrm{d}u_i}{\mathrm{d}s} = \frac{e}{c n_l u^l(0)} \left(\varepsilon_i n_k u^k(0) - n_i \varepsilon_k u^k \right) f'(s) . \tag{5}$$

The last equation is simplified after its multiplication by ε^i: $m\mathrm{d}(u_i \varepsilon^i)/\mathrm{d}s = -\left(\frac{e}{c}\right) f'(s)$; then,

$$u_i \varepsilon^i = u_i(0)\varepsilon^i - \frac{e}{mc}[f(s) - f(0)] . \tag{6}$$

Substituting (6) into the right-hand side of (5) yields a simple equation with the known right-hand part:

$$\frac{\mathrm{d}u_i}{\mathrm{d}s} = \frac{e}{mc}\left[\varepsilon_i - \frac{n_i}{n_l u^l(0)}\left(\varepsilon^k u_k(0) - \frac{e}{mc}(f(s) - f(0))\right)\right] f'(s) . \tag{7}$$

After integration, the particle's 4-velocity as a function of variable s is

$$\begin{aligned} u_i(s) = u_i(0) &+ \frac{e}{mc}\left[\varepsilon_i - n_i \frac{\varepsilon^k u_k(0)}{n^l u_l(0)}\right][f(s) - f(0)] \\ &+ \frac{e^2}{2m^2c^2} \frac{n_i}{n_l u^l(0)} [f(s) - f(0)]^2 , \end{aligned} \tag{8}$$

and the 4-coordinates are

$$x_i(s) = x_i(0) + \int_0^s u_i(s')\mathrm{d}s' .$$

The last two equations give the law of the particle's motion in the parametric form.

Assuming $i = 0$ in equality (8) yields the energy of the particle in the flat wave field:

$$\mathcal{E}(s) = \mathcal{E}_0 + e\left[\varepsilon_0 - n_0 \frac{\varepsilon^k u_k(0)}{n^l u_l(0)}\right][f(s) - f(0)] + \frac{e^2}{2mc}\frac{n_0}{n^l u_l(0)}[f(s) - f(0)]^2 . \quad (9)$$

The formula becomes simplified if the particle was at rest at the initial moment, that is, $\mathcal{E}_0 = mc^2$ and $u_l(0) = (c, 0)$:

$$\mathcal{E}(s) = mc^2 + \frac{e^2}{2mc^2}\left[f(s) - f(0)\right]^2 . \quad (10)$$

If a particle undergoes the action of a wave packet of finite extension, that is, $f(0) = 0$ and $f(s) \to 0$ at $s \to \infty$, it fails to accumulate the summarized energy during the time of wave action, $\mathcal{E}(\infty) = mc^2$. In a periodic field, the particle's energy oscillates about the mean value

$$\bar{\mathcal{E}} = mc^2 + \frac{e^2}{2mc^2}\left[\bar{f}^2 + f^2(0)\right] . \quad (11)$$

The average momentum of an initially resting particle subject to the action of a periodic field differs from zero in the initial frame of reference:

$$\bar{\boldsymbol{p}} = \frac{e}{c}\boldsymbol{\varepsilon} f(0) + \frac{e^2\boldsymbol{n}}{2mc^3}\left[\bar{f}^2 + f^2(0)\right] . \quad (12)$$

The problem can be solved by integrating the equation of motion in the three-dimensional form or by the Hamilton–Jacobi method.

4.58*

$$u^i(s) = u^i(0) + \frac{e}{mc}\left[A^i(s) - A^i(0) + \frac{n^i}{n^l u_l(0)} u_j(0)\left(A^j(s) - A^j(0)\right)\right] - \frac{e^2}{2m^2c^2}\frac{n^i}{n^l u_l(0)}\left[A^j(s) - A^j(0)\right]\left[A_j(s) - A_j(0)\right] ,$$

$$x^i(s) = \int_0^s u^i(s')\mathrm{d}s' + x^i(0) .$$

In the three-dimensional form,

$$r_{\|}(s) = r_{\|}(0) + \frac{c}{w}\int_0^s p_{\|}(s')\mathrm{d}s' ,$$

$$\boldsymbol{r}_{\perp}(s) = \boldsymbol{r}_{\perp}(0) + \frac{c}{w}\int_0^s \boldsymbol{p}_{\perp}(s')\mathrm{d}s' , \quad t = \frac{1}{cw}\int_0^s \mathcal{E}(s')\mathrm{d}s',$$

$$p_{\parallel}(s) = p_{\parallel}(0) + \frac{(\mathcal{E}(s) - \mathcal{E}(0))}{c}, \quad \boldsymbol{p}_{\perp}(s) = \boldsymbol{p}_{\perp}(0) + \frac{e}{c}\int_0^s \boldsymbol{E}(s')\mathrm{d}s',$$

$$\mathcal{E}(s) = \mathcal{E}(0) + \frac{ec}{w}\int_0^s \boldsymbol{E}(s') \cdot \boldsymbol{p}_{\perp}(s')\mathrm{d}s', \quad w = \mathcal{E}(0) - cp_{\parallel}(0).$$

Here, indices $\parallel$ and $\perp$ label parallel and perpendicular directions with respect to $\boldsymbol{n}$.

4.59

$$x = x_0 \cos \omega t, \quad y = y_0 \cosh \omega t, \quad z = v_0 t + z_0, \quad \text{where}$$
$$\omega^2 = \frac{2ek}{m}.$$

The dependencies $x(t)$ and $y(t)$ obtained earlier indicate that a lens of the type under consideration can be used to form a beam of charged particles in the form of a flat band.

4.60

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m\dot{r}}{\sqrt{1 - v^2/c^2}}\right) = \frac{mr\dot{\alpha}^2}{\sqrt{1 - v^2/c^2}} + eE_r + \frac{e}{c}\left(- H_\alpha \dot{z} + H_z r\dot{\alpha}\right),$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{mr^2\dot{\alpha}}{\sqrt{1 - v^2/c^2}}\right) = e\left[E_\alpha + \frac{1}{c}\left(H_r\dot{z} - H_z\dot{r}\right)\right]r,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m\dot{z}}{\sqrt{1 - v^2/c^2}}\right) = e\left[E_z + \frac{1}{c}\left(H_\alpha\dot{r} - H_r\dot{\alpha}\right)\right].$$

4.61* At $H = 0$, the trajectories of the electrons are rectilinear. When the magnetic field is increased, they become increasingly curved in the plane perpendicular to the axis. Let us introduce cylindrical coordinates (r, α, z), where z coincides with the cylinder axis. The electrons cease reaching the anode when their velocity at $r = b$ becomes parallel to the node surface, that is, at $\dot{r}|_{r=b} = 0$. In this case, $\dot{\alpha}|_{r=b} = v_{\max}/b$. Let us use one of the equations from the preceding problem, which in the present case takes the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{mr^2\dot{\alpha}}{\sqrt{1 - v^2/c^2}}\right) = -\frac{e}{c}H(r)r\frac{\mathrm{d}r}{\mathrm{d}t}.$$

Integration of both parts of this equality along the particle path from $r = a$ to $r = b$ yields

$$\left.\frac{mr^2\dot{\alpha}}{\sqrt{1 - v^2/c^2}}\right|_{r=a}^{r=b} = -\frac{e}{2\pi c}\int_a^b 2\pi H r\mathrm{d}r = -\frac{e\Phi}{2\pi c}.$$

Hence,

$$\Phi_{\mathrm{c}} = \frac{2\pi c b}{|e|} p_{\max} = 2\pi c b \sqrt{\frac{2mV}{|e|}\left(1 + \frac{|e|V}{2mc^2}\right)}$$

if the momentum is expressed through the kinetic energy and $T_{\max} = |e|V$.

At a small potential difference $|e|V \ll mc^2$ (or $v \ll c$, which is the same), the result is simplified:

$$\Phi_{\mathrm{c}} = 2\pi c b \sqrt{\frac{2mV}{|e|}} .$$

4.62 The potential difference must be greater than

$$V_{\mathrm{c}} = \sqrt{\frac{4\mathcal{J}^2}{c^2} \ln^2 \frac{b}{a} + \frac{m^2 c^4}{e^2}} - \frac{mc^2}{|e|} .$$

At $|e|V \ll mc^2$ (nonrelativitsic electrons), it follows from the general formula that

$$V_{\mathrm{c}} = \frac{2|e|\mathcal{J}^2}{mc^4} \ln^2 \frac{b}{a} .$$

4.63

$$b = a \exp \frac{p_0 c^2}{|e|\mathcal{J}} , \quad \text{where} \quad p_0 = \frac{m v_0}{\sqrt{1 - v_0^2/c^2}} .$$

4.65•

1. In a nonrelativistic case, the kinetic energy T is a quadratic function of particle velocities. According to the Euler theory of homogeneous functions,

$$2T = \sum_a \frac{\partial T}{\partial \boldsymbol{v}_a} \cdot \boldsymbol{v}_a = \sum_a \boldsymbol{p}_a \cdot \boldsymbol{v}_a ,$$

where summation is performed over all particles of the system. Let us rewrite this equality in the form

$$2T = \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_a \boldsymbol{p}_a \cdot \boldsymbol{r}_a\right) - \sum_a \boldsymbol{r}_a \cdot \dot{\boldsymbol{p}}_a = \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_a \boldsymbol{p}_a \cdot \boldsymbol{r}_a\right) + \sum_a \boldsymbol{r}_a \cdot \frac{\partial U}{\partial \boldsymbol{r}_a}$$

and perform averaging according to the formula

$$\bar{f} = \lim_{\delta t \to \infty} \frac{1}{\delta t} \int_0^{\delta t} f(t) \mathrm{d}t .$$

Because

$$\overline{\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_a \boldsymbol{p}_a \cdot \boldsymbol{r}_a\right)} = \frac{1}{\delta t}\left[\sum_a \boldsymbol{p}_a \cdot \boldsymbol{r}_a \left.\right|_{t=\delta t} - \sum_a \boldsymbol{p}_a \cdot \boldsymbol{r}_a \left.\right|_{t=0}\right]_{\delta t \to \infty} \to 0$$

and by reason of the limitedness of particle momenta and coordinates,

$$2\bar{T} = \sum_a \overline{\boldsymbol{r}_a \cdot \frac{\partial U}{\partial \boldsymbol{r}_a}} . \tag{1}$$

The potential energy during Coulomb interaction is a homogeneous function of coordinates of degree -1; therefore,

$$\sum_a \boldsymbol{r}_a \cdot \frac{\partial U}{\partial \boldsymbol{r}_a} = -U \quad \text{and} \quad 2\bar{T} = -\bar{U} . \tag{2}$$

In this case, the conserved total energy of the system of charged particles is negative, $E = \bar{T} + \bar{U} = -\bar{T} < 0$, because finite motion is feasible only when the total energy is negative.

2. Relation (1) is called the *virial theorem*. It relates $\bar{T}$ to $\bar{U}$ in all cases when the potential energy is a homogeneous function of the coordinates:

$$2\bar{T} + \frac{e}{mc}\bar{\boldsymbol{L}} \cdot \boldsymbol{H} = -\bar{U} , \tag{3}$$

where $\boldsymbol{L} = \sum_a m_a[\boldsymbol{r}_a \times \boldsymbol{v}_a]$ is the total angular momentum of the system.

4.66 The trajectory lies in the plane perpendicular to the angular momentum. $\boldsymbol{l}$ conserved during motion. At $ee' < 0$ (attraction) and $0 > \mathcal{E} \geq -me^2e'^2/(2l^2)$, where $m = m_1m_2/(m_1+m_2)$ is the reduced mass, the motion is finite (the reduced state) and the trajectory is an ellipse described by the equation

$$r = \frac{a}{1 + \epsilon \cos\varphi} . \tag{1}$$

Here,

$$a = \frac{l^2}{m|ee'|} , \quad \epsilon = \sqrt{1 + \frac{2\mathcal{E}l^2}{me^2e'^2}} < 1 , \tag{2}$$

and the angle φ is counted from the line connecting particles at the time of their closest approach. At $\mathcal{E} \to 0(\epsilon \to 1)$, the trajectory takes the form of a parabola and the motion becomes infinite.

At $\mathcal{E} > 0$, the trajectory resembles a hyperbola in both cases, $ee' < 0$ and $ee' > 0$,

$$r = \frac{a}{\mp 1 + \epsilon \cos\varphi} , \tag{3}$$

where $\epsilon > 1$. The plus sign corresponds to attraction, and the second particle is in the inner focus of the hyperbola (Figure 4.10a). The minus sign corresponds to repulsion, and the second particle is in the outer focus of the hyperbola (Figure 4.10b).

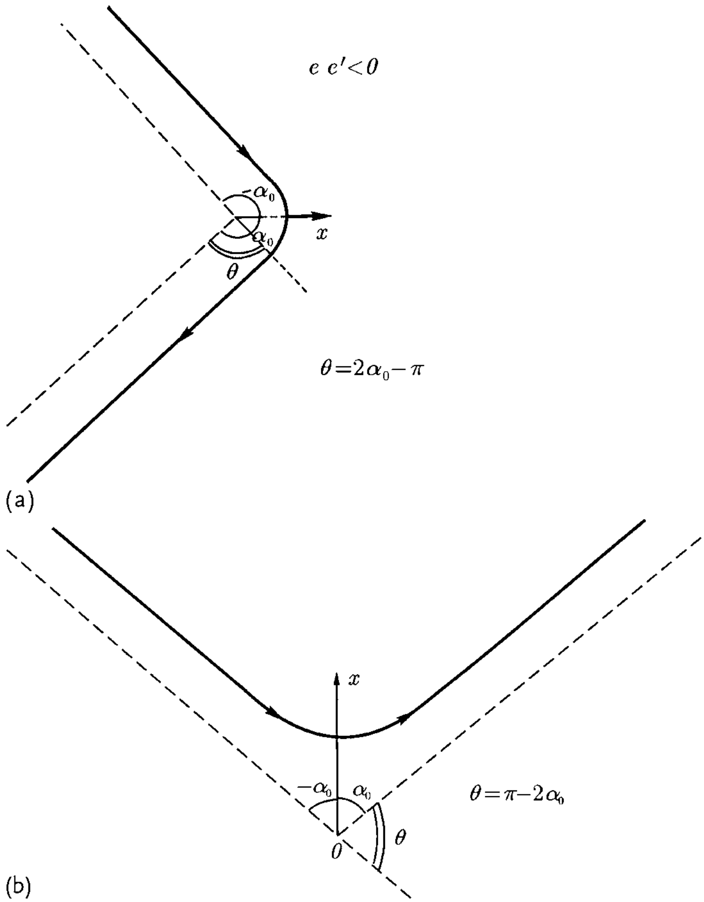

Figure 4.10 The trajectories of infinite motion in a Coulomb field: attraction (a) and repulsion (b).

4.67* The differential scattering cross-section can be calculated from the formula

$$\frac{d\sigma(\theta)}{d\Omega} = \frac{s ds}{\sin\theta \, d\theta}, \tag{1}$$

where θ is the particle scattering angle corresponding to a given value s of the collision parameter (impact parameter). In the present case, θ is the angle between the directions of a particle approaching the scattering center and departing from it; the angle is determined by the asymptotes of the hyperbola. It follows from Figure 4.10a (attraction) and Figure 4.10b (repulsion) that $\theta = 2\alpha_0 - \pi$ and $\theta = \pi - 2\alpha_0$, respectively. The two cases are integrated into the single formula $\theta/2 = \pm(\pi/2 \mp \alpha_0)$. This and the trajectory equation (3) from the previous problem lead to

$$\cot^2\left(\frac{\theta}{2}\right) = \sin^{-2}\left(\frac{\theta}{2}\right) - 1 = \cos^{-2}\alpha_0 - 1 = \epsilon^2 - 1 = \frac{2\mathcal{E} l^2}{m e^2 e'^2}.$$

The angular momentum is expressed through the impact parameter by the formula $l = mv_0 s$. Thus,

$$s^2 = \frac{e^2 e'^2}{m^2 v_0^4} \cot^2 \frac{\theta}{2} \,. \tag{2}$$

Differentiation and substitution into (1) yields the famous *Rutherford*[21] *formula*, with the help of which the atomic nucleus was discovered:

$$\frac{\mathrm{d}\sigma(\theta)}{\mathrm{d}\Omega} = \left(\frac{ee'}{2mv_0^2}\right)^2 \frac{1}{\sin^4(\theta/2)} \,. \tag{3}$$

The cross-section is the same in the cases of attraction and repulsion between the particles.

4.68 In the center of mass of the system,

$$\mathrm{d}\sigma = \left(\frac{e^2}{4\mathcal{E}_c}\right)^2 \left[\frac{1}{\sin^4 \theta/2} + \frac{1}{\cos^4 \theta/2}\right] \mathrm{d}\Omega_c \,,$$

where $\mathcal{E}_c = mv_0^2/2$ and $m = m_1/2$ is the reduced mass.

In the laboratory frame of reference,

$$\mathrm{d}\sigma = \left(\frac{e^2}{2\mathcal{E}_l}\right) \left[\frac{1}{\sin^4 \vartheta} + \frac{1}{\cos^4 \vartheta}\right] 4\cos\vartheta \,\mathrm{d}\Omega_l \,,$$

where $\mathcal{E}_l = m_1 v_0^2/2$ and ϑ is the scattering angle in the laboratory reference frame, $0 \le \vartheta \le \pi/2$.

4.69 Through the area $\mathrm{d}\sigma = s\mathrm{d}s\mathrm{d}\alpha$ of the plane perpendicular the particles' motion, $nv\mathrm{d}\sigma$ particles fly per unit time. They transmit to a particle s at rest momentum equaling

$$m\Delta v_z nv\mathrm{d}\sigma \,, \tag{1}$$

where Δv_z is the change in the z component of the velocity of a single particle undergoing scattering from the motionless particle.

The force being sought and equaling the total momentum transmitted per unit time is found by the integration of (1) over the entire cross-section of the particle beam. Δv_z should be expressed through the impact parameter s. The collisions being elastic,

$$\Delta v_z = -2v \sin^2 \frac{\theta}{2} \,, \tag{2}$$

21) Ernest Rutherford (1871–1937) was an outstanding British physicist, one of the founders of the nuclear physics, and the head of numerous and very fruitful schools of physics.

where θ is the scattering angle. Its relationship with the impact parameter was found in the solution of Problem 4.67* (formula (2)). When using this formula and the preceding relation (2) in conjunction with integration over s, we obtain with formula (1) the following expression for the force:

$$\mathcal{F} = \frac{4\pi}{m} e^2 e'^2 n\lambda \frac{\boldsymbol{v}}{v^3} , \tag{3}$$

where

$$\lambda = \ln\left(s_m \frac{m v^2}{e e'} \right) .$$

At $s_m \to \infty$ corresponding to an unconstrained beam, the value of λ diverges owing to the long-range character of Coulomb forces. Practically speaking, any charge in a neutral system is shielded by charges with the opposite sign; therefore, a given particle interacts only with those particles that pass less than the shielding radius away. That is why s_m under real conditions should be understood as the shielding radius (Debye radius in a plasma, radius of a neutral atom during particle scattering in nonionized matter, etc.).

The quantity λ is called the Coulomb logarithm. It is usually believed that $\lambda =$ const, disregarding its weak dependence on the particle's velocity.

4.70

$$\bar{\mathcal{F}}(\boldsymbol{v}) = -\frac{4\pi}{\mu} e^2 e'^2 \lambda \int \frac{\boldsymbol{v} - \boldsymbol{v}'}{|\boldsymbol{v} - \boldsymbol{v}'|^3} f(\boldsymbol{v}) \mathrm{d}^3 v ,$$

where $\mu = m m'/(m + m')$ is the reduced mass.

The following analogy should be borne in mind: the above expression can be written in the form of the electric force $\mathcal{F} = q\boldsymbol{E}$ acting on the charge $q = -4\pi e^2 e'^2 \lambda/\mu$ in the "electrostatic field"

$$\boldsymbol{E}(\boldsymbol{v}) = -\mathrm{grad}_v \varphi(\boldsymbol{v}) ,$$

where

$$\varphi(\boldsymbol{v}) = \int \frac{f(\boldsymbol{v}')\mathrm{d}^3 v}{|\boldsymbol{v} - \boldsymbol{v}'|}$$

is the "electrostatic potential" satisfying the Poisson equation

$$\Delta_v \varphi(\boldsymbol{v}) = -4\pi f(\boldsymbol{v}) .$$

4.71 The energy of a test particle does not change in collisions with motionless infinitely heavy particles. The alteration of the average momentum is described by the equation

$$\frac{\mathrm{d}\bar{\boldsymbol{p}}}{\mathrm{d}t} = \mathcal{F} , \tag{1}$$

where $\boldsymbol{\mathcal{F}}$ is the average force. It is convenient to perform the calculation with the help of an electrostatic analogy specified in the solution of the previous problem. The distribution by particle velocities is described in terms of the function $f(\boldsymbol{v}) = n\delta(\boldsymbol{v})$. Therefore, $\varphi(\boldsymbol{v}) = n/v$, $\boldsymbol{E}(\boldsymbol{v}) = n\boldsymbol{v}/v^3$, and

$$\boldsymbol{\mathcal{F}} = -\frac{4\pi}{m} e^2 e'^2 n\lambda \frac{\boldsymbol{v}}{v^3} . \tag{2}$$

$\boldsymbol{\mathcal{F}}$ has the character of the "friction force" seeking to diminish the directed particle velocity. But this friction is weaker the greater the particle's velocity ($\mathcal{F} \propto 1/v^2$, "declining friction").

Integration of equation (1) yields

$$\boldsymbol{v}(t) = \boldsymbol{v}_0 \exp\left(-\frac{t}{\tau}\right) ,$$

where $\tau = m^2 v^3 / 4\pi e^2 e'^2 n\lambda$ is the characteristic time during which a particle loses its directed velocity.

4.72

$$\boldsymbol{\mathcal{F}} = \begin{cases} 0 & \text{at } v < v_0 , \\ 4\pi n e^2 e'^2 \lambda \left(\frac{1}{m} + \frac{1}{m'}\right) \frac{\boldsymbol{v}}{v^2} & \text{at } v > v_0 . \end{cases}$$

$$\boldsymbol{\mathcal{F}} = \begin{cases} -4\pi n e^2 e'^2 \lambda \left(\frac{1}{m} + \frac{1}{m'}\right) \frac{\boldsymbol{v}_0}{v_0} & \text{at } \boldsymbol{v} \cdot \boldsymbol{v}_0 > v_0^2 , \\ 4\pi n e^2 e'^2 \lambda \left(\frac{1}{m} + \frac{1}{m'}\right) \frac{\boldsymbol{v}_0}{v_0} & \text{at } \boldsymbol{v} \cdot \boldsymbol{v}_0 < v_0^2 . \end{cases}$$

4.73* An electron traveling with velocity $\boldsymbol{v}$ in a medium of motionless singly charged ions is subject to the friction force

$$\bar{\boldsymbol{\mathcal{F}}} = -\frac{4\pi e^4 n\lambda}{m} \frac{\boldsymbol{v}}{v^3} \tag{1}$$

(see the solution of Problem 4.71).

We note that the velocity dependence of the friction force can also be obtained from the following semiqualitative considerations. The friction force can be defined as the loss of momentum by a particle per unit time in the process of collisions. If the mean time between collisions is τ, and each collision results in the loss of total momentum of roughly mv (resulting in electron deflection by a large angle), then

$$\mathcal{F} \approx \frac{mv}{\tau} . \tag{2}$$

In such a collision, the electron approaches an ion and its kinetic energy becomes close to the potential energy:

$$\frac{mv^2}{2} \approx \frac{e^2}{r} . \tag{3}$$

This approximate equality allows the collision cross-section to be estimated as

$$\sigma \approx \pi r^2 \approx \frac{4\pi e^4}{m^2 v^4} \tag{4}$$

and the mean time between collisions to be estimated as

$$\tau \approx \frac{1}{n\sigma v} \approx \frac{m^2 v^3}{4\pi n e^4} \,. \tag{5}$$

The substitution of τ into (2) yields, taking into account the braking character of the force,

$$\boldsymbol{\mathcal{F}} \approx -\frac{4\pi n e^4 \boldsymbol{v}}{m v^3} \,, \tag{6}$$

which differs from (1) in the absence of the Coulomb logarithm λ. It appears quite natural because estimation by formulas (2)–(5) disregarded distant collisions with minor momentum transmissions whose contribution is given by the Coulomb logarithm.

Let us average formula (1) over possible electron velocities. To this end, we assume

$$\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{v}_T \,, \tag{7}$$

where $\boldsymbol{v}_T$ is the thermal velocity and $\boldsymbol{u}$ is the velocity acquired under the effect of the electric field $\boldsymbol{E}$. At $u \ll v_T$, the denominator in expression (1) may be taken as $v^3 \approx v_T^3$; in the numerator, one cannot disregard $\boldsymbol{u}$ in comparison with $\boldsymbol{v}_T$ because averaging of the directions of the thermal velocity yields $\bar{\boldsymbol{v}}_T = 0$. As a result,

$$\bar{\boldsymbol{\mathcal{F}}} = \frac{4\pi n e^4 \lambda}{m v_T^3} u \,, \tag{8}$$

where $v_T^2 = 3T/m$, provided the Maxwell distribution (T is the temperature in energy units) is applicable. Thus, we obtain $\boldsymbol{\mathcal{F}} \propto u$ at $u \ll v_T$.

At $u \gg v_T$ we assume $v \approx u$ and have

$$\bar{\boldsymbol{\mathcal{F}}} \approx \frac{4\pi n e^4 \lambda}{m u^2} \,, \tag{9}$$

that is, $\bar{\boldsymbol{\mathcal{F}}} \propto 1/u^2$. Obviously, the maximum of $\bar{\boldsymbol{\mathcal{F}}}$ corresponds to the value of $u \approx v_T$, and the formulas (8) and (9) give the same value of

$$\bar{\boldsymbol{\mathcal{F}}}_{\max} \approx \frac{4\pi n e^4 \lambda}{m v_T^2} \,. \tag{10}$$

The approximate shape of the function $\bar{\boldsymbol{\mathcal{F}}}(u)$ is presented in Figure 4.11.

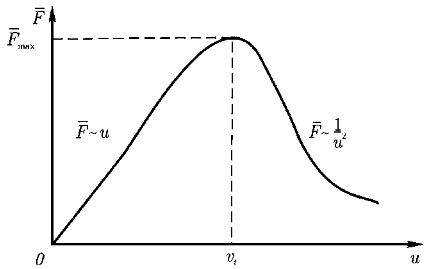

Figure 4.11 Dependence of Coulomb friction on the velocity of a charged particle. Explanation of the phenomenon of runaway electrons.

If $E < \bar{\mathcal{F}}_{\max}/e = E_D$ in the plasma, the decelerating force at a certain value of u satisfying the equation $\bar{\mathcal{F}}(u) = eE$ exceeds the accelerating electric force eE and electrons cannot be accelerated any longer. This is the region of field E values at which the conventional Ohm's law works. In the case of $E > E_D$, the accelerating force overcomes deceleration and electrons can be accelerated unreservedly.[22] This phenomenon is called "runaway electrons."

Substituting the Maxwell value of v_T^2 into formula (10) yields

$$E_D = \frac{e\lambda}{3D^2}\,, \qquad D^2 = \frac{T}{4\pi n e^2}\,. \tag{11}$$

The exact calculation gives a similar value:

$$E_D = 0.214\frac{e\lambda}{D^2}\,. \tag{12}$$

4.74* The problem can be solved in a variety of ways, for example, with the help of equations of motion in cylindrical coordinates (see the answer to Problem 4.60). In what follows, we use the Hamilton–Jacobi method, which is convenient for finding the path of the particle.

Let us write down the Hamilton–Jacobi equation in plane polar coordinates with the polar angle α counted around the direction of the angular momentum vector $\boldsymbol{l}$ conserved in the centrally symmetric field $U(r) = -Ze^2/r$:

$$\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S}{\partial \alpha}\right)^2 - \frac{1}{c^2}\left(\frac{\partial S}{\partial t} - \frac{Ze^2}{r}\right)^2 + m^2c^2 = 0\,. \tag{1}$$

To solve the problem, we need to find action S depending on coordinates r and α, time t, and two independent nonadditive integration constants. Separating variables,

$$S(r,\alpha,t) = W(r) + F(\alpha) + f(t)\,,$$

22) This possibility is as a rule realized only partly owing to collective effects in the plasma.

we find from (1)

$$f(t) = -\mathcal{E}t\,, \quad F(\alpha) = l\alpha\,,$$
$$W(r) = \pm \int \left[\frac{\mathcal{E}^2}{c^2} - m^2c^2 - \frac{l^2c^2 - Z^2e^4}{c^2r^2} + \frac{2Ze^2\mathcal{E}}{c^2r}\right]^{1/2} \mathrm{d}r\,. \tag{2}$$

Here $\mathcal{E}$ and l are the constants introduced in the separation of variables and stand for the total energy and angular momentum of the particle, respectively. In the present case, the total energy $\mathcal{E}$ includes not only the particle's rest and kinetic energies but also the potential energy of its interaction with the Coulomb center:

$$\mathcal{E} = \sqrt{c^2\frac{p_r^2 + p_\alpha^2}{r^2} + m^2c^2} - \frac{Ze^2}{r}\,. \tag{3}$$

It follows from equation (1) if the definitions of generalized momenta, $p_r = \partial S/\partial r$ and $p_\alpha = \partial S/\partial\alpha = l$, are used.

To find the particle's trajectory by the Hamilton–Jacobi method, it is necessary to differentiate the action S with respect to l and equate this derivative to a certain constant, $\partial S/\partial l = \alpha_0$, which should be found from the initial conditions. This leads to the trajectory equation in the form

$$l\int \frac{\mathrm{d}r}{r^2}\left[\frac{\mathcal{E}^2}{c^2} - m^2c^2 - \frac{l^2c^2 - Z^2e^4}{c^2r^2} + \frac{2Ze^2\mathcal{E}}{c^2r}\right]^{-1/2} = \pm(\alpha - \alpha_0)\,. \tag{4}$$

It is convenient to introduce more constants to calculate the integral and classify the trajectories; these constants will generalize to the relativistic cases the quantities used in the corresponding nonrelativistic problem (Problem 4.66):

$$a = \frac{Ze^2}{\mathcal{E}}\left(\frac{1}{\rho^2} - 1\right)\,, \quad \varepsilon = \frac{1}{\rho}\sqrt{1 - \frac{m^2c^4}{\mathcal{E}^2}(1-\rho^2)}\,,$$
$$\text{where} \quad \rho = \frac{Ze^2}{lc}\,. \tag{5}$$

Here, ε and ρ are dimensionless parameters and a has the dimension of length. In what follows, various relationships between the parameters are considered.

Case 1. $\rho = Ze^2/lc < 1, a > 0$. Passing in (4) to the integration over variable $x = a/r$ and choosing $\alpha = 0$ at one of the points where the particle comes closest to the center, we bring the integral into a tabulated form:

$$\int_{1+\varepsilon}^{a/r} \frac{\mathrm{d}x}{\sqrt{\varepsilon^2 - (x-1)^2}} = -\arccos\frac{a/r - 1}{\varepsilon} = \pm\sqrt{1-\rho^2}\,\alpha\,. \tag{6}$$

In this way, the trajectory equation is obtained:

$$r = \frac{a}{1 + \varepsilon \cos \sqrt{1-\rho^2}\,\alpha} . \tag{7}$$

Here again a few variants are feasible. At $\varepsilon < 1$, or, according to (5), at $mc^2 > \mathcal{E} \geq mc^2\sqrt{1-\rho^2}$, the motion is finite. In the general case, the trajectory has the form of an open rosette enclosed between two circles whose radius of motion is finite. In the general case, the trajectory has the form of an open rosette enclosed between two circles of radii $a/(1+\varepsilon)$ and $a/(1-\varepsilon)$ (Figure 4.12). In a nonrelativistic case, such a trajectory is analogous to an ellipse. It can be created by rotating (precession) the ellipse in its own plane. The total variation of the radius from the minimum value $r_{\min} = a/(1+\varepsilon)$ (perigee) to the maximum one $r_{\max} = a/(1-\varepsilon)$ (apogee)[23] and back to a new minimum occurs as α increases by $2\pi/\sqrt{1-\rho^2}$. Thus, orbital perigee turns through the angle $2\pi[(1-\rho^2)^{-1/2}-1]$ during a single period of change of r. If $\sqrt{1-\rho^2}$ is a rational number, the trajectory closes on itself after a certain number of turns.

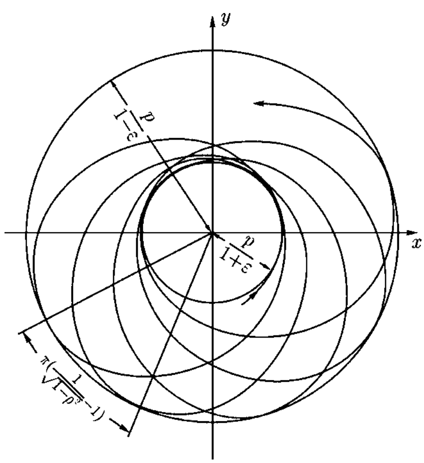

Figure 4.12 Trajectories of a relativistic particle's finite motion in a Coulomb field.

At $\varepsilon > 1$ corresponding to $\mathcal{E} > mc^2$, the motion is infinite. The trajectory resembles a hyperbola (it is obtained from a hyperbola by enlarging the polar angles $(1-\rho^2)^{-1/2}$ times). It has two branches tending toward infinity at $\alpha = \pm\alpha_0$, where $\alpha_0 = (1-\rho^2)^{-1/2} \arccos(-1/\varepsilon)$. A particle approaching the center of one of these branches can make several turns around it before it goes to infinity along the other branch (Figure 4.13). At $\varepsilon = 1(\mathcal{E} = mc^2)$, the motion is infinite too, but the trajectory exhibits a parabola-like shape.

23) Here, astronomy terminology is used. These points could be just as well be called perihelion and aphelion.

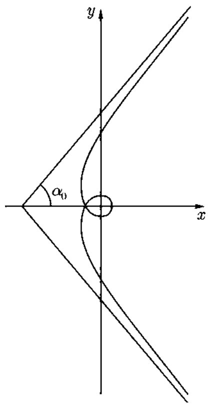

Figure 4.13 The distorted hyperbolic trajectory of a relativistic particle in a Coulomb field.

At $\rho \ll 1$, these trajectories turn into the usual ellipse ($\varepsilon < 1$), hyperbola ($\varepsilon > 1$), and parabola ($\varepsilon = 1$), of the nonrelativistic Kepler problem. This is natural because the condition $\rho \ll 1$ is fulfilled at $v/c \ll 1$.[24]

Case 2. $\rho = Ze^2/lc > 1$, $a = -|a| < 0$. Because the signs in the expression under the radical in the integral of (3) are changed, it needs to be calculated anew; the resulting expression,

$$\int \frac{d(|a|/r)}{\sqrt{(1-|a|/r)^2 - \varepsilon^2}} = Ar\cosh\frac{1+|a|/r}{\varepsilon} = \pm\sqrt{\rho^2-1}\alpha\,, \tag{8}$$

corresponds to the simplest choice of integration constants. Hence, we have the trajectory equation

$$r = \frac{|a|}{-1+\varepsilon\cosh\sqrt{\rho^2-1}\alpha}\,. \tag{9}$$

The helical trajectories wind around the origin of the coordinates at $\alpha \to \pm\infty$. A particle falls onto the center of force (in the nonrelativistic case, a fall onto the center is possible only at $l = 0, \rho = \infty$). At $\mathcal{E} > mc^2$, the $\varepsilon < 1$ and the trajectory has two branches tending toward infinity at $\alpha = \pm\alpha_0$, where $\alpha_0 = (\rho^2-1)^{-1/2}Ar\cosh(1/\varepsilon)$ (Figure 4.14). At $\mathcal{E} < mc^2$, the $\varepsilon > 1$ and the trajectory exhibits the shape depicted in Figure 4.15.

24) An approximate estimate of the ρ value for the bound state in a nonrelativistic case is

$$\rho = \frac{Ze^2}{lc} \approx \frac{Ze^2}{rmvc} \approx \frac{|U|}{mvc}\,.$$

According to the virial theorem (Problem 4.65•), $|\bar{U}| = 2\bar{T} \approx mv^2$, so $\rho \approx v/c \ll 1$. In the case of infinite motion, $|U| \ll 2T$ over most of the trajectory; therefore, ρ is even smaller.

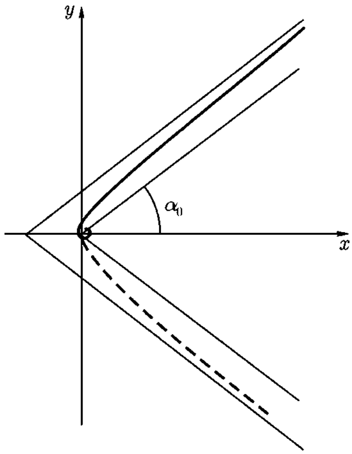

Figure 4.14 The fall of a relativistic particle onto the center (two branches).

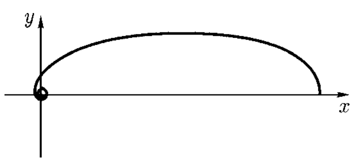

Figure 4.15 The fall of a relativistic particle onto the center (one branch).

Case 3. $\rho = Ze^2/lc = 1$, $a = 0$. Recalculation of the integral leads to the trajectory equation

$$r = \frac{2Ze^2/\mathcal{E}}{\alpha^2 - 1 + m^2c^4/\mathcal{E}^2} .$$

This trajectory is also helical and winds around the center at $\alpha \to \pm\infty$, but more slowly than in the case of $\rho > 1$. The overall character of the trajectory is the same as in case 2.

4.75* At $\mathcal{E} > mc^2$ (infinite motion)

$$\Delta t = \frac{1}{c\sqrt{\mathcal{E}^2 - m^2c^4}} \left\{ \mathcal{E} \left[\sqrt{(r+b)^2 - d^2} - \sqrt{b^2 - d^2} \right] - \frac{Ze^2 m^2 c^4}{\mathcal{E}^2 - m^2c^4} \ln \frac{r + b + \sqrt{(r+b)^2 - d^2}}{b + \sqrt{b^2 - d^2}} \right\} ,$$

where

$$b = \frac{Ze^2\mathcal{E}}{\mathcal{E}^2 - m^2c^4} , \quad d^2 = \frac{l^2c^2}{\mathcal{E}^2 - m^2c^4} + \frac{Z^2e^4m^2c^4}{(\mathcal{E}^2 - m^2c^4)^2} .$$

4.76* In the notation of Problem 4.74* at $\rho < 1$

$$r = \frac{a}{-1 + \varepsilon \cos \sqrt{1-\rho^2}\,\alpha} \,. \tag{4.140}$$

The trajectory resembles a hyperbola (Figure 4.16).

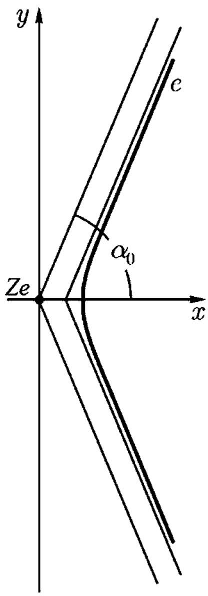

Figure 4.16 Hyperbola-like trajectory of a particle in the repulsion Coulomb field.

Two of its branches tend toward infinity at $\alpha = \pm\alpha_0$, where $\alpha_0 = (1-\rho^2)^{-1/2} \arccos(1/\varepsilon)$. At $\rho \ll 1$ the particle performs hyperbolic motion. This case corresponds to nonrelativistic motion, $v \ll c$. At $\rho > 1$

$$r = \frac{|a|}{1 - \varepsilon \cosh \sqrt{\rho^2 - 1}\,\alpha} \,,$$

where $\varepsilon < 1$. The character of the trajectory is exactly the same as in the first case. Two of its branches tend toward infinity at $\alpha = \pm(\rho^2-1)^{-1/2} a r \cosh(1/\varepsilon)$. In the case of $\rho = 1$,

$$r = \frac{2Ze^2/\mathcal{E}}{1 - m^2c^4/\mathcal{E}^2 - \alpha^2} \,.$$

Two branches tend toward infinity at $\alpha = \pm\sqrt{1 - m^2c^4/\mathcal{E}^2}$.

4.77 In the case of attraction,

$$\theta = \left(\frac{2lc}{\sqrt{l^2c^2 - Ze^2}} - 1 \right) \pi - \frac{2lc}{\sqrt{l^2c^2 - Z^2e^4}} \arctan \frac{v_0\sqrt{l^2c^2 - Z^2e^4}}{Ze^2c} \,,$$

where v_0 is the particle's velocity far from the scattering center. In the case of repulsion,

$$\theta = \pi - \frac{2lc}{\sqrt{l^2c^2 - Z^2e^4}} \arctan \frac{v_0\sqrt{l^2c^2 - Z^2e^4}}{Ze^2c} \,.$$

4.78 Small scattering angles correspond to large impact parameters s. It permits us, assuming $l = p_0 s$, where p_0 is the particle's momentum at $r \to \infty$, to find the sought s dependence of the scattering angle θ by means of limiting transition s (evidently, $l > Ze^2/c$) in the general formulas of the preceding problem. The limiting transition gives the same result in both cases (attraction and repulsion):

$$\theta = \pi - 2\arctan \frac{v_0 p_0 s}{Ze^2} = \frac{2Ze^2}{v_0 p_0 s} \ll 1 \,,$$

whence $s = 2Ze^2/v_0 p_0 \theta$ and

$$\frac{d\sigma(\theta)}{d\Omega} = \left|\frac{s\,ds}{\theta\, d\theta}\right| = 4\left(\frac{Ze^2}{v_0 p_0}\right)^2 \frac{1}{\theta^4} \,.$$

4.79* The accelerating electric field

$$E_\alpha = \frac{1}{2\pi rc}\frac{d\Phi}{dt} \,,$$

where r is the electron's orbital radius, Φ is the magnetic flux penetrating the orbit, and α is the azimuthal angle.

As an electron covers the distance $r d\alpha$ during its orbital motion, the field E_α performs the work

$$\delta A = E_\alpha r d\alpha \,.$$

The electron is accelerated along the orbit with constant radius $r = cp/eH_0$, where H_0 is the magnetic field perpendicular to the orbital plane and increases with time. It is found from the expression $dr = 0$ that $dp = p\,dH_0/H_0$. The electron's energy $\mathcal{E} = c\sqrt{p^2 + m^2c^2}$ increases by $d\mathcal{E} = c^2 p\,dp/\mathcal{E} = c^2p^2 dH_0/\mathcal{E} H_0$ as the field grows by dH_0. Evidently, $\delta A = d\mathcal{E}$. The use of the previous equalities and the relation $c^2 p/\mathcal{E} = v = r d\alpha/dt$ leads, after integration, to

$$\Phi = 2\Phi_0 \,,$$

where $\Phi_0 = \pi r^2 H_0$. The last equality expresses the "2 : 1" rule being sought.

4.80* It follows from the form of Lagrangian function (4.46) that the interaction energy of two charged particles, U, is defined by the formula

$$U = -\frac{e}{c}\boldsymbol{A}\cdot\boldsymbol{v} + e\varphi \,,$$

into which the charge e_1 of one of the particles should be introduced together with the retarded potentials[25] φ_2 and $\boldsymbol{A}_2$ of the other particle's field. Expansion of retarded potentials in powers of delay time yields

$$\varphi_2 = \frac{e_2}{R} + \frac{e_2}{2c^2}\frac{\partial^2 R}{\partial t^2}\,, \quad \boldsymbol{A}_2 = \frac{e_2 \boldsymbol{v}_2}{cR}\,,$$

where R is the distance between the particles. Let us choose the gauge function χ in the form

$$\chi = \frac{e_2}{2c}\frac{\partial R}{\partial t}\,,$$

and perform gradient transformation of the potentials. The new potentials take the form

$$\varphi_2' = \varphi_2 - \frac{1}{c}\frac{\partial \chi}{\partial t} = \frac{e_2}{R}\,, \quad \boldsymbol{A}_2' = \boldsymbol{A}_2 + \nabla\chi = \frac{e_2[\boldsymbol{v}_2 + (\boldsymbol{n}\cdot\boldsymbol{v}_2)\boldsymbol{n}]}{2cR}\,,$$

where $\boldsymbol{n} = \boldsymbol{R}/R$. Hence, we have the *Breit formula* for the interaction energy

$$U = e_1\varphi_2' - \frac{e_1}{c}\boldsymbol{v}_1\cdot\boldsymbol{A}_2' = \frac{e_1 e_2}{R}\left\{1 - \frac{1}{2c^2}[\boldsymbol{v}_1\cdot\boldsymbol{v}_2 + (\boldsymbol{v}_1\cdot\boldsymbol{n})(\boldsymbol{v}_2\cdot\boldsymbol{n})]\right\}\,.$$

This formula approximately takes into account the fact that the force applied to one of the two interacting particles localized R apart is determined by the position and the motion status of the other charge. Both the energy and the momentum are transferred to the field by the charges and are transmitted by the field between the charges during a time interval R/c. The particles and the field make up an integral system which precludes an exact description of the motion of the system of interacting particles without resorting to the degrees of freedom of the field.

4.81

$$L = \frac{m_1 v_1^2}{2} + \frac{m_1 v_1^4}{8c^2} + \frac{m_2 v_2^2}{2} + \frac{m_2 v_2^4}{8c^2} - \frac{e_1 e_2}{R} + \frac{e_1 e_2}{2c^2 R}[\boldsymbol{v}_1\cdot\boldsymbol{v}_2 + (\boldsymbol{v}_1\cdot\boldsymbol{n})(\boldsymbol{v}_2\cdot\boldsymbol{n})]\,.$$

4.82 In the Cartesian system of coordinates with the Oz axis along $\boldsymbol{H}$

$$m_{lz}(t) = m_{lz}(0) = \text{const}\,, \quad m_{lx}(t) = m_{l\perp}(0)\cos(\omega t + \alpha)\,,$$
$$m_{ly}(t) = m_{l\perp}(0)\sin(\omega t + \alpha)\,,$$

where $\omega = \kappa H, |\boldsymbol{m}_l(t)| = |\boldsymbol{m}_l(0)|$.

4.83* The instantaneously accompanying system rotates by virtue of Thomas precession. It has, according to (4.70), at $\gamma \approx 1$

$$\boldsymbol{H}' = -\frac{1}{c}\boldsymbol{v}\times\boldsymbol{E}\,,$$

25) See Chapter 5 for the definition of retarded potentials.

where $\boldsymbol{E}$ is the electric field in the laboratory frame of reference. The spin mechanical moment in the accompanying system varies in conformity with the law

$$\left(\frac{\mathrm{d}\boldsymbol{s}}{\mathrm{d}t}\right)_{\text{curl}} = \boldsymbol{m}_s \times \boldsymbol{H}' \,.$$

Passing to the laboratory system, one arrives at

$$\left(\frac{\mathrm{d}\boldsymbol{s}}{\mathrm{d}t}\right)_{\text{lab}} = \left(\frac{\mathrm{d}\boldsymbol{s}}{\mathrm{d}t}\right)_{\text{curl}} + \boldsymbol{\omega}_T \times \boldsymbol{s} = \boldsymbol{m}_s \times \boldsymbol{H}_{\text{eff}} \,,$$

where

$$\boldsymbol{H}_{\text{eff}} = \boldsymbol{H}' - \frac{m}{ceg}\dot{\boldsymbol{v}} \times \boldsymbol{v}$$

plays the role of an effective magnetic field acting on the spin magnetic moment. The interaction energy has the traditional form

$$U = -\boldsymbol{m}_s \cdot \boldsymbol{H}_{\text{eff}} \,.$$

Substituting

$$\dot{\boldsymbol{v}} = \frac{e}{m}\boldsymbol{E} \quad \text{and} \quad \boldsymbol{E} = -\frac{\mathrm{d}\varphi}{\mathrm{d}r}\frac{\boldsymbol{r}}{r} \,,$$

as well as $\boldsymbol{l} = \boldsymbol{r} \times \boldsymbol{p} = \boldsymbol{r} \times m\boldsymbol{v}$, yields

$$U = \frac{e(g-1)}{2m^2c^2}\frac{1}{r}\frac{\mathrm{d}\varphi}{\mathrm{d}r}\boldsymbol{l} \cdot \boldsymbol{s} \,. \tag{4.141}$$

This expression is used in quantum atomic theory under the name of spin–orbital interaction energy.

4.84*

$$U = -\frac{1}{2m^2c^2}\frac{1}{r}\frac{\mathrm{d}V}{\mathrm{d}r}\boldsymbol{l} \cdot \boldsymbol{s} \,.$$

The energy of interaction between the nucleon spin and the nucleus is due to the Thomas precession alone and must be added to the main interaction energy $V(r)$. The spin–orbital interaction of nucleons should be taken into consideration in the calculation of nuclear levels.

4.85** The covariant equation being sought may include 4-vectors, S^k and u^k, characterizing the particle and the field tensor F_{ik}. In a rest system, $S^k = (0, \boldsymbol{s})$, $u^k = (1, 0)$, which means that $S^k u_k = 0$. This invariant equals zero in any system; therefore, its differentiation with respect to proper time yields $(\mathrm{d}S^k/\mathrm{d}\tau)u_k + S^k(\mathrm{d}u^k/\mathrm{d}\tau) = 0$ or, taking into consideration the equation of motion (4.53),

$$\frac{\mathrm{d}S^k}{\mathrm{d}\tau}u_k = -\frac{e}{mc}S^k F_{kl}u^l \,. \tag{1}$$

The equation being sought must be correlated with (1) and take the form of (4.80) in the particle's rest system, Let us seek the equation in the form

$$\frac{\mathrm{d}S^k}{\mathrm{d}\tau} = aF^{kl}S_l + bu^kF^{il}u_iS_l\ , \tag{2}$$

where a and b are the invariant constants. We took advantage of the fact that vector S_l and tensor F^{kl} may enter the right-hand side only linearly. Using (1), we find the relationship between constants a and b:

$$a + bc^2 = \frac{e}{mc}\ . \tag{3}$$

Let us pass in (2) to the accompanying system of coordinates. At $k = 1, 2, 3$ and $\tau = t$, we have

$$\frac{\mathrm{d}\boldsymbol{s}}{\mathrm{d}t} = a\boldsymbol{s} \times \boldsymbol{H}\ . \tag{4}$$

But this equation must coincide with the nonrelativistic three-dimensional equation of spin motion (4.80). The comparison yields

$$a = g\kappa_0 = \frac{ge}{2mc}\ . \tag{5}$$

Constant b is found from (3):

$$b = \frac{e}{mc^3}\left(1 - \frac{g}{2}\right)\ . \tag{6}$$

Thus, the covariant quasi-classical equation of motion for the spin moment has the form

$$\frac{\mathrm{d}S^k}{\mathrm{d}t} = \frac{ge}{2mc}F^{kl}S_l + \frac{e}{mc^3}\left(1 - \frac{g}{2}\right)u^kF^{il}u_iS_l\ . \tag{7}$$

Yakov Frenkel[26] was the first to derive an equation of this type, as early as 1926. It was later rediscovered by a few other authors (see Ternov and Bordovitsyn, 1980, for more details and references). This equation permits us to describe the evolution of the particle's mean spin (also known as the polarization vector) in gradually changing electromagnetic fields during the quasi-classical motion of the particle. Some new data pertaining to this issue can be found in Pomeranskii *et al.* (2000).

In the case of charged leptons (electrons and positrons, muons, and taons), the spin magnetic moment $\boldsymbol{m}_s = e\hbar/2mc(g = 2,\ s = \hbar/2)$ is described by the quantum relativistic Dirac equation (see Chapter 6). For electrons $m_s = m_\mathrm{B} = e\hbar/2m_\mathrm{e}c = 9.27 \times 10^{-24}$ J/T $= 0.927 \times 10^{-20}$ erg/G is the Bohr magneton. This value of the spin magnetic moment is not exact. The addition to the spin magnetic moment,

$$m_s = \frac{e\hbar}{2mc} + \frac{e\hbar}{4mc}(g - 2)\ ,$$

26) Yakov Il'itch Frenkel (1894–1952) was an outstanding Soviet theoretical physicist and the author of numerous works and the first complete course on theoretical physics.

stemming from g not being equal to 2 is called the *anomalous moment*. Its origin is clarified in Chapters 6 and 7. Equation (7) from Problem 4.85** takes into account both anomalous and normal magnetic moments.

4.86

$$S^0 = \gamma s \cdot \frac{v}{c}\,, \quad S = s + \frac{\gamma^2}{\gamma+1}\frac{(s\cdot v)v}{c^2}\,,$$

where γ is the relativistic factor of the particle.

4.87** Let us use equation (7) from Problem 4.85** and write it down for the spatial and temporal components of S^k in three-dimensional notation:

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{a}{c\gamma}(S\cdot v)E + \frac{a}{\gamma}S\times H + b\gamma v[cE\cdot S - \frac{1}{c}(E\cdot v)(S\cdot v) - H\cdot(v\times S)],$$

$$\frac{\mathrm{d}S^0}{\mathrm{d}t} = \frac{a}{\gamma}E\cdot S + b\gamma[c^2E\cdot S - (E\cdot v)(S\cdot v) - cH\cdot(v\times S)]\,.$$

Let us further express S and S^0 through s and use the values of a and b from Problem 4.85** to obtain after simple but cumbersome calculations (to be done with great care and attention) the equations describing evolution of spin s:

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{e}{2mc}\left(g-2+\frac{2}{\gamma}\right)s\times H + \frac{e\gamma}{2mc^3(\gamma+1)}(g-2)(v\cdot H)(v\times s) + \frac{e}{2mc^2}\left(g-\frac{2\gamma}{\gamma+1}\right)s\times[E\times v]\,, \tag{1}$$

$$\frac{\mathrm{d}s_{\parallel}}{\mathrm{d}t} = \frac{e}{2mcv}(g-2)s_{\perp}\cdot(H\times v) + \frac{e}{2mv}\left(\frac{g}{\gamma^2}+2-g\right)(E\cdot s_{\perp}). \tag{2}$$

Here $s_{\parallel}$ and $s_{\perp}$ are the components parallel and perpendicular to v. The last equation loses validity at $v \to 0$ because of the impossibility to determine $s_{\parallel}$ and $s_{\perp}$.

At $E = 0$ the angle between the particle's spin and momentum can change (in accordance with (2)) only in the presence of the anomalous magnetic moment, that is, at $g \neq 2$.

1. At $v \perp H$ and $E = 0$, $\gamma = \text{const}$. The equation of spin motion

   $$\frac{\mathrm{d}s}{\mathrm{d}t} = \Omega_s \times s\,, \quad \Omega_s = -\frac{ecH}{\mathcal{E}} - \frac{e}{2mc}(g-2)H\,,$$

   coincides with the equation of motion for the particle's momentum (see Problem 4.52*):

   $$\frac{\mathrm{d}p}{\mathrm{d}t} = \Omega_p \times p\,, \quad \Omega_p = \frac{ecH}{\mathcal{E}}\,,$$

and differs from it only in the angular velocity of rotation:

$$\boldsymbol{\Omega}_s = \boldsymbol{\Omega}_p - \frac{e}{2mc}(g-2)\boldsymbol{H} .$$

Spin precesses around the magnetic field direction. The angular velocity of precession differs from the momentum rotation velocity in the presence of the anomalous magnetic moment.

2.

$$\frac{d\boldsymbol{s}}{dt} = \boldsymbol{\Omega} \times \boldsymbol{s} , \quad \boldsymbol{\Omega} = -\frac{ecg\boldsymbol{H}}{\mathcal{E}} .$$

The momentum does not precess, whereas spin precession is due to the total magnetic moment.

3. Given the Ox axis is aligned with $\boldsymbol{E}$, and the particle's motion occurs in the xOy plane, (1) gives

$$\dot{s}_z = 0 , \quad \dot{s}_x = -\Omega s_y , \quad \dot{s}_y = \Omega s_x , \quad \Omega = -\frac{eEv_y}{2mc^2}\left(g - \frac{2\gamma}{\gamma+1}\right) .$$

Spin precession occurs around the direction perpendicular to the plane of motion (i.e., around the orbital angular momentum preserving its direction but not magnitude) with variable angular velocity Ω. Spin projection onto the xOy plane, $\boldsymbol{\zeta} = s_x\boldsymbol{e}_x + s_y\boldsymbol{e}_y$, in turn rotates with velocity $\dot{\varphi}$, which can be found from equation (7) in the solution of Problem 4.85**. Let us write (Figure 4.17)

$$s_\parallel = \zeta_\parallel = \zeta\cos\varphi ,$$
$$\boldsymbol{s}_\perp \cdot \boldsymbol{E} = \boldsymbol{\zeta}_\perp \cdot \boldsymbol{E} = -\zeta E \sin\varphi \sin\alpha = -\zeta E \left(\frac{v_y}{v}\right)\sin\varphi .$$

Substituting these expressions into (2) yields

$$\dot{\varphi} = \frac{eEv_y}{2mv^2}\left(\frac{g}{\gamma^2} + 2 - g\right) .$$

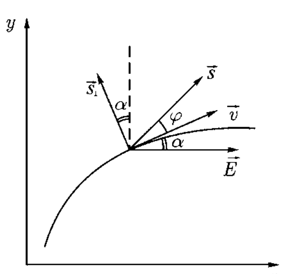

Figure 4.17 Precession of the spin in the xOy plane.

4.88 In the absence of a magnetic field ($\boldsymbol{H} = 0$) in the nonrelativistic approximation ($\gamma = 1$), one has from (1) of the preceding problem

$$\dot{\boldsymbol{s}} = \boldsymbol{m}_s \times \boldsymbol{H}_{\text{eff}} , \quad \boldsymbol{H}_{\text{eff}} = -\frac{1}{c}\boldsymbol{v} \times \boldsymbol{E} - \frac{2mc}{eg}\boldsymbol{\omega}_T .$$

The last term describes the Thomas precession.

4.89 Reflection occurs in the case of antiparallel orientation of the field and magnetic moment provided the sliding angle α is sufficiently small so that $\sin\alpha \le \sqrt{m_n H/T}$.

4.90 The neutron uniformly moves along the wire. The motion in the plane perpendicular to the wire occurs in the potential field $U = \pm 2m_n\mathcal{J}/cr$. As a result, the projections of neutron trajectories onto this plane have the same form as those of the trajectories of two particles with charges e and e' moving relative to each other and interacting according to the Coulomb law (see Problem 4.66). In the solution of this problem, ee' needs to be substituted with $\pm 2m_n\mathcal{J}/c$, and $\mathcal{E} = M\dot{r}^2/2 + l^2/2Mr^2 + U(r)$ should be understood as the transverse motion energy ($l = Mr^2\dot{\alpha}$ is the angular momentum). Specifically, at $\mathcal{E} < 0$, the neutrons perform finite motion near the wire.

4.91

$$l(\alpha) = \frac{m_n\mathcal{J}}{cMv_0^2\sin^2\alpha/2} .$$

4.94 The particle drifts under the action of the effective electric field $\boldsymbol{E}_{\text{eff}} = \boldsymbol{\mathcal{F}}/e$ with velocity

$$\boldsymbol{v}_{\text{d}} = \frac{c}{eH^2}\boldsymbol{\mathcal{F}} \times \boldsymbol{H} .$$

4.95 In the approximation of zero Larmor radius, the particle moves along a curved line of force with velocity $v_\parallel$ undergoing the action of the centripetal force

$$\boldsymbol{\mathcal{F}}_{\text{c}} = -\frac{\mathcal{E}v_\parallel^2}{c^2\rho}\boldsymbol{n} , \tag{1}$$

where $\mathcal{E}$ is the total energy of the particle, ρ is the local radius of curvature of the force line, and $\boldsymbol{n}$ is the unitary vector of the principal normal. We use the relation $\boldsymbol{n}/\rho = (\boldsymbol{h}\cdot\nabla)\boldsymbol{h}$ from differential geometry, where $\boldsymbol{h}$ is the unit tangent vector, and the formula for the drift velocity obtained in the preceding problem to find the curvature drift velocity

$$\boldsymbol{v}_{\text{d}c} = v_\parallel R_\parallel \boldsymbol{h} \times (\boldsymbol{h}\cdot\nabla)\boldsymbol{h} , \tag{2}$$

where $R_\parallel = cp_\parallel/eH$. In the order of magnitude, $v_{\text{d}e} \approx v_\parallel R/\rho \ll v_\parallel$.

Consideration of electric field nonuniformity leads to the second-order terms of smallness $ER/HL \ll 1$. The terms with $\dot{\boldsymbol{H}} = -c\,\text{curl}\,\boldsymbol{E}$ are of the same order.

4.96 Variations of the magnetic field in time give rise to an electric field in accordance with the Maxwell equation $c\,\mathrm{curl}\,\boldsymbol{E} = -\dot{\boldsymbol{H}}$. The change in the particle's energy during one Larmor rotation is

$$\Delta\mathcal{E} = \oint e\boldsymbol{E}\cdot \mathrm{d}\boldsymbol{l} = e\int \mathrm{curl}\,\boldsymbol{E}\cdot\boldsymbol{n}\mathrm{d}S = e\boldsymbol{n}\cdot\mathrm{curl}\,\boldsymbol{E}\pi R_\perp^2 = -\frac{e}{c}\pi R_\perp^2\dot{\boldsymbol{H}}\cdot\boldsymbol{n}$$
$$= +\frac{e}{c}\pi R_\perp^2\dot{H}\ .$$

The change of the sign in the last equality is because the normal $\boldsymbol{n}$ to the Larmor circle matching the sense of rotation by the right-hand screw rule is directed against the magnetic field. Let us divide both parts of the last equality by the Larmor period T_L. Then, on the left-hand side, $\Delta\mathcal{E}/T_\mathrm{L} = v_\perp\Delta p_\perp/T_\mathrm{L} = v_\perp\dot{p}_\perp$.

In an analogous fashion, $(e/2c)(2\pi R_\perp/T_\mathrm{L})R_\perp\dot{H} = 2^{-1}v_\perp p_\perp\dot{H}/H$ is obtained on the right-hand side. In the end, we arrive at the constant adiabatic invariant

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{p_\perp^2}{H} = 0\ .$$

4.97 The total velocity of the particle is the sum of the guiding center velocity $\boldsymbol{v}_\mathrm{c}$ (4.90) and the velocity of the Larmor rotation with respect to the guiding center, $\dot{\boldsymbol{r}}_\perp$. The instantaneous speed of the change in the particle's energy $\dot{\tilde{\mathcal{E}}}$ is expressed through the usual formula:

$$\dot{\tilde{\mathcal{E}}} = e\boldsymbol{E}\cdot(\boldsymbol{v}_\mathrm{c} + \dot{\boldsymbol{r}}_\perp)\ .$$

This expression needs to be averaged over the Larmor period: $\dot{\mathcal{E}} = T_\mathrm{L}^{-1}\oint\dot{\tilde{\mathcal{E}}}\mathrm{d}t$. The first term, $e\boldsymbol{E}\cdot\boldsymbol{v}_\mathrm{c}$, contains only smoothly varying quantities and does not change on averaging. The second term undergoes the following transformation:

$$T_\mathrm{L}^{-1}\oint e\boldsymbol{E}\cdot\dot{\boldsymbol{r}}_\perp\mathrm{d}t = eT_\mathrm{L}^{-1}\oint\boldsymbol{E}\cdot\mathrm{d}\boldsymbol{r}_\perp = -\left(\frac{e}{2}\right)v_\perp R_\perp\boldsymbol{h}\cdot\mathrm{curl}\,\boldsymbol{E}\ .$$

4.98 The adiabatic invariant for a relativistic particle is the quantity $\gamma\mu$, where $\gamma = (1-v^2/c^2)^{1/2}$ is the Lorentz factor and $\mu = p_\perp v_\perp/2H$ is the magnetic moment. Given the kinetic energy of the particle is conserved, $\gamma = \mathrm{const}$ and $\mu = \mathrm{const}$. The last relation is fulfilled for a nonrelativistic particle in which $\gamma\approx 1$ even if its energy is not conserved.

4.99 $\mathcal{F} = -\boldsymbol{\mu}\cdot\nabla H$, where $\boldsymbol{\mu} = \boldsymbol{h}p_\perp v_\perp/2H$ is the magnetic moment created by the rotation of the particle.

4.100

$$M\dot{\boldsymbol{V}} = -\nabla U_\mathrm{eff}(\boldsymbol{R})\ ,\quad U_\mathrm{eff}(\boldsymbol{R}) = -\frac{\beta}{2}E^2(\boldsymbol{R})\ ,$$

where $\boldsymbol{R}$ and $\boldsymbol{V} = \dot{\boldsymbol{R}}$ are the radius vector and the center-of-mass velocity respectively, and M is the total mass of the system.

4.101

$$M\dot{V} = -U_{\text{eff}}(R) + \frac{\beta}{c}[(V \cdot \nabla)E(R) \times H(R)] \,.$$

4.102* Let us choose the vector potential of the magnetic field in the form $A = H \times r/2$, and write down the Lagrangian function of the nonrelativistic system as

$$L = \sum_a \frac{mv_a^2}{2} + \frac{e}{2c}\sum_a v_a \cdot [H \times r_a] - U \,. \quad (1)$$

In the absence of a magnetic field but in the frame of reference (primed) rotating with angular velocity Ω, $r_a = r'_a + \Omega \times r'_a$, and the Lagrangian function takes the form

$$L_\Omega = \sum_a \frac{mv_a'^2}{2} + \sum_a mv'_a \cdot [\Omega \times r'_a] - U + \sum_a \frac{m}{2}[\Omega \times r'_a]^2 \,. \quad (2)$$

Given the condition $|\Omega \times r'_a| \ll v'_a$ in (2) is fulfilled, the last term may be disregarded. The potential energy of the system does not change at all after transition to the rotating system. Therefore, functions L and L_Ω become identical at $\Omega = eH/2mc$ and a sufficiently weak field. But this means that the behavior of the particles is similar too: a system of particles in the laboratory frame of reference acquires angular velocity $\Omega_L = -\Omega$ after the application of a magnetic field. The minus sign appears by virtue of opposite directions of rotation of the particles with respect to the coordinate axes and rotation of the axes with respect to the particles. The smallness condition of the field means that the Larmor precession frequency is low compared with the eigenfrequencies of the quasi-periodic motion of the particles in the system.

4.103 $\sin\vartheta > \sqrt{H/H_m}$.

4.104 $R = 1 - H/H_m$.

4.105 $r = r_0\sqrt{H_0/H}$, where r_0 is the distance between the guiding center and the trap axis in the field H_0, and r is the distance after a change of the field up to H. The field enhancement causes compression of plasma toward the trap axis.

4.106 The guiding center shifts onto the line of force $r = l$, $\varphi = 2cq/Hv_\| l^2$.

4.107 The proton's guiding center moves uniformly in a circle of radius $r = 2r_\star$ lying in the equatorial plane with angular velocity

$$\omega_\text{d} = \frac{3c\mathcal{E}}{e\mu}r - \frac{3GmM}{e\mu} \,,$$

where G is the gravitational constant. $R \approx 226$ km. $T \approx 14.9$ s.

4.108*

1. The calculation of the products of $\boldsymbol{h} \times \nabla H$ and $\boldsymbol{h} \times (\boldsymbol{h} \cdot \nabla)\boldsymbol{h}$ for the magnetic dipole field from (4.90) shows that the motion across the magnetic lines of force reduces to the azimuthal drift in which neither the distance toward the Earth's center nor the latitudinal angle varies. Moreover, the guiding center moves along the line of force for which the equation has the form

$$r = r_0 \cos^2 \lambda \,, \tag{1}$$

where r_0 is the distance between the line of force and the center in the equatorial plane. The particle's energy remains constant since the gravitational field is disregarded.
The use of the known expressions for the intensity of the magnetic dipole field in conjunction with (4.89), (1), and (4.90) permits us to find the angular velocity of the azimuthal drift:

$$\begin{aligned} \omega_\mathrm{d} = \frac{v_{\mathrm{d}\alpha}}{r} = &-\frac{3cpvr_0 \sin^2 \alpha}{2e\mu} \frac{1 + \sin^2 \lambda}{\cos^3 \lambda (3 \sin^2 \lambda + 1)} \\ &- \frac{cpvr_0 \cos^3 \lambda (3 \sin^2 \lambda - 1)}{e\mu (3 \sin^2 \lambda + 1)^2} \,. \end{aligned} \tag{2}$$

Here p and v are the proton's momentum and velocity, respectively.
2. Find the condition determining $\lambda_m > 0$ using (4.89):

$$\frac{\cos^6 \lambda_m}{\sqrt{3 \sin^2 \lambda + 1}} = \sin^2 \alpha \,. \tag{3}$$

The particles travel in the region $-\lambda_m \leq \lambda \leq \lambda_m$.
3. The proton reaches the Earth's surface on the condition that

$$r_0 \cos^2 \lambda_m \leq r_\star \,,$$

where $r_\star$ is the radius of the Earth.

4.109 The particle performs strictly periodic longitudinal motion in the axially symmetric trap at $L =$ const. Therefore, a slow change of L is associated with the longitudinal adiabatic invariant

$$I_\parallel = \frac{1}{2\pi} \oint P_\parallel \mathrm{d}l = \mathrm{const} \,,$$

where

$$P_\parallel = \gamma m v_\parallel + \left(\frac{e}{c}\right) A_\parallel$$

is the longitudinal component of the generalized momentum; integration is performed along a line of force.

The longitudinal part of the vector potential $A_{\|}$ does not influence the strength of the magnetic field $\boldsymbol{H} = \text{curl}\,\boldsymbol{A}$, and it can be regarded as equaling zero, $A_{\|} = 0$. Therefore,

$$I_{\|} = \frac{1}{2\pi} \oint p_{\|} \mathrm{d}l\,,$$

where $p_{\|} = \gamma m v_{\|}$ is the longitudinal component of the ordinary momentum.

Given the uniform magnetic field inside most of the trap and the small size of the regions where the field increases, $p_{\|} \approx$ const and $I_{\|} \approx p_{\|} L/\pi$ for a single oscillation period. The adiabatic invariance condition takes the form

$$p_{\|}(t) L(t) = \text{const}\,.$$

The condition of slow field variation is $|\dot{L}| \ll |v_{\|}|$. The change in the energy with time is

$$\mathcal{E}^2(t) = \mathcal{E}_0^2 + \frac{c^2 p_{0\|}^2 (L_0 - L(t))}{L(t)}\,,$$

where index 0 denotes the initial values.

4.110**

$$\mathcal{E}_2 = \sqrt{m^2 c^4 + \frac{c^2 p_1^2 H_2}{H_1}}\,.$$

This result can be obtained if it is proved that the magnetic flux across the area enclosed by the particle's trajectory during the Larmor period is constant (see Toptygin, 1985, Appendixes III, IV)

4.111*** See Toptygin (1985, Appendix IV)

4.112

$$\frac{m\ddot{X}}{(1 - \dot{X}^2/c^2)^{3/2}} = -\frac{\mathrm{d}U_{\text{eff}}}{\mathrm{d}X}\,, \quad U_{\text{eff}}(X, \dot{X}) = \frac{e^2(1 - \dot{X}^2/c^2)^{3/2}}{4m\omega^2} E^2(X)\,.$$

4.113*

1. $U_{\text{eff}}(X) = U_0(1 - \cos\varphi) + \frac{1}{2} U_1 \sin^2\varphi$, where $\varphi = 2\pi X/L$, $U_1 = e^2 E_0^2/2m\omega^2$, $|e| E_0/m\omega^2 \ll L$.
2. At $U_1 < U_0$ the centers are localized at points $X_n = nL$, and the saddles are localized at $\tilde{X}_n = (n + 1/2)L$, $n = 0, \pm 1, \pm 2, \ldots$ The frequency of natural oscillations near all centers is similar,

$$\omega_0 = \frac{2\pi}{L} \sqrt{\frac{U_0 + U_1}{m}}\,.$$

The phase portrait is shown in Figure 4.18a.

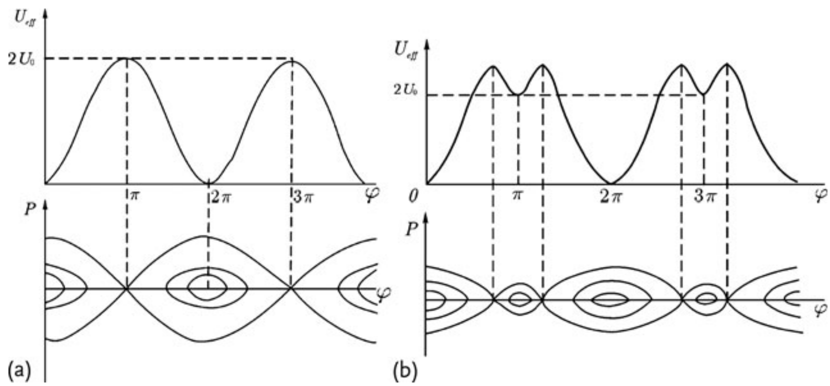

Figure 4.18 Phase portrait of the particle's oscillations at $U_1 < U_0$ (a), and phase portrait of the particle's oscillations at $U_1 > U_0$ (b).

At $U_1 > U_0$, the centers are localized at points $X_{1n} = nL$ (the frequency of natural oscillations $\omega_{01} = (2\pi/L)[(U_0 + U_1)/m]^{1/2}$) and $X_{2n} = (n + 1/2)L$ (the frequency of natural oscillations $\omega_{02} = (2\pi/L)[(U_1 - U_0)/m]^{1/2}$), $n = 0, \pm 1, \ldots$ The saddles occupy points $\tilde{X}_n$ given by the condition $\cos\tilde{\varphi}_n = -U_0/U_1$. Essentially, new points of stable equilibrium, $X = X_{2n}$, appear in this case that were unstable at $U_1 < U_0$; moreover, the phase portrait undergoes substantial changes (Figure 4.18b).

4.114 At $U_0 > U_1$ the centers are localized at points $X_n = nL$, and the saddles are localized at $\tilde{X}_n = (n + 1/2)L$, $n = 0, \pm 1, \pm 2, \ldots$ At $U_0 < U_1$, points X_n become saddles, the saddles at points $\tilde{X}_n$ are preserved, and centers at the points given by the condition $\cos\tilde{\varphi}_n = U_0/U_1 < 0$ appear.

4.115 In the Gaussian system of units, the dimensions of E and H coincide; the same is true of e_0 and g:

$$g_{\min} = \frac{e_0}{2}\left(\frac{\hbar c}{e_0^2}\right) \approx \frac{137}{2} e_0 \gg e_0 .$$

$$\frac{d\mathcal{E}}{dl} = gH \text{dyn/cm} = (300 \times 137/2) H \text{eV/cm} .$$

At $H = 10^3$Oe, $d\mathcal{E}/dl = 20.55$ MeV/cm.

4.116* The calculation of $H = \text{curl}\, \boldsymbol{A}$ makes it possible to find in the region where $r \neq 0$, $\vartheta \neq \pi$, and the magnetic field $H_r = g/r^2$, $H_\vartheta = H_\alpha = 0$, or $\boldsymbol{H} = g\boldsymbol{r}/r^3$, that is, the "Coulomb" field of a point magnetic charge. At $r \to 0$, this field exhibits the same specific feature as the field of a point electric charge.

However, the Dirac potential has a peculiarity also at $\vartheta \to \pi (r \to z)$, that is, over the entire negative semiaxis $0 > z > -\infty$. The differential expression for curl $\boldsymbol{A}$ remains undetermined at this peculiarity.

To find the field at this singular Dirac "string," it is necessary to calculate its flux through a small circle of radius $s \to 0$, the plane of which is perpendicular to the Oz axis and the center lies on the axis:

$$\int\limits_{\pi s^2} \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{S} = \oint \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{l} = \begin{cases} 4\pi g\,, & z < 0 \quad (\vartheta \to \pi)\,; \\ 0\,, & z > 0 \quad (\vartheta \to 0)\,. \end{cases}$$

The finite value of the magnetic flux through an infinitely small area suggests singularity of the magnetic field at the negative semiaxis Oz:

$$H_z = \begin{cases} 4\pi g \delta(x)\delta(y)\,, & z < 0\,; \\ 0 & z > 0\,. \end{cases}$$

The total magnetic flux across the closed surface surrounding the origin of the coordinates is equal to zero.

Thus, the Dirac potential describes the singular "string", besides the point-like magnetic charge. To make it unobservable, Dirac imposed the requirement of $\psi = 0$ at the singular string on the particle's wave functions. This requirement looks artificial and hardly ensures unobservability of the strings. The reader is referred to the collected works of Dirac (Dirac's monopol, 1970) for the discussion of the problem of a magnetic monopole in the framework of traditional electrodynamics and quantum mechanics together with original articles by Dirac. Further investigations into the properties of the Young–Mills gauge fields are highlighted in the works of Sokolov *et al.* (1986), Gal'tsov *et al.* (1991), and Rubakov (2002).

However, there is a simple macroscopic model of a Dirac monopole with a singular string. One end of a long thin solenoid creates a magnetic field similar to the monopole field at distances greater than the diameter of the solenoid but less than its length.

4.117* The equations of motion are

$$m\dot{\varphi} = \frac{eg}{cr^3}[\dot{\boldsymbol{r}} \times \boldsymbol{r}]\,. \tag{1}$$

The motion integrals are

$$c^2 p^2 + m^2 c^4 = \mathcal{E}^2 = \text{const}\,, \quad [\boldsymbol{r} \times \varphi] - \frac{eg}{c}\frac{\boldsymbol{r}}{r} = \boldsymbol{J} = \text{const}\,. \tag{2}$$

The law of motion for a nonrelativistic particle in spherical coordinates (r, ϑ, φ) with the polar axis directed along $\boldsymbol{J}$ is

$$r^2 = r_0^2 + v^2(t - t_0)^2\,, \quad \varphi = \frac{1}{\sin\vartheta} \arctan \frac{v(t - t_0)}{r_0}\,,$$
$$\tan\vartheta = -\frac{mvcr_0}{eg} = \text{const}\,. \tag{3}$$

4.118

$$\mathcal{L} = \frac{\rho}{2}(q^\alpha_{,t})^2 - \frac{\mu}{4}\left(q^\alpha_{\beta} + q^\beta_{,\alpha} - \frac{2}{3}\delta^\alpha_\beta q^\gamma_{,\gamma}\right)^2 - \frac{K}{2}(q^\alpha_{,\alpha})^2 \,;$$

$$\mathcal{H} = \frac{1}{2\rho}(\pi^\alpha)^2 + \frac{\mu}{4}\left(q^\alpha_{\beta} + q^\beta_{,\alpha} - \frac{2}{3}\delta^\alpha_\beta q^\gamma_{,\gamma}\right)^2 + \frac{K}{2}(q^\alpha_{,\alpha})^2 \,,$$

where $\pi^\alpha = \rho q^\alpha_{,t}$. The equation of motion specified in the statement of the problem corresponds to various Lagrangians differing in transformation (4.94). The expressions given in the answer are chosen so that the Hamiltonian has the sense of the energy density of a deformed elastic body.

4.119

$$S = S_g + S_m + S_{\text{int}} \,,$$

where

$$S_g = -\frac{1}{8\pi c G}\int_\Omega (\nabla\varphi)^2 \mathrm{d}^4 x$$

is the action for a gravitational field, $G = 6.67 \times 10^{-8}\ \mathrm{cm}^3/(\mathrm{g\,s})$ is the gravitational constant,

$$S_m = \int_{t_1}^{t_2} \sum_{b=1}^{N} \frac{m_b}{2}(\dot{\boldsymbol{r}}_b)^2 \mathrm{d}t$$

is the action for free particles,

$$S_{\text{int}} = -\sum_{b=1}^{N}\int_{t_1}^{t_2} m_b \varphi(\boldsymbol{r}_b, t)\mathrm{d}t = \rho\varphi \mathrm{d}^4 x$$

is the interaction term, and $\rho(\boldsymbol{r}, t) = \sum_b m_b \delta(\boldsymbol{r} - \boldsymbol{r}_b(t))$ is mass density of point-like particles written via the delta function.

The equation for the potential is derived by varying the value of $S_g + S_{\text{int}}$ in φ at the fixed particle coordinates $\boldsymbol{r}_a$; this gives

$$\Delta\varphi = 4\pi G \rho(\boldsymbol{r}, t) \,.$$

The equations of particle motion are obtained by varying the radius vectors $\boldsymbol{r}_b$ in $S_{\text{int}} + S_m$:

$$\ddot{\boldsymbol{r}}_a = -\nabla\varphi(\boldsymbol{r}_a, t) \,.$$

The last equation is incorrect in the case of point-like particles because the potential on the right-hand side contains the infinite self-action term (in the form

$-Gm_a/|\boldsymbol{r}_a - \boldsymbol{r}_a|$). The correct equation of motion of the ath particle under the effect of all the other ones can be obtained by excluding from the total potential φ the potential created by the ath particle:

$$\ddot{\boldsymbol{r}}_a = -\nabla\varphi'(\boldsymbol{r}_a, t) ,$$

where

$$\varphi'(\boldsymbol{r}_a, t) = -\sum_{b=1}^{N}{}' \frac{Gm_b}{|\boldsymbol{r}_a - \boldsymbol{r}_b|} ;$$

the prime next to the summation sign indicates the absence of a term with $b = a$. We note that the particle mass does not enter the equation of motion. This means that all the bodies regardless of their mass and nature travel in exactly the same manner in a given gravitational field.

4.120

$$S = S_f + S_{\text{part}} + S_{\text{int}} ,$$

where

$$S_f = \frac{1}{8\pi c} \int\limits_{\Omega} (\nabla\varphi)^2 \mathrm{d}^4 x$$

is the action for an electric field, $\varphi(\boldsymbol{r}, t)$ is the electrostatic potential,

$$S_{\text{part}} = \sum_{b=1}^{N} \int\limits_{t_1}^{t_2} \frac{m_b}{2} (\dot{\boldsymbol{r}}_b)^2 \mathrm{d}t$$

is the action for free particles,

$$S_{\text{int}} = -\sum_{b=1}^{N} \int\limits_{t_1}^{t_2} e_b \varphi(\boldsymbol{r}_b, t) \mathrm{d}t = -\frac{1}{c} \int\limits_{\Omega} \rho\varphi \mathrm{d}^4 x$$

is the interaction between the charged particles and the field; e_b is the particle charge, and

$$\rho(\boldsymbol{r}, t) = \sum_{a=1}^{N} e_a \delta(\boldsymbol{r} - \boldsymbol{r}_a(t))$$

is the bulk density of the electric charge.

The equation for the field and the particles is

$$\Delta\varphi = -4\pi\rho , \quad m_a \ddot{\boldsymbol{r}}_a = -e_a \nabla\varphi' .$$

The divergent contribution to the potential of the ath particle describing the self-action effect should be excluded from the right-hand side of the last equation.

4.121

$$S = \int_{\Omega} \left[-\frac{(\mathrm{curl}\, \boldsymbol{A})^2}{8\pi c} + \frac{1}{c^2} \boldsymbol{A} \cdot \boldsymbol{j} \right] \mathrm{d}^4 x \ ,$$

where $\boldsymbol{A}(\boldsymbol{r})$ is the vector potential determining the intensity of the magnetic field.

4.122*

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + U\psi$$

is the Schrödinger equation for the wave function of a nonrelativistic particle in the potential field $U(\boldsymbol{r}, t)$.

4.123*

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial \psi_{,t}} \psi_{,t} + \frac{\partial \mathcal{L}}{\partial \psi^*_{,t}} \psi^*_{,t} - \mathcal{L} = \frac{\hbar^2}{2m} |\nabla \psi|^2 + U|\psi|^2 \ .$$

The total field energy is

$$H = \int \mathcal{H} \mathrm{d}^3 x = \int \psi \left(-\frac{\hbar^2}{2m} \nabla^2 + U \right) \psi \mathrm{d}^3 x \ ;$$

the last expression is obtained by partial integration on the assumption that $\psi \to 0$ at $r \to \infty$. It coincides with the quantum mechanical average value of the particle's energy. If the potential energy $U(\boldsymbol{r})$ is time independent and the particle is in a state with a definite energy, that is, $\widehat{H}\psi = E\psi$, where $\widehat{H} = -\frac{\hbar^2}{2m}\nabla^2 + U$ is the quantum mechanical Hamiltonian operator, then $H = E$. This means that the energy calculated by the formulas of field theory as the integral over the entire three-dimensional space coincides with the energy of a quantum particle.

4.124*

$$\mathcal{L} = \frac{i\hbar}{2} (\psi^* \psi_{,t} - \psi \psi^*_{,t}) - \frac{\hbar^2}{2m} \left| \nabla \psi + \frac{ie}{\hbar c} \boldsymbol{A} \psi \right|^2 - e\varphi |\psi|^2 \ .$$

Equation (4.101) is derived from the Lagrangian using algorithm(4.95) :

$$\frac{\partial \mathcal{L}}{\partial \psi^*} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial \psi^*_{,t}} - \frac{\mathrm{d}}{\mathrm{d}x_\alpha} \frac{\partial \mathcal{L}}{\partial \psi^*_{,\alpha}} = 0 \ .$$

4.125*

$$\Box \varphi - \frac{m^2 c^2}{\hbar^2} \varphi = 0 \tag{1}$$

(the *Klein–Gordon–Fock equation*). Writing the solution in the form of a flat wave with the invariant phase, $\varphi = A \exp(-i k^l x_l)$, where $A = \mathrm{const}$ and $k^l = (\omega/c, \boldsymbol{k})$, gives the following dispersion relation from the Klein–Gordon–Fock equation:

$$\frac{\omega^2}{c^2} = \boldsymbol{k}^2 + \frac{m^2 c^2}{\hbar^2} \ . \tag{2}$$

The de Broglie interpretation of the product of $\hbar\omega$ and $\hbar\boldsymbol{k}$ as the particle's energy $\mathcal{E}$ and momentum $\boldsymbol{p}$ leads to the relativistic relation between the energy and the momentum

$$\mathcal{E}^2 = c^2\boldsymbol{p}^2 + m^2c^4 \,. \tag{3}$$

Because the wave function $\varphi(x^k)$ is a single-component quantity and a scalar depending on the spatial 4-coordinates alone, it can describe only particles devoid of internal degrees of freedom, that is, particles with zero spin.

4.126* The use of the above Lagrangian and (4.95) yields the equation of motion

$$\Box\varphi + \mu^2\varphi - \lambda\varphi^3 = 0 \,. \tag{1}$$

At small φ and disregarding the nonlinear term instead of dispersion relation (2) from the preceding problem, we obtain

$$\frac{\omega^2}{c^2} = -\mu^2 + k^2 \tag{2}$$

and have, at $k^2 < \mu^2$, the purely imaginary frequencies $\omega = ic\sqrt{\mu^2 - k^2}$.

This means that the solutions $\varphi \propto \exp(\pm\sqrt{\mu^2 - k^2}ct)$ with the plus sign will grow exponentially even though the growth cannot be infinite because the nonlinear term limits the amplitude φ.

Let us find the value of $\varphi = \varphi_0$ at which $V(\varphi)$ has a minimum. It follows from the condition $\partial V/\partial\varphi^2 = 0$ that $\varphi_0 = \pm\mu/\sqrt{\lambda}$. It can be expected from physical considerations that the field of φ will increase up to φ_0 and will oscillate thereafter around this value; these oscillations will involve quanta, that is, particles having a certain real and positive mass.

To verify this hypothesis, we substitute $\varphi(x) = \varphi_0 + \eta(x), |\eta| \ll |\varphi_0|$ into the Lagrangian and retain only terms not higher than second order:

$$\mathcal{L} = \frac{1}{2}\left(\partial^k\eta\partial_k\eta\right) - \mu^2\eta^2 - \frac{\mu^4}{4\lambda} \,.$$

The fixed term can be omitted. Comparison of it with the respective term in equation (1) in the preceding problem makes it possible to find the mass of the quanta corresponding to excitations η: $m_\eta = \sqrt{2}\mu\hbar/c$. As a result, the following physical picture emerges: the potential $V(\varphi)$ specified in the statement of the problem corresponds to the two lowest energy states (vacuum solutions) with nonzero values of the field $\varphi_0 = \pm\mu/\sqrt{\lambda}$. In this situation, excited states are possible with a certain number of quanta, that is, particles with mass m_η and the energy–momentum relation typical of relativistic particles:

$$\mathcal{E} = \sqrt{c^2p^2 + m_\eta^2c^4} \,, \quad p = \hbar k \,, \quad \mathcal{E} = \hbar\omega \,.$$

The wave functions of particles at low field excitation levels may be regarded as superpositions of flat waves with $\omega = \mathcal{E}/\hbar, \boldsymbol{k} = \boldsymbol{p}/\hbar$.

4.128 There is a single scalar transformation parameter α. At small α, $\psi' = \psi + i\alpha\psi$, $x'^k = x^k$; therefore, it should be assumed in the general formulas (4.115) that $\Gamma_a^k = 0$, $G_{\mathrm{Ba}}^A = i$, and $\delta\lambda^a = \alpha$. The generalized current is a 4-vector: $J_a^k = j^k$, according to (4.119),

$$j^0 = \frac{\partial \mathcal{L}}{\partial \psi^*_{,t}} i c \psi^* - \frac{\partial \mathcal{L}}{\partial \psi_{,t}} i c \psi = c\hbar |\psi|^2 \; ; \tag{1}$$

$$j^\beta = \frac{\partial \mathcal{L}}{\partial \psi^*_{,\beta}} i \psi^* - \frac{\partial \mathcal{L}}{\partial \psi_{\ \beta}} i \psi = \frac{i\hbar^2}{2m}(\psi \psi^*_{\ \beta} - \psi^* \psi_{\ \beta}) \, . \tag{2}$$

The quantity $\rho(\boldsymbol{r}, t) = j^0/c\hbar = |\psi|^2$ is nonnegative and can be normalized to unity (for the states of finite motion in which a particle is all the time localized in a finite region of space) and can be interpreted in quantum mechanics as the probability density. The 3-vector $\boldsymbol{j}/\hbar$ is the probability flow density. The continuity equation (4.120) accounts for the conservation of probability in time: $\int \rho(\boldsymbol{r}, t) \mathrm{d}^3 x =$ const. A different and more adequate interpretation of the conservation law for charged particles is equally feasible. Multiplying equalities (1) and (2) by the particle charge e yields the electric charge density $e\rho = \rho_\mathrm{e}$ and the electric field density $e\boldsymbol{j}/\hbar = \boldsymbol{j}_\mathrm{e}$, which is created in the usually three-dimensional space by the motion of the particle in accordance with the laws of quantum mechanics. In such an interpretation, the Noether current associated with phase transformation of the charged particle's wave function is a four-dimensional electric current density.

4.129

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \boldsymbol{j} = 0 \, , \quad \text{where}$$

$$\rho(\boldsymbol{r}, t) = \frac{i\hbar}{2mc^2}(\varphi^* \varphi_{,t} - \varphi \varphi^*_{,t}) \, , \quad \boldsymbol{j} = \frac{i\hbar}{2m}(\varphi \nabla \varphi^* - \varphi^* \nabla \varphi) \, .$$

Interpretation of quantity ρ as the probability density is impracticable because ρ may have both signs. But the products $e\rho = \rho_\mathrm{e}$ and $e\boldsymbol{j} = \boldsymbol{j}_\mathrm{e}$ can be regarded as the electric charge and electric current density provided φ describes such a field whose quanta are particles and antiparticles having charges of different signs. See Bjorken and Drell (1964, 1965) and Berestetskii *et al.* (1982) for more details concerning interpretation of the Klein–Gordon–Fock equation.

4.130

$$\boldsymbol{j} = \frac{i\hbar}{2m}(\psi \nabla \psi^* - \psi^* \nabla \psi) - \frac{e}{mc} \boldsymbol{A} \psi^* \psi \, , \quad \rho = \psi^* \psi \, .$$

4.131* For the observed quantities ρ and $\boldsymbol{j}$ to be unaltered during gauge transformation of potentials

$$\boldsymbol{A} = \tilde{\boldsymbol{A}} - \nabla f(\boldsymbol{r}, t) \, , \quad \varphi = \tilde{\varphi} + \frac{1}{c} \frac{\partial f(\boldsymbol{r}, t)}{\partial t} \, , \tag{1}$$

the phase of the wave function needs to be transformed too:

$$\psi = \tilde{\psi} \exp\left(\frac{ie}{\hbar c} f(\boldsymbol{r}, t)\right) . \tag{2}$$

Transformation by means of (1) and (2) differs from phase transformation with $\alpha = \text{const}$ considered in Problems 4.126*–4.128 by virtue of its local character, with the phase factor of the wave function depending on both coordinates and time. The invariance of the quantities being observed with respect to the local phase transformation is conserved only in the presence of an additional compensating field, whose role is played by the electromagnetic field.

Generalization of this simple idea gave rise in the 1960s to a new research area, gauge field theory, which in turn ensured great progress in elementary particle physics.

4.132* Transformation to new field functions $A^k(x)$, $\zeta(x)$, and $\eta(x)$ yields the Lagrangian

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}(\partial_k A_i - \partial_i A_k)(\partial^k A^i - \partial^i A^k) + \frac{g^2}{2}(\varphi_0 + \eta)^2 A_k A^k \\ &+ \frac{1}{2}(\partial_k \eta \partial^k \eta) + \frac{\mu^2}{2}(\varphi_0 + \eta)^2 - \frac{\lambda}{4}(\varphi_0 + \eta)^4 .\end{aligned} \tag{1}$$

In the absence of the vector field, $A_i = 0$, we have the Lagrangian of the vector field considered earlier in Problem 4.126*. It describes the scalar field with vacuum states corresponding to the nonzero values of $\varphi_0 = \pm\mu/\sqrt{\lambda}$. Field excitations (quanta) occur at $\eta \neq 0$ and have mass $m_\eta = \sqrt{2}\mu\hbar/c$.

In the absence of the scalar field, $\varphi_0 = \eta = 0$, the Lagrangian of the vector field A_k assumes the same form as that of a free electromagnetic field:

$$\mathcal{L}_v^{(0)} = -\frac{1}{4}(\partial_k A_i - \partial_i A_k)(\partial^k A^i - \partial^i A^k) . \tag{2}$$

It contains no constants through which the particle mass could be expressed and describes a field with a zero quantum mass. For the vacuum state of the scalar field, $\varphi_0^2 = \mu^2/\lambda$ and $\eta = 0$, the Lagrangian assumes the form

$$\mathcal{L}_v = -\frac{1}{4}(\partial_k A_i - \partial_i A_k)(\partial^k A^i - \partial^i A^k) + \frac{g^2\mu^2}{2\lambda} A_k A^k + \text{const} . \tag{3}$$

Under an additional Lorentz-type condition, $\partial_i A^i = 0$, the equation of motion has the form

$$\Box A^l - \frac{g^2\mu^2}{\lambda} A^l = 0 . \tag{4}$$

Comparison of (4) with (1) from Problem 4.124* allows the mass of vector particles to be found in the model above:

$$m_\text{v} = \frac{g\mu\hbar}{c\sqrt{\lambda}} . \tag{5}$$

This mechanism of the appearance of mass in the initially massless particles is referred to as *the Higgs mechanism*. It is supposed to explain the property of mass in intermediate vector bosons ($W^{\pm}$ and Z^0; see Table 3.1) observed in experiments, and other elementary particles. Still, one of the intermediate bosons, the photon, remains massless. Scalar Higgs particles, that is, quanta of field η with mass $m_H \equiv m_\eta \approx 126\,\mathrm{GeV}/c^2$, were discovered (with high probability equal to $1 - 6 \times 10^{-7}$) in 2012 with the Large Hadron Collider. See the comprehensive reviews by Okun (1988) and Rubakov (2002) and the articles by Okun (2012) and Rubakov (2012) for more details.

4.133* As is known from classical mechanics, the invariance of the Lagrangian function with respect to the time shift accounts for the conservation of the total energy and that with respect to the spatial shift for the conservation of the total momentum of the system. By analogy, in the case of a field, the integral over the entire three-dimensional space $\mathcal{E} = \int J^{00}\mathrm{d}^3x$ should be identified as the total field energy and $P^\alpha = (1/c)\int J^{0\alpha}\mathrm{d}^3x$ should be identifed as its total momentum. However, it should be recalled that generalized currents are defined ambiguously. Therefore, additional considerations are needed to find acceptable expressions for the densities of the respective quantities. See Examples 4.21 and 4.22 and Problems 4.135 and 4.139*.

4.134 It follows from the form of the energy–momentum tensor written in terms of intensities of the electromagnetic field (see Example 4.21) that its nondiagonal elements become zero when and only when the strength $\boldsymbol{E}$ is either parallel or antiparallel to $\boldsymbol{H}$ or when one of them equals zero. In this case,

$$T^{00} = -T^{11} = T^{22} = T^{33} = \frac{1}{8\pi}\left(E^2 + H^2\right)$$

if the field vector is directed along Ox. All frames of reference exhibiting such a property were found in Problem 4.28*. Generally speaking, if the field varies in space and time, the tensor can be diagonalized by transition to the corresponding reference frame only at one point in space and at a definite moment of time.

Diagonalization is impracticable if the initial frame of reference has $\boldsymbol{E} \perp \boldsymbol{H}$ and $E = H$.

4.135 The introduction of the term $-mc^2\psi^*\psi$ into Lagrangian (4.100) yields

$$T^0{}_0 = T^{00} = \frac{\partial\mathcal{L}}{\partial\psi_{,t}}\psi_{,t} + \frac{\partial\mathcal{L}}{\partial\psi^*_{,t}}\psi^*_{,t} - \mathcal{L} = \frac{\hbar^2}{2m}\psi^{,\alpha}\psi^*_{,\alpha} + (U + mc^2)\psi^*\psi\,, \quad (1)$$

which is identical with the Hamilton function density computed in Problem 4.119 with the addition of the particle's rest energy. This leads to

$$\frac{1}{c}T^{0\alpha} = -\frac{1}{c}T^0{}_\alpha = -\frac{\partial\mathcal{L}}{\partial\psi_{,t}}\psi_{,\alpha} - \frac{\partial\mathcal{L}}{\partial\psi^*_{,t}}\psi^*_{,\alpha} = -\frac{i\hbar}{2}\left(\psi^*\psi_{,\alpha} - \psi\psi^*_{,\alpha}\right)\,. \quad (2)$$

The total momentum related to field ψ is calculated by partial integration of the second term on the assumption of the field vanishing at infinity:

$$P^\alpha = \frac{1}{c}\int T^{0\alpha}\mathrm{d}^3x = \int \psi^*\left(-i\hbar\frac{\partial}{\partial x^\alpha}\right)\psi\mathrm{d}^3x = -\int \psi^*\widehat{p}_\alpha\psi\mathrm{d}^3x$$
$$= \int \psi^*\widehat{p}^\alpha\psi\mathrm{d}^3x \,. \tag{3}$$

It coincides with the mean quantum mechanical momentum of the particle.

It is instructive to check the energy–momentum tensor for symmetry. We have

$$\frac{1}{c}T^{\alpha 0} = \frac{1}{c}T^\alpha{}_0 = \frac{1}{c^2}\left(\frac{\partial\mathcal{L}}{\partial\psi_{,\alpha}}\psi_{,t} + \frac{\partial\mathcal{L}}{\partial\psi^*_{,\alpha}}\psi^*_{,t}\right)$$
$$= \frac{\hbar^2}{2mc^2}\left[(\partial^\alpha\psi^*)\psi_{,t} + (\partial^\alpha\psi)\psi^*_{,t}\right] \,. \tag{4}$$

In appearance, this expression is markedly different from $T^{0\alpha}/c$ defined by equation (2). However, if the main term (rest energy), $\psi_{,t} \approx -imc^2\psi/\hbar$, is taken into consideration, $T^{\alpha 0}/c \approx i\hbar\psi^*\partial^\alpha\psi/2 - i\hbar\psi\partial^\alpha\psi^*/2$, which coincides with $T^{0\alpha}$. Finally, the three-dimensional tensor $T^{\alpha\beta}$ gives

$$\int T^{\alpha\beta}\mathrm{d}^3x = \int \psi^*\widehat{p}^\alpha\widehat{v}^\beta\psi\mathrm{d}^3x \tag{5}$$

that is, the mean quantum mechanical value of the momentum flux density.

4.136 If an observer is in system S' and the energy density $\mathcal{H} = T'^{00} = T'^{ik}u'_iu'_k/c^2 = T^{ik}u_iu_k/c^2$, the momentum density projected onto the space-like direction defined by the unity vector $n_k(n_kn^k = -1)$ is written in the form $P'_n = T^{ik}u_in_k/c$.

4.137 In the general case,

$$\oint\limits_S \sigma^{\alpha\beta}\mathrm{d}S_\beta = \int\limits_V \left(\frac{\partial \boldsymbol{g}}{\partial t} + \rho\boldsymbol{E} + \frac{1}{c}\boldsymbol{j}\times\boldsymbol{H}\right)^\alpha \mathrm{d}^3x \,.$$

The right-hand side of this equality contains, besides the force, a change of the field momentum per unit time in the volume being considered.

4.138 It is natural to choose as independent parameters of transformation six independent components of the antisymmetric matrix of the Lorentz transformation, $\delta\Omega_{li} = -\delta\Omega_{il}$, introduced in the solution of Problem 4.20. Index a entering the general expression (4.119) is a set of vector indices, $a = (i, l)$. Then,

$$x'^l = x^l + \delta\Omega^l{}_ix^i \,,$$
$$\delta x^l = \delta\Omega^l{}_ix^i = \Gamma^l_{ij}\delta\Omega^{ij}/2 \,, \quad \Gamma^l_{ij} = \delta^l_ix_j - \delta^l_jx_i \,. \tag{1}$$

(It is convenient to begin by introducing factor 1/2, that is, choosing $\delta\lambda = \delta\Omega/2$, to be in compliance with the definition of the moment known from mechanics.)

The use of (1) allows the Noether current density to be obtained from (4.119):

$$J^k{}_{mn} = J^k{}_m x_n - J^k{}_n x_m - \frac{\partial \mathcal{L}}{\partial q^A{}_{,k}} G^A_{Bmn} q^B . \tag{2}$$

Here, $J^k{}_l$ is the canonical energy–momentum tensor (4.124), and the form of the matrix G^A_{Bmn} is specified by the nature of the field functions q^A.

1. Let $q^A = \psi$ be a scalar complex wave function of a nonrelativistic spinless particle satisfying the Schrödinger equation. Using (4.100) and (4.124), we arrive at

$$J^0{}_\alpha = \frac{\partial \mathcal{L}}{\partial q^A{}_{,0}} q^A{}_{,\alpha} = \frac{\partial \mathcal{L}}{\partial \psi^*_{,0}} \psi^*_{,\alpha} + \frac{\partial \mathcal{L}}{\psi_{,0}} \psi_{,\alpha} = \frac{i\hbar c}{2}\left(\psi^*\psi_{,\alpha} - \psi\psi^*_{,\alpha}\right) . \tag{3}$$

The scalar remains unaltered upon spatial turns; therefore, $G^A B m n = 0$. The use of (1) and (2) leads to

$$\begin{aligned} \frac{1}{c}\int J^0{}_{\alpha\beta} \mathrm{d}^3x &= \frac{1}{c}\int \left(J^0{}_\alpha x_\beta - J^0{}_\beta x_\alpha\right) \mathrm{d}^3x \\ &= i\hbar \int \psi^*(x_\beta \psi_{,\alpha} - x_\alpha \psi_{\,\beta}) \mathrm{d}^3x \\ &= \int \psi^*[\boldsymbol{r} \times \widehat{\boldsymbol{p}}]_{\alpha\beta} \psi \mathrm{d}^3x \equiv \bar{l}_{\beta\alpha} , \end{aligned} \tag{4}$$

where $\boldsymbol{r} = (-x_1, -x_2, -x_3)$ and $\widehat{\boldsymbol{p}} = (-i\hbar\partial_1, -i\hbar\partial_2, -i\hbar\partial_3)$.
Thus, the integration of $J^0{}_{\alpha\beta}/c$ components over the 3-space yields the mean quantum mechanical value of the particle's orbital moment $\bar{\boldsymbol{l}}$ expressed through the components of the dual 3-tensor ($\bar{l}_{12} \to \bar{l}_3$, etc.). Special attention should be given to the altered order of indices α and β on the right-hand and left-hand sides of the above chain of equalities. In the general case, the above expression at $G^A_{Bmn} = 0$ can be represented in the form

$$\bar{L}_{\beta\alpha} = \int (x_\beta \mathrm{d}P_\alpha - x_\alpha \mathrm{d}P_\beta) , \tag{5}$$

where $\mathrm{d}P_\alpha = J^0{}_\alpha \mathrm{d}^3x/c$ is the field momentum applied to the volume element d^3x. This quantity is called the *orbital moment of the field* for its obvious similarity with the angular momentum of a continuous medium. For a scalar field, it is the integral of motion.

2. Let a nonrelativistic particle have spin 1/2. Its wave function $\Psi(x,\xi) = \psi(x)\chi(\xi)$ is the product of scalar ψ and spinor χ, a two-component quantity. The spinor undergoes transformation in accordance with the (Landau and Lifshitz, 1977) during rotation through a small $\delta\varphi$ angle around direction $\boldsymbol{n}$ in the 3-space

$$\chi' = \left(1 + i\boldsymbol{n}\cdot\widehat{\boldsymbol{\sigma}}\frac{\delta\varphi}{2}\right)\chi , \tag{6}$$

where $\widehat{\boldsymbol{\sigma}}$ is a vector whose components are Pauli spin matrices (see Appendix C). Comparison of (6) with (1) gives the form of matrix $\widehat{G}$:

$$\widehat{G} = \left(\frac{i}{2}\right) \boldsymbol{n} \cdot \widehat{\boldsymbol{\sigma}} \,. \tag{7}$$

Calculating the last term in (2) and integrating it over the 3-space yields an additional term for the orbital moment (4), that is, the *spin moment*:

$$S_n = \left(\chi, \frac{\hbar}{2} \boldsymbol{n} \cdot \widehat{\boldsymbol{\sigma}} \chi\right) \int |\psi(x)|^2 \mathrm{d}^3 x = \frac{\hbar}{2} (\chi, \boldsymbol{n} \cdot \widehat{\boldsymbol{\sigma}} \chi) \,. \tag{8}$$

In this way, the mean quantum mechanical value is obtained for the projection of the spin moment onto direction $\boldsymbol{n}$. The integral of the square of the coordinate wave function modulus gives unity by use of the normalization condition. In the general case, the *field spin* is described by the expression

$$S_{\alpha\beta} = -\int \frac{\partial \mathcal{L}}{\partial q^A{}_{,0}} G^A{}_{B\beta\alpha} q^B \,. \tag{9}$$

The conserved quantity in the general case is the total moment $J_{\alpha\beta} = L_{\alpha\beta} + S_{\alpha\beta}$.

4.139* The transformation of coordinates is described by formulas (1) from the solution of the preceding problem. In electrodynamics, the components of the 4-potential, $q^A(x) = A^l(x)$, serve as the field functions. The argument of the 4-potential and its projections onto the coordinate axes change upon a small turn of the four-dimensional system of coordinates. A change of the argument means a shift in space unaccompanied by alteration of the field functions owing to the uniformity of the 4-space (see the consideration of the shift in Problem 4.132*). Changes in the projections of the 4-potential obey the same law as transformations of any 4-vector, such as a radius vector, that is,

$$A'^l = A^l + \delta\Omega^l{}_i A^i = A^l + \left(\delta^l_i A_j - \delta^l_j A_i\right) \frac{\delta\Omega^{ij}}{2} \,. \tag{1}$$

Thus, the second term in formula (2) from the solution of the preceding problem admits the substitution

$$G^A_{\mathrm{Ba}} q^B(x) \to \delta^l_i A_j(x) - \delta^l_j A_i(x) \,, \tag{2}$$

and the derivative $\partial\mathcal{L}/\partial A^l{}_{,k}$ was calculated in Wxample 4.21. As a result, the generalized Noether current that is obtained that in the present case is a rank 3 tensor antisymmetric with respect to the two subscripts i and j:

$$J_{kij} = J_{ki} x_j - J_{kj} x_i + \frac{1}{4\pi} (F_{ki} A_j - F_{kj} A_i) \,. \tag{3}$$

This tensor satisfies the differential law of conservation $\partial^k J_{kij} = 0$ on the strength of Noether's theorem (as can be easily seen directly). Two of its serious disadvantages include the absence of gauge invariance and the lack of the mechanical moment structure (i.e., the vector product of the radius vector and the momentum). It prompts the idea of building up the moment density tensor on the basis of the gauge-invariant and symmetric energy–momentum tensor T_{ik} (see (4.126)):

$$M_{kij} = T_{ki}x_j - T_{kj}x_i \,. \tag{4}$$

Let us calculate the divergence of this tensor:

$$\partial^k M_{kij} = T_{ji} - T_{ij} + x_j \partial^k T_{ki} - x_i \partial^k T_{kj} = 0 \,.$$

This result is related to symmetry ($T_{ji} = T_{ij}$) and the differential conservation law ($\partial^k T_{kl} = 0$) for the energy–momentum tensor. Thus, tensor M_{kij}, like J_{kij}, satisfies the continuity equation (4.116) ensuing from the Noether theorem; consequently, the two tensors must differ in the divergence of a certain rank 4 antisymmetric tensor. It is easy to show that

$$\begin{aligned} M_{kij} - J_{kij} &= \frac{1}{4\pi}\left(F_{kl}F^l{}_i x_j - F_{kl}F^l{}_j x_i\right) + \frac{1}{4\pi}F_{kl}\left(A^l{}_{,i}x_j - A^l{}_{,j}x_i\right) \\ &\quad - \frac{1}{4\pi}(F_{ki}A_j - F_{kj}A_i) \\ &= \frac{1}{4\pi}F_{kl}\left(A^{,l}_{i}x_j - A^{,l}_{j}x_i\right) + \frac{1}{4\pi}(F_{kj}A_i - F_{ki}A_j) \\ &= \partial^l\left[\frac{1}{4\pi}F_{kl}(A_i x_j - A_j x_i)\right] \,. \end{aligned}$$

Thus, both momentum density tensors, M_{kij} and J_{kij}, are compatible with Noether's theorem and lead to identical integral conserved quantities, that is, components of the rank 2 antisymmetric angular momentum tensor:

$$M_{ij} = \frac{1}{c}\int M_{0ji}\mathrm{d}^3x = \frac{1}{c}\int J_{0ji}\mathrm{d}^3x \tag{5}$$

(with the reverse order of indices on the right-hand side). But tensor M_{0ij} leads to more symmetric and demonstrative expressions:

$$M_{ij} = \frac{1}{c}\int (x_i T_{0j} - x_j T_{0i})\mathrm{d}^3x = \int (x_i \mathrm{d}P_j - x_j \mathrm{d}P_i) \,, \tag{6}$$

where $\mathrm{d}P_i = T_{0i}\mathrm{d}^3x/c$ is the field momentum enclosed in the three-dimensional volume d^3x. The spatial part of the tensor M_{ij} makes up a three-dimensional antisymmetric rank 2 tensor dual to the pseudovector of the angular momentum of the field $\boldsymbol{M}$:

$$\boldsymbol{M} = \int \boldsymbol{r} \times \boldsymbol{g}\mathrm{d}^3x \,, \tag{7}$$

where

$$c g^\alpha = T^{0\alpha} = -T_{0\alpha} \,, \quad \boldsymbol{g} = \frac{1}{4\pi c}\boldsymbol{E} \times \boldsymbol{H} \,.$$

4.140*

$$M_{kji} = x_i T_{kj} - x_j T_{ki} \,, \quad \text{where} \quad T_{kl} = T_{kl}^{\text{em}} + T_{kl}^{\text{part}}$$

or the total energy–momentum determined in Example 4.22. For the spatial components $M_{\alpha\beta} = (1/c) \int M_{0\alpha\beta} \mathrm{d}^3 x$ of the system's angular momentum we find

$$\boldsymbol{M} = \int \boldsymbol{r} \times \boldsymbol{g} \mathrm{d}^3 x + \sum_a \boldsymbol{r} \times \boldsymbol{p}_a = \text{const} \,. \tag{1}$$

Let us introduce time t common for all particles to interpret mixed components $M_{0\alpha}$. Then, the Noether theorem predicts the conservation of quantities

$$M_{0\alpha} = ct(G_\alpha + P_\alpha) - \frac{1}{c} \left\{ \int x_\alpha w \mathrm{d}^3 x + \sum_a x_\alpha^a \mathcal{E}_a \right\} = \text{const} \,, \tag{2}$$

where $\boldsymbol{G}$ and $\boldsymbol{P}$ are the total momenta of the field and particles respectively, and w is the electromagnetic field's density. Let us introduce into the consideration a point in the 3-space with radius vector

$$\boldsymbol{R} = \frac{\int \boldsymbol{r} w \mathrm{d}^3 x + \sum_a \boldsymbol{r}_a \mathcal{E}_a}{\int w \mathrm{d}^3 x + \sum_a \mathcal{E}_a} \,. \tag{3}$$

The constancy of the $M_{0\alpha}$ components means that this point moves according to the $\boldsymbol{R} = \boldsymbol{R}_0 + \boldsymbol{V} t$ law with velocity

$$\boldsymbol{V} = \frac{c^2(\boldsymbol{P} + \boldsymbol{G})}{\mathcal{E}_{\text{field}} + \mathcal{E}_{\text{part}}} \,, \quad \text{where} \quad \mathcal{E}_{\text{field}} = \int w \mathrm{d}^3 x \,, \quad \mathcal{E}_{\text{part}} = \sum_a \mathcal{E}_a \,, \tag{4}$$

that is, constant owing to the conservation of the system's total momentum and total energy. The relations thus obtained are the generalized theorems of mechanics for the uniform motion of the center of mass in a closed system for the case of relativistic objects.

4.141* The Lagrangian function has the form

$$L(\boldsymbol{v}, \boldsymbol{r}) = \frac{m v^2}{2} + \frac{e}{c} \boldsymbol{v} \cdot \boldsymbol{A}(\boldsymbol{r}) \,, \tag{1}$$

where $\boldsymbol{A}(\boldsymbol{r}) = \boldsymbol{H} \times \boldsymbol{r}/2$ is the vector potential. The lack of the explicit time dependence of the Lagrangian function accounts for the conservation of the particle's energy

$$\mathcal{E} = \boldsymbol{v} \cdot \frac{\partial L}{\partial \boldsymbol{v}} - L = \frac{m v^2}{2} \,. \tag{2}$$

The invariance of the Lagrangian function with respect to rotation around the $\boldsymbol{H} \parallel Oz$ direction accounts for the conservation of the canonical momentum

$$P_\alpha = \left[\boldsymbol{r} \times \frac{\partial L}{\partial \boldsymbol{v}} \right]_z = m[\boldsymbol{r} \times \boldsymbol{v}]_z + \frac{e}{2c}[\boldsymbol{r} \times [\boldsymbol{H} \times \boldsymbol{r}]]_z \,, \tag{3}$$

corresponding to the rotation around the Oz axis.

Derivation of the conservation law related to the spatial shift requires great care because the Lagrangian function is not invariant with respect to the shift. Calculation of a change in the Lagrangian function on the substitution of $\boldsymbol{r}$ with $\boldsymbol{r} + \delta\boldsymbol{r}$ gives from (1)

$$\delta L = \frac{\partial L}{\partial \boldsymbol{r}} \cdot \delta\boldsymbol{r} = \frac{e}{2c}[\boldsymbol{v} \times \boldsymbol{H}] \cdot \delta\boldsymbol{r} = \delta\boldsymbol{r} \cdot \frac{\mathrm{d}}{\mathrm{d}t}\frac{e}{2c}\boldsymbol{r} \times \boldsymbol{H} , \tag{4}$$

that is, the variation is expressed through the total time derivative from a certain coordinate function. On the other hand, it follows from the Lagrange equation that

$$\frac{\partial L}{\partial \boldsymbol{r}} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \boldsymbol{v}} = \frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t} , \tag{5}$$

where $\boldsymbol{P} = m\boldsymbol{v} + (e/2c)\boldsymbol{H} \times \boldsymbol{r}$ is the canonical momentum conjugate to the radius vector that is not conserved. Substituting (5) into (4) yields the conserved generalized momentum

$$\boldsymbol{\mathcal{P}} = \boldsymbol{P} + \frac{e}{2c}\boldsymbol{H} \times \boldsymbol{r} = m\boldsymbol{v} - \frac{e}{c}\boldsymbol{r} \times \boldsymbol{H} . \tag{6}$$

The constancy of this quantity also follows directly from the equation of motion

$$m\dot{\boldsymbol{v}} = \frac{e}{c}\boldsymbol{v} \times \boldsymbol{H} \tag{7}$$

at $\boldsymbol{H} = \text{const.}$

Let us calculate, in addition, the electromagnetic momentum created by the particle's Coulomb field and the external magnetic field. Using formula (4.128), we find

$$\boldsymbol{p}_{\text{em}} = \frac{1}{4\pi c}\int \boldsymbol{E}(\boldsymbol{r}, t) \times \boldsymbol{H}\mathrm{d}^3x = \frac{1}{2\pi c}\int \boldsymbol{A}\,\mathrm{div}\,\boldsymbol{E}\mathrm{d}^3x = \frac{2e}{c}\boldsymbol{A}(\boldsymbol{r}_0(t)) . \tag{8}$$

Here, $\boldsymbol{E} = e(\boldsymbol{r} - \boldsymbol{r}_0(t))/|\boldsymbol{r} - \boldsymbol{r}_0(t)|^3$, $\boldsymbol{r}_0(t)$ is the radius vector of the particle, and $\mathrm{div}\,\boldsymbol{E} = 4\pi e\delta(\boldsymbol{r} - \boldsymbol{r}_0(t))$. The integral is transformed with the help of formula (1.89) for $\nabla(\boldsymbol{E} \cdot \boldsymbol{A})$ and the equations $\mathrm{curl}\,\boldsymbol{E} = 0$, and $\mathrm{div}\,\boldsymbol{A} = 0$. In the end, the field's momentum is found to be localized at the point of the particle's location. For the angular moment of the system's full momentum, $m\boldsymbol{v} + \boldsymbol{p}_{\text{em}}$, composed of the particle's momentum and the field's momentum, we find

$$\boldsymbol{L} = \boldsymbol{r}_0 \times (m\boldsymbol{v} + \boldsymbol{p}_{\text{em}}) , \tag{9}$$

where $\boldsymbol{r}$ is the particle's radius vector $\boldsymbol{r}_0(t)$. The projection of (9) onto the direction of the magnetic field is conserved and coincides with that of (2).

4.142 The momentum and energy of the field in volume V at the moment $t = x^0/c$ can be expressed through the integrals $\int T^{\alpha 0}\mathrm{d}V$ and $\int T^{00}\mathrm{d}V$, respectively, where integration is performed over the entire three-dimensional space. Let us combine them and write the result in the covariant form and introduce the unit 4-vector $n_i = (1, 0, 0, 0)$: $\int_{\Sigma(t)} T^{ki} \mathrm{d}S_i$, where $\mathrm{d}S_i = n_i \mathrm{d}V$ is the element of the hypersurface determined by the condition $t = x^0/c = x_0/c =$ const; in other words, it is an element of three-dimensional volume. Let us generalize this integral and extend it to the closed hypersurface Σ_{tot} surrounding the 4-volume Ω:

$$\Sigma_{\text{tot}} = \Sigma(t) + \Sigma(t') + \Sigma' . \tag{1}$$

Here, Σ' is the lateral cylindrical surface, the generatrices of which are parallel to the time axis x^0 (see Figure 4.19).

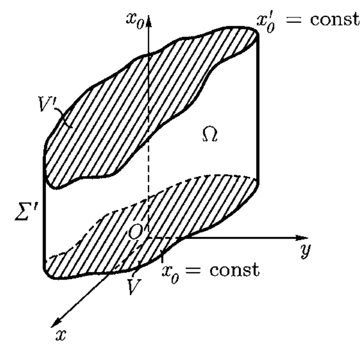

Figure 4.19 Conventional image of a three-dimensional hypersurface.

Let us apply the four-dimensional Gauss–Ostrogradskii theorem to the integral over this hypersurface:

$$\oint_{\Sigma_{\text{tot}}} T^{ki} \mathrm{d}S_i = \int_{\Omega} \frac{\partial T^{ki}}{\partial x^i} \mathrm{d}\Omega , \tag{2}$$

where $\mathrm{d}S_i$ at the lateral surface has a form different from $n_i dV$. Because Ω and $\mathrm{d}\Omega$ are invariants and $\partial T^{ki}/\partial x^i$ is a 4-vector, the integral on the left-hand side is a 4-vector as well. Its individual terms $T^{ki}\mathrm{d}S_i$ are vectors too regardless of the orientation of the $\mathrm{d}S_i$ elements, specifically the integral $\int_{\Sigma(t)} T^{k0} \mathrm{d}V$.

There is no field, $T^{ki} = 0$, at the distant lateral surface Σ'. It follows from this fact, equation (1), and the continuity equation $\partial T^{ki}/\partial x^i = 0$ that $\int T^{k0}(t)\mathrm{d}V = \int T^{k0}(t')\mathrm{d}V$. This means that the 4-vector of interest does not change with time, as must be the case.

4.143 The total angular momentum of the particles and the field in the finite 3-volume being considered

$$L^{\alpha\beta}(t) = \sum l^{\alpha\beta} - \frac{1}{c} \int_{\Sigma(t)} \left(x^\alpha T^{\beta\gamma} - x^\beta T^{\alpha\gamma} \right) \mathrm{d}S_\gamma , \tag{1}$$

where $l^{\alpha\beta} = x^\alpha p^\beta - x^\beta p^\alpha$ is the angular momentum of a single particle, and the sum is taken over the totality of all particles. Unlike the case in the preceding problem, the hypersurface $\Sigma(t)$ perpendicular to the t axis is the finite three-dimensional volume V. The decrease of the system's angular momentum for time $\mathrm{d}t$ is

$$-\mathrm{d}L^{\alpha\beta} = L^{\alpha\beta}(t) - L^{\alpha\beta}(t+\mathrm{d}t) = -\sum \mathrm{d}l^{\alpha\beta} + \frac{1}{c}\int\limits_{\Sigma(t+\mathrm{d}t)} \cdots - \frac{1}{c}\int\limits_{\Sigma(t)} \cdots \quad (2)$$

Let us pass in (2) to the integration over the closed cylindrical hypersurface Σ_{tot} and write down $\int_{\Sigma(t+\mathrm{d}t)} + \int_{\Sigma(t)} + \int_{\Sigma'} = \oint_{\Sigma_{\text{tot}}}$. Here, Σ' is the lateral cylindrical surface, the generatrices of which are parallel to the time axis (see Figure 4.20 bearing in mind the arbitrary character of the 4-space image on a sheet of paper).

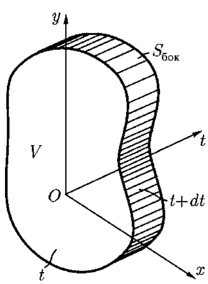

Figure 4.20 Conventional image of a three-dimensional cylindrical hypersurface.

Using the Gauss–Ostrogradskii theorem, we arrive at

$$\oint\limits_{\Sigma_{\text{tot}}} \left(x^\alpha T^{\beta i} - x^\beta T^{\alpha i}\right) \mathrm{d}S_i = \int\limits_{\Omega} \frac{\partial}{\partial x^i} \left(x^\alpha T^{\beta i} - x^\beta T^{\alpha i}\right) \mathrm{d}\Omega \; . \quad (3)$$

The use of (4.133) and (4.53) makes it certain that the integral over the 4-volume in (3) is transformed into a change of the particles' angular momentum $\sum \mathrm{d}l^{\alpha\beta}$. As a result, (2) is transformed to assume the form

$$-\mathrm{d}L^{\alpha\beta} = \frac{1}{c}\int\limits_{\Sigma'} \left(x^\alpha T^{\beta\gamma} - x^\beta T^{\alpha\gamma}\right) \mathrm{d}S_\gamma \; . \quad (4)$$

Elements of the hypersurface Σ' are normal to the time axis and can be written in the form $\mathrm{d}S_\gamma = c\mathrm{d}t n_\gamma \mathrm{d}f$, where $\mathrm{d}f$ is the element of the usual two-dimensional surface enclosing volume V and $\boldsymbol{n}$ is the unity vector of the normal to this element. It allows the expression for a decrease of the angular momentum per unit time to be obtained from (4):

$$-\frac{\mathrm{d}L^{\alpha\beta}}{\mathrm{d}t} = \oint \left(-x^\alpha T^{\beta\gamma} + x^\beta T^{\alpha\gamma}\right) n_\gamma \mathrm{d}f \; . \quad (5)$$

Let us introduce tensor $\mathcal{R}^{\alpha\beta\gamma} = x^\beta T^{\alpha\gamma} - x^\alpha T^{\beta\gamma}$ antisymmetric with respect to the subscripts α and β. This tensor can be interpreted as the angular momentum flux density as follows from (5). The component $\mathcal{R}^{\alpha\beta\gamma}$ is equal to the quantity of the $\alpha\beta$ component of the total angular momentum $L^{\alpha\beta}$ passing per unit time through a unit surface perpendicular to the axis labeled γ. Let us denote by $\boldsymbol{L}$ and $\boldsymbol{\mathcal{R}}$ 3-vectors dual to antisymmetric tensors $L^{\alpha\beta}$ and $\mathcal{R}_{\alpha\beta\gamma} n_\gamma$. Then equality (5) takes the form

$$-\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} = \oint \boldsymbol{\mathcal{R}} \mathrm{d} f \, , \tag{6}$$

where

$$\boldsymbol{\mathcal{R}} = \frac{E^2 + H^2}{8\pi} \boldsymbol{r} \times \boldsymbol{n} - \frac{1}{4\pi} \boldsymbol{r} \times [\boldsymbol{E}(\boldsymbol{n} \cdot \boldsymbol{E}) + \boldsymbol{H}(\boldsymbol{n} \cdot \boldsymbol{H})] \, . \tag{7}$$

The last formula is obtained with the use of expression (4.126) for the components of the energy–momentum tensor and table (4.68).

4.144• In accordance with (4.128) and (2.153)–(2.154),

$$\boldsymbol{P} = \frac{1}{4\pi c} \boldsymbol{E} \times \boldsymbol{H} = -i \sum_{\boldsymbol{k}\sigma} \boldsymbol{k} \dot{q}^*_{\boldsymbol{k}\sigma}(t) q_{\boldsymbol{k}\sigma}(t) \, . \tag{1}$$

Substituting $q_{\boldsymbol{k}\sigma}(t) = b_{\boldsymbol{k}\sigma}(t) + b^*_{-\boldsymbol{k}\sigma}(t)$ into (1) in conformity with (2.150) and (2.151) yields

$$\boldsymbol{P} = 2 \sum_{\boldsymbol{k}\sigma} \boldsymbol{k} \omega_k b^*_{\boldsymbol{k}\sigma}(t) b_{\boldsymbol{k}\sigma}(t) + \sum_{\boldsymbol{k}\sigma} \boldsymbol{k} \omega_k (b^*_{\boldsymbol{k}\sigma}(t) b^*_{-\boldsymbol{k}\sigma}(t) - b_{\boldsymbol{k}\sigma}(t) b_{-\boldsymbol{k}\sigma}(t)) \, . \tag{2}$$

The second sum changes sign in the case of $\boldsymbol{k} \to -\boldsymbol{k}$ transition and therefore equals zero. Using (2.157), we arrive at the formula being sought.

5
Emission and Scattering of Electromagnetic Waves

5.1
Green's Functions and Retarded Potentials

It is convenient to begin studying electromagnetic waves with reference to the electromagnetic potentials satisfying the inhomogeneous d'Alembert equations (2.107) and (2.108)

$$\left.\begin{aligned} \Delta \boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} &= -\frac{4\pi}{c}\boldsymbol{j}(\boldsymbol{r}, t)\ , \\ \Delta\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} &= -4\pi\rho(\boldsymbol{r}, t)\ , \end{aligned}\right\} \tag{5.1}$$

and the Lorentz condition (2.105)

$$\operatorname{div}\boldsymbol{A} + \frac{1}{c}\frac{\partial\varphi}{\partial t} = 0\ . \tag{5.2}$$

The solution of these equations in an unbounded space is possible on the assumption that field sources $\boldsymbol{j}$ and ρ are distributed within a finite space region as the known functions of the coordinates and time. It should be emphasized that such a statement of the problem has an approximate character. Indeed, a field created by charged particles influences their movements; therefore, the right-hand sides of (5.1) are, strictly speaking, nonlinear functionals of the potentials being sought. However, in many (even if not in all – see Section 5.4) cases the feedback effect of the proper field on the motion of the charge is small and can be disregarded. The problem becomes linearized. It is such a problem that we have to address.

Foundations of Classical and Quantum Electrodynamics, First Edition. Igor N. Toptygin.

5.1.1 The Green's Function of a Wave Equation

Let us define the Green's function $G(\boldsymbol{r}, t; \boldsymbol{r}', t')$ as the solution of a wave equation for the unbounded space with a delta-shaped right-hand side:

$$\Delta G - \frac{1}{c^2}\frac{\partial^2 G}{\partial t^2} = -4\pi\delta(\boldsymbol{r} - \boldsymbol{r}')\delta(t - t') \,. \tag{5.3}$$

The solution of an inhomogeneous wave equation can be written with the help of the Green's function in the form of the integral

$$\boldsymbol{A}(\boldsymbol{r}, t) = \frac{1}{c}\int G(\boldsymbol{r}, t; \boldsymbol{r}', t')\boldsymbol{j}(\boldsymbol{r}', t')\mathrm{d}^3 r'\mathrm{d}t' \,. \tag{5.4}$$

Applying the d'Alembert operator to $\boldsymbol{A}(\boldsymbol{r}, t)$ and using (5.3), we come to the conclusion that (5.1) is satisfied.

Example 5.1

Show that the Green's function of a wave equation depends only on the differences $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}'$ and $\tau = t - t'$.

Solution. The right-hand side of (5.3) depends only on $\boldsymbol{R}$ and τ, whereas the derivatives on the left-hand side at fixed $\boldsymbol{r}'$ and t' can be taken in $\boldsymbol{R}$ and τ. Then, (5.3) assumes the form

$$\Delta G - \frac{1}{c^2}\frac{\partial^2 G}{\partial \tau^2} = -4\pi\delta(\boldsymbol{R})\delta(\tau) \,, \tag{5.5}$$

where operator Δ acts on the coordinates of $\boldsymbol{R}$. It follows from the form of the equation that $G(\boldsymbol{r}, t; \boldsymbol{r}', t') = G(\boldsymbol{R}, \tau)$. □

Example 5.2

Find from (5.5) the most general form of the Fourier transform $\tilde{G}(\boldsymbol{k}, \omega)$ for expansion of the Green's function in terms of flat monochromatic waves (see formulas (2.114)).

Solution. The expansion of both sides of (5.5) in flat monochromatic waves in accordance with the second formula in (2.114) leads to the algebraic equation

$$(k^2 - \frac{\omega^2}{c^2})\tilde{G}(\boldsymbol{k}, \omega) = 4\pi \,.$$

Its formal solution has the form

$$\tilde{G}(\boldsymbol{k}, \omega) = \frac{4\pi}{k^2 - \omega^2/c^2} \,. \tag{5.6}$$

The solution is formal because it has a singularity at $k^2 = \omega^2/c^2$ and requires additional information to perform an inverse Fourier transform, that is, to stipulate the rule for singular function integration. Moreover, the solution of (5.6) is not general: it lacks the solution of the homogeneous wave equation having the form $F(\boldsymbol{k}, \omega)\delta(k^2 - \omega^2/c^2)$, where function F must be bounded at $\omega^2 = k^2c^2$. This function is the solution of the homogeneous wave equation by virtue of an equality of type $x\delta(x) = 0$. Thus, the most general form of the Fourier transform of the Green's function is

$$\tilde{G}(\boldsymbol{k}, \omega) = \frac{4\pi}{k^2 - \omega^2/c^2}\omega^2/c^2 + F(\boldsymbol{k}, \omega)\delta\left(k^2 - \frac{\omega^2}{c^2}\right), \tag{5.7}$$

from which different Green's functions are obtained depending on the integration rule for a singular fraction and the form of the F function (see Example 5.3 and Problems 5.2•–5.4). □

Example 5.3

The retarded Green's function $G^{\mathrm{R}}(R, \tau)$ of a wave equation is the solution of (5.5) satisfying the condition

$$G^{\mathrm{R}}(R, \tau) = 0 \quad \text{at} \quad \tau < 0\,. \tag{5.8}$$

Clarify the rule for a detour around the poles in the computation of the Fourier integral that ensures fulfillment of the above condition and calculate $G^{\mathrm{R}}(R, \tau)$.

Solution. The solution of a homogeneous equation is a set of flat monochromatic waves and cannot satisfy condition (5.8). For this reason, assume $F = 0$ in (5.7) and deform the integration path C_{R} in the complex plane to integrate over $\mathrm{d}\omega$ bypassing from above the poles of the integrand $\omega = \pm\omega_k = \pm ck$ (Figure 5.1). At $\tau > 0$, close the integration path with an arc of large radius in the lower half-plane on which the integrand is exponentially small; this leads to the sum of residues with respect to the two poles:

$$G^{\mathrm{R}}(\boldsymbol{k}, \tau) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi}\tilde{G}(\boldsymbol{k}, \omega)\exp(-i\omega\tau) = \frac{4\pi c^2\sin(\omega_k\tau)}{\omega_k}\,, \quad \tau > 0\,.$$

At $\tau < 0$, close the integration path with an arc of large radius in the upper half-plane; this gives zero because the poles are absent inside the contour: $G(\boldsymbol{k}, \tau) = 0$ at $\tau < 0$.

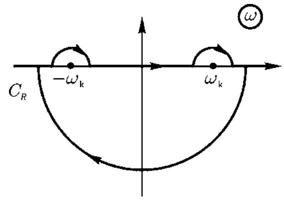

Figure 5.1 The rule for a detour around the poles in the Green's function calculation.

Spherical coordinates with a polar axis along $\boldsymbol{R}$ are used in integration over wave vectors. Use $\mathrm{d}^3k = k^2 \mathrm{d}k \mathrm{d}\Omega_k$ and calculate the integral

$$\int \exp(i\boldsymbol{k}\cdot\boldsymbol{R})\mathrm{d}\Omega_k = 2\pi \int_0^\pi \exp(ikR\cos\vartheta)\sin\vartheta \mathrm{d}\vartheta = \frac{4\pi}{kR}\sin(kR) .$$

As a result,

$$G^{\mathrm{R}}(R,\tau) = \int \frac{\mathrm{d}^3k}{(2\pi)^3} G^{\mathrm{R}}(\boldsymbol{k},\tau) = \frac{2c}{\pi R}\int_0^\infty \sin(kR)\sin(kc\tau)\mathrm{d}k .$$

Substituting the product of sines in the form of the difference between cosines and using representation (1.219) for the delta function finally yields

$$G^{\mathrm{R}}(R,\tau) = \frac{1}{R}\delta\left(\tau - \frac{R}{c}\right) . \tag{5.9}$$

Condition (5.8) is supported by the argument of the delta function. The solution of (5.9) is an infinitely thin spherical wave packet emitted at instant t' from point $R = 0$ and propagating with speed c. □

Other Green's functions also satisfy (5.9); see Problems 5.2•–5.9.

5.1.2 Retarded Potentials

The retarded Green's function plays an important role in classical electrodynamics since it provides a basis for the *causality principle*: the cause (motion of the charge in the source) must precede the effect (variation of the field at the observation point). Find the retardation potentials using (5.4) and (5.9):

$$\boldsymbol{A}(\boldsymbol{r},t) = \frac{1}{c}\int \frac{\boldsymbol{j}(\boldsymbol{r}', t - |\boldsymbol{r}-\boldsymbol{r}'|/c)}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}V', \tag{5.10}$$

$$\varphi(\boldsymbol{r},t) = \int \frac{\rho(\boldsymbol{r}', t - |\boldsymbol{r}-\boldsymbol{r}'|/c)}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}V' . \tag{5.11}$$

The time argument in the charge and current distribution shows that the field at point $\boldsymbol{r}$ and moment t depends on the $\boldsymbol{j}$ and ρ values at point $\boldsymbol{r}'$ at the preceding moment $t' = t - R/c$. Electromagnetic perturbations in a vacuum propagate with velocity c.

Example 5.4

Write down the Fourier transforms of the expansion of retarded potentials $A^{\mathrm{R}}_{\omega}(\boldsymbol{r})$ and $\varphi^{\mathrm{R}}_{\omega}(\boldsymbol{r})$ and the electromagnetic field strengths in monochromatic components.

Solution. It follows from the definition of the Fourier components (see formula (1.250)) that in the case of aperiodic particle motion that

$$A^{\mathrm{R}}_{\omega}(\boldsymbol{r}) = \int\limits_{-\infty}^{\infty} A^{\mathrm{R}}(\boldsymbol{r}, t)e^{i\omega t}\mathrm{d}t$$
$$= \frac{1}{c}\int \frac{\exp(i\omega|\boldsymbol{r}-\boldsymbol{r}'|/c)}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}V' \int\limits_{-\infty}^{\infty} \boldsymbol{j}(\boldsymbol{r}', \tau)e^{i\omega\tau}\mathrm{d}\tau \;, \tag{5.12'}$$

where $\tau = t - |\boldsymbol{r}-\boldsymbol{r}'|/c$, or in the final form

$$\boldsymbol{A}^{\mathrm{R}}_{\omega}(\boldsymbol{r}) = \frac{1}{c}\int \frac{\exp(i\omega|\boldsymbol{r}-\boldsymbol{r}'|/c)}{|\boldsymbol{r}-\boldsymbol{r}'|}\boldsymbol{j}_{\omega}(\boldsymbol{r}')\mathrm{d}V' \;, \tag{5.12}$$

and, similarly,

$$\varphi^{\mathrm{R}}_{\omega}(\boldsymbol{r}) = \int \frac{\exp(i\omega|\boldsymbol{r}-\boldsymbol{r}'|/c)}{|\boldsymbol{r}-\boldsymbol{r}'|}\rho_{\omega}(\boldsymbol{r}')\mathrm{d}V' \;. \tag{5.13}$$

Factor

$$G^{\mathrm{R}}_{\omega}(R) = \frac{1}{R}\exp\left(\frac{i\omega R}{c}\right) \tag{5.14}$$

under the integrals in (5.12) and (5.13) is a Fourier harmonic of the retarded Green's function. Field strengths are expressed as

$$\boldsymbol{H}_{\omega}(\boldsymbol{r}) = \nabla \times \boldsymbol{A}^{\mathrm{R}}_{\omega}(\boldsymbol{r}) \;, \quad \boldsymbol{E}_{\omega} = -\frac{i\omega}{c}\boldsymbol{A}^{\mathrm{R}}_{\omega}(\boldsymbol{r}) - \nabla\varphi^{\mathrm{R}}_{\omega}(\boldsymbol{r}) \;. \tag{5.15}$$

In the case of periodic particle motion with period $T = 2\pi/\omega$, both the field potentials and the field intensities can be expanded in a Fourier series. It is convenient to perform the expansion in exponents with imaginary indices like (1.145). Such an expansion contains frequencies $\omega_m = m\omega$, $m = 0, \pm 1, \pm 2 \ldots$:

$$\boldsymbol{A}^{\mathrm{R}}_{m}(\boldsymbol{r}) = \frac{1}{c}\int \frac{\exp(i\omega_m|\boldsymbol{r}-\boldsymbol{r}'|/c)}{|\boldsymbol{r}-\boldsymbol{r}'|}\boldsymbol{j}_{m}(\boldsymbol{r}')\mathrm{d}V' \;,$$
$$\varphi^{\mathrm{R}}_{m}(\boldsymbol{r}) = \int \frac{\exp(i\omega_m|\boldsymbol{r}-\boldsymbol{r}'|/c)}{|\boldsymbol{r}-\boldsymbol{r}'|}\rho_{m}(\boldsymbol{r}')\mathrm{d}V' \;, \tag{5.16}$$

where $\boldsymbol{j}_m = (1/T)\int_0^T \boldsymbol{j}(\boldsymbol{r}', t)e^{i\omega_m t}\mathrm{d}t$ and ρ_m is expressed analogously. The field strengths take the form

$$\boldsymbol{H}_m(\boldsymbol{r}) = \nabla \times \boldsymbol{A}_m^{\mathrm{R}}(\boldsymbol{r}) \,, \quad \boldsymbol{E}_m = -\frac{i\omega_m}{c}\boldsymbol{A}_m^{\mathrm{R}}(\boldsymbol{r}) - \nabla\varphi_m^{\mathrm{R}}(\boldsymbol{r}) \,. \tag{5.17}$$

□

Emission of electromagnetic waves by a system of charged particles in a free space is a process in which the electromagnetic field breaks away from the source and propagates in the form of waves over arbitrary distances. It necessitates studying the field at a distance greater from the source than both the size of the source ($r \gg l$) and the wavelength of the wave emitted ($r \gg \lambda$, *wave zone*) in order to calculate the process of emission. The field structure in the wave zone becomes simplified and resembles a flat wave field with the characteristic relation (2.122) between vectors $\boldsymbol{E}$ and $\boldsymbol{H}$:

$$\boldsymbol{H} = \boldsymbol{n} \times \boldsymbol{E} \,, \quad \boldsymbol{E} = \boldsymbol{H} \times \boldsymbol{n} \,, \quad E = H \,, \tag{5.18}$$

with $\boldsymbol{n} = \boldsymbol{r}/r$ if the source of emitted waves is close to the origin of the coordinates. The validity of relations (5.18) will be confirmed below by direct calculation.

The energy $\mathrm{d}I/\mathrm{d}\Omega$ emitted in the direction of n per unit solid angle (differential intensity of the radiation) is expressed through the Pointing vector $\boldsymbol{\gamma}$ in the form

$$\frac{\mathrm{d}I}{\mathrm{d}\Omega} = \boldsymbol{\gamma}\cdot\boldsymbol{n}r^2 = \frac{cr^2}{4\pi}\boldsymbol{H}^2(\boldsymbol{r}, t) \,. \tag{5.19}$$

The summed (total) intensity of emission in all directions is calculated by integration of (5.19) over the solid angle:

$$I(r, t) = \frac{cr^2}{4\pi}\int \boldsymbol{H}^2(\boldsymbol{r}, t)\mathrm{d}\Omega = \frac{cr^2}{4\pi}\int \boldsymbol{E}^2(\boldsymbol{r}, t)\mathrm{d}\Omega \,. \tag{5.20}$$

The dependence of I on r is related to the retardation effect alone; the entire energy emitted eventually passes through any sphere with center in the source of the emission.

5.1.3
The Spectral Composition of Emission

Given the periodic motion of particles, the period-averaged intensity of emission in a given direction $\overline{\mathrm{d}I}/\mathrm{d}\Omega = (1/T)\int_0^T(\mathrm{d}I/\mathrm{d}\Omega)\mathrm{d}t$ takes the following form in the case when the Fourier expansion of the magnetic field is substituted into (5.19):

$$\begin{aligned}\frac{\overline{\mathrm{d}I}}{\mathrm{d}\Omega} &= \frac{cr^2}{4\pi T}\int_0^T \boldsymbol{H}(\boldsymbol{r}, t)\cdot\boldsymbol{H}(\boldsymbol{r}, t)\mathrm{d}t = \frac{cr^2}{4\pi}\sum_{m=-\infty}^{\infty}|\boldsymbol{H}_m(\boldsymbol{r})|^2 \\ &= \frac{cr^2}{2\pi}\sum_{m=1}^{\infty}|\boldsymbol{H}_m(\boldsymbol{r})|^2\end{aligned}$$

(the zeroth harmonic of the emission field is absent, $\boldsymbol{H}_{-m} = \boldsymbol{H}_m^*$). The structure of the expression thus obtained suggests that certain terms can be interpreted as differential intensities of emission at the respective frequencies $\omega_m = m\omega$, $m = 1, 2, \ldots$:

$$\frac{\mathrm{d}I_m}{\mathrm{d}\Omega} = \frac{cr^2}{2\pi}|\boldsymbol{H}_m(\boldsymbol{r})|^2 \,. \tag{5.21}$$

The emission spectrum is continuous when particles perform aperiodic motion and the emission lasting for a finite time ceases at $t \to \pm\infty$. The total energy $\mathrm{d}\mathcal{E}^{\mathrm{rad}}/\mathrm{d}\Omega$ emitted in the specified direction $\boldsymbol{n}$ is expressed as the integral

$$\frac{\mathrm{d}\mathcal{E}^{\mathrm{rad}}}{\mathrm{d}\Omega} = \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}I(t)}{\mathrm{d}\Omega}\mathrm{d}t \,.$$

Let us expand the emission field into a Fourier integral and use formula (5.19) again. This gives

$$\begin{aligned}\frac{\mathrm{d}\mathcal{E}^{\mathrm{rad}}}{\mathrm{d}\Omega} &= \frac{cr^2}{4\pi}\int\limits_{-\infty}^{\infty} \boldsymbol{H}(\boldsymbol{r},t)\cdot\boldsymbol{H}(\boldsymbol{r},t)\mathrm{d}t \\ &= \frac{cr^2}{8\pi^2}\int\limits_{-\infty}^{\infty} |\boldsymbol{H}_\omega(\boldsymbol{r})|^2\mathrm{d}\omega = \frac{cr^2}{4\pi^2}\int\limits_{0}^{\infty} |\boldsymbol{H}_\omega(\boldsymbol{r})|^2\mathrm{d}\omega \,.\end{aligned}$$

The quantity

$$\frac{\mathrm{d}^2 I_\omega}{\mathrm{d}\omega\mathrm{d}\Omega} = \frac{cr^2}{4\pi^2}|\boldsymbol{H}_\omega(\boldsymbol{r})|^2 \,, \tag{5.22}$$

independent of r in the wave zone, can be interpreted as the energy emitted during the entire time of the process in a given direction at frequency ω (per unit solid angle and unit frequency range). Formulas (5.21) and (5.22) are alike, but their numerical coefficients and dimensions differ.

Recommended literature:
Landau and Lifshitz (1975); Gal'tsov *et al.* (1991); Bredov *et al.* (2003); Jackson (1999); Kolokolov *et al.* (2000)

Problems

5.1. Show that the retarded potentials (5.10) and (5.11) satisfy the Lorentz condition (3.5).

5.2•. The advanced Green's function $G^{\mathrm{A}}(R,\tau)$ is the solution of equation (5) satisfying the condition $G^{\mathrm{A}}(R,\tau) = 0$ at $\tau > 0$. Find the shape of the integration contour in the plane of the complex frequency ω leading to this condition and construct the advanced Green's function $G^{\mathrm{A}}(R,\tau)$ in the explicit form.

5.3. Show that both the retarded (5.3) and the advanced (see the preceding problem) Green's functions can be written in the Lorentz-invariant form $G^{\mathrm{R,A}}(\boldsymbol{R}, \tau) = 2c\Theta(\pm\tau)\delta(X_i X^i)$, where $X^i = (c\tau, \boldsymbol{R})$, Θ is the step function (1.212).

Hint: Use the property of the delta function (1.209).

5.4. Show that the Fourier transforms of the retarded and the advanced Green's functions can be written in the form

$$\tilde{G}^{\mathrm{R,A}}(\boldsymbol{k}, \omega) = -4\pi\frac{\mathcal{P}}{k_i k^i} \pm i4\pi^2\epsilon(k^0)\delta(k_i k^i)\ ,$$

where $\epsilon(x) = \Theta(x) - \Theta(-x)$ is the sign function, $k^i = (\omega/c, \boldsymbol{k})$.

Hint: Use integration rules (1.223) for singular expressions.

5.5. Write down the Fourier components $\boldsymbol{A}^{\mathrm{A}}_{\omega}(\boldsymbol{r})$ and $\varphi^{\mathrm{A}}_{\omega}(\boldsymbol{r})$ of the advanced potentials derived from formula (5.4) after the advanced Green's function G^{A} has been substituted into it (see Problem 5.2•). Compare them with (5.12) and (5.13). What physical sense can be ascribed to either of them?

5.6. Show that the Fourier harmonics $G^{\mathrm{R,A}}_{\omega}(R)$ of the retarded and the advanced Green's functions satisfy the equation

$$(\widehat{H}_{\omega} \pm i\omega 0)G^{\mathrm{R,A}}_{\omega}(R) = -4\pi\delta(\boldsymbol{R})\ , \qquad (1)$$

where operator

$$\widehat{H}_{\omega} = \Delta + \left(\frac{\omega}{c}\right)^2$$

acts on the $\boldsymbol{R}$ coordinates. The infinitesimally small addition $\pm i\omega 0$ lays down the rules for a detour around the poles in the plane of complex ω.

Hint: Pass to the Fourier expansion in coordinates in (5.1) and compare the result with the Fourier transform obtained before (see formula (5.6) and Problem 5.2•). Take account of the rules for a detour around the poles depicted in Figures 5.1 and 5.3.

5.7*. Show that the solution of (5.1) in the preceding problem can be written in the form

$$G^{\mathrm{R,A}}_{\omega}(\boldsymbol{R}) = \pm\frac{4\pi i}{\omega}\int_0^{\infty} \mathrm{d}\tau \exp\left[\pm\frac{i}{\omega}(\widehat{H}_{\omega} \pm i\omega 0)\tau\right]\delta(\boldsymbol{R})\ .$$

Show also that the integrand

$$\psi^{\mathrm{R,A}}(\boldsymbol{R}, \tau) = \exp\left[\pm\frac{i}{\omega}(\widehat{H}_{\omega} \pm i\omega 0)\tau\right]\delta(\boldsymbol{R})$$

satisfies the "Schrödinger equation"

$$\mp i\frac{\partial\psi^{\mathrm{R,A}}}{\partial t} = (\widehat{H}_{\omega} \pm i\omega 0)\psi^{\mathrm{R,A}}$$

and the initial condition $\psi^{\mathrm{R,A}}(\boldsymbol{R}, 0) = \delta(\boldsymbol{R})$, where $t = \tau/\omega$ plays the role of time.

5.8*. Calculate the function $\psi^{\mathrm{R,A}}(\boldsymbol{R}, \tau)$, defined in the preceding problem.

5.9. Define the function $G^{\pm}(\boldsymbol{R}, \tau)$ by the following equalities:

$$G^{+}(\boldsymbol{R}, \tau) = \frac{1}{2}(G^{\mathrm{R}}(\boldsymbol{R}, \tau) + G^{\mathrm{A}}(\boldsymbol{R}, \tau)) ,$$
$$G^{-}(\boldsymbol{R}, \tau) = G^{\mathrm{A}}(\boldsymbol{R}, \tau) - G^{\mathrm{R}}(\boldsymbol{R}, \tau) .$$

Find the Fourier transforms and specify the differential equations which these functions satisfy.

5.10. Write down the equations satisfied by electromagnetic potentials $\varphi(\boldsymbol{r}, t)$ and $\boldsymbol{A}(\boldsymbol{r}, t)$ if the condition div $\boldsymbol{A} = 0$ (the Coulomb gauge) is imposed on them instead of the Lorentz condition (5.8). Represent these equations in a form such that each of them contains one of the potentials and field sources.

5.11. Write the equations satisfied by the vortex-free (potential) and solenoidal parts of electromagnetic field vectors $\boldsymbol{E}(\boldsymbol{r}, t)$ and $\boldsymbol{H}(\boldsymbol{r}, t)$. Show that the potential part of the electric field describes the instantaneous (nondelayed) Coulomb field created by the charge distribution at the moment for which $\boldsymbol{E}^{\|}(\boldsymbol{r}, t)$ is determined.

5.12*. Expand the retarded Green's function (5.14) in a spherical harmonic $Y_{lm}(\vartheta, \alpha)$ series where angles ϑ and α set the direction of vector $\boldsymbol{r}$.

5.13. Represent the Fourier transforms of retarded potentials $\boldsymbol{A}_{\omega}(\boldsymbol{r})$ and $\varphi_{\omega}(\boldsymbol{r})$ in the wave zone ($kr \gg 1$) in the form of an expansion in Legendre spherical harmonics $Y_{lm}(\vartheta, \alpha)$. How are potential expansion coefficientsrelated if the potentials themselves satisfy the Lorentz condition (5.3)?

Hint: Use the expansion of the Green's function $G^{R}_{\omega}(\boldsymbol{r} - \boldsymbol{r}')$ from the previous problem.

5.14*. Build up three sequences of vector spherical functions *(spherical vectors)* $\boldsymbol{\mathcal{Y}}^{(k)}_{lm}(\vartheta, \alpha)$, defining them through the Legendre spherical functions by the relations

$$\boldsymbol{\mathcal{Y}}^{(1)}_{lm}(\vartheta, \alpha) = [l(l+1)]^{-1/2} \nabla_{\vartheta\alpha} Y_{lm}(\vartheta, \alpha),$$
$$\boldsymbol{\mathcal{Y}}^{(2)}_{lm}(\vartheta, \alpha) = [l(l+1)]^{-1/2} \hat{\boldsymbol{l}} Y_{lm}(\vartheta, \alpha) , \quad l = 1, 2, \ldots , \quad \boldsymbol{\mathcal{Y}}^{(1)}_{00} = \boldsymbol{\mathcal{Y}}^{(2)}_{00} = 0 ,$$
$$\boldsymbol{\mathcal{Y}}^{(3)}_{lm}(\vartheta, \alpha) = \boldsymbol{n} Y_{lm}(\vartheta, \alpha) ,$$

where $\boldsymbol{n} = \boldsymbol{r}/r$, $\hat{\boldsymbol{l}} = -i\boldsymbol{r} \times \nabla = -i\boldsymbol{n} \times \nabla_{\vartheta\alpha}$, and $\nabla_{\vartheta\alpha} = \boldsymbol{e}_{\vartheta}\,\partial/\partial\vartheta + (\boldsymbol{e}_{\alpha}/\sin\vartheta)(\partial/\partial\alpha)$. $\hat{\boldsymbol{l}}$ is the Hermitian operator of angular momentum in quantum mechanics. Prove that spherical vectors with different k are mutually perpendicular and form a complete orthonormal system of functions on a sphere:

$$\int \boldsymbol{\mathcal{Y}}^{(k')*}_{l'm'}(\vartheta, \alpha) \cdot \boldsymbol{\mathcal{Y}}^{(k)}_{lm}(\vartheta, \alpha) \mathrm{d}\Omega = \delta_{ll'} \delta_{mm'} \delta_{kk'} .$$

Also, show that inversion of the coordinates (substitution $\boldsymbol{n} \to -\boldsymbol{n}$) results in the acquisition of the factor $(-1)^{l+1}$ by $\boldsymbol{\mathcal{Y}}^{(1)}_{lm}$ and $\boldsymbol{\mathcal{Y}}^{(3)}_{lm}$ and the acquisition of the factor $(-1)^{l}$ by $\boldsymbol{\mathcal{Y}}^{(2)}_{lm}$.

5.15. Show that in the wave zone ($r \gg \lambda$) with the Lorentz condition (5.3) being fulfilled, the scalar potential of a restricted emitting system is related to its vector potential by the expression

$$\varphi(\boldsymbol{r}, t) = \frac{\boldsymbol{r} \cdot \boldsymbol{A}(\boldsymbol{r}, t)}{r} + \frac{q}{r} ,$$

where $q = \text{const}$ is the total electric charge of the system.

5.2
Emission in Nonrelativistic Systems of Charges and Currents

Investigation of emission is simplified if the propagation time of electromagnetic perturbations within an emitting system is shorter than the characteristic time T of the motion of charged particles in this system:

$$l/c \ll T . \tag{5.23}$$

In the case of periodic motion, T is the period which allows inequality (5.23) to be represented in the form

$$l \ll \lambda , \tag{5.24}$$

where λ is the radiation wavelength. Finally, $l/T \approx v$ is the characteristic velocity of the particles and condition (5.23) reduces to the requirement that the particle's speed must be nonrelativistic:

$$v \ll c . \tag{5.25}$$

Let us consider the emission field of a system of charges satisfying conditions (5.23)–(5.25). To this effect, it is enough to calculate the vector potential in the wave zone ($r \gg \lambda$) taking into account only the terms inversely proportional to the distance r from the system because only these terms make a contribution to the energy emitted by the system.

We proceed on the assumption of the retarded potential (5.10) and expand $|\boldsymbol{r} - \boldsymbol{r}'|$ in a series in terms of the relationship between the system's size and distance r: $|\boldsymbol{r} - \boldsymbol{r}'| \approx r - \boldsymbol{n} \cdot \boldsymbol{r}'$, where $\boldsymbol{n} = \boldsymbol{r}/r$. Let us represent the current density under the sign of integral in the form of the expansion

$$\boldsymbol{j}\left(\boldsymbol{r}', t - \frac{r}{c} + \frac{\boldsymbol{n} \cdot \boldsymbol{r}'}{c}\right) = \boldsymbol{j}\left(\boldsymbol{r}', t - \frac{r}{c}\right) + \dot{\boldsymbol{j}}\left(\boldsymbol{r}', t - \frac{r}{c}\right) \frac{\boldsymbol{n} \cdot \boldsymbol{r}'}{c} + \dots , \tag{5.26}$$

where the dot denotes the time derivative. The denominator of the integrand continues to contain the zero approximation term when $|\boldsymbol{r} - \boldsymbol{r}'|$ is replaced by r.

5.2.1 Electric Dipole Emission

Taking the first nonvanishing term from expansion (5.26) leads to

$$\boldsymbol{A}(\boldsymbol{r}, t) = \frac{1}{cr} \int \boldsymbol{j}\left(\boldsymbol{r}', t - \frac{r}{c}\right) \mathrm{d}V' .$$

The use of the identity

$$\boldsymbol{a}\cdot\boldsymbol{j} = \boldsymbol{j}\cdot\nabla'(\boldsymbol{a}\cdot\boldsymbol{r}') = \nabla'\cdot[\boldsymbol{j}(\boldsymbol{a}\cdot\boldsymbol{r}')] - \boldsymbol{a}\cdot\boldsymbol{r}'(\nabla'\cdot\boldsymbol{j}) ,$$

where $\boldsymbol{a}$ is a constant vector, yields

$$\boldsymbol{a}\cdot \int \boldsymbol{j}\left(\boldsymbol{r}', t - \frac{r}{c}\right) \mathrm{d}V' = \boldsymbol{a}\cdot\frac{\partial}{\partial t} \int \boldsymbol{r}'\rho\left(\boldsymbol{r}', t - \frac{r}{c}\right) \mathrm{d}V' .$$

Here, the continuity equation (2.47) is used. The definition of the electric dipole moment (2.23) permits us to write $\int \boldsymbol{r}'\dot{\rho}(\boldsymbol{r}', t - r/c)\mathrm{d}V' = \dot{\boldsymbol{p}}(t - r/c)$ and

$$\boldsymbol{A}(\boldsymbol{r}, t) = \frac{\dot{\boldsymbol{p}}(t - r/c)}{cr} . \tag{5.27}$$

Example 5.5

Calculate the fields $\boldsymbol{E}(\boldsymbol{r}, t)$ and $\boldsymbol{H}(\boldsymbol{r}, t)$ in the wave zone of an electric dipole radiator. Calculate in addition the energy emitted per unit time in the $\boldsymbol{n}$ direction (recalculated per unit solid angle) and the total energy emitted in all directions.

Solution. Retain only the terms of order r^{-1} arising from the differentiation with respect to argument $\dot{\boldsymbol{p}}$ but not the denominator:

$$\boldsymbol{H} = \operatorname{curl} \boldsymbol{A} = \frac{\ddot{\boldsymbol{p}} \times \boldsymbol{n}}{c^2 r} , \quad \boldsymbol{n} = \frac{\boldsymbol{r}}{r} . \tag{5.28}$$

The electric vector $\boldsymbol{E}$ can be expressed through the magnetic vector $\boldsymbol{H}$ using formulas (5.19) for a flat wave:

$$\boldsymbol{E} = \boldsymbol{H} \times \boldsymbol{n} = \frac{(\ddot{\boldsymbol{p}} \times \boldsymbol{n}) \times \boldsymbol{n}}{c^2 r} . \tag{5.29}$$

Characteristically, the Poynting vector in the wave zone is directed along the radius from the source and is in inverse ratio to the square of the distance:

$$\boldsymbol{\gamma} = \frac{c}{4\pi} \boldsymbol{E} \times \boldsymbol{H} = \frac{c}{4\pi} H^2 \boldsymbol{n} .$$

This means that one and the same energy flows through an element of the spherical surface $r^2\mathrm{d}\Omega$ within the solid angle $\mathrm{d}\Omega$ (with regard for delay) irrespective of the distance r. The differential intensity of emission takes the form

$$\frac{\mathrm{d}I}{\mathrm{d}\Omega} = \frac{c}{4\pi} H^2 r^2 = \frac{[\ddot{\boldsymbol{p}} \times \boldsymbol{n}]^2}{4\pi c^3} = \frac{\ddot{\boldsymbol{p}}^2 \sin^2\theta}{4\pi c^3} , \tag{5.30}$$

where θ is the angle between directions $\ddot{\boldsymbol{p}}$ and $\boldsymbol{n}$ at moment $t - r/c$. Integration over the entire solid angle yields the total intensity of emission

$$I = \frac{2\ddot{\boldsymbol{p}}^2}{3c^3} . \tag{5.31}$$

For an individual charged particle, $\boldsymbol{p} = e\boldsymbol{r}$, and (5.31) gives the *Larmor formula*

$$I = \frac{2e^2\dot{\boldsymbol{v}}^2}{3c^3} . \tag{5.32}$$

This formula implies that only a rapidly moving particle with $\dot{\boldsymbol{v}} \neq 0$ can emit. □

The emission considered above is called electric dipole emission because all quantities are expressed through the derivatives of the electric dipole moment. The intensities of emission of individual Fourier harmonics can be calculated using formulas (5.21) and (2.28). They have the form

$$I_m = \frac{4\omega_m^4}{3c^3}|\boldsymbol{p}_m|^2 . \tag{5.33}$$

If a dipole oscillates with sole frequency ω, that is, $\boldsymbol{p}(t) = \boldsymbol{p}_0 \cos \omega t = (\boldsymbol{p}_0/2)(e^{-i\omega t} + e^{i\omega t})$, then $\boldsymbol{p}_1 = \boldsymbol{p}_0/2$ and the intensity of emission averaged over the oscillation period is $\bar{I} = I_1 = \omega^4 \boldsymbol{p}_0^2/3c^3$.

5.2.2
Quadrupole and Magnetic Dipole Emission

Given a constant or zero electric dipole moment, the electric dipole emission is absent, and the following terms of expansion (5.26) need to be considered. Such a situation occurs, for example, in a system of charged particles with identical ratios $e_a/m_a = \eta$:

$$\boldsymbol{p} = \sum_a e_a \boldsymbol{r}_a = \eta \sum_a m_a \boldsymbol{r}_a = \eta \boldsymbol{R} \sum_a m_a .$$

Here $\boldsymbol{R}$ is the center-of-mass radius vector; in the absence of external forces $\dot{\boldsymbol{R}} = \boldsymbol{V} = \text{const}$, $\ddot{\boldsymbol{R}} = 0$, the dipole emission vanishes.

Example 5.6

Calculate the vector potential of a restricted system of charges in the absence of an electric dipole moment taking into account the terms of the next order of smallness of the parameter l/λ in (5.26). Express the result through magnetic dipole and electric quadrupole moments. Calculate the differential and total intensities of magnetic dipole and electric quadrupole emission.

Solution. Modify the second term on the right-hand side of (5.26) using the identity

$$(\boldsymbol{n}\cdot\boldsymbol{r}')\dot{\boldsymbol{j}} = \frac{1}{2}[(\boldsymbol{n}\cdot\boldsymbol{r}')\dot{\boldsymbol{j}} + (\boldsymbol{n}\cdot\dot{\boldsymbol{j}})\boldsymbol{r}'] + \frac{1}{2}(\boldsymbol{r}' \times \dot{\boldsymbol{j}}) \times \boldsymbol{n} \,.$$

Substituting this expression into (5.10) yields the vector potential in the form

$$\boldsymbol{A} = \frac{\dot{\boldsymbol{m}} \times \boldsymbol{n}}{cr} + \frac{1}{2c^2 r}\frac{\partial}{\partial t}\int [(\boldsymbol{n}\cdot\boldsymbol{r}')\boldsymbol{j} + (\boldsymbol{n}\cdot\boldsymbol{j})\boldsymbol{r}']\mathrm{d}V' \,. \tag{5.34}$$

Here $\dot{\boldsymbol{m}}$ is the time derivative of the system's magnetic moment (2.59). Transform the second integral in (5.34) using the identity $\int \boldsymbol{P}\mathrm{d}V = -\int \boldsymbol{r}\mathrm{div}\,\boldsymbol{P}\mathrm{d}V$ and the continuity equation in which $\boldsymbol{P}$ is the arbitrary nonzero vector in the finite region and perform integration over the entire space. As a result,

$$\boldsymbol{A} = \frac{\dot{\boldsymbol{m}} \times \boldsymbol{n}}{cr} + \frac{1}{2c^2 r}\frac{\partial^2}{\partial t^2}\int \boldsymbol{r}'(\boldsymbol{n}\cdot\boldsymbol{r}')\rho\mathrm{d}V' \,. \tag{5.35}$$

The integral in the last equality can be written through the electric quadrupole moment (2.24). The introduction of tensor notation leads to

$$\int x_\alpha(\boldsymbol{n}\cdot\boldsymbol{r}')\rho\mathrm{d}V' = \frac{1}{3}Q_{\alpha\beta}n_\beta + \frac{1}{3}n_\alpha\int \rho r'^2\mathrm{d}V' \,.$$

This eventually brings us to the vector potential in the wave zone:

$$\boldsymbol{A} = \frac{\dot{\boldsymbol{m}} \times \boldsymbol{n}}{cr} + \frac{\ddot{\boldsymbol{Q}}}{6c^2 r} + \frac{\boldsymbol{n}}{6c^2 r}\int \ddot{\rho}\left(\boldsymbol{r}', t - \frac{r}{c}\right) r'^2\mathrm{d}V' \,, \quad \text{where}$$
$$Q_\alpha = Q_{\alpha\beta}n_\beta \,. \tag{5.36}$$

In the calculation of field strengths, the last term in (5.36) makes no contribution, and the emission field is expressed only through the magnetic dipole and electric quadrupole moments:

$$\boldsymbol{H} = \frac{1}{c^2 r}\left\{(\ddot{\boldsymbol{m}} \times \boldsymbol{n}) \times \boldsymbol{n} + \frac{1}{6c}\dddot{\boldsymbol{Q}} \times \boldsymbol{n}\right\} \,,$$
$$\boldsymbol{E} = \frac{1}{c^2 r}\left\{\boldsymbol{n} \times \ddot{\boldsymbol{m}} + \frac{1}{6c}(\dddot{\boldsymbol{Q}} \times \boldsymbol{n}) \times \boldsymbol{n}\right\} \,. \tag{5.37}$$

The magnetic dipole field arises from the electric dipole field by the substitutions $\ddot{\boldsymbol{p}} \to \ddot{\boldsymbol{m}}$, $\boldsymbol{E} \to \boldsymbol{H}$, and $\boldsymbol{H} \to -\boldsymbol{E}$. During the harmonic motion of charges $\ddot{\boldsymbol{p}} = -\omega^2\boldsymbol{p}$, $\ddot{\boldsymbol{m}} = -\omega^2\boldsymbol{m}$, and $\dddot{\boldsymbol{Q}} = i\omega^3\boldsymbol{Q}$; because of this, the intensities of electric and magnetic dipole emission are proportional to ω^4, whereas the intensity of electric quadrupole emission is proportional to ω^6.

The field strength being known, we can easily calculate the differential intensity of quadrupole and magnetic dipole emission in a given direction, $\boldsymbol{n}$:

$$\frac{\mathrm{d}I}{\mathrm{d}\Omega} = \frac{1}{4\pi c^3}(\ddot{\boldsymbol{m}} \times \boldsymbol{n})^2 + \frac{1}{144\pi c^5}(\dddot{\boldsymbol{Q}} \times \boldsymbol{n})^2 - \frac{1}{12\pi c^4}\ddot{\boldsymbol{m}}\cdot(\dddot{\boldsymbol{Q}} \times \boldsymbol{n}) \,. \tag{5.38}$$

When integrating (5.38) over vector directions $\boldsymbol{n}$, we must write all vector products in tensor notation and use the formulas for averaging unit vector components obtained in the solution of Problem 1.33. The intensity of emission summed over all directions has the form

$$I = \frac{2}{3c^3}\ddot{\boldsymbol{m}}^2 + \frac{1}{180c^5}\dddot{Q}^2_{\alpha\beta} \,. \tag{5.39}$$

Unlike the intensity of electric dipole emission (5.31), the terms in (5.39) contain a small multiplier $(l/\lambda)^2$. For this reason, they play an important role only in the absence of electric dipole emission. □

5.2.3
The Hertz Vector and Antenna Radiation

Given the radiator is a macroscopic body of size $l \geq \lambda$, the expansion of the emission field in multipoles is inapplicable; the exact expressions (5.10) and (5.11) for retarded potentials should be used. The number of unknown functions can be reduced by introducing one vector function $\boldsymbol{Z}(\boldsymbol{r}, t)$ *(the Hertz vector or polarization potential)* through which electromagnetic potentials are expressed instead of potentials $\boldsymbol{A}(\boldsymbol{r}, t)$ and $\varphi(\boldsymbol{r}, t)$ coupled by the Lorentz condition (5.2). There are two types of Hertz vector, one electrical $\boldsymbol{Z}^{(e)}$, the other magnetic $\boldsymbol{Z}^{(m)}$. The former is realized when the source consists of dipole radiators distributed with volumetric density $\boldsymbol{P}$ referred to as the *electric polarization vector*. Electromagnetic potentials are related to the Hertz electric vector by the expressions

$$\varphi = -\operatorname{div} \boldsymbol{Z}^{(e)} \,, \quad \boldsymbol{A} = \frac{1}{c}\frac{\partial \boldsymbol{Z}^{(e)}}{\partial t} \,. \tag{5.40}$$

The charge and current distributions in an electroneutral system are expressed through the electric polarization vector in the formulas

$$\rho = -\operatorname{div} \boldsymbol{P} \,, \quad \boldsymbol{j} = \frac{\partial \boldsymbol{P}}{\partial t} \,. \tag{5.41}$$

In this case, the continuity equation is fulfilled and the total charge of a restricted system must be equal to zero:

$$q = \int\limits_{V\to\infty} \rho \mathrm{d}V = -\oint\limits_{S\to\infty} \boldsymbol{P}\cdot \mathrm{d}\boldsymbol{S} = 0 \,.$$

The total electric dipole moment is expressed in the form of the integral of vector $\boldsymbol{P}$ over the system volume.

For electromagnetic potentials to satisfy (5.1) and relations (5.40), the Hertz vector should obey the inhomogeneous d'Alembert equation:

$$\Delta \boldsymbol{Z}^{(e)} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{Z}^{(e)}}{\partial t^2} = -4\pi \boldsymbol{P} \,. \tag{5.42}$$

Alternate application of the operators $-\text{div}$ and $\partial/c\partial t$ to both sides of (5.42) leads to the equations for potentials (5.1). The solution of (5.42) can be written by analogy with retarded potentials:

$$Z^{(e)}(\boldsymbol{r}, t) = \int \frac{\boldsymbol{P}(\boldsymbol{r}', t - |\boldsymbol{r} - \boldsymbol{r}'|/c)}{|\boldsymbol{r} - \boldsymbol{r}'|} \mathrm{d}V' . \tag{5.43}$$

The Hertz magnetic vector should be introduced when the source contains magnetic dipole radiators with density $\boldsymbol{M}(\boldsymbol{r}, t)$ called the *magnetic polarization vector*. In this case, expression (5.41) is replaced by

$$\rho = 0 , \quad \boldsymbol{j} = c\,\text{curl}\boldsymbol{M} , \tag{5.44}$$

the integral of $\boldsymbol{M}$ over the system volume gives the total magnetic dipole moment, and the Hertz magnetic vector is determined from the inhomogeneous wave equation

$$\Delta \boldsymbol{Z}^{(m)} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{Z}^{(m)}}{\partial t^2} = -4\pi \boldsymbol{M} . \tag{5.45}$$

Electromagnetic potentials are expressed through the formulas

$$\varphi = 0 , \quad \boldsymbol{A} = \text{curl}\, \boldsymbol{Z}^{(m)} . \tag{5.46}$$

Example 5.7

The Principle of Reversibility: Two independent sources of emission are characterized by the distribution of currents $\boldsymbol{j}_1$ and $\boldsymbol{j}_2$ and create two monochromatic fields of equal frequency ω. Show that the field currents and strengths are related by the expression

$$\int \boldsymbol{j}_1 \cdot \boldsymbol{E}_2 \mathrm{d}V = \int \boldsymbol{j}_2 \cdot \boldsymbol{E}_1 \mathrm{d}V , \tag{5.47}$$

where integration is performed over the entire space.

Solution. By virtue of the superposition principle, the field of each source satisfies its own system of Maxwell equations:

$$\text{curl}\, \boldsymbol{H}_1 = \frac{4\pi}{c}\boldsymbol{j}_1 - \frac{i\omega}{c}\boldsymbol{E}_1 , \quad \text{curl}\, \boldsymbol{H}_2 = \frac{4\pi}{c}\boldsymbol{j}_2 - \frac{i\omega}{c}\boldsymbol{E}_2 ,$$

$$\text{curl}\, \boldsymbol{E}_1 = \frac{i\omega}{c}\boldsymbol{H}_1, \qquad \text{curl}\, \boldsymbol{E}_2 = \frac{i\omega}{c}\boldsymbol{H}_2 .$$

Simple transformations such as those in the solution of Example 2.15 and the use of the Gauss–Ostrogradskii theorem yield

$$\int \boldsymbol{j}_1 \cdot \boldsymbol{E}_2 \mathrm{d}V = \int \boldsymbol{j}_2 \cdot \boldsymbol{E}_1 \mathrm{d}V + \frac{c}{4\pi} \oint_{S\to\infty} (\boldsymbol{H}_1 \times \boldsymbol{E}_2 - \boldsymbol{H}_2 \times \boldsymbol{E}_1) \cdot \mathrm{d}\boldsymbol{S} .$$

The expression under the sign of the integral vanishes at an infinitely distant surface on the strength of relations (5.18) because vectors $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ from the sources positioned at a finite distance from each other coincide. This leads to (5.47). □

It is easy to show by simple calculations that relation (5.47) takes the following form for two small electric dipoles:

$$\boldsymbol{p}_1 \cdot \boldsymbol{E}_2 = \boldsymbol{p}_2 \cdot \boldsymbol{E}_1 \ . \tag{5.48}$$

If dipole radiators are quasi-linear conductors of length shorter than the radiation wavelength, the substitution $\boldsymbol{j}\,\mathrm{d}V \rightarrow J\,\mathrm{d}\boldsymbol{l}$ yields from (5.47)

$$J_1 U_2(1) = J_2 U_1(2) \ , \tag{5.49}$$

where $U_i(k) = \int \boldsymbol{E}_k \cdot \mathrm{d}\boldsymbol{l}_i$ is the potential differences created by i sources at the ends of conductors k, and J_i are the current strengths in the radiators. Equality (5.49) does not depend on the dipole operating regime, either active (transmitter) or passive (receiver). Only current strengths change, whereas angle patterns in both regimes remain unaltered.

This property persists in arbitrary antenna systems. The angle patterns of any antenna working in the reception and transmission modes coincide.

Recommended literature:
Landau and Lifshitz (1975); Jackson (1999); Bredov *et al.* (2003); Stratton (1948); Medvedev (1977); Panofsky and Phillips (1963)

Problems

5.16•. Calculate the electromagnetic potentials and field intensities at distances satisfying the condition $l \ll r \ll \lambda$. Take account of electric dipole, quadrupole, and magnetic dipole terms.

5.17•. Calculate the electric field for an electric dipole radiator in the wave zone using the formula $\boldsymbol{E} = -(1/c)\partial \boldsymbol{A}/\partial t$ and show that it can be written in the form $\boldsymbol{E} = \boldsymbol{H} \times \boldsymbol{n}$, where $\boldsymbol{H}$ is given by formula (5.28).

5.18. Calculate the field strength of a point-like electric dipole oscillator with moment $\boldsymbol{p} = \boldsymbol{p}_0 \cos \omega t$ at distances satisfying the condition $l \ll r$ regardless of the relationship between r and λ.

5.19. Find the equation for the force lines of the electric and magnetic fields of an electric oscillator from the preceding problem. Consider qualitative variations of the field pattern in the zone adjacent to the oscillator and in the wave zone.

5.20. Find the expression for the loss of angular momentum per unit time $-\mathrm{d}\boldsymbol{L}/\mathrm{d}t$ by a system emitting like an electric dipole.

Hint: Use the results obtained in Problem (4.142) (formulas (6) and (7)).

5.21*. Find the electric field $\boldsymbol{H}$, $\boldsymbol{E}$ of charge e moving uniformly around a circle of radius a. The motion is nonrelativistic with angular velocity ω. The distance to the observation point is $r \gg a$. Find the time-averaged angular distribution $\overline{\mathrm{d}I/\mathrm{d}\Omega}$ and the total radiation intensity $\bar{I}$; study its polarization.

5.22. Two identical metallic sheets of radius R serve as the plates of a flat capacitor with plate spacing $h \ll R$. The potential difference at the plates varies in accordance with the rule $U(t) = U_0 \cos \omega t$, with $R \gg 2\pi c/\omega$. Calculate the time-averaged intensity of radiation.

5.23. As an electron in the hydrogen atom passes from the $2p$ state to the $1s$ state, the effective density of the electron's charge changes by the law

$$\rho(r, \vartheta, \alpha, t) = \frac{\sqrt{2}e}{\sqrt{3}a_\mathrm{B}^4} r e^{-3r/2a_\mathrm{B}} Y_{00} Y_{10}(\vartheta, \alpha) e^{-i\omega_0 t} ,$$

where $a_\mathrm{B} = \hbar^2/m_\mathrm{e} e^2$ is the Bohr radius, $\omega_0 = 3e^2/8\hbar a_\mathrm{B}$ is the transition rate between electron states, and Y_{lm} is the Legendre spherical functions. Calculate the time-averaged radiation intensity in the electric dipole approximation. Compare the result with the result of quantum calculation (see Section 6.2, Problem 6.49).

5.24. Four point charges $\pm q$ located at the vertices of a square with side a make up a flat quadrupole that rotates with velocity ω about the axis passing through the center of the square perpendicular to its plane. Calculate the angular distribution of the time-averaged radiation intensity and the total intensity on the condition that $a \ll 2\pi c/\omega$.

5.25. An electric field $\boldsymbol{E} = \boldsymbol{E}_0 e^{-\alpha t} \cos \omega_0 t$ with constant α and amplitude $\boldsymbol{E}_0$ begins to act on a resting particle with charge e and mass m at moment $t = 0$. Find the spectral radiation density $\mathrm{d}I_\omega/\mathrm{d}\omega$.

5.26. The dipole moments of a certain restricted system of charges change with time as $\boldsymbol{p}(t) = \boldsymbol{p}_0 e^{-t^2/\tau^2}$, $\boldsymbol{m}(t) = \boldsymbol{m}_0 e^{-t^2/\tau^2}$, where $\boldsymbol{p}_0$, $\boldsymbol{m}_0$, and τ are constant quantities. Calculate the spectral density of radiation $\mathrm{d}I_\omega/\mathrm{d}\omega$ in the dipole approximation (specify the applicability criterion for the approximation).

5.27. Current $J(t) = J_0 \cos \omega t$ flows in a rectangular wire loop of size $a \times b$, with $a, b \ll 2\pi c/\omega$. Calculate the time-averaged intensity of radiation.

5.28. Alternating current $J(t) = J_0 \tau t/(\tau^2 + t^2)$ with constant τ and J_0 flows in a closed wire loop encompassing area S. Calculate the spectral radiation density in the dipole approximation. Specify the criterion for the applicability of this approximation.

5.29. A particle has internal magnetic $\boldsymbol{m}$ and mechanical $\boldsymbol{s}$ moments related by the expression $\boldsymbol{m} = \eta \boldsymbol{s}$. It moves in a uniform magnetic field $\boldsymbol{H}$, with the angle between $\boldsymbol{m}$ and $\boldsymbol{H}$ equaling β. Calculate the time-averaged intensity of radiation caused by precession of the magnetic moment.

5.30. A charge q and mass m are uniformly distributed over a sphere volume of radius R. The sphere rotates with constant angular velocity ω around its diameter, making angle β with the external uniform magnetic field $\boldsymbol{H}$. Calculate the coefficient of proportionality between the magnetic and mechanical moments of the sphere and the time-averaged intensity of radiation induced by precession of the magnetic moment.

5.31. Examine the influence of interference on the emission of electromagnetic waves by a system of charges described in the following example: two identical electric charges e propagate uniformly with a nonrelativistic velocity and frequency ω in a circular orbit of radius a, remaining at the same time at the opposite ends of the diameter. Determine the polarization, angular distribution $\overline{\mathrm{d}I/\mathrm{d}\Omega}$, and radiation intensity $\overline{I}$. How will the intensity of radiation change if one of the charges is removed (cf. the results obtained in Problem 5.6).

5.32. How should the position of charges in the preceding problem differ from the diametric one to ensure the equality of intensities of electrical dipole and quadrupole radiation?

5.33. Vibrations of two electric dipole oscillators occur with a similar frequency ω and a $\pi/2$ phase shift. The amplitudes of the dipole moments are equal in magnitude p_0 and directed at an angle φ to each other. The distance between the oscillators is small compared with the wavelength. Find the field $\boldsymbol{H}$ in the wave zone, the angular distribution $\overline{\mathrm{d}I/\mathrm{d}\Omega}$, and the total radiation intensity $\overline{I}$.

5.34. Consider polarization of the emission field for the system of oscillators presented in the previous problem using the method described in Section 2.3.

5.35. Find the time-averaged energy flux density $\overline{\gamma}$ at large distances from the charge considered in Problem 5.6 taking into account the terms of order r^{-3}. Find the rotary moment $\boldsymbol{N}$ applied to a fully absorbing large-radius spherical screen assuming that the charge moves near its center.

5.36. The electric and magnetic dipoles located in one place of space are mutually perpendicular and oscillate with frequency ω_0. Find the angular distribution $\overline{\mathrm{d}I/\mathrm{d}\Omega}$ and the total radiation intensity $\overline{I}$ averaged over time.

5.37*. Given the condition $l \ll \lambda$ is fulfilled, calculate the vector potential of an emitting system in the wave zone taking into account the terms of order $(l/\lambda)^2$. Also, calculate the radiation intensity on the assumption that $\ddot{\boldsymbol{p}} \neq 0$, $\ddot{\boldsymbol{m}} \neq 0$, and $\dddot{Q}_{\alpha\beta} \neq 0$.

5.38*. A particle with charge e oscillates about the Oz axis according to $z(t) = a \sin \omega t$. Calculate the radiation intensities $\mathrm{d}I_m/\mathrm{d}\Omega$ at multiple frequencies $\omega_m = m\omega$, $m = 1, 2, \ldots$ without assuming the a/λ ratio to be small.

5.39. Calculate the angular distribution $\mathrm{d}I/\mathrm{d}\Omega$ of the total radiation with all frequencies of the oscillator from the previous problem. Also, calculate the total radiation intensity I in all directions.

Hint: Use the relation from the theory of Bessel functions:

$$\sum_{m=1}^{\infty} m^2 J_m^2(mx) = \frac{x^2(4+x^2)}{16(1-x^2)^{7/2}} \, .$$

5.40*. The simplest model of radiation emission by neutron stars (pulsars) is an oblique rotator, that is, a sphere with a magnetic moment $\boldsymbol{m}$ rotating in a vacuum with angular velocity ω about the axis making an angle φ with the direction $\boldsymbol{m}$.

1. Calculate the angular distribution $\overline{\mathrm{d}I/\mathrm{d}\Omega}$ and the total intensity of radiation $\overline{I}$ averaged over time.
2. Estimate numerically the order of magnitude of the magnetic moment of the pulsar taking the characteristic value of the magnetic field at the surface of neutron stars as $H_0 \approx 2 \times 10^{12}$ Oe from observations and the radius of the star as $R \approx 10$ km from theory.
3. Estimate numerically the intensity of pulsar radiation $\overline{I}$ and compare it with solar luminosity $L_{\odot} \approx 4 \times 10^{33}$ erg/s taking the rotation period $T \approx 0.033$ s from the observations of the pulsar in the Crab Nebula.
4. Compare the pulsar radiation intensity thus obtained with the decreasing rate of the stellar rotational energy assessed from the observational data on the increasing rotation period of the pulsar in the Crab Nebula, $\dot{T}/T \approx 1.3 \times 10^{-11}\,\mathrm{s}^{-1}$.

5.41. A droplet that is uniformly charged over its volume undergoes pulsation with constant density. The droplet surface is described by the equation

$$R(\vartheta) = R_0[1 + a P_2(\cos\vartheta)\cos\omega t] \, ,$$

where $a \ll 1$. The droplet charge is q. Find the angular distribution $\overline{\mathrm{d}I/\mathrm{d}\Omega}$ and the total intensity of radiation $\overline{I}$ averaged over time.

5.42. An electric charge e is distributed in a spherically symmetric fashion within a restricted region and undergoes radial oscillations. Find the electromagnetic field $\boldsymbol{E}$, $\boldsymbol{H}$ outside the charge distribution.

5.43. Find the expressions in vector form for the strengths of the electromagnetic fields of the electric $\boldsymbol{p}$ and magnetic $\boldsymbol{m}$ dipole oscillators at distances greater than their size.

Hint: When differentiating with respect to $\boldsymbol{r}$, make sure that the dipole moments are taken at the retarded moment $t' = t - r/c$ and therefore depend on $\boldsymbol{r}$.

5.44*. The centers of two electric dipole oscillators having frequency ω and similar amplitudes $\boldsymbol{p}_0 \parallel Ox$ lie on the Oz axis at equal distances from the origin of the coordinates and at distance $\lambda/4$ from each other. Vibrations of the oscillators are phase shifted by $\pi/1$. Find the angular distribution of radiation $\overline{\mathrm{d}I/\mathrm{d}\Omega}$.

5.45. Show that the total electric dipole moment of a restricted system of charges and currents equals the integral of the electric polarization vector over the volume introduced through equalities (5.41).

5.46. Show that the total magnetic dipole moment of a restricted system of charges and currents equals the integral of the magnetic polarization vector over the volume introduced through equalities (5.44).

5.47. Show that field strengths are expressed through the Hertz vectors $\boldsymbol{Z}^{(e)}$ and $\boldsymbol{Z}^{(m)}$ in the following form:

$$\boldsymbol{E} = \operatorname{curl}\operatorname{curl}\boldsymbol{Z}^{(e)} - \operatorname{curl}\frac{1}{c}\frac{\partial \boldsymbol{Z}^{(m)}}{\partial t} - 4\pi\boldsymbol{P};$$

$$\boldsymbol{H} = \operatorname{curl}\frac{1}{c}\frac{\partial \boldsymbol{Z}^{(e)}}{\partial t} + \operatorname{curl}\operatorname{curl}\boldsymbol{Z}^{(m)} - 4\pi\boldsymbol{M}\ .$$

5.48. Let two independent restricted monochromatic sources having equal frequencies create the fields $\boldsymbol{E}_1, \boldsymbol{H}_1$ and $\boldsymbol{E}_2, \boldsymbol{H}_2$. Show that the following relation is fulfilled for any closed surface S inside which these sources are located:

$$\int_S [\boldsymbol{E}_1 \times \boldsymbol{H}_2]\cdot \mathrm{d}\boldsymbol{S} = \int_S [\boldsymbol{E}_2 \times \boldsymbol{H}_1]\cdot \mathrm{d}\boldsymbol{S}\ .$$

5.49. Show that the reciprocity relation (5.47) takes the form $\boldsymbol{p}_1\cdot\boldsymbol{E}_2 = \boldsymbol{p}_2\cdot\boldsymbol{E}_1$ for electric dipole monochromatic radiators.

5.50. Find the expression for the electric dipole $\boldsymbol{Z}_\mathrm{p}$ and quadrupole $\boldsymbol{Z}_\mathrm{Q}$ terms and for the magnetic dipole $\boldsymbol{Z}_\mathrm{m}$ term of the expansion of the Hertz electric vector holding for the arbitrary time dependence of currents and charges. These quantities must be applicable at distances $r \gg a$ and $\lambda \gg a$ (the condition $r \gg \lambda$ is not necessarily fulfilled). Here a is the size of the system and the superscript (e) on the Hertz vector is omitted.

5.51. The moments of two identical electric dipoles are aligned along the same straight line and oscillate in counter-phase with frequency ω (amplitude $\boldsymbol{p}_0$). The distance between the centers a, $\lambda \gg a$. Find the electromagnetic field at distances $r \gg a$. Find also the angular distribution of radiation $\overline{\mathrm{d}I/\mathrm{d}\Omega}$ and its total intensity $\overline{I}$.

5.52*. A standing wave of current $\mathcal{I}$ with amplitude $\mathcal{I}_0$, frequency ω, and nodes at the ends is excited in a linear antenna of length l. The number of current half-waves falling within the antenna length is equal to m. Find the angular distribution of radiation $\overline{\mathrm{d}I/\mathrm{d}\Omega}$.

5.53. Find the total radiation $\overline{I}$ and radiation resistance $R = \overline{I/\mathcal{I}_0^2}$ of the antenna considered in the previous problem.

Hint: The result is expressed through the integral cosine

$$Ci(x) = C + \ln x + \int_0^x \frac{\cos t - 1}{t} \mathrm{d}t ,$$

where $C = 0.577$ is the Euler constant.

5.54. A running wave of current $\mathcal{I} = \mathcal{I}_0 e^{i(k\xi - \omega t)}$, where $k = \omega/c$, ξ is the coordinate of a point at the antenna, propagates in an l-long antenna.[1] Find the angular distribution $\overline{\mathrm{d}I/\mathrm{d}\Omega}$ and the total radiation intensity $\overline{I}$.

5.55*. A standing wave of the current $\mathcal{I} = \mathcal{I}_0 \sin n\alpha' e^{-i\omega t}$ is excited in a round wire loop of radius a. Find the electromagnetic field $\boldsymbol{H}$, $\boldsymbol{E}$ in the wave zone.

5.56. A damped current flows in a linear antenna (see Problem 5.52*)

$$J = J_0 \sin[k_m(\xi + l/2)] e^{-\gamma t} \cos \omega_m t \Theta(t) , \quad -l/2 \le \xi \le k/2 ,$$

where $\omega_m = ck_m = m\pi c/l$, Θ is the step function, and $m = 1, 2, \ldots$ is the number of half-waves that fall within the antenna length. Calculate the spectral radiation density $\mathrm{d}^2 I_\omega/\mathrm{d}\omega \mathrm{d}\Omega$ in the plane of symmetry perpendicular to the antenna axis.

5.57. N antennas of length l each parallel to the Ox axis are arranged in the xz plane. The distance between neighboring antennas is a. Current $J = J_0 \sin k_m z \cos \omega_m t$ flows in each antenna. Calculate the angular distribution of radiation $\overline{\mathrm{d}I/\mathrm{d}\Omega}$ averaged over time.

5.58. Solve the previous problem for the case of densely arranged antennas forming a thin plate of width $2b$ and length l with surface current density i_0.

5.59. The reflection of system B of charges $\rho(\boldsymbol{r}, t)$ and currents $\boldsymbol{j}(\boldsymbol{r}, t)$ in the $z = 0$ plane is such that (i) each point $\boldsymbol{r} = (x, y, z)$ passes into position $\boldsymbol{r}' = (x, y, -z)$ and (ii) the charge density changes sign: $\rho(\boldsymbol{r}, t) = -\rho'(\boldsymbol{r}', t)$, where ρ' is the charge density in the reflected system B'. Find how the current density $\boldsymbol{j}(\boldsymbol{r}, t)$, electric $\boldsymbol{p}$, Q and magnetic $\boldsymbol{m}$ moments of the system, and its electromagnetic field $\boldsymbol{E}$, $\boldsymbol{H}$ transform during reflection.

5.60. Prove that the electromagnetic field of an arbitrary system of charges B near an ideally conducting plane can be obtained as the superposition of the fields of system B and system B' reflected in this plane (see the preceding problem). Consider, in particular, emission of radiation from an electric dipole oscillator with moment $\boldsymbol{p}(t) = \boldsymbol{p}_0 f(t)$ ($|\boldsymbol{p}_0| = 1$) (where $f(t)$ is an arbitrary function) placed at distance $b \ll \lambda$ from such a plane and making with it the angle $\varphi_0 = \text{const}$ (confine yourself to the electric dipole approximation).

Hint: On an ideally conducting plane the boundary conditions $H_n = 0$ and $\boldsymbol{E}_\tau = 0$ must be fulfilled.

1) The ends of the antenna must be loaded so as to prevent the production of a reflected wave.

5.61. An electric dipole with moment amplitude $\boldsymbol{p}_0$ and frequency ω is at distance $a/2$ from the ideally conducting plane ($a \ll \lambda$, vector $\boldsymbol{p}_0$ is parallel to the plane). Find the electromagnetic field $\boldsymbol{E}, \boldsymbol{H}$ at distances $r \gg \lambda$ and the angular distribution of radiation $\overline{\mathrm{d}I/\mathrm{d}\Omega}$.

5.62.

1. Show that if the function $u(r, \vartheta, \alpha)$ satisfies the Helmholtz equation $\triangle u + k^2 u = 0$, then the Hertz potential for an electric-type ($H_\mathrm{r} = 0$) monochromatic field with frequency $\omega = ck$ in a space free of the field sources can be represented in the form $\boldsymbol{Z} = u\boldsymbol{r} + \nabla\chi$, $\chi = (1/k^2)\partial(ru)/\partial r$.
2. Find the expressions for the constituent strengths of the electromagnetic field $\boldsymbol{E}, \boldsymbol{H}$ along the axes of the spherical system of coordinates through $u(r, \vartheta, \alpha)$ (function u is termed the Debye potential).

Hint: When proving the equality $\triangle \boldsymbol{Z} + k^2 \boldsymbol{Z} = 0$, bear in mind the existence of the relation $\triangle\chi + k^2\chi + 2u = 0$.

5.63. Show that the field of a point-like electric dipole oscillator with moment $\boldsymbol{p}_0 e^{-i\omega t}$ placed at point $\boldsymbol{r}_0$ ($\boldsymbol{r}_0 \parallel \boldsymbol{p}_0$) can be described by the Debye potential (see the preceding problem) in the form $u = p_0 e^{ikR}/r_0 R$, where $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}_0$.

5.64. The point-like electric dipole oscillator with moment $p_0 e^{-i\omega t}$ is placed at distance b from the center of an ideally conducting sphere of radius a. The moment is directed along the line connecting the dipole with the center of the sphere. Find the electromagnetic field $\boldsymbol{E}, \boldsymbol{H}$ making use of the Debye potential u (see Problem 5.62). Find the angular distribution of the emission $\overline{\mathrm{d}I/\mathrm{d}\Omega}$.

5.3
Emission by Relativistic Particles

5.3.1
The Electromagnetic Field of a Propagating Charged Particle

Example 5.8

Calculate the electromagnetic potentials of a particle propagating in an arbitrary mode (*Lienard–Wiechert potentials*) based on the general expressions for retarded potentials (5.10) and (5.11) representing charge ρ and current $\boldsymbol{j}$ densities through the delta function.

Solution. Let the particle's charge e, radius vector $\boldsymbol{s}(t)$, velocity $\boldsymbol{v}(t) = \dot{\boldsymbol{s}}(t)$, and acceleration $\dot{\boldsymbol{v}}(t) = \ddot{\boldsymbol{s}}(t)$ be known. The charge and current densities are expressed through these quantities in the form

$$\rho(\boldsymbol{r}, t) = e\delta(\boldsymbol{r} - \boldsymbol{s}(t)) , \quad \boldsymbol{j}(\boldsymbol{r}, t) = e\boldsymbol{v}(t)\delta(\boldsymbol{r} - \boldsymbol{s}(t)) . \tag{5.50}$$

To calculate the potentials, it is convenient to use the general formula (5.4) with the retarded Green's function (5.9):

$$A(\boldsymbol{r}, t) = \frac{e}{c}\int G^{\mathrm{R}}(\boldsymbol{r} - \boldsymbol{r}', t - t')\boldsymbol{v}(t')\delta(\boldsymbol{r}' - \boldsymbol{s}(t))\mathrm{d}V'\mathrm{d}t'$$
$$= \frac{e}{c}\int \frac{\boldsymbol{v}(t')}{R(t')}\delta\left(t' - t + \frac{R(t')}{c}\right)\mathrm{d}t' , \qquad (5.51)$$

where $\boldsymbol{R}(t') = \boldsymbol{r} - \boldsymbol{s}(t')$. Integration over the volume is performed with the help of $\delta(\boldsymbol{r}' - \boldsymbol{s}(t'))$. The next integration over time is performed using formula (1.209), where (in the present case) $g(t') = t' + R(t')/c - t$ and $\mathrm{d}g(t')/\mathrm{d}t' = 1 - \boldsymbol{n}\cdot\boldsymbol{v}/c$, $\boldsymbol{n} = \boldsymbol{R}(t')/R(t')$. The use of these equalities yields

$$A(\boldsymbol{r}, t) = \left.\frac{e\boldsymbol{v}}{cR(1 - \boldsymbol{n}\cdot\boldsymbol{v}/c)}\right|_{t'} . \qquad (5.52)$$

The scalar potential is found in the same way:

$$\varphi(\boldsymbol{r}, t) = \left.\frac{e}{R(1 - \boldsymbol{n}\cdot\boldsymbol{v}/c)}\right|_{t'} . \qquad (5.53)$$

The value of t' at which the right-hand sides of (5.52) and (5.53) are taken must be found from the condition $g(t') = 0$, that is, from the equation

$$c(t - t') = |\boldsymbol{r} - \boldsymbol{s}(t')| , \qquad (5.54)$$

where $s(t')$ represents the particle's coordinates and $\boldsymbol{r}$ represents the coordinates of the observation point. The difference $t - t' = |\boldsymbol{r} - \boldsymbol{s}(t')|/c$ is the time during which the electromagnetic perturbation propagates from the particle to the observation point in the field. □

Example 5.9

Calculate the strengths of the electromagnetic field $\boldsymbol{E}$, $\boldsymbol{H}$ created by an arbitrarily propagating charged particle.

Solution. Calculate $\boldsymbol{H} = \nabla \times \boldsymbol{A}$ making use of representation (5.51). The operator ∇ acts on coordinates $\boldsymbol{r}$, which enter only as $R = |\boldsymbol{r} - \boldsymbol{s}|$ under the integral over $\mathrm{d}t'$. Differentiate under the sign of the integral using the formula

$$\nabla f(R) = \frac{\partial f}{\partial R}\nabla R = \frac{\partial f}{\partial R}\boldsymbol{n} .$$

This gives

$$\boldsymbol{H}(\boldsymbol{r}, t) = \frac{e}{c}\int (\boldsymbol{r} \times \boldsymbol{n})\left\{\frac{1}{R^2}\delta(t' + \frac{R}{c} - t) - \frac{1}{cR}\delta'(t' + \frac{R}{c} - t)\right\}\mathrm{d}t' .$$

Thereafter, integrate over $\mathrm{d}g = (\mathrm{d}g/\mathrm{d}t')\mathrm{d}t' = (1 - \boldsymbol{n}\cdot\boldsymbol{v}/c)\mathrm{d}t'$ and find with the use of the integration rules for the expressions with the delta function (1.209) and (1.210)

$$\boldsymbol{H}(\boldsymbol{r}, t) = \frac{e\boldsymbol{v} \times \boldsymbol{n}}{cR^2(1 - \boldsymbol{n}\cdot\boldsymbol{v}/c)} + \frac{e}{c(1 - \boldsymbol{n}\cdot\boldsymbol{v}/c)}\frac{\mathrm{d}}{\mathrm{d}t'}\frac{\boldsymbol{v} \times \boldsymbol{n}}{cR(1 - \boldsymbol{n}\cdot\boldsymbol{v}/c)}\,, \tag{5.55}$$

where the right-hand side is taken with the t', a value defined by (5.54). Calculation of the derivative of t' leads to

$$\frac{\mathrm{d}\boldsymbol{v}(t')}{\mathrm{d}t'} = \dot{\boldsymbol{v}}(t')\,, \quad \frac{\mathrm{d}R(t')}{\mathrm{d}t'} = -\boldsymbol{n}\cdot\boldsymbol{v}\,,$$
$$\frac{\mathrm{d}\boldsymbol{n}(t')}{\mathrm{d}t'} = -\frac{\boldsymbol{v}}{R} + \frac{\boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{v})}{R} = \frac{\boldsymbol{n} \times (\boldsymbol{n} \times \boldsymbol{v})}{R}\,.$$

Differentiation of (5.55) with the help of these formulas after the reduction of similar terms yields the final expression for the magnetic field:

$$\boldsymbol{H} = \frac{e(1 - v^2/c^2)(\boldsymbol{v} \times \boldsymbol{n})}{cR^2(1 - \boldsymbol{n}\cdot\boldsymbol{v}/c)^3} + \frac{e\{c\dot{\boldsymbol{v}} \times \boldsymbol{n} + \boldsymbol{n} \times [(\boldsymbol{v} \times \dot{\boldsymbol{v}}) \times \boldsymbol{n}]\}}{c^2R(1 - \boldsymbol{n}\cdot\boldsymbol{v}/c)^3}\,. \tag{5.56}$$

Calculate the electric field $\boldsymbol{E} = -\nabla\varphi - \partial\boldsymbol{A}/c\partial t$ in a similar fashion, that is, represent the scalar potential in the form

$$\varphi(\boldsymbol{r}, t) = \int \frac{e}{R(t')}\delta\left(t' + \frac{R(t')}{c} - t\right)\mathrm{d}t' \tag{5.57}$$

and differentiate quantities (5.51) and (5.57) under the sign of the integral. This gives the electric field

$$\boldsymbol{E} = \frac{e(1 - v^2/c^2)(\boldsymbol{n} - \boldsymbol{v}/c)}{R^2(1 - \boldsymbol{n}\cdot\boldsymbol{v}/c)^3} + \frac{e\boldsymbol{n} \times [(\boldsymbol{n} - \boldsymbol{v}/c) \times \dot{\boldsymbol{v}}]}{c^2R(1 - \boldsymbol{n}\cdot\boldsymbol{v}/c)^3}\,. \tag{5.58}$$

In the formulas for field strengths (5.56) and (5.58), all the quantities entering the right-hand sides are taken at moment t' found from (5.54). □

Comparison of intensities (5.56) and(5.58) suggests that they are related by the expression

$$\boldsymbol{H}(\boldsymbol{r}, t) = \boldsymbol{n}(t') \times \boldsymbol{E}(\boldsymbol{r}, t)\,, \tag{5.59}$$

that is, vector $\boldsymbol{H}$ is perpendicular to $\boldsymbol{E}$ and the line connecting the charge at moment t' with the observation point (but vector $\boldsymbol{E}$ has projection on this line). Each of the intensities consists of two terms, one changing in space as R^{-2} and containing no charge acceleration, the other proportional to the acceleration $\dot{\boldsymbol{v}}$ and decreasing as R^{-1}. The terms of the former type give rise to a quasi-stationary field inherent also in a uniformly moving charge and obeying the same law of decrease with distance as a static field. The terms of the second type describe the radiation wave field because it slowly collapses with distance and generates a finite flux of electromagnetic energy across the closed surface of large radius surrounding the particle. The radiation field arises only in the case of accelerated motion of the charged particle ($\dot{\boldsymbol{v}} \neq 0$).

Example 5.10

Express through velocity and acceleration the radiation energy flux $\mathrm{d}I/\mathrm{d}\Omega$ into a unit solid angle from an arbitrarily moving particle. Analyze the angular distribution with special reference to the following three cases: (i) emission of radiation by a nonrelativistic ($v \ll c$) particle; (ii) emission by an ultrarelativistic ($\gamma = \mathcal{E}/mc^2 \gg 1$) particle at $\dot{\boldsymbol{v}} \parallel \boldsymbol{v}$; (iii) emission by an ultrarelativistic ($\gamma = \mathcal{E}/mc^2 \gg 1$) particle at $\dot{\boldsymbol{v}} \perp \boldsymbol{v}$. In the latter two cases, represent in simplified form the radiation intensity in the direction making a small angle θ with the particle's velocity, using expansion in θ. Interpret the results thus obtained on the basis of the transformation law for the distribution function (see Problem 3.34$^\bullet$).

Solution. The energy flux into the unit solid angle is calculated through the field strengths in the wave zone with the use of formula (5.19). Using instead of $\boldsymbol{H}^2$ the equivalent quantity $\boldsymbol{E}^2$ and formula (5.58) yields

$$\frac{\mathrm{d}I}{\mathrm{d}\Omega} = \frac{e^2 \{\boldsymbol{n} \times [(\boldsymbol{n} - \boldsymbol{v}/c) \times \dot{\boldsymbol{v}}]\}^2}{4\pi c^3 (1 - \boldsymbol{n}\cdot\boldsymbol{v}/c)^6} , \tag{5.60}$$

where all the quantities on the right-side side are taken at moment $t - R/c$.

The energy flux into the unit solid angle (5.60) depends on R only via the time argument. This means that the energy flux across the $R^2\mathrm{d}\Omega$ areas inside the chosen solid angle $\mathrm{d}\Omega$ lying at different distances from the particle will be the same at the respective time moments – the electromagnetic perturbations move off to infinity from the charged particle that generated them. They give rise to the radiation field, which eventually separates from its source.

A quasi-stationary field (i.e., the terms in (5.56) and (5.58) containing no $\dot{\boldsymbol{v}}$ and proportional to R^{-2}) is devoid of such a property. The flux of quasi-stationary field energy inside a given solid angle decreases as R^{-2} with the growth of R. This means that the quasi-stationary field remains coupled to the particle at all times and does not create a flux at infinity.

Let us consider three cases:

1. A nonrelativistic particle, $v \ll c$. Disregarding the v/c-order terms gives from (5.60)

$$\frac{\mathrm{d}I}{\mathrm{d}\Omega} = \frac{e^2}{4\pi c^3}[\dot{\boldsymbol{v}}^2 - (\boldsymbol{n}\cdot\dot{\boldsymbol{v}})^2] = \frac{e^2 \dot{\boldsymbol{v}}^2}{4\pi c^3} \sin^2\theta , \tag{5.61}$$

where θ is the angle between the acceleration at moment t' and the direction of observation. Emission is uniformly distributed with respect to the direction $\dot{\boldsymbol{v}}$ and reaches a maximum in the direction perpendicular to $\dot{\boldsymbol{v}}$. Integration (5.61) over the solid angle gives the Larmor formula (5.32).

2. An ultrarelativistic particle, $\gamma = (1 - v^2/c^2)^{-1/2} \gg 1$, acceleration is directed parallel to the velocity: $\dot{\boldsymbol{v}} \parallel \boldsymbol{v}$. Denoting the angle between $\boldsymbol{n}$ and $\boldsymbol{v}$ by θ yields from (5.60)

$$\frac{\mathrm{d}I}{\mathrm{d}\Omega} = \frac{e^2\dot{\boldsymbol{v}}^2 \sin^2\theta}{4\pi c^3[1 - (v/c)\cos\theta]^6} . \tag{5.62}$$

The large exponent in the denominator (very small at $\cos\theta \approx 1$) is responsible for the strong anisotropy of radiation. The expansion $\sin\theta \approx \theta$, $\cos\theta \approx 1 - \theta^2/2$ in the most interesting region of small angles (taking into account that $v \approx c$) leads to

$$\frac{\mathrm{d}I}{\mathrm{d}\Omega} = \frac{16e^2\dot{\boldsymbol{v}}^2\gamma^{10}(\gamma\theta)^2}{\pi c^3(1+\gamma^2\theta^2)^6} . \tag{5.63}$$

Radiation is concentrated within a cone with an opening angle on the order of several $1/\gamma$. The property of emission by ultrarelativistic particles is attributable to the relativistic transformation of the angles and was already mentioned in Section 3.1.

3. An ultrarelativistic particle, acceleration is perpendicular to the velocity, $\dot{\boldsymbol{v}} \perp \boldsymbol{v}$. The general formula (5.60) gives

$$\frac{\mathrm{d}I}{\mathrm{d}\Omega} = \frac{e^2\dot{\boldsymbol{v}}^2}{4\pi c^3}\left\{\frac{1}{[1-(v/c)\cos\theta]^4} - \frac{(1-v^2/c^2)\sin^2\theta\cos^2\varphi}{[1-(v/c)\cos\theta]^6}\right\} . \tag{5.64}$$

Here, φ is the angle between planes $(\boldsymbol{v}, \boldsymbol{n})$ and $(\boldsymbol{v}, \dot{\boldsymbol{v}})$. The distribution of (5.64) like (5.62) is concentrated in the forward direction. At small θ, it has the form

$$\frac{\mathrm{d}I}{\mathrm{d}\Omega} = \frac{4e^2\dot{\boldsymbol{v}}^2\gamma^8}{\pi c^3(1+\gamma^2\theta^2)^4}\left[1 - \frac{4\gamma^2\theta^2\cos^2\varphi}{(1+\gamma^2\theta^2)^2}\right] . \tag{5.65}$$

□

5.3.2
The Loss of Energy and Momentum of a Charged Particle

The quantity $\mathrm{d}I/\mathrm{d}\Omega$ is the flux of electromagnetic energy inside a unit solid angle that can be measured by a stationary observer placed in a laboratory frame of reference. This quantity differs from the rate at which a particle loses energy due to emission into a unit solid angle $\mathrm{d}^2\mathcal{E}/\mathrm{d}t'\mathrm{d}\Omega$, where $\mathrm{d}t'$ is the "retarded" time interval during which a certain portion of radiation was emitted. This difference is due to the difference between the intervals $\mathrm{d}t$ and $\mathrm{d}t$ even though they are determined in the laboratory system. They are related by the equality ensuing from (5.54). Differentiation of this expression yields

$$\mathrm{d}t = \left(1 - \frac{\boldsymbol{n}\cdot\boldsymbol{v}(t')}{c}\right)\mathrm{d}t' . \tag{5.66}$$

Equating the energy emitted by a particle in time $\mathrm{d}t'$ and the energy that passed the observer in time $\mathrm{d}t$ gives, with the help of (5.66),

$$-\frac{\mathrm{d}^2\mathcal{E}}{\mathrm{d}t'\mathrm{d}\Omega} = \frac{\mathrm{d}I}{\mathrm{d}\Omega}\frac{\mathrm{d}t}{\mathrm{d}t'} = \left(1 - \frac{\boldsymbol{n}\cdot\boldsymbol{v}(t')}{c}\right)\frac{\mathrm{d}I}{\mathrm{d}\Omega}\,. \tag{5.67}$$

According to (5.60), the quantity $\mathrm{d}I/\mathrm{d}\Omega$ depends on argument t'. This relationship links the loss of the energy by a particle and the intensity of radiation registered by the observer.

Integrating (5.67) over the entire solid angle yields the total (or summed over all directions) loss of the energy by the emitting particle:

$$-\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t'} = \int\left(1 - \frac{\boldsymbol{n}\cdot\boldsymbol{v}(t')}{c}\right)\frac{\mathrm{d}I}{\mathrm{d}\Omega}\mathrm{d}\Omega\,. \tag{5.68}$$

It does not coincide with the total intensity

$$I = \int\frac{\mathrm{d}I}{\mathrm{d}\Omega}\mathrm{d}\Omega\,. \tag{5.69}$$

The quantity $-\mathrm{d}\mathcal{E}/\mathrm{d}t'$ is a more convenient experimental characteristic of radiation emitted by a relativistic particle than the total intensity I owing to its relativistic invariance. The intensity is not an invariant quantity.

Example 5.11

Prove the relativistic invariance of quantity $-\mathrm{d}\mathcal{E}/\mathrm{d}t'$. Express the rate of loss of momentum $-\mathrm{d}\boldsymbol{p}/\mathrm{d}t'$ by an emitting particle through the energy loss rate summed over all directions, $-\mathrm{d}\mathcal{E}/\mathrm{d}t'$.

Solution. Compare the rate of energy loss in an instantaneously accompanying system S_0 in which the particle is at rest at a given moment and in a laboratory frame of reference S in which the particle has velocity $\boldsymbol{v}$. The intensity of radiation in the former system (coinciding with the rate of energy loss $-\mathrm{d}\mathcal{E}_0/\mathrm{d}t_0'$) is given by formula (5.61). Because it is an even function of the angle, the particle does not lose momentum in this system, $\mathrm{d}\boldsymbol{p}_0 = 0$. The energy loss for time $\mathrm{d}t_0' = \mathrm{d}\tau$ is $-\mathrm{d}\mathcal{E}_0$.

The loss of energy by a particle in the laboratory system is calculated using the Lorentz transform (3.9) for the 4-vector $p^k = (\mathrm{d}\mathcal{E}/c, \mathrm{d}\boldsymbol{p})$: $-\mathrm{d}\mathcal{E} = -\mathrm{d}\mathcal{E}_0(1 - v^2/c^2)^{-1/2}$. It takes time $\mathrm{d}t' = \mathrm{d}\tau(1 - v^2/c^2)^{-1/2}$. Hence, the rate of energy loss is

$$-\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t'} = -\frac{\mathrm{d}\mathcal{E}_0}{\mathrm{d}\tau}\,, \tag{5.70}$$

which proves the relativistic invariance of this quantity.

The loss of momentum is also calculated with the use of the Lorentz transform:

$$\mathrm{d}\boldsymbol{p} = \frac{\boldsymbol{v}\,\mathrm{d}\mathcal{E}_0}{c^2\sqrt{1 - v^2/c^2}}\,.$$

Let us refer this loss to the "particle's time" $\mathrm{d}t'$ in the laboratory frame of reference:

$$-\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t'} = \frac{\boldsymbol{v}}{c^2}\left(-\frac{\mathrm{d}\mathcal{E}_0}{\sqrt{1-v^2/c^2}\mathrm{d}t'}\right) .$$

However, $\sqrt{1-v^2/c^2}\mathrm{d}t' = \mathrm{d}\tau$ is the differential of the particle's eigentime; therefore, the use of (5.70) leads to a simple relationship between the losses of energy and momentum in one and the same laboratory frame of reference:

$$-\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t'} = \frac{\boldsymbol{v}}{c^2}\left(-\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t'}\right) . \tag{5.71}$$

This formula holds regardless of the particle's velocity. □

Example 5.12

Express the rate of loss of energy and momentum by a relativistic particle per unit eigentime in a covariant form through the four-dimensional velocity and acceleration as well as through the respective three-dimensional quantities.

Solution. Use the definition of 4-acceleration (3.23) and express the invariant quantity $\mathrm{d}\mathcal{E}/\mathrm{d}t'$ through a relativistic invariant, the square of 4-acceleration:

$$-\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t'} = -\frac{2e^2}{3c^3} w^k w_k . \tag{5.72}$$

At $v \to 0$, relation (5.72) becomes the Larmor formula (5.32). Combining (5.71) and (5.72) and introducing the eigentime interval $\mathrm{d}\tau = \sqrt{1-v^2/c^2}\mathrm{d}t'$ and the 4-velocity (3.23) allows both of the formulas, (5.71) and (5.72), to be written in the form of equality of 4-vectors:

$$-\frac{\mathrm{d}p^i}{\mathrm{d}\tau} = -\frac{2e^2}{3c^5} w^k w_k u^i , \tag{5.73}$$

where $p^i = (\mathcal{E}/c, \boldsymbol{p})$ is the four-dimensional vector of energy–momentum of the relativistic particle.

Four-acceleration can be expressed through the three-dimensional quantities:

$$w^k = \frac{\mathrm{d}u^k}{\mathrm{d}\tau} = \gamma\frac{\mathrm{d}}{\mathrm{d}t'}\gamma(c, \boldsymbol{v}) = \left(\gamma^4\frac{\boldsymbol{v}\cdot\dot{\boldsymbol{v}}}{c}, \gamma^2\dot{\boldsymbol{v}} + \gamma^4\frac{\boldsymbol{v}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})}{c^2}\right) .$$

Writing the square of 4-acceleration,

$$w^k w_k = -\gamma^6\left[\dot{\boldsymbol{v}}^2 - \frac{(\boldsymbol{v}\times\dot{\boldsymbol{v}})^2}{c^2}\right] ,$$

we find the relativistic generalization of the Larmor formula (5.72) through three-dimensional quantities:

$$-\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t'} = -\frac{2e^2\gamma^6}{3c^2}\left[\dot{\boldsymbol{v}}^2 - \frac{(\boldsymbol{v}\times\dot{\boldsymbol{v}})^2}{c^2}\right] . \tag{5.74}$$

□

5.3.3
The Spectral Distribution of Radiation Emitted by Relativistic Particles

Express the spectral radiation density in a given direction (5.22) through the velocity and radius vector of a relativistic particle. In the wave zone, $R(\tau) \approx r - \boldsymbol{n} \cdot \boldsymbol{s}(\tau)$, where $\boldsymbol{n} = \boldsymbol{r}/r$ and $\boldsymbol{s}(\tau)$ is the particle's radius vector. The use of (5,12') and integration over $\mathrm{d}V'$ yields in the wave zone

$$\boldsymbol{A}_\omega(\boldsymbol{r}) = \frac{e}{c}\frac{\exp(ikr)}{r}\int_{-\infty}^{\infty} \boldsymbol{v}(\tau)\exp(i\omega\tau - i\boldsymbol{k}\cdot\boldsymbol{s}(\tau))\mathrm{d}\tau\,, \tag{5.75}$$

where $\boldsymbol{k} = \omega\boldsymbol{n}/c$ is the wave vector of the emitted wave. The Fourier harmonic of the magnetic field strength is found from formula (5.15):

$$\boldsymbol{H}_\omega = i\boldsymbol{k}\times\boldsymbol{A}_\omega = \frac{ie\omega}{c^2}\frac{\exp(ikr)}{r}\int_{-\infty}^{\infty} \boldsymbol{n}\times\boldsymbol{v}(\tau)\exp(i\omega\tau - i\boldsymbol{k}\cdot\boldsymbol{s}(\tau))\mathrm{d}\tau\,. \tag{5.76}$$

The relationship between the intensity of the electric and magnetic fields in the wave zone (5.18) follows from the Maxwell equations. Substituting (5.76) into the general expression (5.22) yields the spectral radiation density in the specified direction:

$$\frac{\mathrm{d}^2 I_\omega}{\mathrm{d}\omega\mathrm{d}\Omega} = \frac{e^2\omega^2}{4\pi^2c^3}\left|\int_{-\infty}^{\infty} \boldsymbol{n}\times\boldsymbol{v}(\tau)\exp(i\omega\tau - i\boldsymbol{k}\cdot\boldsymbol{s}(\tau))\mathrm{d}\tau\right|^2\,. \tag{5.77}$$

Example 5.13

Perform a semiquantitative analysis of the spectral radiation density for an ultrarelativistic ($\gamma \gg 1$) particle. Consider two limiting cases: (i) the total angle of deviation of the particle in the external field is large compared with the radiation angle $1/\gamma$; (ii) the total angle of deviation of the particle is smaller than or of the same order of magnitude as the radiation angle $1/\gamma$. Estimate the characteristic spectral radiation intervals for both cases.

Solution.

1. The total angle of deviation of the particle is large compared with the radiation angle. An example of such a situation is radiation of an ultrarelativistic particle in a magnetic field. The particle emits radiation in the specified direction $\boldsymbol{n}$ from a small segment of its trajectory within which the direction of velocity changes through an angle of the order of $1/\gamma$. The length of this segment (the length of radiation formation or coherent length) $l_{\mathrm{coh}} \approx \rho/\gamma$, where ρ is the instantaneous radius of curvature. The wave phase must change in accordance with (5.76) by $\Delta\varphi = \omega\tau - \boldsymbol{n}\cdot\Delta\boldsymbol{s}(\tau)\omega/c$, where $\Delta\boldsymbol{s}(\tau)$ is the displacement of the

particle for time τ. Because the angle between $\boldsymbol{n}$ and $\boldsymbol{v}$ does not exceed $1/\gamma$ in terms of the order of magnitude, $\boldsymbol{n}\cdot\Delta\boldsymbol{s}(\tau)/c \approx v\tau/c \approx \rho/c\gamma$ and $\Delta\varphi \approx (\omega\rho/v\gamma)(1 - v/c) \approx \omega\rho/2\gamma^3 c$. Only waves of such frequency can be emitted for which $\Delta\varphi < 1$; at $\Delta\varphi > 1$, the waves inside the radiation cone have large phase shifts and cancel one another out. The range of the emitted frequencies is given by the condition

$$\Delta\omega < \omega_c = \frac{c}{\rho}\gamma^3 . \tag{5.78}$$

The radiation intensity decreases drastically at $\omega \gg \omega_c$.

If a particle propagates in a uniform magnetic field normally to the direction of the strength (vector!) of the field, then $\rho = cp/eH \approx \gamma mc^2/eH$, in conformity with the data presented in Section 4.2. Then,

$$\omega_c = \frac{eH}{mc}\gamma^2 . \tag{5.79}$$

The motion in a uniform magnetic field is strictly periodic; therefore, the spectrum consists of discrete lines separated by a distance equaling the rotation rate of a relativistic particle: $\Omega = ecH/\mathcal{E} = eH/mc\gamma$. At $\gamma \gg 1$, the spectrum becomes quasi-continuous. The radiation maximum occurs at frequencies on the order of ω_c (see Problems 5.83** and 5.85*).

2. The total angle of deviation of a particle during the entire time of its acceleration is either small or on the order of the radiation angle. All radiation is emitted into a narrow cone and depends on the path segment over which the particle's velocity varies. Such a situation occurs, for example, when an ultrarelativistic particle is scattered through a small angle in the Coulomb field of the nucleus. Let the external field exist in a region of size a (by way of example, it may be the radius of the shielding of the Coulomb field). The particle is accelerated during time $\Delta t' \approx a/v \approx a/c$. A stationary observer in a laboratory frame of reference sees this process for

$$\Delta t \approx \frac{\Delta t}{\Delta t'}\Delta t' \approx \left(1 - \frac{\boldsymbol{n}\cdot\boldsymbol{v}}{c}\right)\frac{a}{c} \approx \frac{a}{\gamma^2 c} .$$

The spectral interval is found by using the relation $\Delta\omega\cdot\Delta t \approx 1$:

$$\Delta\omega \approx \omega_c \approx \frac{c}{a}\gamma^2 . \tag{5.80}$$

The characteristic frequency is proportional to the square of the particle's energy, as in (5.79).

□

5.3.4
Radiation from Colliding Particles

Colliding charged particles move with acceleration; because of this, they emit electromagnetic waves (*bremsstrahlung*). The law of motion for colliding particles and the energy they emit depend on the form of the interaction and the impact parameter ρ (if the potential interaction parameter of the colliding particles depends on the distance between them alone). It is convenient to characterize the energy emitted in all directions when the particle flux is scattered by a certain scattering center by the differential effective radiation

$$\frac{d\kappa}{d\Omega} = 2\pi \int_0^\infty \frac{d\mathcal{E}^{\mathrm{rad}}(\rho)}{d\Omega}\rho d\rho \tag{5.81}$$

and the total effective radiation

$$\kappa = 2\pi \int_0^\infty \mathcal{E}^{\mathrm{rad}}(\rho)\rho d\rho \, . \tag{5.82}$$

Here, $d\mathcal{E}^{\mathrm{rad}}(\rho)/d\Omega$ is the energy emitted from a single collision in the $\boldsymbol{n}$ direction per unit solid angle with impact parameter ρ and averaged over the azimuth in the plane perpendicular to the stream of particles.

If electric dipole radiation plays a key role in the collision, then (5.81) assumes the form

$$\frac{d\kappa}{d\Omega} = \frac{1}{4\pi c}[A + B P_2(\cos\theta)] \, , \tag{5.83}$$

where $P_2(\cos\theta)$ is the Legendre polynomial, and θ is the angle between the direction of observation $\boldsymbol{n}$ and the direction Oz of the stream of incident particles,

$$A = \frac{2}{3}\int_0^\infty 2\pi\rho d\rho \int_{-\infty}^{\infty} \ddot{\boldsymbol{p}}^2 dt \, , \quad B = \frac{1}{3}\int_0^\infty 2\pi\rho d\rho \int_{-\infty}^{\infty} (\ddot{\boldsymbol{p}}^2 - 3\ddot{p}_z^2) dt \, . \tag{5.84}$$

Example 5.14

Derive formulas (5.83) and (5.84) from the general expression (5.30) for electric dipole radiation.

Solution. Choose the Oz axis of the coordinate system along the particle flux and the Ox axis in the plane determined by vector $\boldsymbol{n}$ and axis Oz (Figure 5.2). In this case, $\boldsymbol{n} = \boldsymbol{e}_x \sin\theta + \boldsymbol{e}_z \cos\theta$ and

$$\begin{aligned}\overline{(\boldsymbol{n} \times \ddot{\boldsymbol{p}})^2} &= \ddot{\boldsymbol{p}}^2 - \overline{(\boldsymbol{n}\cdot\ddot{\boldsymbol{p}})^2} \\ &= \ddot{\boldsymbol{p}}^2 - \ddot{p}_\perp^2 \overline{\cos^2\phi} \sin^2\theta \\ &\quad - \ddot{p}_z \cos^2\theta - 2\ddot{p}_\perp \ddot{p}_z \overline{\cos\phi} \sin\theta \cos\theta \\ &= \ddot{\boldsymbol{p}}^2 - \frac{1}{2}\ddot{p}_\perp^2 \sin^2\theta - \ddot{p}_z^2 \cos^2\theta \ ,\end{aligned}$$

where the bar denotes averaging over the azimuthal angle ϕ. Introduce $\ddot{p}_\perp^2 = \ddot{\boldsymbol{p}}^2 - \ddot{p}_z^2$ and $\cos^2\theta = (2P_2(\cos\theta) + 1)/3$. Substituting the expression $(\boldsymbol{n} \times \ddot{\boldsymbol{p}})^2$ transformed in this way into (5.30) and integrating over the impact parameters and time yields (5.83) and (5.84).

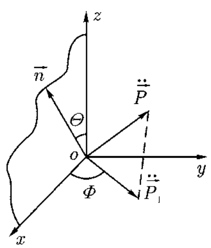

Figure 5.2 Diagram for the calculation of the effective radiation arising from collision of charged particles.

□

Example 5.15

Derive the formula for the spectral density of effective radiation in the case of scattering of a particle beam emitted along the Oz axis.

Solution. The Fourier expansion

$$\ddot{\boldsymbol{p}}(t) = -\int_{-\infty}^{\infty} \omega^2 \boldsymbol{p}_\omega e^{-i\omega t} \frac{\mathrm{d}\omega}{2\pi}$$

and the use of (1.219) gives

$$\int_{-\infty}^{\infty} \ddot{\boldsymbol{p}}^2(t)\mathrm{d}t = \frac{1}{\pi}\int_0^{\infty} \omega^4 |\boldsymbol{p}_\omega|^2 \mathrm{d}\omega \ .$$

Similar transformations of the integral $\int_{-\infty}^{\infty} \ddot{p}_z^2(t)\mathrm{d}t$ and substitution of these results into (5.83) and (5.84) yields

$$\frac{\mathrm{d}\kappa}{\mathrm{d}\Omega} = \int_0^\infty \frac{\mathrm{d}^2\kappa_\omega}{\mathrm{d}\omega\mathrm{d}\Omega}\mathrm{d}\omega \ ,$$

where

$$\frac{\mathrm{d}^2\kappa_\omega}{\mathrm{d}\omega\mathrm{d}\Omega} = \frac{1}{4\pi^2 c^3}[A(\omega) + B(\omega)P_2(\cos\theta)] \tag{5.85}$$

is the spectral density of effective radiation,

$$A(\omega) = \frac{2\omega^4}{3}\int_0^\infty |\boldsymbol{p}_\omega|^2 2\pi\rho\mathrm{d}\rho \ ,$$

$$B(\omega) = \frac{\omega^4}{3}\int_0^\infty \left(|\boldsymbol{p}_\omega|^2 - 3|p_{z\omega}|^2\right) 2\pi\rho\mathrm{d}\rho \ . \tag{5.86}$$

□

5.3.5
Radiation from Particle Decays and Transformations

Particle decays and transformations into other particles are accompanied by the sudden (very fast, to be precise) disappearance of the moving charged particles and the appearance of new particles propagating with certain (frequently relativistic) velocities. These processes give rise to bremsstrahlung, also called *internal bremsstrahlung* to distinguish it from the radiation accompanying collisions of particles with external objects. The duration of such transformations can be estimated from the order of magnitude using the quantum mechanical energy–time uncertainty relation:

$$\tau_0 \approx \frac{\hbar}{\Delta\mathcal{E}} \ , \tag{5.87}$$

where $\Delta\mathcal{E}$ is the characteristic energy of the process.

The methods of classical electrodynamics may be used to calculate the spectrum of internal bremsstrahlung in the approximation $\tau \to 0$, that is, for relatively low frequencies

$$\omega \ll \frac{2\pi}{\tau} \ . \tag{5.88}$$

This constraint refers to a stationary object in which particle transformation occurs in a laboratory system. If the object travels with a relativistic velocity, the Lorentz transform of the time interval and frequency leads to the condition

$$\omega\tau \ll \frac{2\pi}{1-(v/c)\cos\theta} \ , \tag{5.89}$$

where θ is the angle between the directions of the wave velocity and the observation.

One more limitation is imposed on the frequency by the laws of conservation of energy and momentum in the case of photon emission (see Chapters 6 and 7) that are considered in quantum theory. They restrict the frequency to a certain $\omega_{\max}$ value. Generation of quanta with energies $\hbar\omega$ on the order of maximum $\hbar\omega_{\max}$ is calculated by the methods of quantum mechanics.

Suggested literature:
Landau and Lifshitz (1975); Jackson (1999); Bredov *et al.* (2003); Baier *et al.* (1973); Medvedev (1977); Ginzburg (1979a); Panofsky and Phillips (1963); Ternov and Mikhailin (1986); Bolotovskii *et al.* (1978); Alferov *et al.* (1989); Fleishman (2008); Fleishman and Toptygin (2013)

Problems

5.65. Show that in the case of aperiodic motion the Fourier components of the Lienard–Wiechert potentials can be represented in the form

$$A_\omega(\mathbf{r}) = \frac{e}{c}\int_{-\infty}^{\infty} \frac{\mathbf{v}(\tau)\exp(i\omega\tau + i\omega R(\tau)/c}{R(\tau)}\mathrm{d}\tau, \tag{5.90}$$

$$\varphi_\omega(\mathbf{r}) = e\int_{-\infty}^{\infty} \frac{\exp(i\omega\tau + i\omega R(\tau)/c)}{R(\tau)}\mathrm{d}\tau\ , \tag{5.91}$$

where $R(\tau) = |\mathbf{r} - \mathbf{s}(\tau)|$. If a particle performs periodic motion with period $T = 2\pi/\omega_0$, the Fourier harmonics with frequency $n\omega_0$ (n being an integer) are expressed in the form

$$A_n = \frac{e}{cT}\int_0^T \frac{\mathbf{v}(\tau)}{R(\tau)}\exp[in\omega_0(\tau - R(\tau)/c)]\mathrm{d}\tau\ . \tag{5.92}$$

The scalar potential $\varphi_n(\mathbf{r})$ differs from the last expression in the absence of the factor $\mathbf{v}(\tau)/c$.

Hint: Use expression (5.12) and perform integration over $\mathrm{d}V'$.

5.66*. Perform expansion in powers of R/c in the general formulas for retarded potentials (5.10) and (5.11) and find in this way the expansion of the Lienard–Wiechert potentials in powers of $1/c$.

5.67. Calculate the field potentials of a uniformly propagating point charge from the Lienard–Wiechert potentials (5.52) and (5.53), expressing in them the retarded time t' through the field observation time t. Also, calculate the field intensities of a uniformly moving point charge.

5.68. A point charge e propagates with low velocity $\boldsymbol{v}$ and acceleration $\dot{\boldsymbol{v}}$ within a bounded region. Find the approximate expressions for the electromagnetic field $\boldsymbol{E}$, $\boldsymbol{H}$ of a particle at the points spaced from the particle by a distance r larger that the size of the region in which the charge propagates. Determine the positions of the boundaries of quasi-stationary and wave zones.

5.69. Calculate the angular distribution of radiation by the charge, $\mathrm{d}I/\mathrm{d}\Omega$, considered in the preceding problem. Find the total radiation I.

5.70. Find geometrically the relation (5.67) between the rate of energy losses by a particle per unit solid angle in a given direction and the radiation intensity by considering the shape of the space region in which the electromagnetic energy emitted by a particle for time $\mathrm{d}t'$ is located.

5.71. Prove that if a particle performs periodic motion, the rate of energy losses averaged over the period coincides with the mean radiation intensity.

5.72. Show that if a particle is accelerated under the action of an external electromagnetic field with intensities $\boldsymbol{E}$ and $\boldsymbol{H}$, then the rate of energy losses summed over all directions can be written in the form

$$-\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t'} = -\frac{2e^2\gamma^2}{3m^2c^2}\left[\left(\boldsymbol{E} + \frac{1}{c}\boldsymbol{v} \times \boldsymbol{H}\right)^2 - \frac{1}{c^2}(\boldsymbol{E}\cdot\boldsymbol{v})^2\right] . \tag{5.93}$$

Analyze this result by considering the cases of longitudinal and transverse (with respect to velocity $\boldsymbol{v}$) acceleration.

5.73. The velocity $\boldsymbol{v}$ of a relativistic particle is parallel to its acceleration $\dot{\boldsymbol{v}}$ at a certain moment of retarded time t'. Find the instantaneous angular distribution of the radiation intensity $\mathrm{d}I/\mathrm{d}\Omega$ and the total instantaneous radiation intensity I as well as the rate of energy loss summed over all directions, $-\mathrm{d}\mathcal{E}/\mathrm{d}t'$. What is the character of the angular distribution of radiation intensity in the relativistic case?

5.74. The particle's velocity decreases from v_0 to 0 for a span of time τ. Find the angular distribution of bremsstrahlung emitted during the entire time of the particle's motion assuming the acceleration to be constant. What momentum duration Δt will be recorded by a resting measuring instrument?

5.75. A relativistic particle with charge e, mass m, and momentum p propagates in a circular orbit in a constant uniform magnetic field $\boldsymbol{H}$. The orbit has radius $a = cp/eH$. Find the rate of loss of energy by the particle summed over all directions, $-\mathrm{d}\mathcal{E}/\mathrm{d}t'$.

5.76. An ultrarelativistic electron propagates along a helical line in a uniform magnetic field of strength $\boldsymbol{H}$. Its velocity $\boldsymbol{v}$ makes angle θ with the $\boldsymbol{H}$ vector. Find the energy $-d\mathcal{E}/\mathrm{d}t'$ lost by the electron per unit time. Also, find the radiation energy flux I through a motionless sphere of large radius that encloses the electron.

5.77. Find the instantaneous angular distribution of the radiation intensity $\mathrm{d}I/\mathrm{d}\Omega$ of a relativistic particle, the velocity of which is perpendicular to the acceleration at a retarded moment of time. Draw a polar plot for the cases of $v \ll c$ and $v \sim c$. Find the directions in which radiation fails to be emitted.

5.78*. A particle with charge e and mass m propagates with velocity v around a circle in a constant uniform magnetic field $\boldsymbol{H}$. Find the angular distribution of the radiation intensity $\overline{\mathrm{d}I/\mathrm{d}\Omega}$ averaged over the particle's orbital period in the magnetic field. What is the character of this angular distribution in the ultrarelativistic case $v \sim c$?

Hint: Use the results from the preceding problem. Pass to the spherical coordinates with the pole in the center of the circular trajectory and the polar axis along $\boldsymbol{H}$. Use the relevant formulas from reference books to calculate the integral over the azimuthal angle.

5.79*. Find the Fourier components of the radiation field $\boldsymbol{A}_n$, $\boldsymbol{H}_n$ of charge e propagating in a circular orbit with radius a and relativistic velocity v. Study the character of the polarization of the Fourier components.

Hint: Use formulas from the theory of Bessel functions.

5.80. Explain the presence of higher harmonics in the field spectrum of a charge propagating with constant velocity in a circular orbit (see the preceding problem). How do intensities of these harmonics change when $\beta = v/c \to 0$? What is the shape of the radiation field in this case?

5.81*. A charge propagates around a circle of radius a with velocity $v = \beta c$. Find the spectral expansion of the radiation intensity $\mathrm{d}I_n/\mathrm{d}\Omega$ in a given direction. Also, calculate the radiation at individual harmonics summed over all directions. Use integrals of the Bessel functions to perform integration over the angles

$$\int_0^{\pi} \mathrm{d}\vartheta \sin\vartheta \cot^2\vartheta\, J_n^2(n\beta\sin\vartheta) = 2\int_0^{2n\beta} \frac{J_{2n}(x)}{x}\mathrm{d}x - \frac{1}{n\beta}\int_0^{2n\beta} J_{2n}(x)\mathrm{d}x\,,$$

$$\int_0^{\pi} \mathrm{d}\vartheta \sin\vartheta \cot^2\vartheta\, J_n'^2(n\beta\sin\vartheta) = \frac{2}{n\beta}J_{2n}'^2(2n\beta) - \frac{2}{\beta^2}\int_0^{2n\beta} \frac{J_{2n}(x)}{x}\mathrm{d}x + \frac{1}{n\beta}\int_0^{2n\beta} J_{2n}(x)\mathrm{d}x\,.$$

5.82. Find the angular distribution of the total radiation from a charged particle propagating in a circular orbit in a uniform magnetic field by summation of the radiation at individual harmonics found in the previous problem. Undertake a detailed study of an ultrarelativistic case ($\gamma = \mathcal{E}/mc^2 \gg 1$), and write down the simplified formula for the angular distribution of radiation.

Hint: Use the following formula from the theory of Bessel functions

$$\sum_{m=1}^{\infty} m^2 J_m'^2(mx) = \frac{4 + 3x^2}{16(1 - x^2)^{5/2}}$$

and an analogous formula from the condition for Problem 5.39.

5.83**. Calculate the spectral radiation power in a given direction $\boldsymbol{n}$ generated by an ultrarelativistic charged particle propagating around a circle in a uniform magnetic field. Regard the frequency ω as a continuous quantity, bearing in mind the presence of a huge number ($N \sim \gamma^3 \gg 1$) of similar harmonics in the spectrum. Make a rough calculation in the first nonvanishing approximation of γ^{-2} and θ^2, where γ is the relativistic factor and θ is the angle between the orbital plane and the direction of radiation. Express the result through the modified Bessel functions

$$K_{2/3}(\xi) = \sqrt{3} \int_0^{\infty} x \sin\left[\frac{3}{2}\xi \left(x + \frac{1}{3}x^3\right)\right] dx \ ,$$

$$K_{1/3}(\xi) = \sqrt{3} \int_0^{\infty} \cos\left[\frac{3}{2}\xi \left(x + \frac{1}{3}x^3\right)\right] dx \ .$$

Analyze the spectral and angular distributions at low and high frequencies. Radiation by an ultrarelativistic particle in a magnetic field is referred to as *synchrotron radiation*.

5.84. Calculate (using formula (3) obtained in solving the preceding problem) the spectral power of synchrotron radiation summed over all directions. Analyze the result for low and high frequencies. Use the formulas from the theory of Bessel functions to integrate over angle θ

$$\int_{-\infty}^{\infty} dx (1 + x^2)^2 K_{2/3}^2 \left[\left(\frac{y}{2}\right)(1 + x^2)^{3/2}\right] = \frac{\pi}{y\sqrt{3}} \left\{\int_y^{\infty} K_{5/3}(x)dx + K_{2/3}(y)\right\} \ ;$$

$$\int_{-\infty}^{\infty} dx x^2 (1 + x^2) K_{1/3}^2 \left[\left(\frac{y}{2}\right)(1 + x^2)^{3/2}\right] = \frac{\pi}{y\sqrt{3}} \left\{\int_y^{\infty} K_{5/3}(x)dx - K_{2/3}(y)\right\} \ .$$

5.85*. Calculate the spectral power of synchrotron radiation in a given direction $\boldsymbol{n}$ generated by an ultrarelativistic charged particle propagating along a helical line in a uniform magnetic field. The angle between the direction of the particle's velocity and the magnetic field is equal to $\pi/2 - \alpha$; therefore, the circular motion corresponds to the value $\alpha = 0$. Use the same approximation as in Problem 5.83**. Also, determine the period between radiation pulses that a distant observer would register and the boundary frequency ω_c of the spectrum.

5.86. Find the synchrotron radiation spectrum recorded by an observer from a cloud of cosmic electrons with energy $\mathcal{E}_0 \gg mc^2$. The electrons reside in a uniform magnetic field, and their momentum distribution function is isotropic:

$$f(p)\mathrm{d}^3p = \frac{N_0}{4\pi}\delta(p - p_0)\mathrm{d}p\,\mathrm{d}\Omega_p \ .$$

The direction of the magnetic field makes angle Θ with the direction of observation, $\boldsymbol{n}$. The size of the electron cloud is small in comparison with the distance to it.

5.87*. Let the particle distribution by energies in the cloud of cosmic electrons (considered in the preceding problem) obey the power law as is frequently the case in cosmic sources of radio-frequency radiation:

$$f(\mathcal{E})\mathrm{d}\mathcal{E} = (\nu - 1)N_0\mathcal{E}_*^{\nu-1}\frac{\mathrm{d}\mathcal{E}}{\mathcal{E}^\nu} \ , \quad \mathcal{E} \geq \mathcal{E}_* \ , \quad \nu > 1 \ .$$

Calculate the synchrotron radiation spectrum of this source in a frequency range depending on the boundary energy $\mathcal{E}_*$ only via the normalization factor. Find the relationship between the radiation spectrum and the spectral index ν of the energy spectrum of relativistic electrons.

5.88. The Crab Nebula in our galaxy is believed to be one of the main sources of synchrotron radiation in a broad frequency range. According to observations, the synchrotron radiation spectrum index $\alpha \approx 0.3$ at $f = \omega/2\pi < 10^{14}$ Hz and $\alpha \approx 1.0$ at $f > 10^{14}$ Hz. The Crab Nebula is supposed to have a magnetic field $H \approx 10^{-4}$ Oe. Estimate the electron energy spectrum indices and find the spectral break energy $\mathcal{E}_*$.

5.89*. An ultrarelativistic particle enters an electrical "undulator," a device where it is subjected to the action of a periodically changing electric field $\boldsymbol{E}(t) = \boldsymbol{E}_0 \cos\omega_0 t$ perpendicular to its starting velocity $\boldsymbol{v}_0 \perp \boldsymbol{E}_0$. The field is assumed to be rather weak so that the particle's trajectory is only slightly deflected from the straight line. Calculate the angular distribution $\mathrm{d}\mathcal{E}^{\mathrm{rad}}/\mathrm{d}\Omega$ and the total energy $\mathcal{E}^{\mathrm{rad}}$ emitted by the particle during its time of flight through an undulator of length L. Estimate the order of magnitude of the characteristic frequencies of the emitted waves for electrons with energy $\mathcal{E} = 5$ GeV propagating in a radio-frequency field with $\lambda_0 = 2\pi/\omega_0 = 3$ cm.

5.90. A transverse static magnetic field in the form of a circularly polarized wave is created in a magnetic undulator of length L:

$$\boldsymbol{H}(z) = H_0\left(\boldsymbol{e}_x \sin\frac{2\pi}{\lambda_0}z + \boldsymbol{e}_y\cos\frac{2\pi}{\lambda_0}z\right) \ , \quad \lambda_0 \ll L \ .$$

A relativistic electron with starting coordinate and velocity values $x = a$, $y = z = 0$, $\dot{x} = 0$, $\dot{y} = -\beta_\perp c$, and $\dot{z} = \beta_\| c$, where $\beta_\perp = \lambda_0\omega_{\mathrm{H}}/2\pi c\gamma \ll 1$, but $\beta_\perp \gg \gamma^{-1}$, $\beta_\|^2 = \beta^2 - \beta_\perp^2 \approx 1$, $\omega_H = eH_0/mc$, $a = \beta_\perp c/\omega_0$, and

$\omega_0 = 2\pi\beta_{\|}c/\lambda_0$, enters the undulator at moment $t = 0$. Calculate the spectral distribution of the total energy emitted by the electron. Compare the result with the synchrotron radiation spectrum obtained in Problems 5.84 and 5.86.

5.91*. N electrons are simultaneously present on a circular orbit (see Problem 5.79*). Study the influence of interference between the fields created by these electrons on the radiation intensity of the nth Fourier harmonic. Consider specific cases: (i) the totally haphazard electron distribution over the orbit and (ii) the orderly distribution of electrons spaced $2\pi/N$ from one another.

5.92. Consider in the preceding problem emission of Fourier harmonics by an electron bunch of a smaller size than the radius of the orbit. Perform calculations for the two functions of the particle distribution inside the bunch: (i) the uniform distribution within the sector with angular size φ,

$$f(\psi) = \begin{cases} 1/\varphi\,, & -\varphi/2 \le \psi \le \varphi/2\,; \\ 0\,, & |\psi| > \varphi/2\,; \end{cases}$$

(ii) the Gaussian distribution

$$f(\psi) = \frac{1}{\varphi\sqrt{\pi}} \exp\left(-\frac{\psi^2}{\varphi^2}\right), \quad \varphi \ll 2\pi\,.$$

Calculate the radiation power of the coherent and incoherent bunches.

5.93. A nonrelativistic particle with charge q and mass m undergoes a head-on collision (impact parameter $\rho = 0$) with the scattering center; this interaction is described by the potential energy $U(r)$. The particle is reflected at a distance $r_{\min}$ from the center. Express the energy $\mathcal{E}^{\text{rad}}$ of electromagnetic radiation by the particle through the potential energy $U(r)$ and the total nonrelativistic energy $\mathcal{E}$ of the particle.

5.94. Calculate $\mathcal{E}^{\text{rad}}$ (see the previous problem) for the case of Coulomb repulsion: $U(r) = Zeq/r$, $eq > 0$. What part of the particle's energy is spent on emission?

5.95. Two particles with charges e_1 and e_2 and masses m_1 and m_2 ($e_1/m_1 \neq e_2/m_2$) perform elliptical motion. Find the total time-averaged intensity of radiation $\overline{I}$.

5.96. Find the average loss of momentum $\overline{\mathrm{d}\boldsymbol{K}}/\mathrm{d}t$ for a period by a system of two particles orbiting in an ellipse (see the preceding problem).

Hint: The general formula for the loss of momentum was obtained in Problem 5.5.

5.97*. Find the differential effective radiation $\mathrm{d}\kappa_n/\mathrm{d}\Omega$ associated with the scattering of a flow of particles with charges e_1, masses m_1, and velocities v_0 from a charged particle having charge e_2 and mass m_2. The charges e_1 and e_2 have the same signs.

Hint: When calculating integrals A and B in formula (5.83) pass from the integration over $\mathrm{d}t$ to integration over $\mathrm{d}r$, $\mathrm{d}t = \mathrm{d}r/\dot{r}$, where $\dot{r} = \sqrt{1 - 2a/r - s^2/r^2}$,

s is the impact parameter, and $2a$ is the minimal distance at which the particles can approach each other (it is reached at $s = 0$). Integrate first over ds, then over dr. Use the equation for the trajectory of relative motion (Problem 4.66) to calculate B.

5.98*. A particle having charge e_1 and mass m collides with another particle having a mass significantly greater than m and with charge e_2; the impact parameter is s. The kinetic energy of the projectile particle is high in comparison with the potential interaction energy of the particles, $e_1 e_2/r$. For this reason, the velocity v of the projectile particle may be regarded as constant during the entire collision event; it is not necessarily low compared with the speed of light. Find the angular distribution of the total radiation $d\Delta W_{\boldsymbol{n}}/d\Omega$. Consider, in particular, the case of $\beta = v/c \ll 1$.

Hint: Make use of the general formula for the angular distribution of total radiation (5.60). Express the particle's acceleration $\dot{\boldsymbol{v}}$ through the Coulomb force acting on it and the velocity $\boldsymbol{v}$ using the formula $\boldsymbol{v} = c^2 \boldsymbol{p}/\mathcal{E}$ and $\dot{\boldsymbol{p}} = e_1 e_2 \boldsymbol{r}/r^3$.

5.99. Calculate the total radiation ΔW and momentum $\Delta \boldsymbol{p}$ of the particle considered in the previous problem during the entire period of its motion. Perform calculations either directly, that is, by integrating the angular distribution found in the preceding problem, or by using the formulas obtained in Problems (5.68) and (5.71).

5.100*. A particle with charge e_1 and mass m collides with a heavy particle with charge e_2. The impact parameter s is high so that the kinetic energy of the particle during the entire period of its motion is greater than its potential energy. The velocity of the particle is $v \ll c$. Find the particle's bremsstrahlung spectrum $d\Delta W_\omega/d\omega$.

Use the formula

$$\int_0^\infty \frac{\cos px\, dx}{(q^2 + x^2)^{s+1}} = \sqrt{\pi} \left(\frac{p}{2q} \right)^{s+1/2} \frac{K_{s+1/2}(pq)}{\Gamma(s+1)} .$$

5.101*. A flow of particles having charges e_1 and masses m_1 is scattered from a particle with charge e_2 and mass m_2 ($e_1/m_1 = e_2/m_2$). Express the differential effective radiation $d\kappa_{\boldsymbol{n}}/d\omega$ through the components $Q_{\alpha\beta}$ of the system's quadrupole moment. Represent the result in a form analogous to (5.83) and (5.84).

5.102*. Find the total effective radiation κ associated with the scattering of a flow of charged particles (with charge e, mass m, and velocity v_0) from a particle with the identical characteristics.

5.103. A flow of particles with charge e and velocity $v \ll c$ is scattered from an absolutely hard sphere of radius a. Find the effective radiation $d\kappa_\omega$ in the frequency range $d\omega$. What is the level of total effective radiation κ? What is the differential cross-section $d\sigma/d\hbar\omega$ of photon generation with frequency ω per photon unit energy $\hbar\omega$?

5.104*. Solve the preceding problem for the scattering of relativistic particles.

5.105. Find the intensity of Fourier harmonic emission at frequencies $\omega_n = n\omega_0$, multiples of the fundamental frequency ω_0, for the nonrelativistic elliptical motion of two charged particles (see Problem 5.95).

Hint: The relative coordinates of particles performing elliptical motion can be represented in the parametric form

$$x = a(\cos u - \epsilon)\,, \quad y = a\sqrt{1-\epsilon^2}\sin u\,, \quad \omega_0 t = u - \epsilon \sin u\,,$$

here u $(0 \le u \le 2\pi)$ is the parameter, $\omega_0 = (2|\mathcal{E}|)^{3/2}/e_1^2 e_2^2 \mu^{1/2}$ is the circulation frequency of a particle orbiting an ellipse, $\epsilon = \sqrt{1 - 2|\mathcal{E}|L^2/e_1^2 e_2^2 \mu}$ is its eccentricity, $a = |e_1 e_2|/2|\mathcal{E}|$ is the large semiaxis, $\mathcal{E} < 0$ is the total nonrelativistic energy of the system (coupling energy), L is the angular momentum, and μ is the reduced mass.

5.106. The dipole moment of a small ("point-like") charge system located at the origin of the coordinates changes instantaneously from $\boldsymbol{p}_1$ to $\boldsymbol{p}_2 = \boldsymbol{p}_1 + \Delta\boldsymbol{p}$. Calculate the electromagnetic field in the entire space and the spectral radiation density in a given direction $\mathrm{d}^2 I_\omega/\mathrm{d}\omega\mathrm{d}\Omega$.

Hint: Write down the dipole moment density (electric polarization vector) in the form

$$\boldsymbol{P}(\boldsymbol{r}, t) = [\boldsymbol{p}_1 \Theta(-t) + \boldsymbol{p}_2 \Theta(t)]\delta(\boldsymbol{r})\,,$$

where $\Theta(t)$ is the step function, and use formulas (5.41) and the Hertz electric vector to calculate the field.

5.107. A positive point charge (i) moved uniformly with velocity v along the Oz axis and instantaneously stopped at $t = 0$ at the origin of the coordinates, and (ii) was at rest at the origin of the coordinates and suddenly gained constant velocity v at $t = 0$. Draw a picture of the electric field for the moment $t > 0$. How is the magnetic field distributed in space?

5.108. The beta decay of atomic nuclei results in a change of their energy and charge number with the emission of an electron (positron) and a neutrino: $N_Z \to N_{Z\pm1} + e^{\mp} + \nu$. The emerging electron may have relativistic energy. Consider this process as a jump start of a charged particle with constant velocity and calculate the quantities $\mathrm{d}^2 I_\omega/\mathrm{d}\omega\mathrm{d}\Omega$ and $\mathrm{d}I_\omega/\mathrm{d}\omega$ as well as the number of photons N_ω with a given frequency per unit interval of the quantum energy.

5.109*. A beta electron (see the preceding problem) has not only a charge but also has a spin magnetic moment (in the electron's rest system, it is a Bohr magneton μ_{B}, see Section 4.2). Clarify the results of the preceding problem by taking into account the sudden appearance not only of the electron's charge but also of its magnetic moment. Bearing in mind the relativistic motion of the electron, one should consider the moment transformation rule too (see Problem 4.37•). Calculate quantities the $\mathrm{d}^2 I_\omega/\mathrm{d}\omega\mathrm{d}\Omega$ and $\mathrm{d}I_\omega/\mathrm{d}\omega$ assuming the magnetic moment in the electron's rest system is directed along the velocity (longitudinally polarized electrons) and regarding it as a classical quantity.

5.110. An excited atomic nucleus can transfer the excitation energy to an orbital electron and thereby ionize the atom. Find the number of quanta N_ω with a given frequency per unit energy interval on the assumption that the conversion process is instantaneous and the conversion electron is free and nonrelativistic.

5.111*. A nucleus with charge Ze captures orbital electrons and undergoes transformation into a different nucleus with charge $(Z-1)e$: $N_Z \to N_{Z-1} + \nu$. The difference in energy between nuclear level energies is transferred to a neutrino. Assume that an electron moves in the atom in a circular orbit of radius a with frequency ω_0 and find the number of quanta N_ω with a given frequency per unit energy arising from the sudden disappearance of the electron together with its energy and magnetic moment.

Hint: The electron may be captured from any point of its orbit. The electron spin has an irregular direction. It dictates the necessity of averaging over the initial stages of the electron's motion and the directions of its spin.

5.112. A pion decays into two particles, a muon and a muon neutrino, $\pi^\pm \to \mu^\pm + \nu_\mu$. The kinetic energy of particle formation in the pion's rest system $T = (m_\pi - m_\mu)c^2 \approx 34$ MeV. Find the number of quanta of a given frequency per unit energy. Determine the maximally possible quantum energy $\hbar\omega_{\max}$ from the laws of conservation (on the assumption of zero mass for the neutrino).

5.113. A kaon decays as $K^+ \to \pi_+ + \pi_+ + \pi^-$, with pions being nonrelativistic particles. Compute the distribution of radiation intensity by frequencies and angles in the kaon's rest system. Also, calculate the distribution of soft quanta by frequencies regardless of the exit angles.

5.4 Interaction of Charged Particles with Radiation

5.4.1 Interaction of a Charged Particle with its Own Electromagnetic Field

It was mentioned in Section 2.1 that difficulty is encountered in the determination of the proper electrostatic energy of a charged particle. For a point particle, this energy is infinite. The introduction of the "classical" radius of the particle (see Problem 3.120•)

$$r_0 = \frac{e^2}{mc^2} \tag{5.94}$$

and the ensuing assumption of its having an internal structure permit us to make the proper energy finite. This, however, leads to controversy with the theory of relativity (see Problems 5.114*–5.117*). Moreover, an object of size r_0 is not described by the classical theory: the quantum effects become significant at much greater distances equal to the Compton wavelength:

$$\Lambda = \frac{\hbar}{mc} \approx 137 r_0 \, . \tag{5.95}$$

Unfortunately, the quantum theory does not resolve the difficulties encountered in the computation of the proper energy. Other infinite quantities, such as an addition to the particle's charge, appear, besides the infinite proper energy (see Berestetskii *et al.*, 1982; Akhiezer and Berestetskii, 1981; Peskin and Schroeder, 1995; Weinberg, 2000).

An accelerated motion of a particle is accompanied by its additional interaction with the intrinsic field. The accelerated motion gives rise to the emission of electromagnetic waves, which leads to the loss of energy and momentum by the particle. This means that the particle's motion in itself depends on the electromagnetic waves it emits. Therefore, the correct statement of the problem of charged particle motion requires that the terms taking into account the influence of radiation on the motion be included in the equation.

However, in many cases this influence is small, which allows the motion of an emitting particle to be given. The quantitative criterion for the smallness of radiation reaction can be obtained by comparing the energy losses due to radiation for a certain time Δt with variations of the particle's kinetic energy under the action of external forces during the same time. Let us estimate the two energies in a frame of reference in which a particle performs nonrelativistic motion within the time period of interest. The Larmor formula (5.32) gives

$$\Delta \mathcal{E}^{\text{rad}} \approx \frac{2e^2 \dot{\boldsymbol{v}}^2 \Delta t}{3c^3} \approx \frac{2e^2 \dot{\boldsymbol{v}} \Delta \boldsymbol{v}}{3c^3} \, ,$$

where $\Delta \boldsymbol{v}$ is the change of velocity for time Δt. On the other hand, $\Delta \mathcal{E}^{\text{kin}} \approx m \Delta \boldsymbol{v} \cdot \boldsymbol{v}$. The inequality $\Delta \mathcal{E}^{\text{rad}} \ll \Delta \mathcal{E}^{\text{kin}}$ gives

$$\Delta t \approx \left| \frac{\boldsymbol{v}}{\dot{\boldsymbol{v}}} \right| \gg \frac{2e^2}{3mc^3} = \frac{2r_0}{3c} = \tau_0 \, , \tag{5.96}$$

where τ_0 is the time during which electromagnetic perturbations propagate over a distance on the order of the particle's classical radius. Time τ_0 is greatest for the lightest elementary particle, the electron, $\tau_e = 0.63 \times 10^{-23}$ s.

In the case of quasi-periodic motion of a particle, its average velocity and acceleration return to the initial values after every period $T_0 = 2\pi/\omega_0$, and the loss of energy by radiation for the period should be compared with the mean kinetic energy $\Delta \mathcal{E}^{\text{rad}} \approx (2e^2 l^2 \omega_0^4 / 3c^3) T_0$, $\overline{\mathcal{E}}^{\text{kin}} \approx m \omega_0^2 l^2$, where l is the size of the region in which the particle propagates. The comparison yields $T_0 \gg \tau_0$ or $\omega_0 \tau_0 \ll 1$, that is, time Δt in inequalities (5.96) should be replaced by period T_0.

Thus, the radiation reaction in the case of nonrelativistic motion may be regarded as a small effect provided the particle's motion is smooth and its state changes insignificantly for time τ and at distances $c\tau \approx r_0$. This equation is practically always fulfilled in the case of relativistic motion. In ultrarelativistic motion (see, e.g., Problem 5.138*), the radiation damping force may become the main factor governing the particle's motion.

5.4.2
Renormalization of Mass. The Radiation Damping Force in the Relativistic Case

Let us turn now to the derivation of the explicit and relativistic covariant expression for the radiation damping force (Baier *et al.*, 1973; Sokolov and Ternov, 1986; Gal'tsov *et al.*, 1991). This issue is closely related to the relativistically invariant representation of the particle's intrinsic electromagnetic mass.

On the basis of the covariant equation of motion (4.53) for a point-like particle with charge e and certain mass m_0, we arrive at

$$m_0 \frac{\mathrm{d}^2 x_i}{\mathrm{d}\tau^2} = \frac{e}{c} F_{ik} u^k + \mathcal{F}_i \,, \tag{5.97}$$

where the derivative is taken with respect to invariant eigentime τ. The right-hand side contains, besides the external electromagnetic force (the first term), force $\mathcal{F}_i$ with which the proper electromagnetic field created by the particle acts on it. We write this force through the 4-potential $\mathcal{A}_i$ of the particle's proper field in the same form as the external force:

$$\mathcal{F}_i = \frac{e}{c} \frac{\mathrm{d}x^k}{\mathrm{d}\tau} \left(\frac{\partial \mathcal{A}_k}{\partial x^i} - \frac{\partial \mathcal{A}_i}{\partial x^k} \right) . \tag{5.98}$$

The potential $\mathcal{A}_i$ satisfies (5.1), which can be written in the four-dimensional form:

$$\Box \mathcal{A}_i(x) = -\frac{4\pi}{c} j_i(x) = -4\pi e \int \mathrm{d}\tau' \frac{\mathrm{d}x_i(\tau')}{\mathrm{d}\tau'} \delta^4(x - x(\tau')) . \tag{5.99}$$

Here, x represents the totality of four space–time moving coordinates and $x(\tau') \equiv x'$ represents the four-dimensional coordinates of the particle in its eigentime function. Representation (4.110) is used for the current density. The solution of (5.99) can be written in the form (5.4) through the retarded Green's function:

$$\mathcal{A}_i(x) = \frac{1}{c^2} \int G^{\mathrm{R}}(x-y) j_i(y) \mathrm{d}^4 y = \frac{e}{c} \int \mathrm{d}\tau' \frac{\mathrm{d}x_i(\tau')}{\mathrm{d}\tau'} G^{\mathrm{R}}(x - x(\tau')) . \tag{5.100}$$

The last equation allows us to find the force acting on the particle from its own field:

$$\mathcal{F}_i(x) = \frac{e^2}{c^2} \int \frac{\mathrm{d}x^k(\tau)}{\mathrm{d}\tau} \times \left(\frac{\mathrm{d}x_k(\tau')}{\mathrm{d}\tau'} \frac{\partial}{\partial x^i} - \frac{\mathrm{d}x_i(\tau')}{\mathrm{d}\tau'} \frac{\partial}{\partial x^i} \right) G^{\mathrm{R}}(x(\tau) - x(\tau')) \mathrm{d}\tau' . \tag{5.101}$$

Here, x, entering the argument of force $\mathcal{F}_i$, has the sense of the particle's four-dimensional coordinates considered as the functions of its eigentime.

The next important step in the calculation of the self-action force relates to the representation of the retarded Green's function as the sum of two terms and their interpretation (see Problems 5.3 and 5.9):

$$G^{\mathrm{R}}(x - x') = G^{+}(x - x') - \frac{1}{2} G^{-}(x - x') , \quad \text{where}$$
$$G^{+}(x - x') = c\delta(s^2) , \tag{5.102}$$

$$G^{-}(x-x') = 2c\left[\Theta\left(x_0'-x_0\right)-\Theta\left(x_0-x_0'\right)\right]\delta(s^2)$$
$$= -2c\frac{x_0-x_0'}{\left|x_0-x_0'\right|}\delta(s^2)\,, \quad s^2 = \left(x_k-x_k'\right)\left(x^k-x'^k\right)\,. \quad (5.103)$$

According to the results obtained in Problem 5.9, function G^- is non-singular at $R \to 0$ and satisfies the homogeneous d'Alembert equation, whereas the field associated with this function can be interpreted as the particle's radiation field. The corresponding fraction of the self-action field represents the radiation reaction. Function G^+ satisfies the equation with a delta-shaped source; it is responsible for the quasi-stationary field connected with the particle and for creating its electromagnetic mass. These two functions depend on the differences $x_k(\tau) - x_k(\tau') = s_k$. In the case of fixed t',

$$\frac{\partial}{\partial x^k} = \frac{\partial}{\partial s^k} = 2\left(x_k - x_k'\right)\frac{\partial}{\partial(s^2)}\,. \quad (5.104)$$

Because τ and τ' are different moments of the particle's eigentime, they must be similar, and their difference even for an extended particle is on the order of τ_0 defined in accordance with (5.96). Evidently, $\tau_0 \to 0$ for a point-like particle. It is therefore desirable to expand all quantities under integral (5.101) over the small difference $\tau' - \tau = \nu$ and pass to the integration over ν:

$$x_k - x_k' \approx -\nu\frac{\mathrm{d}x_k}{\mathrm{d}\tau} - \frac{\nu^2}{2}\frac{\mathrm{d}^2x_k}{\mathrm{d}\tau^2} - \frac{\nu^3}{6}\frac{\mathrm{d}^3x_k}{\mathrm{d}\tau^3} + \mathcal{O}(\nu^4)\,, \quad s^2 \approx c^2\nu^2 + \mathcal{O}(\nu^4)\,. \quad (5.105)$$

These expansions lead to

$$G^{\mathrm{R}}(x-x') = \left(1-\frac{\nu}{|\nu|}\right)\frac{\delta(\nu)}{s|\nu|}\,,$$
$$\frac{\partial}{\partial x^k} = -\frac{1}{c^2}\left(\frac{\mathrm{d}x_k}{\mathrm{d}\tau} + \frac{\nu}{2}\frac{\mathrm{d}^2x_k}{\mathrm{d}\tau^2} + \frac{\nu^2}{6}\frac{\mathrm{d}^3x_k}{\mathrm{d}\tau^3}\right)\frac{\mathrm{d}}{\mathrm{d}\nu}\,. \quad (5.106)$$

Single integration by parts is used to calculate the self-action force in formula (5.101):

$$\mathcal{F}_i = -\left\{\frac{e^2}{2c^2}\int\frac{\delta(\nu)}{|\nu|}\mathrm{d}\nu\right\}\frac{\mathrm{d}^2x_i}{\mathrm{d}\tau^2} + \frac{2e^2}{3c^3}\left(\frac{\mathrm{d}^3x_i}{\mathrm{d}\tau^3} - \frac{1}{c^2}\frac{\mathrm{d}x_i}{\mathrm{d}\tau}\frac{\mathrm{d}x^k}{\mathrm{d}\tau}\frac{\mathrm{d}^3x_k}{\mathrm{d}\tau^3}\right)\,. \quad (5.107)$$

The first negative term containing the divergent integral can be interpreted as part of the inertial force containing the particle's infinite electromagnetic mass:

$$\Delta m = \frac{e^2}{2c^2}\int\frac{\delta(\nu)}{|\nu|}\mathrm{d}\nu\,. \quad (5.108)$$

Taken together with mass m_0 of a "naked" (not interacting with the radiation field) particle, it makes up the total experimentally observable mass $m = m_0 + \Delta m$ of the particle. It is in this sense that the analogous divergent quantity is interpreted in quantum electrodynamics, and the procedure for its inclusion in the observed mass is called "renormalization of mass." Such an interpretation is possible because the "naked" mass is an unobservable quantity to which an infinite negative constituent can be ascribed. However, the very appearance of a divergent quantity provides sure evidence for imperfection of the theory.

The second term in (5.108) is the particle's decelerating force caused by the loss of momentum due to radiation:

$$\mathcal{F}_i^{\text{rad}} = \frac{2e^2}{3c^3}\left(\frac{\mathrm{d}^3 x_i}{\mathrm{d}\tau^3} - \frac{1}{c^2}\frac{\mathrm{d}x_i}{\mathrm{d}\tau}\frac{\mathrm{d}x^k}{\mathrm{d}\tau}\frac{\mathrm{d}^3 x_k}{\mathrm{d}\tau^3}\right) . \tag{5.109}$$

The equation for the particle's motion taking account of the radiation force (*Dirac–Lorentz equation*) is written in the form

$$m\frac{\mathrm{d}^2 x_i}{\mathrm{d}\tau^2} = \frac{e}{c}F_{ik}\frac{\mathrm{d}x^k}{\mathrm{d}\tau} + \frac{2e^2}{3c^3}\left(\frac{\mathrm{d}^3 x_i}{\mathrm{d}\tau^3} - \frac{1}{c^2}\frac{\mathrm{d}x_i}{\mathrm{d}\tau}\frac{\mathrm{d}x^k}{\mathrm{d}\tau}\frac{\mathrm{d}^3 x_k}{\mathrm{d}\tau^3}\right) , \tag{5.110}$$

where m is the experimentally observable finite mass of the particle. The Lorentz radiation friction force can be written in a more compact form after differentiation of the identity $u^k w_k = 0$, where u^k and w^k are the four-dimensional velocity and acceleration respectively, relative to the eigentime:

$$\mathcal{F}_i^{\text{rad}} = \frac{2e^2}{3c^3}\left[\frac{\mathrm{d}w_i}{\mathrm{d}\tau} + \frac{1}{c^2}u_i\left(w_k w^k\right)\right] . \tag{5.111}$$

Example 5.16

Study the role of individual terms on the right-hand side of (5.110). Consider to this effect a situation in which a particle is subject to the external field F_{ik} for a finite time and calculate the change Δp_i of its 4-momentum during the period of field action. Find the relationship between the radiation friction force and the loss of the particle's energy and momentum due to radiation.

Solution. Let the particle propagate without acceleration at $\tau \le \tau_1$ and $\tau \ge \tau_2$ Integration of (5.110) over the $[\tau_1, \tau_2]$ interval yields $\Delta p_i = (\Delta p_i)^{\text{ext}} = (\Delta p_i)^{\text{rad}}$, where the first term

$$(\Delta p_i)^{\text{ext}} = \int_1^2 \frac{e}{c}F_{ik}\mathrm{d}x^k$$

is the change of the 4-momentum with respect to the external field. The second term is written using the formula (5.111) for the radiation force:

$$(\Delta p_i)^{\text{rad}} = \int_1^2 \mathcal{F}_i \mathrm{d}\tau = \frac{2e^2}{3c^3}w_i\bigg|_1^2 + \frac{2e^2}{3c^5}\int_1^2 w_k w^k u_i \mathrm{d}\tau .$$

The first term on the right-hand side is zero owing to the absence of acceleration. The remaining integral coincides with the result of integration of quantity (5.73) and therefore represents variations of the particle's 4-momentum due to radiation. □

Example 5.17

Write down the radiation friction force in a nonrelativistic case. What difficulties are encountered in the study of particle motion after inclusion of the radiation force in the equation of motion?

Solution. In the nonrelativistic approximation $\mathrm{d}/\mathrm{d}\tau = \mathrm{d}/\mathrm{d}t$ and $\gamma = 1$. Use formula (5.111) to find, in accordance with the general relation (4.60) connecting the four-dimensional and three-dimensional forces, $\mathcal{F}_i = (\boldsymbol{F}\cdot\boldsymbol{v}/c, \boldsymbol{F})$, where

$$\boldsymbol{F} = \frac{2e^2}{3c^3}\ddot{\boldsymbol{v}} \tag{5.112}$$

is the nonrelativistic three-dimensional radiation friction force.

Write down the equation for the particle's motion in the absence of external fields but taking into consideration the radiation force:

$$m\dot{\boldsymbol{v}} = \frac{2e^2}{3c^3}\ddot{\boldsymbol{v}}\,. \tag{5.113}$$

An important feature of this equation (and the much more complicated (5.110)) is the higher (third) order of time derivatives of the coordinates in comparison with the ordinary equations of motion in relativistic (and nonrelativistic) classical mechanics. Integration of (5.113) yields its general solution

$$\boldsymbol{v}(t) = \boldsymbol{a} + \boldsymbol{b}e^{t/\tau_0}\,,$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are integration constants and τ_0 is defined in accordance with (5.96). Pay attention to the fact that the decay constant is positive and the second term is the "self-accelerating" solution, that is, the velocity builds up in the absence of the accelerating force. It should be remembered, however, that equation (5.113) is of the second order with respect to velocity and therefore requires that two initial conditions, velocity and acceleration, be specified. In the absence of external forces, acceleration is zero; hence, $\boldsymbol{v}(0) = \boldsymbol{v}_0$, $\dot{\boldsymbol{v}}(0) = 0$. Determining the integration constants from these conditions yields $\boldsymbol{b} = 0$ and the correct (for this simplest case) solution $\boldsymbol{v}(t) = \boldsymbol{v}_0 = \text{const}$. □

Example 5.18

Express the radiation friction force through the velocity of the particle and the external electromagnetic field. Use for this purpose the expression for the particle's motion disregarding radiation and assuming the radiation force to be insignificant. Consider in detail cases of nonrelativistic and ultrarelativistic motion.

Solution. Find from the equation of motion (4.53)

$$w_i = \frac{\mathrm{d}u_i}{\mathrm{d}\tau} = \frac{e}{mc} F_{ik} u^k , \qquad \frac{\mathrm{d}^2 u_i}{\mathrm{d}\tau^2} = \frac{e}{mc}\frac{\partial F_{ik}}{\partial x^l} u^k u^l + \frac{e^2}{(mc)^2} F_{ik} F^{kl} u_l$$

and use these results to obtain from (5.111) the radiation force expressed through the external field:

$$\mathcal{F}_i = \frac{2e^3}{3mc^4}\frac{\partial F_{ik}}{\partial x^l} u^k u^l + \frac{2e^4}{3m^2c^5}(F_{ks}u^s)(F^{kl}u_l)u_i \,. \tag{5.114}$$

Substitute $u^k = (c, \boldsymbol{v})$ into (5.114) in the nonrelativistic limit and use tables (4.68) to obtain the three-dimensional radiation force

$$\boldsymbol{F} = \begin{cases} \frac{2e^3}{3mc^3}\dot{\boldsymbol{E}} + \frac{2e^4}{3m^2c^4}\boldsymbol{E}\times\boldsymbol{H} & \text{at } E \gg vH/c\,; \\ \frac{2e^3}{3mc^4}\boldsymbol{v}\times\dot{\boldsymbol{H}} + \frac{2e^4}{3m^2c^5}\boldsymbol{H}\times(\boldsymbol{v}\times\boldsymbol{H}) & \text{at } E \ll vH/c\,. \end{cases} \tag{5.115}$$

Here

$$\dot{\boldsymbol{E}} = \frac{\partial \boldsymbol{E}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{E}$$

and $\dot{\boldsymbol{H}}$ is written in a similar form. The smallness of the radiation force compared with the Lorentz force leads in both cases to the limitation of the external field

$$H \ll \frac{|e|}{r_0^2} \,. \tag{5.116}$$

However, this limitation is of no practical importance because quantum corrections become significant at lower field values.

In the ultrarelativistic case $u^k = \gamma(c, \boldsymbol{v})$, $\gamma \gg 1$, this implies the necessity of taking into account in (5.114) only the term containing the product of three 4-velocities. Find from (5.114) with the help of (4.68) the three-dimensional force

$$\boldsymbol{F} = -\frac{2e^4\gamma^2}{3m^2c^7}\{(\boldsymbol{E}\cdot\boldsymbol{v})^2 + (c\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{H})^2\}\boldsymbol{v} \,, \tag{5.117}$$

which is directed oppositely to the particle's velocity. Here, everywhere except γ, $|\boldsymbol{v}| = c$. The choice of the Oz axis along $\boldsymbol{v}$ leads to a simpler expression for the radiation force:

$$F_z = -\frac{2e^4\gamma^2}{3m^2c^4}\{(E_x - H_y)^2 + (E_y + H_x)^2\} \,. \tag{5.118}$$

It follows from (5.117) and (5.118) that the force is proportional to the square of the particle' energy and the squared external field. □

5.4.3
Scattering of Electromagnetic Waves by Particles

Apart from interaction with the self-radiation field, charged particles can interact with electromagnetic waves generated by external sources. This results in the accelerated motion of the particles and the appearance of secondary (scattered) radiation. The process of scattering is characterized by differential

$$d\sigma_s = \frac{dI(\theta, \alpha)}{\overline{\gamma_0}} \quad \text{and total} \quad \sigma_s = \int d\sigma_s \tag{5.119}$$

cross-sections. Here $dI(\theta, \alpha) = \overline{\gamma} r^2 d\Omega$ is the time-averaged intensity of emission into the solid angle $d\Omega$, and $\overline{\gamma}$ and $\overline{\gamma_0}$ are the mean energy flux densities in the scattered and incident waves defined via the Poynting vector.

The field oscillator method is an efficacious approach to the investigation of interaction between particles and electromagnetic waves; this method is related to the Fourier expansion, the essence of which was expounded at the end of Section 2.3. The application of this method to more complicated cases is described in Problems 5.143 and 5.149*–5.155.

Suggested literature:
Landau and Lifshitz (1975); Jackson (1999); Ginzburg (1979a); Frenkel (1926); Baier *et al.* (1973); Bredov *et al.* (2003); Fleishman (2008); Fleishman and Topty-gin (2013)

Problems

5.114*. Find the momentum of the particle's electromagnetic field with charge e moving uniformly with velocity v. Consider the particle in its rest system S' as a small hard ball of radius r_0 (in a system where the particle's velocity equals v the Lorentz contraction occurs). Introduce the particle's electromagnetic rest mass m_0 related to its field energy at rest by the Einstein relation. What difficulties can be anticipated?

5.115. Find the energy W_m of a magnetic field and the total electromagnetic energy W of the particle considered in the preceding problem.

5.116. Find the force $\boldsymbol{F}$ with which a charged spherically symmetric particle acts on itself (the interaction force) during accelerated translational movement at low speed $v \ll c$. Disregard the delay and the Lorentz contraction.

Hint: Calculate the resultant of the forces applied to the minor elements de of the particle's charge by making use of the expression for the field strength of the point charge.

5.117*. Find the corrected expression for the self-action force $\boldsymbol{F}$ of a charged spherically symmetric particle (see the preceding problem). For the solution, take account of the effect of the finite propagation velocity of interaction up to the first order in time $t' - t$ of the propagation of interaction between the particle's elements. Consider, in particular, the limiting case of a point particle. Estimate the contribution of the disregarded terms of higher order in $t' - t$ in this limiting case.

5.118•*. It was shown in Problems 5.114*–5.117* that the "naïve" definition of the proper energy and momentum of a charged particle having finite dimensions (the Abraham–Lorentz model, see formulas (1) and (2) from the solution of Problem 5.114*) leads to the wrong relationship between them. Introduce the covariant definition of the same quantities using the energy–momentum tensor of the electromagnetic field (4.126) for the purpose and integrate over a certain space-like hypersurface chosen such that the conditions

$$\mathcal{E}_0 = \frac{1}{8\pi c}\int E'^2 \mathrm{d}V' , \quad \boldsymbol{p}_0 = 0$$

are fulfilled in the particle's rest system.

Show that in this case the particle's energy and momentum make up the 4-vector with components $p^i = (\gamma mc, \gamma m\boldsymbol{v})$, where

$$m = \frac{\gamma}{8\pi c}\int (E^2 - H^2)\mathrm{d}^3x = \frac{1}{8\pi c}\int E'^2 \mathrm{d}V'$$

is the invariant mass. Integration in the first integral is performed over the three-dimensional volume in an arbitrary inertial reference frame and in the second integral in the particle's rest system.

5.119•. For what time T would the Rutherford hydrogen atom survive if its electron moved and emitted as a classical particle? Assume that the electron losing energy moves toward the proton along a gently sloping helix so that at each moment it emits as a charge on the circular orbit the radius of which slowly varies with time. Under what condition does this assumption hold? The initial atomic radius is $a = 0.5 \times 10^{-8}$ cm.

5.120. A relativistic particle with charge e and mass m moves along a circular orbit in a constant uniform magnetic field $\boldsymbol{H}$ and losses energy by radiation. Find the law for the change of the energy and the orbital radius in time, $\mathcal{E}(t)$ and $r(t)$, respectively. The energy of the particle at the initial moment $t = 0$ is $\mathcal{E}_0$ (cf. Problem 5.89*).

5.121. An electron is accelerated in a betatron by a vortex electric field along the orbit of a constant radius a. The vortex electric field is induced by a variable magnetic field having frequency ω. Find the critical value of the electron energy $\mathcal{E}_\mathrm{c}$ at which its losses by radiation are equivalent to the energy acquired by the electron due to the work of the vortex electric field.

5.122*. A particle with charge e and mass m is attracted to a certain center by the quasi-elastic force $-m\omega_0^2\boldsymbol{r}$. Free oscillations develop in this harmonic oscillator at a certain moment of time $t = 0$. Find the law of damping of these oscillations, taking into account the radiation reaction but assuming it to be small. Determine the shape of the spectrum of such an oscillator and the width of the spectral line ("natural width"). How is the uncertainty of the energy $\hbar\omega$ of emitted photons related to the oscillator's lifetime?

5.123•. A gas consists of atoms of mass m. A stationary atom of this gas emits light with frequency ω_0 (the natural width of the emission line is disregarded). Owing to the thermal atomic motion and the Doppler effect, an observer motionless with respect to a vessel containing the gas will record a frequency other than ω_0. Find the shape of the gas radiation spectrum $\mathrm{d}I_\omega/\mathrm{d}\omega$ if the gas is heated up to temperature T.

Hint: The velocities of gas atoms are distributed in conformity with the Maxwell law:

$$\frac{\mathrm{d}N}{N} = \left(\frac{m}{2\pi T}\right)^{3/2} e^{-mv^2/2T} \mathrm{d}v_x \mathrm{d}v_y \mathrm{d}v_z \ ,$$

where $\mathrm{d}N/N$ is the fraction of molecules whose velocity $\boldsymbol{v}$ falls in the $\mathrm{d}v_x \mathrm{d}v_y \mathrm{d}v_z$ range and T is the absolute temperature expressed in energy units. Because the condition $v \ll c$ is fulfilled, it is possible to drop all terms of order higher than v/c in the formula for the Doppler frequency shift.

5.124. An emitting atom described by the harmonic oscillator model moves in a gas and undergoes collisions with other atoms that change the character of its oscillations in a jump-like mode. The probability that the time of the atom's free motion varies from τ to $\tau + \mathrm{d}\tau$ is expressed by the formula $\mathrm{d}W(\tau) = (\Gamma/2)e^{-\Gamma\tau/2}\mathrm{d}\tau$ (mean time interval between collisions, $\overline{\tau} = 2/\Gamma$). Find the shape of the radiation spectrum $\mathrm{d}I_\omega/\mathrm{d}\omega$ of such an oscillator, disregarding the natural line width.

5.125*. A group of waves characterized by the spectral intensity distribution S_ω and the total intensity $S = \int_0^\infty S_\omega \mathrm{d}\omega$ (S is the amount of energy passing through an area of 1 cm^2 for the entire period of group propagation) is incident on a three-dimensional isotropic oscillator. The spectral distribution width of the group is large compared with the natural width of the spectral line of the oscillator γ. The electron velocity $v \ll c$. Find the energy absorbed by the oscillator from an optical wave taking into consideration the radiation damping force. How do the character of the polarization and the direction of propagation of the waves composing the group affect the result?

5.126. Find the total amount of energy ΔW absorbed by a one-dimensional oscillator with eigenfrequency ω_0 from a group of waves with spectral distribution S_ω in the following three cases: (i) a linearly polarized flat group of waves in which the direction of the vector $\boldsymbol{E}$ oscillations makes angle ϑ with the axis of the oscillator, (ii) a nonpolarized flat group of waves propagating at angle θ to the axis of the oscillator; (iii) an isotropic radiation field (flat waves with any polarization direction and any direction of propagation are incident with equal probability on the oscillator).

5.127*. A linearly polarized wave is incident on an isotropic harmonic oscillator. The electron velocity $v \ll c$. Find the differential $\mathrm{d}\sigma/\mathrm{d}\Omega$ and total σ wave scattering cross-sections taking into account the radiation friction force. Consider, in particular, the cases of a weakly and strongly bound electron.

5.128. A circularly polarized flat electromagnetic wave is scattered by a free charge. Determine the scattered field $\boldsymbol{H}$ and study the character of its polarization. Find the differential $\mathrm{d}\sigma/\mathrm{d}\Omega$ and total σ wave scattering cross-sections.

5.129. A nonpolarized flat wave is scattered by a free charge. Find the degree ρ of depolarization of the scattered wave depending on the scattering angle ϑ.

5.130*. A linearly polarized wave is scattered by a free charge. The charge moves with relativistic velocity $\boldsymbol{v}$ in the direction of wave propagation. Find the differential scattering cross-section. In addition, consider the case of scattering of a nonpolarized wave

Hint: Use formula (5.60) and express $\dot{\boldsymbol{v}}$ through $\boldsymbol{E}$ and $\boldsymbol{H}$.

5.131*. An isotropic harmonic oscillator with frequency ω_0, charge e, and mass m is placed in a free uniform constant magnetic field $\boldsymbol{H}$. Determine the motion of the oscillator. Study the character of the polarization of oscillator radiation.[2)]

5.132. A system of particles comprises N one-dimensional harmonic oscillators, whose equilibrium positions are at points with radius vectors $\boldsymbol{r}_j$, $j = 1, 2, \ldots, N$. Calculate the differential cross-section of a flat monochromatic wave having a small amplitude ($eE_0/m\omega c \ll 1$) for this particle system. Analyze the various relationships between the wavelength and the linear dimension of the region in which these particles exist.

5.133. A small-amplitude electromagnetic wave ($eE_0/m\omega c \ll 1$) is scattered from a free nonrelativistic electron. Calculate the force averaged over the wave period that acts on the electron. Consider various polarizations of the incident wave.

5.134. A particle having charge e moves with velocity $\boldsymbol{v}$ in the field of a flat monochromatic wave propagating in the $\boldsymbol{n}$ direction. The energy density w of the electromagnetic wave is known. Find the force averaged over the wave period that acts on the particle. How does the force express itself in the presence of a non-monochromatic packet of flat waves propagating in one and the same direction?

Hint: Use the force (3) from the solution of the preceding problem in the particle's rest system. Thereafter, convert it (and all the quantities on which it depends) into a system moving with velocity $\boldsymbol{v}$ relative to the initial one. On the basis the results obtained in Problem 3.67, make sure that the Thomson cross-section remains unaltered.

5.135*. An infinite flat surface emits electromagnetic radiation with energy density w and the directional diagram is defined by the formula $\psi(\mu) = 3\mu^2/2\pi$, $\mu = \cos\theta > 0$, where θ is the angle between the normal to the surface and

2) Such a harmonic oscillator provides the model of an atom in an external magnetic field. Thus, it is proposed in this problem to develop the classical theory of the Zeeman effect. See also Problem 6.73 and 6.74.

the direction of radiation. Find the limiting velocity to which an electron can be accelerated in the radiation field. Why can the electron not acquire infinite energy?

5.136*. A hot flat spot of radius a emits radiation with density w_0 near its surface and with an isotopic directional diagram. An electron is located at the axis of symmetry perpendicular to the spot plane. Introduce and study the differential equation describing variations of the relativistic factor γ with distance. Find its numerical solutions. Determine the limiting Lorentz factor up to which acceleration is possible.

5.137. In astronomy, by luminosity L is meant the total energy emitted by a source per unit time. The luminosity is measured in erg per second or in units of solar luminosity, $L_\odot = 3.86 \times 10^{33}$ erg/s. The critical (Eddington)[3] luminosity L_c is the limiting luminosity at which the gravitational force is still able to equilibrate the radiation pressure force on the plasma envelope of a star. At $L > L_c$, the radiation pressure causes the plasma envelope to disperse. Calculate the critical luminosity of a spherical star and express it through the stellar mass on the assumption that the radiation pressure acts on the quasi-neutral hydrogen (electron–proton) plasma.

5.138*. An electron in a rarefied plasma is subjected to the action of a strong ($eE_0/mc\omega \gg 1$) circularly polarized electromagnetic wave in which

$$E_x = E_0 \sin\omega\left(t - \frac{z}{c}\right) , \qquad E_y = E_0 \cos\omega\left(t - \frac{z}{c}\right) ,$$
$$H_x = E_y , \qquad H_y = -E - x .$$

Consider the scattering of the wave by an electron in the stationary regime where its velocity in the direction of wave propagation $v_z = 0$ and the electron moves in a circular orbit lying in the xy plane with the circulation frequency equaling the wave frequency ω. The motion of the electron in the longitudinal direction is hampered by the electric field resulting from plasma polarization. Stationarity of the movement is due to the fact that the entire energy brought in by the wave is scattered from the electron. Show that such a regime of the electron motion actually exists and find the cross-section of scattering of a strong wave by the electron. By means of numerical calculation construct the curve describing the dependence of the total scattering cross-section on the incident wave amplitude in the nonlinear regime.

5.139**. Let a strong circularly polarized electromagnetic wave (see the preceding problem) propagate in the direction of a uniform and constant magnetic field. Study the stationary regime of electron motion; in particular, study in detail the case of resonance when the direction of electron rotation by the wave and the wave frequency coincide with the direction and the circulation frequency of the electron in the magnetic field. Calculate numerically the cross-section of scattering of a strong

3) Arthur Stanley Eddington (1882–1944) was an outstanding English physicist and astronomer. He made numerous discoveries in astrophysics.

wave by the electron in the magnetic field, and construct the resonance curve describing the dependence of the scattering cross-section on the wave frequency and the strength of the longitudinal magnetic field.

5.140**. A free electron is located in the field of a flat monochromatic wave of arbitrary strength. Calculate the intensity of scattered radiation and the scattering cross-section without regard for the radiation reaction. Consider various polarizations of the flat wave incident on the electron.

5.141*. A relativistic particle with charge e and mass m propagates in a constant and uniform electric field $\boldsymbol{E} = \text{const}$ at an arbitrary angle. Calculate the spectral radiation density in a given direction, $\boldsymbol{n} = \boldsymbol{k}/k$, without regard for the radiation reaction.

5.142. A nonrelativistic charged particle propagates with velocity $\boldsymbol{v}_0$ and undergoes the action of the electric field $E(t) = E_0 T_0 \delta(t)$ in the direction of its movement. Analyze the solutions of the particle's equation of motion taking into consideration the decelerating force due to radiation. Is it possible to choose integration constants so as to avoid self-acceleration? Calculate the total energy emitted by the particle.

5.143. Expand the potential $\varphi(\boldsymbol{r})$ and the electric field $\boldsymbol{E}(\boldsymbol{r})$ of an immobile point charge e in flat waves.

5.144. A point charge propagates in a vacuum with velocity $\boldsymbol{v} = \text{const}$. Expand the charge field φ, $\boldsymbol{A}$, $\boldsymbol{E}$, $\boldsymbol{H}$ in flat monochromatic waves.

5.145*. Find the potentials $\varphi(\boldsymbol{r}, t)$ and $\boldsymbol{A}(\boldsymbol{r}, t)$ of the field of a uniformly propagating point charge e (see the answer to Problem 4.31*) using the expansion of these potentials in flat waves obtained in the preceding problem.

Hint: To calculate the integral over $\mathrm{d}^3 k$ change the variables $k_x \to k_x/\sqrt{1 - v^2/c^2}$, $k_y \to k_y$, $k_z \to k_z$ (axis $Ox \parallel v$) and make use of the expansion for the field of a motionless point charge in flat waves (see Problem 5.143).

5.146*. A neutral point system of charges uniformly propagates in a vacuum with velocity $\boldsymbol{v}$. Find the electromagnetic field $\varphi(\boldsymbol{r}, t)$, $\boldsymbol{A}(\boldsymbol{r}, t)$ using the Fourier expansion in flat monochromatic waves for the case of electric $\boldsymbol{p}$ and magnetic $\boldsymbol{m}$ dipole moments stipulated in a laboratory frame of reference.

Hint: The densities of the electric field and current of the system are expressed through the formulas

$$\boldsymbol{j} = c\nabla \times [\boldsymbol{m}\delta(\boldsymbol{r} - \boldsymbol{v}t)] + \frac{\partial}{\partial t}[\boldsymbol{p}\delta(\boldsymbol{r} - \boldsymbol{v}t)],$$

$$\rho = -\nabla \cdot [\boldsymbol{p}\delta(\boldsymbol{r} - \boldsymbol{v}t)] .$$

5.147*. Find the potentials of the field of a uniformly propagating magnetic dipole (the $\boldsymbol{m}_0$ moment in the dipole rest system). The velocity of the dipole is $\boldsymbol{v}$. Confine yourself to the following specific cases: (i) when $\boldsymbol{m}_0 \parallel \boldsymbol{v}$ and (ii) when $\boldsymbol{m}_0 \perp \boldsymbol{v}$. Make use of the formulas for moment transformation derived in Problem 4.37$^{\bullet}$.

5.148*. Find the field of a uniformly propagating electric dipole (the $\boldsymbol{p}_0$ moment in the dipole rest system) based on the results obtained in Problem 5.146* (see the answer to Problem 4.36•).

5.149*. The electromagnetic field of radiation is described by the oscillatory coordinates $q_{k\lambda}$ (see formula (2.144)). Write the differential equations describing the interaction of the radiation field in variables $q_{k\lambda}$ with a charged nonrelativistic particle.

5.150. Find the change in the radiation field energy $\mathrm{d}W/\mathrm{d}t$ per unit time as a result of particle–field interaction. Express this quantity through the oscillatory coordinates $q_{k\lambda}$ and forces $F_{k\lambda}(t)$ (see the solution of the preceding problem).

5.151*. A particle with charge e performs a simple harmonic oscillation in accordance with a given law: $\boldsymbol{r} = \boldsymbol{r}_0 \sin \omega_0 t$, where $\boldsymbol{r}_0 = \text{const}$. Find the angular distribution and the total intensity of radiation I by the field oscillator method (see Problem 5.149*).[4)]

5.152. A charge e propagates with constant angular velocity ω_0 round a circle of radius a_0. Elucidate the character of the polarization of the charge radiation field, and find the angular distribution and the total radiation intensity with the use of the field oscillator method (see Problem 5.21*).

5.153*. A linearly polarized wave with frequency ω is incident on a harmonic oscillator having eigenfrequency ω_0. Find the differential $\mathrm{d}\sigma/\mathrm{d}\Omega$ and total σ scattering cross-sections by the field oscillator method (disregarding radiation friction). Study polarization of the scattered radiation.

5.154. Find the differential $\mathrm{d}\sigma/\mathrm{d}\Omega$ and total σ scattering cross-sections of linearly polarized, circularly polarized, and nonpolarized monochromatic waves from a free nonrelativistic charge making use of the field oscillator method (see Problems 5.127* and 5.128).

5.155. A free charge scatters (i) a nonpolarized wave with frequency ω and (ii) a circularly polarized wave. Study the character of the polarization of the radiation field using the field oscillator method (see Problems 5.127 and 5.128).

5.5 Answers and Solutions

5.2• $G^{\mathrm{A}}(R, \tau) = \delta(\tau + R/c)/R$. The integration contour C_{A} is shown in Figure 5.3.

4) Certainly, this problem can be solved in a much simpler way (see Section 5.2). The proposed method is interesting in that it is closely related to the methods for the solution of an analogous problem in quantum electrodynamics.

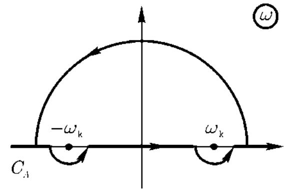

Figure 5.3 Integration contour for the calculation of the advanced Green's function.

5.5

$$A_\omega^A(\boldsymbol{r}) = \frac{1}{c}\int \frac{\exp[-i\omega|\boldsymbol{r}-\boldsymbol{r}'|/c]}{|\boldsymbol{r}-\boldsymbol{r}'|}\boldsymbol{j}_\omega(\boldsymbol{r}')\mathrm{d}V' ,$$
$$\varphi_\omega^A(\boldsymbol{r}) = \int \frac{\exp[-i\omega|\boldsymbol{r}-\boldsymbol{r}'|/c]}{|\boldsymbol{r}-\boldsymbol{r}'|}\rho_\omega(\boldsymbol{r}')\mathrm{d}V' .$$

Fourier harmonics of the retarded potentials contain exponents $\exp[i(kR-\omega t)]$, $k = \omega/c$, which correspond to divergent spherical waves transferring perturbations from an emitting charge system into the surrounding space. Fourier harmonics of the advanced potentials contain exponents $\exp[-i(kR+\omega t)]$ that describe the spherical waves converging to the center. Such waves could be generated by a certain source at infinity but not stipulated by the charge system. For this reason, the advanced potentials are not suitable for the description of the radiation process in an unbounded space.

5.8*

$$\psi^{\mathrm{R,A}}(R,\tau) = -i^{1/2}\left(\frac{\omega}{4\pi\tau c^2}\right)^{3/2}\exp\left[\pm\frac{i\tau}{\omega}(\omega^2 \pm i\omega 0) + \frac{i\omega}{4\tau c^2}R^2\right] .$$

5.9

$$\tilde{G}^+(k) = -4\pi\frac{\mathcal{P}}{k_i k^i} , \quad \tilde{G}^-(k) = -i8\pi^2\epsilon(k^0)\delta(k_i k^i) , \tag{5.120}$$

$$\left.\begin{aligned} \Delta G^+ - \tfrac{1}{c^2}\tfrac{\partial^2 G^+}{\partial t^2} &= -\tfrac{4\pi}{c}\boldsymbol{j}(\boldsymbol{r},t) , \\ \Delta G^- - \tfrac{1}{c^2}\tfrac{\partial^2 G^-}{\partial t^2} &= 0 . \end{aligned}\right\} \tag{5.121}$$

5.10

$$\Delta\varphi(\boldsymbol{r},t) = -4\pi\rho(\boldsymbol{r},t) ,$$
$$\Delta\boldsymbol{A}(\boldsymbol{r},t) - \frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2} = -\frac{4\pi}{c}\boldsymbol{j}(\boldsymbol{r},t) + \frac{1}{c}\int\left\{\frac{\boldsymbol{j}(\boldsymbol{r}',t)}{R^3} - \frac{3\boldsymbol{R}(\boldsymbol{j}(\boldsymbol{r}',t)\cdot\boldsymbol{R})}{R^5}\right\}\mathrm{d}V' ,$$

where $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}'$. The equation for φ contains time as a parameter. Therefore, the scalar potential describes the Coulomb field determined by an instantaneous (not retarded) charge distribution.

5.11 By definition of solenoidal and potential quantities,

$$\nabla \cdot \boldsymbol{H}^{\perp} = \nabla \cdot \boldsymbol{E}^{\perp} = \nabla \cdot \boldsymbol{j}^{\perp} = 0\,, \quad \nabla \times \boldsymbol{H}^{\parallel} = \nabla \times \boldsymbol{E}^{\parallel} = \nabla \times \boldsymbol{j}^{\parallel} = 0\,.$$

From the Maxwell equation $\nabla \cdot \boldsymbol{H} = 0$, we find $\boldsymbol{H} = \boldsymbol{H}^{\perp}$, $\boldsymbol{H}^{\parallel} = 0$.

The potential electric field satisfies the equations

$$\nabla \cdot \boldsymbol{E}^{\parallel}(\boldsymbol{r}, t) = 4\pi\rho(\boldsymbol{r}, t)\,, \quad \nabla \times \boldsymbol{E}^{\parallel}(\boldsymbol{r}, t) = 0\,.$$

These are the equations of electrostatics containing time as a parameter. They describe the instantaneous Coulomb interaction. The solenoidal constituents satisfy the equations

$$\nabla \times \boldsymbol{H}^{\perp} = \frac{1}{c}\frac{\partial \boldsymbol{E}^{\perp}}{\partial t} + \frac{4\pi}{c}\boldsymbol{j}^{\perp}\,, \quad \nabla \times \boldsymbol{E}^{\perp} = -\frac{1}{c}\frac{\partial \boldsymbol{H}^{\perp}}{\partial t}\,.$$

5.12* Use the differential equation for the function $G^R_\omega(\boldsymbol{r} - \boldsymbol{r}')$:

$$\begin{aligned}(\Delta_r + k^2)G^R_\omega &= -4\pi\delta(\boldsymbol{r} - \boldsymbol{r}') \\ &= -\frac{4\pi}{r^2}\delta(r - r')\delta(\cos\vartheta - \cos\vartheta')\delta(\alpha - \alpha')\,, \\ k &= \frac{|\omega|}{c}\,. \end{aligned} \tag{1}$$

Because the spherical Legendre functions $Y_{lm}(\vartheta, \alpha)(l = 0, 1, 2 \ldots,\ m = -l, -l + 1, \ldots, +l)$ make up a total orthonormal system (see Section 1.3), they satisfy the condition

$$\sum_{l,m} Y^*_{lm}(\vartheta', \alpha')\, Y_{lm}(\vartheta, \alpha) = \delta(\cos\vartheta - \cos\vartheta')\delta(\alpha - \alpha')\,. \tag{2}$$

Thus, sum (2) can be introduced into the right-hand side of equation (1). Its left-hand side contains the Laplace operator

$$\Delta_r = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{\vartheta,\varphi}\,. \tag{3}$$

According to the result obtained in Problem 1.119,

$$\begin{aligned}\left\{\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\sin\vartheta\frac{\partial}{\partial\vartheta} + \frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial\alpha^2}\right\} Y_{lm}(\vartheta, \alpha) &= \Delta_{\vartheta,\varphi}\, Y_{lm}(\vartheta, \varphi) \\ &= -l(l + 1)\, Y_{lm}(\vartheta, \alpha)\,. \end{aligned}$$

This permits us to seek the solution of equation (1) in the form of a series

$$G^R_\omega = \sum_{l=0}^{\infty} f_l(r, r') \sum_{m=-l}^{l} Y^*_{lm}(\vartheta', \alpha')\, Y_{lm}(\vartheta, \alpha) \tag{4}$$

and obtain for unknown functions $f_l(r, r')$ the equation

$$\left[\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr} - \frac{l(l+1)}{r^2} + k^2\right] f_l = -\frac{4\pi}{r^2}\delta(r - r') . \tag{5}$$

The solutions of the homogeneous equation (5) have the form of various spherical Bessel functions (see Section 1.3 for their definition and properties). In the case of a fixed finite r', the solution of equation (5) must be constrained at $r \to 0$ and describe a divergent spherical wave at $r \to \infty$. To fulfill these conditions, the solution should be sought in the form

$$f_l(r, r') = \begin{cases} B\, j_l(kr) , & r < r' ; \\ C h_l^{(1)}(kr) , & r > r' , \end{cases} \tag{6}$$

where the spherical Bessel j_l and Hankel $h_l^{(1)}$ functions have the necessary asymptotes.

Coefficients B and C should be found from the boundary conditions at $r = r'$. Naturally, the function $f_l(r, r')$ itself must be continuous at $r = r'$, $f_l(r, r')|_{r=r'+0} = f_l(r, r')|_{r=r'-0}$. In the opposite case, the equation is violated. The condition for the first derivative can be obtained by integrating the two sides of equation (5) over the small interval $[r' - \epsilon,\ r' + \epsilon]$ and passing to the $\epsilon \to 0$ limit. This gives (taking into account the continuity and finiteness of $f_l(r, r')$)

$$\left.\frac{d f_l}{dr}\right|_{r=r'+0} - \left.\frac{d f_l}{dr}\right|_{r=r'-0} = -\frac{4\pi}{r'^2} . \tag{8}$$

The use of (6)–(8) and the value of the Wronskian determinant for the spherical Bessel functions,

$$W(j_l(z), h_l^{(1)}(z)) = j_l(z)h_l^{(1)'}(z) - h_l^{(1)}(z)j_l'(z) = \frac{i}{z^2} ,$$

yields

$$f_l(r, r') = 4\pi i k \begin{cases} j_l(kr)h_l^{(1)}(kr') , & r < r' ; \\ j_l(kr')h_l^{(1)}(kr) , & r > r' . \end{cases} \tag{9}$$

This function should be substituted into expansion (4).

5.13

$$\boldsymbol{A}_\omega(\boldsymbol{r}) = \frac{e^{ikr}}{r}\sum_{l,m} \boldsymbol{Q}_{lm} Y_{lm}(\vartheta, \alpha) , \quad \varphi_\omega(\boldsymbol{r}) = \frac{e^{ikr}}{r}\sum_{l,m} Q'_{lm} Y_{lm}(\vartheta, \alpha) ,$$

where

$$\boldsymbol{Q}_{lm} = 4\pi(-i)^{l+2}\int \boldsymbol{j}_\omega(r', \vartheta', \alpha') j_l(kr') Y^*_{lm}(\vartheta', \alpha') r'^2 dr' d\Omega',$$

$$Q'_{lm} = 4\pi(-i)^{l+2}\int \rho(r', \vartheta', \alpha') j_l(kr') Y^*_{lm}(\vartheta', \alpha') r'^2 dr' d\Omega';$$

$$Q'_{lm} = \frac{\boldsymbol{r}}{r}\cdot\boldsymbol{Q}_{lm} .$$

5.16•

$$\varphi(\boldsymbol{r},t) = \frac{q}{r} + \frac{\boldsymbol{p}(t)\cdot\boldsymbol{r}}{r^3} + \frac{Q_{\alpha\beta}(t)x_\alpha x_\beta}{2r^5}\,, \qquad \boldsymbol{E}(\boldsymbol{r},t) = -\nabla\varphi(\boldsymbol{r},t)\,.$$

$$\boldsymbol{A}(\boldsymbol{r},t) = \frac{\dot{\boldsymbol{p}}(t)}{cr} + \frac{\boldsymbol{m}(t)\times\boldsymbol{n}}{r^2}\,,$$

$$\boldsymbol{H}(\boldsymbol{r},t) = \nabla\times\boldsymbol{A} = \frac{\dot{\boldsymbol{p}}\times\boldsymbol{n}}{cr^2} + \frac{3\boldsymbol{n}(\boldsymbol{m}\cdot\boldsymbol{n})}{r^3} - \frac{\boldsymbol{m}}{r^3}\,.$$

The electric field is expressed through the static formulas with the time-dependent dipole and quadrupole moments. The magnetic field contains an additional term by virtue of the Biot–Savart law: if the elementary dipole $\boldsymbol{p}(t) = q(t)\boldsymbol{l}$, then $\boldsymbol{H}_{\mathrm{BS}} = \dot{\boldsymbol{p}}\times\boldsymbol{n}/cr^2 = J(t)[\boldsymbol{l}\times\boldsymbol{r}]/cr^3$ is the field of elementary current $J(t) = \dot{q}(t)$ flowing in the l segment.

5.18 In the spherical coordinates with the polar axis along $\boldsymbol{p}_0$

$$E_{\mathrm{r}} = \frac{2p_0}{r^2}\cos\vartheta\left[\frac{1}{r}\cos(kr-\omega t) + k\sin(kr-\omega t)\right],$$

$$E_\vartheta = \frac{p_0}{r}\sin\vartheta\left[\left(\frac{1}{r^2}-k^2\right)\cos(kr-\omega t) + \frac{k}{r}\sin(kr-\omega t)\right],$$

$$H_\alpha = -\frac{p_0k^2}{r}\sin\vartheta\left[\cos(kr-\omega t) - \frac{1}{kr}\sin(kr-\omega t)\right],$$

$$E_\alpha = H_{\mathrm{r}} = H_\vartheta = 0\,.$$

5.19 The magnetic lines of force have the form of circumferences whose planes are perpendicular to the Oz axis, with the centers lying on this axis. The electric lines of force are described by the following equations:

$$C_1 = \sin^2\vartheta\left[\frac{1}{r}\cos(kr-\omega t) + k\sin(kr-\omega t)\right], \qquad C_2 = \alpha\,,$$

where C_1 and C_2 are constants.

5.20 The angular momentum flux density is

$$\mathcal{R} = \frac{(\boldsymbol{n}\times\ddot{\boldsymbol{p}})(\boldsymbol{n}\cdot\dot{\boldsymbol{p}})}{2\pi c^3 r^2}\,.$$

It is helpful to use the formula $\overline{n_i n_k} = \delta_{ik}/3$ (see Chapter 1) to calculate the quantities $-\mathrm{d}\boldsymbol{L}/\mathrm{d}t = \int \mathcal{R}r^2\mathrm{d}\Omega$. This leads to

$$-\frac{\mathrm{d}\boldsymbol{L}(t)}{\mathrm{d}t} = \frac{2}{3c^2}\dot{\boldsymbol{p}}\times\ddot{\boldsymbol{p}}\Big|_{t'=t-r/c}\,.$$

5.21*

$$\boldsymbol{H} = \frac{1}{c}\cdot\frac{\partial\operatorname{curl}\boldsymbol{Z}}{\partial t}$$
$$= ea\left[\boldsymbol{e}_\vartheta\left(-i\frac{\omega^2}{c^2r} + \frac{\omega}{cr^2}\right) + \boldsymbol{e}_\alpha\left(\frac{\omega^2}{c^2r} + i\frac{\omega}{cr^2}\right)\cos\vartheta\right]e^{i(kr-\omega t+\alpha)}\,,$$

$$\begin{aligned}\boldsymbol{E} = \operatorname{curl}\operatorname{curl}\boldsymbol{Z} = ea\Bigg[&\boldsymbol{e}_\mathrm{r}\left(-\frac{i\omega}{cr^2}+\frac{1}{r^3}\right)2\sin\vartheta\\&+\boldsymbol{e}_\vartheta\left(\frac{\omega^2}{c^2r}+i\frac{\omega}{cr^2}-\frac{1}{r^3}\right)\cos\vartheta\\&+\boldsymbol{e}_\alpha\left(i\frac{\omega^2}{c^2r}-\frac{\omega}{cr^2}-\frac{i}{r^3}\right)\Bigg]e^{i(kr-\omega t+\alpha)}\,.\end{aligned}$$

The expressions $\boldsymbol{E}$ and $\boldsymbol{H}$ are simplified in the wave zone $\boldsymbol{r} \gg \lambda = 2\pi c/\omega$:

$$\begin{aligned}\boldsymbol{H} &= ea\frac{\omega^2}{c^2r}(-i\boldsymbol{e}_\vartheta+\boldsymbol{e}_\alpha\cos\vartheta)e^{i(kr-\omega t+\alpha)}\,,\\ \boldsymbol{E} &= ea\frac{\omega^2}{c^2r}(\boldsymbol{e}_\vartheta\cos\vartheta+i\boldsymbol{e}_\alpha)e^{i(kr-\omega t+\alpha)} = \boldsymbol{H}\times\boldsymbol{n}\,.\end{aligned}$$

Emission into the upper semisphere ($\cos\vartheta > 0$) results in left-handed elliptical polarization, specifically left circular polarization at $\vartheta = \pi$. Emission into the lower semisphere ($\cos\vartheta < 0$) leads to right-handed elliptical polarization turning into circular polarization at $\vartheta = \pi$. The waves emitted in the equatorial plane have linear polarization. The angular distribution and the total radiation intensity are

$$\overline{\frac{\mathrm{d}I}{\mathrm{d}\Omega}} = \overline{\boldsymbol{\gamma}}\cdot\boldsymbol{n}r^2 = \frac{e^2\omega^4a^2}{8\pi c^3}(1+\cos^2\vartheta)\,,\quad \overline{I} = \frac{2\omega^4e^2a^2}{3c^3}\,.$$

This case is realized, for instance, when a charge propagates in a uniform magnetic field.

5.22 $\bar{I} = \frac{U_0^2R^4\omega^4}{48h^2c^3}$.

5.23 $\bar{I} = \frac{2^{17}(ea_\mathrm{B})^2\omega_0^4}{3^{11}c^3}$.

5.24

$$\overline{\frac{\mathrm{d}I}{\mathrm{d}\Omega}} = \frac{4q^2a^4\omega^6}{\pi c^5}\sin^2\vartheta\left(\cos^2\vartheta+\frac{1}{2}\right)\,,\quad \overline{I} = \frac{112q^2a^4\omega^6}{15c^5}\,.$$

5.25

$$\frac{\mathrm{d}I_\omega}{\mathrm{d}\omega} = \frac{2e^4E_0^2}{3\pi m^2c^3}\frac{\omega^2+\alpha^2}{[(\omega+\omega_0)^2+\alpha^2][(\omega-\omega_0)^2+\alpha^2]}\,.$$

5.26

$$\frac{\mathrm{d}I_\omega}{\mathrm{d}\omega} = \frac{2\tau^2\omega^4}{3c^3}(p_0^2+m_0^2)e^{-(\omega\tau)^2/2}\,.$$

The criterion of applicability is the size of the system, $l \ll c\tau$.

5.27 $\bar{I} = \frac{S^2J_0^2\omega^4}{3c^5}$, $\quad S = ab$.

5.28

$$\frac{\mathrm{d}I_\omega}{\mathrm{d}\omega} = \frac{2\pi J_0^2 S^2 \tau^2 \omega^4}{3c^5} e^{-2\omega\tau} .$$

The criterion of applicability is the size of the system, $\sqrt{S} \ll c\tau$.

5.29 $\bar{I} = 2\eta^4 H^4 m^2 \sin^2\beta/3c^3$.

5.30

$$\eta = \frac{q}{2mc} , \quad \bar{I} = \frac{q^2\omega^2}{600c}\left(\frac{qHR}{mc^2}\right)^4 \sin^2\beta .$$

5.31 $\boldsymbol{p} = \boldsymbol{m} = 0$, $\boldsymbol{Q} \neq 0$,

$$\begin{aligned} \boldsymbol{H} &= \frac{1}{c}\dot{\boldsymbol{A}} \times \boldsymbol{n} \\ &= -\frac{4ea^2\omega^3}{c^3 r}\sin\vartheta[\boldsymbol{e}_\vartheta\cos(2\omega t' - 2\alpha) + \boldsymbol{e}_\alpha \cos\vartheta\sin(2\omega t' - 2\alpha)] . \end{aligned}$$

The variation rate of the charge and current distribution (hence, the field frequency) is twice the circulation frequency ω of each of the charges in the orbit. Polarization of the emission is elliptical, approximating circular polarization at $\vartheta \to 0$, π and turning into linear polarization at $\vartheta = \pi/2$;

$$\overline{\frac{\mathrm{d}I}{\mathrm{d}\Omega}} = \frac{2e^2a^4\omega^6}{\pi c^5}\sin^2\vartheta(1+\cos^2\vartheta) , \quad \overline{I} = \frac{32}{5}\cdot\frac{e^2a^4\omega^6}{c^5} .$$

Elimination of one of the charges results in a $(\lambda/a)^2$-fold increase of radiation intensity in terms of the order of magnitude, that is, a rather high one, owing to the fulfillment of the condition $a/\lambda \ll 1$.

5.32 If the angle between the radius vectors of charges equals $\pi - \varphi$, then

$$\varphi = \sqrt{\frac{12}{5}}\cdot\frac{a\omega}{c} .$$

5.33 Arrange the Ox axis along the amplitude of the moment of an oscillator that is phase advanced compared with another oscillator and choose the xy plane as the plane in which the moments of both oscillators lie. Denote by ϑ and α the polar angles of the unit vector $\boldsymbol{n}$ indicating the direction of wave propagation. This leads to

$$\begin{aligned} \boldsymbol{H}(r,t) = \boldsymbol{H}e^{-i\omega t'} &= \frac{\omega^2 p}{c^2 r}\{\boldsymbol{e}_\vartheta[\sin\alpha + i\sin(\alpha-\varphi)] \\ &\quad + \boldsymbol{e}_\alpha[\cos\alpha + i\cos(\alpha-\varphi)]\cos\vartheta\}\, e^{-i\omega t'} , \end{aligned}$$

$$\overline{\frac{\mathrm{d}I}{\mathrm{d}\Omega}} = \frac{p^2\omega^4}{8\pi c^3}\left\{2 - [\cos^2\alpha + \cos^2(\alpha-\varphi)]\sin^2\vartheta\right\} , \quad \overline{I} = \frac{2p^2\omega^4}{3c^3} .$$

Radiation is maximum in the directions $\vartheta = 0$ and $\vartheta = \pi$, perpendicular to the moments of both oscillators; it is nonuniformly distributed over the azimuth as illustrated in Figure 5.4 by polar diagrams for the case of $\varphi = 45°$. Figures 5.4a and 5.4b show the angular distribution in planes $\varphi = 90°$ and $\alpha = \varphi/2 = 22.5°$, respectively.

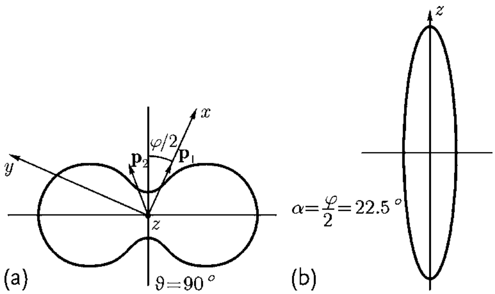

Figure 5.4 Angular distribution of emission from two near oscillators: for $\vartheta = 90°$ (a) and for $\alpha = \varphi/2 = 22.5°$ (b).

5.34 Moving the phase reference point by γ yields the new field amplitude $\boldsymbol{H}e^{-i\gamma} = \boldsymbol{H}_1 - i\boldsymbol{H}_2$. Assuming that $\boldsymbol{H}_1 \cdot \boldsymbol{H}_2 = 0$ leads to

$$\tan 2\gamma = 2\frac{\sin\alpha\sin(\alpha-\varphi) + \cos\alpha\cos(\alpha-\varphi)\cos^2\theta}{\sin^2\alpha - \sin^2(\alpha-\varphi) + [\cos^2\alpha - \cos^2(\alpha-\varphi)]\cos^2\theta} . \tag{1}$$

Determination of $\cos\gamma$ and $\sin\gamma$ with the help of (1) gives $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$ depending on θ, α, and φ.

Let us consider some specific cases. The case of $\theta = 90°$ is associated with linear polarization, with the plane of polarization being perpendicular to the xy plane. Elliptical polarization occurs at $\theta = 0, \pi$ with the ratio of ellipse semiaxes $\tan(\varphi/2)$. Circular polarization occurs in the cases of $\varphi = \pi/2$ and $\theta = 0, \pi$. It is equally easy to study the cases of $\alpha = \varphi/2$, $\varphi/2 \pm \pi/2$, and $\varphi/2 + \pi$. In all these cases, polarization is, generally speaking, elliptical. Circular polarization is associated with the case of $\alpha = \varphi/2$, $\varphi/2 + \pi$ in directions determined by the condition $\tan\varphi/2 = |\cos\theta|$.

At $\alpha = \varphi/2 \pm \pi/2$, the directions of circular polarization are specified by the equation $\cot(\varphi/2) = |\cos\theta|$.

5.35

$$\overline{\gamma} = \frac{e^2a^2\omega^4}{8\pi c^3 r^2}(1+\cos^2\vartheta)\boldsymbol{e}_r + \frac{e^2a^2\omega^3}{4\pi c^2 r^3}\sin\vartheta\,\boldsymbol{e}_\alpha , \quad \boldsymbol{N} = \frac{2}{3}\frac{e^2a^2\omega^3}{c^3}\boldsymbol{e}_z .$$

The last result can be obtained either by taking into account that the angular momentum $\mathrm{d}\boldsymbol{K}/\mathrm{d}t = -\frac{2}{3c^3}\dot{\boldsymbol{p}} \times \ddot{\boldsymbol{p}}$ (see Problem 5.5) is equal to the rotary moment $\boldsymbol{N}$

applied to the screen or directly from the formula

$$N = \frac{1}{c} \int\limits_{r \gg a} \boldsymbol{r} \times \overline{\boldsymbol{\gamma}} r^2 \mathrm{d}\Omega \ .$$

5.36

$$\overline{\frac{\mathrm{d}I}{\mathrm{d}\Omega}} = \frac{\omega_0^4}{8\pi c^3} \left\{ p_0^2 (1 - \sin^2 \vartheta \cos^2 \alpha) + m_0^2 \sin^2 \vartheta \right\} \ ,$$

$$\overline{I} = \frac{\omega_0^4}{3c^3} \left(p_0^2 + m_0^2 \right) \ .$$

Here, the system of coordinates is used in which the Ox axis is directed along $\boldsymbol{p}$ and the Oz axis is directed along $\boldsymbol{m}$.

5.37*

$$\begin{aligned} \boldsymbol{A}(\boldsymbol{r}, t) = {} & \frac{\dot{\boldsymbol{p}}}{cr} + \frac{\dot{\boldsymbol{m}} \times \boldsymbol{n}}{cr} + \frac{\ddot{\boldsymbol{Q}}}{6c^2 r} + \frac{\boldsymbol{n}}{6c^2 r} \int r'^2 \ddot{\rho} \mathrm{d}V' \\ & + \frac{1}{2c^3 r} \int \boldsymbol{r}' (\boldsymbol{n} \cdot \boldsymbol{r}')^2 \dddot{\rho} \mathrm{d}V' - \frac{1}{c^3 r} \int \boldsymbol{r}' (\boldsymbol{n} \cdot \boldsymbol{r}') (\boldsymbol{n} \cdot \ddot{\boldsymbol{j}}) \mathrm{d}V' \ , \end{aligned}$$

$$I = \frac{2}{3c^3} (\ddot{\boldsymbol{p}}^2 + \ddot{\boldsymbol{m}}^2) + \frac{1}{180 c^5} \dddot{Q}_{\alpha\beta}^2 + \frac{2}{15 c^5} \ddot{\boldsymbol{p}} \cdot \dddot{\boldsymbol{L}} \ ,$$

where

$$\boldsymbol{L} = \int \dot{\rho} r'^2 \boldsymbol{r}' \mathrm{d}V' + \int [r'^2 \boldsymbol{j} - 3\boldsymbol{r}' (\boldsymbol{r}' \cdot \boldsymbol{j}) \mathrm{d}V' \ ,$$

all the other notations are universally accepted ones and all time-dependent quantities are taken with a delay $t' = t - r/c$. The reader's attention is drawn to the last term in the expression for I (Baranova and Zel'dovich, 1977), which is absent in most textbooks on electrodynamics.

5.38* Using the Fourier series expansion of potentials (5.16) and calculating the Fourier harmonic of the magnetic field strength (5.17) yields

$$\boldsymbol{H}_m = i \frac{e m \omega^2}{2\pi c^2} \frac{e^{i k_m r}}{r} \int\limits_0^T \boldsymbol{n} \times \boldsymbol{v}(\tau) \exp i (m\omega\tau - \boldsymbol{k}_m \cdot \boldsymbol{s}(\tau)) \mathrm{d}\tau \ ,$$

where $\boldsymbol{k}_m = m\omega \boldsymbol{n}/c$ and $\boldsymbol{s}(\tau) = \boldsymbol{e}_z z(\tau)$. The time integral is expressed through the Bessel function in accordance with (1.150). As a result.

$$\frac{\mathrm{d}I_m}{\mathrm{d}\Omega} = \frac{c}{2\pi} |\boldsymbol{H}_m|^2 r^2 = \frac{e^2 \omega^2}{2\pi c} \tan^2 \theta \, m^2 J_m^2 (m\beta \cos\theta) \ ,$$

where $\beta = a\omega/c$ and θ is the angle between the direction of radiation $\boldsymbol{n}$ and the Oz axis. Here, it is assumed that the harmonics with numbers m and $-m$ make a similar contribution to the radiation. At $\beta \ll 1$,

$$\frac{\mathrm{d}I_1}{\mathrm{d}\Omega} \approx \frac{e^2 a^2 \omega^4}{8\pi c^3} \sin^2\theta$$

for dipole radiation and

$$\frac{\mathrm{d}I_2}{\mathrm{d}\Omega} \approx \frac{e^2 a^4 \omega^6}{2\pi c^5} \sin^2\theta \cos^2\theta$$

for quadrupole radiation, containing, unlike dipole radiation, the small multiplier $(a\omega/c)^2$.

5.39

$$\frac{\mathrm{d}I}{\mathrm{d}\Omega} = \frac{e^2\omega^2\beta^2(4+\beta^2\cos^2\theta)}{32\pi c(1-\beta^2\cos^2\theta)^{7/2}} \sin^2\theta\,,$$

$$I = \frac{e^2\omega^2\beta^2(4-3\beta^2)}{12c(1-\beta^2)^{3/2}}\,, \quad \beta < 1\,.$$

5.40*

1. The field strength in the wave zone is calculated from formulas (5.37) assuming in them $\boldsymbol{Q} = 0$. The angular distribution of radiation

$$\overline{\frac{\mathrm{d}I}{\mathrm{d}\Omega}} = \frac{1}{4\pi c^3}\overline{|\boldsymbol{n}\times\ddot{\boldsymbol{m}}|^2}\,. \tag{1}$$

The right-hand side is calculated using the equation of motion for the magnetic moment $\dot{\boldsymbol{m}} = \boldsymbol{\omega}\times\boldsymbol{m}$. This gives $(\boldsymbol{n}\times\ddot{\boldsymbol{m}})^2 = \omega^4\boldsymbol{m}_\perp^2(1-\sin^2\vartheta\cos^2(\omega t-\alpha))$, where $\boldsymbol{m}_\perp$ is the constituent perpendicular to the rotation axis, ϑ is the polar angle counted from direction $\boldsymbol{\omega}$, and ωt and α are the azimuths of vectors $\boldsymbol{m}_\perp$ and $\boldsymbol{n}$ in the plane perpendicular to $\boldsymbol{\omega}$. Substitution into (1) and averaging over time yields

$$\overline{\frac{\mathrm{d}I}{\mathrm{d}\Omega}} = \frac{\omega^4 m^2\sin^2\varphi}{8\pi c^3}(1+\cos^2\vartheta)\,, \quad \overline{I} = \frac{2\omega^4 m^2\sin^2\varphi}{3c^3}\,. \tag{2}$$

2. Assuming the pulsar magnetic field to be dipole gives the order of magnitude of the quantity $m \approx H_0 R^3 \approx 2\times 10^{30}$ Oe cm^3.
3. Substituting the necessary quantities and $\sin^2\varphi \approx 1$ leads to $\overline{I} \approx 1.3\times 10^{38}$ erg/s, which corresponds to approximately $3\times 10^4 L_\odot$.
4. A decrease in the rotational energy is calculated using the formula $\dot{\mathcal{E}}_{\text{curl}} = I\omega\dot{\omega} = -2\mathcal{E}_{\text{curl}}\dot{T}/T$, where $I = (2/5)MR^2$ is the moment of inertia of the sphere and $M \approx 1.3M_\odot \approx 2.6\times 10^{33}$ g is the mass of the star (on the order of the solar mass). Thus, $\dot{\mathcal{E}}_{\text{curl}} \approx -5\times 10^{38}$ erg/s.

The similarity of the estimates of pulsar magnetic dipole radiation and the decrease of mechanical rotation energy appears to confirm the validity of the model. The observed luminosity of the Crab Nebula is roughly 4×10^{37} erg/s and roughly 2×10^{36} erg/s in the X-ray and optical ranges, respectively. These data are also in agreement with the model since almost 10% of the energy of long-wavelength primary radiation is converted into secondary radiation in the plasma surrounding the star. For more detailed information about neutron stars, see Fleishman and Toptygin (2013).

5.41

$$\overline{\frac{dI}{d\Omega}} = \frac{9}{800\pi} \cdot \frac{\omega^6 q^2 R_0^4 a^2}{c^5} \sin^2\vartheta \cos^2\vartheta\,, \quad \overline{I} = \frac{3}{500} \cdot \frac{\omega^6 q^2 R_0^4 a^2}{c^5}\,.$$

5.42 $E = \frac{q\boldsymbol{r}}{r^3}\,, \quad H = 0\,.$

5.43 The field of magnetic dipole:

$$\boldsymbol{E}_m(\boldsymbol{r}, t) = -\frac{1}{c}\dot{\boldsymbol{A}}_m = \frac{\boldsymbol{n} \times \ddot{\boldsymbol{m}}(t')}{c^2 r} + \frac{\boldsymbol{n} \times \dot{\boldsymbol{m}}(t')}{c^2 r}\,,$$

$$\boldsymbol{H}_m(\boldsymbol{r}, t) = \operatorname{curl} \boldsymbol{A}_m = \frac{3\boldsymbol{n}(\boldsymbol{m} \cdot \boldsymbol{n}) - \boldsymbol{m}}{r^3} + \frac{3\boldsymbol{n}(\dot{\boldsymbol{m}} \cdot \boldsymbol{n}) - \dot{\boldsymbol{m}}}{c r^2} + \frac{\boldsymbol{n} \times (\boldsymbol{n} \times \ddot{\boldsymbol{m}})}{c^2 r}\,.$$

The field of the electric dipole is derived from the field of the magnetic dipole by the substitutions $\boldsymbol{m} \to \boldsymbol{p}$, $\boldsymbol{H}_m \to \boldsymbol{E}_e$, and $\boldsymbol{E}_m \to -\boldsymbol{E}_e$.

5.44*

$$\overline{\frac{dI}{d\Omega}} = \frac{\omega^4 p_0^2}{8\pi c^3}(1 - \sin^2\vartheta \cos^2\alpha)\cos^2\left(\frac{\pi}{2}\cos^2\frac{\vartheta}{2}\right)\,,$$

where ϑ and α are the polar angles characterizing the direction of emission (see the polar diagrams in Figure 5.5). The advanced oscillator is located above along the Oz axis.

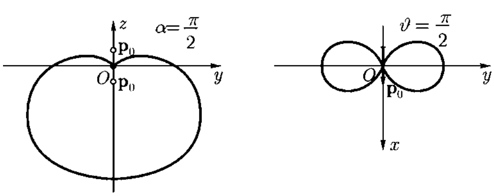

Figure 5.5 Angular distribution of emission from two dipole oscillators for a quarter of the wavelength.

5.50 Expanding the Hertz vector $Z(\boldsymbol{r}, t)$ in monochromatic components and using the series expansion (Problem 5.12*) yields

$$Z_p(\boldsymbol{r}, t) = \frac{\boldsymbol{p}(t')}{r}\,, \tag{1}$$

where $t' = t - \frac{r}{c}$,

$$Z_Q(\boldsymbol{r}, t) = \frac{1}{2r^2}\boldsymbol{Q}(t') + \frac{1}{2rc}\dot{\boldsymbol{Q}}(t')\,, \tag{2}$$

$$Z_m(\boldsymbol{r}, t) = \frac{\boldsymbol{m}(t') \times \boldsymbol{n}}{r} + \frac{c}{r^2}\left[\int \boldsymbol{m}(t')\mathrm{d}t'\right] \times \boldsymbol{n}\,. \tag{3}$$

These formulas hold at $r \gg a$, where a is the size of the system. The arbitrary constant arising in the calculation of the integral entering equation (3) does not affect the field strength.

5.51 The dipole moments of the system equal zero and the electric quadrupole moment has one nonzero component Q_{zz} (given the Oz axis is directed along $\boldsymbol{p}_0$). Because of this, the $\boldsymbol{Q}$ vector is parallel to the Oz axis and is equal to $\boldsymbol{Q}(t') = Q_0 \cos\vartheta \cos\omega t' \boldsymbol{e}_z$ if the time reference point is adequately chosen; here $Q_0 = 2p_0 a$.

It is convenient to perform calculations in a complex form by projecting $\boldsymbol{Z}$ onto the axis of the spherical system of coordinates. Separation of the material part yields

$$\begin{aligned}
H_\alpha &= \frac{Q_0 \sin 2\vartheta}{4}\left[\left(\frac{k^3}{r} - \frac{3k}{r^3}\right)\sin(\omega t - kr) - \frac{3k^2}{r^2}\cos(\omega t - kr)\right], \\
E_r &= \frac{Q_0(3\cos^2\vartheta - 1)}{2} \\
&\quad \times \left[\left(\frac{3}{r^4} - \frac{k^2}{r^2}\right)\cos(\omega t - kr) - \frac{3k}{r^3}\sin(\omega t - kr)\right], \\
E_\vartheta &= \frac{Q_0 \sin 2\vartheta}{4} \\
&\quad \times \left[\left(\frac{6}{r^4} - \frac{3k^2}{r^2}\right)\cos(\omega t - kr) + \left(\frac{k^3}{r} - \frac{6k}{r^3}\right)\sin(\omega t - kr)\right], \\
\overline{\frac{\mathrm{d}I}{\mathrm{d}\Omega}} &= \frac{Q_0^2\omega^6}{32\pi c^5}\sin^2\vartheta\cos^2\vartheta\,, \quad \overline{I} = \frac{Q_0^2\omega^6}{60c^5}\,,
\end{aligned}$$

where $Q_0 = 2p_0 a$.

5.52* Let us choose the coordinate system as shown in Figure 5.6. The distribution of current in the antenna is described by the formula

$$\mathcal{I} = \mathcal{I}_0 \sin[k(\xi + l/2)]e^{-i\omega t}\,,$$

where $k = \omega/c = m\pi/l$.

The electric dipole moments of the antenna unit length $P = (i/\omega)\mathcal{I}$, in accordance with (5.41). Element $\mathrm{d}\xi$ of the antenna can be regarded as an electric dipole oscillator with moment $\mathrm{d}p = P\mathrm{d}\xi$. The inequality $\mathrm{d}\xi \ll \lambda$ being fulfilled, we can compute the magnetic field created by the $\mathrm{d}\xi$ element at point A using formulas from the electric dipole approximation:

$$\mathrm{d}\boldsymbol{H}_0(\boldsymbol{r}_0, t) = -\frac{\omega^2}{c^2 r}\boldsymbol{e}_\alpha \sin\vartheta P\left(t - \frac{r}{c}\right)\mathrm{d}\xi\,,$$

where $r = r_0 - \xi \cos\vartheta$. Because we are interested only in the field in the wave zone, the quantity $\sin\vartheta/r$ undergoing small variations in the region $r \gg l$ can be taken outside the sign of the integral. Thus,

$$H_r = H_\vartheta = 0\,,$$

$$H_\alpha = -\frac{i\omega\sin\vartheta}{c^2 r_0}\mathcal{I}_0 e^{i(kr_0-\omega t)}\int\limits_{-l/2}^{l/2} e^{ik\xi\cos\vartheta}\sin m\pi\left(\frac{\xi}{l}+\frac{1}{2}\right)\mathrm{d}\xi\,.$$

The completed the integration, we find the angular distribution using the formula $\overline{\mathrm{d}I}/\mathrm{d}\Omega = c\overline{H_\alpha^2}r_0^2/4\pi$:

$$\frac{\overline{\mathrm{d}I}}{\mathrm{d}\Omega} = \begin{cases} \dfrac{\mathcal{I}_0^2}{2\pi c}\cdot\dfrac{\cos^2\left(\frac{m\pi}{2}\cos\vartheta\right)}{\sin^2\vartheta} & \text{for } m \text{ odd}\,,\\[2ex] \dfrac{\mathcal{I}_0^2}{2\pi c}\cdot\dfrac{\sin^2\left(\frac{m\pi}{2}\cos\vartheta\right)}{\sin^2\vartheta} & \text{for } m \text{ even}\,. \end{cases}$$

The character of the angular distribution is illustrated by the polar diagrams presented in Figure 5.7. The dashed line shows the distribution of current along the antenna and the solid line demonstrates the angular distribution of radiation.

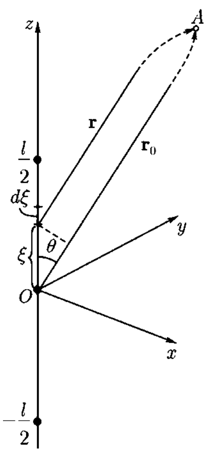

Figure 5.6 Diagram for the calculation of the emission by a linear antenna.

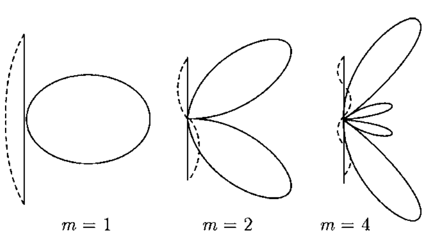

Figure 5.7 Angular emission at different numbers of half-waves on the length of an antenna.

5.53

$$\overline{I} = \frac{\mathcal{I}_0^2}{2c}[\ln(2\pi m) + C - Ci(2\pi m)]\,,$$
$$R = 2\frac{\overline{I}}{\mathcal{I}_0^2} = \frac{1}{c}[\ln(2\pi m) + C - Ci(2\pi m)]\,.$$

5.54

$$\overline{\frac{\mathrm{d}I}{\mathrm{d}\Omega}} = \frac{\mathcal{I}_0^2}{2\pi c}\frac{\sin^2\vartheta\,\sin^2[(kl/2)(1-\cos\vartheta)]}{(1-\cos\vartheta)^2}\,,$$
$$\overline{I} = \frac{\mathcal{I}_0^2}{c}\left[C - 1 + \ln\frac{4\pi l}{\lambda} - Ci\left(\frac{4\pi l}{\lambda}\right) + \frac{\sin(4\pi l/\lambda)}{4\pi l/\lambda}\right]\,,$$

where $\lambda = 2\pi/k$ is the emitted wavelength and ϑ is the polar angle counted from the coordinate axis ξ.

It is easy to see that a traveling wave emits more intensely than a standing one with the same l, λ, and $\mathcal{I}_0$ values.

5.55* If the distance r between the observation point $A(r_0, \vartheta, \alpha)$ (Figure 5.8) and the loop is sufficiently large ($r \gg a$), the radius vectors $\boldsymbol{r}$ from all $\mathrm{d}\boldsymbol{l}$ elements of the ring can be regarded as being parallel, with $r = r_0 - a\cos\varphi = r_0 - a\sin\vartheta\cos(\alpha' - \alpha)$. The $\mathrm{d}\boldsymbol{l}$ element possess the electric dipole moment $\mathrm{d}\boldsymbol{p} = P\mathrm{d}\boldsymbol{l} = (i/\omega)\mathcal{I}\mathrm{d}\boldsymbol{l}$, where P denotes the electric dipole moment of the wire unit length, and creates a magnetic field at point A.

$$\begin{aligned}\mathrm{d}\boldsymbol{H}(\boldsymbol{r}_0, t) &= -\frac{\omega^2\mathrm{d}\boldsymbol{p}(t')\times\boldsymbol{n}}{c^2 r}\,,\\ &= -i\frac{\omega a\mathcal{I}_0}{c^2 r_0}e^{-i\omega t + ikr_0 - iak\sin\vartheta\cos(\alpha'-\alpha)}\\ &\quad\times\sin n\alpha'[\cos(\alpha'-\alpha)\boldsymbol{e}_\vartheta + \cos\vartheta\sin(\alpha'-\alpha)\boldsymbol{e}_\alpha]\mathrm{d}\alpha'\,.\end{aligned}$$

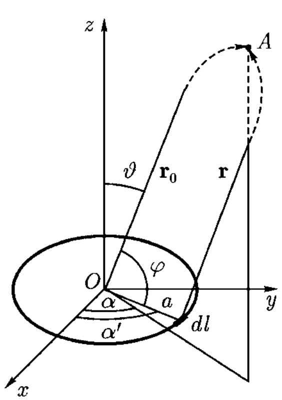

Figure 5.8 Diagram for the calculation of the emission by a round wire loop with current.

Disregard the a-order quantity compared with r_0 in the denominator of the last expression. It is impermissible to do this in the exponent because the value of ak is, generally speaking, not small and substantially affects the phase.

The problem of finding the field reduces to the integration

$$H_\vartheta = -\frac{i\omega a}{c^2}\frac{\mathcal{I}_0}{r_0}e^{i(kr_0-\omega t)}\int_{-\pi}^{\pi}\cos(\alpha'-\alpha)\sin n\alpha' e^{-ika\sin\vartheta\cos(\alpha'-\alpha)}\mathrm{d}\alpha' .$$

The expression for H_α differs from that for H_ϑ in the replacement of $\cos(\alpha'-\alpha)$ by $\sin(\alpha'-\alpha)$ in the pre-exponential factor.

Introducing the integration variable $\beta = \alpha' - \alpha$ yields

$$H_\vartheta = -\frac{i\omega a}{c^2}\cdot\frac{\mathcal{I}_0}{r_0}e^{i(kr_0-\omega t)}\left(\cos n\alpha\int_{-\pi}^{\pi}\cos\beta\sin n\beta e^{-ika\sin\vartheta\cos\beta}\mathrm{d}\beta\right.$$
$$\left.+\sin n\alpha\int_{-\pi}^{\pi}\cos\beta\cos n\beta e^{-ika\sin\vartheta\cos\beta}\mathrm{d}\beta\right) .$$

The first of the integrals in the parentheses vanishes owing to the oddness of the integrand, whereas the second one can be transformed toward the interval 0, π (the even integrand) and can be expressed through the derivative of the Bessel function. Thus,

$$H_\vartheta(r_0,t) = -E_\alpha = \frac{2\pi\omega a}{c^2}\cdot\frac{\mathcal{I}_0}{r_0}e^{i\left(kr_0-\omega t-n\frac{\pi}{2}\right)}\sin n\alpha J_n'(ka\sin\vartheta) .$$

Similar calculations with the use of the formula $J_{n-1}(x)+J_{n+1}(x) = (2n/x)J_n(x)$ yield

$$H_\alpha(r_0,t) = E_\vartheta = \frac{2\pi\omega a n\mathcal{I}_0 e^{i(kr_0-\omega t-n\pi/2)}}{c^2 r_0}\cos n\alpha\frac{J_n(ka\sin\vartheta)}{ka\tan\vartheta} .$$

5.56 Introduce the Hertz electric vector according to (5.43) and calculate the magnetic field using the formula presented in Problem 5.47:

$$\boldsymbol{H} = \operatorname{curl}\frac{1}{c}\frac{\partial \boldsymbol{Z}^{(e)}}{\partial t} = \operatorname{curl}\frac{1}{c}\int\frac{\dot{\boldsymbol{P}}(\boldsymbol{r}',t-R/c)}{R}\mathrm{d}V'$$
$$= \operatorname{curl}\frac{\boldsymbol{e}_z J_0}{c}$$
$$\times\int_{-l/2}^{l/2}\frac{1}{R}\sin\left[k_m\left(\xi+\frac{l}{2}\right)\right]\cos\omega_m\left(t-\frac{R}{c}\right)\exp\left[-\gamma\left(t-\frac{r}{c}\right)\right]\mathrm{d}\xi .$$

Find $R \approx r_0 - \xi\cos\vartheta \approx r_0$ using Figure 5.6 because $\vartheta = \pi/2$. The calculation of the curl and the Fourier integral yields for odd m values the Fourier component of the magnetic field

$$\boldsymbol{H}_\omega = \frac{J_0[\omega_m^2 + \gamma(i\omega - \gamma)]}{r_0 c^2 k_m[\omega_m^2 + (i\omega - \gamma)^2]}[\boldsymbol{n}_0 \times \boldsymbol{e}_z] e^{i\omega r_0/c}$$

and the spectral radiation density in the $\vartheta = \pi/2$ plane of symmetry

$$\frac{d^2 I_\omega}{d\omega d\Omega} = \frac{J_0^2[(\omega_m^2 - \gamma^2)^2 + \gamma^2\omega^2]}{4\pi^2 c\omega_m^2[(\omega_m^2 - \omega^2 + \gamma^2)^2 + 4\gamma^2\omega^2]} .$$

In the case of even m, the spectral radiation density in the plane of symmetry vanishes. At $\gamma^2 \ll \omega_m^2$ the radiation spectrum has a typical resonance shape with a sharp peak at a frequency of $\omega^2 = \omega_m^2 + \gamma^2$.

5.57 The magnetic field is created by N sources. Its calculation via the Hertz vector as in the previous problem yields

$$\boldsymbol{H}(\boldsymbol{r}_0, t) = -\mathrm{Re}\left\{ i\frac{\boldsymbol{n}_0 \times \boldsymbol{e}_z J_0 k_m}{c r_0} \exp\left[i\omega_m \left(t - \frac{R}{c} \right) \right] \right.$$
$$\times \sum_{s=0}^{N-1} \exp[i s k_m a \sin\vartheta \cos\varphi]$$
$$\left. \times \int_{-l/2}^{l/2} \sin\left[k_m \left(\xi + \frac{l}{2} \right) \right] \exp[i k_m \xi \cos\vartheta] d\xi \right\} .$$

The integral is taken without difficulty and the sum is calculated by the formula for geometric progression. The intensity of radiation averaged over time is found from the formula

$$\overline{\frac{dI}{d\Omega}} = \frac{c r_0^2}{8\pi}|\boldsymbol{H}|^2$$
$$= \frac{J_0^2}{2\pi c \sin^2\vartheta} \frac{\sin^2[(N/2)k_m a \sin\vartheta \cos\varphi]}{\sin^2[(1/2)k_m a \sin\vartheta \cos\varphi]} \begin{Bmatrix} \cos^2[(\frac{m\pi}{2})\cos\vartheta] , \\ \sin^2[(\frac{m\pi}{2})\cos\vartheta] , \end{Bmatrix}$$

where the upper value in parentheses refers to odd m and the lower one refers to even m.

5.58

$$\overline{\frac{dI}{d\Omega}} = \frac{2 i_0^2}{2\pi c k_m^2} \frac{\sin^2[k_m b \sin\vartheta \cos\varphi]}{\sin^4\vartheta \cos^2\varphi} \begin{Bmatrix} \cos^2[(\frac{m\pi}{2})\cos\vartheta] , \\ \sin^2[(\frac{m\pi}{2})\cos\vartheta] , \end{Bmatrix}$$

where the upper value in parentheses refers to the odd m and the lower one refers to even m.

5.59 Because $\boldsymbol{j} = \rho\boldsymbol{v} = \rho \mathrm{d}\boldsymbol{r}/\mathrm{d}t$, $(j_x, j_y, j_z) \to (-j_x, -j_y, j_z)$, in this case, the reflected currents are computed at the reflected points: $\boldsymbol{j}_x(\boldsymbol{r}) = -j'_x(\boldsymbol{r}')$, and so on.

Similarly, using the conventional definitions and formulas for retarded potentials written in Cartesian coordinates yields $(p_x, p_y, p_z) \to (-p_x, -p_y, p_z)$, $(Q_x, Q_y, Q_z) \to (-Q_x, -Q_y, -Q_z)$, $(m_x, m_y, m_z) \to (m_x, m_y, -m_z)$, $(E_x, E_y, E_z) \to (-E_x, -E_y, E_z)$, and $(H_x, H_y, H_z) \to (H_x, H_y, -H_z)$.

5.60 The boundary conditions $H_n = 0$ and $E_\tau = 0$ at the surface ($z = 0$) of a conductor are fulfilled as follows directly from the results obtained in Problem 5.44*. In the specific case of an electric dipole oscillator, the electromagnetic field in the $z > 0$ semispace coincides with the field of the electric dipole oscillator having moment $\boldsymbol{p} = 2\boldsymbol{e}_z f(t) \sin\varphi_0$. It vanishes at $\varphi_0 = 0$ (the dipole is parallel to the plane) and reaches a maximum at $\varphi_0 = \pi/2$ (the dipole is perpendicular to the plane). In the last case, the total radiation energy emitted into the semispace $z > 0$ is four times that emitted by a similar oscillator located far from the conducting plane.

5.61

$$E_\vartheta = H_\alpha = \frac{\omega^3 p_0 a}{2c^3 r}\cos 2\vartheta \cos\alpha \cos\omega t' ,$$

$$E_\alpha = -H_\vartheta = -\frac{\omega^3 p_0 a}{2c^3 r}\cos\vartheta \sin\alpha \cos\omega t' ,$$

$$\overline{\frac{\mathrm{d}I}{\mathrm{d}\Omega}} = \frac{p_0^2 a^2 \omega^6}{32\pi c^6}(\cos^2 2\vartheta \cos^2\alpha + \cos^2\vartheta \sin^2\alpha) .$$

5.62

$$H_r = 0 , \quad H_\vartheta = -\frac{ik}{\sin\vartheta}\frac{\partial u}{\partial\alpha} , \quad H_\alpha = ik\frac{\partial u}{\partial\vartheta} ;$$

$$E_r = k^2 r u + \frac{\partial^2(ru)}{\partial r^2} ,$$

$$E_\vartheta = \frac{1}{r}\frac{\partial^2(ru)}{\partial r\partial\vartheta} , \quad e_\alpha = \frac{1}{r\sin\vartheta}\frac{\partial^2(ru)}{\partial r\partial\alpha} .$$

5.64

$$u = \frac{p_0}{bR}e^{ikR} - \frac{ikp_0}{b}\sum_{l=0}^{\infty}(2l+1)h_l^{(1)}(kb)\frac{\mu_l}{\nu_l}h_l^{(1)}(kr)P_l(\cos\vartheta) ,$$

where

$$\mu_l = \left.\frac{d(rj_l(kr))}{\mathrm{d}r}\right|_{r=a} , \quad \nu_l = \left.\frac{d(rh_l^{(1)}(kr))}{\mathrm{d}r}\right|_{r=a} .$$

Hence, the fields $\boldsymbol{E}$ and $\boldsymbol{H}$ are expressed through the formulas obtained in Problem 5.62. The angular distribution should be found with the use of the asymptotic expression for the spherical Hankel functions (see (1.173)). It this case,

$$E_\alpha = H_\vartheta = 0\,, \quad H_\alpha = ik\frac{\partial u}{\partial\vartheta} = F(\vartheta)\frac{e^{ikr}}{r} = E_\vartheta\,,$$

where

$$F(\vartheta) = \frac{p_0 k^2}{b}\sum_{l=0}^{\infty}\frac{2l+1}{i^{l-1}}\left\{j_l(kb) - h_l^{(1)}(kb)\frac{\mu_l}{\nu_l}\right\}\frac{\mathrm{d}P_l(\cos\vartheta)}{\mathrm{d}\vartheta}\,;$$

$$\overline{\frac{\mathrm{d}I}{\mathrm{d}\Omega}} = \frac{c}{8\pi}|H_\alpha|^2 r^2 = \frac{c}{8\pi}|F(\vartheta)|^2\,.$$

5.66*

$$\varphi(\boldsymbol{r}, t) = \sum_{n=0}^{\infty}\frac{(-1)^n}{c^n n!}\frac{\partial^n}{\partial t^n}\int R^{n-1}\rho(\boldsymbol{r}', t)\mathrm{d}V' = e\sum_{n=0}^{\infty}\frac{(-1)^n}{c^n n!}\frac{\mathrm{d}^n R_0^{n-1}}{\mathrm{d}t^n}\,,$$

$$\boldsymbol{A}(\boldsymbol{r}, t) = e\sum_{n=0}^{\infty}\frac{(-1)^n}{c^n n!}\frac{\mathrm{d}^n(\boldsymbol{v}(t)R_0^{n-1})}{\mathrm{d}t^n}\,,$$

where $R = |\boldsymbol{r} - \boldsymbol{r}'|$ and $R_0 = |\boldsymbol{r} - \boldsymbol{r}_0(t)|$. All quantities on the right-hand sides of these equalities are taken at the same moment of time as the left-hand ones. The delayed interaction formally reduces to instantaneous interaction. The expansions thus obtained can be used in the case of a sufficiently slow ($v \ll c$) and smooth (with limited acceleration and its derivatives of all orders) motion for not very large R_0.

5.67 See the answer to Problem 4.34•.

5.68 At small v/c, formulas (5.29) assume the form

$$\boldsymbol{E} = \frac{e\boldsymbol{r}}{r^3} + 3\frac{e\boldsymbol{r}(\boldsymbol{r}\cdot\boldsymbol{v})}{cr^4} - \frac{e\boldsymbol{v}}{cr^2} + \frac{e\boldsymbol{r}\times(\boldsymbol{r}\times\dot{\boldsymbol{v}})}{c^2r^3}\bigg|_{t'=t-r/c}\,,$$

$$\boldsymbol{H} = \frac{e\boldsymbol{v}\times\boldsymbol{r}}{cr^3} + \frac{e\dot{\boldsymbol{v}}\times\boldsymbol{r}}{c^2r^2}\bigg|_{t'=t-r/c}\,.$$

Here, r is the distance from a point in the region in which the charge propagates to the point of observation.

The first three terms in the expression for $\boldsymbol{E}$ and the first term in the expression for $\boldsymbol{H}$ are proportional to $1/r^2$ and predominate at relatively short distances from the charge (in the quasi-stationary zone). The electric field in this zone is largely reduced to the Coulomb field $\boldsymbol{E} = e\boldsymbol{r}/r^3$; the magnetic field is described by the Biot–Savart formula, $\boldsymbol{H} = e\boldsymbol{v}\times\boldsymbol{r}/cr^3$. At large distances from the charge (in the wave zone), the last terms of $v\,E$ and $v\,H$ diminishing by the $1/r$ law prevail. These terms describe the radiation field and have the form

$$\boldsymbol{E} = \frac{e\boldsymbol{n}\times(\boldsymbol{n}\times\dot{\boldsymbol{v}})}{c^2r}\,, \quad \boldsymbol{H} = \frac{e\dot{\boldsymbol{v}}\times\boldsymbol{n}}{c^2r}\,,$$

where $\boldsymbol{n} = \boldsymbol{r}/r$. The position of the borderline between the quasi-stationary and wave zones is specified by the condition $e/r_b^2 \sim e|\dot{\boldsymbol{v}}|/c^2 r_b$, whence $r_b \approx a(c^2/v^2)$, taking into consideration that $|\dot{\boldsymbol{v}}| \sim v^2/a$, where a is a quantity on the order of the size of the region in which the charge propagates.

5.69

$$\frac{\mathrm{d}I}{\mathrm{d}\Omega} = \frac{e^2}{4\pi c^3}(\dot{\boldsymbol{v}} \times \boldsymbol{n})^2 , \quad I = \frac{2e^2}{3c^3}\dot{\boldsymbol{v}}^2 , \quad \boldsymbol{n} = \frac{\boldsymbol{r}}{r} .$$

5.70 The radiation energy emitted by a charge during a time period $\mathrm{d}t'$ is enclosed between two spheres. One of them has the center at point O, where the charge happens to be at moment t'. The other has the center at point O', where the charge occurs at moment $t' + \mathrm{d}t'$ (Figure 5.9). The radius of the first sphere is R and the radius of the second sphere is $R + c\mathrm{d}t'$. Let us consider an element of volume $\mathrm{d}V = \mathrm{d}S\mathrm{d}R = R^2\mathrm{d}\Omega\,(c - \boldsymbol{n}\cdot\boldsymbol{v})\mathrm{d}t'$. This volume encompasses the electromagnetic energy

$$\mathrm{d}W = \frac{E^2}{4\pi}\mathrm{d}V = \frac{cE^2}{4\pi}\left(1 - \frac{\boldsymbol{n}\cdot\boldsymbol{v}}{c}\right) R^2\mathrm{d}\Omega\,\mathrm{d}t' .$$

It gives the value of the rate of the energy loss $-\mathrm{d}^2\mathcal{E}/\mathrm{d}t'\mathrm{d}\Omega = \mathrm{d}^2 W/\mathrm{d}t'\mathrm{d}\Omega$ contained in the condition for the problem.

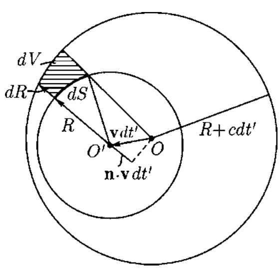

Figure 5.9 Diagram for the calculation of the rate of energy losses by a charged particle.

5.72 The purely longitudinal acceleration is realized at $\boldsymbol{E} \parallel \boldsymbol{v}$, $\boldsymbol{H} = 0$. In this case,

$$-\left(\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t'}\right)_{\parallel} = \frac{2e^4E^2}{3m^2c^3}$$

does not depend on the energy itself. If $\boldsymbol{E} = 0$ and $\boldsymbol{H} \neq 0$, then the acceleration is perpendicular to the velocity and

$$-\left(\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t'}\right)_{\perp} = \frac{2e^4\gamma^2H^2v^2}{3m^2c^5} .$$

At similar field strengths ($E = H$), the ratio

$$\frac{(\mathrm{d}\mathcal{E}/\mathrm{d}t')_{\perp}}{(\mathrm{d}\mathcal{E}/\mathrm{d}t')_{\parallel}} = \gamma^2\frac{v^2}{c^2}$$

is too high for an ultrarelativistic particle. This means that the main role during the arbitrary ultrarelativistic motion of the particle is played by its emission due to the transverse force component.

5.73

$$\frac{\mathrm{d}I(t)}{\mathrm{d}\Omega} = \frac{c}{4\pi} E^2 R^2 = \frac{e^2 \dot{v}^2 \sin^2\vartheta}{4\pi c^3 (1-\beta\cos\vartheta)^6} ,$$

where ϑ is the angle between the directions of the velocity $\boldsymbol{v}$ and emission $\boldsymbol{n}$, $\beta = v/c$. The angular radiation pattern is presented in Figure 5.10. When the particle's velocity v is low, the intensities of forward and backward emission are similar. When v is comparable with c, the forward emission prevails in proportion to the closeness of v to c. The maximum radiation occurs in the ϑ_0 direction described by the following equation:

$$\cos\vartheta_0 = \frac{1}{4\beta}\left(\sqrt{1+24\beta^2} - 1\right) .$$

At $\beta \to 0$, $\vartheta_0 \to \pi/2$; at $\beta \to 1$, $\vartheta_0 \to 0$. Thus, emission in the ultrarelativistic limit largely occurs at small angles to the direction of the particle's velocity. On the assumption of $\vartheta \ll 1$, $\mathrm{d}I/\mathrm{d}\Omega$ can be represented in the form

$$\frac{\mathrm{d}I}{\mathrm{d}\Omega} = \frac{e^2 \dot{v}^2 \vartheta^2}{2\pi c^3 [(mc^2/\mathcal{E})^2 + \vartheta^2]^6} .$$

It follows from this formula that an ultrarelativistic particle emits largely inside the cone with opening angle $\psi = mc^2/\mathcal{E}$.

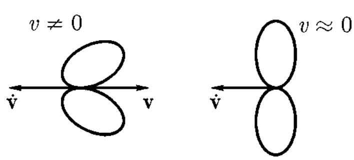

Figure 5.10 Angular diagrams of a particle's emission if the braking is antiparallel to the velocity.

The total intensity of radiation

$$I = \int \frac{\mathrm{d}I}{\mathrm{d}\Omega}\mathrm{d}\Omega = \frac{2e^2\dot{v}^2}{3c^3}\frac{1+\beta^2/5}{(1-\beta^2)^4} .$$

The total rate of energy loss

$$-\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t'} = \frac{2e^2}{3c^3}\cdot\frac{\dot{v}^2}{(1-\beta^2)^3} .$$

5.74 The total bremsstrahlung in the $\mathrm{d}\Omega$ direction for the entire particle's time of flight

$$\frac{\mathrm{d}\Delta W}{\mathrm{d}\Omega} = \int \frac{\mathrm{d}I}{\mathrm{d}\Omega}\mathrm{d}t = \int \left(-\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}\Omega\,\mathrm{d}t'}\right)\mathrm{d}t'$$
$$= \frac{e^2 v_0^2}{16\pi c^3 \tau} \cdot \frac{\sin^2\vartheta}{\cos\vartheta}\left\{\frac{1}{[1-(v_0/c)\cos\vartheta]^4} - 1\right\} ,$$

where ϑ is the angle between the direction of the particle's velocity and the direction of emission $\boldsymbol{n}$.

The observed pulse length depends on the angle ϑ between the directions of the particle's velocity and the emission:

$$\Delta t = \tau\left[1 - \frac{v_0}{2c}\cos\vartheta\right] .$$

5.75

$$-\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t'} = \frac{2e^4 H^2 p^2}{3m^4 c^5} .$$

5.76

$$-\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t'} = \frac{2e^4 H^2 \sin^2\theta}{3m^2 c(1-\beta^2)} .$$

At $\theta \gg \sqrt{1-v^2/c^2}$ a motionless observer positioned far from an electron will record individual pulses of radiation emitted at the moments when the velocity of the electron is directed toward itself (within a cone with the opening angle $\psi \approx \sqrt{1-v^2/c^2}$, see Problem 5.54). The time between pulses (Figure 5.11)

$$\tau = T\left(1 - \frac{v_{\parallel}\cos\theta}{c}\right) \approx T\sin^2\theta ,$$

where $T = 2\pi\mathcal{E}/ecH$ is the cyclotron rotation period, $\mathcal{E}$ is the particle's energy, and $v_{\parallel} = v\cos\theta$ is the projection of the velocity onto the field direction. Thus, owing to the translational motion of the electron with velocity v, the radiation emitted during time T will pass through a motionless sphere for time τ. Hence,

$$I = -\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t'}\frac{T}{\tau} = \frac{2e^4 H^2}{3m^2 c(1-v^2/c^2)} .$$

At $\theta \le \psi \ll 1$,

$$I = \frac{2e^4 H^2}{3m^2 c(1-v^2/c^2)}\frac{2\theta^2}{[(mc^2/\mathcal{E})^2 + 2\theta^2]} .$$

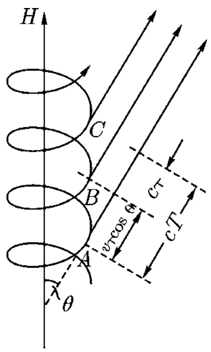

Figure 5.11 Particle emission in a uniform magnetic field.

5.77

$$\frac{\mathrm{d}I}{\mathrm{d}\Omega} = \frac{e^2|\dot{\boldsymbol{v}}|^2}{4\pi c^3} \cdot \frac{(1-\beta\cos\vartheta)^2 - (1-\beta^2)\sin^2\vartheta\cos^2\alpha}{(1-\beta\cos\vartheta)^6}\,, \quad \beta = \frac{v}{c}\,.$$

The polar axis is directed along the velocity and the azimuthal angle α is counted from the direction of the acceleration. The condition for the distribution of radiation is presented in Figure 5.12. The radiation is absent in the direction described by the equation $\gamma(1 - v\cos\vartheta/c) = \sin\vartheta|\cos\vartheta|$. Specifically, there is no radiation in the direction $\vartheta = \arccos(v/c)$ at $\alpha = 0, \pi$ (Figure 5.12a). At $\alpha = \pi/2, 3\pi/2$ (Figure 5.12b) the intensity of radiation differs from zero regardless of ϑ.

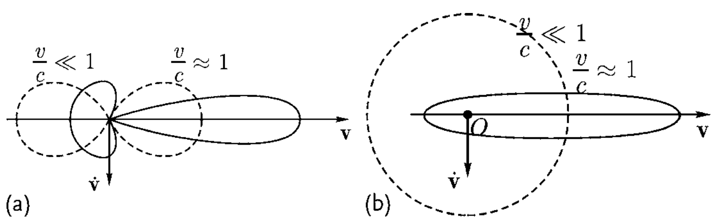

Figure 5.12 Angular diagrams of a particle's emission if its acceleration is perpendicular to the velocity. (a) The case $\alpha = 0, \pi$, and $\vartheta = \arccos(v/c)$. (b) The case $\alpha = \pi/2, 3\pi/2$, and angle ϑ is arbitrary.

5.78*

$$\begin{aligned}
\overline{\frac{\mathrm{d}I}{\mathrm{d}\Omega}} &= -\frac{\overline{\mathrm{d}\mathcal{E}}}{\mathrm{d}\Omega\,\mathrm{d}t'} \\
&= \frac{e^4 H^2 \beta^2}{8\pi^2 m^2 c^3}(1-\beta^2)\int_0^{2\pi} \frac{(1-\beta^2)\cos^2\vartheta + (\beta - \sin\vartheta\cos\alpha)^2}{(1-\beta\sin\vartheta\cos\alpha)^5}\mathrm{d}\alpha \\
&= \frac{e^4 H^2 \beta^2 (1-\beta^2)}{8\pi m^2 c^3} \cdot \frac{1+\cos^2\vartheta - \frac{1}{4}\beta^2(1+3\beta^2)\sin^4\vartheta}{(1-\beta^2\sin^2\vartheta)^{7/2}}\,,
\end{aligned}$$

where $\beta = v/c$.

The reference azimuthal angle α entering the integrand is chosen such that the direction of vector $\boldsymbol{n}$ is characterized by the polar angles ϑ, $\pi/2$. In the ultrarelativistic case $v \approx c$, radiation is concentrated near the orbital plane in the interval of angles $\Delta\vartheta \approx \sqrt{1-\beta^2}$.

5.79*

$$A_{n\vartheta} = \frac{e\beta e^{ikR_0}}{2\pi R_0} \cos\vartheta \int\limits_0^{2\pi} \cos\alpha' e^{i(n\alpha' - n\beta \sin\vartheta \sin\alpha')} \mathrm{d}\alpha' ,$$

$$A_{n\alpha} = \frac{e\beta e^{ikR_0}}{2\pi R_0} \int\limits_0^{2\pi} \sin\alpha' e^{i(n\alpha' - n\beta \sin\vartheta \sin\alpha')} \mathrm{d}\alpha' ,$$

where the wave vector $\boldsymbol{k} = \boldsymbol{n}\omega/c$, the origin of the coordinates is the center of the orbit, the Oz axis is perpendicular to the orbital plane, direction $\boldsymbol{k}$ is characterized by the polar angles ϑ, $\pi/2$, and R_0 is the distance between the center of the orbit and the observation point. Hence,

$$H_{n\alpha} = i\frac{\omega}{c} n A_{n\vartheta} \approx i\frac{\beta e n e^{ikR_0}}{aR_0} \cot\vartheta J_n(n\beta\sin\vartheta) ,$$

$$H_{n\vartheta} = -i\frac{\omega}{c} n A_{n\alpha} \approx \frac{e\beta^2 n e^{ikR_0}}{aR_0} J_n'(n\beta\sin\vartheta) .$$

Polarization of radiation is, generally speaking, elliptical with the principal axis in the directions $\boldsymbol{e}_\alpha$ and $\boldsymbol{e}_\vartheta$, and ratio of the $H_{n\vartheta}$ and $H_{n\alpha}$ semiaxes is $\beta\tan\vartheta J_n' (n\beta\sin\vartheta)/J_n(n\beta\sin\vartheta)$. The direction of traversal of the ellipse is determined by the sign of this ratio. Polarization is circular at $\vartheta = 0$ and linear at $\vartheta = \pi/2$. Moreover, linear polarization is observed at large enough n and β in those directions to which zeros and poles of the function J_n'/J_n correspond.

5.80 The presence of higher harmonics in the field spectrum is attributable to the fact that the field propagation time between different points of the orbit is finite and, generally speaking, comparable with the charge circulation period along the orbit if the charge velocity matches the speed of light c. Because of this, the time needed to pass the observation point by the field emitted from a particle during the half-period when it approaches this point is smaller than the time taken to pass this point by the field emitted during the second half-period. Thus, a certain complicated periodic time dependence of the field depicted by the superposition of the Fourier series harmonics corresponds to the simple harmonic dependence of the particle's coordinates on time.

It can be expected that higher harmonics will disappear at $\beta \to 0$. Indeed, at $x \approx 0$ and $n > 0$, we have $J_n(x) \approx x^n/2^n n!$ and $J_n'(x) \approx x^{n-1}/2^n(n-1)!$ As follows from these formulas, only those harmonics are essential at $\beta \to 0$ that have the least possible values of $|n| = 1$. In this case,

$$H_\alpha = H_{1\alpha} + H_{-1\alpha} = -\frac{e\beta^2}{a}\frac{\cos\vartheta\sin(kR_0)}{R_0} ,$$

$$H_\vartheta = H_{1\vartheta} + H_{-1\vartheta} = \frac{e\beta^2}{a}\frac{\cos(kR)_0}{R_0} .$$

5.81*

$$\frac{\mathrm{d}I_n}{\mathrm{d}\Omega} = \frac{c}{2\pi}|\boldsymbol{H}_n|^2 R_0^2 = \frac{cn^2e^2\beta^2}{2\pi a^2}\left[\cot^2\vartheta J_n^2(n\beta\sin\vartheta) + \beta^2 J_n'^2(n\beta\sin\vartheta)\right] .$$

If the circular motion occurs under the action of a constant and uniform magnetic field H,

$$a = \frac{mc^2\beta}{eH\sqrt{1-\beta^2}} ,$$

$$I_n = \frac{cne^2\beta}{a^2}\left\{2\beta^2 J_{2n}'(2n\beta) - (1-\beta^2)\int\limits_0^{2n\beta} J_{2n}(x)\mathrm{d}x\right\} .$$

5.82 Summation of harmonics yields the angular distribution of radiation $\overline{\mathrm{d}I/\mathrm{d}\Omega}$ averaged over time and calculated earlier by a different method (see Problem 5.78*). In the strongly relativistic case ($\gamma \gg 1$), a sharp emission anisotropy emerges, with the radiation being concentrated in the orbital plane: the ratio

$$\frac{(\overline{\mathrm{d}I/\mathrm{d}\Omega})_{\vartheta=\pi/2}}{(\overline{\mathrm{d}I/\mathrm{d}\Omega})_{\vartheta=0}} \approx \frac{7}{8}\gamma^5 \gg 1 .$$

Introduce the angle $\theta = \pi/2 - \vartheta \ll 1$ between the direction of observation and the orbital plane and write down the angular distribution in the form

$$\overline{\frac{\mathrm{d}I}{\mathrm{d}\Omega}} = \frac{e^4H^2\gamma^3(7+12\gamma^2\theta^2)}{16\pi m^2c^3(1+\gamma^2\theta^2)^{7/2}} .$$

5.83** Use formula (5.77) and choose the system of coordinates such that the particle's orbit is in the xy plane and the line of sight is in the xz plane (Figure 5.3). At $\tau = 0$, the particle is at the origin of the coordinates. Then,

$$\boldsymbol{s}(\tau) = \boldsymbol{e}_x a\sin\left(\frac{v\tau}{a}\right) - \boldsymbol{e}_y a\left[\cos\left(\frac{v\tau}{a}\right) - 1\right] ,$$

$$\boldsymbol{v}(\tau) = \dot{\boldsymbol{s}}(\tau) = \boldsymbol{e}_x v\cos\left(\frac{v\tau}{a}\right) + \boldsymbol{e}_y v\sin\left(\frac{v\tau}{a}\right) ,$$

where $a = cp/eH$ is the Larmor radius (see Section 4.2). Take account of the relativistic effect of forward emission within the angle $\theta_0 \approx \gamma^{-1} \ll 1$ and leave powers of the angle θ between the line of sight and the Ox axis not higher than the second power. In this approximation,

$$\boldsymbol{n} \times \boldsymbol{v}(\tau) = \boldsymbol{e}^{(1)} v\theta \cos\left(\frac{v\tau}{a}\right) + \boldsymbol{e}^{(2)} v \sin\left(\frac{v\tau}{a}\right) ,$$

$$\omega\tau - \boldsymbol{k}\cdot\boldsymbol{s}(\tau) = \omega\tau - \frac{\omega a}{c}\left(1 - \frac{\theta^2}{2}\right)\sin\left(\frac{v\tau}{a}\right) . \tag{1}$$

Here, $\boldsymbol{e}^{(1)} = \boldsymbol{e}_y$, $\boldsymbol{e}^{(2)} = \boldsymbol{n} \times \boldsymbol{e}_y$, and $\boldsymbol{n}$ make up a triad of mutually perpendicular vectors.

Because the directional pattern of radiation is very narrow, the wave packet emitted in the $\boldsymbol{n}$ direction is collected from the small arc of the circle on the order a/γ passed by the particle for time $\Delta\tau \approx a/v\gamma$. For this reason, the main contribution to the time integrals comes from the times on the order of $\Delta\tau$, and series expansion of trigonometric function is possible in equalities (1). If is sufficient to take account of the first nonvanishing terms in the pre-exponent; however, the cubic term should be taken into consideration too in the sine-series expansion in the exponent. Moreover, the expansion in $\gamma^{-2} \ll 1$ needs to be performed. After all these expansions have been performed, the time integrals are reduced to the integrals specified in the statement of the problem by the substitution of variables:

$$x = \frac{c\tau}{a(\gamma^{-2} + \theta^2)^{1/2}} , \quad \xi = \frac{\omega a}{3c}(\gamma^{-2} + \theta^2)^{3/2} . \tag{2}$$

As a result, the spectral–angular distribution of radiation is obtained from (5.77):

$$\frac{d^2 I'_\omega}{d\omega d\Omega} = \frac{e^2}{3\pi^2 c}\left(\frac{a\omega}{c}\right)^2 (\gamma^{-2} + \theta^2)^2 \left[K^2_{2/3}(\xi) + \frac{\theta^2}{\gamma^{-2} + \theta^2} K^2_{1/3}(\xi)\right] . \tag{3}$$

The resultant quantity is the spectral radiation density of a particle from the small element of the trajectory at which a radiation pulse in a given direction $\boldsymbol{n}$ is formed. It has the dimension of energy times time. The time-averaged spectral radiation power is more convenient to measure; it is obtained by means of division of the spectral radiation density obtained by the particle's orbital circulation period T (i.e., by the time between short bursts visible to the observer):

$$T = \frac{2\pi}{\Omega} = \frac{2\pi\gamma}{\omega_c} = \frac{2\pi mc\gamma}{eH} \tag{4}$$

(see Problem 4.52*. The period T is written here through the angular velocities $\omega_H = eH/mc$ and $\Omega = \omega_H/\gamma$ used in this problem, where the former was denoted by ω_c and referred to as the cyclotron frequency). Therefore,

$$\frac{d^2 I_\omega}{d\omega d\Omega} = \frac{e^2 \omega_H}{6\pi^3 c\gamma}\left(\frac{a\omega}{c}\right)^2 (\gamma^{-2} + \theta^2)^2 \left[K^2_{2/3}(\xi) + \frac{\theta^2}{\gamma^{-2} + \theta^2} K^2_{1/3}(\xi)\right] . \tag{5}$$

This rather complicated function is simplified in the limiting cases. Specifically, the use of (1.170) at $\xi \gg 1$ yields

$$\frac{d^2 I_\omega}{d\omega d\Omega} = \frac{e^2\omega}{4\pi^2 c}(\gamma^{-2}+\theta^2)^{1/2} \times \left(1+\frac{\theta^2}{\gamma^{-2}+\theta^2}\right)\exp\left[-\frac{2a\omega}{3c}(\gamma^{-2}+\theta^2)^{3/2}\right] . \tag{6}$$

At $\xi \to 0$, the spectral power tends to vanish but reaches the maximum value at $\xi \approx 1$. Find, on the assumption of $\theta = 0$ in the last condition, the critical frequency value at which exponential breakdown of the spectrum occurs:

$$\omega_c = \frac{3eH}{mc}\gamma^2 . \tag{7}$$

Certainly, this value is determined only in terms of the order of magnitude. It is in agreement with estimate (5.78). Because the angular velocity of the particle (fundamental velocity) $\omega_0 = \Omega = ecH/\mathcal{E} = eH/mc\gamma$ (see Problem 4.52*), the critical frequency value thus obtained corresponds to the Fourier harmonic with number

$$n_c = \frac{\omega_c}{\omega_0} = 3\gamma^3 \gg 1 . \tag{8}$$

5.84

$$\frac{dI_\omega}{d\omega} = \frac{\sqrt{3}e^2\omega_H}{2\pi c}\frac{2\omega}{\omega_c}\int\limits_{2\omega/\omega_c}^{\infty} K_{5/3}(x)dx .$$

Use the asymptotic formulas (1.168) and (1.170) to find the spectral distribution of synchrotron radiation in the limiting cases:

$$\frac{dI_\omega}{d\omega} = \begin{cases} \frac{3^{3/2}}{2\pi}\Gamma\frac{5}{3}\frac{e^2\omega_H}{c}\left(\frac{\omega}{\omega_c}\right)^{1/3} , & \omega \ll \omega_c ; \\ \frac{\sqrt{3}}{2\sqrt{\pi}}\frac{e^2\omega_H}{c}\left(\frac{\omega}{\omega_c}\right)^{1/2}\exp\left(-\frac{2\omega}{\omega_c}\right) , & \omega \gg \omega_c . \end{cases}$$

5.85* An ultrarelativistic particle performs helical motion and emits within the radiation angle γ^{-1} along the generatrix of a cone having opening angle $\pi - 2\alpha$. Repeat in modified geometry the calculation analogous with that performed for Problem 5.83** to obtain the spectral radiation power in a given direction:

$$\frac{d^2 I_\omega}{d\omega d\Omega} = \frac{1}{6\pi^3}\frac{e^2\omega_H}{\gamma c\cos^2\alpha}\left(\frac{a\omega}{c\cos\alpha}\right)^2 \times (\gamma^{-2}+\theta^2)^2\left[K_{2/3}^2(\xi)+\frac{\theta^2}{\gamma^{-2}+\theta^2}K_{1/3}^2(\xi)\right] . \tag{1}$$

Here, θ is the angle between the direction of observation and the generatrix of the radiation cone, $\pi/2-\alpha \gg \theta$, $\xi = (a\omega/3c\cos\alpha)(\gamma^{-2}+\theta^2)^{3/2}$. The period between radiation pulses was determined in Problem 5.76 as

$$T = \frac{2\pi\mathcal{E}}{ecH}\cos^2\alpha = \frac{2\pi\gamma}{\omega_H}\cos^2\alpha . \tag{2}$$

It was used to calculate (1). The critical frequency is found from the condition $\xi = 1$ at $\theta = 0$:

$$\omega_c = \frac{3c\gamma^3 \cos\alpha}{a} = 3\omega_H \gamma^2 \cos\alpha \,. \tag{3}$$

5.86 The spectral radiation power of an electron cloud is written in the form of the integral of the product of the distribution function and the spectral function of a single electron found in Problem 5.85*:

$$\frac{dP_\omega}{d\omega} = \int \frac{d^2 I_\omega}{d\omega d\Omega} f(p) d^3 p \,.$$

The integration should actually be performed over the direction of electron momenta, which is equivalent to the integration over the angle θ and is possible in the same way as in Problem 5.84 using the formulas indicated in its statement. For the purpose of integration, consider the angle α as being constant and replace it by $\pi/2 - \Theta$, in accordance with the condition for the problem. This yields

$$\frac{dP_\omega}{d\omega} = \frac{\sqrt{3}}{8\pi^2} \frac{N_0 e^2 \omega_H}{c \sin^2\Theta} \frac{2\omega}{\omega_c} \int\limits_{2\omega/\omega_c}^{\infty} K_{5/3}(x) dx \,,$$

where $\omega_c = 3eH\gamma_0^2 \sin\Theta / mc$. The limiting cases of low and high frequencies are presented in the answer to Problem 5.84.

5.87* The spectral radiation power of the monoenergetic electron cloud from the preceding problem should be integrated over electron energies $d\mathcal{E} = mc^2 d\gamma$ with the corresponding distribution function:

$$\frac{dP_\omega}{d\omega} = \frac{\sqrt{3}}{8\pi^2}(\nu - 1) N_0 \gamma_*^{\nu-1} \frac{e^2 \omega_H}{c \sin^2\Theta} \int\limits_{\gamma_*}^{\infty} \frac{d\gamma}{\gamma^\nu} \frac{2\omega}{\omega_c} \int\limits_{2\omega/\omega_c}^{\infty} K_{5/3}(x) dx \,. \tag{1}$$

Passing from integration over the energies to integration over the variable $d\eta = d(2\omega/\omega_c) = -2\eta d\gamma/\gamma$ leads to

$$\int\limits_{\gamma_*}^{\infty} \frac{d\gamma}{\gamma^\nu} \eta \int\limits_{\eta}^{\infty} K_{5/3}(x) dx = \frac{1}{2} \left(\frac{3\omega_H \sin\Theta}{2\omega} \right)^{(\nu-1)/2} \times \int\limits_0^{\eta_*} \eta^{(\nu-1)/2} d\eta \int\limits_\eta^\infty K_{5/3}(x) dx \,. \tag{2}$$

The integral is reduced to the tabulated one at $\eta_* \gg 1$, that is, at the frequencies $\omega \gg \gamma_*^2 \omega_H \sin\Theta$. In this cases, the η_* limit is replaced by ∞ and formula 11.4.22

from Abramovitz and Stegun (1965) is used after integration by parts. As a result,

$$\frac{\mathrm{d}P_\omega}{\mathrm{d}\omega} = \frac{3^\nu(\nu-1)}{8\pi^2(\nu+1)}\Gamma\left(\frac{3\nu+19}{12}\right)\Gamma\left(\frac{3\nu-1}{12}\right) \times N_0\gamma_*^{\nu-1}(\sin\Theta)^{(\nu-5)/2}\frac{e^2\omega_\mathrm{H}}{c}\left(\frac{\omega_\mathrm{H}}{\omega}\right)^{(\nu-1)/2} . \tag{3}$$

Observation of cosmic sources of radio-frequency radiation not infrequently makes it possible to determine the spectral indices of such emission α: $\mathrm{d}P_\omega/\mathrm{d}\omega \propto \omega^{-\alpha}$. Comparison with (3) allows the spectral index of emitting electrons to be found:

$$\nu = 2\alpha + 1 , \quad \alpha = \frac{\nu-1}{2} . \tag{4}$$

5.88 $\nu \approx 1.6$ at $\mathcal{E} < \mathcal{E}_*$ and $\nu \approx 3.0$ at $\mathcal{E} > \mathcal{E}_*$. The spectrum break energy can be estimated bearing in mind that the maximum radiation of an electron with energy $\mathcal{E}$ occurs at frequency $\omega \approx 3\omega_\mathrm{H}(\mathcal{E}/mc^2)^2$ (to be precise, at $0.45\omega_\mathrm{H}(\mathcal{E}/mc^2)^2$, according to numerical calculation). For the purpose of estimation, each electron can be regarded as creating a discrete radiation line with such frequency. Then, $\omega_* \approx 0.45\omega_\mathrm{H}\gamma_*^2$. Substituting the numerical values yields $\mathcal{E}_* \approx 5\times 10^{11}$ eV.

5.89* The differential emission is

$$\frac{\mathrm{d}^2\mathcal{E}^{\mathrm{rad}}}{\mathrm{d}t'\mathrm{d}\Omega} = -\frac{\mathrm{d}^2\mathcal{E}}{\mathrm{d}t'\mathrm{d}\Omega} = \frac{e^2}{4\pi c^3}\frac{\{\boldsymbol{n}\times[(\boldsymbol{n}-\boldsymbol{v}/c)\times\dot{\boldsymbol{v}}]\}^2}{(1-\boldsymbol{n}\cdot\boldsymbol{v}/c)^5} \tag{1}$$

(formulas (5.67) and (5.60) are used). Quantities $\boldsymbol{v}(t)$ and $\dot{\boldsymbol{v}}(t)$ need to be calculated. Use the equation of motion for a relativistic particle (4.58) written via the velocity (see the answer to Problem 4.41) to obtain

$$\begin{cases} m\gamma\dot{\boldsymbol{v}}_\perp + m\gamma^3\frac{\boldsymbol{v}_\perp}{c^2}(v_\parallel\dot{v}_\parallel + \boldsymbol{v}_\perp\cdot\dot{\boldsymbol{v}}_\perp) & = e\boldsymbol{E}_0\cos\omega_0 t , \\ m\gamma\dot{v}_\parallel + m\gamma^3\frac{v_\parallel}{c^2}(v_\parallel\dot{v}_\parallel + \boldsymbol{v}_\perp\cdot\dot{\boldsymbol{v}}_\perp) & = 0 , \end{cases} \tag{2}$$

where both parts of the equation of motion are projected onto the direction perpendicular and parallel to the particle's initial velocity $\boldsymbol{v}_0$. Thereafter, it is assumed that $v_\perp \ll v_\parallel$, $v_\parallel \approx v_0 \approx c$, $\gamma = (1-v_0^2/c^2)^{-1/2} \gg 1$. It follows from the second equation in (2) that $\dot{v}_\parallel \approx -\boldsymbol{v}_\perp\cdot\dot{\boldsymbol{v}}_\perp/c$, that is, $\dot{v}_\parallel \approx v_\perp\dot{v}_\perp/c \ll \dot{v}_\perp$. The first equation in (2) yields with an accuracy up to the terms $(v_\perp/c)^2 \ll 1$

$$\boldsymbol{v}_\perp(t) = \boldsymbol{u}\sin\omega_0 t , \quad \boldsymbol{u} = \frac{e\mathcal{E}_0}{m\omega_0\gamma} . \tag{3}$$

The substitution of the $\boldsymbol{v}$ and $\dot{\boldsymbol{v}}$ values thus obtained into (1) gives

$$\frac{\mathrm{d}^2\mathcal{E}^{\mathrm{rad}}}{\mathrm{d}t'\mathrm{d}\Omega} = \frac{e^2\omega_0^2\cos^2\omega_0 t'}{4\pi c^3} \times \frac{u^2(1-\boldsymbol{n}\cdot\boldsymbol{v}/c)^2 + 2(\boldsymbol{v}\cdot\boldsymbol{u})(\boldsymbol{n}\cdot\boldsymbol{u})(1-\boldsymbol{n}\cdot\boldsymbol{v}/c) - \gamma^{-2}(\boldsymbol{n}\cdot\boldsymbol{u})^2}{(1-\boldsymbol{n}\cdot\boldsymbol{v}/c)^5} . \tag{4}$$

The angular distribution of total radiation ensues from the integration of (4) over $\mathrm{d}t'$ within $[0, L/c]$, where $L/c \gg 2\pi/\omega_0$ is the time of the particle's motion in the undulator. This leads to substitution of $\cos^2 \omega_0 t'$ with 1/2 and multiplication of (4) by L/c Choose the Oz axis of the Cartesian system of coordinates along the undulator length and count the polar angle θ of the $\boldsymbol{n}$ vector from the Oz axis and its azimuthal angle φ from the direction of $\boldsymbol{E}_0$. This permits us to write $\boldsymbol{n}\cdot\boldsymbol{v} = v_0 \cos\theta$, $\boldsymbol{n}\cdot\boldsymbol{u} = u \sin\theta \cos\varphi$. and

$$\frac{\mathrm{d}\mathcal{E}^{\mathrm{rad}}}{\mathrm{d}\Omega} = \frac{e^4 E_0^2 L}{8\pi c^4 m^2 \gamma^2} \frac{(1-\beta_0\cos\theta)^2 - \gamma^{-2}\sin^2\theta\cos^2\varphi}{(1-\beta_0\cos\theta)^5} . \tag{5}$$

In the relativistic case, the characteristic angles $\theta \approx \gamma^{-1} \ll 1$; therefore, the angular distribution can be simplified as

$$\frac{\mathrm{d}\mathcal{E}^{\mathrm{rad}}}{\mathrm{d}\Omega} = \frac{e^4 E_0^2 L}{\pi c^4 m^2 \gamma^2} \frac{1}{(\gamma^{-2} + \theta^2)^3} . \tag{6}$$

Integration of (6) over $\mathrm{d}\Omega = 2\pi\theta\,\mathrm{d}\theta$ gives the total radiation:

$$\mathcal{E}^{\mathrm{rad}} = \frac{e^4 E_0^2 L}{2c^4 m^2}\gamma^2 . \tag{7}$$

The characteristic frequencies at which undulator radiation occurs can be estimated from the following consideration. Transformation of the initial field $\boldsymbol{E} = \boldsymbol{E}_0 \cos\omega_0 t$ into a frame of reference moving along the axis of the undulator with the unperturbed particle's velocity $\boldsymbol{v}_0$, using formulas (4.70), yields

$$\boldsymbol{E}' = \gamma \boldsymbol{E}_0 \cos(\omega'\tau + k'\zeta) , \quad \boldsymbol{H}' = -\frac{1}{c}\boldsymbol{v}_0 \times \boldsymbol{E}' , \tag{8}$$

where $\omega' = \gamma\omega_0$ and $k' = \omega' v_0/c^2$ are the frequency and the wave vector of the transformed field, respectively, andτ and ζ are the time and the coordinate in the accompanying system in which the particle performs nonrelativistic motion. In this system, the relations characteristic of flat monochromatic waves in a vacuum are satisfied with accuracy up to the $\gamma^{-2} \ll 1$ order terms: the dispersion law $\omega'^2 - c^2 k'^2 \approx 0$, the relationship between field intensities $\boldsymbol{E}' \perp \boldsymbol{H}'$, $E' \approx H'$. This means that in the system accompanying the particle, it is subject to the action of the field of a flat monochromatic wave (or "equivalent photons," in the quantum language, see Section 6.4). The emission by the particle can be interpreted as scattering of this wave.

The frequency of the wave ω' emitted by a particle in the accompanying system during transition to the laboratory frame of reference undergoes one more Doppler shift (see the formula presented in the statement of Problem 3.35$^\bullet$) and acquires the value

$$\omega = \frac{\omega'}{\gamma(1 - v_0\cos\theta/c)} \approx \frac{2\omega_0}{\gamma^{-2} + \theta^2} \approx \omega_0\gamma^2 . \tag{9}$$

At $\mathcal{E} = 5$ GeV the electron relativistic factor $\gamma \approx 10^4$, the emitted frequencies $\omega \approx 2\pi c/\lambda_0\gamma^2 \approx 6 \times 10^{18}\,\mathrm{s}^{-1}$, and the wavelengths $\lambda \approx \lambda_0/\gamma^2 = 3 \times 10^{-8}$ cm (X-ray range).

5.90 The system of equations of motion with the initial conditions specified in the statement of the problem is readily solved, yielding

$$s(t) = a(e_x \cos\omega_0 t - e_y \sin\omega_0 t) + \beta_{\parallel} c t e_z \ .$$

The trajectory is helical and the velocity of the electrons makes angle $\Theta = \beta_\perp \ll 1$ with the undulator axis. The character of the radiation depends on the relationship between Θ and characteristic angle θ between the particle's velocity and the direction of emission. At $1 \gg \Theta \gg \theta \approx \gamma^{-1}$, the emission is directed (as in Problem 5.85*) along the generatrix of the cone determined by the electron velocity within the γ^{-1} angle. An observer can see short bursts lasting $a/c\beta_\perp\gamma$ each. The calculation of the spectral–angular distribution of the radiation power is possible in the same fashion as in Problems 5.83** and 5.85*, assuming the undulator to be infinitely long on the strength of the condition $L \gg \lambda_0$. The calculation is followed by integration over the solid angle (as in Problem 5.84) and multiplication of the result by the time of electron motion through the undulator L/c. This leads to

$$\frac{\mathrm{d}\mathcal{E}^{\mathrm{rad}}}{\mathrm{d}\omega} = \frac{\sqrt{3}e^2\omega_0 L}{2\pi c^2\beta_\perp^2}\frac{2\omega}{\omega_\mathrm{c}}\int\limits_{2\omega/\omega_\mathrm{c}}^{i} n\,f\,t\,y\,K_{5/3}(x)\mathrm{d}x \ ,$$

where $\omega_\mathrm{c} = 3\omega_0\gamma^3\beta_\perp$.

At $\Theta \approx \gamma^{-1}$ radiation fills the entire cone, and the method used to calculate the spectrum becomes invalid.

5.91* The solution of Problem 5.60 brought about the expressions for the nth harmonic of the radiation field from a single charge. Evidently, these expressions for different charges differ from each other only in the initial phases. Denote by ψ_l the phase shift of the field of the lth electron with respect to the field of that electron to which the first number is ascribed and write down the resultant field through the real functions:

$$H_{n\vartheta} = \frac{e\beta^2 n}{a R_0} J_n'(n\beta\sin\vartheta)\sum_{l=1}^{N}\cos n\left(\omega t - \frac{\omega R_0}{c} + \psi_l\right) \ .$$

The expression for $H_{n\alpha}$ is analogous. The mean radiation intensity for the period $T = 2\pi/\omega$ is equal to

$$\mathrm{d}I_{nN} = \frac{c}{4\pi}\cdot\frac{1}{T}\int\limits_0^T \left(H_{n\vartheta}^2 + H_{n\alpha}^2\right)\mathrm{d}t\, R_0^2\mathrm{d}\Omega = S_N \mathrm{d}I_n \ ,$$

where $\mathrm{d}I_n$ is the intensity of emission by a single electron determined in the previous problem and S_N is the coefficient taking account of interference between electron fields ("coherence factor"):

$$S_N = N + \sum_{\substack{l,l'=1\\(l\neq l')}}^{N} \cos n(\psi_l - \psi_{l'}) \ .$$

Let us consider the following specific cases:

1. The totally disordered arrangement of electrons in the orbit

$$\sum \cos n(\psi_l - \psi_{l'}) = 0 \, .$$

2. The regular arrangement of electrons in the orbit

$$\psi_l = \frac{2\pi}{N}(l-1)$$

and

$$S_N = N \sum_{l=2}^{N} \cos 2\pi(l-1)\frac{n}{N} = \frac{N}{2}\left[\sum_{l=1}^{N} e^{2\pi(l-1)\frac{n}{N}i} + \sum_{l=1}^{N} e^{-2\pi(l-1)\frac{n}{N}i}\right]$$
$$= N(-1)^n \frac{\sin n\pi}{\tan(n\pi/N)} = \begin{cases} 0\,, & \text{if } n/N \text{ is not an integer}\,, \\ N^2\,, & \text{if } n/N \text{ is an integer}\,. \end{cases}$$

3. Electrons gather in bunches and all differences $\psi_l - \psi_{l'}$ are small. For not very large n at which the bunch size is small compared with the respective wavelength, all $\cos n(\psi_l - \psi_{l'})$ in the expression for S_N can be substituted with unities. Then, $S_N = N^2$. The factor S_N decreases as n increases; its value depends on the character of the arrangement of the electrons within the bunch and cannot be specified in a general form.

The appearance of the factor N^2 is not, generally speaking, a definitive sign of high radiation intensity because it is associated with a higher harmonic number. Let $n = kN$, where k is an integer. In the nonrelativistic case, at $2kN\beta \ll 1$, the use of the asymptotic formulas (1.155) for the Bessel functions leads (even at $k = 1$) to

$$I_{NN} = \frac{2e^2 c N(N+1)(\beta N)^{N+2}}{a^2(2N+1)(2N)!}\,,$$

which in turn gives

$$I_{11} = \frac{2e^2 c\beta^4}{3a^2}\,, \quad I_{22} = \frac{8e^2 c\beta^6}{5a^2}\,, \quad I_{33} = \frac{243 e^2 c\beta^8}{70a^2}\,,$$

that is, the addition of each electron increases the multipolarity of radiation and adds the small multiplier β^2.

In the opposite, strongly relativistic, case the results obtained in Problems 5.83** and 5.84 should be used after the substitution of $\omega \to n\omega_0$, $I_n = \omega_0 \mathrm{d}I_\omega/\mathrm{d}\omega$. At $n = N$ (i.e., $k = 1$),

$$I_{NN} = \frac{e^2 c}{\pi\sqrt{3}a^2\gamma^2} N^3 \int_{x_N}^{\infty} K_{5/3}(x)\mathrm{d}x\,,$$

where $x_N = 2N/3\gamma^3$. In specific cases,

$$I_{NN} \approx \begin{cases} \frac{3^{1/6}\Gamma(2/3)}{\pi}\frac{e^2c}{a^2}N^{7/3}\,, & N \ll 3\gamma^3/2\,; \\ \frac{1}{\sqrt{2\pi}}\frac{e^2c}{a^2\gamma^{1/2}}N^{5/2}\exp\left(-\frac{2N}{3\gamma^3}\right)\,, & N \gg 3\gamma^3/2\,. \end{cases}$$

The comparison of these results with the case of disordered arrangement of electrons in the orbit leads to the conclusion that the factor of coherent amplification of radiation at $N \approx \gamma^3$ has a value on the order of γ^2.

5.92 Assume that electrons gather in a bunch and substitute the coherence factor in the form

$$S_N = N + \sum_{l=1}^{N}\left\{\cos n\psi_l \sum_{l'=1}^{N}{}' \cos n\psi_{l'} + \sin n\psi_l \sum_{l'=1}^{N}{}' \sin n\psi_{l'}\right\}\,,$$

where the prime symbol above the sum designates the absence of the term with $l' = l$. In the case of symmetric arrangement of electrons within a bunch with respect to the relative zero azimuth, the average of the sine is zero, and the coherence factor transforms into

$$S_N = N + N(N-1)\left(\overline{\cos n\psi}\right)^2 \approx N + N^2\left(\overline{\cos n\psi}\right)^2\,,$$

where the first and the second terms correspond to incoherent and coherent radiation, respectively. Averaging yields

$$I_{nN} = N I_n + N^2 I_n \times \begin{cases} \frac{\sin^2(n\varphi/2)}{(n\varphi/2)^2}\,, \\ \exp\left[-(n\varphi)^2/2\right]\,. \end{cases}$$

Here, the upper and the lower rows correspond to the uniform and the Gaussian distributions of electrons in the bunch, respectively. I_n is the intensity of emission by a single electron.

5.93

$$\mathcal{E}^{\text{rad}} = \frac{2\sqrt{2}q^2}{3m^{3/2}c^3}\int_{r_{\min}}^{\infty}\left(\frac{\mathrm{d}U}{\mathrm{d}r}\right)^2\frac{\mathrm{d}r}{\sqrt{\mathcal{E} - U(r)}}\,,$$

where $U(r_{\min}) = \mathcal{E}$.

5.94

$$\frac{\mathcal{E}^{\text{rad}}}{\mathcal{E}} = \frac{16q}{45Ze}\left(\frac{v}{c}\right)^3\,.$$

5.95 Let us choose the origin of the coordinates in the charge system's center of inertia. Then, the electric dipole moment of the system

$$\boldsymbol{p} = e_1\boldsymbol{r}_1 + e_2\boldsymbol{r}_2 = \mu\left(\frac{e_1}{m_1} - \frac{e_2}{m_2}\right)\boldsymbol{r}\,, \tag{1}$$

where $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$, $\mu = m_1 m_2/(m_1 + m_2)$.

The e/m ratios of charges being different, $\boldsymbol{p} \neq 0$ and the system largely emits as an electric dipole ($v/c \ll 1$). The instantaneous intensity

$$I(t) = \frac{2\ddot{\boldsymbol{p}}^2}{3c^3} = \frac{2\mu^2}{3c^3}\left(\frac{e_1}{m_1} - \frac{e_2}{m_2}\right)^2 \ddot{\boldsymbol{r}}^2(t')\,.$$

According to the equations of charge motion, $\mu\ddot{\boldsymbol{r}} = e_1 e_2 \boldsymbol{r}/r^3$; therefore,

$$I = \frac{2e_1^2 e_2^2}{3c^3}\left(\frac{e_1}{m_1} - \frac{e_2}{m_2}\right)^2 \frac{1}{r^4}\,.$$

When calculating the time-averaged intensity of radiation $\overline{I} = (1/T)\int_0^T I\,dt'$, one should perform the integration over the α angle instead of integration over t' according to the equation $dt' = \mu r^2 d\alpha/K$ (K is the system's angular momentum) with the use of the trajectory equation. This yields

$$\overline{I} = \frac{2^{3/2}}{3c^3}\left(\frac{e_1}{m_1} - \frac{e_2}{m_2}\right)^2 \frac{\mu^{5/2}|e_1 e_2|^3|\mathcal{E}|^{3/2}}{K^5}\left(3 - \frac{2|\mathcal{E}|K^2}{\mu e_1^2 e_2^2}\right)\,.$$

5.96

$$\overline{\frac{d\boldsymbol{K}}{dt}} = -\frac{2^{7/2}\mu^{3/2}|\mathcal{E}|^{3/2}}{3c^3}\left(\frac{e_1}{m_1} - \frac{e_2}{m_2}\right)^2 \frac{\boldsymbol{K}}{K^3}\,.$$

5.97* By acting as before in the solution of Problem 5.95, write down the second derivative of the dipole momentum in the form:

$$\ddot{\boldsymbol{p}} = \left(\frac{e_1}{m_1} - \frac{e_2}{m_2}\right)\frac{e_1 e_2 \boldsymbol{r}}{r^3}\,. \tag{1}$$

No difficulites are encountered in the calculation of A in (5.83) and (5.84), whereas calculation of B implies knowledge of $\ddot{p}_z$, that is, the projection of $\ddot{\boldsymbol{p}}_z$ on the direction of the initial motion of particles undergoing scattering, in the form of a coordinate function (coordinates r, α are the polar coordinates in the plane of the particles' relative motion). It should be taken into consideration that the angle α in the equation for the trajectory of the relative motion $-1+\epsilon\cos\alpha = a(\epsilon^2-1)/r$ is counted from the axis of symmetry of the trajectory (Oz' axis). Thus, $y' = r\sin\alpha$ and $z' = r\cos\alpha$. The angle between the Oz and Oz' axes equals $\pi - \alpha_0$ ($\cos\alpha_0 = 1/\epsilon$); therefore, $z = -z'\cos\alpha_0 - y'\sin\alpha_0 = -r[(1/\epsilon)\cos\alpha + \sqrt{\epsilon^2-1}\sin\alpha/\epsilon)$. Using equation (1) and bearing in mind that $\sin\alpha$ is an odd function leads to

$$\int_0^{\infty}\int_{-\infty}^{+\infty} \ddot{p}_z{}^2 \mathrm{d}t s\mathrm{d}s = e_1^2 e_2^2 \left(\frac{e_1}{m_1} - \frac{e_2}{m_2}\right)^2 \times \int_0^{\infty}\int_{-\infty}^{+\infty} \frac{\cos^2\alpha + (\epsilon^2 - 1)\sin^2\alpha}{\epsilon^2 r^4} \mathrm{d}t s\mathrm{d}s\,.$$

Express $\cos^2\alpha$ and $\sin^2\alpha$ through r and ϵ using the trajectory equation and substitute $\epsilon^2 = u$, $s\mathrm{d}s = \frac{a^2}{2}\mathrm{d}u$. After this, the written integral transforms to the form:

$$\frac{a}{v_0}\int_{2a}^{\infty} \frac{\mathrm{d}r}{r^3} \int_1^{(r/a-1)^2} \left[-\frac{a^2}{r^2}u + \left(4\frac{a^2}{r^2} - 2\frac{a}{r} + 1\right) + \left(-5\frac{a^2}{r^2} + 6\frac{a}{r} - 2\right)\frac{1}{u} + 2\left(\frac{a}{r} - 1\right)^2 \frac{1}{u^2}\right] \frac{\mathrm{d}u}{\sqrt{(r/a-1)^2 - u}}\,.$$

The calculation of the integral over $\mathrm{d}u$ is associated with the emergence of a logarithmic term modified by integration by parts. The calculation of the outer integral over $\mathrm{d}r$ requires the substitution $x = 2a/r$, which reduces this integral to the sum of several B functions:

$$B(k,l) = \int_0^1 x^{k-1}(1-x)^{l-1}\mathrm{d}x = \frac{\Gamma(k)\Gamma(l)}{\Gamma(k+l)}\,.$$

Finally, we arrive at

$$A = \frac{8\pi}{9} e_1 e_2 \left(\frac{e_1}{m_1} - \frac{e_2}{m_2}\right)^2 \mu v_0\,, \quad B = 0\,.$$

5.98* Considering the approximation $v = \text{const}$, the particle's trajectory is a straight line. Let the particle's motion occur in the xz plane parallel to the Ox axis. In these coordinates, $\boldsymbol{n} = (n_x, n_y, n_z)$, where

$$n_x = \sin\vartheta\cos\alpha\,, \quad n_y = \sin\vartheta\sin\alpha\,,$$
$$n_z = \cos\vartheta\,, \quad \boldsymbol{r} = (s, 0, vt')\,, \quad r = \sqrt{s^2 + v^2 t'^2}\,, \quad \boldsymbol{v} = (0,0,v)\,.$$

The known formula $\boldsymbol{v} = c^2\boldsymbol{p}/\mathcal{E}$, where $\mathcal{E} = mc^2/\sqrt{1-\beta^2}$, $\beta = v/c$, is used to obtain $\dot{\boldsymbol{v}} = c^2\dot{\boldsymbol{p}}/\mathcal{E} - c^2\boldsymbol{p}\dot{\mathcal{E}}/\mathcal{E}^2$. According to the equation of particle motion, $\dot{\boldsymbol{p}} = e_1 e_2 \boldsymbol{r}/r^3$. The law of energy conservation requires that $\mathcal{E} + e_1 e_2/r = \text{const}$. Differentiation of the last equality with respect to t' yields

$$\dot{\mathcal{E}} = \frac{e_1 e_2 \dot{r}}{r^2} = \frac{e_1 e_2 \boldsymbol{r}\cdot\boldsymbol{v}}{r^3}\,,$$

so

$$\dot{\boldsymbol{v}} = \frac{e_1 e_2 c^2}{\mathcal{E}}\left[\boldsymbol{r} - \frac{\boldsymbol{p}(\boldsymbol{r}\cdot\boldsymbol{v})}{\mathcal{E}}\right] = \frac{e_1 e_2 c^2}{\mathcal{E} r^3}[s\boldsymbol{e}_x + v t'(1-\beta^2)\boldsymbol{e}_z]\,.$$

Substituting these expressions into (5.60) gives

$$\begin{aligned}\frac{\mathrm{d}\Delta W_{\boldsymbol{n}}}{\mathrm{d}\Omega} &= \frac{e_1^4 e_2^2 c^4}{4\pi c^3 \mathcal{E}^2 (1-\beta n_z)^5}\left\{ s^2\left[(1-\beta n_z)^2 - n_x^2(1-\beta^2)\right]\right.\\ &\quad\times \int_{-\infty}^{\infty} \frac{\mathrm{d}t'}{(s^2+v^2t'^2)} + c^2\beta^2(1-\beta^2)^2(1-n_z)^2\\ &\quad\left.\times \int_{-\infty}^{\infty} \frac{t'^2\mathrm{d}t'}{(s^2+v^2t'^2)^3}\right\}\,.\end{aligned}$$

Integration leads to

$$\begin{aligned}\frac{\mathrm{d}\Delta W_{\boldsymbol{n}}}{\mathrm{d}\Omega} &= \frac{e_1^4 e_2^2(1-\beta^2)}{32 m^2 c^3 s^3 v (1-\beta n_z)^5}\\ &\quad\times\left[4 - 3n_x^2 - n_z^2 - 6\right.\\ &\quad\left.+\ \beta n_z + \beta^2\left(-2 + 3n_x^2 + 5n_z^2\right) + \beta^4\left(1-n_z^2\right)\right]\,.\end{aligned} \tag{1}$$

In the nonrelativistic limit, $\beta \to 0$ and

$$\frac{\mathrm{d}\Delta W_{\boldsymbol{n}}}{\mathrm{d}\Omega} = \frac{e_1^4 e_2^2}{32 m^2 c^4 s^3 v}\left(4 - 3n_x^2 - n_z^2\right)\,. \tag{2}$$

In ultrarelativistic case $\beta \approx 1$ and

$$\frac{\mathrm{d}\Delta W_{\boldsymbol{n}}}{\mathrm{d}\Omega} = \frac{3 e_1^4 e_2^2 (1-\beta)}{2^9 m^2 c^4 s^3 \sin^4\frac{\vartheta}{2}}\,. \tag{3}$$

At $\vartheta \le \sqrt{1-\beta}$, the last formula is invalid and the exact expression (1) needs to be used.

5.99

$$\Delta W = \frac{\pi e_1^4 e_2^2}{12 m^2 c^3 s^3 v}\cdot\frac{4-\beta^2}{1-\beta^2}\,,\qquad \Delta\boldsymbol{p} = \frac{\boldsymbol{v}\Delta W}{c^2}\,.$$

5.100*

$$\frac{\mathrm{d}\Delta W_\omega}{\mathrm{d}\omega} = \frac{8 e_1^4 e_2^2 \omega^2 c}{3\pi v^4}\left[K_1^2\left(\frac{\omega s}{v}\right) + K_0^2\left(\frac{\omega s}{v}\right)\right]\,.$$

5.101* Formula (5.81) for differential effective radiation can be written in the form:

$$\frac{d\kappa_n}{d\Omega} = 2\pi \int_0^\infty \int_{-\infty}^\infty \overline{\frac{dI}{d\Omega}} dt s ds \, . \tag{1}$$

The intensity of radiation $dI/d\Omega = cH^2r^2/4\pi$, where $\boldsymbol{H} = \dot{\boldsymbol{A}} \times \boldsymbol{n}/c$. Averaging of radiation intensity in formula (1) must be performed over all directions in the plane perpendicular to the incident particle flow. To this effect, it is convenient to represent the vector product entering the expression for $\boldsymbol{H}$ in the form of $H_\alpha = \frac{1}{c} e_{\alpha\beta\gamma} \dot{A}_\beta n_\gamma$, where $e_{\alpha\beta\gamma}$ is the antisymmetric unit pseudotensor; the summation is performed over the repeated indices. Components of the vector potential A_β are expressed through the components of the quadrupole moment $Q_{\beta\epsilon}$, defined by formula (2.24):

$$A_\beta = \frac{1}{2c^2 r} \ddot{Q}_{\beta\epsilon} n_\epsilon \, .$$

Thus,

$$H_\alpha = \frac{1}{2c^3 r} e_{\alpha\beta\gamma} Q'''_{\beta\epsilon} n_\gamma n_\epsilon$$

and

$$\overline{\frac{dI}{d\Omega}} = \frac{1}{16\pi c^5} Q'''_{\beta\epsilon} Q'''_{\beta'\epsilon'} e_{\alpha\beta\gamma} e_{\alpha\beta'\gamma'} \overline{n_\gamma n_\epsilon n_{\gamma'} n_{\epsilon'}} \, .$$

Make use of the polar system of coordinates with the polar axis directed along the incident flux and the pole at the point where the particle having charge e_2 and mass m_2 is located. The averaging must be performed at a fixed value of the constituent $n_z \equiv n_3 = \cos\vartheta$ (ϑ is the direction of emission). It is easy to see that

$$\begin{cases} \overline{n_i n_k} &= \frac{1}{2}\delta_{ik}\left(1 - n_3^2\right) \, , \\ \overline{n_i n_k n_l n_m} &= \frac{1}{8}\left(\delta_{ik}\delta_{lm} + \delta_{il}\delta_{km} + \delta_{im}\delta_{kl}\right)\left(1 - n_3^2\right)^2 \, , \\ \overline{n_i} &= \overline{n_i n_k n_l} = 0 \, , \end{cases} \tag{2}$$

where indices i, k, and l take the values 1 and 2.

The use of (2) and the formula

$$e_{\alpha\beta\gamma} e_{\alpha\beta'\gamma'} = \delta_{\beta\beta'}\delta_{\gamma\gamma'} - \delta_{\beta\gamma'}\delta_{\gamma\beta'}$$

leads to

$$\begin{aligned} \overline{\frac{dI}{d\Omega}} = \frac{1}{16\pi c^5} \Bigg\{ & \left[\left(Q'''_{\beta 3}\right)^2 - \left(Q'''_{33}\right)^2\right] \cos^4\vartheta \\ & + \frac{1}{2}\left[\left(6 Q'''_{33}\right)^2 - 2 Q'''_{33} Q'''_{\beta\beta} + \left(Q'''_{\beta\beta'}\right)^2 - 3\left(Q'''_{\beta 3}\right)^2\right] \sin^2\vartheta \cos^2\vartheta \\ & + \frac{1}{8}\left[2\left(Q'''_{\beta\beta'}\right)^2 - \left(Q'''_{\beta\beta}\right)^2 - 3\left(Q'''_{33}\right)^2 + 2 Q'''_{33} Q'''_{\beta\beta}\right] \sin^4\vartheta \Bigg\} \, . \end{aligned} \tag{3}$$

Substituting (3) into (1) finally yields

$$\frac{d\kappa_n}{d\Omega} = A + B P_2(\cos\vartheta) + C P_4(\cos\vartheta) , \tag{4}$$

where P_2 and P_4 are the Legendre polynomials,

$$A = \frac{1}{120c^5} \int_{-\infty}^{\infty} \int_{0}^{\infty} \left[3\left(Q'''_{\beta\beta'}\right)^2 - \left(Q'''_{\beta\beta}\right)^2 \right] s\,ds\,dt, \tag{5}$$

$$B = \frac{1}{168c^5} \int_{-\infty}^{\infty} \int_{0}^{\infty} \left[-3\left(Q'''_{\beta\beta'}\right)^2 + 2\left(Q'''_{\beta\beta}\right)^2 + 9\left(Q'''_{\beta 3}\right)^2 \right.$$
$$\left. - 6 Q'''_{33} Q'''_{\beta\beta} \right] s\,ds\,dt , \tag{6}$$

$$C = \frac{1}{280c^5} \int_{-\infty}^{\infty} \int_{0}^{\infty} \left[-2\left(Q'''_{\beta\beta'}\right)^2 + 2\left(Q'''_{\beta 3}\right)^2 \right.$$
$$\left. - \left(Q'''_{\beta\beta}\right)^2 - 35\left(Q_{33}\right)^2 + 10 Q'''_{33} Q'''_{\beta\beta} \right] s\,ds\,dt . \tag{7}$$

The quantities $Q'''_{\alpha\beta}$ denote the third time derivatives of the components of the quadrupole moment.

5.102* The total effective radiation

$$\kappa = \int \frac{d\kappa_n}{d\Omega} d\Omega .$$

On the basis of (4) and (5) obtained in the preceding problem, one can write

$$\kappa = 4\pi A = \frac{\pi}{30c^5} \int_{-\infty}^{\infty} \int_{0}^{\infty} \left[3\left(Q'''_{\alpha\beta}\right)^2 - \left(Q'''_{\beta\beta}\right)^2 \right] s\,ds\,dt . \tag{1}$$

Denote Cartesian components of the particle radius vector by x_α and Cartesian components of the particle relative velocity by $v_\alpha = \dot{x}_\alpha$ leads, taking into consideration the equation of particle relative motion, to

$$\ddot{x}_\alpha = \frac{2e^2 x_\alpha}{m r^3} , \quad x'''_\alpha = \frac{2e^2}{m} \cdot \frac{r x_\alpha - 3 x_\alpha v_r}{r^6} ,$$

where

$$v_r = \dot{r} .$$

Substituting these expressions into (1) and introducing the azimuthal component of the particle relative velocity v_α ($v^2 = v_\alpha^2 + v_r^2$) yields

$$\kappa = \frac{4\pi e^6}{15m^2c^5}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{v^2+11v_\alpha^2}{r^4}s\mathrm{d}s\mathrm{d}t\,. \tag{2}$$

Owing to the conservation of energy and angular momentum, $v^2 = v_0^2 - 4e^2/mr$ and $v_\alpha = v_0 s/r$. Integrating in (2), with integration over $\mathrm{d}r$ instead of $\mathrm{d}t$ in accordance with the formula $\mathrm{d}t = \mathrm{d}r/v_r = \mathrm{d}r/\sqrt{v^2 - v_\alpha^2}$ (in any order), finally yields

$$\kappa = \frac{4\pi}{9}\frac{e^4 v_0^3}{mc^5}\,.$$

5.103 The validity limits of formula (5.88) are satisfied at all frequencies ω because the collision time $\tau = 0$. In the case of scattering from a hard sphere the angle of incidence equals the angle of reflection; therefore, $|\boldsymbol{v}_2 - \boldsymbol{v}_1|^2 = 2v\sin\vartheta/2$, where ϑ is the scattering angle. Angle ϑ is related to the impact parameter s by the expression $s = a\sin\vartheta/2$ at $s \leq a$. At $s > a$, a particle is not subject to scattering. Hence,

$$\mathrm{d}\kappa_\omega = \frac{2e^2}{3\pi c^3}4v^2\int_0^a \sin^2\frac{\vartheta}{2}2\pi s\mathrm{d}s\mathrm{d}\omega = \frac{4e^2a^2v^2}{3c^3}\mathrm{d}\omega\,.$$

The differential effective radiation thus found is independent of frequency. Therefore, the total effective radiation

$$\kappa = \int_0^{\infty}\mathrm{d}\kappa_\omega = \infty\,.$$

This divergence occurs because the sphere was considered to be absolutely hard. However, the truth is that absolutely hard bodies do not existent, $\tau \neq 0$, and the expression found for $\mathrm{d}\kappa_\omega$ at high ω values is invalid.

Moreover, the calculation disregarded a decrease of the particle's energy due to radiation, In fact, the emitted energy cannot be greater than the initial kinetic energy of the particle.

The cross-section of photon generation is found as follows. The effective radiation $\mathrm{d}\kappa_\omega/\mathrm{d}\omega$ is first divided by the photon energy $\hbar\omega$ and then by $\hbar$ (to refer it to the energy range):

$$\frac{\mathrm{d}\sigma}{\mathrm{d}(\hbar\omega)} = \frac{1}{\hbar^2\omega}\frac{\mathrm{d}\kappa_\omega}{\mathrm{d}\omega} = \frac{e^2}{\hbar c}\left(\frac{v}{c}\right)^2\frac{4a^2}{3\hbar\omega}\,.$$

5.104*

$$\frac{\mathrm{d}\sigma}{\mathrm{d}(\hbar\omega)} = \frac{e^2}{\hbar c}\left(\frac{1}{\beta}\ln\frac{1+\beta}{1-\beta} - 2\right)\frac{2a^2}{\hbar\omega}\,,\quad \beta = \frac{v}{c}\,.$$

5.105

$$I_n = \frac{64\mathcal{E}^4 n^2}{3e_1^2 e_2^2 c^3}\left(\frac{e_1}{m_1} - \frac{e_2}{m_2}\right)^2 \left[J_n'^2(n\epsilon) + \frac{1-\epsilon^2}{\epsilon^2} J_n^2(n\epsilon)\right] .$$

5.106

$$Z(\boldsymbol{r}, t) = \frac{\boldsymbol{p}_2}{r}\Theta(\tau) + \frac{\boldsymbol{p}_1}{r}\Theta(-\tau) , \quad \tau = t - \frac{r}{c};$$

$$\boldsymbol{H}(\boldsymbol{r}, t) = \frac{\Delta\boldsymbol{p}\times\boldsymbol{r}}{cr^3}\left\{\delta(\tau) + \frac{r}{c}\delta'(\tau)\right\};$$

$$\boldsymbol{E}(\boldsymbol{r}, t) = \begin{cases} r^{-3}(\boldsymbol{n}(\boldsymbol{p}_1\cdot\boldsymbol{n}) - \boldsymbol{p}_1) , & \tau < 0 ; \\ \boldsymbol{H}\times\boldsymbol{n} , & \tau = 0 , \\ r^{-3}(\boldsymbol{n}(\boldsymbol{p}_2\cdot\boldsymbol{n}) - \boldsymbol{p}_2) , & \tau > 0 . \end{cases}$$

These results are in the best way possible to illustrate the finiteness of the propagation velocity of electromagnetic perturbations. A sudden change of the dipole moment results in the propagation of a spherical wave of infinitely small thickness with radius ct. The magnetic field differs from zero (and is singular) only at this particular sphere. The electric field on the light sphere is singular too and is related to the magnetic one by the usual expression (5.18) for the wave zone. However, a static electric field exists, besides the magnetic field. The static electric field of the dipole with moment p_1 is retained in the $r > ct$ region, whereto the perturbation does not reach. The static field of the dipole with moment p_2 exists in the $r < ct$ region.

5.107

1. At $r > ct$ there is the static electric field of a point charge with radial lines of force (Figure 5.13a). At $r < ct$ there is the field of the charge propagating with velocity $v = \text{const}$. This field was calculated in Problems 4.34• and 4.35.
2. These two fields interchange (Figure 5.13b). The thin layer in the vicinity of $r = ct$ undergoes rearrangement of the electric field and has a singular alternating magnetic field (see the preceding problem).

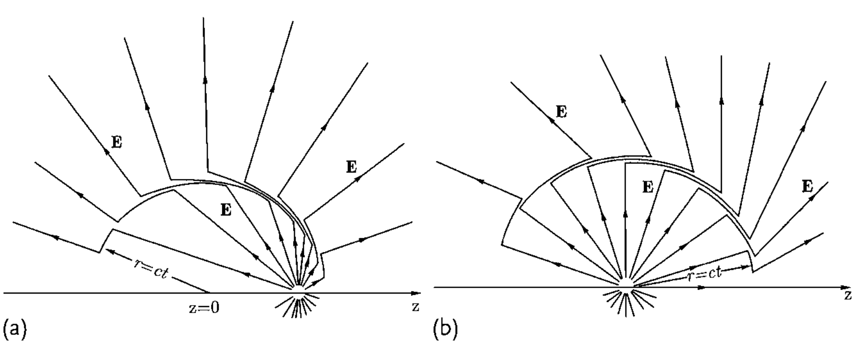

Figure 5.13 Reconstruction of the electric field at the sharp change of the particle's motion: (a) the sudden gaining of a constant velocity; (b) the sudden stopping.

5.108 Use formula (5.77) assuming $\boldsymbol{v}(\tau) = 0$ at $\tau < 0$, $\boldsymbol{v}(\tau) = \boldsymbol{v} = \text{const}$, and $\boldsymbol{s}(\tau) = \boldsymbol{v}\tau$, where the velocity $\boldsymbol{v}$ depends on the energy and the direction of the escaping beta electron:

$$\frac{\mathrm{d}I_\omega}{\mathrm{d}\Omega} = \frac{e^2 v^2}{4\pi^2 c^3} \frac{\sin^2\theta}{(1-\beta\cos\theta)^2}\,, \qquad I_\omega = \frac{e^2}{\pi c}\left(\frac{1}{\beta}\ln\frac{1+\beta}{1-\beta} - 2\right)\,.$$

Here, θ determine the direction of the emission relative to the electron's velocity. The infinitely oscillating exponent $\exp i\omega(1-\beta\cos\theta)\tau|_{\tau\to\infty}$ should be regarded as equaling zero because averaging it over any small interval $\Delta\omega$ yields zero.

The number of quanta N_ω per unit integral of their energy is given by the relation $N_\omega \hbar\omega\, d(\hbar\omega) = I_\omega \mathrm{d}\omega$, that is,

$$N_\omega = \frac{e^2}{\pi c\hbar^2\omega}\left(\frac{1}{\beta}\ln\frac{1+\beta}{1-\beta} - 2\right)\,.$$

The number of quanta grows infinitely at $\omega \to 0$ ("infrared catastrophe"), but their total energy remains finite. The highest quantum energy allowable by the law of energy conservation equals the energy of beta transition of the nucleus E_0 minus the electron rest energy (and neutrino rest energy): $\hbar\omega_{\text{max}} = E_0 - mc^2$.

5.109* In accordance with the dipole moment transformation rule (Problem 4.37•), in the laboratory system

$$\boldsymbol{m} = \frac{\boldsymbol{\mu}}{\gamma}\,, \quad \boldsymbol{p} = \frac{1}{c}\boldsymbol{v}\times\boldsymbol{\mu}\,, \tag{1}$$

where $\boldsymbol{\mu}$ is the magnetic moment in the electron rest system. The magnetic and electric dipole moments create vectors of magnetic and electric polarization:

$$\boldsymbol{M} = \frac{1}{\gamma}\boldsymbol{\mu}\,\Theta(t)\delta(\boldsymbol{r}-\boldsymbol{v}t)\,, \quad \boldsymbol{P} = \frac{1}{c}\boldsymbol{v}\times\boldsymbol{\mu}\,\Theta(t)\delta(\boldsymbol{r}-\boldsymbol{v}t)\,. \tag{2}$$

Find the current produced by a moving electron with the help of formulas (5.41) and (5.44):

$$\begin{aligned} \boldsymbol{j}(\boldsymbol{r},t) = &\left\{e\boldsymbol{v} - \frac{1}{\gamma}\boldsymbol{\mu}\times\nabla - \frac{1}{c}(\boldsymbol{v}\times\boldsymbol{\mu})(\boldsymbol{v}\cdot\nabla)\right\}\delta(\boldsymbol{r}-\boldsymbol{v}t)\Theta(t) \\ &+ \frac{1}{c}\boldsymbol{v}\times\boldsymbol{\mu}\,\delta(t)\delta(\boldsymbol{r})\,. \end{aligned} \tag{3}$$

The first term in parentheses is the current produced by the electron charge. The vector potential in the wave zone is calculated using formula (5.4) written for the Fourier transform:

$$\boldsymbol{A}_\omega = \frac{1}{c}\int G_\omega(R)e^{i\omega t'}\boldsymbol{j}(\boldsymbol{r}',t')\mathrm{d}^3r'\mathrm{d}t'\,, \quad G_\omega(R) = \frac{e^{ikR}}{R}\,, \quad \boldsymbol{R} = \boldsymbol{r}-\boldsymbol{r}'\,. \tag{4}$$

Integration by parts is used to transfer the derivatives from the delta function to the smooth function $G_\omega(R)$, and only the exponent is differentiated. As a result,

$$A_\omega(\boldsymbol{r}) = \frac{1}{i c\omega(1-\boldsymbol{n}\cdot\boldsymbol{v}/c)}\left\{e\boldsymbol{v} - \frac{i\omega}{\gamma}\boldsymbol{\mu}\times\boldsymbol{n} - \frac{i\omega}{c^2}(\boldsymbol{v}\times\boldsymbol{\mu})(\boldsymbol{n}\cdot\boldsymbol{v})\right\}\frac{e^{ikr}}{r} + \frac{1}{c^2}\boldsymbol{v}\times\boldsymbol{\mu}\frac{e^{ikr}}{r}\,. \tag{5}$$

If $\boldsymbol{\mu}$ is considered to be an ordinary vector parallel to the velocity, $\boldsymbol{v}\times\boldsymbol{\mu} = 0$. In this model,

$$\frac{d^2 I_\omega}{d\omega d\Omega} = \frac{1}{4\pi^2 c^3(1-\beta\cos\theta)^2}\left\{e^2(\boldsymbol{n}\times\boldsymbol{v})^2 + \frac{\omega^2}{\gamma^2}(\boldsymbol{\mu}\times\boldsymbol{n})^2\right\}, \tag{6}$$

$$\frac{dI_\omega}{d\omega} = \left(\frac{e^2}{\pi c} + \frac{\omega^2\mu^2}{\pi c^3\beta^2\gamma^2}\right)\left[\frac{1}{\beta}\ln\frac{1+\beta}{1-\beta} - 2\right]. \tag{7}$$

The second term in the parentheses making a contribution to radiation from the magnetic moment is proportional to the square of the frequency. Substituting $\mu = \mu_B = e\hbar/m_e c$, that is, the Bohr magneton, yields the ratio of two terms $(\hbar\omega/2m_e c)^2(c/v\gamma)^2$. The role of the magnetic moment becomes apparent at $\hbar\omega \sim m_e c^2$.

It should be borne in mind, however, that the electron spin is a quantum phenomenon. For this reason, the magnetic moment $\boldsymbol{\mu}$ should be interpreted even in classical electrodynamics as the operator $\widehat{\boldsymbol{\mu}} = \mu_B\widehat{\boldsymbol{\sigma}}$, where $\widehat{\boldsymbol{\sigma}}$ is the Pauli matrix operator (see Appendix C and Chapter 6). Finally, the quantum mechanical averaging of radiation intensity is performed over the electron spin states. Denote such averaging by angle brackets to have

$$\langle[\widehat{\boldsymbol{\mu}}\times\boldsymbol{v}]^2\rangle = \mu_B^2\langle[\widehat{\boldsymbol{\sigma}}\times\boldsymbol{v}]^2\rangle = \mu_B^2\langle[v^2\widehat{\boldsymbol{\sigma}}^2 - (\widehat{\boldsymbol{\sigma}}\cdot\boldsymbol{v})^2]\rangle = 2(\mu_B v)^2\,. \tag{8}$$

Here, the relations $\widehat{\sigma}_x^2 = \widehat{\sigma}_y^2 = \widehat{\sigma}_z^2 = 1$ and $\widehat{\sigma}_\alpha\widehat{\sigma}_\beta + \widehat{\sigma}_\beta\widehat{\sigma}_\alpha = 2\delta_{\alpha\beta}$ are used. As follows from (8), the quantum mechanical averaging does not coincide with classical averaging, which gives $\boldsymbol{\mu}\times\boldsymbol{v} = 0$ at $\boldsymbol{\mu}\parallel\boldsymbol{v}$. The quantum magnetic moment has no definite direction in space. Similarly, $\langle[\widehat{\boldsymbol{\mu}}\times\boldsymbol{n}]^2\rangle = 2\mu_B^2$. As a result, expressions (6) and (7) are supplemented by the terms containing the electric dipole moment.

5.110 If the energy transmitted to an electron is much higher than its coupling energy in the atom, the electron prior to the conversional transition may be regarded as motionless. In this approximation,

$$N_\omega = \frac{2}{3\pi}\frac{e^2}{\hbar c}\frac{\beta^2}{\hbar\omega}\,.$$

5.111*

$$N_\omega = \frac{1}{2\pi}\frac{e^2}{\hbar c}\left[\frac{4}{3}\left(\frac{a\omega_0}{c}\right)^2\frac{\omega^2(\omega^2+\omega_0^2)}{(\omega^2-\omega_0^2)^2} + \left(\frac{\hbar\omega}{m_e c^2}\right)^2\right]\frac{1}{\hbar\omega}\,. \tag{1}$$

For comparison of the result obtained with the experimental data, Jackson (1999) identified ω_0 as the frequency of the $2p \to 1s$ transition (the rotation in a circular orbit being possible only in the case of nonzero orbital moment) and used the relations for a hydrogen-like atom: the distance between the levels $\hbar\omega_0 = 3Z^2e^2/8a_B$, $a = a_B/Z$, $a_B = \hbar^2/m_e e^2$. Moreover, Jackson introduced the correction factor taking into account the decrease of the neutrino's energy with increasing energy of the emitted quantum: $(1 - \hbar\omega/E_0)^2$, where E_0 is the nuclear transition energy (equaling the neutrino energy at $\hbar\omega = 0$). The last factor is essential only for hard quanta emitted by virtue of the magnetic moment (the second term in (1)). As a result, the working formula assumes the form

$$N_\omega = \frac{3}{32\pi} Z^2 \left(\frac{e^2}{\hbar c}\right)^3 \left[\frac{\omega^2(\omega^2+\omega_0^2)}{(\omega^2-\omega_0^2)^2}\right] \frac{1}{\hbar\omega} + \frac{1}{2\pi}\frac{e^2}{\hbar c}\frac{\hbar\omega}{(m_e c^2)^2}\left(1 - \frac{\hbar\omega}{E_0}\right)^2 . \tag{2}$$

5.112 $\hbar\omega_{\text{max}} \approx 17$ MeV.

$$N_\omega = \frac{2}{3\pi}\frac{e^2}{\hbar c}\frac{\beta^2}{\hbar\omega} + \frac{1}{2\pi}\frac{e^2}{\hbar c}\frac{\hbar\omega}{(m_\mu c^2)^2} .$$

5.113

$$\frac{d^2 I_\omega}{d\omega d\Omega} = \frac{2e^2}{\pi^2 c}\frac{T_-}{m_\pi c^2}\sin^2\theta , \quad N_\omega = \frac{16}{3\pi}\frac{e^2}{\hbar c}\frac{T_-}{m_\pi c^2 \hbar\omega} ,$$

where $T_- = m_\pi v^2/2$ is the kinetic energy of a negative pion and θ is the angle between the directions of its momentum and quantum escape.

5.114* The field momentum of a moving particle

$$\boldsymbol{G} = \int \boldsymbol{g} dV ,$$

where $\boldsymbol{g} = 1/(4\pi c)\boldsymbol{E}\times\boldsymbol{H}$, and integral is taken over the entire space. The magnetic field of the propagating particle is $\boldsymbol{H} = \boldsymbol{v}/c \times \boldsymbol{E}$, because the magnetic field is absent in the particle's rest system (S'). Hence,

$$\boldsymbol{g} = \frac{1}{4\pi c}[\boldsymbol{v}E^2 - \boldsymbol{E}(\boldsymbol{v}\cdot\boldsymbol{E})] .$$

The Lorentz transform (4.69) gives

$$E_x = E'_x , \quad E_y = \frac{E'_y}{\sqrt{1-\beta^2}} , \quad E_z = \frac{E'_z}{\sqrt{1-\beta^2}} .$$

(the Ox axis is directed along $\boldsymbol{v}$). The volume element $dV = dV'\sqrt{1-\beta^2}$ (as a result of Lorentz contraction). Thus,

$$\boldsymbol{G} = \frac{\boldsymbol{v}}{4\pi c^2\sqrt{1-\beta^2}} \int \left(E_y'^2 + E_z'^2\right) dV' = \frac{\boldsymbol{v}}{4\pi c^2\sqrt{1-\beta^2}}\frac{2}{3}\int E'^2 dV' . \tag{1}$$

The last transformation follows from spherical symmetry of the field in system S'.

If the particle's rest mass is assumed to have purely electromagnetic origin, that is, it is the mass of its electric field defined by the Einstein relation $W' = m_0c^2$, it must be

$$m_0 = \frac{1}{c^2} \cdot \frac{1}{8\pi} \int E'^2 \mathrm{d}V' \,. \tag{2}$$

Also, the field momentum must be equal to $m_0\boldsymbol{v}/\sqrt{1-\beta^2}$; however, it can be seen from formula (1) that it is not the case.[5] The field momentum depends on velocity $\boldsymbol{v}$ exactly as it should do in the case of a particle:

$$\boldsymbol{G} = \frac{m_0'\boldsymbol{v}}{\sqrt{1-\beta^2}} \,. \tag{3}$$

However, the "mass" $m_0' = \frac{4}{3}m_0 \neq m_0$ does not coincide with the particle's rest mass m_0 defined by formula (2).

The coefficient 4/3 in the expression for $\boldsymbol{G}$ means that the momentum and energy of the particle's electromagnetic field do not create the 4-vector and cannot be identified with the momentum and energy of the particle itself.

We note that the electromagnetic mass defined by formula (2) becomes infinite in the case of a point-like particle.

5.115

$$W_m = \frac{1}{8\pi} \int H^2 \mathrm{d}V = \frac{1}{2} \cdot \frac{m_0' v^2}{\sqrt{1-\beta^2}} \,,$$

where the quantity m_0' is defined in the solution of the previous problem.

The total energy of the particle's electromagnetic field

$$W = \frac{1}{8\pi} \int (E^2 + H^2) \mathrm{d}V = m_0' c^2 \left(\frac{1}{\sqrt{1-\beta^2}} - \frac{1}{4}\sqrt{1-\beta^2} \right)$$

exhibits no dependence on the velocity $m_0c^2/\sqrt{1-\beta^2}$, that must occur for the energy of the particle (see Problem 5.88).

5.116 Omit the v/c and higher-order terms and consider the action of a certain element $\mathrm{d}e_1$ on another element $\mathrm{d}e_2$. The Coulomb part of the electric field is spherically symmetric and makes no contribution to the self-action force, nor does the quasi-stationary magnetic field make any contribution. It is therefore sufficient to consider only that part of the intensity of the electric field $\mathrm{d}\boldsymbol{E}$ of the $\mathrm{d}e_1$ element that depends on acceleration. The $\mathrm{d}e_2$ element is subject to the force

$$\mathrm{d}\boldsymbol{F} = -\mathrm{d}e_2 \mathrm{d}\boldsymbol{E} = \frac{\mathrm{d}e_1 \mathrm{d}e_2}{c^2 \boldsymbol{r}} [\dot{\boldsymbol{v}} - \boldsymbol{r}_0(\boldsymbol{r}_0 \cdot \dot{\boldsymbol{v}})] \,,$$

5) Under this assumption, the field energy must be equal to $m_0c^2/\sqrt{1-\beta^2}$, but it is not so either as shown in the next problem.

where $\boldsymbol{r}_0 = \boldsymbol{r}/r$, $\boldsymbol{r}$ being the radius vector directed from element $\mathrm{d}e_1$ to element $\mathrm{d}e_2$. The particle as a whole experiences the action of field

$$\boldsymbol{F} = \int \mathrm{d}\boldsymbol{F} = -\frac{4}{3} \cdot \frac{W_0}{c^2} \dot{\boldsymbol{v}} ,$$

where $W_0 = (1/2) \int \mathrm{d}e_1 \mathrm{d}e_2 / r$ is the energy of the electromagnetic field of a particle at rest and the factor 4/3 arises from the integration over the direction of $\boldsymbol{r}_0$. Defining the particle's rest mass as $m_0' = 4\,W_0/3c^2$ (see Problem 5.88) yields the following expression for the self-action force:

$$\boldsymbol{F} = -m_0' \dot{\boldsymbol{v}} .$$

This means that the self-action force of the particle coincides with its inertial force, provided the delay is disregarded.

5.117* The force acting on the $\mathrm{d}e_2$ charge element from the $\mathrm{d}e_1$ element depends on the latter's acceleration $\dot{\boldsymbol{v}}$ at time t':

$$\mathrm{d}\boldsymbol{F}(t) = -\frac{\mathrm{d}e_1 \mathrm{d}e_2}{c^2 r} [\dot{\boldsymbol{v}} - \boldsymbol{r}_0(\boldsymbol{r}_0 \cdot \dot{\boldsymbol{v}})]\Big|_{t'=t-\frac{r}{c}} .$$

The power series expansion of acceleration $\dot{\boldsymbol{v}}$ in $t' - t = -r/c$ leads to

$$\dot{\boldsymbol{v}}(t') = \dot{\boldsymbol{v}}(t) + (t' - t)\ddot{\boldsymbol{v}}(t) = \dot{\boldsymbol{v}}(t) - \frac{r}{c}\ddot{\boldsymbol{v}}(t) .$$

Integration over elements $\mathrm{d}e_1$ and $\mathrm{d}e_2$ yields (see the preceding problem) the self-action force:

$$\boldsymbol{F} = -m_0' \dot{\boldsymbol{v}} + \frac{2}{3} \frac{e^2}{c^3} \ddot{\boldsymbol{v}} .$$

The second term on the right-hand side is the radiation friction force. It is independent of the particle's structure and does not change its form in the limiting case of a point-like particle. The proper energy W_0 and therefore the electromagnetic mass m_0 become infinite in this limiting case. Evidently, the disregarded $(t' - t)^n$-order terms, where $n \geq 2$, are proportional to r_0^{n-1} (r_0 is the particle radius) and disappear in the point particle limit.

5.118•* Let us use the symmetric tensor (4.126) of the energy–momentum of an electromagnetic field. The integral

$$p^i = \frac{1}{c} \int T^{ik} \mathrm{d}^3 S_k , \quad \text{where} \quad \mathrm{d}^3 S_k = n_k \mathrm{d}^3 S \tag{1}$$

is the covariant element of the three-dimensional hypersurface, is the 4-vector. The tensor components T^{00} and $T^{\alpha 0}/c(\alpha = 1, 2, 3)$ make up energy density (4.127) and momentum density (4.128) of the electromagnetic field, respectively. Let integral (1) in the particle's rest system S' transform into the integral over the three-dimensional volume $\mathrm{d}V'$, that is, over the hypersurface $\mathrm{d}V'$. In this system, E' is the Coulomb field of the particle, $H' = 0$, and $\mathrm{d}^3 S_k = n'_k \mathrm{d}V'$, where the normal unit 4-vector $n'_k = (1, 0, 0, 0)$. For this reason, integral (1) gives the energy and momentum values of a particle in its rest system:

$$p'^0 = \frac{1}{8\pi c} \int E'^2 \mathrm{d}V' = \frac{\mathcal{E}}{c} , \quad p'^\alpha = (\boldsymbol{p}_0)^\alpha = 0 . \tag{2}$$

In an arbitrary inertial frame of reference S, the field is expressed in accordance with (4.69) and (4.70), for example,

$$\boldsymbol{H} = \frac{\gamma}{c} \boldsymbol{v} \times \boldsymbol{E}' = \frac{\boldsymbol{v} \times \boldsymbol{E}}{c} , \quad \text{because} \quad \boldsymbol{E}_\perp = \gamma \boldsymbol{E}'_\perp . \tag{3}$$

The normal unit vector n_k is expressed through n'_k with the help of Lorentz transforms: $n_k = (\gamma, -\gamma \boldsymbol{v}/c)$. In accordance with (3.13), the invariant element of the hypersurface can be written in the form $\mathrm{d}^3 S = \mathrm{d}V' = \gamma \mathrm{d}^3 x$, where $\mathrm{d}^3 x$ is an element of the three-dimensional volume in system S. This result and expressions (4.127) and (4.128) give

$$\begin{aligned} \frac{1}{c} T^{0k} n_k = \frac{\gamma}{c} (T^{00} - \boldsymbol{v} \cdot \boldsymbol{g}) &= \frac{\gamma}{8\pi c} \left[E^2 + H^2 - 2 \left(\frac{\boldsymbol{v}}{c} \times \boldsymbol{E} \right) \cdot \boldsymbol{H} \right] \\ &= \frac{\gamma}{8\pi c} \left[E^2 - H^2 \right] . \end{aligned} \tag{4}$$

Thus,

$$p^0 = \frac{\gamma}{8\pi c} \int \left(E^2 - H^2 \right) \mathrm{d}^3 S = \gamma m c , \tag{5}$$

whence the particle's electromagnetic mass

$$m = \frac{1}{8\pi c^2} \int \left(E^2 - H^2 \right) \mathrm{d}^3 S = \frac{\gamma}{8\pi c^2} \int \left(E^2 - H^2 \right) \mathrm{d}^3 x \tag{6}$$

is the relativistic invariant. Find in a similar way,

$$\frac{1}{c} T^{\alpha k} n_k = \frac{\gamma v^\alpha}{8\pi c^2} \left(E^2 - H^2 \right) \tag{7}$$

and obtain an adequate expression for the particle's momentum $\boldsymbol{p} = \gamma m \boldsymbol{v}$. The above line of reasoning allows the invariant electromagnetic mass of the particle to be computed but does not give the true solution of the problem for a point-like particle: for such a particle, the mass is a divergent quantity (see (5.108)) and the internal structure of an extended particle must be described by quantum mechanics.

5.119• $T = m^2 c^3 a_0^3 / (4 e^4) \approx 10^{-11}$ s.

The above conjectures concerning the character of electron motion hold if the energy loss for the orbital period τ is small compared with the total electron energy, that is, $\tau |\mathrm{d}\mathcal{E}/\mathrm{d}t| \ll |\mathcal{E}|$, whence $ac/v \gg r_0 = e^2/mc^2$ (r_0 is the classical electron radius). This condition starts to be violated only at very short distances on the order of 10^{-13} cm, at which classical electrodynamics is, generally speaking, inapplicable because quantum effects play an important role in this region.

In consequence, the result for the problem (the very short lifetime of an atom) gives definitive evidence that the classical concept of electron motion of in an atom (e.g., the notion of trajectory) is wrong. In fact, quantum mechanics came into being in the course of overcoming this and other fundamental difficulties encountered in classical physics.

5.120 $\mathcal{E}(t) = mc^2 \coth\left[\frac{2e^4H^2}{3m^3c^5}t + \frac{1}{2}\ln\frac{\mathcal{E}_0+mc^2}{\mathcal{E}_0-mc^2}\right]$.

At $t \to \infty$, $\mathcal{E}(t) \to mc^2$, that is, the particle stops. The orbit radius can be expressed through $\mathcal{E}(t)$ in the formula

$$r(t) = \frac{cp}{eH} = \frac{1}{eH}\sqrt{\mathcal{E}^2(t) - m^2c^4}\ .$$

At $t \to \infty$, $r(t) \to 0$, that is, the particle propagates along a twisted helical line.

5.121

$$\mathcal{E}_c = mc^2\sqrt[3]{\frac{3a^2\omega}{2cr_0}}\ , \qquad r_0 = \frac{e^2}{mc^2}\ .$$

5.122* The equation of motion for a harmonic oscillator has the following form, taking into account the radiation friction forces:

$$\ddot{\boldsymbol{r}} + \omega_0^2\boldsymbol{r} = \frac{2e^2}{3mc^3}\frac{\mathrm{d}}{\mathrm{d}t}\ddot{\boldsymbol{r}}\ . \tag{1}$$

Equation (1) corresponds to the cubic characteristic equation

$$k^2 + \omega_0^2 = \cdot\frac{2e^2}{3mc^3}k^3\ . \tag{2}$$

The condition for the smallness of the radiation friction force in comparison with the quasi-elastic force permits us to solve (2) by means of successive approximations omitting the right-hand side in the zero approximation; in this case, $k \approx k_0 = \pm i\omega_0$. In the first approximation, the substitution of k with k_0 on the right-hand side of (2) and the introduction of the notation

$$\gamma = \frac{2e^2\omega_0^2}{3mc^3} \tag{3}$$

yields $k \approx k_1 = \pm i\omega_0 - \gamma/2$. It is possible to choose only one of the solutions, for example, that to which the sign $\ll - \gg$ corresponds:

$$\boldsymbol{r} = \boldsymbol{r}_0 e^{-\gamma t/2} \cdot e^{-i\omega_0 t} \quad (t > 0)\ . \tag{4}$$

This solution holds at $\gamma \ll \omega_0$ and has the character of damped oscillations.

The oscillator energy decreases as the square of the modulus of its amplitude:

$$W = W_0 e^{-\gamma t} . \tag{5}$$

It is natural to call the quantity $1/\gamma$ the lifetime of the excited state of the oscillator.
The strength of the electric field of radiation is proportional to $\ddot{\boldsymbol{r}}$, so

$$\boldsymbol{E} = \int_{-\infty}^{+\infty} \boldsymbol{E}_\omega e^{-i\omega t} \mathrm{d}\omega = \begin{cases} \boldsymbol{E}_0 e^{-i\omega_0 t} e^{-\gamma \frac{t}{2}} & \text{at} \quad t > 0 , \\ 0 & \text{at} \quad t < 0 \end{cases}$$

and

$$\boldsymbol{E}_\omega = \frac{\boldsymbol{E}}{2\pi} \int_0^{\infty} e^{-(\gamma/2 + i\omega_0)t + i\omega t} \mathrm{d}t = \frac{\boldsymbol{E}_0}{2\pi[i(\omega - \omega_0) - \gamma/2]} .$$

Hence, the spectral distribution of radiation intensity

$$\frac{\mathrm{d}I_\omega}{\mathrm{d}\omega} = \frac{I_0 \gamma}{2\pi} \frac{1}{(\omega - \omega_0)^2 + \gamma^2/4} , \tag{6}$$

where $I_0 = \int_{-\infty}^{+\infty} \mathrm{d}I_\omega$ is the overall radiation intensity. The spectral distribution (6) is described by a resonance curve (Figure 5.14).

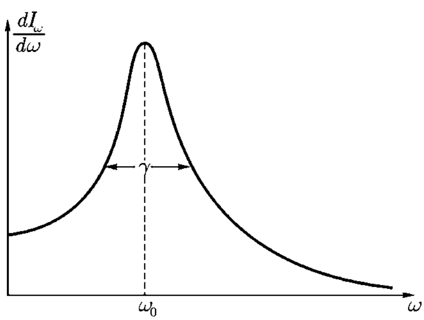

Figure 5.14 The shape of the oscillator's spectrum with account of the radiation force.

The width of the spectral line is characterized by the quantity $\Delta\omega = \gamma$.

The natural width of the line is very small (it would be $\Delta\lambda = \Delta(2\pi c/\omega) = 4\pi r_0/3 = 1.17 \times 10^{-12}$ cm on the wavelength plot).

If radiation is emitted in discrete portions rather than continuously (evidently, this premise extends outside the scope of classical electrodynamics), the indeterminacy of photon energy $\Delta\mathcal{E} = \hbar\Delta\omega = \hbar\gamma$ is related to the lifetime of the excited state $\tau = 1/\gamma$ by the expression

$$\Delta\mathcal{E} \cdot \tau = \hbar . \tag{7}$$

This is a specific case of the very general quantum mechanical uncertainty principle for energy–time.

5.123•

$$\frac{dI_\omega}{d\omega} = I_0 \exp\left[-\frac{(\omega-\omega_0)^2}{\gamma_D^2}\right] ,$$

where $\gamma_D = \sqrt{2T\omega_0^2/(mc^2)}$ is the *Doppler width of the spectral line* and I_0 denotes the intensity at $\omega = \omega_0$. The line width shows a temperature dependence and may serve as a measure of gas temperature.

5.124 $\frac{dI_\omega}{d\omega} = \frac{I\Gamma}{2\pi} \cdot \frac{1}{(\omega-\omega_0)^2+\Gamma^2/4}$, where $I = \int_{-\infty}^{+\infty} dI_\omega$.

5.125* Given a wave polarized along the Ox axis,

$$x_\omega = \frac{eE_{x\omega}}{m}\frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma} , \tag{1}$$

where

$$\gamma = \frac{2}{3} \cdot \frac{e^2\omega_0^2}{mc^3} .$$

The energy absorbed by an oscillating electron

$$\Delta W = \int_{-\infty}^{+\infty} eE_x(t)\dot{x}(t)dt = \frac{2\pi e^2}{m}\int_0^\infty |E_{x\omega}|^2 \frac{2\omega^2\gamma}{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}d\omega ,$$ [6)]

because $(\dot{x})_\omega = -i\omega x_\omega$. The subintegral function in the last expression describes the spectral distribution of absorption intensity. It follows from the form of this function that quantity γ is a measure of the line width as in the case of emission. The width of the group spectral distribution being, by the condition, large in comparison with the natural line width γ,

$$\Delta W = \frac{2\pi e^2}{m}|E_{x\omega_0}|^2 2\omega_0^2\gamma \int_{-\omega_0}^{\infty} \frac{d\xi}{(2\omega_0\xi)^2 + \omega_0^2\gamma^2} ,$$

where $\xi = \omega - \omega_0$. The lower limit can be replaced by $-\infty$, since $\gamma \ll \omega_0$. The integration finally yields

$$\Delta W = \frac{2\pi e^2}{m}|E_{x\omega_0}|^2 = 2\pi^2 r_0 c S_{\omega_0} ,$$

where $r_0 = e^2/(mc^2)$ is the classical electron radius.

6) It is easy to check that

$$\int_{-\infty}^{+\infty} A(t)\cdot B(t)dt = 2\pi\int_0^\infty \left(A_\omega B_\omega^* + A_\omega^* B_\omega\right)d\omega .$$

The result does not depend on γ. The frequency dependence is indirect, that is, the quantity ΔW is proportional to the spectral density S_{ω_0} at the oscillator's resonance frequency ω_0. Clearly, the same result could be obtained in the case of incidence of a nonpolarized and nonplanar wave group onto an isotropic oscillator. In this case, S_ω would be the sum of the intensities of all polarized waves with frequency ω composing this group.

5.126

$$\text{(i)}\quad \Delta W = 2\pi^2 r_0 c S_{\omega_0} \cos^2\vartheta\ ; \qquad \text{(ii)}\quad \Delta W = \pi^2 r_0 c S_{\omega_0} \sin^2\vartheta\ ;$$

$$\text{(iii)}\quad \Delta W = \frac{2}{3}\pi^2 r_0 c S_{\omega_0}\ .$$

5.127* In this case, the equation of motion for a harmonic oscillator assumes the form

$$\ddot{\boldsymbol{r}} + \omega_0^2 \boldsymbol{r} = \frac{2e^2}{3mc^3}\frac{\mathrm{d}}{\mathrm{d}t}\ddot{\boldsymbol{r}} + \frac{e}{m}\boldsymbol{E}_0 e^{-i\omega t} \tag{1}$$

provided the nonuniformity of the electric field in the region occupied by the oscillator and the action of the magnetic force, that is, v/c-order effects, are disregarded.

The solution of (1) corresponding to forced oscillations is expressed through the formula

$$\boldsymbol{r} = \frac{e}{m}\cdot\frac{\boldsymbol{E}}{\omega_0^2 - \omega^2 - i\omega\gamma}\ .$$

Hence, for the time-averaged intensity of the light scattered in a given direction,

$$\overline{\frac{\mathrm{d}I}{\mathrm{d}\Omega}} = \frac{1}{4\pi c^3}\overline{|e\ddot{\boldsymbol{r}}\times\boldsymbol{n}|^2} = \frac{cE_0^2 r_0^2}{8\pi}\cdot\frac{\omega^4\sin^2\vartheta}{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}\ ,$$

where ϑ is the angle between the direction $\boldsymbol{n}$ of the distribution of the scattered radiation and the direction of polarization of an incident wave. The energy flow density (averaged over time) in the incident wave $\overline{\gamma}_0 = cE_0^2/(8\pi)$. The scattering cross-section

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{\overline{\gamma}_0}\overline{\frac{\mathrm{d}I}{\mathrm{d}\Omega}} = r^2\frac{\omega^4\sin^2\vartheta}{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}\ .$$

The overall scattering cross-section is derived from this equation by integration over angles:

$$\sigma = \int\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\mathrm{d}\Omega = \frac{8\pi}{3}r_0^2\frac{\omega^4}{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}\ .$$

In the case of a bound electron, when $\omega_0 \gg \omega$,

$$\sigma = \frac{8\pi}{3}\cdot\frac{r_0^2\omega^4}{\omega_0^4}\ .$$

There is the characteristic frequency dependence of the cross-section: $\sigma \sim \omega^4$.

In the case of a weakly bound electron and small radiation friction, $\gamma \approx 0$, $\omega_0 \approx 0$ and

$$\sigma = \frac{8\pi r_0^2}{3}$$

(the Thomson cross-section).

5.128 $\boldsymbol{H} = -Ae^2/(mc^2 r)(\boldsymbol{e}_\alpha \cos\vartheta - i\boldsymbol{e}_\vartheta)e^{-i(\omega t' - \alpha)}$, where ϑ and α are the polar angles of the direction of scattered wave propagation $\boldsymbol{n}$ (the direction of propagation of the incident wave along the Oz axis) and A is the amplitude of the incident wave.

It follows from the equation for $\boldsymbol{H}$ that the scattered wave, generally speaking, is elliptically polarized. The waves scattered forward and backward are circularly polarized, whereas those scattered in the xy plane are linearly polarized. The differential and total scattering cross-sections are

$$\frac{d\sigma}{d\Omega} = r_0^2 \frac{(1 + \cos^2\vartheta)}{2}\,, \qquad \sigma = \frac{8\pi}{3} r_0^2\,.$$

5.129 $\rho = \cos^2\vartheta$.

5.130* In the case of a linearly polarized wave,

$$d\sigma_p = r_0^2 \frac{(1-\beta^2)(1-\beta)^2}{(1-\beta\cos\vartheta)^6}[(1-\beta\cos\vartheta)^2 - (1-\beta^2)\sin^2\vartheta\cos^2\alpha]\,,$$

where ϑ and α are the polar angles of the direction of scattered wave propagation, the Oz axis is parallel to the charge velocity $\boldsymbol{v}$, $\beta = v/c$, and azimuthal angle α is counted from the direction of vector $\boldsymbol{E}$ in the incident wave.

In the case of a nonpolarized wave,

$$d\sigma_{np} = r_0^2 \frac{(1-\beta^2)(1-\beta)^2}{(1-\beta\cos\vartheta)^6}\left[\frac{1+\beta^2}{2}(1+\cos^2\vartheta) - 2\beta\cos\vartheta\right]\,.$$

5.131* The solution of the equation of oscillator motion in the magnetic field $\boldsymbol{H} \parallel Oz$ in the manner in which it was done in Problem 4.52* yields

$$\boldsymbol{r} = A_1(\boldsymbol{e}_x + i\boldsymbol{e}_y)e^{-i(\omega_0 - \omega_L)t} + A_2(\boldsymbol{e}_x - i\boldsymbol{e}_y)e^{-i(\omega_0 + \omega_L)t} + A_3\boldsymbol{e}_z e^{-i\omega t}$$

at $\omega_0 \gg eH/2mc = \omega_L$, where A_1, A_2, and A_3 are the integration constants found from the initial conditions.

It can be seen from the expression for $\boldsymbol{r}$ that an oscillator placed in a magnetic field becomes anisotropic and the frequency of its oscillations splits into three components: ω_0 and $\omega_0 \pm \omega_L$. Observation of radiation in any direction reveals, generally speaking, elliptical polarization of each of the monochromatic components. Specifically, there are two spectral lines circularly polarized in opposite directions along the Oz axis (along field $\boldsymbol{H}$). All three monochromatic components are linearly polarized in the direction perpendicular to the field. In this case, the electric field vector of the undisplaced spectral line oscillates in the direction of the magnetic field, whereas the electric field vectors of both shifted lines oscillate in the perpendicular direction.

5.132 The condition of amplitude smallness far from the resonance frequency ($|\omega - \omega_0| \sim \omega_0$) implies that the wave field in the region of oscillations of an individual oscillator can be regarded as uniform. In resonance ($\omega \approx \omega_0$), a more stringent condition is needed, that is, $eE_0/mc\gamma \ll 1$, where γ is the resonance line width instead of the particle's Lorentz factor (see Problems 5.122*–5.124). These conditions fulfilled, we may write the field acting on the jth particle in the form $\boldsymbol{E} = \boldsymbol{E}_0 \exp(i\boldsymbol{k}\cdot\boldsymbol{r}_j - \omega t)$ and we find its radius vector $\boldsymbol{s}_j(t)$ counted from the equilibrium position from the equation of motion by disregarding the magnetic field ($v/c \ll 1$): $\boldsymbol{s}_j = \boldsymbol{a}\exp(i\boldsymbol{k}\cdot\boldsymbol{r}_j - \omega t)$, where the amplitude of oscillations

$$\boldsymbol{a} = \frac{eE_0}{m\left(\omega_0^2 - \omega^2 - i\omega\gamma\right)} . \tag{1}$$

Calculation of the scattered wave field in the wave zone under the superposition principle with the use of formulas for electric dipole radiation (see Section 5.2) yields

$$\begin{aligned}\boldsymbol{H} &= \sum_{j=1}^{N} \frac{e\boldsymbol{s}_j(t - R_j/c) \times \boldsymbol{n}'}{c^2 R_j} \\ &\approx -\frac{\exp[-i\omega(t - r/c)]}{r}\frac{\omega^2 \boldsymbol{p} \times \boldsymbol{n}'}{c^2} \sum_j e^{-i\boldsymbol{q}\cdot\boldsymbol{r}_j} .\end{aligned} \tag{2}$$

Here $R_j = |\boldsymbol{r} - \boldsymbol{r}_j| \approx r - \boldsymbol{n}'\cdot\boldsymbol{r}_j$, $\boldsymbol{p} = e\boldsymbol{a}$, and $\boldsymbol{r}$ is the radius vector counted from a certain origin of the coordinates in the region occupied by the particles. The vector $\boldsymbol{q} = \omega\boldsymbol{n}'/c - \boldsymbol{k} = \boldsymbol{k}' - \boldsymbol{k}$ is a change of the wave vector during scattering. Calculation of the differential cross-section from formula (5.119) yields

$$\frac{\mathrm{d}\Sigma}{\mathrm{d}\Omega} = F(\boldsymbol{q})\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} , \tag{3}$$

where

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\omega^4(\boldsymbol{n}' \times \boldsymbol{p})^2}{c^4|\boldsymbol{E}_0|^2} \tag{4}$$

is the cross-section of scattering either by an individual oscillator or by a free electron (at $\omega_0 = \gamma = 0$) that was calculated earlier (see Problems 5.127*–5.129). The multiplier

$$F(\boldsymbol{q}) = \left|\sum_{j=1}^{N} e^{-i\boldsymbol{q}\cdot\boldsymbol{r}_j}\right|^2 \tag{5}$$

is the coherence factor showing how scattering by a system of charges differs from the scattering by a single particle. It shows the strong dependence on the relationship between the reciprocal wave vector q^{-1} transferred to the scatterer and the size of the region in which the particles propagate. The limiting cases are as follows:

1. The transferred wave vector is small, and $|\boldsymbol{q}\cdot\boldsymbol{r}_j| \ll 1$ for all j. Replacing the exponents in equation (5) by unities results in $F = N^2$, where N is the total number of scatterers. This is the case of totally coherent scattering when the fields of individual particles combine in the same phase (compared with the case of coherent radiation by a bunch of particles; Problem 5.92). The scattering is proportional to the squared number of scatterers:

$$\frac{\mathrm{d}\Sigma}{\mathrm{d}\Omega} = N^2 \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} . \tag{6}$$

When long wavelengths ($\lambda \gg r_j$) are scattered, (6) holds at all scattering angles. If $\lambda \le r_j$, (6) is valid only for small-angle scattering at which $q r_j \ll 1$ even though $k r_j \ge 1$. Because $q = 2k \sin(\theta/2)$, where θ is the scattering angle, the range of coherent scattering angles is given by the inequality

$$\theta \ll \theta_\mathrm{c} \approx \frac{1}{kl} , \tag{7}$$

where l is the size of the scattering system.

2. The transferred wave vector is large, and $|\boldsymbol{q}\cdot\boldsymbol{r}_j| \gg 1$. Write down (5) in the form

$$F = \sum_{j=1}^{N} \exp[i\boldsymbol{q}\cdot(\boldsymbol{r}_j - \boldsymbol{r}_j)] + \sum_{j\neq m} \exp[i\boldsymbol{q}\cdot(\boldsymbol{r}_j - \boldsymbol{r}_m)] .$$

Clearly, the first sum is equal to N. Generally speaking, the value of the second sum depends on the position of the charges. In the case of a random charge arrangement and large enough N, the oscillating items cancel each other out and $F = N$. Hence,

$$\frac{\mathrm{d}\Sigma}{\mathrm{d}\Omega} = N \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} . \tag{8}$$

Here, the intensities of waves scattered by individual particles rather than the amplitudes are summed up and the effect is proportional to the number of the scatterers and not to its square.

It should be borne in mind that we disregarded secondary scattering of the already scattered waves. This means that the path of the waves (or photons) must exceed the size of the scattering system. In other words, its "optical thickness" must be small.

5.133 The force depends on the momentum transferred to the particle by an incident wave per unit time. It follows from symmetry considerations that the force as well as the momentum being transferred is oriented along the direction of primary wave propagation $\boldsymbol{k}/k$. Such momentum is equal to the difference in projections onto $\boldsymbol{k}$ of the momenta of the primary wave and all scattered waves, the intensity of which is determined by the incident wave and the scattering cross-section. It leads to the general formula for the force averaged over the wave period:

$$\overline{\mathcal{F}} = \frac{\overline{\gamma}_0}{c} \int (1 - \cos\theta) \left(\frac{d\sigma_s(\theta, \alpha)}{d\Omega} \right) d\Omega \ . \tag{1}$$

The differential scattering cross-section depends on the primary wave polarization and takes the following forms for linearly and circularly polarized and also unpolarized waves, respectively:

$$\begin{aligned}
\frac{d\sigma_s(\theta, \alpha)}{d\Omega} &= r_0^2 (1 - \sin^2\theta \cos^2\alpha) \ , \\
\frac{d\sigma_s(\theta, \alpha)}{d\Omega} &= \frac{1}{2} r_0^2 (1 + \cos^2\theta) \ , \\
\frac{d\sigma_s(\theta, \alpha)}{d\Omega} &= \frac{1}{2} r_0^2 (1 + \cos^2\theta) \ ,
\end{aligned} \tag{2}$$

where r_0 is the particle's classical radius, and θ and α are the angles determining the direction of the scattered wave distribution. Despite all this, the force is insensitive to polarization and can be represented in the above three cases as

$$\mathcal{F} = w_0 \sigma_{\mathrm{T}} \ , \tag{3}$$

where $w_0 = \overline{\gamma}_0/c = (1/8\pi)|E_0|^2$ is the energy density in the primary wave and

$$\sigma_{\mathrm{T}} = \frac{8\pi r_0^2}{3} \tag{4}$$

is the Thomson cross-section of wave scattering by a free charge.

5.134

$$\boldsymbol{F} = w\sigma_{\mathrm{T}} \left(1 - \frac{\boldsymbol{n}\cdot\boldsymbol{v}}{c}\right) \left\{ \boldsymbol{n} - \gamma^2 \left(1 - \frac{\boldsymbol{n}\cdot\boldsymbol{v}}{c}\right) \frac{\boldsymbol{v}}{c} \right\} \ .$$

This expression for the force also holds for nonmonochromatic radiation propagating in the $\boldsymbol{n}$ direction.

5.135* As follows from symmetry considerations, the resultant force acting on an electron should be directed normally to the surface. Averaging the force found in the preceding problem over all possible directions of radiation leads to

$$\langle F_n \rangle = \frac{3}{4} w\sigma_{\mathrm{T}} \gamma^2 \left(1 + \beta^2 - \frac{32}{15}\beta\right) \ .$$

The force is positive at $\beta < \beta_*$ and negative at $\beta > \beta_*$, where β_* is the root of the equation

$$\beta^2 - \frac{32}{15}\beta + 1 = 0$$

satisfying the condition $0 \leq \beta_* \leq 1$, that is, $\beta_* \approx 0.7$. The particle is accelerated to velocity $v_* \approx 0.7c$ and thereafter propagates with this speed without further acceleration. It can be accounted for by an increasingly high number of photons *in the particle's rest system* participating in head-on collisions with the particle as its velocity increases. This results in a decrease of the accelerating force and it finally vanishes at $v = v_*$. At $v > v_*$, head-on collisions prevail over overtaking ones and the particle slows down. A change in the direction of quantum motion is due to the transformation of angles (aberration of light).

5.136* Radiation propagating inside the solid angle $\Omega_0 = 2\pi(1 - \mu_0)$, where $\mu_0 = \cos\theta_0 = (1 + a^2/z^2)^{-1/2}$, comes up to an electron located at distance z from the spot (see Figure 5.15). In the case of isotropic emission from each point of the spot, its distribution by angles

$$\psi(\mu)\mathrm{d}\Omega = \frac{\Theta(\mu - \mu_0)\mathrm{d}\mu}{1 - \mu_0} \tag{1}$$

depends on the z coordinate. The radiation energy density is coordinate dependent too in accordance with the law

$$w(z) = w_0 \frac{\Omega(z)}{\Omega(0)} = w_0(1 - \mu_0(z)) \, . \tag{2}$$

Equation (2) should be substituted in the expression for the force found in Problem 5.133. Its averaging from the distribution function (1) gives

$$f(\gamma, z) = \langle F_z \rangle = \frac{1}{6}\gamma^2 w_0 \sigma_\mathrm{T}(1 - \mu_0)[\beta^2(1 + \mu_0 - 2\mu_0^2) - 6\beta + 3(1 + \mu_0)] \, . \tag{3}$$

Create the equation describing a change of the particle's energy and pass to the differentiation with respect to $\mathrm{d}z = v\mathrm{d}t$ to obtain

$$\frac{\mathrm{d}\gamma}{\mathrm{d}z} = \frac{f(\gamma, z)}{mc^2} \, . \tag{4}$$

In this equation, the variables are inseparable, but it is possible to study the sign of the derivative $\mathrm{d}\gamma/\mathrm{d}z$ and the asymptotes. At large distances from the spot, $z \gg z_0 \gg a$, equation (4) can be solved in the analytical form. The maximum possible Lorentz factor is determined from the transcendental equation

$$\gamma\left(1 + \sqrt{1 - \frac{1}{\gamma^2}} - \arcsin\frac{1}{\gamma}\right) = \gamma_0\left(1 + \sqrt{1 - \frac{1}{\gamma_0^2}} - \arcsin\frac{1}{\gamma_0}\right) + \frac{w_0 \sigma_\mathrm{T} a^2}{2mc^2 z_0} \, ,$$

where γ_0 is the Lorentz factor at $z = z_0$, which is to be found from the numerical solution of equation (4). It is essentially dependent on radiation density w_0.

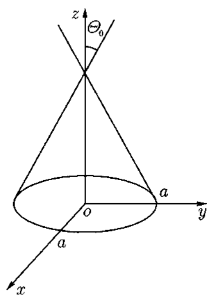

Figure 5.15 Acceleration of an electron by radiation of a hot plane spot.

5.137 Use the result obtained in Problem 5.133. The radiation energy density decreases with the distance from the star's center in inverse proportion to r^2 : $w(r) = L/4\pi c r^2$. The light pressure force acting on an electron

$$F_r = \sigma_T w(r) = \frac{2e^4 L}{3m_e^2 c^5} \frac{1}{r^2} .$$

A proton is subject to an $(m_e/m_p)^2$ smaller force. The gravitational force largely acts on the proton and exhibits the same distance dependence: $F_g = GMm_e/r^2$, where $G = 6.67 \times 10^{-8}\,\mathrm{cm^3/(g\,s^2)}$ is the gravitational constant and M is the stellar mass. Both forces act on the quasi-neutral plasma as a whole owing to Coulomb interaction between protons and electrons. The equality of the forces accounts for

$$L_c = \frac{4\pi c m_p G M}{\sigma_T} \approx 10^{38} \frac{M}{M_\odot} \mathrm{erg/s} ,$$

where $M_\odot \approx 2 \times 10^{33}$ g is the Sun's mass.

5.138* Make use of the Dirac–Lorentz equation (5.110) written via momentum and acceleration

$$\frac{dp_i}{d\tau} = \frac{e}{mc} F_{ik} p^k + \frac{2e^2}{3mc^3} \left\{ \frac{d^2 p_i}{d\tau^2} + \frac{1}{c^2} p_i \left(w_k w^k \right) \right\} . \quad (1)$$

The F_{ik} tensor must contain both the wave field and the polarization field E_z that impedes the longitudinal motion of electrons. Let us examine the stationary state in which the particle's energy $\mathcal{E} = mc^2\gamma =$ const and the three-dimensional momentum rotates in the xy plane with wave frequency ω:

$$\boldsymbol{p}(t) = p[\boldsymbol{e}_x \cos(\omega t - \varphi) - \boldsymbol{e}_y \sin(\omega t - \varphi)] , \quad p_z = 0 , \quad (2)$$

where t is the coordinate time and φ is the phase shift of electron rotation with respect to the electric field of the wave.

Component $i = 0$ in (1) gives the energy balance equation after the substitution of the quantities above:

$$e\boldsymbol{E}\cdot\boldsymbol{v} \equiv eE_0 v \sin\varphi = \frac{2e^2\omega^2}{3c} \beta^2 \gamma^4 . \quad (3)$$

Here, the left-hand side of the equality contains the power transferred by the wave to the electron; the right-hand side is exactly equal to the energy loss (5.72) due to emission by the particle rotating around the circle with a given frequency. Pass to differentiation with respect to coordinate time $d\tau = \gamma^{-1} dt$ in components $i = 1, 2, 3$ and distinguish projection on the direction of the particle's acceleration multiplying scalarwise both parts of equation (1) by the corresponding unit vector $\boldsymbol{e}(t) = \boldsymbol{e}_x \sin(\omega t - \varphi) + \boldsymbol{e}_y \cos(\omega t - \varphi)$. This allows the momentum to be expressed through the wave amplitude and the phase shift:

$$\omega p = e E_0 \cos\varphi \ . \tag{4}$$

The last equation describes a balance between centrifugal and electric forces without participation of the radiation force because the angular radiation distribution is symmetric with respect to the plane of the circular orbit. Finally, the projection of equation (1) onto the Oz axis makes it possible to express the longitudinal electric field E_z through the wave field:

$$E_z = E_0 \beta \sin\varphi \ . \tag{5}$$

Equations (3) and (4) permit us to calculate the dependence of the cross-section of the wave amplitude and analyze the limiting cases. To this effect, the dimensionless parameters (Zel'dovich, 1975) $b = e E_0 / m c \omega$ and $\kappa = 3c / r_0 \omega$ should be introduced and the phase φ should be excluded from these equations:

$$(\gamma^2 - 1)\left(\frac{\gamma^6}{\kappa^2} + 1\right) = b^2 \ . \tag{6}$$

The algebraic equation thus obtained describes the particle's energy $\mathcal{E} = mc^2\gamma$ as a function of the parameters b and κ. In the domain of applicability of classical electrodynamics, $\kappa \gg 1$. This inequality is violated only at quantum energies $\hbar\omega > 137 mc^2 \approx 70$ MeV, that is, in the far quantum region. According to (5.119), the total cross-section is determined by the ratio of the energy emitted per unit time (the right-hand side of (3)) to the energy flow density in the incident wave $\gamma_0 = c E_0^2 / 4\pi$:

$$\sigma = \frac{8\pi e^2 \omega^2}{3 c^2 E_0^2} \beta^2 \gamma^4 = \sigma_{\mathrm{T}} \frac{\gamma^2 \kappa^2}{\gamma^6 + \kappa^2} \ . \tag{7}$$

The limiting cases are as follows:

1. $\gamma \ll \kappa^{1/3}$. Expression (6) gives $\gamma^2 \approx 1 + b^2$; therefore, $b \ll \kappa^{1/3}$. This is the case when the radiation reaction in (1) plays but a minor role. It follows from (7) that

$$\sigma \approx \sigma_{\mathrm{T}} \quad \text{at} \quad b \ll 1 \ , \qquad \sigma \approx \sigma_{\mathrm{T}} b^2 \quad \text{at} \quad b \gg 1 \ . \tag{8}$$

2. $\gamma \gg \kappa^{1/3}$; where $\gamma \approx (\kappa b)^{1/4}$ and $b \gg \kappa^{1/3}$. These inequalities suggest that the radiation reaction outweighs the external force. The cross-section is estimated as

$$\sigma \approx \sigma_{\mathrm{T}} \frac{\kappa}{b} \ . \tag{9}$$

The cross-section reaches a maximum value at $b \approx \kappa^{1/3}$:

$$\sigma_{\max} \approx \sigma_T \kappa^{2/3} \approx \sigma_T \left(\frac{\lambda}{2\pi r_0}\right)^{2/3} . \tag{10}$$

The classical theory is applicable if the characteristic quantum energy makes up a small fraction of the electron energy: $\hbar\omega_c \ll mc^2\gamma$. The characteristic energy is the quantum energy in the maximum radiation, that is, at frequencies $\omega_c \approx \omega\gamma^3$ (see formula (8) from the solution of Problem 5.83** following the substitution $\omega_0 \to \omega$). The borderline between the classical and quantum domains is found from the above inequalities: $E_0 \approx e/\Lambda_C^2$, where Λ_C is the Compton wavelength of the electron.

5.139** The general scheme for solution is the same as in the preceding task. Equation (3) preserves the form and the equation of motion (4) from the preceding problem becomes

$$\omega p = \pm eH\beta + eE_0 \cos\varphi ,$$

where the plus sign refers to an inordinary wave for which the signs $\omega_H = eH/mc$ and ω coincide and the minus sign pertains to the ordinary wave with the opposite frequency signs. In the inordinary wave, the directions of rotation of an electron by the wave and the magnetic field coincide. The equations of motion with the notation from the preceding task yield

$$\gamma\beta = \pm\frac{\omega_H}{\omega}\beta + b\cos\varphi , \quad \kappa b\beta \sin\varphi = \beta^2\gamma^4 , \quad \sigma = \sigma_T \frac{\gamma^2(\gamma^2-1)}{b^2} .$$

Calculation of the electron energy and cross-section depending on the input parameters (ω_H, ω, E_0) in the general case is possible only by numerical methods. See the original work of Zel'dovich and Illarionov (1972) for the analysis of limiting cases.

5.140** The motion of an electron in a flat wave of arbitrary force and with arbitrary polarization without regard for the radiation reaction was studied in Problems 4.57 and 4.58. Radiation by a particle performing a given motion is calculated using the formulas from Section 5.3. See the original work of Sarachik and Schappert (1970)

5.141*

$$\frac{d^2\mathcal{E}^{\text{rad}}}{d\omega d\Omega} = \frac{m^2 c\omega^2}{4\pi^2 E^2} \left\{ \left[\left(1 - \frac{\nu^2}{z^2}\right)\gamma_\perp^2 - 1\right] R_\nu^2(z) + \gamma_\perp^2 R_\nu'^2(z) \right\} ,$$

where

$$\gamma_\perp = \frac{\sqrt{m^2c^2 + p_\perp^2}}{mc} , \quad \nu = \frac{cp_\perp k_\perp}{eE}\cos\varphi , \quad z = \frac{mc}{eE} k_\perp \gamma_\perp .$$

Index $\perp$ denotes the component perpendicular to the constant electric field $\boldsymbol{E} =$ const, and φ is the angle between $\boldsymbol{p}_\perp$ and $\boldsymbol{k}_\perp$. Calculation, analysis of specific cases, and discussion of physical aspects of the problem can be found in the original article (Nikishov and Ritus, 1969).

5.142 Solution of the one-dimensional equation taking account of the radiation force,

$$m\dot{v} = \frac{2e^2}{3c^3}\ddot{v} + eE_0\delta\frac{t}{T_0}\,, \tag{1}$$

yields acceleration of the particle

$$\dot{v}(t) = w(t) = \left(w_0 - \frac{eE_0T_0}{m\tau_0}\Theta\frac{t}{T_0}\right)e^{t/\tau_0}\,, \tag{2}$$

where w_0 is the integration constant and τ_0 is given by formula (5.96). Imposing the initial condition $w = 0$ at $t < 0$, that is, before the force begins to act, yields $w_0 = 0$ and self-acceleration

$$w(t) = -\frac{eE_0T_0}{m\tau_0}e^{t/\tau_0}\,, \tag{3}$$

that is, the physically senseless result at $t > 0$. However, if the "terminal condition" is imposed and acceleration becomes zero after the force ceases to act, $w(t) = 0$ at $t > 0$, then $w_0 = eE_0T_0/(m\tau_0)$ and the solution takes the form

$$w(t) = w_0e^{t/\tau_0} \quad \text{at } t < 0\,, \quad v(t) = \begin{cases} v_0 + \tau_0w_0e^{t/\tau_0}\,, & t < 0\,; \\ v_0 + \tau_0w_0\,, & t > 0\,. \end{cases} \tag{4}$$

The solution thus obtained does not contain self-acceleration; it has another drawback, that is, the principle of causality is violated and the particle begins "feeling" the force before it starts to act. This result may be interpreted as "incompressibility" of the particle r_0 in size. Because of this, the perturbation is transferred over such a distance instantaneously, as soon as the forward edge of the particle senses it. The total radiation calculated with regard for the particle's acceleration (4) is

$$\mathcal{E}^{\text{rad}} = \frac{e^2w_0^2\tau_0}{3c^3}\,. \tag{5}$$

The difficulties illustrated in the foregoing discussion indicate that the radiation force does not quite correctly account for the radiation reaction. It appears that difficulties do not arise when this force is small compared with the external force in the particle's rest system. These difficulties stem from the production of infinite intrinsic energy and other divergences in the current theories.

5.143

$$\varphi_k = \frac{e}{2\pi^2k^2}\,, \quad \boldsymbol{E}_k = -\frac{ie\boldsymbol{k}}{2\pi^2k^2}\,.$$

5.144 The volumetric density $\rho(\boldsymbol{r}, t) = e\delta(\boldsymbol{r} - \boldsymbol{v}t)$ is expressed through the delta function, and its Fourier component is

$$\rho_{k\omega} = \frac{e}{(2\pi)^4} \int \delta(\boldsymbol{r} - \boldsymbol{v}t) e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} \mathrm{d}^3 r \mathrm{d}t$$

$$= \frac{e}{(2\pi)^4} \int_{-\infty}^{+\infty} e^{i(\omega - \boldsymbol{k}\cdot\boldsymbol{v})t} \mathrm{d}t = \frac{e}{8\pi^3} \delta(\boldsymbol{k}\cdot\boldsymbol{v} - \omega) .$$

It is found from this expression based on the results of the preceding problem that

$$\varphi_{k\omega} = \frac{e}{2\pi^2} \cdot \frac{\delta(\boldsymbol{k}\cdot\boldsymbol{v} - \omega)}{k^2 - \omega^2/c^2} .$$

The use of a similar approach yields

$$\boldsymbol{A}_{k\omega} = \frac{e\boldsymbol{v}}{2\pi^2 c} \frac{\delta(\boldsymbol{k}\cdot\boldsymbol{v} - \omega)}{k^2 - \omega^2/c^2} .$$

As follows from the expressions for field strength components,

$$\boldsymbol{E}_{k\omega} = i\frac{e}{2\pi^2} \cdot \frac{\delta(\boldsymbol{k}\cdot\boldsymbol{v} - \omega)}{k^2 - \omega^2/c^2} \left(-\boldsymbol{k} + \boldsymbol{v}\frac{\omega}{c^2}\right) ,$$

$$\boldsymbol{H}_{k\omega} = i\boldsymbol{k} \times \boldsymbol{A}_{k\omega} = i\frac{e}{2\pi^2 c} (\boldsymbol{k} \times \boldsymbol{v}) \frac{\delta(\boldsymbol{k}\cdot\boldsymbol{v} - \omega)}{k^2 - \omega^2/c^2} .$$

All field components contain the factor $\delta(\boldsymbol{k}\cdot\boldsymbol{v} - \omega)$, owing to the dispersion equation, $\omega = \boldsymbol{k}\cdot\boldsymbol{v}$. Because of this, all Fourier expansions of the electromagnetic field in this case are actually three dimensional rather than four dimensional. For example, in the case of potential φ,

$$\varphi = \int\limits_{(k)} \int\limits_{-\infty}^{\infty} \frac{e}{2\pi^2} \cdot \frac{\delta(\omega - \boldsymbol{k}\cdot\boldsymbol{v})}{k^2 - \frac{\omega^2}{c^2}} e^{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)} \mathrm{d}^3 k \mathrm{d}\omega = \int \varphi_k(t) e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \mathrm{d}^3 k ,$$

where

$$\varphi_k(t) = \frac{e}{2\pi^2} \cdot \frac{e^{-i(\boldsymbol{k}\cdot\boldsymbol{v})t}}{k^2 - \omega^2/c^2} .$$

5.146* Let us consider the calculation of the scalar potential. According to the equations from the solution of Problem 2.126

$$\varphi_{k\omega} = \frac{4\pi\rho_{k\omega}}{k^2 - \omega^2/c^2} .$$

The Fourier component of the charge density is

$$\rho_{k\omega} = -\frac{1}{(2\pi)^4} \int [\boldsymbol{p}\cdot\nabla\delta(\boldsymbol{r} - \boldsymbol{v}t)] e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} \mathrm{d}^3 r \mathrm{d}t$$

$$= \frac{1}{(2\pi)^4} \int [\boldsymbol{p}\cdot\nabla e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}] \delta(\boldsymbol{r} - \boldsymbol{v}t) \mathrm{d}^3 r \mathrm{d}t = -i\frac{\boldsymbol{p}\cdot\boldsymbol{k}}{(2\pi)^3} \delta(\omega - \boldsymbol{k}\cdot\boldsymbol{v}) .$$

The dispersion equation $\omega = \boldsymbol{k} \cdot \boldsymbol{v}$ has the same form as in the case of the field of a uniformly moving point charge (see Problem 5.144). Calculating $\varphi(\boldsymbol{r}, t))$ as suggested in Problem 5.145* yields

$$\varphi(\boldsymbol{r}, t) = -\boldsymbol{p} \cdot \nabla \frac{1}{r^*} = \frac{\boldsymbol{p} \cdot \boldsymbol{r}_0}{r^{*3}} , \tag{1}$$

where

$$\boldsymbol{r}_0 = \left(x - v t, \frac{y}{\gamma^2}, \frac{z}{\gamma^2} \right) , \quad r^* = \sqrt{(x - v t)^2 + \frac{1}{\gamma^2}(y^2 + z^2)} .$$

Analogous calculations give for the vector potential

$$\boldsymbol{A}(\boldsymbol{r}, t) = \frac{\boldsymbol{m} \times \boldsymbol{r}^*}{r^{*3}} + \frac{\boldsymbol{v}(\boldsymbol{p} \cdot \boldsymbol{r}_0)}{c r^{*3}} . \tag{2}$$

5.147* (i) $\boldsymbol{A} = \frac{\boldsymbol{m}_0 \times \boldsymbol{r}^*}{\gamma r^{*3}}$, $\varphi = 0$; (ii) $\boldsymbol{A} = \frac{\boldsymbol{m}_0 \times \boldsymbol{r}^*}{r^{*3}}$, $\varphi = \frac{\boldsymbol{v} \cdot \boldsymbol{A}}{c}$.

5.149*

$$\ddot{q}_{k\lambda}(t) + \omega_k^2 q_{k\lambda}(t) = F_{k\lambda}(t) , \tag{1}$$

where

$$F_{k\lambda}(t) = \frac{ec}{2\pi^2} [\boldsymbol{e}_{k\lambda}^* \cdot \boldsymbol{v}(t)] e^{-i\boldsymbol{k} \cdot \boldsymbol{r}_0(t)} , \tag{2}$$

$\boldsymbol{r}_0(t)$ is the radius vector of the particle at moment t, and $\boldsymbol{v}$ is its velocity at this moment. In the nonrelativistic case,

$$m\ddot{\boldsymbol{r}}_0 = \boldsymbol{F} + e\boldsymbol{E}(\boldsymbol{r}_0) , \tag{3}$$

where m is the particle's mass and $\boldsymbol{F}$ is the force of nonelectromagnetic origin acting on the particle.

$$\boldsymbol{E}(\boldsymbol{r}_0) = -\frac{1}{\pi\sqrt{2}} \int \boldsymbol{e}_{k\lambda} \dot{q}_{k\lambda} e^{i\boldsymbol{k} \cdot \boldsymbol{r}_0} (\mathrm{d}\boldsymbol{k}) \tag{4}$$

is the radiation field strength at the point where the particle is located. We do not take into account the force acting on the particle from the side of the magnetic field on the assumption that $v \ll c$. Equation (1) is the equation for forced oscillations of the oscillator under the effect of the external force $F_{k\lambda}(t)$. The motion of the particle and the magnetic field interacting with each other is described the set of equations (1) and (3).

5.150 The change in the energy of one oscillator

$$\frac{\mathrm{d}W_{k\lambda}}{\mathrm{d}t} = \frac{1}{2}(F_{k\lambda}\dot{q}_{k\lambda}^* + F_{k\lambda}^* \dot{q}_{k\lambda}) .$$

The rate of field energy variations

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \frac{1}{2} \int (F_{k\lambda}\dot{q}_{k\lambda}^* + F_{k\lambda}^* \dot{q}_{k\lambda})(\mathrm{d}\boldsymbol{k}) .$$

5.151* In this case, the force $F_{k\lambda}(t)$ assumes the form

$$F_{k\lambda}(t) = b_{k\lambda} \cos \omega_0 t \,,$$

where

$$b_{k\lambda} = \frac{e}{\pi\sqrt{2}}(\boldsymbol{v}_0 \cdot \boldsymbol{e}_{k\lambda}) \,, \quad \boldsymbol{v}_0 = \omega_0 \boldsymbol{r}_0$$

(for simplicity, we consider linearly polarized field oscillators so that the unit vectors $\boldsymbol{e}_{k\lambda}$ are real). Integration of equation (1) from Problem 5.149* yields

$$q_{k\lambda} = \frac{b_{k\lambda}}{\omega_k^2 - \omega_0^2}(\cos \omega_0 t - \cos \omega_k t) \,,$$

provided the field oscillators were unexcited at the initial moment of time $t = 0$. This value of $q_{k\lambda}$ is substituted into the expression for the rate of radiation field energy variations $\mathrm{d}W_{k\lambda}/\mathrm{d}t$, which was found in Problem 5.150:

$$\frac{\mathrm{d}W_{k\lambda}}{\mathrm{d}t} = \frac{b_{k\lambda}^2}{\omega_k^2 - \omega_0^2}(\omega_k \cos \omega_0 t \sin \omega_k t - \omega_0 \cos \omega_0 t \sin \omega_0 t) \,.$$

Integration of the last expression over t from 0 to t, gives the amount of energy transferred by the particle to the field oscillator $(\boldsymbol{k}, \lambda)$ for time t:

$$W_{k\lambda} = \int_0^t \frac{\mathrm{d}W_{k\lambda}}{\mathrm{d}t}\mathrm{d}t = \frac{b_{k\lambda}^2}{\omega_k^2 - \omega_0^2}\left[\frac{\omega_k}{2} \cdot \frac{1 - \cos(\omega_k + \omega_0)t}{\omega_k + \omega_0} + \right.$$
$$\left. + \frac{\omega_k}{2} \cdot \frac{1 - \cos(\omega_k - \omega_0)t}{\omega_k - \omega_0} - \frac{\omega_0}{4} \cdot \frac{1 - \cos 2\omega_0 t}{\omega_0}\right] .$$

At $\omega_k = \omega_0$ and $t \to \infty$, the second term in the square brackets is too large compared with the first and third terms. This means that oscillators are excited in a resonant manner: those field oscillators whose the frequency is close to that of the exciting force $F_{k\lambda}$ are excited in the first place. Therefore, we leave only the resonance term and sum up the energies gained by the field oscillators whose frequencies are not significantly different from ω_0, the $\boldsymbol{k}$ direction lies within the solid angle $\mathrm{d}\Omega$, and the unit polarization vector $\boldsymbol{e}_{k1}$ $(\boldsymbol{e}_{k2})$ has one and the same direction:

$$\mathrm{d}W = \sum_{k,\lambda} W_{k\lambda} = \frac{\mathrm{d}\Omega}{2c^3} \int_{\omega_0-\delta}^{\omega_0+\delta} \sum_{\lambda} \frac{\omega_k^3 b_{k\lambda}^2}{\omega_k + \omega_0} \cdot \frac{1 - \cos(\omega_k - \omega_0)t}{(\omega_k - \omega_0)^2}\mathrm{d}\omega_k \,.$$

The subintegral function in the last expression shows a sharp peak at $\omega_k = \omega_0$. The larger t, the narrower this maximum. At sufficiently large t, the gradually changing factor $\sum_\lambda(\omega_k^3 b_{k\lambda}^2/(\omega_k + \omega_0))$ can be taken outside the sign of the integral assuming that $\omega_k = \omega_0$. In the remaining integral, δ can go to infinity.

Then, it takes the form (see formula (2.18)):

$$\int_{-\infty}^{\infty} \frac{1-\cos\alpha t}{\alpha^2}\,\mathrm{d}\alpha = \pi t\,, \quad t \to \infty\,.$$

Thus,

$$\frac{\mathrm{d}W}{\mathrm{d}\Omega} = \frac{\pi(b_{k1}^2 + b_{k2}^2)\omega_0^2}{2c^3}t\,,$$

whence the well-known result emerges for the intensity of radiation in a given direction:

$$\overline{\frac{\mathrm{d}I}{\mathrm{d}\Omega}} = \frac{1}{t}\frac{\mathrm{d}W}{\mathrm{d}\Omega} = \frac{e^2\omega_0^2\overline{v^2}\sin^2\vartheta}{4\pi c^3}\,,$$

where $\overline{v^2} = v_0^2/2$ denotes the mean velocity of an oscillating particle and ϑ is the angle between the directions of $\boldsymbol{v}_0$ and $\boldsymbol{k}$. We used the following readily available expression to derive the last formula:

$$(\boldsymbol{v}_0 \cdot \boldsymbol{e}_{k1})^2 + (\boldsymbol{v}_0 \cdot \boldsymbol{e}_{k2})^2 = v_0^2 \sin^2\vartheta\,.$$

Integration over angles yields the overall radiation intensity

$$\overline{I} = \frac{2e^2\omega_0^2\overline{v^2}}{3}\,.$$

5.153* Let us solve approximately the system of (1) and (3) from Problem 5.149*. We disregard the radiation reaction and substitute the intensity of the field $\boldsymbol{E} = \boldsymbol{E}_0 \cos\omega t$ of the incident wave into (3). Its solution corresponding to forced oscillations has the form

$$\boldsymbol{r}(t) = \frac{e}{m}\boldsymbol{E}_0\frac{\cos\omega t}{\omega_0^2 - \omega^2}\,. \tag{1}$$

The motion of the particle under the action of an incident wave excites radiation field oscillators in accordance with (1) from Problem 5.149* in which the force $F_{k\lambda}$ should be expressed through $\boldsymbol{r}(t)$:

$$F_{k\lambda} = \frac{e^2\omega}{m\pi\sqrt{2}} \cdot \frac{\boldsymbol{e}_{k\lambda}\cdot\boldsymbol{E}_0}{\omega^2 - \omega_0^2}\sin\omega t\,.$$

Unit polarization vectors are chosen so as to be material. The solution of (1) from Problem 5.149* with the initial condition $q_{k\lambda}(0) = 0$ yields

$$q_{k\lambda}(t) = \frac{e^2}{m\pi\sqrt{2}} \cdot \frac{\omega(\boldsymbol{E}_0\cdot\boldsymbol{e}_{k\lambda})}{(\omega_k^2 - \omega^2)(\omega^2 - \omega_0^2)}(\sin\omega t - \sin\omega_k t)\,.$$

Acting further as in Problem 5.151*, we obtain the intensity of radiation in the $\boldsymbol{k}$ direction with polarization characterized by the unit vector $\boldsymbol{e}_{k\lambda}$:

$$\frac{\mathrm{d}I_{k\lambda}}{\mathrm{d}\Omega} = \frac{1}{t}\frac{\mathrm{d}W_{k\lambda}}{\mathrm{d}\Omega} = \frac{e^4}{8\pi m^2 c^3} \cdot \frac{\omega^4(\boldsymbol{E}_0 \cdot \boldsymbol{e}_{k\lambda})^2}{(\omega^2 - \omega_0^2)^2} \ . \tag{2}$$

It follows from (2) that scattered radiation is polarized in the plane passing through $\boldsymbol{E}_0$ and $\boldsymbol{k}$. Introducing the angle ϑ between the vectors $\boldsymbol{E}_0$ and $\boldsymbol{k}$ yields

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{8\pi}{cE_0^2}\frac{\mathrm{d}I}{\mathrm{d}\Omega} = \left(\frac{e^2}{mc^2}\right)^2 \frac{\omega^4}{(\omega_0^2 - \omega^2)^2}\sin^2\vartheta \ ,$$

in excellent agreement with the results obtained in Problem 5.127*. Integration over angles gives the total scattering cross-section:

$$\sigma = \frac{8\pi}{3}\left(\frac{e^2}{mc^2}\right)^2 \frac{\omega^4}{(\omega_0^2 - \omega^2)^2} \ .$$

6
Quantum Theory of Radiation Processes. Photon Emission and Scattering

Quantum electrodynamics is one of the fundamental parts of modern physics; it explores a wide area of physical phenomena. These are related to various applied and fundamental areas such as the physics of elementary particles, quantum optics and electronics, and quantum information science, and also astrophysics and cosmology. Quantum electrodynamics was the first quantum field theory in the history of physics. This theory further played a significant role in the formation of the modern quantum field theories, namely, the general theory of the electromagnetic and weak interactions, and the theory of quantum chromodynamics describing strong hadron interactions. In this sense, we introduce briefly the principles of quantum electrodynamics mostly in examples of interaction of light with nonrelativistic systems, but in Section 6.3 and Chapter 7 we consider also relativistic problems and use the invariant theory of perturbations.

We assume that the reader is experienced in the standard methods of quantum theory. However in Appendix C, we briefly describe the basic principles of quantum mechanics. The most important formulas appearing in the main text are derived there.

6.1
Quantum Theory of the Free Electromagnetic Field

6.1.1
Field Oscillators

Let us begin our consideration from the Hamiltonian of the free electromagnetic field written in the classical form and considered at the end of Section 2.3. It was shown there that the field can be described as equivalent to a mechanical system comprising an ensemble of an infinite number of noninteracting harmonic oscillators. Each oscillator can be associated with own field eigenmode, which is a plane wave with a specific wave vector $\boldsymbol{k}$, polarization σ, and frequency $\omega_k = ck$. According to (2.145), the wave vectors are quantized by periodic boundary conditions imposed on the field eigenmodes. In following we will denote these eigenmode as $s = (\boldsymbol{k}, \sigma)$. The field energy $\mathcal{H}$ and momentum $\boldsymbol{P}$ in the quantization volume V

Foundations of Classical and Quantum Electrodynamics, First Edition. Igor N. Toptygin.

are expressed as canonical variables of position Q_s and momentum P_s for the each field oscillator defined by (2.160), (4.128), and (4.130):

$$\mathcal{H} = \sum_s \mathcal{H}_s , \quad \boldsymbol{P} = \sum_s \frac{\boldsymbol{k}_s \mathcal{H}_s}{\omega_s} , \quad \mathcal{H}_s = \frac{1}{2}\left(P_s^2 + \omega_s^2 Q_s^2\right) , \tag{6.1}$$

where $\mathcal{H}_s$ is the classical Hamiltonian (energy) of the sth field oscillator.

The quantization procedure transforms the classical description of the free electromagnetic field to the quantum description as a system of quantized and non-interacting oscillators. The procedure was introduced by Dirac (1927) (see Dirac, 1990).

6.1.2
Photons

Let us define the stationary states of the electromagnetic field, which has accessible values of the field energy and whose wave functions are the eigenstates of the field Hamiltonian.[1] The quantum operators describing the electromagnetic field contribute to the quantum Hamiltonian similarly as their classical counterparts contribute to the classical Hamiltonian function (6.1). Then the quantum Hamiltonian has the following form:

$$\widehat{\mathcal{H}} = \sum_s \widehat{\mathcal{H}}_s , \quad \widehat{\mathcal{H}}_s = \frac{1}{2}\left(\widehat{P}_s^2 + \omega_s^2 \widehat{Q}_s^2\right) . \tag{6.2}$$

The operators $\widehat{Q}_s$ and $\widehat{P}_s$ obey the standard position–momentum Heisenberg permutation relation: $[\widehat{Q}_s, \widehat{P}_s] = i\hbar$. For the set of uncoupled field modes, their entire wave function is written as a direct product of the eigenstates of all oscillators, $\Phi = \prod_s \Phi_s$, and it is necessary only to solve the Schrödinger equation for a single oscillator:

$$\widehat{\mathcal{H}}_s \Phi_s = \mathcal{E}_s \Phi_s . \tag{6.3}$$

The details of the derivation and the solution are presented in Appendix C. Finally, the energy spectrum is given by (C61)

$$\mathcal{E}_s = \hbar\omega_s \left(N_s + \frac{1}{2}\right) , \quad N_s = 0, 1, \ldots , \tag{6.4}$$

and the wave function in the coordinate representation is expressed by the Hermitian polynomials, see (C65), by substituting $n = N_s$ and $\xi = Q_s\sqrt{\omega_s/\hbar}$.

The full energy of the electromagnetic field localized in volume V is given by the sum of the oscillator energies:

$$\mathcal{E}_{\{s\}} = \sum_s \hbar\omega_s \left(N_s + \frac{1}{2}\right) . \tag{6.5}$$

1) Such states are commonly known as Fock states. Vladimir Aleksandrovich Fock (1898–1974) was an outstanding Soviet theoretician who investigated the mathematical properties of the Hilbert space composed by these states.

The operator of the field momentum $\widehat{\boldsymbol{P}} = \sum_s \boldsymbol{k}_s \widehat{\mathcal{H}}_s/\omega_s$ commutes with the Hamiltonian, which justifies that the momentum can be observed simultaneously with the energy and has the value

$$\boldsymbol{P}_{\{s\}} = \sum_s \hbar \boldsymbol{k}_s \left(N_s + \frac{1}{2} \right) = \sum_s \hbar \boldsymbol{k}_s N_s \,, \tag{6.6}$$

where in the second equality we assumed isotropy of the space and the vacuum state for all modes. As we see, the energy and momentum of each field state are distributed among individual contributions parameterized by frequency $\hbar\omega_s$ and momentum $\hbar\boldsymbol{k}_s = \hbar\omega_s \boldsymbol{n}_s/c$. These contributions can be associated with zero-mass quasi-particles propagating with the speed of light, which we call photons. These particles were considered for the first time by Planck[2] but finally the photon received the status of elementary particle after the works of Einstein (1905) and Compton[3] (1922).

The absence of photons in all modes ($N_s = 0$) creates a physical vacuum and corresponds to the ground state of the field subsystem. In the latter case, the field momentum vanishes, $\boldsymbol{P}_0 = 0$, but the field energy (ground-stated oscillations) does not:

$$\mathcal{E}_0 = \frac{1}{2} \sum_s \hbar\omega_s \,. \tag{6.7}$$

This quantity is infinitely large, because of the infinite number of contributing modes, whose frequencies are not limited. This is an example of nonovercome nonconversary in quantum electrodynamics (similarly to what we encounter for the infinite self-energy of a point-like particle, i.e, for the problem transferred from the classical theory to the quantum theory, see Sections 2.1 and 5.4). The presence of ground-stated oscillations is a rigorous consequence of the theory and can be observed. In particular, it is observed in atomic oscillations in crystals in the low-temperature regime. The small shift of atomic levels (Lamb shift, see Problem 6.18$^{\bullet *}$) is also caused by the vacuum oscillations of the electromagnetic field. Finally, recent cosmological observations indicate that the vacuum energy provides an essential contribution to the matter density of the universe. Although the vacuum energy is well established, its infinite value undoubtedly reflects a certain imperfection of the theory. Fortunately, this difficulty does not affect the standard problems concerning the changes of the electromagnetic field energy, that is, the processes of photon absorption and emission.

2) Max Planck (1858–1947) was outstanding German theoretician, a Nobel prize recipient, and one of the founder of quantum theory. In 1900 he proposed his famous formula explaining the spectrum of blackbody radiation.

3) Compton Arthur Holly (1892–1962) was an American physicist, Nobel Prize Winner. The main works in the field of atomic and nuclear physics and cosmic ray physics.

6.1.3
Occupation Number Representation and Operators of the Electromagnetic Field

In the classical theory, the vector components of the electromagnetic field, $\boldsymbol{A}$, $\boldsymbol{E}$, and $\boldsymbol{H}$ (see (2.151), (2.153), and (2.154)), can be expanded in the basis of plane waves with arbitrary polarization:

$$A(\boldsymbol{r}, t) = \sum_s \left\{ b_s(t) \boldsymbol{\mathcal{A}}_s(\boldsymbol{r}) + b_s^*(t) \boldsymbol{\mathcal{A}}_s^*(\boldsymbol{r}) \right\} , \quad \text{where}$$

$$\boldsymbol{\mathcal{A}}_s(\boldsymbol{r}) = \boldsymbol{e}_s \sqrt{\frac{4\pi c^2}{\mathcal{V}}} e^{i\boldsymbol{k}\cdot\boldsymbol{r}} , \tag{6.8}$$

$$\boldsymbol{E}(\boldsymbol{r}, t) = i \sum_s \frac{\omega_s}{c} \left\{ b_s(t) \boldsymbol{\mathcal{A}}_s(\boldsymbol{r}) - b_s^*(t) \boldsymbol{\mathcal{A}}_s^*(\boldsymbol{r}) \right\} , \tag{6.9}$$

$$\boldsymbol{H}(\boldsymbol{r}, t) = i \sum_s \boldsymbol{k} \times \left\{ b_s(t) \boldsymbol{\mathcal{A}}_s(\boldsymbol{r}) - b_s^*(t) \boldsymbol{\mathcal{A}}_s^*(\boldsymbol{r}) \right\} . \tag{6.10}$$

Here, the complex amplitudes

$$b_s = \frac{1}{2} \left(Q_s + \frac{i}{\omega_s} P_s \right) , \quad b_s^* = \frac{1}{2} \left(Q_s - \frac{i}{\omega_s} P_s \right) \tag{6.11}$$

are expressed by canonical variables Q_s and P_s given by (2.157). As a result of the quantization procedure, these variables become quantum operators acting on generalized field coordinates Q_s (Q_s representation). However, it is more convenient to express them via so-called raising (creation) and lowering (annihilation) operators acting in the Fock photon number representation (see (C54)):

$$b_s(t) \to \widehat{b}_s = \sqrt{\frac{\hbar}{2\omega_s}} \widehat{c}_s , \quad b_s^*(t) \to \widehat{b}_s^\dagger = \sqrt{\frac{\hbar}{2\omega_s}} \widehat{c}_s^\dagger , \quad [\widehat{c}_s, \widehat{c}_s^\dagger] = 1 . \tag{6.12}$$

Then the quantum operator components of the electromagnetic field (6.8)–(6.10) have the following forms:

$$\widehat{\boldsymbol{A}}(\boldsymbol{r}) = \sum_s \left\{ \widehat{c}_s \boldsymbol{A}_s(\boldsymbol{r}) + \widehat{c}_s^\dagger \boldsymbol{A}_s^*(\boldsymbol{r}) \right\} , \tag{6.13}$$

$$\widehat{\boldsymbol{E}}(\boldsymbol{r}) = i \sum_s \frac{\omega_s}{c} \left\{ \widehat{c}_s \boldsymbol{A}_s(\boldsymbol{r}) - \widehat{c}_s^\dagger \boldsymbol{A}_s^*(\boldsymbol{r}) \right\} , \tag{6.14}$$

$$\widehat{\boldsymbol{H}}(\boldsymbol{r}) = i \sum_s \boldsymbol{k} \times \left\{ \widehat{c}_s \boldsymbol{A}_s(\boldsymbol{r}) - \widehat{c}_s^\dagger \boldsymbol{A}_s^*(\boldsymbol{r}) \right\} . \tag{6.15}$$

Here, the space coordinates $\boldsymbol{r}$ play the role of the indices parameterizing the field operators in three-dimensional space. In the Schrödinger picture, after transformation (6.12) there is no time dependence of the field amplitudes. The mode functions are normalized as follows:

$$\boldsymbol{A}_s(\boldsymbol{r}) = \sqrt{\frac{\hbar}{2\omega_s}} \boldsymbol{\mathcal{A}}_s(\boldsymbol{r}) = \boldsymbol{e}_s \sqrt{\frac{2\pi\hbar c^2}{\mathcal{V}\omega_s}} e^{i\boldsymbol{k}_s\cdot\boldsymbol{r}} . \tag{6.16}$$

According to (C63), the operators $\hat{c}_s^\dagger$ and $\hat{c}_s$ raise and lower, respectively, by unity the excitation level of the field oscillator. In terms of photons these operators either create or annihilate one photon in mode s. Therefore, in quantum electrodynamics they are commonly called creation and annihilation operators.

When these operators are applied it is more natural and convenient to use the field mode occupation numbers N_s, representing the number of photons in mode s, instead of coordinate representation Q_s.

Any general quantum state of the field subsystem can be expanded over the basis of the photon number states. For the complete set of modes s, a general stationary state of the field can be written as

$$\Phi_{\{N_s\}} \equiv |\{N_s\}\rangle = \prod_s |N_s\rangle \,. \tag{6.17}$$

The occupation numbers N_s are the eigenvalues of the photon number operator given by

$$\hat{N}_s = \hat{c}_s^\dagger \hat{c}_s \,. \tag{6.18}$$

As shown in Appendix C (see (C63)), the following relations are fulfilled:

$$\begin{aligned} \hat{c}_s^\dagger |N_s\rangle &= \sqrt{N_s + 1}|N_s + 1\rangle \,, \quad \hat{c}|N_s\rangle = \sqrt{N_s}|N_s - 1\rangle \,, \\ \hat{N}_s|N_s\rangle &= N_s|N_s\rangle \,. \end{aligned} \tag{6.19}$$

Any stationary state in the photon number representation can be simply written as

$$|N_s\rangle = \Phi_{N_s}(N_s') = \delta_{N_s N_s'} \,, \tag{6.20}$$

where N_s stands for an eigenvalue for mode s, and N_s' is the wave function argument, expressing the variable quantity.

Example 6.1

Verify that the field operators (6.14) and (6.15) do not commute with the Hamiltonian. As a consequence, show that the field amplitudes do not have precisely defined values in the Fock states, including the vacuum state.

Solution. The field Hamiltonian can be expressed via the photon number operators with the help of (C55) and (6.18):

$$\hat{\mathcal{H}} = \sum_s \hbar\omega_s \left(\hat{N}_s + \frac{1}{2}\right) \,. \tag{6.21}$$

The operators $\hat{N}_s$ and $\hat{c}_s$ (or $\hat{c}_s^\dagger$) for a given mode s do not commute with each other, but all the operators belonging to different modes do commute. Thus, the operators $\hat{\boldsymbol{A}}$, $\hat{\boldsymbol{E}}$, and $\hat{\boldsymbol{H}}$ do not commute with the Hamiltonian. As a consequence, the field components $\boldsymbol{E}$ and $\boldsymbol{H}$ do not have certain values in the states with a fixed number of photons. In particular, in the vacuum state the field exhibits uncertainty associated with its *vacuum fluctuations*. □

Example 6.2

Write the single mode operators $\widehat{\mathcal{H}}_s$, $\widehat{N}_s$, $\widehat{c}_s$, and $\widehat{c}_s^\dagger$ in the photon number representation.

Solution. The operators have the following matrix form:

$$N_{N'N} = \langle N'|\widehat{N}|N\rangle = N\delta_{N'N}\ ; \quad \mathcal{H}_{N'N} = \mathcal{E}\delta_{N'N}\ ;$$
$$c_{N'N} = \sqrt{N}\delta_{N',N-1}\ ; \quad c^\dagger_{N'N} = \sqrt{N+1}\delta_{N',N+1}\ , \tag{6.22}$$

where $\mathcal{E}_N$ is given by (6.4) and the mode index s can be disregarded. □

Recommended literature:

Landau and Lifshitz (1977); Berestetskii *et al.* (1982); Heitler (1954); Dirac (1990); Feynman (1998); Cohen-Tannoudji *et al.* (1992); Itzukson and Zuber (1980)

Problems

6.1. Starting from the classical expression for the field energy (2.152) and using the operators $\widehat{\boldsymbol{E}}$ and $\widehat{\boldsymbol{H}}$, introduce the Hamiltonian $\widehat{\mathcal{H}}$ of the electromagnetic field in terms of annihilation $\widehat{c}$ and creation $\widehat{c}^\dagger$ operators.

6.2•. Verify that operators $\widehat{Q}_s$ and $\widehat{P}_s$ originally introduced in the Schrödinger picture and transformed to the Heisenberg picture fulfill the canonical Hamiltonian equations. Use the Hamiltonian (6.2) and the equations for the time derivative of operators, (C34) and (C35).

6.3•. Consider the photon annihilation $\widehat{c}$ and creation $\widehat{c}^\dagger$ operators written in the Heisenberg picture and show that they have the following dependence on time:

$$\widehat{c}^\dagger(t) = \widehat{c}^\dagger e^{i\omega t}\ , \quad \widehat{c}(t) = \widehat{c}e^{-i\omega t}\ , \tag{6.23}$$

where on the right side the same operators are taken in the Schrödinger representation.

6.4. Consider a similar problem for the Heisenberg operators $\widehat{Q}(t)$ and $\widehat{P}(t)$ of the harmonic oscillator and find their dependence on time.

6.5. Evaluate the expectation values in the nth stationary state of the arbitrary field mode for the following operators: (i) $\overline{c}$, $\overline{c^\dagger}$; (ii) $\overline{P}$, $\overline{Q}$; (iii) $\overline{c^2}$, $\overline{(c^\dagger)^2}$; (iv) $\overline{\Delta Q^2}\cdot\overline{\Delta P^2}$.

6.6. In the P representation (P is an independent variable) consider the problem of the energy spectrum and of a single-mode stationary state of the electromagnetic field.

6.7•. Evaluate the square variances of the uncertainties for variables Q and P in the nth state of the single mode and verify the Heisenberg uncertainty principle (C23).

6.8. Find the commutators $[\widehat{c}^n, \widehat{c}^\dagger]$ and $[\widehat{c}^{\dagger n}, \widehat{c}]$, where n is an arbitrary positive integer.

6.9. Prove the operator identities (C72). The derivative can be formally evaluated by assuming the operators as equivalent to the complex numbers. But the ordering form of the operators' product for noncommuting operators $\widehat{c}$ and $\widehat{c}^\dagger$ should be preserved.

6.10*. Prove that

$$[\widehat{c}^n, \widehat{f}] = \left(\left(\widehat{c} + \frac{\partial}{\partial \widehat{c}^\dagger}\right)^n - \widehat{c}^n\right)\widehat{f}\,,$$

$$[\widehat{f}, \widehat{c}^{\dagger n}] = \left(\left(\widehat{c}^\dagger + \frac{\partial}{\partial \widehat{c}}\right)^n - \widehat{c}^{\dagger n}\right)\widehat{f}\,,$$

where $\widehat{f}$ is an arbitrary operator function (see the previous problem). Consider the special example when $\widehat{f} = \widehat{c}^l$.

6.11*. Consider the field operators (6.13)–(6.15) in the Heisenberg picture and show that the equations of motion for $\widehat{\boldsymbol{E}}$ and $\widehat{\boldsymbol{H}}$ transform to the Maxwell equations.

Hint: Use the result obtained in Problem 6.4 and (C35).

6.12*. Evaluate the commutators for the operators of the field components in the Heisenberg picture:

$$\left[\widehat{E}_\alpha(\boldsymbol{r}_1, t_1), \widehat{E}_\beta(\boldsymbol{r}_2, t_2)\right], \quad \left[\widehat{H}_\alpha(\boldsymbol{r}_1, t_1), \widehat{H}_\beta(\boldsymbol{r}_2, t_2)\right], \quad \left[\widehat{E}_\alpha(\boldsymbol{r}_1, t_1), \widehat{H}_\beta(\boldsymbol{r}_2, t_2)\right].$$

On the basis of the relations obtained, draw a conclusion about ability for simultaneous observation of the electromagnetic field components in different spatial points.

Hint: For the polarization summation, use (2.165).

6.13$^\bullet$. Show that the Heisenberg operators $\widehat{E}_\alpha(\boldsymbol{r}_1, t_1)$ and $\widehat{A}_\beta(\boldsymbol{r}_2, t_2)$ do not commute if points 1 and 2 are separated by a spatial-type interval. Does this result compromise the causality principle?

6.14$^\bullet$. Consider the classical expression (4.130) for the electromagnetic field momentum and express the quantum operator of the field momentum $\widehat{\boldsymbol{P}}$ in terms of the photon creation and annihilation operators.

6.15. Evaluate the square variances of the fluctuations of the field amplitudes $\overline{\Delta E^2}$ and $\overline{\Delta H^2}$ in the vacuum state.

6.16*. Consider a quantum device which measures the vacuum fluctuations of the electromagnetic field. The measurement is bounded by the spatial scale and limited by the time interval. The device averages the detected signals over this scale and this interval and its sensitivity is described by the following filtering function:

$$g(\boldsymbol{r}, t) \geq 0\,, \quad \int g(\boldsymbol{r}, t) \mathrm{d}^3 r \mathrm{d}t = 1\,.$$

The integrand area expands on the infinite four-dimensional space. Evaluate the square variances of the vacuum fluctuations of the electric and magnetic components measured by such a device. As a weighting function use $g(\boldsymbol{r}, t) = Ce^{-r^2/2l^2}\delta(t)$ (C is a normalization constant, $l = \text{const}$).

6.17*. Two parallel noncharged metal plates of size $L \times L$ are placed in a vacuum and are separated by distance z. Make a quantitative estimate for the force attracting these plates by assuming that the force is caused by the vacuum fluctuations of the electromagnetic field existing in the area bounded by such a cavity (Casimir effect). The main contribution comes from the fluctuations with wavelength scales comparable with the separation distance z.

6.18$^{\bullet *}$. The vacuum fluctuations of the electromagnetic field disturb the electronic states of atoms. Estimate, in order of magnitude, the energy shift between the 2s and 2p states of hydrogen (Lamb shift) caused by these fluctuations.

6.19$^{\bullet *}$. Consider the total angular momentum $\boldsymbol{J}$ of the free electromagnetic field and decompose it into two parts consisting of (i) orbital momentum $\boldsymbol{L}$ referred to a spatial point $\boldsymbol{r}_0$ and (ii) spin momentum $\boldsymbol{S}$ which is independent on the origin of the coordinates. Express the operators of the orbital and spin momenta through the vector potential and electric field operators.

6.20*. Express the operator of the field spin momentum via the photon creation and annihilation operators. Which quantum states, having definite numbers of photons, make the diagonal representation of the spin operator? For these particular states, express the spin of the field via the photon numbers.

6.21$^{\bullet}$. Express the spectral intensity of the radiation emission $I_\sigma(\omega, \boldsymbol{n})$ parameterized by the frequency ω, polarization σ, and direction $\boldsymbol{n}$ via the number of quanta $N_{\boldsymbol{k}\sigma}$, where $\boldsymbol{k} = \omega \boldsymbol{n}/c$.

The thermal or statistically equilibrium states of the electromagnetic field can be realized when the radiation is in thermal balance with matter (e.g., if the radiation is contained in an impenetrable cavity having a certain temperature T). In this configuration, certain mean values $\overline{N}_s$ of the Fock photon numbers occur.

Example 6.3

Planck distribution

Find the mean number of photons in a mode of the electromagnetic field $s = (\boldsymbol{k}, \sigma)$ by using of Gibbs canonic statistical averaging.[4]

4) The equilibrium electromagnetic field spectrum (i.e., distribution of the radiation energy with frequency) was experimentally known by the end of the nineteenth century (its spectral shape is derived in Problem 6.23$^{\bullet}$). This spectrum could not be described on the basis of only the classical description of electrodynamics and statistical physics. To explain the observed properties of the equilibrium radiation, Max Planck (1900) introduced the concept of the quantum scale for mechanical action $\hbar$ and quantization of the electromagnetic field $\hbar\omega$. That event can be considered as the creation of the quantum theory and quantum electrodynamic field's energy $\hbar\omega$.

Solution. Each mode (oscillator) of the electromagnetic field can be considered as an independent thermodynamic system which interacts only with a heat bath (reservoir) having temperature T. This is because of the absence of direct interaction among photons. The emitting, absorbing, and scattering matter, which exists in the equilibrium state, plays the role of such a heat bath. The Gibbs distribution (C44) straightforwardly allows us to introduce the probability for N_s photons occupying a given mode:

$$w_{N_s} = \frac{1}{Z_s} \exp\left(-\frac{\mathcal{E}_s}{T}\right) , \quad \mathcal{E}_s = \hbar\omega \left(N_s + \frac{1}{2}\right) , \tag{1}$$

where $\mathcal{E}_s$ is the energy of the mode in accordance with (6.4). The partition statistical sum (see (C45)) is given by

$$Z_s = \sum_{N_s=0}^{\infty} \exp\left(-\frac{\mathcal{E}_s}{T}\right) = e^{-\alpha/2} \sum_{n=0}^{\infty} e^{-\alpha n} = \frac{e^{-\alpha/2}}{1 - e^{-\alpha}} , \quad \text{where}$$
$$\alpha = \frac{\hbar \omega_s}{T} . \tag{2}$$

Here, the temperature T is scaled by energy units. With use of (1), one can express the mean number of photons in a specific mode as the following derivative of the partition sum:

$$\overline{N}_s = \sum_{N_s=0}^{\infty} N_s w_{N_s} = (1 - e^{-\alpha}) \left(-\frac{\partial}{\partial \alpha}\right) \sum_{n=0}^{\infty} e^{-\alpha n} = \frac{1}{e^{\alpha} - 1} . \tag{3}$$

The Planck distribution of the mean photon number as a function of mode frequency ω_s is given by

$$\overline{N}_s = \frac{1}{\exp(\hbar \omega_s / T) - 1} \tag{6.24}$$

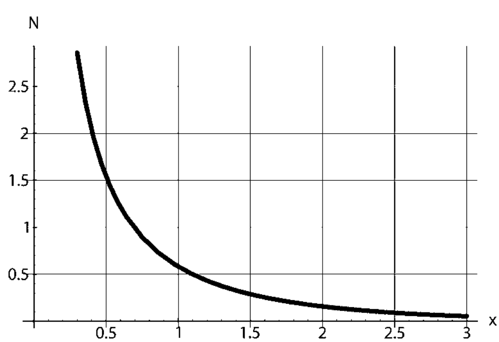

Figure 6.1 Planck distribution (average number of photons per mode).

(see Figure 6.1, where the horizontal axis corresponds to the dimensionless parameter $x = \hbar\omega/T$). For a fixed temperature, the number of quanta infinitely increases for low frequency ω, but the energy accumulated in any spectral interval $\Delta\omega$ remains finite (see Problem 6.22). The Planck distribution gives us a particular example of the Bose–Einstein distribution for an ideal quantum gas consisting of particles with integer spin. □

Suggested literature:
Landau and Lifshitz (1980); Rumer and Ryvkin (1977)

Problems

6.22. For the equilibrium state of a thermal electromagnetic field, calculate the mean energy of a single-mode oscillator.

6.23•. Let the light source existing in the thermal equilibrium state emit radiation. Using the Planck distribution, calculate the spectral distribution of the radiation intensity and energy density and analyze the result. Explain why the classical description of the spectral density of radiation fails at high frequencies ("ultraviolet catastrophe"phenomenon).

6.24. Show that the intensity of the "blackbody" radiation emitted from a unit surface area (the thermal light source) is described by the *Stefan–Boltzmann law:*[5)]

$$I = \int I(\omega)\cos\theta\,\mathrm{d}\Omega\,\mathrm{d}\omega = \sigma T^4\,, \quad \text{where}$$

$$\sigma = \frac{\pi^2 k_B^4}{60\hbar^3 c^2} \approx 5.67\times 10^{-5}\ \mathrm{g/(s^3K^4)} \tag{6.25}$$

is so-called the Stefan–Boltzmann constant. Temperature T is measured on the Kelvin scale. Angle θ is counted from a normal to the surface and integration over a solid angle is expanded on the hemisphere. The total radiation energy distributed in volume $\mathcal{V}$ (internal energy) is given by

$$\mathcal{E} = \frac{4\sigma}{c}T^4\mathcal{V}\,. \tag{6.26}$$

Hint: Use the tabulated integral

$$\int_0^\infty \frac{x^3\mathrm{d}x}{e^x-1} = \frac{\pi^4}{15}\,.$$

6.25. For the thermal equilibrium radiation, find the density of the energy distribution $\rho(\lambda)$ on the wavelength scale. On the basis of the spectral distribution obtained, formulate Wien's[6)] displacement law which connects temperature with the wavelength corresponding to the maximum radiation.

5) Joseph Stefan (1835–1893) was an Austrian physicist who had a wide range of scientific interests. Ludwig Boltzmann (1844–1906) was an outstanding Austrian physicist. His main scientific works were in the fields of statistical physics and kinetics.

6) Wilhelm Wien (1864–1928) was a German physicist and a Nobel Prize recipient.

6.26. Reconsider the Planck distribution obtained in Problem 6.23• as a function of the dimensionless ratios λ/λ_m and λ/λ'_m, where λ_m and λ'_m are the values of the wavelengths associated with the maximum of the spectral distribution as a function of either the frequency or the wavelength. Estimate λ_m and λ'_m for the following cosmic objects: (i) relict radiation ($T = 3$ K); (ii) the Earth's surface ($T = 300$ K); (iii) Mu Cephei (also known as Herschel's Garnet Star) ($T = 2000$ K); (iv) the Sun's surface ($T = 6000$ K); (v) Sirius's surface ($T = 11000$ K); (vi) Beta Centauri ($T = 22\,500$ K); (vii) the central star of the planetary nebula in Lyra ($T = 75\,000$ K); (viii) the surface of a neutron star ($T = 250\,000$ K); (ix) the interior areas of common stars such as the Sun ($T = 25 \times 10^6$ K).

6.27. Calculate the Helmholtz free energy F, the radiation pressure $\mathcal{P}$, the entropy S, and the thermal capacity C_V for equilibrium radiation with temperature T emitted in volume $\mathcal{V}$.

Hint: Use the thermodynamic relations given in Appendix C ((C46) and (C46′)).

6.1.4
Coherent States

In the general theoretical description of the electromagnetic field, it is important to define a convenient representation for the field states. For the small number of photons when the methods of perturbation theory are applicable, the Fock photon number representation seems quite relevant. It is parameterized by the definite number of photons N_s per mode with fully uncertain phase of the basis state. If the radiation consists of many photons per coherence volume and generally demonstrates classical behavior such that the effects of coherency become crucially important, it is better to use another representation. This is known as the basis of coherent states or the Glauber[7] representation (see Glauber, 1965, 1969, but such states were considered for the first time by Schrödinger in 1926). In this representation the description of light emitted by intense laser beams can be represented semiclassically and this simplifies further calculations. In addition to describing of the electromagnetic field, coherent states are a convenient tool for studying other quantum systems (see, e.g., Malkin and Man'ko, 1979).

The coherent state $|z_s\rangle$ for a particular mode s is defined as the eigenstate of the non-Hermitian annihilation operator $\widehat{c}_s$:

$$\widehat{c}|z_s\rangle = z_s|z_s\rangle \,. \tag{6.27}$$

The eigenvalue z_s of such a non-Hermitian operator can be an arbitrary complex number. Below we point out some important properties of these states for our consideration.

7) The research of Glauber on coherent states was recognized by the award of the Nobel Prize.

Example 6.4

Find the eigenstates of the annihilation operator (i.e., coherent states) as an expansion in the Fock basis set. Normalize the expansion obtained to unit norm. What is the probability of observing n photons in the coherent state?

Solution. Let us introduce the scalar product of both parts of (6.27) on $\langle n|$ (the mode index s is disregarded):

$$\langle n|\widehat{a}|z\rangle = \langle z|\widehat{a}^\dagger|n\rangle^* = \sqrt{n+1}\langle n+1|z\rangle = z\langle n|z\rangle \ .$$

The last equality lets us build the recurrence relation and link $\langle n|z\rangle$ with $\langle 0|z\rangle$:

$$\langle n|z\rangle = \frac{z^n}{\sqrt{n!}}\langle 0|z\rangle \ . \tag{6.28}$$

Applying the normalization condition $\langle z|z\rangle = 1$ and the above equality, we obtain

$$\langle z|z\rangle = \sum_{n=0}^{\infty}\langle z|n\rangle\langle n|z\rangle = |\langle 0|z\rangle|^2 \sum_{n=0}^{\infty}\frac{(z^* z)^n}{n!} = |\langle 0|z\rangle|^2 e^{|z|^2} = 1 \ .$$

Choosing the arbitrary phase factor as $+1$ leads to $\langle 0|z\rangle = \exp(-|z|^2/2)$, and the normalized eigenvector of coherent state is given by

$$|z\rangle = \exp(-|z|^2/2)\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}|n\rangle \ . \tag{6.29}$$

This expression gives us the probability of detecting n photons via measurement of the coherent field intensity. In accordance with the superposition principle, the probability is given by the squared modulus of the expansion coefficient in the set of the Fock states:

$$w(n) = |\langle n|z\rangle|^2 = \frac{|z|^{2n}}{n!}\exp(-|z|^2) \ . \tag{6.30}$$

This gives us a *Poisson distribution*, where the mean value of the number of photons per mode is given by

$$\langle n\rangle = |z|^2 \ . \tag{6.31}$$

□

Example 6.5

Expand the Fock state $|n\rangle$ in the set of coherent states $|z\rangle$. The parameter z is a complex continuous value.

Solution. Let us multiply both parts of (6.29) by $z^{*m}\exp(-|z|^2/2)/(\pi\sqrt{m!})$ and integrate the product over the variable z in infinite limits. The elementary integrand cell can be expressed in polar coordinates as $\mathrm{d}^2z = \mathrm{d}z'\mathrm{d}z'' = r\mathrm{d}r\mathrm{d}\varphi$, where z' and z'' are the real and imaginary parts of the complex number z, and r and φ are its modulus and phase, respectively. After integration, the right side of (6.29) transforms to

$$\sum_{n=0}^{\infty}\frac{1}{\pi\sqrt{m!n!}}|n\rangle\int_0^{\infty} r\mathrm{d}r e^{-r^2}r^{m+n}\int_0^{2\pi}\mathrm{d}\varphi e^{i(n-m)\varphi} = |m\rangle\ .$$

This implies

$$|n\rangle = \frac{1}{\pi}\int\exp\left(-\frac{1}{2}|z|^2\right)\frac{z^{*n}}{\sqrt{n!}}|z\rangle\mathrm{d}^2z\ . \tag{6.32}$$

□

Problems

6.28. Verify that the coherent states are not orthogonal at $z \neq z'$ and prove the following relations:

$$\langle z|z'\rangle = e^{z^*z'-|z|^2/2-|z'|^2/2}, \tag{6.33}$$

$$|\langle z|z'\rangle|^2 = e^{-|z-z'|^2}\ . \tag{6.34}$$

Verify that the coherent states are a complete but overfull basis set such that

$$\sum_z|z\rangle\langle z| \equiv \int|z\rangle\langle z|\mathrm{d}^2z = \pi\sum_n|n\rangle\langle n| = \pi\cdot\widehat{1}\ , \tag{6.35}$$

with the extra factor π on the right side.

6.29. Prove that the creation operator $\widehat{c}^\dagger$ has no eigenfunctions $|\beta\rangle$ for any finite complex eigenvalues β.

6.30. Analyze the time dynamics of the coherent state. Prove that for the coherent state evolving in time, its complex amplitude rotates in the complex plane with the mode frequency.

6.31. Find the wave function of a coherent state in the coordinate representation. Prove that it is a Gaussian wave packet, which oscillates periodically and reproduces the motion of a classical particle in a parabolic (harmonic oscillator) potential.

6.32. Prove that the coherent state can be generated by the action of the so -called displacement operator

$$\widehat{D}(\alpha) = \exp(\alpha\widehat{c}^\dagger - \alpha^*\widehat{c}) \tag{6.36}$$

from the vacuum state $|vac\rangle \equiv |0\rangle$ (the ground state of the mode oscillator).

6.33. Prove that the coherent state is described by the minimum uncertainty with regard to the coordinate and momentum of the field oscillator, that is, $\langle \Delta Q^2 \rangle \langle \Delta P^2 \rangle = \hbar^2/4$. Applying this to the dimensionless quadrature components given by[8]

$$\widehat{X}_1 = \sqrt{\frac{\omega}{\hbar}}\widehat{Q} \equiv \xi \quad \widehat{X}_2 = \frac{1}{\sqrt{\hbar\omega}}\widehat{P} = -\frac{i\partial}{\partial\xi} \tag{6.37}$$

leads to $\langle \Delta X_1^2 \rangle = \langle \Delta X_2^2 \rangle = 1/2$.

6.34. Calculate the square variances of the coordinate and momentum for free dynamics of the field oscillator which starts from an arbitrary initial state. Prove that the variances $D_1(t)$ and $D_2(t)$ of the dimensionless quadrature components (defined in the previous problem) generally oscillate in antiphase. The oscillation frequency is equal to the doubled mode frequency and the sum of variances is an integral of motion. Prove that for the Fock and coherent states the oscillation amplitude equals zero.

Hint: Consider the operators of the coordinate and momentum in the Heisenberg representation (see Problem 6.4).

6.35. Calculate the variance of the photon number for the single-mode coherent state of the electromagnetic field.

6.1.5
Representation of the Quantum States and the Operators in the Basis of Coherent States

As shown above, the coherent states are a complete but overfull basis set, which can be used to expand the arbitrary state vectors and operators:

$$|f\rangle = \frac{1}{\pi}\int \langle\alpha|f\rangle|\alpha\rangle \mathrm{d}^2\alpha\,, \tag{6.38}$$

$$\widehat{O} = \frac{1}{\pi^2}\int\int \mathrm{d}^2\alpha \mathrm{d}^2\beta \langle\alpha|\widehat{O}|\beta\rangle|\alpha\rangle\langle\beta|\,. \tag{6.39}$$

A similar expansion can be introduced for the density operator as well, but in this case it is more convenient to use the Glauber–Sudarschan P representation

$$\widehat{\rho} = \int \mathrm{d}^2\alpha \mathcal{P}(\alpha)|\alpha\rangle\langle\alpha|\,, \tag{6.40}$$

which has diagonal form. The normalization condition $\mathrm{Tr}(\widehat{\rho}) = 1$ and the Hermitian nature of the density operator justify the following criteria for the $\mathcal{P}$ function:

$$\int \mathrm{d}^2\alpha \mathcal{P}(\alpha) = 1\,, \quad \mathcal{P}(\alpha)^* = \mathcal{P}(\alpha)\,, \tag{6.41}$$

8) The phase difference between quadrature components is $\pi/2$.

that is, $\mathcal{P}$ is a real and normalized function. However, $\mathcal{P}(\alpha)$ does not have the precise meaning of the probability density. This function can be negative and singular and can even have stronger singular features than the Dirac delta function. Therefore, $\mathcal{P}$ is normally referred as quasi-probability. The intrinsic properties of this function are similar but not identical to those of the density operator defined in the Wigner[9] representation (see Landau and Lifshitz, 1980). The Glauber–Sudarschan $\mathcal{P}$ function allows us to calculate the expectation value of any normally ordered operators' product (all creation operators for each mode are placed before annihilation operators), which is simplified to evaluation of the integral. For example,

$$\mathrm{Tr}(\widehat{\rho}\widehat{a}^{+n}\widehat{a}^{m}) = \int \mathrm{d}^2\alpha \mathcal{P}(\alpha)\alpha^{*n}\alpha^{m} .$$

Here, we used the identities $\widehat{a}^n|\alpha\rangle = \alpha^n|\alpha\rangle$ and $\langle\alpha|\widehat{a}^{\dagger n} = \langle\alpha|\alpha^{*n}$.

The quasi-probability $\mathcal{P}(\alpha)$ is not an experimentally measurable quantity. The real and imaginary parts of its complex argument α are associated with the values of the noncommuting canonical variables $(\widehat{a} + \widehat{a}^\dagger)/2$, $(\widehat{a} - \widehat{a}^\dagger)/2i$. These variables are not simultaneously observable and their distribution on the phase plane cannot be expressed by a probabilistic measure. However, for many physical situations the Glauber–Sudarschan quasi-probability can be known and expansion (6.40) can be used.

Example 6.6

Using the overlap integral for coherent states (6.33), prove that the relation that is the inverse of (6.40) can be written in the following form:

$$\mathcal{P}(z)e^{-|z|^2} = \frac{1}{\pi^2}\int e^{|u|^2}\langle -u|\widehat{\rho}|u\rangle e^{u^*z-uz^*}\mathrm{d}^2u . \tag{6.42}$$

This allows us to build the Glauber–Sudarschan quasi-probability for any given density operator $\widehat{\rho}$.

Solution. Let us project the operator equality (6.40) from the left on $\langle -u|$ and that from the right by $|u\rangle$:

$$\langle -u|\widehat{\rho}|u\rangle = \int \mathcal{P}(z)\langle -u|z\rangle\langle z|u\rangle \mathrm{d}^2z . \tag{1}$$

The matrix elements in the integrand can be rearranged with the help of relation (6.33):

$$\langle -u|\widehat{\rho}|u\rangle = e^{-|u|^2}\int \mathcal{P}(z)e^{-|z|^2}e^{uz^*-u^*z}\mathrm{d}^2z . \tag{2}$$

9) Wigner, Eugene Paul (1902–1985) – outstanding American physicist–theorist, Nobel Prize winner.

The expression obtained can be considered as a two-dimensional Fourier transform. Indeed, the complex variables u and z can be substituted with their real components: $u = p + iq$, $z = P + iQ$, $uz^* - u^*z = 2i(qP - pQ)$, $\mathrm{d}^2z = \mathrm{d}P\mathrm{d}Q$. Using the inverse Fourier transform, we straightforwardly derive expression (6.42). In some cases the integral (6.42) does not converge to a regular function. Then it should be treated as a singular generalized function on argument z (e.g., it can be the nth derivative of the Dirac delta function). □

Instead of singular quasi-probability, we can consider its two-dimensional Fourier transform:

$$\theta[\kappa, \kappa^*] = \mathrm{Tr}(\widehat{\rho} \exp(\kappa \widehat{a}^\dagger) \exp(-\kappa^* \widehat{a})) = \int \mathrm{d}^2\alpha \mathcal{P}(\alpha) \exp(\kappa\alpha^* - \kappa^*\alpha) . \tag{6.43}$$

The function obtained is a regular function of two complex parameters κ and κ^* and is known as a normally ordered characteristic function.

Problems

6.36. Find the Glauber–Sudarschan quasi-probability $\mathcal{P}(\alpha)$ in the case when the density operator can be written in a polynomial form: $\widehat{\rho} = \sum_{n,m} C_{n,m} \widehat{c}^n \widehat{c}^{\dagger m}$.

6.37•. Using the approach presented in Example 6.6, find the $\mathcal{P}$ function for a particular mode of the thermal equilibrium state of the electromagnetic field. Show that in this case the quasi-probability is a Gaussian distribution, whose half-width is given by the mean value of the photon number $\overline{n}$ in the mode and which has zero mean amplitude z:

$$\mathcal{P}(z) = \frac{1}{\pi\overline{n}} e^{-|z|^2/\overline{n}} . \tag{6.44}$$

6.38*. Derive the normally ordered characteristic function (6.43) and density matrix in the Fock representation for the state with quasi-probability $\mathcal{P}(\alpha) = \delta(|\alpha| - |\alpha_0|)/2\pi|\alpha|$ (This state, modeling ideal laser radiation, has a fixed amplitude and a fully undefined and homogeneously distributed phase).

6.39. Derive the normally ordered characteristic function and the quasi-probability $\mathcal{P}(\alpha)$ for the state formed from the superposition of Gaussian (thermal) and coherent $|\alpha_0\rangle$ states.

6.40. Prove that the transition probability for an m-quantum excitation process in a thermal single mode field is $m!$ times more effective than that in the coherent single-mode field with the same mean number of photons.

Hint: The probability of an m-photon excitation process is proportional to the normally ordered correlation function $G^{m,m} = \langle \widehat{a}^{\dagger m} \widehat{a}^m \rangle$.

6.1.6 Squeezed States

The coherent state of light is unique in its minimal quantum uncertainty with regard to the coordinate and momentum of the field oscillator (see the uncertainty relation (C32)). The variances of both the coordinate and the momentum are conserved in the time dynamics of the field and are given by $\langle \Delta Q^2 \rangle = \hbar/2\omega$ and $\langle \Delta P^2 \rangle = \hbar\omega/2$, respectively (see Problem 6.34). Let us turn to another example of so-called *squeezed states* of light. For these states the variance of one of the field quadratures (either Q or P) is essentially reduced below the standard quantum limit estimated above, but the variance of the other quadrature is enhanced. The product of quadrature variance has the last value $\hbar^2/4$ (in accordance with the Heisenberg uncertainty principle). The term "the squeezed states" designates on the oscillator phase plane such transformation which corresponds to a reduction on one axis and a stretching on another (see below the formulas and Figure 6.2).

Squeezed states (we designate them as $|\mu, \nu, \beta\rangle$, see Problem 6.41) can be rigorously introduced as the eigenstates of the following operator:

$$\widehat{b} = \mu \widehat{c} + \nu \widehat{c}^{\dagger} \; . \tag{6.45}$$

The complex parameters μ and ν satisfy the relation $|\mu|^2 - |\nu|^2 = 1$. This preserves the commutation relation for the transformed operators $[\widehat{b}, \widehat{b}^{\dagger}] = 1$.

The fluctuations of the quadrature components are unequal now, and it is possible to reducce of one of them by increasing other. For an estimation of their variance we will use the operators $\widehat{X}_1$ and $\widehat{X}_2$ given by equalities (6.37). Within a factor, the quadrature components coincide with the coordinate and momentum operators of the oscillator:

$$\widehat{X}_1 = (\widehat{a} + \widehat{a}^{\dagger})/2 \, , \quad \widehat{X}_2 = (\widehat{a} - \widehat{a}^{\dagger})/2i \; . \tag{6.46}$$

As a consequence of the commutation rule $[\widehat{X}_1, \widehat{X}_2] = i/2$, the quadrature components satisfy the uncertainty relation

$$\left\langle \Delta \widehat{X}_1^2 \right\rangle \left\langle \Delta \widehat{X}_2^2 \right\rangle \geq \frac{1}{16} \, , \tag{6.47}$$

and their variances are given by

$$\left\langle \Delta \widehat{X}_1^2 \right\rangle \equiv \langle \mu, \nu, \beta | \Delta \widehat{X}_1^2 | \mu, \nu, \beta \rangle = \frac{|\mu - \nu|}{4} \, , \tag{6.48}$$

$$\left\langle \Delta \widehat{X}_2^2 \right\rangle \equiv \langle \mu, \nu, \beta | \Delta \widehat{X}_1^2 | \mu, \nu, \beta \rangle = \frac{|\mu + \nu|}{4} \; . \tag{6.49}$$

As we see, the quantum noise of the first quadrature component can be suppressed if the condition $|\mu - \nu| < 1$ is fulfilled, but the second component becomes noisier in this case because of the basic requirement that $|\mu|^2 - |\nu|^2 = 1$. If we relevantly change the phases for the complex numbers μ and ν, we can reduce the noise for the second quadrature component. Squeezed states do not have a classical analogy and for these states the quasi-probability $\mathcal{P}(\alpha)$ can be negative.

Example 6.7

Prove the latter statement: in the case of a squeezed state for some values of the complex amplitude α the quasi-probability $\mathcal{P}(\alpha)$ becomes negative.

Solution. Let us transform the operators' product to the normally ordered form in the expression for the expectation value of the fluctuation $\langle \mu, \nu, \beta | \Delta \widehat{X}_1^2 | \mu, \nu, \beta \rangle$:

$$\langle \mu, \nu, \beta | \Delta \widehat{X}_{1(2)}^2 | \mu, \nu, \beta \rangle = \langle \mu, \nu, \beta | \widehat{\mathcal{N}} (\Delta \widehat{X}_{1(2)})^2 | \mu, \nu, \beta \rangle + \frac{1}{4} ,$$

$$\langle \mu, \nu, \beta | \widehat{\mathcal{N}} (\Delta \widehat{X}_{1(2)})^2 | \mu, \nu, \beta \rangle = \frac{\pm(\langle \widehat{a}^2 \rangle - \langle \widehat{a} \rangle^2) + \langle \widehat{a}^\dagger \widehat{a} \rangle - \langle \widehat{a}^\dagger \rangle \langle \widehat{a} \rangle + c.c.}{4} ,$$

where $\widehat{\mathcal{N}}$ denotes the normally ordering operator and $c.c.$ means the complex conjugate term.

For the coherent state, the expectation values of the normally ordered quadrature fluctuations are equal to zero. But for the squeezed states, one of them becomes negative, which leads to the following inequality:

$$\langle \mu, \nu, \beta | \widehat{\mathcal{N}} (\Delta \widehat{X}_i)^2 | \mu, \nu, \beta \rangle = \int \mathcal{P}(\alpha)(\alpha + \alpha^* - \langle \alpha \rangle - \langle \alpha^* \rangle)^2 \mathrm{d}^2\alpha < 0 .$$

This inequality can be fulfilled if the quasi-probability $\mathcal{P}(\alpha)$ has a negative value in some area of the complex α plane. □

Let us consider how the behavior of squeezing depends on the phase relation. The complex parameters μ and ν can be written as

$$\mu = \coth r ; \quad \nu = \sinh r \exp(2i\vartheta) , \tag{6.50}$$

and the squeezed state can be generally written in the following form: $|\gamma\beta\rangle \equiv |\cosh r, \sinh r e^{2i\vartheta}, \beta\rangle$, where $\gamma = r e^{2i\vartheta}$ expresses the level of squeezing. For such parameterizations the condition $|\mu|^2 - |\nu|^2 = 1$ is surely satisfied. The variance of quadrature component X_1 will be suppressed if $r \to \infty$ when $\exp(i\vartheta) = 1$ and that of quadrature component X_2 will be suppressed when $\exp(i\vartheta) = -1$. If the phase ϑ does not satisfy these special values, we can defined the following general quadrature components:

$$Y_\vartheta = \frac{\widehat{a} e^{-i\vartheta} + \widehat{a}^\dagger e^{i\vartheta}}{2} , \quad Y_{\vartheta+\pi/2} = \frac{\widehat{a} e^{-i\vartheta} - \widehat{a}^\dagger e^{i\vartheta}}{2} . \tag{6.51}$$

Their variances obey the following relations:

$$\left\langle \Delta \widehat{Y}_\vartheta^2 \right\rangle = \frac{\exp(-2r)}{4} , \quad \left\langle \Delta \widehat{Y}_{\vartheta+\pi/2}^2 \right\rangle = \frac{\exp(2r)}{4} . \tag{6.52}$$

The absolute value of the squeezing parameter γ determines the noise reduction for one of the quadrature components (and noise increases for the other). The phase of γ shows the directions along which the squeezing is observed (see Figure 6.2).

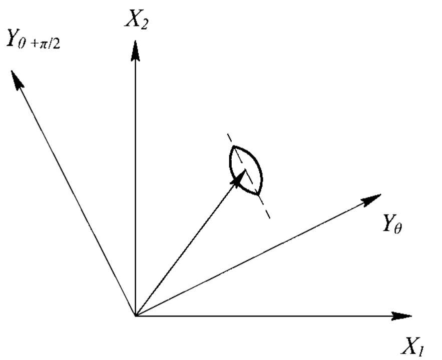

Figure 6.2 The evident image of the squeezed state.

Figure 6.2 shows the area in the complex α plane where the density distribution $\langle\alpha|\widehat{\rho}|\alpha\rangle$ is essentially different from zero and indicates the uncertainty domain for complex amplitude. The domain size does not depend on the displacement of the field amplitude and so the squeezing phenomenon can be observed for any amplitude. In the case of a coherent state, the uncertainty area reduces to a circle-type domain with the origin at point α whose distribution is independent on α. The coherent state can be also considered as a partial case of an ideal squeezed state when $\nu = 1$ and $\mu = 0$.

With referring to Figure 6.2, we can illustrate qualitatively the noise reduction when the squeezed light is detected. It is clear (and can be confirmed by a rigorous theoretical treatment) that in experiments the main role for the measurement precision comes from the fluctuations of the absolute value of the complex field amplitude. Therefore, the fluctuations will be suppressed if the minor axis of the squeezing ellipse is orientated in a radial direction. However, for phase-sensitive measurements (e.g., in an interferometer scheme) it is more preferable that the ellipse is antisqueezed in a radial direction.

Noise reduction in measurements is very important, and in this sense squeezed states have great potential in practice. The realization of dependable sources of such light has some technical difficulties but essential experimental efforts have been applied in this direction and there are many laboratory demonstrations of the squeezing effect.

Problems

6.41. Show that the squeezed states can be obtained as a result of the consecutive action of displacement and squeezing operators on the vacuum state (see (6.36)):

$$|\mu, \nu, \beta\rangle = |r, \vartheta, \beta\rangle = |\gamma, \beta\rangle = \widehat{S}\,\widehat{D}|\mathrm{vac}\rangle \,,$$

where $\widehat{D} = \widehat{D}(\beta) = \exp(\beta\widehat{a}^\dagger - \beta^*\widehat{a})$ is the displacement operator defined in Problem 6.32 and $\widehat{S} = \widehat{S}(\gamma) = \exp(\gamma^*\widehat{a}^2/2 - \gamma\widehat{a}^{\dagger 2}/2)$ is the squeezing operator; $\gamma = re^{2i\vartheta}$ (see (6.50)). Find the connection between the complex parameters μ and ν, which parameterize the squeezed state in the general case, and the pair of real parameters r and ϑ and the complex parameter γ.[10]

6.42. Expand the squeezed states in the basis of coherent states. Show that in the case when $|\mu|^2 - |\nu|^2 = 1$, the expansion coefficients can be expressed as follows:

$$\langle\alpha|\mu, \nu, \beta\rangle = \frac{1}{\sqrt{\mu}} \exp\left(-\frac{|\alpha|^2}{2} - \frac{|\beta|^2}{2} - \frac{\nu\alpha^{*2}}{2\mu} + \frac{\nu^*\beta^2}{2\mu} + \frac{\alpha^*\beta}{\mu}\right) .$$

6.43. Expand the squeezed states in the basis of the Fock photon number states. Show that the expansion coefficients can be expressed as follows:

$$\langle n|\mu, \nu, \beta\rangle = \frac{1}{\sqrt{n!\mu}} \left(\frac{\nu}{2\mu}\right)^{n/2} \exp\left(-\frac{|\beta|^2}{2} - \frac{\nu^*\beta^2}{2\mu}\right) H_n\left(\frac{\beta}{\sqrt{2\mu\nu}}\right) ,$$

where $H_n(x)$ is a Hermitian polynomial.

6.44. Prove that for an ideal single-mode squeezed state the normally ordered characteristic function (6.43) can be expressed as

$$\theta[\kappa, \kappa^*] = \exp\left(\kappa\alpha^* - \kappa^*\alpha - \kappa\kappa^* \sinh^2 r \right.$$
$$\left. - \frac{\kappa^2}{4} e^{-2i\vartheta} \sinh 2r - \frac{\kappa^{*2}}{4} e^{2i\vartheta} \sinh 2r\right) ,$$

where the squeezed state is described by the following parameters: $\mu = \cosh r$, $\nu = e^{2i\vartheta}\sinh r$, and $\gamma = re^{2i\vartheta}$, and the amplitude β is expressed by amplitude α as $\beta = \mu\alpha + \nu\alpha^*$.

6.45. Show that for thermal radiation the probability densities of the coordinate and momentum quadratures have a Gaussian distribution with zero mean value and with the following variances:

$$\left\langle \Delta\widehat{X}_1^2 \right\rangle = \left\langle \Delta\widehat{X}_2^2 \right\rangle = \frac{1}{2}\coth\left(\frac{\hbar\omega}{2T}\right) .$$

10) Originally such states were considered by Yuen in 1976, who called them as "two-photon coherent states." Similar states $\widehat{D}\widehat{S}|\mathrm{vac}\rangle$ were considered by Caves in (1981), and formally they are not identical to "two-photon coherent states" because of noncommutativity of the operators $\widehat{D}$ and $\widehat{S}$. The latter states are referred as ideal squeezed states. The statistical properties of the states of both classes (visualized by the uncertainty of quadrature components) are identical. In the literature, these classes are equally considered and are normally referred to as "squeezed states."

6.46. Prove that the electromagnetic field in the coherent state can be considered as being emitted by a classical nonfluctuating current.

Hint: Compare the evolution operator transforming the vacuum state with the classical current and displacement operator (6.36) producing the coherent state.

Suggested literature:
Glauber (1965, 1969); Mandel and Wolf (1995); Klyshko (1988, 2011); Bykov (1993, 2006); Klyshko (1994, 1996)

6.1.7 Entangled States

In 1935, Einstein, Podolsky, and Rosen published their famous work (Einstein *et al.*, 1935) concerning the problem of nonlocality in quantum mechanics. Applying the principle of causality and making the assumption that certain hidden variables exist, Bell in an entire physical description formulated some propositions. They are represented by a set of inequalities for the correlation of the measurement events between the points separated by a spatial-type interval (see, e.g, Belinskii and Klyshko (1993)). Violation of these inequalities indicates the inconsistency of the hidden variable theory and provides evidence of nonlocality in quantum theory. A possible manifestation of such a violation is the experimental observation of the correlation coincidence in photodetection of photons prepared in an entangled state.[11)]

Let us consider the following example of the polarization-entangled state of light:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|x\rangle_1|y\rangle_2 - |y\rangle_1|x\rangle_2) \,. \tag{1}$$

This state is described by the wave function for the system of two photons propagating along the z axis. The basic states $|x\rangle$ and $|y\rangle$ correspond to the linearly polarized photon along the x and y directions, respectively, and the subscript indices 1 and 2 numerate the photons. Such states for the field can be generated in parametric processes in nonlinear crystals.

Both photons are assumed to be detected by photodetectors after they have passed the polarization analyzers. Let the optical axes of the analyzers be at angle φ_1 to the x axis and at angle φ_2 to the y axis for the first and second photodetectors, respectively. For the 100% effective detection scheme, the probability of a photon passing through the analyzer is given by

$$W_i = \langle\psi|\hat{c}_i^\dagger \hat{c}_i|\psi\rangle \,, \tag{2}$$

11) Entangled states are the states of a microscopic system when the different but strongly correlated magnitudes of the quantum variable related to different objects can be observed. In the example given below, each photon can be detected in state $\sigma = 1$ (polarization along the x axis) or in state $\sigma = 2$ (polarization along the y axis). For more information about entangled states, see Bargatin *et al.* (2001).

where $\widehat{c}_i^\dagger$ and $\widehat{c}_i$ are the creation and annihilation operators of the photon at the photodetector. These operators can be expressed by the operators related to the detection of the x and y polarization modes (Mandel and Wolf, 1995):

$$\widehat{c}_i = \widehat{c}_{ix} \cos\varphi_i + \widehat{c}_{iy} \sin\varphi_i \ , \quad i = 1, 2 \ , \tag{3}$$

where $\widehat{c}_{ix}$ and $\widehat{c}_{iy}$ are the annihilation operators for the linearly polarized photons along directions x and y, respectively.

By substituting (1) and (3) into (2), we obtain the detection probability for both detectors $W_i = 1/2$, which is independent of the orientation of the analyzers. This means that the light incident on each photodetector is unpolarized. At the same time, there is a strong mutual correlation in the photocounts. The quantity W_i gives only the unconditional probability for the photodetection events. But the correlations are described by the probability that both detectors make a click coincidentally and detect each photon in a certain polarization mode:

$$\begin{aligned} W_{12} &= \langle\psi|\widehat{c}_1^\dagger\widehat{c}_2^\dagger\widehat{c}_1\widehat{c}_2|\psi\rangle \\ &= \frac{1}{2}\left(\sin^2\varphi_1\cos^2\varphi_2 + \cos^2\varphi_1\sin^2\varphi_2 - 2\sin\varphi_1\cos\varphi_2\sin\varphi_2\cos\varphi_1\right) \\ &= \frac{1}{2}\sin^2(\varphi_1 - \varphi_2) \ . \end{aligned} \tag{4}$$

This coincidence probability depends on the mutual orientation of the analyzers, which depends on the difference $\varphi_1 - \varphi_2$. For $\varphi_1 - \varphi_2 = \pi/2$, the above expression leads to $W_{12} = 1/2$, but if the analyzers are parallel such that $\varphi_1 - \varphi_2 = 0$, then the coincidence probability becomes zero, $W_{12} = 0$. This is a direct consequence of strong mutual correlation between the photons' polarization in the entangled state described above.

After removal of the factor 1/2, (4) gives us the conditional probability of the photodetection events:

$$W_{\text{cond}}(2|1) = \frac{W_{12}}{W_1} = \sin^2(\varphi_1 - \varphi_2) \ . \tag{5}$$

We surprisingly see that the result of one measurement depends on the result of other; the measurements are separated by a distant spatial-type interval such that there is no causal connection between the devices involved in the measurements. It is also intriguing that such nonlocality does not contradict the normal causality principle as far as the observer cannot affect the result of the other measurement. Indeed, W_{12} is the probability of detecting both photons with two clicks. As directly follows from definition (2), we cannot affect the detection probability of any *given* photodetector, which is independent of the analyzer orientation.

There have been many demonstrations of experiments of such a type and in all of them the quantum nonlocality as well as violation of Bell's inequalities were surly confirmed. We considered here the simplest example of the polarization-entangled two-photon state. Clearly, that there are many states of such a type. Instead of the basic functions $|x\rangle$ and $|y\rangle$ in (1), we could use $|\phi\rangle$ and $|v\rangle$, which specify photons with different spectral modes, and the correlations will be observed by means of the time-dependent coincidence spectroscopy technique.

6.1.8 Beamsplitters

A beamsplitter is an example of a system which performs an unitary operation and transforms two input states into two output states. This operation can be represented as a coherent scattering process and is conveniently described by the $\widehat{S}$-matrix formalism. In its standard definition, the scattering matrix $\widehat{S} = \widehat{S}(-\infty, \infty)$ operates with the interaction representation and transforms an initial state $|\Psi_{\text{in}}\rangle$ to a final state $|\Psi_{\text{out}}\rangle$, which are associated with the states of the system in the "infinite past" and in the "infinite future," respectively:

$$|\Psi_{\text{out}}\rangle = \widehat{S}|\Psi_{\text{in}}\rangle \,. \tag{1}$$

We extend this transformation by a generally defined time parameter that approaches infinity:

$$\left|\Psi_{\text{out}}\left(\frac{T}{2}\right)\right\rangle = \widehat{S}\left|\Psi_{\text{in}}\left(-\frac{T}{2}\right)\right\rangle \,. \tag{2}$$

The scattering matrix in the interaction representation is then given by

$$\widehat{S} = \widehat{S}(\infty, -\infty) = \lim_{T\to\infty} e^{\frac{i}{\hbar}H_0\frac{T}{2}} e^{-\frac{i}{\hbar}HT} e^{-\frac{i}{\hbar}H_0\frac{T}{2}} \,, \tag{3}$$

where H_0 is a free part of the total Hamiltonian H of the scattering system, which is beamsplitter in our case. In the final relations $T \to \infty$, but it is convenient to keep the time parameter T and consider the states at a certain moment of time.

Let us consider a single photon as input, which arrives at the beamsplitter at either port 1 or port 2 (see Figure 6.3a). The input state can be represented as an action of the creation operator on the vacuum state:

$$|\Psi_{\text{in}}\rangle = \left|\Psi_{\text{in}}\left(-\frac{T}{2}\right)\right\rangle = \widehat{c}_j^{\dagger}|\text{vac}\rangle \,, \tag{4}$$

where index $j = 1, 2$ numerates the beamsplitter's input ports. Then the wave function of the output state can be expressed as follows:

$$|\Psi_{\text{out}}\rangle = \widehat{S}\widehat{c}_j^{\dagger}\widehat{S}^{-1}\widehat{S}|\text{vac}\rangle \,, \tag{5}$$

The transformation $\widehat{S}|\text{vac}\rangle \to |\text{vac}\rangle$ modifies the vacuum state in such way that it remains a vacuum state but is represented by the modes in the output channel. We denote the Schrödinger operators of annihilation and creation of the output modes as $\widehat{b}_J$ and $\widehat{b}^\dagger_J$, where $J = \mathrm{I}, \mathrm{II}$ numerates the output ports of the beamsplitter. Then the output state can be expressed as follows:

$$|\Psi_{\text{out}}\rangle = (\alpha^* \widehat{b}^\dagger_{\mathrm{I}} + \beta^* \widehat{b}^\dagger_{\mathrm{II}})|\text{vac}\rangle \ , \tag{6}$$

where $|\alpha|^2 + |\beta|^2 = 1$. This relation describes transformation of the input state of a single photon to the output state of a single photon with preservation of the spatial and temporal profile of the wavepacket, which is the crucial property of the coherent scattering process (transition and reflection) realized on a beamsplitter.

Basic relation (6) allows us to link the operators $\widehat{a}_j$ and $\widehat{b}_J$, which correspond to the input and output modes of the beamsplitter. We can easily find that the operators should express one another by a unitary matrix $\widehat{U}$:

$$\begin{pmatrix} \widehat{S}\widehat{c}_1\widehat{S}^{-1} \\ \widehat{S}\widehat{c}_2\widehat{S}^{-1} \end{pmatrix} = \widehat{U}\begin{pmatrix} \widehat{b}_{\mathrm{I}} \\ \widehat{b}_{\mathrm{II}} \end{pmatrix} \equiv \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} \widehat{b}_{\mathrm{I}} \\ \widehat{b}_{\mathrm{II}} \end{pmatrix} , \tag{7}$$

The parameters of matrix $\widehat{U}$ are determined by the specific properties of the beamsplitter. Each line in the transformation matrix expresses the possibility of the photon arriving at different input ports of the beamsplitter. Matrix $\widehat{U}$ represents a so-called "50 : 50" beamsplitter if

$$\widehat{U} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} . \tag{8}$$

This kind of beamslitter can be realized with a dielectric plate made from an isotropic dielectric material.

In accordance with our derivation, transformation (7) links the canonical operators defined in the Schrödinger representation and acting in the subspaces of the input and output modes. But if we want the quantization scheme to extend to the whole space, then a relation similar to (7) can be introduced for the Heisenberg operators $\widehat{a}_j(-T/2)$ and $\widehat{b}_J(T/2)$ considered in the infinite past and in the infinite future as follows:

$$\begin{pmatrix} \widehat{c}_1\left(-\frac{T}{2}\right) \\ \widehat{c}_2\left(-\frac{T}{2}\right) \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} \widehat{b}_{\mathrm{I}}\left(\frac{T}{2}\right) \\ \widehat{b}_{\mathrm{II}}\left(\frac{T}{2}\right) \end{pmatrix} . \tag{9}$$

In the latter form of the beamsplitter transformation, the time moments $\pm T/2$ are strongly correlated with the spatial arguments of the field operators such that $T = L/c$, where L is the spatial distance between the points where the field operators are considered, such that that transformation (9) is insensitive to the formally defined time parameter T.

Hadamard transform A beamsplitter allows us to prepare a single-photon quantum state known as a quantum bit (qubit), which is a quantum generalization of the classical unit of information known as a bit. Such a state can be prepared if a single photon arrives at either port 1 or port 2 of the beamsplitter. Let us associate the numbers 0 and 1 with the events for the input photon occurring at port 1 (with mode $|1\rangle$) and at port 2 (with mode $|2\rangle$), respectively. Then in the case of a 50 : 50 beamsplitter, the output photon's states are given by

$$|\text{out}\rangle_0 = \widehat{S}|1\rangle = \frac{1}{\sqrt{2}}(|\text{I}\rangle + |\text{II}\rangle) \tag{10}$$

for state 0 and

$$|\text{out}\rangle_1 = \widehat{S}|2\rangle = \frac{1}{\sqrt{2}}(|\text{I}\rangle - |\text{II}\rangle) \tag{11}$$

for state 1. If we now associate the numbers 0 and 1, respectively, with states $|\text{I}\rangle$ and $|\text{II}\rangle$ in the output ports, then we can see that in the output states (10) and (11) both numbers are simultaneously encoded. Such states prepared by the beamsplitter do not have a classical analogue, and this opens up new possibilities of processing information which would not be available with classical algorithms.

In the theory of quantum information, the above relations are known as the Hadamard transform[12] and are considered as a quantum logical operation. Formally this transform is defined as follows:

$$\mathcal{H}|0\rangle \to \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) ,$$

$$\mathcal{H}|1\rangle \to \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) ,$$

where the basis states of the input and output ports of the beamsplitter are parameterized by the numbers 0 and 1 and in our case the Hadamard transform $\mathcal{H}$ can be identified by the $\widehat{S}$ matrix.

The Hadamard transform is a real, unitary (orthogonal), and symmetric operation and its double action on any qubit state represents an identical operation:

$$\mathcal{H}|\text{out}\rangle = \mathcal{H}^2|\text{in}\rangle = |\text{in}\rangle . \tag{14}$$

Let us consider the following general single-photon input state:

$$|\text{in}\rangle = \alpha|0\rangle + \beta|1\rangle . \tag{15}$$

Then the output state of the beamsplitter is given by

$$|\text{out}\rangle = \mathcal{H}|\text{in}\rangle = \frac{1}{\sqrt{2}}[(\alpha + \beta)|0\rangle + (\alpha - \beta)|1\rangle] . \tag{16}$$

If $\alpha = 0$ or $\beta = 0$, the photon can be detected in any of output ports with equal probability. In the alternative situation when $\alpha = \beta$, the photon will always be detected in port $|0\rangle_{\text{out}}$ and will never be detected in port $|1\rangle_{\text{out}}$. A system consisting of

12) Jacques Salomon Hadamard (1865–1963) was a French mathematician.

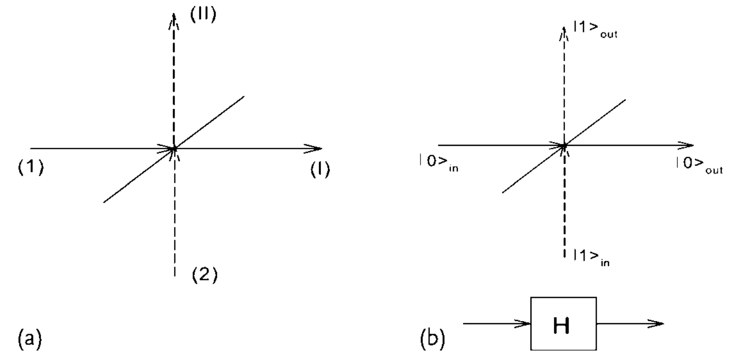

Figure 6.3 Beamsplitter: (a) A light beam incident on input port (1) or input port (2) splits into two beams and exits via ports (I) and (II). (b) In a single-photon case, the arrival of the photon at one of input ports can be associated with a quantum state, either $|0\rangle_{\text{in}}$ or $|1\rangle_{\text{in}}$, such that its superposition in the output basis with states $|0\rangle_{\text{out}}$ and $|1\rangle_{\text{out}}$ is an example of the Hadamard transform (see the text).

two beamsplitters is known as a Mach–Zender interferometer. This interferometer has two identical beamsplitters and corresponds to a double Hadamard transform. The double Hadamard transform is an identical operation which converts the output state into an input one; see (14).

Photon bunching and antibunching The transformation derived for the canonical operators (7) and (9) is applicable to follow the transmission/reflection not only of a single photon but for any quantum state of light. As an important example let us consider two photons with the same polarizations and spatial and temporal profiles arriving simultaneously at two different input ports of the beamsplitter. The output state can be found straightforwardly :

$$\begin{aligned}|\Psi_{\text{out}}\rangle &= S\widehat{c}_1^\dagger\widehat{c}_2^\dagger|\text{vac}\rangle = \frac{1}{2}\left(\widehat{b}_{\text{I}}^\dagger + \widehat{b}_{\text{II}}^\dagger\right)\left(\widehat{b}_{\text{I}}^\dagger - \widehat{b}_{\text{II}}^\dagger\right)|\text{vac}\rangle \\ &= \frac{1}{2}\left(\widehat{b}_{\text{I}}^{\dagger 2} - \widehat{b}_{\text{II}}^{\dagger 2}\right)|\text{vac}\rangle\ . \end{aligned} \tag{17}$$

The result obtained shows us that both photons exit from the same port, either I or II. This effect is known as photon bunching and it is closely connected with the bosonic nature of light quanta as indistinguishable quasi-particles. The final state is an example of an entangled state.

Could the photons with overlapping wavepacket profiles exit from different ports of the beamsplitter and demonstrate an antibunching effect? The answer is yes, but in this case they have to have different and always orthogonal polarizations in the exit channel. The above consideration has to be reconsidered and generalized if we keep the polarization degrees of freedom. We skip the details of the derivation and point out that in this case the output state will be the entangled state

$$|\Psi_{\text{out}}\rangle = \frac{1}{2}\left(\widehat{b}_{\text{I}x}^\dagger\widehat{b}_{\text{II}y}^\dagger - \widehat{b}_{\text{I}y}^\dagger\widehat{b}_{\text{II}x}^\dagger\right)|\text{vac}\rangle\ , \tag{18}$$

which is defined in the basis of orthogonal x and y polarizations, and represents the polarization-entangled state of light; see (1). In 1998, Anton Zeilinger experimentally demonstrated the quantum teleportation phenomenon (Zeilinger, 1997), which was crucially based on this effect.

Quantum teleportation In their famous experiment, Zeilinger and coworkers used a nonlinear crystal and parametric down-conversion process to generate a sequence of photon pairs sharing entangled state (1), where x and y are associated with two orthogonal polarizations. Then by manipulation of these photons and with a third signal photon originally prepared in an arbitrary and unknown quantum state $\alpha|x\rangle + \beta|y\rangle$, the latter can be teleported in space.

The teleportation scheme works as follows. We send the signal photon and one of the photons from the entangled pair onto the beamsplitter and perform the detection by a double click at the output ports. In accordance with the discussion above, there is 25% probability that there will be clicks at two different ports of the beamsplitter and the photons detected will have orthogonal polarizations. That means that we made a projective measurement of the two-photon system, now consisting of the signal photon and one of the photons generated in the down-conversion process, onto state 1. In such a measurement we did not obtain any information about the photons' polarizations but because of strong quantum correlations we know that the remaining photon will possess the same polarization as the signal photon had. This is an example of a so-called Bell-type measurement. Although the polarization is unknown to us, we find that the remaining photon can be considered as existing in the pure quantum state described by the wave function $\alpha|x\rangle + \beta|y\rangle$.

As we see, this experiment involves instant transfer (teleportation) of a microscopic quantum state between arbitrary remote spatial locations. However, there is no contradiction with the special theory of relativity in such an experiment. The teleportation process cannot be considered as an information transfer since the required measurement (double click at different ports of the beamsplitter) is performed only at the point of detection and not at the point of teleportation. To transfer the information to the detector of the remaining photon, we need an additional classical channel, which would tell us if the measurement was successful. But nevertheless we see that quantum theory gives us a quite subtle and new interpretation of the relativistic restrictions in information processing.

6.2 Quantum Theory of Photon Emission, Absorption, and Scattering by Atomic Systems

Most of the problems related to the interaction of charged particles and photons can be resolved in the frame of the perturbation theory approach. This is mainly supported by the fact that the dimensionless expansion parameter of the perturbation series

$$\alpha = \frac{e^2}{\hbar c} \approx \frac{1}{137} \tag{6.53}$$

(*fine structure constant*) is small enough. Then the probabilities of basic quantum electrodynamic processes calculated in the first nonvanishing order of perturbation theory normally have the required accuracy. But there are various examples requiring more rigorous analysis.

6.2.1 Interaction of the Quantized Electromagnetic Field with a Nonrelativistic System

The dynamics of a charged particle system is described by the Schrödinger equation extended by Pauli spin terms. This includes the interactions of the particles with the external classical and quantized field, internal interactions, and spin dynamics. The Pauli equation will be obtained from the relativistic Dirac equation in Example 6.14. For a single particle, the Pauli Hamiltonian is given by

$$\widehat{\mathcal{H}} = \frac{1}{2m}\left(\widehat{\boldsymbol{p}} - \frac{e}{c}\widehat{A}\right)^2 + e\varphi - \widehat{\boldsymbol{\mu}}\cdot\boldsymbol{H} , \tag{6.54}$$

where $\widehat{\boldsymbol{p}} = -i\hbar\nabla$ is the momentum operator, $\widehat{\boldsymbol{\mu}}$ is the operator of the spin magnetic angular momentum, and $\varphi(\boldsymbol{r})$ is the scalar potential of the external longitudinal static field. These operators are coupled with coordinate and spin variables of the particle. The operators $\widehat{\boldsymbol{A}}(\boldsymbol{r})$ and $\widehat{\boldsymbol{H}}(\boldsymbol{r}) = \nabla \times \widehat{\boldsymbol{A}}(\boldsymbol{r})$ depend only on the particle coordinates and on the canonical variables of the electromagnetic field. The field state is described by the occupation numbers N_s, where s is the set of the quantum numbers parameterizing the photon state (see the Fock representation in Section 6.1).

For a many-particle system, Hamiltonian (6.54) is generalized as the sum of terms over all the particles and their cross-interaction terms:

$$\widehat{\mathcal{H}} = \sum_{a=1}^{N}\left\{\frac{1}{2m_a}\left[\widehat{\boldsymbol{p}_a} - \frac{e_a}{c}\widehat{A}(\boldsymbol{r}_a)\right]^2 + e_a\varphi(\boldsymbol{r}_a) - \widehat{\boldsymbol{\mu}_a}\cdot\boldsymbol{H}(\boldsymbol{r}_a)\right\} . \tag{6.55}$$

Below we simplify our consideration with a single-particle configuration.

All the operators are considered in the Schrödinger representation and do not depend on time. The vector potential operator is given by

$$\widehat{\boldsymbol{A}}(\boldsymbol{r}) = \sum_s \{\widehat{a}A_s(\boldsymbol{r}) + \widehat{a}^\dagger A_s^*(\boldsymbol{r})\} , \tag{6.56}$$

and the other operators can be similarly expressed.

Let us decompose Hamiltonian (6.54) into the sum $\widehat{\mathcal{H}} = \widehat{\mathcal{H}}_0 + \widehat{V}$, where we incorporated all the terms with $\widehat{\boldsymbol{A}}$ into the interaction Hamiltonian $\widehat{V}$:

$$\widehat{V} = -\frac{e}{mc}\widehat{\boldsymbol{p}}\cdot\widehat{\boldsymbol{A}} + \frac{e^2}{2mc^2}\widehat{\boldsymbol{A}}^2 - \widehat{\boldsymbol{\mu}}\cdot\widehat{\boldsymbol{H}} . \tag{6.57}$$

For the transverse photon modes the vector potential and momentum operators commute such that $\widehat{\boldsymbol{p}}\cdot\widehat{\boldsymbol{A}} = \widehat{\boldsymbol{A}}\cdot\widehat{\boldsymbol{p}}$. The first and third terms in (6.57) describe single-photon emission and absorption, but the second, diamagnetic term $\widehat{\boldsymbol{A}}^2$ is responsible for two-photon processes.

If both processes are manifestable, the contribution of two-photon transitions is usually weaker than that of single-photon processes. Let us estimate the relative order of magnitude for the terms in (6.57). For this we replace the operators with representative values of the corresponding physical quantities:

$$R_A \approx \frac{(e^2/2mc^2)A^2}{(e/mc)pA} \approx \frac{eA}{pc} \approx \frac{eE\lambda}{pc} \approx \frac{v}{c}\frac{eE\lambda}{\mathcal{E}_{\text{kin}}} ,$$

where p and $\mathcal{E}_{\text{kin}}$ are estimations of the momentum and kinetic energy of the particle. For an electron in an atom, we apply the estimate $v/c \approx 10^{-2}$, $\mathcal{E}_{\text{kin}} \approx 5\,\text{eV}$, and $\lambda \approx 10^{-5}$ cm. As we see, the ratio R_A approaches unity when $E > 10^7$ V/cm. Such fields can only be generated by high-powered lasers. For an atomic nucleus, $R_A \ll 1$ because the coherent radiation in the gamma frequency spectrum is not attainable in experiment.

For the third term in (6.57), we have the estimate

$$R_\mu \approx \frac{\mu k A}{(e/mc)pA} \approx \frac{v}{c}\frac{\hbar\omega}{\mathcal{E}_{\text{kin}}} .$$

The energy of an emitted or absorbed photon is comparable with the level spacing of the radiating system; this leads to $\hbar\omega \approx \mathcal{E}_{\text{kin}}$. For nonrelativistic systems such as atomic nuclei and atomic or molecular electronic shells, $v/c \ll 1$. Then $R_\mu < v/c \ll 1$. But the estimates presented are based on the assumption that the term $(e/mc)\widehat{\boldsymbol{p}}\cdot\widehat{\boldsymbol{A}}$ generally dominates. If for some reason this term is weak, then other terms become significant. Having in mind only the single-photon processes, we will disregard the quadratic term in our further consideration:

$$\widehat{V} = -\frac{e}{mc}\widehat{\boldsymbol{A}}\cdot\widehat{\boldsymbol{p}} - \widehat{\boldsymbol{\mu}}\cdot[\nabla \times \widehat{\boldsymbol{A}}] . \tag{6.58}$$

6.2.2
Spontaneous and Stimulated Emission

Let us consider the interaction of a system of charged particles with the electromagnetic field and calculate the emission probability of a photon with a given wave vector and polarization $s = (\boldsymbol{k}, \sigma)$. The initial state of the field is described by a set of the photon numbers $\{N_s\}$ for all the modes. In the final state the photon number of mode s becomes equal to $N_s + 1$ and for others modes $N_{s'}$, $s' \neq s$ remains unchanged. The particle undergoes a transition between states $|i\rangle$ and $|f\rangle$, both belonging to a discrete spectrum. In the first order of perturbation theory, the transition probability per unit time is expressed by the Fermi golden rule (see Landau and Lifshitz, 1977):

$$\mathrm{d}w^{\text{rad}} = \frac{2\pi}{\hbar}|\langle f|\widehat{V}|i\rangle|^2\delta(\mathcal{E}_i - \mathcal{E}_f - \hbar\omega)\frac{\mathcal{V}k^2\mathrm{d}k\mathrm{d}\Omega_k}{(2\pi)^3} . \tag{6.59}$$

Since the state of the emitted photon belongs to a continuous spectrum, this expression has a differential form. The magnitude $\mathrm{d}\nu = \mathcal{V}\omega^2\mathrm{d}\omega\mathrm{d}\Omega_k/(2\pi c)^3$ is the number of photon quantum states with a given polarization per wave vector interval d^3k (see expression (2.164)).

The wave vectors of the initial and final states are given by the products of the particle and field wave functions:

$$|i\rangle = \psi_i \Phi_{\{N_s\}}\,, \quad |f\rangle = \psi_f \Phi_{\{N_s+1\}}\,. \tag{6.60}$$

The lower index indicates the photon number of the outgoing emission mode. Using (6.60), we can write the matrix element of the interaction operator as

$$\langle f|\widehat{V}|i\rangle = \langle \Phi_{\{N_s+1\}}| \times \int d^3r \left\{-\frac{e}{mc}(\psi_f, \widehat{\boldsymbol{p}}\psi_i)\cdot\widehat{\boldsymbol{A}}(\boldsymbol{r}) - (\psi_f, \widehat{\boldsymbol{p}}\psi_i)\cdot[\nabla \times \widehat{\boldsymbol{A}}(\boldsymbol{r})]\right\} |\Phi_{\{N_s\}}\rangle\,, \tag{6.61}$$

where d^3r is an elementary volume of the three-dimensional space and the matrix elements in parentheses represent the sum over internal spin variables.

Performing partial integration and assuming that the spatial wave functions vanish when $r \to \infty$, we can transform the inner integral to

$$-\frac{1}{c}\int \boldsymbol{j}_{fi}\cdot\widehat{\boldsymbol{A}}^*(\boldsymbol{r})d^3r\,,$$

where

$$\boldsymbol{j}_{fi} = -\frac{ie\hbar}{2m}\left[(\psi_f, \nabla\psi_i) - (\nabla\psi_f, \psi_i)\right] + c\nabla \times (\psi_f, \widehat{\boldsymbol{\mu}}\psi_i) \tag{6.62}$$

is standardly called the *transitional current*. When $\psi_f = \psi_i = \psi$, the transitional current is the electric current created by a charged particle with a magnetic moment in state ψ. Using (6.56) and (C63), we can write the matrix element of operator $\widehat{\boldsymbol{A}}(\boldsymbol{r})$ as follows

$$\langle \Phi_{\{N_s+1\}}|\widehat{\boldsymbol{A}}(\boldsymbol{r})|\Phi_{\{N_s\}}\rangle = \sqrt{N_s+1}\boldsymbol{A}_s^*(\boldsymbol{r})\,. \tag{6.63}$$

In the entire sum (6.56), only one term of the photons created in the specific mode actually contributes, so expression (6.61) simplifies to

$$\langle f|\widehat{V}|i\rangle = -\frac{1}{c}\sqrt{N_s+1}\int \boldsymbol{j}_{fi}(\boldsymbol{r})\cdot\boldsymbol{A}_s^*(\boldsymbol{r})d^3r\,. \tag{6.64}$$

The emission probability can be found via substitution of the above matrix element in (6.59) and integration over the photon frequency:

$$dw^{\text{rad}} = \frac{\mathcal{V}(N_s+1)\omega^2}{4\pi^2\hbar^2c^5}\left|\int \boldsymbol{j}_{fi}(\boldsymbol{r})\cdot\boldsymbol{A}_s^*(\boldsymbol{r})d^3r\right|^2 d\Omega_k\,, \tag{6.65}$$

where $\omega = (\mathcal{E}_i - \mathcal{E}_f)/\hbar$. The emission probability consists of two terms: $dw^{\text{rad}} = dw^{\text{ind}} + dw^{\text{sp}}$. The former is proportional to the number of photons N_s in the mode considered and is commonly known as the *stimulated emission probability*. The latter can exist even in the absence of photons in the initial state. This term is responsible for *spontaneous emission* (Einstein , 1916a):

$$\mathrm{d}w^{\mathrm{sp}} = \frac{\mathcal{V}\omega^2}{4\pi^2\hbar^2 c^5}\left|\int \boldsymbol{j}_{fi}(\boldsymbol{r})\cdot\boldsymbol{A}_s^*(\boldsymbol{r})\mathrm{d}^3 r\right|^2 \mathrm{d}\Omega_k\ . \tag{6.66}$$

Spontaneous emission of atoms, the optical excitation of which can be initiated by collisions with other atoms, plays a role in thermal light sources such as arc-discharge lamps, incandescent lamps, the Sun, and so on. Using the Planck distribution (6.23) with representative temperature $T = (3–6)\times 10^3$ K, we can easily estimate the mean number of photons: $N_s \approx 10^{-3}$–10^{-2}. In such conditions the role of the stimulated emission processes is practically negligible. For lasers, the very high intensity in a narrow spectral band results in strong stimulated emission.

The probability of absorption of a photon by a quantum system for the transition between the same states $f \to i$ (but considered in reverse order) is also expressed by (6.59), where the previous final state becomes the initial one now and vice versa:

$$\mathrm{d}w^{\mathrm{abs}} = \frac{2\pi}{\hbar}|\langle f'|\widehat{V}|i'\rangle|^2\delta(\mathcal{E}_{i'} - \mathcal{E}_{f'} + \hbar\omega)\frac{\mathcal{V}\omega^2\mathrm{d}\omega\mathrm{d}\Omega_k}{(2\pi c)^3}\ , \tag{6.67}$$

where $|i'\rangle = \psi_f\Phi_{\{N_s\}}, |f'\rangle = \psi_i|\Phi_{\{N_s-1\}}$ and $\mathrm{d}\nu$ parameterizes the continuous spectrum of the absorbing photon. Introducing the matrix element

$$\langle\Phi_{\{N_s-1\}}|\widehat{\boldsymbol{A}}(\boldsymbol{r})|\Phi_{\{N_s\}}\rangle = \sqrt{N_s}\boldsymbol{A}_s(\boldsymbol{r})\ ,$$

we obtain

$$\langle f'|\widehat{V}|i'\rangle = -\frac{1}{c}\sqrt{N_s}\int \boldsymbol{j}_{if}(\boldsymbol{r})\cdot\boldsymbol{A}_s(\boldsymbol{r})\mathrm{d}^3 r = -\frac{1}{c}\sqrt{N_s}\left[\int \boldsymbol{j}_{fi}(\boldsymbol{r})\cdot\boldsymbol{A}_s^*(\boldsymbol{r})\mathrm{d}^3 r\right]^*,$$

where because of the Hermitian character of the current operator we have $\boldsymbol{j}_{if} = \boldsymbol{j}^*_{fi}$ and finally the probability is given by

$$\mathrm{d}w^{\mathrm{abs}} = \frac{N_s\omega^2}{4\pi^2\hbar^2 c^5}\left|\int \boldsymbol{j}_{fi}(\boldsymbol{r})\cdot\boldsymbol{A}_s^*(\boldsymbol{r})\mathrm{d}^3 r\right|^2 \mathrm{d}\Omega_k = \mathrm{d}w^{\mathrm{ind}} = N_s\mathrm{d}w^{\mathrm{sp}}\ . \tag{6.68}$$

Here, $\omega = (\mathcal{E}_i - \mathcal{E}_f)/\hbar$. As follows from (6.65) and (6.67), the differential transition probabilities for photon absorption and emission are expressed as

$$\frac{\mathrm{d}w^{\mathrm{rad}}}{\mathrm{d}w^{\mathrm{abs}}} = \frac{N_s + 1}{N_s}\ . \tag{6.69}$$

The average photon number N_s, where $s = (\boldsymbol{k}, \sigma)$, is determined by the intensity of the incident radiation as (see Problem 6.21$^\bullet$):

$$N_s = \frac{8\pi^3 c^2}{\hbar\omega^3} I_\sigma(\omega, \boldsymbol{n})\ . \tag{6.70}$$

For unpolarized light ($I_1(\omega) = I_2(\omega)$) which is propagating isotropically in all directions, its intensity is independent on $\boldsymbol{n}$. Then the total spectral intensity of the radiation is given by $I(\omega) = \int (I_1(\omega) + I_2(\omega))\mathrm{d}\Omega_k = 8\pi I_\sigma(\omega)$ and the photon number $N_s \equiv N_\omega = (\pi^2 c^2/\hbar\omega^3) I(\omega)$ depends only on the radiation frequency. After substituting expression (6.67) into (6.68) and evaluating the integral over $\mathrm{d}\Omega_k$ and summing over polarizations, we obtain the following relation between the total probabilities:

$$w^{\mathrm{abs}} = w^{\mathrm{ind}} = \frac{\pi^2 c^2}{\hbar\omega^3} I(\omega) w^{\mathrm{sp}} , \tag{6.71}$$

where $w^{\mathrm{abs}} = \sum_\sigma \int \mathrm{d}w^{\mathrm{abs}}$. The quantities introduced

$$A_{if} = w^{\mathrm{sp}} , \quad B_{if} = \frac{c}{I(\omega)} w^{\mathrm{ind}} , \quad B_{fi} = \frac{c}{I(\omega)} w^{\mathrm{abs}} \tag{6.72}$$

are referred to as the *Einstein coefficients* (Einstein , 1916b) of spontaneous emission, stimulated emission, and radiation absorption, respectively. The above expressions also give

$$B_{fi} = B_{if} = \frac{\pi^2 c^3}{\hbar\omega^3} A_{if} . \tag{6.73}$$

However, this relation is only valid for nondegenerate energy levels $\mathcal{E}_i$ and $\mathcal{E}_f$. For degenerate levels with degeneracy numbers g_i and g_f, the calculated probabilities should be statistically averaged over the initial states and summed over the final states.

Example 6.8

Prove that for degenerate levels the Einstein coefficients satisfy the following relation:

$$g_f B_{fi} = g_i B_{if} = \frac{\pi^2 c^3}{\hbar\omega^3} g_i A_{if} . \tag{6.74}$$

Solution. Let us write the mean probabilities $\overline{\varpi}$ for degenerate levels:

$$\overline{w}^{abs} = \frac{1}{g_f} \sum w^{\mathrm{abs}} , \quad \overline{w}^{ind} = \frac{1}{g_i} \sum w^{\mathrm{ind}} , \quad \overline{w}^{sp} = \frac{1}{g_i} \sum w^{\mathrm{sp}} .$$

If we sum (6.73) over the initial and final states and express the result via the mean probabilities introduced, we arrive at relation (6.74), where the Einstein coefficients are now given by (6.72) but with the mean probabilities substituted. □

6.2.3
Electric Dipole Radiation

Example 6.9

Consider the radiation process for a particle localized in a finite area of spatial scale a satisfying the condition $ka \ll 1$, where $\boldsymbol{k}$ is the wave vector of the emitted photon. Express the spontaneous emission probability for the photon with a given wave vector and polarization via the matrix element of the electric dipole moment. Sum over all polarizations and derive the intensity of radiation for a given frequency and in direction $\boldsymbol{n}$. Compare the result with its classical counterpart, see (5.30) and (5.31).

Solution. The condition $ka \ll 1$ allows us to approximate the basis function of the field mode $\boldsymbol{e}_s^* \sqrt{2\pi\hbar c^2/\mathcal{V}\omega}\, e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} \approx \boldsymbol{e}_s^* \sqrt{2\pi\hbar c^2/\mathcal{V}\omega}$. Substituting this into matrix element (6.64), we obtain

$$\int \boldsymbol{j}_{fi}\cdot\boldsymbol{A}_s^* \mathrm{d}^3 r = e\sqrt{\frac{2\pi\hbar c^2}{\mathcal{V}\omega}}\boldsymbol{e}_s^*\cdot \int \left(\psi_f, \left(-\frac{i\hbar\nabla}{m}\right)\psi_i\right) \mathrm{d}^3 r \,.$$

where we took into account that the magnetic-type interaction term with spin angular momentum vanished after integration. The operator $-i\hbar\nabla/m$ can be associated with the particle velocity. It can be transformed with the following quantum mechanical relations:

$$\widehat{\boldsymbol{v}} = \widehat{\dot{\boldsymbol{r}}} = \frac{i}{\hbar}\left[\widehat{\mathcal{H}}, \widehat{\boldsymbol{r}}\right] \quad \text{and} \quad \widehat{\mathcal{H}}_0 \psi_{i,f} = \mathcal{E}_{i,f}\psi_{i,f} \,,$$

where $\widehat{\mathcal{H}}_0$ is the internal Hamiltonian of the particle. Also taking into account the Hermitian character of the Hamiltonian

$$\int (\psi_f, \widehat{\mathcal{H}}_0 \boldsymbol{r}\psi_i)\mathrm{d}^3 r = \int (\widehat{\mathcal{H}}_0\psi_f, \boldsymbol{r}\psi_i)\mathrm{d}^3 r = \mathcal{E}_f \int (\psi_f, \boldsymbol{r}\psi_i)\mathrm{d}^3 r \,,$$

we obtain

$$\int \boldsymbol{j}_{fi}\cdot\boldsymbol{A}_s^* \mathrm{d}^3 r = -\frac{i}{\hbar}(\mathcal{E}_i - \mathcal{E}_f)\sqrt{\frac{2\pi\hbar c}{\mathcal{V}\omega}}\boldsymbol{e}_s^*\cdot\boldsymbol{p}_{fi} \,,$$

where $(\mathcal{E}_i - \mathcal{E}_f)/\hbar$ is the transition frequency for the emitted photon and

$$\boldsymbol{p}_{fi} = e\int (\psi_f, \boldsymbol{r}\psi_i)\mathrm{d}^3 r \tag{6.75}$$

is the *dipole moment of the transition* $i \to f$. In the case of an N-particle system, we can generalize the result by replacing $e\boldsymbol{r}$ by $\sum_{a=1}^{N} e_a \boldsymbol{r}_a$, and the wave functions in the matrix element will depend on the coordinates of all the particles.

The transition probability is given by

$$\mathrm{d}w^{\mathrm{sp}} = \frac{\omega^3}{2\pi\hbar c^3}|\boldsymbol{e}_s^*\cdot\boldsymbol{p}_{fi}|^2\mathrm{d}\Omega_k\,. \tag{6.76}$$

This expression gives us the probability of emission of a photon with a particular polarization. The sum over polarization can be made similarly to (2.165) (Problem 2.152•), and after all the derivations, we finally obtain

$$\mathrm{d}w^{\mathrm{sp}} = \frac{\omega^3}{2\pi\hbar c^3}|\boldsymbol{n}\times\boldsymbol{p}_{fi}|^2\mathrm{d}\Omega_k\,. \tag{6.77}$$

By multiplying this equation by the photon energy $\hbar\omega$, we obtain the intensity of radiation in a particular direction, which is given by (see (5.30) and (5.31))

$$\frac{\mathrm{d}I}{\mathrm{d}\Omega} = \frac{\omega^4}{2\pi c^3}|\boldsymbol{n}\times\boldsymbol{p}_{fi}|^2\,. \tag{6.78}$$

After integration over the solid angle, the total intensity emitted in all directions is given by

$$I = \frac{4\omega^4}{3c^3}|\boldsymbol{p}_{fi}|^2\,. \tag{6.79}$$

Note that the quantum mechanical result for the intensity of radiation can be obtained from the classical one (see (5.30) and (5.31)) for a harmonic oscillator, with the replacement $\overline{\ddot{\boldsymbol{p}}^2} \to 4\omega^4|\boldsymbol{p}_{fi}|^2$. □

6.2.4 Electric Quadrupole and Magnetic Dipole Radiation

If electric dipole radiation is forbidden because of transition symmetry, then the nonvanishing contribution to the transition matrix element should be associated with the presence of higher-order terms in the expansion of the exponential factor. In the next order we arrive at the following matrix element:

$$-\frac{1}{c}\int \boldsymbol{j}_{fi}\cdot\boldsymbol{A}_s^*(\boldsymbol{r})\mathrm{d}^3r = \frac{i\omega}{c^2}\sqrt{\frac{2\pi\hbar c^2}{\mathcal{V}\omega}}\boldsymbol{e}_s^*\cdot\int \boldsymbol{j}_{fi}(\boldsymbol{r})(\boldsymbol{n}\cdot\boldsymbol{r})\mathrm{d}^3r\,.$$

This can be transformed as in the classical theory (see, e.g., Problem 5.6) and we get

$$\begin{aligned}-\frac{1}{c}\int \boldsymbol{j}_{fi}\cdot\boldsymbol{A}_s^*(\boldsymbol{r})\mathrm{d}^3r = \frac{i\omega}{c}\sqrt{\frac{2\pi\hbar c^2}{\mathcal{V}\omega}}\boldsymbol{e}_s^*\cdot&\left\{\frac{1}{2c}\left[\int \boldsymbol{r}\times\boldsymbol{j}_{fi}(\boldsymbol{r})\mathrm{d}^3r\right]\times\boldsymbol{n}\right.\\ &\left.-\frac{1}{2c}\int \boldsymbol{r}(\boldsymbol{n}\cdot\boldsymbol{r})\nabla\cdot\boldsymbol{j}_{fi}(\boldsymbol{r})\mathrm{d}^3r\right\}\,.\end{aligned}$$

The first integral in the braces is the transition matrix element of the magnetic moment:

$$\boldsymbol{\mu}_{fi} = \frac{1}{2c} \int \boldsymbol{r} \times \boldsymbol{j}_{fi}(\boldsymbol{r}) \mathrm{d}^3 r \,. \tag{6.80}$$

By applying the continuality equation $\nabla \cdot \boldsymbol{j}_{fi} = -i\omega\rho_{fi}$, we can transform the second integral to the following form:

$$-\frac{1}{c} \int \boldsymbol{j}_{fi} \cdot \boldsymbol{A}_s^*(\boldsymbol{r}) \mathrm{d}^3 r = \frac{i\omega}{c} \sqrt{\frac{2\pi\hbar c^2}{\mathcal{V}\omega}} \boldsymbol{e}_s^* \cdot \left\{ [\boldsymbol{\mu}_{fi} \times \boldsymbol{n}] - \frac{i\omega}{6c} \boldsymbol{Q}_{fi} \right\} \,,$$

where $(\boldsymbol{Q}_{fi})_\alpha = (Q_{fi})_{\alpha\beta} n_\beta$ and $(Q_{fi})_{\alpha\beta}$ is the transition matrix element of the quadrupole tensor,

$$(Q_{fi})_{\alpha\beta} = \int (3x_\alpha x_\beta - r^2 \delta_{\alpha\beta}) \rho_{fi}(\boldsymbol{r}) \mathrm{d}^3 r \,. \tag{6.81}$$

After substituting the transformed transition matrix element into the golden role (6.59), we obtain the probability of emission of a photon with a given polarization and wave vector:

$$\mathrm{d}w^{\mathrm{sp}} = \frac{\omega^3}{2\pi\hbar c^3} \left| \boldsymbol{e}_s^* \cdot \left[\boldsymbol{\mu}_{fi} \times \boldsymbol{n} - \frac{i\omega}{6c} \boldsymbol{Q}_{fi} \right] \right|^2 \mathrm{d}\Omega_k \,. \tag{6.82}$$

After summing over polarizations (using (2.165)) and multiplying by $\hbar\omega$, we obtain the intensity of radiation in an arbitrary direction as

$$\begin{aligned} \frac{\mathrm{d}I}{\mathrm{d}\Omega} = \frac{\omega^4}{2\pi c^3} \Bigg\{ & |\boldsymbol{\mu}_{fi} \times \boldsymbol{n}|^2 + \frac{\omega^2}{36c^2} |\boldsymbol{Q}_{fi} \times \boldsymbol{n}|^2 \\ & + \frac{i\omega}{6c} \left[\boldsymbol{Q}_{fi}^* \cdot [\boldsymbol{\mu}_{fi} \times \boldsymbol{n}] - \boldsymbol{Q}_{fi} \cdot [\boldsymbol{\mu}_{fi}^* \times \boldsymbol{n}] \right] \Bigg\} \,. \end{aligned} \tag{6.83}$$

The analogy with classical expression (5.38) is evident. Integration of this expression over the solid angle gives us the total spectral intensity emitted for the selected quantum transition:

$$I = \frac{4\omega^4}{3c^3} |\boldsymbol{\mu}_{fi}|^2 + \frac{\omega^6}{90c^5} |(Q_{fi})_{\alpha\beta}|^2 \,. \tag{6.84}$$

Suggested literature:
Berestetskii *et al.* (1982); Feynman (1998); Bethe and Salpeter (1957); Akhiezer and Berestetskii (1981); Heitler (1954)

Problems

6.47$^{\bullet}$. The initial state of an electron in the central symmetric field is specified by a set of four quantum numbers: the principal quantum number $n = 1, 2, \ldots$, the orbital quantum number $l = 0, 1, \ldots, n-1$, and two magnetic quantum numbers

$m_l = 0, \pm1, \pm2, \ldots, \pm l$ and $m_s = \pm 1/2$, which are associated with the projection of the orbital and spin angular momenta onto the quantization axis. Show that the electric-dipole-type radiation can link this state with the final electron states, which are specified by the following quantum numbers:

$$m'_s = m_s\,, \quad l' = l \pm 1 \quad (l' = 1 \quad l = 0)\,, \quad m'_l = m_l\,, \quad m_l \pm 1\,, \quad (6.85)$$

where n' is an arbitrary principal number compatible with the radiation process and $\mathcal{E}_f < \mathcal{E}_i$ (*electric dipole emission selection rules*). Because of the selection rule $\Delta l = \pm 1$, the parity of the electron wave function is always changed for any dipole transition (see, e.g., Problem 1.20).

Hint: Use the Clebsch–Gordan expansion for the angular part of the electron wave function (see (C78)).

6.48. Let the electron in the hydrogen atom be in excited state 2p. For the transitions from the upper states specified by different values of the magnetic quantum number $m_l = 0, \pm1$ to the ground state 1s, calculate the probability of spontaneous emission of a photon with linear polarization in an arbitrary direction $\boldsymbol{n}$. Compare the angular distributions of the emission for different transitions. In particular, sum over the photon polarizations and find the angular distribution for nonpolarized atoms.

6.49. Perform the quantum calculation for Problem 5.23. For the hydrogen atom, find the overall emission intensity integrated over all directions.

6.50•. Evaluate the lifetime τ of the excited hydrogen atom in state 2p caused by its spontaneous decay.

6.51. Let the electron in the hydrogen atom be in its ground state, 1s. Would it be possible to excite the atom up to the state 2p, $m_l = +1$, with polarized photons? For what polarization of the photons would only state $m_l = -1$ ($m_l = 0$) be occupied?

Hint: It is useful to use the solution of Problem 6.20*.

6.52. The emission of the Lyman spectral series of the hydrogen atom is a result of electric dipole transitions from states 2p, 3p, ... to the ground state, 1s. Perform a comparative calculation of the spontaneous emission intensities I for first two spectral lines of this series Ly_α (transition 2p $\to$ 1s) and Ly_β (transition 3p $\to$ 1s).

Hint: The radial part of the wave function for the 3p state is given by Appendix C (see expressions (C74) and (C75)).

6.53*. Interaction of the electron with the vacuum fluctuations of the electromagnetic field results in splitting of the energy levels $2s_{1/2}$ and $2p_{1/2}$ (Lamb shift, $\Delta\mathcal{E} \approx 1058\,\mathrm{MHz} \approx 4.4 \times 10^{-6}\,\mathrm{eV}$; see Problem 6.18•*). The lower index indicates the quantum number of the total electron angular momentum $j = 1/2$. Express the probability of the spontaneous decay $2s_{1/2} \to 2p_{1/2}$ in terms of atomic parameters and estimate it numerically (in units of reciprocal seconds).

Hint: In accordance with the basic principle of the angular momentum algebra in quantum mechanics, the eigenstate of the electron with total angular momentum j, its projection m_j, and orbital momentum l and denoted as $|j, m_j, l\rangle$ can be expanded in a Clebsch–Gordan series (see (C78)):

$$\Phi_{l j m_j} = \sum_{m\,\mu} C^{j\,m_j}_{l m 1/2\,\mu} Y_{lm}(\theta,\varphi)\chi_\mu \,,$$

where χ_μ is the spinor wave function, which describes the state with projection of the spin $\mu = \pm 1/2$ on the quantization axis. The expansion coefficients $C^{\cdots}_{\cdots\cdots}$ provide the requirement of triangle-inequality $l + 1/2 \geq j \geq |l - 1/2|$ and the selection rule for the projections of the angular momenta $m_j = m + \mu$.

6.54*. The coupling of the magnetic moments of the electron and the proton splits the ground state of the hydrogen atom into two sublevels (hyperfine splitting). The energy splitting between the hyperfine sublevels is $\Delta\mathcal{E} \approx 1420\,\text{MHz} \approx 5.9 \times 10^{-6}$ eV. The upper level is a triplet (total angular momentum $S = 1$), and the lower level is a singlet (total angular momentum $S = 0$). Calculate the transition probability between the hyperfine components of the hydrogen atom.

6.55*. The hydrogen atom is excited by a monochromatic light beam with frequency !. The energy of the photons satisfies the condition $\hbar\omega \gg I_0$, where I_0 is the binding energy (ionization potential) of the atom. Calculate the photoionization cross-section for the atom ionized by polarized radiation and for the electron exiting in an arbitrary direction. Also calculate the total cross-section in case of non-polarized radiation and for all ionization directions. The final state of the electron can be approximated by a free plane wave.

6.56. Calculate the total cross-section of photorecombination for the hydrogen atom (a process which is the inverse of photoionization). Consider the electron as a free particle in the initial state and the recombined atom in its ground state.

6.57•. Express the transition magnetic dipole moment (6.80) in terms of the matrix element of the total magnetic moment $\widehat{\boldsymbol{M}}$ of the emitting particle:

$$\boldsymbol{\mu}_{fi} = \int \psi_f^* \widehat{\boldsymbol{M}} \psi_i \mathrm{d}^3 r \,, \quad \widehat{\boldsymbol{M}} = \frac{e\hbar}{2mc}\left(\widehat{\boldsymbol{l}} + g\widehat{\boldsymbol{s}}\right) \,, \tag{6.86}$$

where the operators $\widehat{\boldsymbol{l}}$ and $\widehat{\boldsymbol{s}}$, in accordance with (4.76) and (4.79) and the correspondence principle, are the dimensionless operators of the orbital and spin angular momenta of the particle. For the electron factor, $g \approx 2$.

6.2.5
Perturbation Theory for the Density Matrix

Below we consider some special approaches focusing on particular problems in electrodynamic. These approaches are based on the general formalism described above and are applicable to a nonrelativistic system consisting of charged particles (atoms, molecules, solids, atomic nuclei, etc.). For the sake of brevity and clarity, we will use term “atom” for any such system.

Keeping the basic contributions (6.21) and (6.55), we can introduce the total Hamiltonian for the particle system interacting with the quantized electromagnetic field as the sum

$$\widehat{\mathcal{H}} = \widehat{\mathcal{H}}_{\mathrm{f}} + \widehat{\mathcal{H}}_{\mathrm{a}} + \widehat{V} \,, \tag{6.87}$$

where $\widehat{\mathcal{H}}_{\mathrm{f}}$ is the Hamiltonian of the free field (6.21), $\widehat{\mathcal{H}}_{\mathrm{a}}$ is the Hamiltonian of a free atom, which can include the interaction with static electric and magnetic fields, and $\widehat{V}$ is the interaction part of the Hamiltonian between the atom and the quantized electromagnetic field and is given by

$$\widehat{V} = \sum_a \left\{ -\frac{e_a}{m_a c} \widehat{\mathbf{p}}_a \cdot \widehat{\mathbf{A}}(\mathbf{r}_a) - \boldsymbol{\mu}_a \cdot \widehat{\mathbf{H}}(\boldsymbol{r}_a) + \frac{e_a^2}{2 m_a c^2} \widehat{\mathbf{A}}^2(\boldsymbol{r}_a) \right\} \,. \tag{6.88}$$

Note that the first two terms are linear with respect to the field operator and the second has a quadratic dependence.

Example 6.10

Let the initial state of the system be represented by the density operator $\widehat{\rho}(t_0) = \widehat{\rho}_{\mathrm{a}}(t_0)\widehat{\rho}_{\mathrm{f}}(t_0)$, where subscripts a and f are associated with the atomic and field subsystems, respectively. Perform an expansion expressing the density operator dynamics for $t > t_0$ up to the second order of perturbation theory.

Solution. The density operator for the system obeys the following Liouville–Schrödinger equation (C42):

$$\frac{\partial \widehat{\rho}(t)}{\partial t} + \frac{i}{\hbar}[\widehat{\mathcal{H}}, \widehat{\rho}(t)] = 0 \,, \tag{1}$$

where $\widehat{\mathcal{H}}$ is the total Hamiltonian of the system (6.87). If $\widehat{\mathcal{H}}$ does not depend on time, the formal solution of the equation above is given by

$$\widehat{\rho}(t) = \exp\left\{ -\frac{i}{\hbar}\widehat{\mathcal{H}}(t - t_0) \right\} \widehat{\rho}(t_0) \exp\left\{ \frac{i}{\hbar}\widehat{\mathcal{H}}(t - t_0) \right\} \,. \tag{2}$$

But this solution is formal and normally useless for practical calculations, because of the difficulty to find the closed expression for such a general operator. It is more convenient to perform a sequence of iterations and introduce the approximate solution applicable for a finite time increment. Let us make the following substitution into (1):

$$\widehat{\rho}(t) = \exp\left\{ -\frac{i}{\hbar}\widehat{\mathcal{H}}_0(t - t_0) \right\} \widehat{\rho}_I(t) \exp\left\{ \frac{i}{\hbar}\widehat{\mathcal{H}}_0(t - t_0) \right\} \,, \tag{3}$$

where

$$\widehat{\mathcal{H}}_0 = \widehat{\mathcal{H}}_{\mathrm{f}} + \widehat{\mathcal{H}}_{\mathrm{a}} \tag{4}$$

is the free atom and field Hamiltonian *disregarding interaction* between them. The operator

$$\widehat{\rho}_I(t) = \exp\left\{\frac{i}{\hbar}\widehat{\mathcal{H}}_0(t-t_0)\right\}\widehat{\rho}(t)\exp\left\{-\frac{i}{\hbar}\widehat{\mathcal{H}}_0(t-t_0)\right\} \tag{6.89}$$

is the density operator in the *interaction representation*. The time dependence of $\widehat{\rho}_I$ only reveals itself as a result of the interaction process. If interaction vanishes $\widehat{V} = 0$, the density operator $\widehat{\rho}_I(t) = \widehat{\rho}(t_0)$ becomes independent of time. The interaction representation for state vectors is considered in Section 7.7.

As follows from (1) and (6.89), the density operator in the interaction representation obeys the following dynamic equation:

$$\frac{\mathrm{d}\widehat{\rho}_I(t)}{\mathrm{d}t} + \frac{i}{\hbar}\left[\widehat{V}_I(t), \widehat{\rho}_I(t)\right] = 0\,, \tag{5}$$

where

$$\widehat{V}_I(t) = \exp\left\{\frac{i}{\hbar}\widehat{\mathcal{H}}_0(t-t_0)\right\}\widehat{V}\exp\left\{-\frac{i}{\hbar}\widehat{\mathcal{H}}_0(t-t_0)\right\} \tag{6.90}$$

is the interaction Hamiltonian in the interaction representation. Then (5) can transformed to the integral form:

$$\widehat{\rho}_I(t) = \widehat{\rho}_I(t_0) - \frac{i}{\hbar}\int_{t_0}^{t}\left[\widehat{V}_I(\tau), \widehat{\rho}_I(\tau)\right]\mathrm{d}\tau\,. \tag{6}$$

As a first step to find the iterative solution of (6), we can substitute into the right side the zero-order approximation for the density operator at the initial moment of time: $\widehat{\rho}_I^{(0)}(t) = \widehat{\rho}_I(t_0) = \widehat{\rho}(t_0)$. Then its first-order correction is given by

$$\widehat{\rho}_I^{(1)}(t) = \widehat{\rho}(t_0) - \frac{i}{\hbar}\int_{t_0}^{t}\left[\widehat{V}_I(\tau), \widehat{\rho}(t_0)\right]\mathrm{d}\tau\,, \tag{7}$$

and in the second-order approximation we get

$$\widehat{\rho}_I^{(2)}(t) = \widehat{\rho}(t_0) - \frac{i}{\hbar}\int_{t_0}^{t}\left[\widehat{V}_I(\tau), \widehat{\rho}(t_0)\right]\mathrm{d}\tau$$
$$+\left(-\frac{i}{\hbar}\right)^2\int_{t_0}^{t}\mathrm{d}\tau_1\int_{t_0}^{\tau_1}\mathrm{d}\tau_2\left\{\widehat{V}_I(\tau_1), \left[\widehat{V}_I(\tau_2), \widehat{\rho}(t_0)\right]\right\}\,. \tag{8}$$

Let us define the transition probability of observing the system in a particular state $|f\rangle$ in terms of the stationary states for the atom and the field. The probability is given by (C39)

$$W(t) = \langle f|\widehat{\rho}(t)|f\rangle = \langle f|\widehat{\rho}_I(t)|f\rangle \,. \qquad (9)$$

The exponential factors created by the free Hamiltonian after its action on state $|f\rangle$ cancel each other. The iterative solution (8) can be substituted into (9). If state $|f\rangle$ is orthogonal to the initial state (e.g., they have different photon numbers), we obtain $\widehat{\rho}(t_0)|f\rangle = 0$, $\langle f|\widehat{\rho}(t_0) = 0$ and the first two terms in (8) and part of the contributions in the third term vanish. Eventually the probability is given by

$$W(t) = \frac{1}{\hbar^2}\int_{t_0}^{t} d\tau_1 \int_{t_0}^{\tau_1} d\tau_2 \langle f|\widehat{V}_I(\tau_1)\widehat{\rho}(t_0)\widehat{V}_I(\tau_2) + \widehat{V}_I(\tau_2)\widehat{\rho}(t_0)\widehat{V}_I(\tau_1)|f\rangle \,. \qquad (6.91)$$

Because of the Hermitian rule

$$\widehat{V}_I(\tau_2)\widehat{\rho}(t_0)\widehat{V}_I(\tau_1) = \left(\widehat{V}_I(\tau_1)\widehat{\rho}(t_0)\widehat{V}_I(\tau_2)\right)^\dagger ,$$

the two terms in (6.91) are mutually complex conjugated terms. □

6.2.6
Long-Wavelength Dipole Approximation

When a nonrelativistic system interacts with the field modes whose wavelengths exceed the size of the system, then the interaction Hamiltonian (6.58) can be transformed to a simpler form similar to the interaction energy (2.35) between the dipole and the static field (see, e.g., Cohen-Tannoudji *et al.*, 1992). Let us assume that the neutral charge system is small and localized near the origin of the reference frame. This lets us make the replacement $\widehat{\boldsymbol{A}}(\boldsymbol{r}) \to \widehat{\boldsymbol{A}}(0)$. Then, we can apply a unitary transformation expressed by the following operator:

$$\widehat{U} = \exp\left(-\frac{i}{\hbar c}\widehat{\mathbf{d}}\cdot\widehat{\mathbf{A}}(\mathbf{0})\right) = \exp\left[\sum_j \left(\widehat{\lambda}_j^\dagger \widehat{c}_j - \widehat{\lambda}_j \widehat{c}_j^\dagger\right)\right] , \qquad (6.92)$$

where $\widehat{\mathbf{d}}$ is the operator of the system's dipole moment

$$\widehat{\mathbf{d}} = \sum_a e_a \mathbf{r}_a \qquad (6.93)$$

(we changed the notation for the dipole moment to avoid possible confusion with the momentum operator). The combined index $j = \mathbf{k}\alpha$ indicates both the spatial mode and the polarization mode. Operator $\widehat{\lambda}_j$ is given by

$$\widehat{\lambda}_j = i\left(\frac{2\pi}{\mathcal{V}\hbar\omega_j}\right)^{1/2} \mathbf{e}_j^*\cdot\widehat{\mathbf{d}}\,. \tag{6.94}$$

Unitary transformation (6.92) over the particle and field canonical variables leads to

$$\begin{aligned}
\mathbf{r}'_a &= \widehat{U}\mathbf{r}_a\widehat{U}^\dagger = \mathbf{r}_a\,,\\
\widehat{\mathbf{p}}'_a &= \widehat{U}\widehat{\mathbf{p}}_a\widehat{U}^\dagger = \widehat{\mathbf{p}}_a + \frac{e_a}{c}\widehat{\mathbf{A}}_\perp(\mathbf{0})\,,\\
\widehat{c}'_j &= \widehat{U}\widehat{c}_j\widehat{U}^\dagger = \widehat{c}_j + \widehat{\lambda}_j\,,\\
\widehat{c}_j^{\dagger\prime} &= \widehat{U}\widehat{c}_j^\dagger\widehat{U}^\dagger = \widehat{c}_j^\dagger + \widehat{\lambda}_j^\dagger\,.
\end{aligned} \tag{6.95}$$

These transformations follow straightforwardly from the operator identity (C67).

It is instructive to follow this transformation over the field operators. For the vector potential we have

$$\begin{aligned}
\widehat{\boldsymbol{A}}'(\boldsymbol{r}) &= \widehat{U}\widehat{\boldsymbol{A}}(\boldsymbol{r})\widehat{U}^\dagger\\
&= \sum_j\left(\frac{2\pi\hbar c^2}{\omega_j\mathcal{V}}\right)^{1/2}\left[\mathbf{e}_j\left(\widehat{c}_j + \widehat{\lambda}_j\right)e^{i\mathbf{k}_j\cdot\mathbf{r}} + \mathbf{e}_j^*\left(\widehat{c}_j^\dagger + \widehat{\lambda}_j^\dagger\right)e^{-i\mathbf{k}_j\cdot\mathbf{r}}\right]\\
&= \widehat{\boldsymbol{A}}(\boldsymbol{r}) + i\sum_{\mathbf{k}}\frac{2\pi c}{\mathcal{V}\omega(k)k^2}\left[\boldsymbol{k}(\boldsymbol{k}\cdot\widehat{\boldsymbol{d}})e^{i\mathbf{k}\cdot\mathbf{r}} - \boldsymbol{k}(\boldsymbol{k}\cdot\widehat{\boldsymbol{d}})e^{-i\mathbf{k}\cdot\mathbf{r}}\right]\\
&= \widehat{\boldsymbol{A}}(\boldsymbol{r})\,.
\end{aligned} \tag{6.96}$$

To derive this we used (6.95) and simplified the sum of the polarization vectors (2.165) (and we also made the replacement $\boldsymbol{k} \to -\boldsymbol{k}$ in one of the sums over $\boldsymbol{k}$). We see that the vector potential operator does not change under the unitary transformation considered. The magnetic field operator does not change as well: $\widehat{\boldsymbol{H}}'(\boldsymbol{r}) = \widehat{\boldsymbol{H}}(\boldsymbol{r})$. But the electric field operator transforms to a different form, which has the following expansion in the set of canonical operators:

$$\begin{aligned}
\widehat{\mathbf{E}}'(\mathbf{r}) &= \widehat{U}\widehat{\mathbf{E}}_\perp(\mathbf{r})\widehat{U}^\dagger\\
&= \sum_j\left(\frac{2\pi\hbar\omega_j}{\mathcal{V}}\right)^{1/2}\left[i\mathbf{e}_j\left(\widehat{c}_j + \widehat{\lambda}_j\right)e^{i\mathbf{k}_j\cdot\mathbf{r}} - i\mathbf{e}_j^*\left(\widehat{c}_j^\dagger + \widehat{\lambda}_j^\dagger\right)e^{-i\mathbf{k}_j\cdot\mathbf{r}}\right]\\
&= \widehat{\mathbf{E}}(\mathbf{r}) - 4\pi\widehat{\mathbf{P}}(\mathbf{r})\,,
\end{aligned} \tag{6.97}$$

where

$$\boldsymbol{P}(\boldsymbol{r}) = \sum_j\frac{1}{\mathcal{V}}\boldsymbol{e}_j\left(\boldsymbol{e}_j^*\cdot\widehat{\boldsymbol{d}}\right)e^{i\mathbf{k}_j\cdot\mathbf{r}}\,. \tag{6.98}$$

The physical meaning of this quantity can be understood after we perform the sum in (6.98) over all polarizations and integrate it over the wave vectors. Using (2.165) and (2.164) (where for the current case the factor 2 should be omitted), we obtain

$$\widehat{\boldsymbol{P}}(\boldsymbol{r}) = \widehat{\boldsymbol{d}}\delta(\boldsymbol{r}) + \frac{3\boldsymbol{r}(\boldsymbol{r}\cdot\widehat{\boldsymbol{d}})}{4\pi r^5} - \frac{\widehat{\boldsymbol{d}}}{4\pi r^3} . \tag{6.99}$$

The first term on the right side is the vector operator of atomic polarization, which is associated with a point-like dipole located at the origin of the reference frame. The next two terms are the longitudinal electric field operator created by the dipole itself (see the dipole term in (2.21)). Thus, the above unitary transformation displaces the transverse radiation field by the field of the dipole and by its own polarization. When we approach the dipole such that $r \to 0$, all the terms become diverging.

Let us follow how the total Hamiltonian will be transformed. If we consider only the long-wavelength electric dipole approximation, we can disregard the spin Pauli term $\sum_a \widehat{\boldsymbol{\mu}}_a \cdot \widehat{\boldsymbol{H}}_a$ in the exact Hamiltonian. The operator $\widehat{\boldsymbol{A}}(\boldsymbol{r})$ can be replaced by $\widehat{\boldsymbol{A}}(0)$ but we keep the diamagnetic quadratic term in the Hamiltonian. The original Hamiltonian given by (6.21) and (6.55) is transformed as follows:

$$\begin{aligned}
\widehat{\mathcal{H}}' &= \widehat{U} \left\{ \sum_j \hbar\omega_j \left(\widehat{c}_j^\dagger \widehat{c}_j + \frac{1}{2} \right) \right. \\
&\quad \left. + \sum_{a=1}^{N} \left[\frac{1}{2m_a} \left(\widehat{\boldsymbol{p}}_a - \frac{e_a}{c} \widehat{\mathbf{A}}(0) \right)^2 + e_a \varphi(\boldsymbol{r}_a) \right] \right\} \widehat{U}^\dagger \\
&= \sum_j \hbar\omega_j \left(\widehat{c}_j^\dagger \widehat{c}_j + \frac{1}{2} \right) + \sum_a \left\{ \frac{\widehat{\mathbf{p}}_a^2}{2m_a} + e_a \varphi(\boldsymbol{r}_a) \right\} \\
&\quad - \widehat{\mathbf{d}} \sum_j \left(\frac{2\pi\hbar\omega_j}{\mathcal{V}} \right)^{1/2} \left[i\mathbf{e}_j \widehat{c}_j - i\mathbf{e}_j^* \widehat{c}_j^\dagger \right] + \sum_j \frac{2\pi}{\mathcal{V}} \left(\mathbf{e}_j^* \widehat{\mathbf{d}} \right) \left(\mathbf{e}_j \widehat{\mathbf{d}} \right) .
\end{aligned} \tag{6.100}$$

The first two sums describe the Hamiltonian of the free electromagnetic field and the atom. The remaining terms describe the interaction contributions. Applying relations (6.97) and (6.98), we can write the interaction part in the following form:

$$\begin{aligned}
\widehat{\mathcal{H}}_I' &= -\widehat{\boldsymbol{d}}\cdot\widehat{\boldsymbol{E}}(0) + 2\pi\widehat{\boldsymbol{d}}\cdot\boldsymbol{P}(0) \\
&= -\widehat{\boldsymbol{d}}\cdot\widehat{\boldsymbol{E}}(0) + 2\pi \left[\widehat{\boldsymbol{d}}^2 \delta(\boldsymbol{r}) + \frac{3(\boldsymbol{r}\cdot\widehat{\boldsymbol{d}})^2}{4\pi r^5} - \frac{\widehat{\boldsymbol{d}}^2}{4\pi r^3} \right]_{r\to 0} .
\end{aligned} \tag{6.101}$$

The first term is similar to the interaction energy between the static dipole and the external (displaced) field (see the dipole term in (2.35) and (6.97)). It consists of the field canonical variables and is responsible for the processes of photon emission and absorption. The second, diverging term is the self-action energy of the point-like dipole. This term is not linked to the transverse radiation field and appears here because disregarding the actual atomic size in the long-wavelength approximation used is not precisely correct. In calculations this term normally does not contribute to the observable quantities and can be canceled out. Finally, the interaction Hamiltonian between a particular atom located at arbitrary spatial point $\boldsymbol{R}$ is given by

$$\widehat{V} = -\widehat{\boldsymbol{d}}\cdot\widehat{\boldsymbol{E}}(\boldsymbol{R}) \tag{6.102}$$

(in the above expressions we simply assumed that $\boldsymbol{R} = 0$).

In applications the electromagnetic field is usually quasi-resonant to some particular optical transitions in an atom. For a selected transition the dipole moment operator can be expanded as the sum of two terms:

$$\widehat{\mathbf{d}} = \sum_{mn} (d_{nm}|n\rangle\langle m| + d_{mn}|m\rangle\langle n|) = \widehat{\mathbf{d}}_{+} + \widehat{\mathbf{d}}_{-} \,, \tag{6.103}$$

where we assume that $|m\rangle$ and $|n\rangle$ correspond to the lower and upper energy levels, respectively. Let ω_0 denote the frequency for this transition and let the atom be driven by a quasi-resonant field with frequency ω such that $|\omega - \omega_0| \ll \omega_0$. In these conditions, the main contribution to the perturbation series for the interaction part (6.102) comes from the resonance terms which have exponential factors oscillating as the difference of these two frequencies. If we restrict our consideration to such a resonance approximation, the interaction Hamiltonian for the atom driven by the field and located at the origin of the reference frame can be expressed as follows:

$$\widehat{V} \approx -\sum_j \left(\frac{2\pi\hbar\omega_j}{\mathcal{V}}\right)^{1/2} \left[i\left(\mathbf{e}_j\widehat{\mathbf{d}}_{+}\right)\widehat{c}_j - i\left(\mathbf{e}_j^{*}\widehat{\mathbf{d}}_{-}\right)\widehat{c}_j^{\dagger}\right] . \tag{6.104}$$

For arbitrary location $\mathbf{R}$, the interaction Hamiltonian is given by

$$\widehat{V} \approx -\sum_j \left(\frac{2\pi\hbar\omega_j}{\mathcal{V}}\right)^{1/2} \left[i\left(\mathbf{e}_j\widehat{\mathbf{d}}_{+}\right)\widehat{c}_j e^{i\mathbf{k}_j\mathbf{R}} - i\left(\mathbf{e}_j^{*}\widehat{\mathbf{d}}_{-}\right)\widehat{c}_j^{\dagger} e^{-i\mathbf{k}_j\mathbf{R}}\right] . \tag{6.105}$$

The approximation given by (6.104) and (6.105) is known as the *rotating wave approximation*. The main advantage of this interaction Hamiltonian is that even for virtual intermediate states subsequently generated by arbitrary orders of the evolutionary operator expansion, photon emission always follows from the atomic transition from the upper to the lower energy level and photon absorbtion follows from atomic excitation. The rotating wave approximation is a widely applied approach for practical calculations and is a very useful approximation with use beyond the perturbation theory approach.

Suggested literature:
Cohen-Tannoudji *et al.* (1992); Mandel and Wolf (1995); Kilin (1995); Perina (1987); Andreev *et al.* (1988); Klyshko (1996); Akhmanov and Nikitin (1998)

Example 6.11

Up to now we have considered the stationary atomic states with wave functions independent of time (see (C37)). However, the interaction between atomic electrons and the quantized electromagnetic field initiates radiative transitions from the upper to lower atomic energy levels and results in the finite lifetime of the excited states. As a consequence, the real wave function of an excited state is a superposition of states with different energies (quasi-stationary state). Photons emitted from such a quasi-stationary state should have a certain frequency distribution. Calculate the frequency distributions for the photons emitted by a two-level atom in a vacuum under the assumption that the lifetime of its excited state is much longer than the inverse frequency of the atomic transition (*spectral line shape*).

Solution. The wave function of the entire system (two-level atom and quantized field) can be represented as the following superposition:

$$|\psi(t)\rangle = \sum_s a_{1s}(t)|1\rangle|1_s\rangle \exp\left[i\left(\omega_s - \frac{\mathcal{E}}{\hbar}\right)t\right] + a_{20}(t)|2\rangle|\text{vac}\rangle \exp\left(-\frac{i}{\hbar}\mathcal{E}t\right) , \tag{1}$$

where we took into consideration only the vacuum and single-photon states of the field subsystem. At the initial time, the field exists in its vacuum state and the atom is in an excited state such that $a_{20}(0) = 1$, $a_{1s}(0) = 0$. In the final state, when $t \to \infty$, the atom decays to the lower energy level with emission of a photon at frequency ω. In equation (1), the exponential time dependencies correspond to the unperturbed wave functions of the atom and the field. By s we parameterize the mode in which the photon is emitted. The frequency distribution for the emitted quanta can be calculated with the aid of the following relation:

$$\mathrm{d}w(\omega) = \int \sum \frac{|a_{1s}(\infty)|^2 \mathcal{V}\omega^2 \mathrm{d}\omega \mathrm{d}\Omega}{(2\pi c)^3} , \tag{2}$$

where the integral is expanded over the photon's escape directions with the sum over its polarizations.

After substitution of (1) into the Schrödinger equation (C29), we get the system of equations for the expansion coefficients given above:

$$i\hbar \dot{a}_{1q} = \langle 1q|\widehat{V}|20\rangle a_{20}(t) \exp[i(\omega_q - \omega_0)t] , \tag{3}$$

$$i\hbar \dot{a}_{20} = \sum_s \langle 20|\widehat{V}|1s\rangle a_{1s}(t) \exp[i(\omega_0 - \omega_s)t] , \tag{4}$$

where ω_0 denotes the frequency of the atomic transition $(\mathcal{E}_2 - \mathcal{E}_1)/\hbar$. The approximate solution of the system given by (3) and (4) can be found with the substitution $a_{20}(t) = 1$ in the following form:

$$a_{20}(t) = e^{-\gamma t/2} , \tag{5}$$

where γ is an unknown decay constant. Then we can find the solution of equation (3) which satisfies the initial condition given above:

$$a_{1q}(t) = -\langle 1q|\widehat{V}|20\rangle \frac{\exp[i(\omega_q - \omega_0 + i\gamma/2)t] - 1}{\hbar(\omega_q - \omega_0 + i\gamma/2)} . \tag{6}$$

After substitution of this solution into (4), we arrive at the following transcendental equation for parameter γ:

$$-\frac{i\hbar\gamma}{2} = \sum_s \frac{|\langle 1s|\widehat{V}|20\rangle|^2}{\hbar(\omega_0 - \omega_s - i\gamma/2)} \left\{1 - \exp\left[i\left(\omega_0 - \omega_s - \frac{i\gamma}{2}\right)t\right]\right\} . \tag{7}$$

The approximate equation obtained is correct, if its right-hand side is independent of t at reasonable time intervals which are much longer than the oscillation period. This intuitive expectation can be justified by applying the iteration scheme to the solution of (7) under the assumption that γ is a small parameter. Thus, in the zero-order approximation we assume that on the right side $\gamma = 0$ (see, e.g. Heitler, 1954) and consider the right side at the limit of infinitely long time (see (1.216) and (1.224)):

$$\left.\frac{1 - \exp[i(\omega_0 - \omega_s)t]}{\omega_0 - \omega_s}\right|_{t\to\infty} \to \frac{\mathcal{P}}{\omega_0 - \omega_s} - i\pi\delta(\omega_0 - \omega_s) . \tag{8}$$

After substitution of (8) into (7) and after replacement of the sum over the field modes by integrals over the frequencies and radiation directions (which includes the sum over the photon's polarizations as well), we find that decay constant is actually a complex quantity, which has real and imaginary parts $\gamma = \gamma' + i\gamma''$. Its real part is given by

$$\gamma' = \sum_\sigma \mathcal{P} \int \frac{2\pi}{\hbar^2} \left|\langle 1s|\widehat{V}|20\rangle\right|^2 \delta(\omega_0 - \omega) \frac{\mathcal{V}\omega^2 \mathrm{d}\omega \mathrm{d}\Omega}{(2\pi c)^3} . \tag{9}$$

This part corresponds to the rate of spontaneous emission (see (6.59) and the discussion concerning it). For typical dipole-allowed atomic transitions $\gamma' \approx 10^8 - 10^9\ \mathrm{s}^{-1}$ (see, e.g., Problem 6.50•), whereas the optical transition frequencies are on the order of $10^{16}\ \mathrm{s}^{-1}$. Therefore, the condition $\gamma' \ll \omega_0$ is perfectly fulfilled.

The imaginary part of the decay constant expresses the energy shift of the atomic level induced by interaction of the radiation with the vacuum field (Lamb shift, or light shift; see Problem 6.18•*):

$$\gamma'' = \frac{1}{4\pi^3 c^3 \hbar^2} \sum_\sigma \mathcal{P} \int \left|\langle 1s|\widehat{V}|20\rangle\right|^2 \frac{\mathcal{V}\omega^2 \mathrm{d}\omega \mathrm{d}\Omega}{\omega_0 - \omega} . \tag{10}$$

The integral over frequencies in (10) is nonconverging and should be incorporated into the physically observed value of the energy of the upper state of the atom. To perform the correct calculation of the light shift we have to follow special approaches (see Akhiezer and Berestetskii, 1981; Berestetskii *et al.*, 1982; Bogolubov and Shirkov, 1980).

The frequency distribution (spectral line shape) for the emitted photon can be found with the aid of equations (2) and (6):

$$dw(\omega) = \sum \int \frac{\left|\langle 1s|\widehat{V}|20\rangle\right|^2}{\hbar^2[(\omega-\omega_0)^2+\gamma^2/4]} \frac{\mathcal{V}\omega^2 d\omega d\Omega}{(2\pi c)^3} , \tag{11}$$

Here we incorporated the light shift into the physical energy of the excited atom such that $\gamma \approx \gamma'$. Since the spectral line width is small, we can replace ω by ω_0 in (11) (except for in the denominator). Comparing the right side of equations (11) and (9), we obtain

$$dw(\omega) = \frac{\gamma}{2\pi} \frac{d\omega}{(\omega-\omega_0)^2+\gamma^2/4} . \tag{12}$$

As in the classical theory, the spectral profile has a Lorentz-type shape (see Problem 5.122*, (6)), but its line width γ is now determined by the probability of quantum transition (decay rate) between the upper and lower atomic states. □

Problems

6.58•. Derive the operators of the free electromagnetic field in the interaction representation, which is defined by (6.89) and (6.90).

6.59•. Derive the operator of the interaction Hamiltonian between an atomic dipole and the quantized field (6.102) in the interaction representation.

6.60•. Express the wave function $\Psi_I(t)$ in the interaction representation via wave function $\Psi_S(t)$ in the Schrödinger representation and introduce the Schrödinger equation for $\Psi_I(t)$.

6.61. The oscillator strength for atomic electrons defines the following quantity

$$f_{ln} = \frac{2m}{\hbar e^2}\omega_{ln}|(d_z)_{ln}|^2 .$$

Here, l and n are the quantum numbers which parameterize the states of the atomic subsystem, ω_{ln} is the transition frequency, and $(d_z)_{ln}$ is the z Cartesian component of the matrix element of the dipole moment. Verify the following sum rule: $\sum_l f_{ln} = N$. Here, N is the total number of electrons in the system and the sum is extended over the complete set of atomic states.

6.62. Let the system be in its ground state and an external electric static field be applied to it. Find the polarizability $\alpha_{\mu\nu}$ of the system and express it by the matrix elements of its dipole components. The polarizability is defined as a tensor linking the vector of the dipole-type polarization induced by the external field with the vector components of this field.

6.63. Let a two-level atomic system interact with a stochastic stationary electromagnetic field (thermal field). The field is described by its diagonal density matrix components ρ_{nn} in the Fock representation, which are independent of time. Consider an atomic system interacting with such a field in the dipole approximation and find the transition rate, that is, the transition probability per unit time.

6.64*. Calculate the differential cross-section for light scattering on an atom. Consider only a nonresonant interaction process, when the frequencies of the incident and scattered light do not coincide with the transition frequencies between the upper and lower atomic levels.[13)]

6.65*. Perform a similar calculation for the differential cross-section for light scattering on an atom using the interaction Hamiltonian in the dipole approximation (see (6.102)). Compare the result obtained with the solution of the previous problem and show that both the approaches give the same expression for the cross-section.

6.66. Consider (2) given in the solution of the previous problem in the following limits:

1. Let the scattered photon have the same frequency as the incident photon (Rayleigh scattering) and assume that $\omega_2 = \omega_1 = \omega \ll \omega_{li}$ for all intermediate states. Show that in this case the result obtained can be approximately described by the formulas derived in Problem 5.127* for the scattering of light by a classical oscillator.
2. Verify that the cross-section of light scattering is transformed to the Thomson formula (see Problems 5.127* and 5.133) when the energy of incident photons is greater than the binding energy of electrons (but much smaller than the electron self-energy mc^2).

6.67*. In the first nonvanishing order of perturbation theory calculate the probability of two-photon absorption and emission of light by atoms. Express the result in terms of the transition matrix elements and analyze the selection rules for these processes. Perform the calculations in the framework of the dipole approximation.

6.68*. Calculate the probability of the parametric light generation – a process where annihilation of a single photon in mode $\{\omega_1, \boldsymbol{k}_1, \boldsymbol{e}_1\}$ results in the creation of a photon pair in two modes: $\{\omega_2, \boldsymbol{k}_2, \boldsymbol{e}_2\}$ and $\{\omega_3, \boldsymbol{k}_3, \boldsymbol{e}_3\}$. The frequencies of these three quanta obey the phase-matching condition $\omega_1 = \omega_2 + \omega_3$, and the energy of the initial and final states of the scattering medium is unchanged in this process (coherent process).

6.69. Let the quantum state for two closely located "two-level" atoms be described by the following wave function: $\Psi_1 = (|1\rangle_a |2\rangle_b + |2\rangle_a |1\rangle_b)/\sqrt{2}$. Calculate the probability of photon emission on the basis of the perturbation theory approach. Show

13) In this and the following problems we talk about atoms. But in some cases, the expressions derived can also be applied to more complicated system, such as molecules or clusters.

that the spontaneous emission rate for such a system is twice as great as that for each atom considered independently *(superradiance phenomenon)*. Perform a similar calculation for the state $\Psi_2 = (|1\rangle_a|2\rangle_b - |2\rangle_a|1\rangle_b)/\sqrt{2}$ and compare the results.

6.70•. Consider the cooperative radiation by a classical system consisting of harmonic oscillators (see Problem 5.122*). Let N identical oscillators with eigenfrequency !$_0$ and decay constant $\gamma = 2e^2\omega_0^2/3mc^2$ be placed in a small spatial volume and let the volume size be much smaller than the emission wavelength λ. Since the oscillators are physically identical, radiation damping for any oscillator will be induced not only by its own radiation, but also by radiation produced by other partners. Therefore, the equation of motion for the ath specific oscillator is given by

$$\ddot{\boldsymbol{r}}_a + 2\gamma\dot{\boldsymbol{r}}_a + \omega_0^2\boldsymbol{r}_a = -\sum_{b\neq a} 2\gamma\dot{\boldsymbol{r}}_b\ ,$$

where $a, b = 1, 2, \ldots, N$. Let all the oscillators have equal initial amplitudes $\boldsymbol{r}_0$ and phase at the initial moment of time. Calculate the emission spectral profile, the amplitude of the electromagnetic pulse, and the decay constant. Compare the results with the corresponding quantities calculated for a system of independent oscillators.

6.3 Interaction between Relativistic Particles

In this section we will consider a few simple examples and problems concerning relativistic particles. We will illustrate the basic physical ideas in the simplest way, without the use of the rigorous theoretical scheme of quantum electrodynamics in the relativistic covariant formulation. One can find a more detailed treatment in Chapter 7 and in Akhiezer and Berestetskii (1981), Berestetskii *et al.* (1982), and Bogolubov and Shirkov (1982).

6.3.1 The Relativistic Dirac Equation for Fermions

The Schrödinger equation (C29) with the Hamiltonian

$$\widehat{\mathcal{H}} = \frac{\widehat{\boldsymbol{p}}^2}{2m} + U(\boldsymbol{r}, t) \tag{6.106}$$

is not applicable for the description of relativistic particles, since it leads to a nonrelativistic relation between the energy and momentum of a free particle: assuming $U = 0$ and substituting $\Psi(\boldsymbol{r}, t) = A\exp[i(\boldsymbol{p}\cdot\boldsymbol{r} - \mathcal{E}t)/\hbar]$ into (C29), we find

$$\mathcal{E} = \frac{\boldsymbol{p}^2}{2m}\ . \tag{6.107}$$

Besides, the single-component wave function $\Psi(\boldsymbol{r}, t)$ by no means reflects the presence of the spin degree if freedom of a particle. For a description of relativistic electrons it is necessary to consider the relativistic coupling of the particle energy $\mathcal{E}$ with its three-dimensional momentum $\boldsymbol{p}$,

$$\mathcal{E} = \sqrt{m^2 c^4 + c^2 p^2}\,, \tag{6.108}$$

and include in the theory the description of the internal moment (spin). The relativistic wave equation for particles with spin $s = 1/2$ was derived by Dirac in 1928. It can be written in the form of the Schrödinger equation (C29):

$$i\hbar \frac{\partial \Psi}{\partial t} = \widehat{\mathcal{H}} \Psi\,, \tag{6.109}$$

in which the wave function

$$\Psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}, \quad \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \quad \chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \tag{6.110}$$

is a four-component parameter, and the Dirac Hamiltonian of a free particle

$$\widehat{\mathcal{H}} = c(\widehat{\boldsymbol{\alpha}} \cdot \widehat{\boldsymbol{p}}) + \widehat{\beta} m c^2 \tag{6.111}$$

contains the 4×4 Dirac matrices

$$\widehat{\boldsymbol{\alpha}} = \begin{pmatrix} 0 & \widehat{\boldsymbol{\sigma}} \\ \widehat{\boldsymbol{\sigma}} & 0 \end{pmatrix}, \quad \widehat{\beta} = \begin{pmatrix} \widehat{1} & 0 \\ 0 & -\widehat{1} \end{pmatrix}. \tag{6.112}$$

Here we used the 2×2 form, a shorthand form where each element in the (6.112) 2×2 matrices represent the 2×2 Pauli matrix (C79). As a result, matrices $\widehat{\alpha}$ and $\widehat{\beta}$ are 4×4. A similar form is used for the four-component wave function (bispinor). The value m in (6.111) represents the particle mass, and the operator $\widehat{\boldsymbol{p}} = -i\hbar\nabla$ corresponds to its three-dimensional momentum. Pauli matrices $\widehat{\boldsymbol{\sigma}}$ and $\widehat{1}$ consist of the total set of 2×2 Hermitian matrices which may be used for representation of the arbitrary 2×2 matrix. The Dirac matrices in the 4×4 form are

$$\widehat{\alpha}_x = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \widehat{\alpha}_y = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix},$$

$$\widehat{\alpha}_z = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \widehat{\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{6.113}$$

All Dirac matrices are Hermitian and satisfy the anticommutation relations

$$\begin{aligned} &\{\widehat{\alpha}_\mu, \widehat{\alpha}_\nu\} \equiv \widehat{\alpha}_\mu \widehat{\alpha}_\nu + \widehat{\alpha}_\nu \widehat{\alpha}_\mu = 2\delta_{\mu\nu}\widehat{1}\,, \\ &\quad \{\widehat{\boldsymbol{\alpha}}, \widehat{\beta}\} = 0\,, \quad \widehat{\beta}^2 = \widehat{1}\,, \quad \mu, \nu = x, y, z\,. \end{aligned} \tag{6.114}$$

Equations (6.109) and (6.111)–(6.114) are only one of the possible forms of the Dirac equation for a free particle ("standard representation"). Other forms of the equation can be obtained by unitary transformation of the wave function and Dirac matrices.

Example 6.12

Let the solution of the Dirac equation for a free particle describe the state with momentum $\boldsymbol{p}$ and energy $\mathcal{E}$. These values exist simultaneously since their operators commute. Find the dependence $\mathcal{E}(p)$ following from the equation.

Solution. For the states with $\boldsymbol{p}$ and $\mathcal{E}$,

$$\widehat{\boldsymbol{p}}\Psi_p = \boldsymbol{p}\Psi_p\,, \quad \widehat{\mathcal{H}}\Psi_p = \mathcal{E}\Psi_p\,,$$

where $\widehat{\mathcal{H}}$ is given by formula (6.111), and Ψ_p is the four-component wave function (bispinor). From the equalities we have

$$[c(\widehat{\boldsymbol{\alpha}}\cdot\boldsymbol{p}) + \widehat{\beta}mc^2]\Psi_p = \mathcal{E}\Psi_p\,. \tag{1}$$

Multiplying both parts of the last equality by $[c(\widehat{\boldsymbol{\alpha}}\cdot\boldsymbol{p}) + \widehat{\beta}mc^2]$ and using equalities (6.114), we obtain the dependence sought:

$$c^2p^2 + m^2c^4 = \mathcal{E}^2\,. \tag{2}$$

From this equality it follows that for each absolute value of the momentum there are two values of the particle energy:

$$\mathcal{E}(p) = \pm\sqrt{c^2p^2 + m^2c^4}\,. \tag{3}$$

Unlike in classical relativistic mechanics, in a given case it is impossible simply to exclude solutions with negative energy as having no physical sense, since without these solutions the system of Dirac Hamiltonian wave functions will lose the property of completeness. Without this property the whole traditional probabilistic interpretation of the wave function and other quantum mechanical values loses its meaning. Along with this, it is evident that the experimental data do not contain any indications, even in the quantum region, of the existence of particles with negative energy. In this case, in particular, spontaneous transitions from states with positive energy to lower states with negative energy would be allowed. The first interpretation of electrons with negative energy belongs to Dirac. He suggested that in the ground state of the electron system all single-electron states with negative energy are filled and the states with positive energy are free. According to the Pauli exclusion principle, transitions to the filled states are forbidden. Then it should be suggested also that the filled negative "background" neither creates the electric field nor contributes to the energy and momentum of the system.

If the vacuum state of the whole system is electrically neutral, the unfilled state ("hole") in the Dirac background will be perceived by an observer as a particle with positive energy and charge opposite in sign to that of the electron. One can interpret the absorption of a photon with energy $\hbar\omega > 2mc^2$ by an electron with negative energy and the transition of an electron into the free state with $\mathcal{E} > mc^2$ as the production of two particles with equal positive masses $m > 0$ and charges $\pm e$ of opposite sign. Thus, the Dirac theory and the necessity of interpreting of the states with negative energy inspired the prediction of the existence of *antiparticles*, which are particles with positive energy but opposite in sign (with respect to "particles") electric charges and spin magnetic moments. The first antiparticle, discovered experimentally by Anderson in cosmic rays in 1932, was a positive electron, that is, a positron. Then a muon and a pion with charge of both signs followed, and in the 1950s an antiproton and an antineutron. Today the fact that for every kind of particle there is an antiparticle is beyond question (in some cases electrically neutral particles without a magnetic moment are their own antiparticles; an example is the photon). Nowadays the Dirac "hole" interpretation has historical significance but the Dirac theory and his equation were multiply confirmed and are the basis for relativistic quantum mechanics. □

6.3.2
The Klein–Gordon–Fock Equation

Let us return to the relativistic dependence of the energy on the momentum:

$$c^2 p^2 + m^2 c^4 = \mathcal{E}^2 . \tag{6.115}$$

If we consider this dependence as a consequence of substitution of a de Broglie plane wave $\varphi = A \exp\{i/\hbar(\boldsymbol{p}\cdot\boldsymbol{r} - \mathcal{E}t)\}$ of a relativistic particle in some wave equation, it is simple on the basis of relation (6.115) to construct this equation:

$$\Box\varphi - \frac{m^2c^2}{\hbar^2}\varphi = 0 , \quad \text{where} \quad \Box = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \tag{6.116}$$

is the relativistic invariant d'Alembert operator.

Equation (6.116) is the Klein–Gordon–Fock equation[14] and was obtained in the solution of Problem 4.125* of the corresponding Lagrangian. It contains the second derivatives of the coordinates and time. Relation (6.115) is used for any relativistic particle; therefore, any relativistic wave function of a free particle with a certain energy should satisfy (6.116) irrespective of the state of its internal degrees of freedom. Satisfies to it and Dirac's bispinor, which for a free particle is proportional to $\exp\{(i/\hbar)(\boldsymbol{p}\cdot\boldsymbol{r} - \mathcal{E}t)\}$. The Dirac equation can be considered as a replacement of the equation of second order with a system of equations of the first order for the four-component wave function. Such a replacement has allowed numerous physical phenomena to be described by means of the Dirac relativistic equation. It has

14) Oskar Benjamin Klein (1894–1977) was a Swedish theoretical physicist, whose main work was in quantum mechanics and quantum field theory. Walter Gordon (1893–1939) was a German theoretical physicist who formulated the relativistic quantum equation independently of Klein and Fock.

allowed us to include correctly in the equation the spin degree of freedom, and also to come to the rather general concept of an antiparticle.

6.3.3 The Analysis of the Dirac Equation

If a relativistic fermion is in the external electromagnetic field, then the field can be taken into consideration, as in the nonrelativistic case, by the replacement $\widehat{\boldsymbol{p}} \to \widehat{\boldsymbol{p}} - (e/c)\widehat{\boldsymbol{A}}$ in (6.111) and by addition of the term $e\widehat{\varphi}$ to the Hamiltonian, where $\widehat{\varphi}$ is the operator of the scalar potential (the operator of multiplication by the coordinate function in the coordinate representation). Finally, the Hamiltonian takes the form

$$\widehat{\mathcal{H}} = c\widehat{\boldsymbol{\alpha}} \cdot \left(\widehat{\boldsymbol{p}} - \frac{e}{c}\widehat{\mathrm{A}}\right) + \widehat{\beta} mc^2 + e\varphi(\boldsymbol{r}) \,. \tag{6.117}$$

Operator $\widehat{\boldsymbol{A}}$ can include both the classical external magnetic field and the quantized transverse vector potential describing the production and annihilation of photons.

Example 6.13

Derive from the Dirac equation with Hamiltonian (6.117) the equation of continuity for the probability density and probability current density of a relativistic particle. Construct the probability density of a given value of the particle coordinates as a nonnegative value $\rho(\boldsymbol{r}, t) \geq 0$, satisfying the probability conservation law $\int \rho(\boldsymbol{r}, t) \mathrm{d}^3 r = \text{const}$, where the integration is extended over the whole three-dimensional space.

Solution. Write down the Dirac equation for Ψ and the Hermitian conjugated wave function $\Psi^\dagger = (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*)$,

$$i\hbar \frac{\partial \Psi}{\partial t} = \left\{c\widehat{\boldsymbol{\alpha}} \cdot \left(\widehat{\boldsymbol{p}} - \frac{e}{c}\widehat{\mathrm{A}}\right) + \widehat{\beta} mc^2 + e\varphi(\boldsymbol{r})\right\} \Psi , \tag{6.118}$$

$$-i\hbar \frac{\partial \Psi}{\partial t} = \Psi^\dagger \left\{-c\widehat{\boldsymbol{\alpha}} \cdot \left(\widehat{\boldsymbol{p}} + \frac{e}{c}\widehat{\mathrm{A}}\right) + \widehat{\beta} mc^2 + e\varphi(\boldsymbol{r})\right\} \,, \tag{6.119}$$

where in the second equation the differential operator $\widehat{\boldsymbol{p}}$ acts on the function $\Psi^\dagger$ to the left of it. Compose the bilinear value

$$\rho(\boldsymbol{r}, t) = \sum_{\lambda=1}^{4} |\psi_\lambda|^2 = \Psi^\dagger(\boldsymbol{r}, t)\Psi(\boldsymbol{r}, t) \tag{6.120}$$

and differentiate it over time using relations (6.118) and (6.119). We obtain the equation of continuity

$$\frac{\partial \rho}{\partial t} = \frac{\partial \Psi^\dagger}{\partial t}\Psi + \Psi^\dagger \frac{\partial \Psi}{\partial t} = \frac{c}{i\hbar}[(\widehat{\boldsymbol{p}}\Psi^\dagger)\cdot\widehat{\boldsymbol{\alpha}}\Psi + \Psi^\dagger\widehat{\boldsymbol{\alpha}}\cdot\widehat{\boldsymbol{p}}\Psi] = -\nabla\cdot\boldsymbol{j} \,,$$

where the probability current density

$$j(r, t) = c\Psi^\dagger(r, t)\hat{\alpha}\Psi(r, t) \tag{6.121}$$

is a real vector. The total probability is conserved, $\int \Psi^\dagger \Psi d^3 r = \text{const}$, if the current density decreases faster than r^{-2}. The electric current density is expressed as the product of the particle charge and probability current density: $j_e(r, t) = e j(r, t)$. □

Example 6.14

Perform the limiting transition from the Dirac equation in an external field with Hamiltonian (6.117) to the Pauli equation in which the first relativistic correction of the $1/c$ term is taken into account.

Solution. To exclude the rest energy of a particle from the equation of motion, we substitute

$$\Psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix} e^{-imc^2 t/\hbar} ,$$

where $\varphi(r, t)$ and $\chi(r, t)$ are the two-component functions (spinors). Substituting Ψ in equation (6.109) with Hamiltonian (6.117) and using representation (6.112) of Dirac matrices, we obtain the system of equations

$$i\hbar\frac{\partial\varphi}{\partial t} - U\varphi = c\hat{\sigma}\cdot\left(\hat{p} - \frac{e}{c}A\right)\chi ,$$
$$i\hbar\frac{\partial\chi}{\partial t} + (2mc^2 - U)\chi = c\hat{\sigma}\cdot\left(\hat{p} - \frac{e}{c}A\right)\varphi . \tag{1}$$

Here we consider $A(r, t)$ as an ordinary function of the coordinates and time (the multiplying operator), and the scalar potential of the electric field is included in the potential energy U.

In the nonrelativistic case we have the inequality $|U| \ll mc^2$. Besides, the operator $i\hbar\partial/\partial t$, having been applied to the wave functions φ and χ, gives a factor of order $|\mathcal{E} - mc^2| \ll mc^2$. Therefore, in the left side of the second equation in (1), we can leave the largest term $2mc^2\chi$ and obtain

$$\chi = \frac{1}{2mc}\hat{\sigma}\cdot\left(\hat{p} - \frac{e}{c}A\right)\varphi , \tag{2}$$

which gives $|\chi/\varphi| \approx p/mc \ll 1$. Substituting (2) into (1) and transforming the square of the operator, we obtain

$$\left[\hat{\sigma}\cdot\left(\hat{p} - \frac{e}{c}A\right)\right]^2 = \hat{\sigma}_\mu\hat{\sigma}_\nu\left(\hat{p}_\mu - \frac{e}{c}A_\mu\right)\left(\hat{p}_\nu - \frac{e}{c}A_\nu\right)$$
$$= (\delta_{\mu\nu} + ie_{\mu\nu\kappa}\hat{\sigma}_\kappa)\left(\hat{p}_\mu - \frac{e}{c}A_\mu\right)\left(\hat{p}_\nu - \frac{e}{c}A_\nu\right)$$
$$= \left(\hat{p} - \frac{e}{c}A\right)^2 - \frac{e\hbar}{c}\hat{\sigma}\cdot H . \tag{3}$$

Here we used the relations $\boldsymbol{H} = \nabla \times \boldsymbol{A}$ and (C81) for the 2×2 *Pauli matrices*.

On substitution of (3) into (1), we obtain the Pauli equation:

$$i\hbar\frac{\partial\varphi}{\partial t} = \widehat{\mathcal{H}}\varphi \ , \quad \widehat{\mathcal{H}} = \frac{1}{2m}\left(\widehat{\boldsymbol{p}} - \frac{e}{c}\boldsymbol{A}\right)^2 + U - \frac{e\hbar}{2mc}\widehat{\boldsymbol{\sigma}}\cdot\boldsymbol{H} \ . \tag{6.122}$$

The term $-(e\hbar/2mc)\widehat{\boldsymbol{\sigma}}\cdot\boldsymbol{H} = -\widehat{\boldsymbol{\mu}}\cdot\boldsymbol{H}$ describes the interaction of the spin magnetic moment of an electron with an external magnetic field. □

Example 6.15

On the basis of the results of the preceding two examples, write down, in the Pauli approximation, the probability density and the probability current density via the spinor $\varphi(\boldsymbol{r}, t)$.

Solution. Using (6.120), we find

$$\rho(\boldsymbol{r}, t) = \sum_{\lambda=1}^{2} |\varphi_\lambda|^2 = \varphi^\dagger(\boldsymbol{r}, t)\varphi(\boldsymbol{r}, t) \ , \tag{6.123}$$

since the small spinor χ gives the correction of order $1/c^2$. In the same approximation we have

$$\boldsymbol{j}(\boldsymbol{r}, t) = c(\varphi^\dagger\widehat{\boldsymbol{\sigma}}\chi + \chi^\dagger\widehat{\boldsymbol{\sigma}}\varphi) \ . \tag{1}$$

Let us substitute into (1) the expression for χ in the form of (2) from the preceding example, as well as the relation

$$\chi = \frac{1}{2mc}\left(i\hbar\nabla - \frac{e}{c}\boldsymbol{A}\right)\varphi^\dagger\widehat{\boldsymbol{\sigma}} \ . \tag{2}$$

Reducing the similar terms with the use of (6.121), we find

$$\boldsymbol{j}(\boldsymbol{r}, t) = -\frac{i\hbar}{2m}(\varphi^\dagger\nabla\varphi - (\nabla\varphi^\dagger)\varphi) - \frac{e}{mc}\boldsymbol{A}\varphi^\dagger\varphi + \frac{\hbar}{2m}\nabla \times (\varphi^\dagger\widehat{\boldsymbol{\sigma}}\varphi) \ , \tag{6.124}$$

which is in accord with the result obtained in Problem 4.130 (but in that problem the spin was not taken into account and therefore the spin term is absent). □

Example 6.16

Let the motion of a relativistic particle be described by the Dirac equation. Verify that in the potential-center-symmetry field, the orbital moment of a particle $\boldsymbol{r} \times \boldsymbol{p}$ with respect to the field center is not conserved. Prove also that the physical value $\boldsymbol{J}$, the operator of which is expressed by

$$\widehat{\boldsymbol{J}} = -i\boldsymbol{r} \times \nabla + \frac{1}{2}\widehat{\boldsymbol{\Sigma}} \ , \quad \widehat{\boldsymbol{\Sigma}} = -\frac{i}{2}\widehat{\boldsymbol{\alpha}} \times \widehat{\boldsymbol{\alpha}} = \begin{pmatrix} \widehat{\boldsymbol{\sigma}} & 0 \\ 0 & \widehat{\boldsymbol{\sigma}} \end{pmatrix} \ , \tag{6.125}$$

is the integral of motion and, hence, it could be interpreted as the total angular momentum, which is conserved because of the isotropy of the space. Thus, the operator $(1/2)\widehat{\boldsymbol{\Sigma}}$ can be associated with the internal angular momentum (spin) of a particle.

Solution. Both of the operators $\widehat{\boldsymbol{l}} = -i\boldsymbol{r} \times \nabla$ and $\widehat{\boldsymbol{\Sigma}}$ do not depend on time in the Schrödinger representation; therefore the conservation condition of the corresponding physical values is their commutativity with the Dirac Hamiltonian:

$$\widehat{\mathcal{H}} = c(\widehat{\boldsymbol{\alpha}}\cdot\widehat{\boldsymbol{p}}) + \widehat{\beta} mc^2 + U(r) \,. \tag{1}$$

Let us calculate the commutators. They commute with both U and the diagonal matrix $\widehat{\beta}$; therefore,

$$[\widehat{\mathcal{H}}, \widehat{l}_\mu] = \frac{c}{\hbar} e_{\mu\nu\kappa}\widehat{\alpha}_\lambda(\widehat{p}_\lambda x_\nu - x_\nu \widehat{p}_\lambda)\widehat{p}_\kappa = -ice_{\mu\lambda\kappa}\widehat{\alpha}_\lambda \widehat{p}_\kappa = -ic[\widehat{\boldsymbol{\alpha}}\times\widehat{\boldsymbol{p}}]_\mu \neq 0 \,, \tag{2}$$

where the Heisenberg permutation relations (C23) have been used. Thus, the orbital moment is not conserved.

Further, using (6.125) for $\widehat{\boldsymbol{\Sigma}}$, via the matrices $\widehat{\boldsymbol{\alpha}}$, we find

$$\begin{aligned}\frac{1}{2}\left[\widehat{\mathcal{H}}, \widehat{\Sigma}_\mu\right] &= \frac{ic}{4}\widehat{p}_\lambda e_{\mu\nu\kappa}\left(\widehat{\alpha}_\nu\widehat{\alpha}_\kappa\widehat{\alpha}_\lambda - \widehat{\alpha}_\lambda\widehat{\alpha}_\nu\widehat{\alpha}_\kappa\right)\\ &= \frac{ic}{2}\widehat{p}_\lambda e_{\mu\nu\kappa}\left(\widehat{\alpha}_\nu\delta_{\kappa\lambda} - \widehat{\alpha}_\kappa\delta_{\lambda\nu}\right) = ic\left[\widehat{\boldsymbol{\alpha}}\times\widehat{\boldsymbol{p}}\right]_\mu \neq 0 \,. \end{aligned} \tag{3}$$

The spin of a relativistic particle in the central field is also not conserved. But the operator of the total moment $\widehat{\boldsymbol{J}} = \widehat{\boldsymbol{l}} + (1/2)\widehat{\boldsymbol{\Sigma}}$ commutes with the Hamiltonian, and the total moment is the integral of motion (although its projections do not simultaneously have the values determined). □

Example 6.17

Apply the limiting process from the Dirac equation to the weak relativistic case taking account of quadratic relativistic corrections of order $[(\mathcal{E} - mc^2)/mc^2]^2 = (E/mc^2)^2$, where $E = \mathcal{E} - mc^2$ is the nonrelativistic electron energy. An electron moves in the center-symmetry field $U(r)$, independent of time.

Solution. It is convenient to proceed from the stationary Dirac equation, assuming $i\hbar\partial\Psi/\partial t = (E+mc^2)\Psi$. With the two-component spinors φ and χ, we obtain from the Dirac equation the system of equations (compare them with Example 6.14)

$$(E - U(r))\varphi = c(\widehat{\sigma}\cdot\widehat{\boldsymbol{p}})\chi \,, \tag{1}$$

$$(E - U(r) + 2mc^2)\chi = c(\widehat{\sigma}\cdot\widehat{\boldsymbol{p}})\varphi \,, \tag{2}$$

We will calculate the small spinor (in the nonrelativistic approximation) χ taking into account two terms of the decomposition:

$$\chi = \frac{(\widehat{\sigma} \cdot \widehat{\boldsymbol{p}})\varphi}{2mc} - \frac{(E - U)(\widehat{\sigma} \cdot \widehat{\boldsymbol{p}})\varphi}{4m^2c^2} . \tag{3}$$

In the second correction term we will use the equations without relativistic corrections, $E\varphi = U\varphi + \widehat{\boldsymbol{p}}^2\varphi/2m$, and obtain the relation

$$(E - U)(\widehat{\boldsymbol{\sigma}} \cdot \widehat{\boldsymbol{p}})\varphi = \frac{1}{2m}(\widehat{\boldsymbol{\sigma}} \cdot \widehat{\boldsymbol{p}})\varphi + [(\widehat{\sigma} \cdot \widehat{\boldsymbol{p}})\, U]\varphi . \tag{4}$$

Substituting the transformed spinor χ into (1), we find the equation for the spinor φ:

$$(E - U)\varphi = \frac{\widehat{\boldsymbol{p}}^2\varphi}{2m} - \frac{\widehat{\boldsymbol{p}}^4\varphi}{8m^3c^2} - \frac{[\widehat{\boldsymbol{p}}^2 U]}{4m^2c^2}\varphi - \frac{1}{4m^2c^2}\sigma_\nu\sigma_\mu\left(\widehat{p}_\mu U\right)\left(\widehat{p}_\nu\varphi\right) . \tag{5}$$

Here we used the identity $(\widehat{\boldsymbol{\sigma}} \cdot \widehat{\boldsymbol{p}})^2 = \widehat{\boldsymbol{p}}^2$. Further, we take into account the spherical symmetry of the function, which permits us to write

$$\widehat{\boldsymbol{p}}\, U = -i\hbar\nabla U = -\frac{i\hbar}{r}\frac{\mathrm{d}U}{\mathrm{d}r}\boldsymbol{r} ,$$

and use the notation $\hbar\widehat{\boldsymbol{l}} = \boldsymbol{r} \times \widehat{\boldsymbol{p}}$. With a help of identities (6.121), we reduce (5) to the form of the Schrödinger equation:

$$E\varphi = \widehat{\mathcal{H}}'\varphi , \tag{6}$$

where the Hamiltonian

$$\widehat{\mathcal{H}}' = \frac{\widehat{p}^2}{2m} + U(r) + \frac{\hbar^2}{2m^2c^2 r}\frac{\mathrm{d}U}{\mathrm{d}r}\widehat{\boldsymbol{l}}\cdot\widehat{\boldsymbol{s}} - \frac{\widehat{p}^4}{8m^3c^2} + \frac{\hbar^2}{4m^2c^2}\nabla^2 - \frac{1}{4m^2c^2}(\widehat{\boldsymbol{p}}\, U)\cdot\widehat{\boldsymbol{p}} . \tag{7}$$

The Hamiltonian obtained $\mathcal{H}'$ contains all the required corrections, but the two-component wave function φ has nonstandard normalization. Particularly, if the Dirac bispinor is normalized to unity, $\int \Psi^\dagger\Psi \mathrm{d}^3 r = 1$, then in the approximation considered the spinor φ is normalized by condition:

$$\int (\varphi^\dagger\varphi + \chi^\dagger\chi)\mathrm{d}^3 r = \int \varphi^\dagger\left(1 + \frac{\widehat{p}^2}{4m^2c^2}\right)\varphi \mathrm{d}^3 r = 1 . \tag{8}$$

To avoid this inconvenience, let us transform this spinor using the nonunitary operator $\widehat{K}$ in such a way that the new spinor $\Phi = \widehat{K}\varphi$ will be normalized to unity. With the required accuracy we choose $\widehat{K} = 1 + \widehat{p}^2/8m^2c^2$ and obtain

$$\begin{aligned}\int \Phi^\dagger\Phi \mathrm{d}^3 r &= \int \varphi^\dagger\left(1 + \frac{\widehat{p}^2}{8m^2c^2}\right)^2 \varphi \mathrm{d}^3 r \\ &\approx \int \varphi^\dagger\left(1 + \frac{\widehat{p}^2}{4m^2c^2}\right)\varphi \mathrm{d}^3 r = 1 .\end{aligned}$$

But the transformation of the wave function implies the transformation of the Hamiltonian. Applying the operator $\widehat{K}$ to both parts of equation (6), we find

$$E\widehat{K}\varphi = \widehat{K}\widehat{\mathcal{H}}'\widehat{K}^{-1}\widehat{K}\varphi \quad \text{or} \quad E\Phi = \widehat{\mathcal{H}}\Phi \,, \quad \text{where} \quad \widehat{\mathcal{H}} = \widehat{K}\widehat{\mathcal{H}}'\widehat{K}^{-1} \,. \tag{9}$$

The reversed operator in the given approximation is easily determined from the condition $\widehat{K}\widehat{K}^{-1} = 1$: $\widehat{K} = 1 - \widehat{p}^2/8m^2c^2$. In fact, in formula (9) one should transform only the nonrelativistic Hamiltonian $\mathcal{H}_0 = \widehat{p}^2/2m + U$, since the transformation of the small terms would lead to corrections of higher order of smallness. Performing the calculations, we obtain

$$\widehat{\mathcal{H}} = \widehat{\mathcal{H}}_0 + \widehat{V} \,, \quad \widehat{V} = \widehat{V}_1 + \widehat{V}_2 + \widehat{V}_3 \,. \tag{10}$$

Here

$$\widehat{V}_1 = \frac{\hbar^2}{2m^2c^2r}\frac{\mathrm{d}U}{\mathrm{d}r}\widehat{\boldsymbol{l}}\cdot\widehat{\boldsymbol{s}} \tag{11}$$

is the *spin–orbital interaction*, which is the result of the presence of a magnetic field in the noninertial rest frame of reference of an electron. This effect (in the classical interpretation) was studied in Problems 3.17, 3.26, and 4.83. The term

$$\widehat{V}_2 = -\frac{\widehat{p}^4}{8m^3c^2} \tag{12}$$

is the relativistic correction to the electron's kinetic energy. The latter term

$$\widehat{V}_3 = \frac{\hbar^2}{8m^2c^2}\nabla^2 U \tag{13}$$

does not have a classical analogue. For the hydrogen-like atom with $U = -Ze^2/r$, it reduces to the contact interaction with the nucleus:

$$\widehat{V}_3 = \frac{\pi Z e^2\hbar^2}{2m^2c^2}\delta(\boldsymbol{r}) \,. \tag{14}$$

□

Problems

6.71. Using the Pauli equation (6.122), calculate the superfine splitting of the energy level of the ground state of the hydrogen atom. The superfine splitting is caused by interaction of the spin magnetic moments of the electron and the nucleus (proton).

6.72. Calculate, according to perturbation theory, the relativistic corrections for the ground and first excited states of a hydrogen-like atom. Use the results obtained in Example 6.17 and nonperturbed functions (C77).

6.73. The hydrogen atom is in a weak uniform magnetic field $\boldsymbol{H}$, the effect of which is much less than the *spin–orbital interaction* considered in the previous problem. Find the corrections for the energy levels (*anomalous Zeeman effect*[15]).

6.74. Perform the same procedure for the strong magnetic field in the case where the splitting of the energy levels significantly exceeds the spin–orbital interaction (*the Paschen*[16] *and Buck*[17] *effect*).

6.75. The hydrogen atom is in an external constant uniform electric field $\boldsymbol{\mathcal{E}} =$ const, which is weak compared with the nuclear field, but strong compared with the spin–orbital interaction. Find by perturbation theory the corrections for the first excited level ($n = 2, l = 0, 1$). (*Stark effect*[18] for hydrogen).

6.76. A relativistic electron moves in a uniform constant magnetic field $\boldsymbol{H} =$ const. Find the energy spectrum and wave functions using the Dirac equation and the Cartesian frame of reference. Use the vector potential in the form $\boldsymbol{A} = (0, Hx, 0)$, which describes the field oriented along the Oz axis.

Example 6.18

Obtain all solutions of the Dirac equation for a free particle corresponding to a given momentum $\boldsymbol{p}$, but to different signs of the energy and different values of the spin projection. What values of the spin projection are possible at the given momentum and energy?

Solution. We seek the Dirac equation solution for a free particle in the form

$$\Psi_{\boldsymbol{p}} = uA\exp\left[\frac{i(\boldsymbol{p}\cdot\boldsymbol{r} - \mathcal{E}t)}{\hbar}\right] = \begin{pmatrix}\varphi\\ \chi\end{pmatrix} A\exp\left[\frac{i(\boldsymbol{p}\cdot\boldsymbol{r} - \mathcal{E}t)}{\hbar}\right],$$

where A is a normalizing factor of the coordinate wave function, and u is a four-component bispinor independent of the coordinates and time, on which we impose the normalizing condition

$$u^\dagger u = 1\,. \tag{2}$$

$A = (2\pi\hbar)^{-3/2}$ if the normalization is done to $\delta(\boldsymbol{p} - \boldsymbol{p}')$, and $A = \mathcal{V}^{-1/2}$ if the periodicity conditions are used as for photons in Section 6.1. Note that these normalizing conditions are not the only ones possible and sometimes other conditions are applicable. The drawback of the conditions indicated is that they do not posses relativistic invariance. This is clearly seen by the example of the factor $A = \mathcal{V}^{-1/2}$:

15) Pieter Zeeman (1865–1943) was a Dutch physicist and Nobel Prize recipient.

16) Friedrich Paschen (1865–1947) was a German experimental physicist, and was founder of a vast school of spectroscopy.

17) Ernst Buck (1881–1959) was a German experimental physicist, and was one of the members of the Paschen school.

18) Johannes Stark (1874–1957) was a German physicist and Nobel Prize recipient.

the three-dimensional volume changes on going to another inertial system. The same is true for condition (2): the product $u^\dagger u$, as will be seen in Section 7.2, is not an invariant of Lorenz transformations.

Substituting (1) into the Dirac equation, we arrive at the equation which should be satisfied with bispinor u,

$$(c\widehat{\boldsymbol{\alpha}}\cdot\boldsymbol{p} + \widehat{\beta}mc^2 - \mathcal{E})u = 0\,, \tag{3}$$

or using the representation of the Dirac matrices via the Pauli matrices (6.112),

$$\left.\begin{aligned} c(\widehat{\boldsymbol{\sigma}}\cdot\boldsymbol{p})\chi + (mc^2 - \mathcal{E})\varphi &= 0\,, \\ c(\widehat{\boldsymbol{\sigma}}\cdot\boldsymbol{p})\varphi - (mc^2 + \mathcal{E})\chi &= 0\,. \end{aligned}\right\} \tag{4}$$

From system (4), we obtain

$$\varphi = \frac{c(\widehat{\boldsymbol{\sigma}}\cdot\boldsymbol{p})\chi}{\mathcal{E} - mc^2}\,, \quad \chi = \frac{c(\widehat{\boldsymbol{\sigma}}\cdot\boldsymbol{p})\varphi}{\mathcal{E} + mc^2} \tag{5}$$

or

$$\frac{c^2(\widehat{\boldsymbol{\sigma}}\cdot\boldsymbol{p})^2}{\mathcal{E}^2 - m^2c^4} = \frac{c^2p^2}{\mathcal{E}^2 - m^2c^4} = 1\,,$$

and thus the possible values of energy are

$$\mathcal{E}(p) = \pm\sqrt{c^2p^2 + m^2c^4}\,. \tag{6}$$

The interpretation of the negative energy values was discussed in Example 6.12 and will be considered in more detail in Section 7.3.

The Dirac equation allowed only the relationship between spinors χ and φ to be expressed. Now the solution can be written in the form

$$u_+(\boldsymbol{p}) = N\begin{pmatrix} \varphi \\ \frac{c(\boldsymbol{p}\cdot\widehat{\boldsymbol{\sigma}})\varphi}{\mathcal{E}+mc^2} \end{pmatrix}, \quad \text{or} \quad u_-(\boldsymbol{p}) = N\begin{pmatrix} \frac{c(\boldsymbol{p}\cdot\widehat{\boldsymbol{\sigma}})\chi}{\mathcal{E}-mc^2} \\ \chi \end{pmatrix}. \tag{7}$$

For calculation of the remaining spinors it is necessary to fix the spin state of a particle. We require that in the state considered the spin projection on some direction $\boldsymbol{n}$ be a certain value. The operator of the spin projection (see (6.125)) has the form $(1/2)(\boldsymbol{n}\cdot\widehat{\boldsymbol{\Sigma}})$.

But the direction $\boldsymbol{n}$ cannot be arbitrary since the operator $(1/2)(\boldsymbol{n}\cdot\widehat{\boldsymbol{\Sigma}})$ should commute with the Dirac Hamiltonian of a free particle $\widehat{\mathcal{H}} = c\widehat{\boldsymbol{\alpha}}\cdot\boldsymbol{p} + \widehat{\beta}mc^2$ if we want to include the spin projection in the total set of observables. Computing the commutator $[\widehat{\mathcal{H}}, \boldsymbol{n}\cdot\widehat{\boldsymbol{\Sigma}}]$, we conclude that it reduces to zero only if $\boldsymbol{n} = \boldsymbol{p}/p$, that is, only the spin projections on the momentum have the values $\pm 1/2$. The particle property with a certain value of the spin projection on the momentum is referred to as *helicity*. A Dirac particle may have two helicities, positive and negative.

Now we have taken into account all the limitations which the Dirac equation imposes on the spinors φ and χ. These spinors are independent of the momentum of a particle, but can describe only states with a certain value of the spin projection on the momentum. Therefore, we can use the spinors (C85) known from non relativistic quantum mechanics and consider the vector $\boldsymbol{n}$ as the direction of the spin in the rest system of a particle. A more general relativistic covariant description of the spin state will be given in Chapter 7.

From the results obtained, we conclude that four different bispinors correspond to the momentum $\boldsymbol{p}$; they describe different physical states of the relativistic particle. Numbering them with index λ we have,

λ	1	2	3	4
State	$+, 1/2$	$+, -1/2$	$-, 1/2$	$-, -1/2$

The four pairs of quantities in the second line indicate the sign of the energy and the spin projection value in the rest system of a particle. The bispinors are

$$u_\lambda(\boldsymbol{p}) = N \begin{pmatrix} w_\mu \\ \frac{c(\boldsymbol{p}\cdot\widehat{\boldsymbol{\sigma}})w_\mu}{\mathcal{E}+mc^2} \end{pmatrix}, \quad \begin{array}{c} \mathcal{E} \geq mc^2, \\ \lambda = 1, 2\,, \quad \mu = \pm 1/2\,; \end{array} \tag{8}$$

$$u_\lambda(\boldsymbol{p}) = N \begin{pmatrix} \frac{c(\boldsymbol{p}\cdot\widehat{\boldsymbol{\sigma}})w_\mu}{\mathcal{E}-mc^2} \\ w_\mu \end{pmatrix}, \quad \begin{array}{c} \mathcal{E} \leq -mc^2\,, \\ \lambda = 3, 4\,, \quad \mu = \pm 1/2\,. \end{array} \tag{9}$$

Here, w_μ is given by expressions (C85), and

$$N = \sqrt{\frac{|\mathcal{E}| + mc^2}{2|\mathcal{E}|}} \tag{10}$$

provides the fulfillment of the normalization condition (2). Note that all four bispinors correspond to different eigenvalues of the Hermitian operators, which are the Hamiltonian and spin projections on the momentum. Therefore, they are mutually orthogonal and satisfy the condition of completeness:

$$u^\dagger_{\lambda'} u_\lambda = \delta_{\lambda\lambda'}\,, \quad \sum_{\lambda=1}^{4} u^*_{\lambda j} u_{\lambda k} = \delta_{jk}\,, \tag{11}$$

where the bispinor components are numbered with indices $j, k = 1, 2, 3, 4$. In the Dirac notation

$$\langle\lambda'|\lambda\rangle = \delta_{\lambda\lambda'}\,, \quad \sum_{\lambda=1}^{4} |\lambda\rangle\langle\lambda| = \widehat{1}\,. \tag{12}$$

□

6.3.4
The Interaction Operator of a Relativistic Particle with Photons

As in the nonrelativistic system (see Section 6.2), one should select from the Dirac Hamiltonian (6.117) the terms connecting a particle with the photon subsystem regarding the interaction with an external field. Operator (6.117) contains only one such term:

$$\widehat{V} = -e\widehat{\boldsymbol{\alpha}}\cdot\widehat{\boldsymbol{A}}(\boldsymbol{r}) \,, \tag{6.126}$$

where $\widehat{\boldsymbol{\alpha}}$ is the Dirac matrix (6.112), and $\widehat{\boldsymbol{A}}(\boldsymbol{r})$ is the operator of the vector potential (6.13). In the case of a system of Dirac particles, one should sum expressions (6.126) over all particles:

$$\widehat{V} = -\sum_{a=1}^{N} e_a \widehat{\boldsymbol{\alpha}}_a\cdot\widehat{\boldsymbol{A}}(\boldsymbol{r}_a) \,. \tag{6.127}$$

The nonperturbed wave function, as in the nonrelativistic case, is the product of state vectors that do not interact with the electromagnetic field and the particles.

Example 6.19

A photon with wave vector $\boldsymbol{k}_0$ and polarization $\boldsymbol{e}_0$ is scattered from a free electron, which initially had negligible (zero) momentum. Calculate the probability per unit time and the differential cross-section of the process which results in the transition of a photon in the state with wave vector $\boldsymbol{k}$ and polarization $\boldsymbol{e}$. Perform the averaging over the polarizations of the initial photon and over the spin states of the electron. Analyze the different relations between $\hbar\omega_0$ and mc^2 (*Compton effect*).

Solution. The process with two photon states requires use of the second order of perturbation theory. The problem is similar to the previously considered cases of photon scattering by coupled (atomic) electrons. The transition probability per unit time is given by formula (1) obtained in the solution of Problem 6.67*, in which one should use the perturbation operator (6.126):

$$\mathrm{d}w_{fi} = \frac{2\pi}{\hbar}\left|\sum_l \frac{\langle f|\widehat{V}|l\rangle\langle l|\widehat{V}|i\rangle}{\epsilon_i - \epsilon_l}\right|^2 \delta(\epsilon_i - \epsilon_f)\frac{\mathcal{V}k^2\mathrm{d}k\mathrm{d}\Omega}{(2\pi)^3} \,, \tag{1}$$

where the indices i, l, and f denote the initial, intermediate, and final states of the whole system (electron and photons).

For the solution of Problem 6.64*, two types of intermediate states have been already discussed:

1. In the first transition $i \to l$ the electron absorbs a primary photon, and in the second transition $l \to f$ it emits a scattered photon.
2. In the first transition $i \to l$ the electron emits a scattered photon, and in the second transition $l \to f$ it absorbs a primary photon.

In all three states the electron is free and it can be described by wave functions of the plane wave type (see Example 6.18). The structure of the matrix elements $\langle f|\widehat{V}|l\rangle$ and $\langle l|\widehat{V}|i\rangle$ is such that the integration over coordinates is produced from the product of three exponents, and it ensures the fulfillment of the momentum conservation law in each transition:

$$\langle l|\widehat{V}|i\rangle \propto \int \exp\left\{\frac{i}{\hbar}(\boldsymbol{p}_l \mp \hbar\boldsymbol{k} - \boldsymbol{p}_i)\right\} \mathrm{d}^3 r = \mathcal{V}\delta_{\boldsymbol{p}_l, \boldsymbol{p}_i \pm \hbar \boldsymbol{k}} , \tag{2}$$

where the minus sign corresponds to the absorption of a photon and the plus sign corresponds to emission of a photon. Accounting for the momentum conservation, we can represent transitions of types 1 and 2 in graphs (*Feynman*[19] *diagrams* in Figure 6.4).

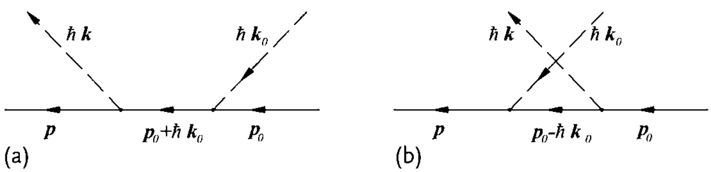

Figure 6.4 Feynman diagrams for the Compton effect: intermediates states of type 1 (a) and type 2 (b) as discussed in the text.

In Figure 6.4, solid lines indicate the states of an electron, and dashed lines represent the states of photons with their momenta. To each apex, in which three lines converge, one should relate the matrix element of the operator $\widehat{V}$. At each apex the momentum conservation law is fulfilled, but the energy conservation law is not. The energy is conserved only for the entire process as a whole.

We compute the values in (1). Evidently, the initial energy of the system is $\epsilon_i = mc^2 + \hbar\omega_0$. We introduce the notation ϵ' for the energy of the intermediate state corresponding to Figure 6.4a, and ϵ'' for the energy of the intermediate state corresponding to Figure 6.4b. Then

$$\begin{aligned} &\epsilon' = \mathcal{E}' , \quad \epsilon'' = \mathcal{E}'' + \hbar\omega_0 + \hbar\omega , \quad \text{where} \\ &\mathcal{E}' = \pm\sqrt{m^2c^4 + (\hbar\omega_0)^2} , \quad \mathcal{E}'' = \pm\sqrt{m^2c^4 + (\hbar\omega)^2} \end{aligned} \tag{3}$$

are the electron energies in the intermediate states. They could have both signs. Finally, for the difference of energies in the denominator in (1) we obtain

$$\epsilon_i - \epsilon' = mc^2 + \hbar\omega_0 - \mathcal{E}' , \quad \epsilon_i - \epsilon'' = mc^2 - \hbar\omega - \mathcal{E}'' . \tag{4}$$

19) Richard Feynman (1918–1988) was the outstanding American physicist-theorist, one of founders of modern quantum electrodynamics (see Chapter 7). Has brought the big contribution to development of physics of elementary particles, the statistical physics and the gravitation theory. Nobel Prize winner.

Further, we compute the matrix elements, denoting, for brevity, by F the sum over l in formula (1):

$$F = \sum_l \frac{\langle f|\widehat{V}|l\rangle\langle l|\widehat{V}|i\rangle}{\epsilon_i - \epsilon_l}$$
$$= \frac{2\pi\hbar c^2 e^2}{\mathcal{V}\sqrt{\omega\omega_0}}$$
$$\times \sum_{\lambda=1}^{4}\left\{\frac{\langle u|\widehat{\boldsymbol{\alpha}}\cdot\boldsymbol{e}^*|u'_\lambda\rangle\langle u'_\lambda|\widehat{\boldsymbol{\alpha}}\cdot\boldsymbol{e}_0|u_0\rangle}{mc^2 + \hbar\omega_0 - \mathcal{E}'_\lambda} + \frac{\langle u|\widehat{\boldsymbol{\alpha}}\cdot\boldsymbol{e}_0|u''_\lambda\rangle\langle u''_\lambda|\widehat{\boldsymbol{\alpha}}\cdot\boldsymbol{e}^*|u_0\rangle}{mc^2 - \hbar\omega - \mathcal{E}''_\lambda}\right\} . \tag{5}$$

Here the summation over λ accounts for the two signs of the energy and two spin states for each bispinor u', u''. It is convenient to perform this summation by transferring the whole dependence on λ to the nominator. Let use the Dirac Hamiltonian (6.111) and write

$$(c\hbar\widehat{\boldsymbol{\alpha}}\cdot\boldsymbol{k}_0 + \widehat{\beta}mc^2)u'_\lambda = \mathcal{E}'_\lambda u'_\lambda\,, \quad (-c\hbar\widehat{\boldsymbol{\alpha}}\cdot\boldsymbol{k} + \widehat{\beta}mc^2)u''_\lambda = \mathcal{E}''_\lambda u''_\lambda\,. \tag{6}$$

Multiplying in (5) the nominator and denominator of the first fraction by $mc^2 + \hbar\omega_0 + \mathcal{E}'_\lambda$, and the nominator and denominator of the second fraction by $mc^2 - \hbar\omega + \mathcal{E}''_\lambda$, and using (6), we obtain

$$F = \frac{2\pi\hbar c^2 e^2}{\mathcal{V}\sqrt{\omega\omega_0}}$$
$$\times \sum_{\lambda=1}^{4}\left\{\frac{\langle u|\widehat{\boldsymbol{\alpha}}\cdot\boldsymbol{e}^*(mc^2 + \hbar\omega_0 + c\hbar\widehat{\boldsymbol{\alpha}}\cdot\boldsymbol{k}_0 + \widehat{\beta}mc^2)|u'_\lambda\rangle\langle u'_\lambda|\widehat{\boldsymbol{\alpha}}\cdot\boldsymbol{e}_0|u_0\rangle}{(mc^2 + \hbar\omega_0)^2 - \mathcal{E}'^2}\right.$$
$$\left. + \frac{\langle u|\widehat{\boldsymbol{\alpha}}\cdot\boldsymbol{e}_0(mc^2 - \hbar\omega - c\hbar\widehat{\boldsymbol{\alpha}}\cdot\boldsymbol{k} + \widehat{\beta}mc^2)|u''_\lambda\rangle\langle u''_\lambda|\widehat{\boldsymbol{\alpha}}\cdot\boldsymbol{e}^*|u_0\rangle}{(mc^2 - \hbar\omega)^2 - \mathcal{E}''^2}\right\} . \tag{7}$$

For simplification of these expressions we will use the completeness condition of bispinors (equation (11) in Example 6.18), and also relations (6) for the initial electron state, which give $\widehat{\beta}u_0 = u_0$, and the anticommutation rule of the Dirac matrices (6.114). Thus, we obtain

$$F = \frac{\pi\hbar e^2}{\mathcal{V}m\sqrt{\omega\omega_0}}\left\{2(\boldsymbol{e}_0\cdot\boldsymbol{e}^*)\langle u|u_0\rangle\right.$$
$$\left. + \langle u|(\widehat{\boldsymbol{\alpha}}\cdot\boldsymbol{e}^*)(\widehat{\boldsymbol{\alpha}}\cdot\boldsymbol{n}_0)(\widehat{\boldsymbol{\alpha}}\cdot\boldsymbol{e}_0) + (\widehat{\boldsymbol{\alpha}}\cdot\boldsymbol{e}_0)(\widehat{\boldsymbol{\alpha}}\cdot\boldsymbol{n})(\widehat{\boldsymbol{\alpha}}\cdot\boldsymbol{e}^*)|u_0\rangle\right\} , \tag{8}$$

where the unit vectors $\boldsymbol{n}_0$ and $\boldsymbol{n}$ indicate the direction of momenta of the primary and scattered photons.

If the directions of the electron spin in the initial and final states are not stated, then the square of the modulus of the matrix element (8) should be averaged over the initial state and summed over the final state. Evidently, the initial electron energy $\mathcal{E}_0 = mc^2$ and the final electron energy $\mathcal{E} = mc^2 + \hbar(\omega_0 - \omega)$ are positive.

Therefore, the averaged square of the modulus

$$\overline{|F|^2} = \frac{1}{2}\left(\frac{\pi\hbar e^2}{\mathcal{V}m\sqrt{\omega\omega_0}}\right)^2 \sum_{\lambda_0=1}^{2}\sum_{\lambda=1}^{2} |\langle u|\widehat{B}|u_0\rangle|^2 \,, \tag{9}$$

where $\widehat{B} = 2(\boldsymbol{e}_0\cdot\boldsymbol{e}^*)\widehat{1} + (\widehat{\boldsymbol{\alpha}}\cdot\boldsymbol{e}^*)(\widehat{\boldsymbol{\alpha}}\cdot\boldsymbol{n}_0)(\widehat{\boldsymbol{\alpha}}\cdot\boldsymbol{e}_0) + (\widehat{\boldsymbol{\alpha}}\cdot\boldsymbol{e}_0)(\widehat{\boldsymbol{\alpha}}\cdot\boldsymbol{n})(\widehat{\boldsymbol{\alpha}}\cdot\boldsymbol{e}^*)$ is the operator in (8). The summation is performed only over two states related to the positive energies. To use again the completeness of the system of bispinors, it is convenient to introduce the projection operators on the states with positive energy:

$$\widehat{\Pi}_0 = \frac{\mathcal{E}_0 + \widehat{\mathcal{H}}_0}{2\mathcal{E}_0}\,, \quad \widehat{\Pi} = \frac{\mathcal{E} + \widehat{\mathcal{H}}}{2\mathcal{E}}\,, \quad \text{where}$$
$$\widehat{\mathcal{H}}_0 = \widehat{\beta}mc^2\,, \quad \widehat{\mathcal{H}} = c\hbar\widehat{\boldsymbol{\alpha}}\cdot(\boldsymbol{k}_0 - \boldsymbol{k}) + \widehat{\beta}mc^2 \tag{10}$$

are the Hamiltonians of the initial and final electron states in the momentum representation. For the states with positive energy we have $\widehat{\Pi}_0 u_0 = u_0$ and $\widehat{\Pi}u = u$, and for the states with negative energy we have $\widehat{\Pi}_0 u_0 = \widehat{\Pi}u = 0$. Using the projection operators, we extend the summation in (9) to all four states to obtain

$$\begin{aligned}\overline{|F|^2} &= \frac{\pi^2\hbar^2 e^4}{2\mathcal{V}^2 m^2\omega\omega_0}\sum_{\lambda_0=1}^{4}\sum_{\lambda=1}^{4}\langle\lambda|\widehat{B}\widehat{\Pi}_0|\lambda_0\rangle\langle\lambda_0|\widehat{B}^\dagger\widehat{\Pi}|\lambda\rangle \\ &= \frac{\pi^2\hbar^2 e^4}{2\mathcal{V}^2 m^2\omega\omega_0}\mathrm{Tr}\left\{\widehat{B}\widehat{\Pi}_0\widehat{B}^\dagger\widehat{\Pi}\right\}\,.\end{aligned} \tag{11}$$

Let us use the value of the trace calculated in Problem 6.83• for the quanta with linear polarization:

$$\mathrm{Tr}\left\{\widehat{B}\widehat{\Pi}_0\widehat{B}^\dagger\widehat{\Pi}\right\} = \frac{8mc^2}{\mathcal{E}}(\boldsymbol{e}_0\cdot\boldsymbol{e})^2 + \frac{2\hbar(\omega_0-\omega)}{\mathcal{E}}(1 - \boldsymbol{n}\cdot\boldsymbol{n}_0)\,. \tag{12}$$

Let us compute the differential cross-section of photon scattering by a free electron. To do this it is necessary to divide (1) by the flux density of incident photons $c/\mathcal{V}$ and integrate over the energies of the scattered photon using delta functions:

$$\mathrm{d}\sigma = \frac{\mathcal{V}}{c}\int \mathrm{d}\overline{w_{fi}}\,. \tag{13}$$

The averaging produced previously over the initial polarization of the electron and the summation over the final polarization are denoted by a bar. From the momentum conservation law we have

$$\begin{aligned}\epsilon_i - \epsilon_f &= mc^2 + \hbar(\omega_0-\omega) - \sqrt{m^2c^4 + \hbar^2\left(\omega_0^2 + \omega^2 - 2\omega_0\omega\cos\theta\right)} \\ &= 0\,,\end{aligned} \tag{14}$$

where θ is the scattering angle of the quantum. At a given scattering angle we obtain

$$\frac{d(\epsilon_i - \epsilon_f)}{d(\hbar\omega)} = \frac{mc^2 + \hbar\omega_0(1-\cos\theta)}{\mathcal{E}} = \frac{mc^2\omega_0}{\mathcal{E}\omega}\,, \tag{15}$$

which gives, with the help of relation (1.209),

$$\delta(\epsilon_i - \epsilon_f) = \frac{\mathcal{E}\omega}{mc^2\omega_0}\delta(\hbar\omega_{\text{sc}} - \hbar\omega) \; . \tag{16}$$

Here,

$$\omega_{\text{sc}} = \frac{\hbar\omega_0}{1 + \hbar\omega_0/(mc^2)(1 - \cos\theta)} \tag{17}$$

is the frequency of the scattered photon expressed by the frequency of the primary photon and scattering angle (see the solution to Problem 6.73). By using sequentially formulas (1), (5)–(12), and (16), we obtain the desired cross-section is (*formula of Klein, Nishina,*[20] *and Tamm*[21]):

$$\mathrm{d}\sigma = \frac{1}{4}\left(\frac{e^2}{mc^2}\right)^2 \frac{\omega^2}{\omega_0^2}\left[\frac{\omega_0}{\omega} + \frac{\omega}{\omega_0} - 2 + 4(\boldsymbol{e}_0\cdot\boldsymbol{e})^2\right]\mathrm{d}\Omega \; . \tag{6.128}$$

Here, ω is the frequency of the scattered quantum, which is given by the right side of equality (17). □

6.3.5
Method of Equivalent Photons

Example 6.20

A particle with charge q and relativistic factor $\gamma = \mathcal{E}/Mc^2 \gg 1$ executes classical motion along a linear trajectory. Show that its electromagnetic field coincides more accurately with a set of plane monochromatic waves in a free space the larger is the particle relativistic factor. Represent the field of a particle as the set of "equivalent photons" and find their density $n(\omega)$ per unit frequency interval.

Solution. Let us consider the properties of Fourier harmonics (see Problem 5.144) in the strong relativistic case:

$$\boldsymbol{E}_{k\omega} = -i8\pi^2 q\frac{\delta(\omega - \boldsymbol{k}\cdot\boldsymbol{v})}{k^2 - \omega^2/c^2}\left(\boldsymbol{k} - \frac{\omega\boldsymbol{v}}{c^2}\right) \; , \quad \boldsymbol{H}_{k\omega} = \frac{\boldsymbol{v}}{c}\times\boldsymbol{E}_{k\omega} \; . \tag{1}$$

The presence of the delta function in (1) gives the dispersion relation $\omega(\boldsymbol{k}) = kv\cos\theta$. The denominator $k^2 - \omega^2/c^2 = k^2[1 - (v^2/c^2)\cos^2\theta]$ indicates that the

20) Uoshio Nishina (1890–1951) was a Japanese physicist, and was the founder of the nuclear physics school in Japan.

21) Igor Evgenievich Tamm (1895–1971) was an outstanding Soviet physicist-theorist, the founder of numerous and fruitful school of researchers. He developed a wide range of problems of theoretical and applied physics. Nobel Prize winner.

harmonics with wave vectors directed along the velocity $\boldsymbol{v}$ in the limits of angle $\theta_c \approx 1/\gamma \ll 1$ have the largest value. At a given $\boldsymbol{k}$ we find the difference between the frequency of the Fourier harmonics of the field and the photon frequency:

$$\omega_{\text{ph}}(k) - \omega(\boldsymbol{k}) = \omega_{\text{ph}}[1 - (v/c)\cos\theta] \approx \omega_{\text{ph}}/\gamma^2 \,. \tag{2}$$

This difference is small in the wide range of angles at $\gamma \gg 1$. Let us also compare the longitudinal and transverse components of the electric field with respect to the particle velocity. We have

$$\left|\frac{\boldsymbol{E}^{\parallel}_{\boldsymbol{k}\omega}}{\boldsymbol{E}^{\perp}_{\boldsymbol{k}\omega}}\right| \approx \frac{1}{\gamma^2\theta_c} \approx \frac{1}{\gamma} \ll 1 \tag{3}$$

at characteristic angles. Hence, at $\gamma \gg 1$ both field intensities, like the vacuum waves, possess the property of transversity.

To compute the density of the equivalent photons let us use the coordinate representation of fields in the plane $z = 0$ perpendicular to the particle velocity (see Problems 4.34$^\bullet$ and 5.145*):

$$\begin{aligned} \boldsymbol{E}(\boldsymbol{r}_\perp, t) &= -\frac{q(\boldsymbol{r}_\perp + \boldsymbol{v}t)}{\gamma^2(v^2t^2 + r_\perp^2/\gamma^2)^{3/2}} \,, \\ \boldsymbol{H}(\boldsymbol{r}_\perp, t) &= -\frac{q\boldsymbol{r}_\perp \times \boldsymbol{v}}{\gamma^2(v^2t^2 + r_\perp^2/\gamma^2)^{3/2}} \,. \end{aligned} \tag{4}$$

We decompose the field intensities into monochromatic components:

$$\begin{aligned} \boldsymbol{E}_\omega(\boldsymbol{r}_\perp) &= \int_{-\infty}^{\infty} \boldsymbol{E}(\boldsymbol{r}_\perp, t)e^{i\omega t}\text{d}t \\ &= -\frac{2q}{\gamma v^2 r_\perp}\left[|\omega|\boldsymbol{r}_\perp K_1\left(\frac{|\omega|r_\perp}{\gamma v}\right) + i\frac{r_\perp\omega\boldsymbol{v}}{\gamma v}K_0\left(\frac{|\omega|r_\perp}{\gamma v}\right)\right] \,, \end{aligned} \tag{5}$$

$$\boldsymbol{H}_\omega(\boldsymbol{r}_\perp) = \frac{2q|\omega|}{\gamma v^2 c r_\perp}[\boldsymbol{r}_\perp \times \boldsymbol{v}]K_1\left(\frac{|\omega|r_\perp}{\gamma v}\right) \,. \tag{6}$$

Here we used the integral representations of modified Bessel functions (Abramovitz and Stegun, 1965) and recurrent relations between them.

The photon number density $n(r_\perp, \omega)$ per unit frequency interval, moving with target parameter $r_\perp$ (i.e., at a distance $r_\perp$ from the classical trajectory of a particle) can be determined from the equality

$$\frac{c}{4\pi v}\int_{-\infty}^{\infty} [\boldsymbol{E} \times \boldsymbol{H}]\cdot\boldsymbol{v}\text{d}t = \int_0^{\infty} n(r_\perp, \omega)\hbar\omega\text{d}\omega \,. \tag{7}$$

Calculating the left part with the help of formulas (5) and (6), we find

$$n(r_\perp, \omega) = \frac{q^2}{\pi^2 c^3\gamma^2\hbar}\omega K_1^2\left(\frac{|\omega|r_\perp}{\gamma v}\right) \,, \quad \omega > 0 \,. \tag{8}$$

The notion of the target parameter of a photon has a sense only under the condition of the smallness of its wave packet compared with the target parameter itself, that is, at $\lambda = 2\pi c/\omega \ll r_\perp$, or at $\omega \gg 2\pi c/r_\perp$. The photon spectrum (8) is truncated at $\omega \gg \gamma c/r_\perp$.

Since the classical description of motion of the particle is approximate, the target parameter $r_\perp$ cannot be arbitrarily small. The localization limit of a particle is its Compton wavelength Λ_C, otherwise the problem becomes a multiparticle problem. Therefore in formula (8) one should use

$$r_\perp \geq r_{\min} = \kappa \Lambda_C = \kappa \hbar / Mc \,, \tag{9}$$

where κ is a dimensionless value of order of unity. The photon spectrum is truncated at energies $\hbar\omega_{\max} \approx \mathcal{E}/\kappa$ on the order of the particle energy.

The total number of equivalent photons of a given frequency (independent of the impact parameter) can be obtained by integrating (8) over the plane perpendicular to the particle velocity:

$$\begin{aligned} n(\omega) &= \int_{r_{\min}}^{\infty} n(r_\perp, \omega) 2\pi r_\perp \mathrm{d}r_\perp \\ &= \frac{2}{\pi} \frac{q^2}{\hbar c \omega} \left\{ z_m K_0(z_m) K_1(z_m) + \frac{1}{2} z_m^2 K_0^2(z_m) - \frac{1}{2} z_m^2 K_1^2(z_m) \right\} \,, \\ z_m &= \kappa \frac{\hbar\omega}{\mathcal{E}} \,. \end{aligned} \tag{10}$$

Integration was done with the help of recurrent relations between McDonald functions. The low-frequency asymptote of the photon distribution has the form

$$n(\omega) = \frac{2}{\pi} \frac{q^2}{\hbar c} \frac{1}{\omega} \ln\left(\frac{\mathcal{E}}{\hbar\omega}\right) \,. \tag{6.129}$$

Here the terms of order unity compared with $\ln(\mathcal{E}/\hbar\omega) \gg 1$ are omitted. □

The method of equivalent photons permits us to consider the various radiation processes involving ultrarelativistic particles as a result of Compton (or Thomson) scattering of pseudophotons by them and the formation of real photons. For the calculation of cross-sections, in some cases, it is necessary to go to the reference system accompanying the radiating particle (see Problem 6.87–6.89).

Suggested literature:
Heitler (1954); Akhiezer and Berestetskii (1981); Berestetskii *et al.* (1982); Bogolubov and Shirkov (1980); Itzukson and Zuber (1980); Feynman (1998); Dirac (1990, 1967)

Problems

6.77. Find the Lagrangian function density (see Section 4.3) which leads to the Dirac equations (6.118)–(6.119). Construct Hamiltonian (6.117) from the Lagrangian.

6.78. Construct the velocity operator of a free relativistic particle with Hamiltonian (6.111) and determine its eigenvalues. Give them a physical interpretation.

6.79. Using the conservation laws of energy and momentum, express the frequency of the scattered photon ω in the Compton effect through the frequency of the primary photon ω_0 and the scattering angle θ. Determine the possible values of energy of the scattered photon. Express also the change of the photon wavelength through the Compton wavelength of an electron Λ_C and the scattering angle. Initially the electron was at rest.

6.80. Show that

$$\begin{aligned} \mathrm{Tr}\,(\widehat{\alpha}_\mu) &= \mathrm{Tr}\,(\widehat{\alpha}_\mu\widehat{\beta}) = \mathrm{Tr}\,(\widehat{\alpha}_\mu\widehat{\alpha}_\nu) = \mathrm{Tr}\,(\widehat{\alpha}_\mu\widehat{\alpha}_\nu\widehat{\beta}) \\ &= \mathrm{Tr}\,(\widehat{\alpha}_\mu\widehat{\alpha}_\nu\widehat{\alpha}_\lambda) = \mathrm{Tr}\,(\widehat{\alpha}_\mu\widehat{\alpha}_\nu\widehat{\alpha}_\lambda\widehat{\beta}) = 0\,, \end{aligned} \tag{6.130}$$

where $\widehat{\alpha}_\mu$ and $\widehat{\beta}$ are the Dirac matrices, and $\mu \neq \nu \neq \lambda$.

6.81. Prove the equalities

$$(\widehat{\boldsymbol{\alpha}}\cdot\boldsymbol{e})(\widehat{\boldsymbol{\alpha}}\cdot\boldsymbol{n}) = -(\widehat{\boldsymbol{\alpha}}\cdot\boldsymbol{n})(\widehat{\boldsymbol{\alpha}}\cdot\boldsymbol{e})\,, \quad (\widehat{\boldsymbol{\alpha}}\cdot\boldsymbol{a})(\widehat{\boldsymbol{\alpha}}\cdot\boldsymbol{a}) = \boldsymbol{a}^2\,, \quad \mathrm{Sp}(\widehat{\boldsymbol{\alpha}}\cdot\boldsymbol{a})(\widehat{\boldsymbol{\alpha}}\cdot\boldsymbol{b}) = 4(\boldsymbol{a}\cdot\boldsymbol{b})\,, \tag{6.131}$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are arbitrary nonoperator vectors, and $\boldsymbol{e}$ and $\boldsymbol{n}$ are mutually perpendicular vectors.

6.82. Compute the traces

$$\begin{aligned} \mathrm{Tr}\,(\widehat{\alpha}_\mu\widehat{\alpha}_\nu) &= 4\delta_{\mu\nu}\,, \\ \mathrm{Tr}\,(\widehat{\alpha}_\mu\widehat{\alpha}_\nu\widehat{\alpha}_\lambda\widehat{\alpha}_\sigma) &= 4(\delta_{\mu\nu}\delta_{\lambda\sigma} + \delta_{\mu\sigma}\delta_{\lambda\nu} - \delta_{\mu\lambda}\delta_{\nu\sigma})\,. \end{aligned} \tag{6.132}$$

Hint: Use the commutation rules of Dirac matrices and their transformation properties as components of three-dimensional vectors.

6.83•. Compute $\mathrm{Tr}\,(\widehat{B}\widehat{\Pi}_0\widehat{B}^\dagger\widehat{\Pi})$ (see Example 6.19) for the case of linearly polarized photons where the unit vectors of polarization are real.

Hint: Use formulas (6.130)–(6.132).

6.84•. Analyze the Klein–Nishina–Tamm formula (6.128):

1. Write down the cross-sections $\mathrm{d}\sigma_\parallel$ and $\mathrm{d}\sigma_\perp$ of scattering of linearly polarized photons for the cases where the scattered photon is polarized in a plane $(\boldsymbol{k}, \boldsymbol{e}_0)$ and perpendicular to this plane, and also for the case where the polarization of the scattered photon is not fixed.
2. Show that the cross-section of scattering of nonpolarized photons $\mathrm{d}\bar{\sigma}$ is

$$\mathrm{d}\bar{\sigma} = \frac{1}{2}r_0^2\frac{\omega^2}{\omega_0^2}\left[\frac{\omega_0}{\omega} + \frac{\omega}{\omega_0} - \sin^2\theta\right]\mathrm{d}\Omega\,. \tag{6.133}$$

3. Apply the limiting process to the nonrelativistic case $\hbar\omega_0 \ll mc^2$ and compare the result with the Thomson formula (see Problems 5.128 and 5.129).
4. Consider the ultrarelativistic case $\hbar\omega_0 \gg mc^2$ and analyze separately the regions of large and small scattering angles.

6.85. Integrate over the scattering angle the cross-section of scattering of nonpolarized photons by an electron and find the total cross-section σ_C of the Compton scattering. Consider, in particular, nonrelativistic and relativistic limiting cases. Plot the dependence of its ratio to the Thomson cross-section σ_C/σ_T on the parameter $x = \hbar\omega_0/mc^2$ (see Problem 5.129).

6.86. Write down the differential cross-section for the Compton scattering of nonpolarized photons (6.133) per unit energy interval of the scattered quantum $\hbar\omega$.

6.87. A heavy ultrarelativistic particle collides with a nonrelativistic electron. By the method of equivalent photons, calculate, summing over all directions, the effective differential cross-section for emission of photons of a given energy.

6.88*. An ultrarelativistic electron collides with an atomic nucleus at rest with charge Ze. By using the method of equivalent photons, calculate the effective differential cross-section for the bremsstrahlung of quanta of a given frequency.

6.89. An ultrarelativistic electron collides with an electron initially at rest. By using the method of equivalent photons, calculate the effective differential cross-section for the bremsstrahlung of quanta of a given frequency.

6.4
Answers and Solutions

6.3• Apply operator expansion (C67) to operator $\widehat{c}_H(t)$ in the Heisenberg representation:

$$\widehat{c}_H(t) = \exp(\alpha\widehat{\mathcal{H}})\widehat{c}\exp(-\alpha\widehat{\mathcal{H}}) \ , \tag{1}$$

where $\alpha = it/\hbar$ and $\mathcal{H} = \hbar\omega(\widehat{c}^\dagger\widehat{c} + 1/2)$. We obtain $[\widehat{\mathcal{H}}, \widehat{c}] = -\hbar\omega\widehat{c}$, $[\widehat{\mathcal{H}}, [\widehat{\mathcal{H}}, \widehat{c}]] = (-\hbar\omega)^2\widehat{c}$, and so on. Finally, $\widehat{c}_H(t) = \widehat{c}\sum_{n=0}^{\infty}(-i\omega t)^n/n! = \widehat{c}\exp(-i\omega t)$. Similarly, $\widehat{c}^\dagger_H(t) = \widehat{c}^\dagger\exp(i\omega t)$.

An alternative solution is to write the direct time derivative of expression (1)

$$\frac{\mathrm{d}\widehat{c}_H}{\mathrm{d}t} = \frac{i}{\hbar}\exp(\alpha\widehat{\mathcal{H}})[\widehat{\mathcal{H}}, \widehat{c}]\exp(-\alpha\widehat{\mathcal{H}}) = -i\omega\widehat{c}_H \ ,$$

which leads to an exponential time dependence of $\widehat{c}_H$.

6.4

$$\widehat{Q}(t) = \widehat{Q}(0)\cos\omega t + \left(\frac{\widehat{P}(0)}{\omega}\right)\sin\omega t \ ,$$

$$\widehat{P}(t) = \widehat{P}(0)\cos\omega t - \omega\widehat{Q}(0)\sin\omega t \ .$$

6.5 (i) $\overline{a} = \overline{a^\dagger} = 0$; (ii) $\overline{P} = \overline{Q} = 0$; (iii) $\overline{c^2} = \overline{c^{\dagger 2}} = 0$; (iv) $\overline{\Delta Q^2} \cdot \overline{\Delta P^2} = \hbar^2(n+1/2)^2$.

6.6 $\mathcal{E}_n = \hbar\omega(n+1/2)$, $\Phi_n(\eta) = A_n e^{-\eta^2/2} H_n(\eta)$, $\eta = P/\sqrt{\hbar\omega}$, $A_n = 1/\sqrt{2^n n! \sqrt{\pi}}$, H_n is Hermitian polynomial.

6.7• $\Delta Q \Delta P = \hbar(n+1/2) \geq \hbar/2$.

6.8 Using the first relation in (C66), $\widehat{A}_n = [\widehat{c}^n, \widehat{c}^\dagger] = [\widehat{c}^{n-1}\widehat{c}, \widehat{c}^\dagger] = \widehat{c}^{n-1}[\widehat{c}, \widehat{c}^\dagger] + \widehat{A}_{n-1}\widehat{c} = 2\widehat{c} + \widehat{A}_{n-2}\widehat{c}^2 = \cdots = n\widehat{c}^{n-1}$. It can be similarly shown that $[\widehat{c}^{\dagger n}, \widehat{c}] = -n(\widehat{c}^\dagger)^{n-1}$.

6.9 According to the conditions we have $\widehat{f} = A_{00} + A_{10}\widehat{a} + A_{01}\widehat{c}^+ + A_{20}\widehat{c}^2 + \ldots$, where A are constants. We avoid here any subtle discussion of the series convergency after the action of operator $\widehat{f}$ on the wave function. Then, as an example, let us prove the first relation. Consider the commutation relation between operator $\widehat{c}^+$ and a typical operator product $A\widehat{c}^k\widehat{c}^{\dagger l}\widehat{c}^m\widehat{c}^{\dagger n}$, which arises in expansion of $\widehat{f}\left(\widehat{c}, \widehat{c}^\dagger\right)$. Using the solution of the previous problem and (C66), we can write $[\widehat{c}^k\widehat{c}^{\dagger l}\widehat{c}^m\widehat{c}^{\dagger n}, \widehat{c}^\dagger] = [\widehat{c}^k\widehat{c}^{\dagger l}\widehat{c}^m, \widehat{c}^\dagger]\widehat{c}^{\dagger n} + \widehat{c}^k\widehat{c}^{\dagger l}\widehat{c}^m[\widehat{c}^{\dagger n}, \widehat{c}^\dagger] = [\widehat{c}^k, \widehat{c}^\dagger]\widehat{c}^{\dagger l}\widehat{c}^m\widehat{c}^{\dagger n} + \widehat{c}^k\widehat{c}^{\dagger l}[\widehat{c}^m, \widehat{c}^\dagger]\widehat{c}^{\dagger n} = k\widehat{c}^{k-1}\widehat{c}^{\dagger l}\widehat{c}^m\widehat{c}^{\dagger n} + \widehat{c}^k\widehat{c}^{\dagger l} m\widehat{c}^{m-1}\widehat{c}^{\dagger n}$. On the other hand, the same result can be derived after formal differentiation with respect to $\widehat{c}$ when $\widehat{c}^\dagger$ is fixed and the order in the operators' product is preserved.

6.11* The field operators in the Heisenberg representation can be derived from (6.13)–(6.15) via the relevant replacement of $\widehat{c}^\dagger$ and $\widehat{c}$ by their Heisenberg counterparts $\widehat{c}^\dagger(t)$ and $\widehat{c}(t)$ introduced in the solution to Problem 6.4.

6.12*

$$\begin{aligned}\left[\widehat{E}_\alpha(\boldsymbol{r}_1, t_1), \widehat{E}_\beta(\boldsymbol{r}_2, t_2)\right] &= \left[\widehat{H}_\alpha(\boldsymbol{r}_1, t_1), \widehat{H}_\beta(\boldsymbol{r}_2, t_2)\right] \\ &= i\hbar\left(\frac{\delta_{\alpha\beta}}{c^2}\frac{\partial^2}{\partial t_1 \partial t_2} - \frac{\partial^2}{\partial x_{1\alpha}\partial x_{2\beta}}\right) \\ &\quad \times G^-(\boldsymbol{r}_1 - \boldsymbol{r}_2, t_1 - t_2)\ ; \\ \left[\widehat{E}_\alpha(\boldsymbol{r}_1, t_1), \widehat{H}_\beta(\boldsymbol{r}_2, t_2)\right] &= -i\hbar e_{\alpha\beta\mu}\frac{\partial^2}{c\partial t_1 \partial x_{1\mu}} G^-(\boldsymbol{r}_1 - \boldsymbol{r}_2, t_1 - t_2)\ .\end{aligned}$$

Here

$$G^-(\boldsymbol{R}, \tau) = \frac{1}{R}\left[\delta\left(\tau + \frac{R}{c}\right) - \delta\left(\tau - \frac{R}{c}\right)\right]$$

is the function defined in the solution to Problem 5.9. Simultaneous measurements of the field components can be achieved only for those spatial points of the four-dimensional space–time manifold which are separated by a spatial-type interval, that is, cannot be communicated by a light signal (Heitler, 1954).

6.13•

$$\left[\widehat{E}_\alpha(\boldsymbol{r}_1, t_1), \widehat{A}_\beta(\boldsymbol{r}_2, t_2)\right]$$
$$= \frac{i\hbar c}{2\pi^2}\int \mathrm{d}^3k \left(\delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}\right) \exp i\boldsymbol{k}\cdot(\boldsymbol{r}_1 - \boldsymbol{r}_2)\cos\omega(t_1 - t_2)\ .$$

This commutation relation is nonvanishing even for points separated by a spatial-type interval, e.g., $t_1 = t_2$, $\boldsymbol{r}_1 \neq \boldsymbol{r}_2$. But vector potential $\boldsymbol{A}$ is not a physically observable quantity and the principle of causality is not applicable in this case.

6.14• In quantum theory an operator of any physical variable should be Hermitian. Since the operators $\widehat{\boldsymbol{E}}$ and $\widehat{\boldsymbol{H}}$ do not commute (see Problem 6.12*), the operator $\widehat{\boldsymbol{P}}$ accepting the classical relation should be additionally symmetrized:

$$\widehat{\boldsymbol{P}} = \frac{1}{8\pi c}\int_V [\widehat{\boldsymbol{E}}(\boldsymbol{r}) \times \widehat{\boldsymbol{H}}(\boldsymbol{r}) - \widehat{\boldsymbol{H}}(t) \times \widehat{\boldsymbol{E}}(\boldsymbol{r})]\mathrm{d}V\ . \tag{1}$$

After substitution of (6.14) and (6.15) into (1) we get the momentum operator expressed in terms of creation and annihilation operators:

$$\widehat{\boldsymbol{P}} = \sum_s \hbar\boldsymbol{k}\left(\widehat{c}_s^\dagger \widehat{c}_s + \frac{1}{2}\right)\ . \tag{2}$$

6.15 The expectation values of the field components in the vacuum state $|0\rangle$ (for all the modes) are $\overline{\boldsymbol{E}} = \langle 0|\widehat{\boldsymbol{E}}|0\rangle = 0$, $\overline{\boldsymbol{H}} = 0$,

$$\overline{\Delta E^2} = \overline{E^2} = \langle 0|\widehat{\boldsymbol{E}}^2(\boldsymbol{r})|0\rangle = \sum_s \frac{2\pi\hbar\omega_s}{V} = \frac{2\hbar}{\pi c^3}\int_0^\infty \omega^3\mathrm{d}\omega \to \infty\ .$$

Similarly, $\overline{\Delta H^2} = \overline{H^2} \to \infty$. This divergency has the same nature as infinite vacuum energy.

6.16* The operator of the electric field in the Heisenberg representation was introduced in Problem 6.12*, and after applying (6.14), we can transform it to the following form:

$$\widehat{\boldsymbol{E}}(\boldsymbol{r}, t) = \sum_s \left\{\widehat{c}_s \boldsymbol{E}(\boldsymbol{r}, t) + \text{h.c.}\right\}\ , \quad \text{where} \quad \boldsymbol{E}(\boldsymbol{r}, t) = i\boldsymbol{e}_s\sqrt{\frac{2\pi\hbar\omega}{V}}e^{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)}\ . \tag{1}$$

In the last equality the operators $\widehat{c}$ and $\widehat{c}^\dagger$ are independent of time and h.c. indicates the Hermitian conjugate term. The Heisenberg operator of the electric field can be further averaged over the space–time domain with a weighting function:

$$\widehat{\boldsymbol{E}}(\boldsymbol{r}, t) = \int \widehat{\boldsymbol{E}}(\boldsymbol{r} - \boldsymbol{r}', t - t')g(\boldsymbol{r}', t')\mathrm{d}^3r'\mathrm{d}t'\ . \tag{2}$$

With the aid of (1) we obtain

$$\widehat{\overline{E}}(\boldsymbol{r}, t) = \sum_s \left\{ \widehat{c}_s \boldsymbol{E}(\boldsymbol{r}, t) G(\boldsymbol{k}) + \text{h.c.} \right\} , \tag{3}$$

where

$$G(\boldsymbol{k}) = \int g(\boldsymbol{r}, t) e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} \mathrm{d}^3 r \mathrm{d}t , \quad \omega = ck . \tag{4}$$

The mean value of the averaged operator over the vacuum state is given by

$$\overline{\boldsymbol{E}} = \langle 0|\widehat{\overline{\boldsymbol{E}}}|0\rangle = 0 , \quad \overline{\Delta E^2} = \overline{E^2} = \langle 0|\widehat{\overline{\boldsymbol{E}^2(\boldsymbol{r}, t)}}|0\rangle = \frac{\hbar}{2\pi} \int G(\boldsymbol{k}) \omega \mathrm{d}^3 k , \tag{5}$$

where in the last transformation we replaced the sum over discrete modes by the integral over continual wave vectors and took into account two possible polarizations per mode. The same transformation can done for the magnetic field. The Fourier image of the weighting function is given by $G(\omega) = \exp(-\omega^2 l^2/2c^2)$. Finally, we obtain the bounded scale for the vacuum fluctuations of the field components:

$$\overline{\Delta E^2} = \overline{\Delta H^2} = \frac{\hbar c}{l^4} . \tag{6}$$

The high-frequency modes resulting in divergency of the vacuum fluctuations have been eliminated by such an averaging procedure.

6.17* Under the condition $L \gg z$ we can calculate the energy variation ΔU for vacuum fluctuations in the presence and absence of the metal plates. The divergency of vacuum fluctuations is not manifestable in this quantity. In the presence of metal plates, the lower limit in the integral over k can be chosen as $k_{\min} \approx 2\pi/z$ (we do not legitimate any field fluctuations with wavelength longer than z). On the basis of the solution of Problem 6.15 we find that ΔU selected per unit surface area of the plates can be expressed as follows:

$$\Delta U \approx \frac{\hbar c}{2\pi^2} z \left\{ \int_{k_{\min}}^{\infty} k^3 \mathrm{d}k - \int_0^{\infty} k^3 \mathrm{d}k \right\} = -\frac{\hbar c}{2\pi^2} z \int_0^{k_{\min}} k^3 \mathrm{d}k = -\frac{2\pi^2 \hbar c}{z^3} .$$

Then the force is given by the derivative over z:

$$\mathcal{F} = -\frac{\mathrm{d}}{\mathrm{d}z} \Delta U \approx -\frac{6\pi^2 \hbar c}{z^4} .$$

The plates attract each other with force decreasing with the separation distance. For a precise calculation, see Lifshitz and Pitaevskii (1980), which gives another numerical coefficient, $\pi^2/240$, instead of ours, $6\pi^2$. Since the calculated force depends on $k_{\min}$ in high (fourth) power, the result is quite sensitive to this parameter, which is known only in the order of its magnitude.

6.18•* In a hydrogen-type atom in addition to the nuclear field and to the field of other electrons, the vacuum fluctuations of the electromagnetic field also interfere with the motion of a valence electron. These fluctuations induce so-called electron vibration (*Zitterbewegung*) near its own atomic orbital and result in an energy level shift (Lamb shift). The variation of the potential energy associated with such an orbital vibration can be expressed as follows:

$$\delta U = U(\boldsymbol{r} + \delta \boldsymbol{r}) - U(\boldsymbol{r}) = \Delta U \cdot \delta \boldsymbol{r} + \frac{1}{2}\left(\frac{\partial^2 U}{\partial x_\alpha \partial x_\beta}\right) \delta x_\alpha \delta x_\beta + \dots \tag{1}$$

By averaging this expression over the random vacuum fluctuations, we obtain the following first-order correction to the potential energy caused by the vacuum fluctuations:

$$V(\boldsymbol{r}) = \langle \delta U \rangle = \frac{1}{6}(\nabla^2 U(\boldsymbol{r}))\langle \delta \boldsymbol{r}^2 \rangle \,. \tag{2}$$

Here we assumed full isotropy of the vacuum fluctuations: $\langle \delta \boldsymbol{r} \rangle = 0$, $\langle \delta x_\alpha \delta x_\beta \rangle = (1/3)\langle \delta \boldsymbol{r}^2 \rangle \delta_{\alpha\beta}$.

Let us estimate the square variance of the electron coordinate near its nondisturbed atomic orbital. We will assume that electron displacement can be described semiclassically. If the vibration frequency is much higher than atomic frequencies, the equation of motion for the electron driven by the field oscillator of the sth mode will be given by

$$m\delta \ddot{\boldsymbol{r}}_s = e\boldsymbol{E}_s \cos \omega_s t \quad (k\delta r \ll 1) \,, \tag{3}$$

This equation has the solution $\delta \boldsymbol{r}_s = -e\boldsymbol{E}_s(t)/m\omega_s^2$, and its square variance can be evaluated similarly to what was done in the solution of Problem 6.15. We have to make the evident replacement $\langle \boldsymbol{E}_s^2(t) \rangle \to \langle 0|\widehat{\boldsymbol{E}}_s^2|0\rangle = 2\pi\hbar\omega_s/V$, where the angle brackets indicate the average over the vacuum state. Finally, we get

$$\langle \delta \boldsymbol{r}_s^2 \rangle = \frac{2\pi e^2 \hbar}{m^2 \omega_s^3 V} \,. \tag{4}$$

The discrete sum over the field modes can be smoothed by an integral over the frequency:

$$\langle \delta \boldsymbol{r}^2 \rangle = \sum_s \langle \delta \boldsymbol{r}_s^2 \rangle = \frac{2e^2\hbar}{\pi m^2 c^3} \int_{\omega_{\min}}^{\omega_{\max}} \frac{d\omega}{\omega} \,. \tag{5}$$

To avoid evident divergence of this integral, we set a cutoff of the lower and the upper limits. In accordance with the above assumption, the lower limit is associated with atomic frequency ω_0. The upper limit is justified by another assumption that for a nonrelativistic atomic system the main contributing frequencies are restricted by the inequality $\hbar\omega \le mc^2$ (m is the electron's mass). Thus, in the assumption $\omega_{\min} = \omega_0$ and $\omega_{\max} = mc^2/\hbar$, we obtain

$$\langle \delta \boldsymbol{r}^2 \rangle = \frac{2e^2\hbar}{\pi m^2 c^3} \ln\left(\frac{mc^2}{\hbar\omega_0}\right) \,. \tag{5}$$

Now we can evaluate the correction to the electron energy, which is caused by vacuum fluctuations. According to the basic statements of perturbation theory (Landau and Lifshitz, 1977) the first-order correction for interaction $\widehat{V}$ is given by

$$\Delta\mathcal{E} = \langle\psi|\widehat{V}|\psi\rangle \,, \tag{6}$$

where ψ is the original stationary wave function of the disturbed state. For the hydrogen atom, where the nucleus is a point-like particle, we have $U(\boldsymbol{r}) = -e^2/r$, $\nabla^2 U(\boldsymbol{r}) = 4\pi e^2 \delta(\boldsymbol{r})$. Using this result and (2) and (5), we can evaluate the energy shift as follows:

$$\Delta\mathcal{E} = \frac{4e^4\hbar}{3m^2c^3}|\psi(0)|^2 \ln\left(\frac{mc^2}{\hbar\omega_0}\right) \,. \tag{7}$$

In our approximation the shift occurs only for those states where $\psi(0) \neq 0$. Only states with zero orbital momentum, $l = 0$ (i.e., s states), posses such a property. For the 2s state of hydrogen, the wave function at the origin of the atom is $|\psi_{200}(0)|^2 = 1/8\pi a_{\mathrm{B}}^3$, where $a_{\mathrm{B}} = \hbar^2/me^2$ is the Bohr radius. Let us use atomic units, where $\hbar\omega_0$ represents the atomic unit of energy: $\hbar\omega_0 = me^4/\hbar^2$. After substituting it into expression (7), we obtain

$$\Delta\mathcal{E}_{201/2} = \frac{1}{6\pi}mc^2\alpha^5 \ln\left(\frac{1}{\alpha^2}\right) \,, \quad \Delta\mathcal{E}_{211/2} = 0 \,, \quad \alpha = \frac{e^2}{\hbar c} \approx \frac{1}{137} \,. \tag{8}$$

Here the lower indices indicate the principal quantum number, orbital angular momentum, and total angular momentum of the electron ($j = 1/2$ in both cases). If the vacuum fluctuations are disregarded, the two atomic levels considered will be degenerate; if they are not disregarded, the $2s$ level will be upshifted. Our estimate gives $\Delta\mathcal{E}_{201/2} = 0.52mc^2\alpha^5$. For a more precise evaluation, see Berestetskii *et al.* (1982), which gives a slightly different value $0.41mc^2\alpha^5$. The corresponding experimental value is 1050 MHz.

6.19•* In classical theory the total angular momentum of the field considered with respect to a point $\boldsymbol{r}_0$ is given by

$$\boldsymbol{J}(\boldsymbol{r}_0) = \int (\boldsymbol{r} - \boldsymbol{r}_0) \times \boldsymbol{\mathcal{P}}\mathrm{d}V = \boldsymbol{J}(0) - \boldsymbol{r}_0 \times \boldsymbol{P} \,, \quad \boldsymbol{\mathcal{P}} = \frac{1}{4\pi c}\boldsymbol{E} \times \boldsymbol{H} \,, \tag{1}$$

where $\boldsymbol{\mathcal{P}}$ is the momentum density. We substitute $\boldsymbol{H} = \nabla \times \boldsymbol{A}$ and transform integral (1) by partial integration under the assumption that the components of the field vanish at infinity (it is useful to apply relation (1.23) (i) and the identity (ii) in the solution of Problem 1.28 and the Maxwell equation $\nabla\mathbf{E} = 0$). This gives us two terms: $\boldsymbol{J}(\boldsymbol{r}_0) = \boldsymbol{L}(\boldsymbol{r}_0) + \boldsymbol{S}$, where

$$\boldsymbol{L}(\boldsymbol{r}_0) = \frac{1}{4\pi c}\int E_\alpha[(\boldsymbol{r} - \boldsymbol{r}_0) \times \nabla]A_\alpha \mathrm{d}V \tag{2}$$

is the field orbital momentum, which *depends* on the origin of the coordinate frame, and

$$S = \frac{1}{4\pi c}\int [E \times A]\mathrm{d}V \,, \tag{3}$$

which is the spin (internal) angular momentum, whose definition does not depend on the origin. Corresponding operators can be obtained from (4) and (5) with the aid of the recipe introduced in the solution of Problem 6.14•:

$$\widehat{L} = \frac{1}{8\pi c}\int \{\widehat{E}_\alpha[(r - r_0) \times \nabla]\widehat{A}_\alpha + \widehat{A}_\alpha[(r - r_0) \times \nabla]\widehat{E}_\alpha\}\mathrm{d}V \,, \tag{4}$$

$$\widehat{S} = \frac{1}{8\pi c}\int [\widehat{E} \times \widehat{A} - \widehat{A} \times \widehat{E}]\mathrm{d}V \,. \tag{5}$$

6.20* With substitution of operators (6.13) and (6.14), and basis functions (6.16) into the expression for the spin angular momentum (5), derived in the solution of previous problem, we obtain

$$\widehat{S} = i\hbar \sum_{k,\sigma,\sigma'} \left(\widehat{a}^\dagger_{k\sigma'}\widehat{a}_{k\sigma} + \frac{1}{2}\delta_{\sigma\sigma'}\right)\left(e_{k\sigma} \times e^*_{k\sigma'}\right) \,.$$

In the case of linear polarization, the polarization vectors are real and their vector product is nonzero only if $\sigma \neq \sigma'$: $e_{k1} \times e^*_{k2} = k/k$. The spin angular momentum takes the form

$$\widehat{S} = i\sum_k \left(\frac{\hbar k}{k}\right)\left(\widehat{a}^\dagger_{k2}\widehat{a}_{k1} - \widehat{a}^\dagger_{k1}\widehat{a}_{k2}\right) \,,$$

and after averaging over the Fock states with fixed photon number states, we find it has zero value.

In the case of circularly polarized light, the polarization vectors are $e_{k,\pm1} = (e_{k1} \pm e_{k2})/\sqrt{2}$ (see Problem 2.133). Their vector product is nonzero only if $\sigma = \sigma'$. This gives $e_{k,\pm1} \times e^*_{k,\pm1} = \mp i k/k$. The spin angular momentum of each k mode is directed along k and is equal to the reduced Planck constant $\hbar$ multiplied by the difference between the number of photons with right-handed and left-handed polarizations (helicities):

$$S = \sum_k \left(\frac{\hbar k}{k}\right)(N_{k,+1} - N_{k,-1}) \,.$$

6.21•

$$I_\sigma(\omega, n) = \frac{\hbar\omega^3}{8\pi^3 c^2} N_{k\sigma} \,.$$

6.22

$$\overline{\mathcal{E}}_s = \hbar\omega_s\left(\overline{N}_s + \frac{1}{2}\right) = \frac{1}{2}\coth\frac{\hbar\omega_s}{2T} \,.$$

6.23• The thermal radiation in the statistically equilibrium state has an isotropic distribution in space and its spectral intensity $I(\omega)$ (the flux of its energy density in a given direction per unit spectral interval) can be expressed by the spectral density of the radiation energy as $\rho(\omega) = 4\pi I(\omega)/c$. If we multiply the energy of a quantum $\hbar\omega$ by the mean number of such quanta in a particular mode, and then by number of modes in the quantization volume $\mathcal{V}$ (2.164), by frequency interval $d\omega$, and by solid angle $d\Omega_k$ we obtain the energy

$$d\mathcal{E}(\omega) = \hbar\omega_s \overline{N} \frac{2\mathcal{V}}{(2\pi c)^3} \omega^3 d\omega d\Omega_k \,. \tag{1}$$

After integration of this differential relation over the full solid angle and normalization of it to the unit volume $\mathcal{V}$ and spectral interval $d\omega$, we get the spectral density of radiation energy:

$$\rho(\omega) = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{e^{(\hbar\omega/T)} - 1} \tag{2}$$

(*Planck distribution* for the spectral density of equilibrium radiation energy).

In Figure 6.5 (in arbitrary units for both axes) the Planck spectral distribution is plotted for three different temperatures related as 1 : 1.5 : 2. Function $\rho(\omega)$ is maximum at frequency

$$\omega_m \approx 2.822\frac{T}{\hbar} \,, \quad \lambda_m = \frac{2\pi c\hbar}{2.822T} \approx \frac{0.511}{T}\ \text{cm} \,, \tag{3}$$

where in the last expression the temperature is in kelvins. The dependence $\omega_m \propto T$ is known as *Wien's displacement law.* This law claims that the maximum of the radiation spectrum shifts to higher frequency (shorter wavelength) with increasing T. As examples, the surface temperature of the yellow Sun (its photosphere) is approximately 6000 K, that of white Sirius is approximately 10 000 K, and that of blue Vega is approximately 20 000 K.

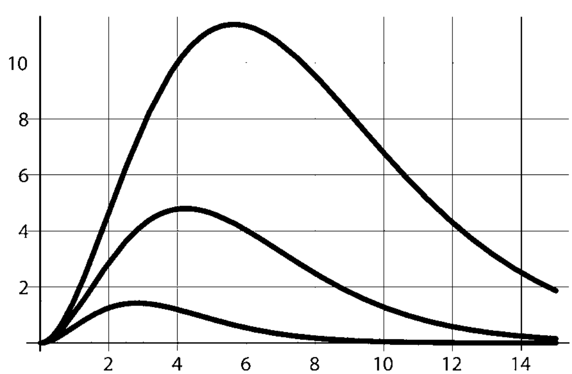

Figure 6.5 Planck distribution. Dependency of spectral density on frequency.

There are two frequency domains where the Planck formula can be simplified. For low frequencies, when $T/\hbar\omega \gg 1$, we have the *Rayleigh–Jeans law*

$$\rho(\omega) \approx \frac{T}{\pi^2 c^3}\omega^2 \propto \omega^2 \,. \tag{4}$$

This law reproduces the fully classical physics *for all frequencies.* It is a direct consequence of the multiplication of an oscillator's mean energy T by the density of the oscillators, which is proportional to ω^2. The spectral density of the radiation energy increases infinitely with increasing frequency of the oscillator ("ultraviolet catastrophe").

For high frequencies, when $T/\hbar\omega \ll 1$, the Planck formula transforms to exponential decay of the spectral density of the radiation energy, *Wien's spectral law*:

$$\rho(\omega) \approx \frac{\hbar}{\pi^2 c^3}\omega^3 e^{-\hbar\omega/T} . \tag{5}$$

The exponential decay of the spectral density is associated with the discrete nature of the oscillator's energy of a particular frequency. When $\hbar\omega > T$, the thermal energy (on the order of T) of atoms is insufficient to initiate transition to a higher level and cause emission of a photon. So the number of quanta with such energy is quite limited.

The Planck formula (1) is relevant for the whole frequency domain and is reliably applicable for a broad range of temperatures.

6.25

$$\rho'(\lambda) = \frac{16\pi^2 c\hbar}{\lambda^5}\frac{1}{\exp(2\pi c\hbar/\lambda T) - 1}$$

(see Figure 6.6, in arbitrary units for wavelength on the horizontal axis and energy density on the vertical axis). This distribution decreases exponentially when $\lambda \to 0$ and as a power of λ when $\lambda \to \infty$. The dependence has a maximum at wavelength

$$\lambda = \lambda'_m = \frac{2\pi\hbar c}{4.99T} = \frac{0.289}{T}\text{cm} ,$$

where in last expression the temperature is in kelvins. The numerical factor 4.99 is the solution of the transcendental equation $e^{-x} = 1 - x/5$.

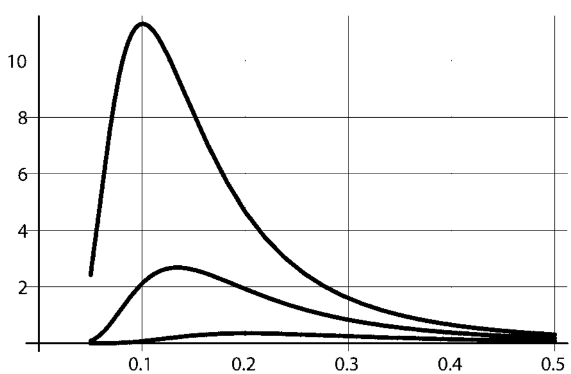

Figure 6.6 Planck distribution. Dependency of spectral density on wavelength.

The maximum of the spectral distribution $\rho'(\lambda)$ shifts to shorter wavelengths with increasing T, $\lambda'_m \propto 1/T$, which is *Wien's law*. We stress that λ'_m and the corresponding frequency $\omega'_m = 2\pi c/\lambda'_m$ do not coincide with λ_m and ω_m obtained from the distribution by frequencies (see equation (3) in the solution of Problem 6.23•). Thus, $\lambda'_m = 0.565\lambda_m < \lambda_m$ and $\omega'_m = 1.77\omega_m$, and this difference is noticeable. For example, for radiation from the Sun ($T = 6000$ K) wavelength obtained from the frequency distribution is $\lambda_m = 8.5 \times 10^{-5}$ cm (near-infrared spectral range) and from the wavelength distribution is $\lambda'_m = 4.82 \times 10^{-5}$ cm (blue spectral range!). The visible yellow-white natural color of the Sun lies between λ_m and λ'_m.

6.26

	Name	T (K)	λ_m (cm)	λ'_m (cm)	Range
(i)	Relict radiation	3	0.17	0.095	Microwave
(ii)	Earth's surface	300	1.7×10^{-3}	0.95×10^{-3}	Infrared
(iii)	Mu Cephei	2000	2.55×10^{-4}	1.43×10^{-4}	Dark red
(iv)	Sun	6000	8.5×10^{-5}	4.82×10^{-5}	Yellow-white
(v)	Sirius	11 000	4.6×10^{-5}	2.57×10^{-5}	White
(vi)	Beta Centauri	22 500	2.27×10^{-5}	1.29×10^{-5}	Blue
(vii)	Lyra nebula	75 000	0.68×10^{-5}	0.38×10^{-5}	Ultraviolet
(viii)	Neutron star	250 000	2×10^{-6}	1.1×10^{-6}	X-ray
(ix)	Inner regions of a star	2.5×10^7	2×10^{-8}	1.1×10^{-8}	X-ray

6.27

$$F = -T \ln Z = -\frac{1}{3}\mathcal{E}\,, \qquad S = \frac{16\sigma}{3c} T^3 \mathcal{V}\,,$$
$$C_V = \frac{16\sigma}{c} T^3 \mathcal{V}\,, \qquad \mathcal{P} = \frac{\mathcal{E}}{3\mathcal{V}}\,,$$

where σ is the Stefan–Boltzmann constant and $\mathcal{E}$ is the internal energy (see the definitions given in the solution to Problem 6.24).

6.28 The photon-number states (Fock states) $|n\rangle$ are discrete and complete sets of the states. Any arbitrary state of the field existing in a particular mode can be expanded in the basis of $|n\rangle$. Coherent states $|z\rangle$ (where z is an arbitrary complex number) are a continuous manifold which is formed by an overfull set of the eigenfunctions. It is sufficient to use only some of them to expand any quantum state of the field mode.

6.29 Let us assume that the following finite-norm states can exist:

$$\hat{c}^\dagger |\beta\rangle = \beta|\beta\rangle\,; \quad \langle\beta|\hat{c} = \langle\beta|\beta^*\,. \tag{1}$$

The state vector $|\beta\rangle$ can be expanded in the series of the photon number states (see Example 6.4):

$$|\beta\rangle = \sum_{n=0}^{\infty} \langle n|\beta\rangle |n\rangle \,. \tag{2}$$

The expansion coefficients are given by the following recurrent relation:

$$\langle n|\beta\rangle = \frac{\sqrt{n}}{\beta} \langle n-1|\beta\rangle \,. \tag{3}$$

With applying this relation n times, we obtain

$$\langle n|\beta\rangle = \frac{\sqrt{n!}}{\beta^n} \langle 0|\beta\rangle \,. \tag{4}$$

and the norm of $|\beta\rangle$ is given by

$$\langle\beta|\beta\rangle = \sum_{n=0}^{\infty} \frac{n!}{|\beta|^{2n}} \,. \tag{5}$$

This series diverges for any finite β and our original assumption becomes self-contradicting, which proves the statement claimed in the condition for this problem.

6.30 The time dynamics of a coherent state is described by the nonstationary Schrödinger equation:

$$i\hbar \frac{\partial |\alpha(t)\rangle}{\partial t} = \widehat{\mathcal{H}} |\alpha(t)\rangle \,, \tag{1}$$

where $\widehat{\mathcal{H}} = \hbar\omega(\widehat{a}^\dagger \widehat{a} + 1/2)$. The straightforward solution of this equation considered with initial condition $|\alpha(t)\rangle|_{t\to 0} = |\alpha_0\rangle$ is given by

$$|\alpha(t)\rangle = \exp\left(-\frac{i}{\hbar}\widehat{\mathcal{H}} t\right) |\alpha_0\rangle \,. \tag{2}$$

After expansion of the coherent state in terms of the photon number states we get

$$\begin{aligned} |\alpha(t)\rangle &= \exp\left(-\frac{|\alpha_0|^2}{2}\right) \sum_{n=0}^{\infty} \exp\left(-i\omega\left(n+\frac{1}{2}\right)t\right) \frac{\alpha_0^n}{\sqrt{n!}} |n\rangle \\ &= \exp\left(-\frac{i\omega t}{2}\right) \exp\left(-\frac{|\alpha_0 \exp(-i\omega t)|^2}{2}\right) \\ &\quad \times \sum_{n=0}^{\infty} \frac{(\alpha_0 \exp(-i\omega t))^n}{\sqrt{n!}} |n\rangle \\ &= \exp\left(-\frac{i\omega t}{2}\right) |\alpha_0 \exp(-i\omega t)\rangle \,, \end{aligned} \tag{6.134}$$

where we took into account that the photon number states are eigenstates of the Hamiltonian $\widehat{\mathcal{H}}$. Thus, the coherent state remains coherent at any time and its amplitude depends on time as $\alpha(t) = \alpha_0 \exp(-i\omega t)$. This can be visualized as rotation of α with frequency ω in the phase complex plane. The phase factor $\exp(-i\omega t/2)$ is associated with the vacuum fluctuations and this factor is the same for all the coherent states. The presence of this factor does not affect any calculation of the expectation values of any observable quantities.

6.31 The coherent state is defined as

$$\widehat{c}|\alpha\rangle = \alpha|\alpha\rangle \,. \tag{1}$$

With application of (C54), the last equation can be introduced in the oscillator's position representation:

$$\left(\xi + \frac{\partial}{\partial\xi}\right)\langle\xi|\alpha\rangle = \sqrt{2}\alpha\langle\xi|\alpha\rangle \,, \tag{2}$$

where $\langle\xi|\alpha\rangle = \psi_\alpha(\xi)$ is the wave function of a coherent state in the position representation and ξ is the dimensionless coordinate. Straightforward solution of differential equation (2) leads to

$$\langle\xi|\alpha\rangle = C(\alpha)\exp\left(\frac{-(\xi - \sqrt{2}\alpha)^2}{2}\right) \,. \tag{3}$$

The normalization constant can be taken as real:

$$C(\alpha) = \left(\frac{1}{\pi}\right)^{1/4} \exp(-2(\mathrm{Im}\alpha)^2) \,. \tag{4}$$

Expression (3) is a Gaussian distribution parameterized by mean value $\langle\xi\rangle = \sqrt{2}\alpha$ and variance $\langle\Delta\xi^2\rangle = 1/2$. For a free field existing in the coherent state, the dependence of its amplitude on time is given by $\alpha(t) = \alpha_0 \exp(-i\omega t)$ (see the solution to previous problem). The probability distribution for the oscillator's positions becomes time dependent:

$$\rho(\xi) = \pi^{-1/2} \exp\left(-(\xi - \xi_0 \cos\omega t)^2\right) \,, \tag{5}$$

where we used $\alpha_0 = \xi_0/\sqrt{2}$. This distribution shows that the mean value of the oscillator's position has the same time dependence as in the dynamics of the classical harmonic oscillator.

6.32 Let us expand a coherent state in the basis of the Fock number states

$$|\alpha\rangle = \exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle \,. \tag{1}$$

Using relation $|n\rangle = \hat{c}^{\dagger n}|0\rangle/\sqrt{n!}$, we can express expansion (1) in the following form:

$$|\alpha\rangle = \exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{n=0}^{\infty}\frac{(\alpha\hat{c}^\dagger)^n}{n!}|0\rangle = \exp\left(-\frac{|\alpha|^2}{2}\right)\exp\left(\alpha\hat{c}^\dagger\right)|0\rangle\ .$$

The above relation can be transformed to a symmetric form if we apply the following operator acting on the vacuum state:

$$\exp\left(-\alpha^*\hat{c}\right)|0\rangle = |0\rangle\ .$$

Then according to theorem (C73) we find

$$\exp\left(-\frac{|\alpha|^2}{2}\right)\exp\left(\alpha\hat{c}^\dagger\right)\exp\left(-\alpha^*\hat{c}\right) = \exp\left(\alpha\hat{c}^\dagger - \alpha^*\hat{c}\right)\ ,$$

which gives us the displacement operator defined in the statement of the problem.

6.33 The quantities needed can be calculated if we express the position and momentum operators in terms of the creation and annihilation operators. For the position quadrature component we find

$$\begin{aligned}\left\langle \Delta X_1^2\right\rangle = \langle\xi^2\rangle - \langle\xi\rangle^2 &= \left\langle\frac{\hat{c}^2 + \hat{c}^{\dagger 2} + 2\hat{c}^\dagger\hat{c} + 1}{2}\right\rangle - \frac{\langle(\hat{c}+\hat{c}^\dagger)\rangle^2}{2}\\ &= \left(\frac{1}{2}\right)(\alpha^2 + \alpha^{*^2} + 2|\alpha|^2 + 1) - \left(\frac{1}{2}\right)(\alpha^2 + \alpha^{*2} + 2|\alpha|^2)\\ &= \frac{1}{2}\ .\end{aligned}$$

Similarly, the square variance of the momentum quadrature component is given by $\langle\Delta X_2^2\rangle = 1/2$. For the standard position and momentum variables we get $\langle\Delta Q^2\rangle = \hbar/(2\omega)$ and $\langle\Delta P^2\rangle = \hbar\omega/2$.

6.34

$$\begin{aligned}D_1(t) = \ &D_1(0)\cos^2(\omega t) + D_2(0)\sin^2(\omega t)\\ &+\left[\left\langle\hat{X}_1\hat{X}_2 + \hat{X}_2\hat{X}_1\right\rangle - 2\left\langle\hat{X}_1\right\rangle^2\left\langle\hat{X}_2\right\rangle^2\right]\cos(\omega t)\sin(\omega t)\ ,\end{aligned}$$

$$\begin{aligned}D_2(t) = \ &D_2(0)\cos^2(\omega t) + D_1(0)\sin^2(\omega t)\\ &-\left[\left\langle\hat{X}_1\hat{X}_2 + \hat{X}_2\hat{X}_1\right\rangle - 2\left\langle\hat{X}_1\right\rangle^2\left\langle\hat{X}_2\right\rangle^2\right]\cos(\omega t)\sin(\omega t)\ .\end{aligned}$$

Both quantities oscillate at double the mode frequency. Their sum $D_1(t) + D_2(t) = D_1(0) + D_2(0)$ is independent of time, and the variance oscillations are shifted by $\pi/2$ such that $D_1(t) = D_2(t - \pi/(2\omega))$.

For the field existing in the Fock states or in the coherent states $\langle \widehat{X}_1 \widehat{X}_2 + \widehat{X}_2 \widehat{X}_1 \rangle - 2\langle \widehat{X}_1 \rangle^2 \langle \widehat{X}_2 \rangle^2 = 0$ (each term is equal to zero for the Fock states; the expectation value of the product is equal to the product of the expectation values for coherent states) and $D_1(0) = D_2(0)$ is also fulfilled. Thus, for these states, the variances of the quadrature components are independent of time.

6.35 Using the Poisson distribution (6.30), we can find the variance of the photon number $\langle \Delta n^2 \rangle = \langle (n - \bar{n})^2 \rangle = \bar{n} = |z|^2$.

6.36 Using the completeness of the coherent state basis set, we get

$$\begin{aligned} \widehat{\rho} &= \sum_{n,m} C_{n,m} \frac{1}{\pi} \int \widehat{c}^n |\alpha\rangle \langle \alpha| \widehat{c}^{\dagger m} \mathrm{d}^2 \alpha \\ &= \int \sum_{n,m} \frac{1}{\pi} C_{n,m} \alpha^n \alpha^{*m} |\alpha\rangle \langle \alpha| \mathrm{d}^2 \alpha \; . \end{aligned}$$

Comparing this expression with (6.38), we find

$$\mathcal{P}(\alpha) = \frac{1}{\pi} C_{n,m} \alpha^n \alpha^{*m} \; .$$

We note that this function is considered as a function of both arguments α and α^*, that is, it is not an analytical function of α.

6.37• Let us expand the density matrix of a single mode in the basis set of the Fock states

$$\widehat{\rho} = \sum_{n=0}^{\infty} \rho_{nn} |n\rangle \langle n| \; , \quad \text{where} \quad \rho_{nn} = (1 - e^{-\alpha}) e^{-\alpha n} \; , \qquad \alpha = \frac{\hbar\omega}{T} \; . \tag{1}$$

Here, ρ_{nn} was defined by formula (1) in Example 6.3 as the Gibbs distribution W_N. The relations given above can be expressed via the mean number of photons, given by (6.24)

$$\widehat{\rho} = \frac{1}{1 + \overline{n}} \sum_{n=0}^{\infty} \left(\frac{\overline{n}}{1 + \overline{n}} \right)^n |n\rangle \langle n| \; , \tag{2}$$

The matrix element contributing to (6.42) is given by

$$\begin{aligned} \langle -u|\widehat{\rho}|u\rangle &= \frac{1}{1 + \overline{n}} \sum_{n=0}^{\infty} \left(\frac{\overline{n}}{1 + \overline{n}} \right)^n \langle -u|n\rangle \langle n|u\rangle \\ &= \frac{1}{1 + \overline{n}} \exp\left(-\frac{2\overline{n} + 1}{1 + \overline{n}} |u|^2 \right) \; . \end{aligned} \tag{3}$$

Here we used (6.28) and performed a power expansion of the exponent. After substitution of (3) into integral (6.42) and evaluation of the integral, we arrive at expression (6.44).

6.38* The density matrix in the Fock representation can be found by straightforward calculation of its matrix elements:

$$\begin{aligned}\rho_{nm} &= \frac{1}{\pi}\int \mathcal{P}(\alpha)\langle n|\alpha\rangle\langle\alpha|m\rangle \mathrm{d}^2\alpha\\ &= \frac{1}{\pi}\int \mathcal{P}(\alpha)\frac{\alpha^n\alpha^{*m}}{\sqrt{n!m!}}\exp(-|\alpha|^2)\mathrm{d}^2\alpha\\ &= \frac{1}{\pi}\int_0^\infty |\alpha|d|\alpha|\int_0^{2\pi}\mathrm{d}\varphi\mathcal{P}(\alpha)\frac{|\alpha|^{n+m}}{\sqrt{n!m!}}\exp(i(n-m)\varphi)\exp(-|\alpha|^2)\,.\end{aligned}$$

Evaluation of the phase integral over φ creates the Kronecker symbol δ_{nm}. The remaining integral over the modulus $|\alpha|$ can be evaluated with the aid of the properties of the delta function. Finally, we obtain

$$\rho_{nm} = \frac{|\alpha_0|^{2n}}{n!}\exp(-|\alpha_0|^2)\delta_{nm}\,.$$

The diagonal elements reproduce the photon number distribution in the coherent state but the off-diagonal elements are equal to zero. We stress that the absence of the off-diagonal components makes this state different from the coherent state.

The normally ordered characteristic function can be found from its definition:

$$\begin{aligned}\theta[\kappa,\kappa^*] &= \mathrm{Sp}(\widehat{\rho}\exp(\kappa\widehat{a}^\dagger)\exp(-\kappa^*\widehat{a}))\\ &= \sum_n \rho_{nn}\langle n|\exp(\kappa\widehat{a}^\dagger)\exp(-\kappa^*\widehat{a})|n\rangle\,. \end{aligned}\tag{6.135}$$

Using the power expansion of the operator $\exp(-\kappa^*\widehat{a})$ in a power series of $\widehat{a}$,

$$\exp(-\kappa^*\widehat{a})|n\rangle = \sum_{m=0}^{\infty}\frac{1}{m!}(-\kappa^*\widehat{a})^m|n\rangle\,,$$

and applying the standard recurrent property of the annihilation operator $\widehat{a}^m|n\rangle = \sqrt{n*(n-1)*\ldots(n-m)}|n-m\rangle$, if $n\ge m$ and $\widehat{a}^m|n\rangle = 0$, if $n<m$, we get

$$\begin{aligned}\theta[\kappa,\kappa^*] &= \sum_n\sum_{l=0}^{n}\sum_{m=0}^{n}\rho_{nn}\kappa^l\frac{\sqrt{n!}}{l!\sqrt{(n-l)!}}\\ &\quad\times(-\kappa^*)^m\frac{\sqrt{n!}}{m!\sqrt{(n-m)!}}\langle n-l|n-m\rangle\\ &= \sum_n\sum_{l=0}^{n}\rho_{nn}(-|\kappa|^2)^l\frac{n!}{(l!)^2(n-l)!}\,.\end{aligned}$$

The sum over l can be recomposed in terms of the Laguerre polynomials (see Abramovitz and Stegun, 1965):

$$\sum_{l=0}^{n}(-1)^l\frac{n!}{l!^2(n-l)!}x^l = L_n(x)\,,$$

which leads to

$$\theta[\kappa,\kappa^*] = \sum_n \rho_{nn} L_n(|\kappa|^2) \,.$$

If we substitute the matrix elements above into this expansion and take into account the identity

$$\sum_{n=0}^{\infty} \frac{y^n}{n!} L_n(x) = e^y J_0(2\sqrt{xy}) \,,$$

we can express the characteristic function via the Bessel function of zero order:

$$\theta[\kappa,\kappa^*] = J_0(2|\kappa||\alpha_0|) \,. \tag{6.136}$$

where on the right side while calculating the derivative over κ or κ^* we assume that $|\kappa| = \sqrt{\kappa\kappa^*}$.

The field state considered in this problem can be generated, for example, by a random classical current $I(t) = |\alpha_0| \exp(i(\omega_0 t + \varphi))$, where α_0 and ω_0 are the amplitude and frequency of this current, but phase φ is a stochastic variable randomly distributed between 0 and 2π.

6.39 For two independent stochastic processes the characteristic function is factorized:

$$\theta[\kappa,\kappa^*] = \theta_1[\kappa,\kappa^*]\theta_2[\kappa,\kappa^*] = \exp\left(\kappa\alpha_0^* - \kappa^*\alpha_0\right)\exp(-\bar{n}\kappa\kappa^*) \tag{6.137}$$

The first equality is a consequence of the superposition principle since it leads to the following decomposition of the statistical moments in terms of the moments associated with the driving processes:

$$\langle a\rangle = \langle a\rangle_1 + \langle a\rangle_2 \,,$$

$$\langle a^\dagger a\rangle = \langle a^\dagger a\rangle_1 + \langle a^\dagger a\rangle_2 + \langle a^\dagger\rangle_1\langle a\rangle_2 + \langle a^\dagger\rangle_2\langle a\rangle_1 \,,$$

an so on. Parameter $\bar{n}$ is us the mean number of the photons in the incoherent Gaussian-type radiation. In accordance with (6.43), the normally ordered characteristic function can be found as a Fourier transform of the Glauber–Sudarshan quasi-probability $\mathcal{P}(\alpha)$. We encourage the reader to verify that in the situation considered quasi-probability is the convolution of the quasi-probabilities of both contributing stochastic processes:

$$\begin{aligned}\mathcal{P}(\alpha) &= \int \mathcal{P}_1(\beta)\mathcal{P}_2(\alpha-\beta)\mathrm{d}^2\beta \\ &= \frac{1}{\pi\bar{n}} \int \delta(\beta-\alpha_0)\exp\left(-\frac{|\alpha-\beta|^2}{\bar{n}}\right)\mathrm{d}^2\beta \\ &= \frac{1}{\pi\bar{n}}\exp\left(-\frac{|\alpha-\alpha_0|^2}{\bar{n}}\right) \,.\end{aligned}$$

The result obtained has a clear physical meaning. The superposed state has a Gaussian-type quasi-probability distribution centered near the mean value α_0 and has uncertainty within a circle of area $\sqrt{\bar{n}}$. This distribution transforms to a delta function when $\bar{n} \to 0$, which corresponds to a coherent state. In the case when $\alpha_0 = 0$, we obtain the Gaussian-type distribution corresponding to incoherent radiation.

6.40 The correlation function can be calculated with the aid of the corresponding Glauber–Sudarshan quasi-probability. For thermal radiation we find

$$\begin{aligned} G^{m,m} &= \int \mathcal{P}(\alpha)\alpha^{*m}\alpha^m \mathrm{d}^2\alpha = \int \frac{1}{\pi\bar{n}} \exp\left(-\frac{|\alpha|^2}{\bar{n}}\right) \alpha^{*m}\alpha^m \mathrm{d}^2\alpha \\ &= m!\bar{n} \end{aligned}$$

and for a single-mode coherent radiation we get

$$G^{m,m} = \int \mathcal{P}(\alpha)\alpha^{*m}\alpha^m \mathrm{d}^2\alpha = \int \delta(\alpha - \alpha_0)\alpha^{*m}\alpha^m \mathrm{d}^2\alpha = |\alpha_0|^2 = \bar{n} .$$

6.41 The squeezed states are defined as eigenfunctions from the following equation:

$$(\mu\hat{c} + \nu\hat{c}^\dagger)|\gamma,\beta\rangle = \beta|\gamma,\beta\rangle , \tag{1}$$

where $|\mu|^2 - |\nu|^2 = 1$. Operator $\mu\hat{c} + \nu\hat{c}^\dagger$ can be expressed as a result of the unitary transform of the annihilation operator with the squeezing operator:

$$\mu\hat{c} + \nu\hat{c}^\dagger = \hat{S}(\gamma)\hat{c}\hat{S}^\dagger(\gamma) . \tag{2}$$

The reader can verify this with the aid of the following operator identity:

$$\exp(\hat{A})\hat{B}\exp(-\hat{A}) = \hat{B} + [\hat{A}, \hat{B}] + \frac{1}{2}[\hat{A}, [\hat{A}, \hat{B}]] + \dots ,$$

where $\hat{B} = \hat{c}$ and $\hat{A} = \gamma^*\hat{c}^2/2 - \gamma\hat{c}^{\dagger 2}/2$, and $\mu = \cosh r$ $\nu = e^{2i\vartheta}\sinh r$.

Then equation (1) can be transformed to the following form:

$$\hat{S}(\gamma)\hat{c}\hat{S}^\dagger(\gamma)|\gamma,\beta\rangle = \beta|\gamma,\beta\rangle , \tag{3}$$

or

$$\hat{c}\hat{S}^\dagger(\gamma)|\gamma,\beta\rangle = \beta\hat{S}^\dagger(\gamma)|\gamma,\beta\rangle . \tag{4}$$

The last relation shows that the function $\hat{S}^\dagger(\gamma)|\gamma,\beta\rangle$ describes a coherent state with eigenvalue β, that is, $|\beta\rangle = \hat{S}^\dagger(\gamma)|\gamma,\beta\rangle$. This leads to

$$|\gamma,\beta\rangle = \hat{S}(\gamma)|\beta\rangle = \hat{S}(\gamma)\hat{D}(\beta)|\mathrm{vac}\rangle .$$

Because of general uncertainty of the phase of the wave function the complex parameters μ and ν can be considered as functions of two real variables r and ϑ. In turn, these variables can be integrated into one complex parameter γ.

6.42 Let us use the definition of squeezed states:

$$(\mu\widehat{c} + \nu\widehat{c}^\dagger)|\mu, \nu, \beta\rangle = \beta|\mu, \nu, \beta\rangle\ . \tag{1}$$

We multiply this equality from the left by an arbitrary bra vector of the coherent state $\langle\alpha|$:

$$\langle\alpha|\mu\widehat{c} + \nu\widehat{c}^\dagger|\mu, \nu, \beta\rangle = \beta\langle\alpha|\mu, \nu, \beta\rangle\ . \tag{2}$$

If we use one of the representations of the coherent state (see Problem 6.32),

$$|\alpha\rangle = \exp\left(-\frac{|\alpha|^2}{2}\right)\exp\left(\alpha\widehat{c}^\dagger\right)|\text{vac}\rangle\ , \tag{3}$$

we can transform the left side of (2) as

$$\begin{aligned}\langle\alpha|\mu\widehat{c} + \nu\widehat{c}^\dagger|\mu, \nu, \beta\rangle &= \mu\langle\alpha|\widehat{c}|\mu, \nu, \beta\rangle + \nu\langle\alpha|\widehat{c}^\dagger|\mu, \nu, \beta\rangle \\ &= \mu\langle\text{vac}|\exp\left(-\frac{|\alpha|^2}{2}\right)\exp\left(\alpha^*\widehat{c}\right)\widehat{c}|\mu, \nu, \beta\rangle \\ &\quad + \nu\alpha^*\langle\alpha|\mu, \nu, \beta\rangle \\ &= \mu\left(\frac{\partial}{\partial\alpha^*} + \frac{\alpha}{2}\right)\langle\text{vac}|\exp\left(-\frac{\alpha\alpha^*}{2}\right) \\ &\quad \times\exp\left(\alpha^*\widehat{a}\right)|\mu, \nu, \beta\rangle + \nu\alpha^*\langle\alpha|\mu, \nu, \beta\rangle\ .\end{aligned} \tag{4}$$

Note that in the last equality the variables α and α^* are considered as independent, and are mutually conjugated only after differentiation. Relation (2) can be written as a first-order differential equation with respect to the desired expansion coefficient of squeezed state in terms of coherent states $|\alpha\rangle$:

$$\frac{\partial}{\partial\alpha^*}\langle\alpha|\mu, \nu, \beta\rangle = \left(\frac{\beta}{\mu} - \frac{\alpha}{2} - \frac{\nu\alpha^*}{\mu}\right)\langle\alpha|\mu, \nu, \beta\rangle\ . \tag{5}$$

Integration gives

$$\langle\alpha|\mu, \nu, \beta\rangle = C\exp\left(\frac{\beta\alpha^*}{\mu} - \frac{|\alpha|^2}{2} - \frac{\nu\alpha^{*2}}{2\mu}\right)\ . \tag{6}$$

The constant of integration can be found from the normalization condition:

$$\langle\mu, \nu, \beta|\mu\nu, \beta\rangle = \frac{1}{\pi}\int d^2\alpha|\langle\mu, \nu, \beta|\alpha\rangle|^2 = 1$$

The normalization integral obtained can be calculated explicitly in the practically important case of $|\mu|^2 - |\nu|^2 = 1$:

$$\begin{aligned}&\frac{1}{\pi}\int d^2\alpha\left|\exp\left(\frac{\beta\alpha^*}{\mu} - \frac{|\alpha|^2}{2} - \frac{\nu\alpha^{*2}}{\mu}\right)\right|^2 \\ &\quad = |\mu|\exp\left(|\beta|^2 - \frac{\beta^{*2}\nu}{2\mu^*} - \frac{\beta^2\nu^*}{2\mu}\right)\ .\end{aligned}$$

Finally we obtain the following representation for the matrix element which is a result of overlap between the squeezed state wave function and the coherent state wave function:

$$\langle\alpha|\mu,\nu,\beta\rangle = \frac{1}{\sqrt{\mu}}\exp\left(-\frac{|\alpha|^2}{2} - \frac{|\beta|^2}{2} - \frac{\nu\alpha^{*2}}{2\mu} + \frac{\nu^*\beta^2}{2\mu} + \frac{\alpha^*\beta}{\mu}\right) . \tag{7}$$

Note that in the particular case when $\alpha = 0$ (vacuum state) and $\beta = 0$ (squeezed vacuum state) the overlap matrix element vanishes, $1/\sqrt{\mu} \to 0$, when $\mu \to \infty$. This result has clear a geometric interpretation: overlap between the Gaussian error circle (coherent state) and the error ellipse (squeezed state) on the α plane reduces with increasing degree of squeezing.

6.43 Let us use the expansion

$$\langle\alpha|\mu,\nu,\beta\rangle = \sum_{n=0}^{\infty}\langle\alpha|n\rangle\langle n|\mu,\nu,\beta\rangle . \tag{1}$$

Next we substitute the expression obtained in the previous problem into the left side of this equality and we substitute the known result of the scalar product between the Fock and the coherent state vectors into the right side:

$$\langle\alpha|n\rangle = \exp\left(-\frac{|\alpha|^2}{2}\right)\frac{\alpha^{*n}}{\sqrt{n!}} .$$

So, we obtain

$$\begin{aligned}\frac{1}{\sqrt{\mu}}&\exp\left(-\frac{\nu\alpha^{*^2}}{2\mu} + \frac{\beta\alpha^*}{\mu}\right)\\ &= \exp\left(\frac{|\beta|^2}{2} - \frac{\beta^2\nu^*}{2\mu}\right)\sum_{n=0}^{\infty}\frac{\alpha^{*n}}{\sqrt{n!}}\langle n|\mu,\nu,\beta\rangle .\end{aligned} \tag{2}$$

This relation can be considered as a series expansion of the left side in powers of α^*. Or the left side can be interpreted as a generating function for the Hermitian polynomials. Really, the generating function is defined as (Abramovitz and Stegun, 1965)

$$\exp(-x^2 + ax) = \sum_{n=0}^{\infty}\frac{x^n}{n!}H_n\left(\frac{a}{2}\right) \tag{3}$$

and when $x = \sqrt{\nu/2\mu}\,\alpha^*$ and $a = \beta\sqrt{2/\mu\nu}$ it coincides with the exponential factor on the left side of (2). Now relation (2) has the following form:

$$\begin{aligned}\frac{1}{\sqrt{\mu}}&\sum_{n=0}^{\infty}\frac{\alpha^{*n}}{n!}H_n\left(\frac{\beta}{\sqrt{2\mu\nu}}\right)\left(\frac{\nu}{2\mu}\right)^{n/2}\\ &= \exp\left(\frac{|\beta|^2}{2} - \frac{\beta^2\nu^*}{2\mu}\right)\sum_{n=0}^{\infty}\frac{\alpha^{*n}}{\sqrt{n!}}\langle n|\mu,\nu,\beta\rangle .\end{aligned} \tag{4}$$

Finally, by equating of the coefficients with the same order of α^*, we find the matrix element:

$$\langle n|\mu,\nu,\beta\rangle = \frac{1}{\sqrt{n!\mu}}\left(\frac{\nu}{2\mu}\right)^{n/2}\exp\left(-\frac{|\beta|^2}{2}+\frac{\nu^*\beta^2}{2\mu}\right)H_n\left(\frac{\beta}{\sqrt{2\mu\nu}}\right) . \quad (5)$$

In the case when the squeezed vacuum state ($\beta \to 0$) is written in terms of the Fock states and in the limit $\mu \to \infty$, the main role is played by projections onto states with large n. This means that the geometric interpretation of the squeezed vacuum as a deformed vacuum state on the α plane is very relative because the squeezed state is formed with the help of a large number of photons.

6.44 Let us use the definition of a normally ordered characteristic function and the result obtained in the previous problem:

$$\begin{aligned}\theta[\kappa,\kappa^*] &= \langle\gamma,\beta|\exp(\kappa\hat{c}^\dagger)\exp(-\kappa^*\hat{c})|\gamma,\beta\rangle \\ &= \langle \mathrm{vac}|D^\dagger(\beta)S^\dagger(\gamma)e^{\kappa a^\dagger}S(\gamma)D(\beta)D^\dagger(\beta) \\ &\quad\times S^\dagger(\gamma)e^{-\kappa^* a}S(\gamma)D(\beta)|\mathrm{vac}\rangle . \end{aligned}\quad (1)$$

The transformation properties of the operators are

$$\begin{aligned}&\hat{S}(\gamma)\hat{c}\hat{S}^\dagger(\gamma) = \mu\hat{c}+\nu\hat{c}^\dagger , \qquad &&\hat{S}(\gamma)\hat{c}^\dagger\hat{S}^\dagger(\gamma) = \mu\hat{c}^\dagger+\nu^*\hat{a} , \\ &\hat{D}(\beta)\hat{c}\hat{D}^\dagger(\beta) = \hat{c}-\beta , &&\hat{D}(\beta)\hat{c}^\dagger\hat{D}^\dagger(\beta) = \hat{c}^\dagger-\beta^* . \end{aligned}\quad (2)$$

Using the properties of the squeeze and displacement operators $\hat{S}^\dagger(\gamma) = \hat{S}(-\gamma)$ and $\hat{D}^\dagger(\beta) = \hat{D}(-\beta)$, we can transform relation (1) into the form

$$\begin{aligned}\theta[\kappa,\kappa^*] &= \exp\left[\kappa(\mu\beta^*-\nu^*\beta)-\kappa^*(\mu\beta-\nu\beta^*)\right] \\ &\quad\times\langle\mathrm{vac}|\exp\left[\kappa(\mu\hat{c}^\dagger-\nu^*\hat{c})\right]\exp\left[-\kappa^*(\mu\hat{c}-\nu\hat{c}^\dagger)\right]|\mathrm{vac}\rangle . \end{aligned}\quad (3)$$

Then we make expand the operator functions in a Taylor series and retain terms that do not vanish after averaging over the vacuum state. Thus, we have

$$\begin{aligned}\theta[\kappa,\kappa^*] &= \exp\left[\kappa(\mu\beta^*-\nu^*\beta)-\kappa^*(\mu\beta-\nu\beta^*)\right] \\ &\quad\times\exp\left(-\kappa\kappa^*\sinh^2 r-\frac{\kappa^2}{4}e^{-2i\vartheta}\sinh 2r-\frac{\kappa^{*2}}{4}e^{2i\vartheta}\sinh 2r\right) . \end{aligned}\quad (4)$$

This result coincides with the solution given in the statement of the problem.

This state describes a field with mean complex amplitude $\langle\hat{a}\rangle = \alpha$. On average, the behavior of this field is similar to the behavior of a coherent state with the same amplitude. The main difference from the coherent state is in the behavior of the uncertainties of quadrature components. Assume for simplicity that β is a real parameter and phase $\vartheta \to 0$. In accordance with the quadrature component definition (see Problem 6.33)

$$\hat{X}_1 = \frac{1}{\sqrt{2}}(\hat{c}^\dagger+\hat{c}) , \quad \hat{X}_2 = \frac{i}{\sqrt{2}}(\hat{c}^\dagger-\hat{c}) .$$

With the help of generating function (4) we can write

$$\left\langle \Delta \widehat{X}_1^2 \right\rangle = \frac{1}{2}\left[\langle \widehat{c}^2 + \widehat{c}^{\dagger 2} + 2\widehat{c}^\dagger \widehat{c} + 1\rangle - (\langle \widehat{c}\rangle^2 + \langle \widehat{c}^\dagger\rangle^2 + 2\langle \widehat{c}\rangle\langle \widehat{c}^\dagger\rangle)\right]$$
$$= \frac{1}{2}[1 - 2e^{-r}\sinh r] \to 0 \quad \text{when} \quad r \to \infty\,,$$

$$\left\langle \Delta \widehat{X}_2^2 \right\rangle = \frac{1}{2}\left[\langle -\widehat{c}^2 - \widehat{c}^{\dagger 2} + 2\widehat{c}^\dagger \widehat{c} + 1\rangle + (\langle \widehat{c}\rangle^2 + \langle \widehat{c}^\dagger\rangle^2 - 2\langle \widehat{c}\rangle\langle \widehat{c}^\dagger\rangle)\right]$$
$$= \frac{1}{2}[1 + 2e^{r}\sinh r] \to \infty \quad \text{when} \quad r \to \infty\,. \tag{5}$$

Thus, the uncertainty of the position quadrature component is decreased and the uncertainty of the momentum quadrature component is increased infinitely. At the same time its product $\langle \Delta \widehat{X}_1^2\rangle\langle \Delta \widehat{X}_2^2\rangle = 1/4$ maintains the same value as in the coherent state. Property (5) defines these states as squeezed states. It will be useful for reader to generalize the result obtained for the case of complex amplitude β and arbitrary phase ϑ.

6.45 To evaluate the mean value of normally ordered products of the operators of the field $\widehat{c}$ and $\widehat{c}^\dagger$, we can use the Glauber–Sudarshan quasi-probability (6.41). For a thermal field this quasi-probability has a Gaussian form (6.44):

$$\mathcal{P}(\alpha) = \frac{1}{\pi \bar{n}} \exp\left(-\frac{|\alpha|^2}{\bar{n}}\right)\,. \tag{1}$$

For quadrature components considered separately (see Problems 6.37• and 6.44) the distributions maintain their own Gaussian forms.

The variance of the position quadrature component can be expressed through annihilation and creation operators:

$$\langle \Delta \widehat{X}_1^2 \rangle = \frac{1}{2}\left[\langle \widehat{c}^2 + \widehat{c}^{\dagger 2} + 2\widehat{c}^\dagger \widehat{c} + 1\rangle - (\langle \widehat{c}\rangle^2 + \langle \widehat{c}^\dagger\rangle^2 + 2\langle \widehat{c}\rangle\langle \widehat{c}^\dagger\rangle)\right]\,. \tag{2}$$

This variance can be calculated by averaging expression (2) with the help of quasi-probability (1):

$$\langle \Delta \widehat{X}_1^2 \rangle = \int d^2\alpha \frac{1}{2\pi \bar{n}} \exp\left(-\frac{|\alpha|^2}{\bar{n}}\right)(\alpha^2 + \alpha^{*2} + 2|\alpha|^2 + 1)$$
$$= \frac{1}{2}\coth\left(\frac{\hbar\omega}{2T}\right)\,. \tag{3}$$

In the last equality we used the formula for the mean photon number for thermal radiation with temperature T. The variance of the momentum quadrature component $\langle \Delta \widehat{X}_2^2\rangle$ is calculated similarly.

6.46 Let us show that the evolution operator "generated" by interaction with a classical current coincides with the amplitude displacement operator $\widehat{D}(\alpha)$ (see formula (6.36)). The operator that describes the interaction between the classical current and the electromagnetic field is given by

$$\widehat{V}(t) = -\frac{1}{c} \int \boldsymbol{j} \widehat{\boldsymbol{A}} \mathrm{d}^3 r, \tag{1}$$

where $\boldsymbol{j} = \boldsymbol{j}(\boldsymbol{r}, t)$ is a numerical function of position and time, and $\widehat{\boldsymbol{A}} = \widehat{\boldsymbol{A}}(\boldsymbol{r})$ is the vector potential operator of the quantized field.

Consider field evolution from the vacuum state over a small time interval δt. The evolution operator is defined as

$$S(t + \delta t, t) = \exp\left(-\frac{i}{\hbar} \widehat{V}_I(t) \delta t\right), \tag{2}$$

where $\widehat{V}_I(t)$ is the interaction Hamiltonian (1) in the interaction representation (see Problem 6.10*). If we express operator $\widehat{\boldsymbol{A}}(\boldsymbol{r})$ in explicit form (see formula (6.13)) we obtain

$$S(t + \delta t, t) = \exp\left\{\delta t \sum_{\boldsymbol{k}\lambda} \left(-\widehat{c}_{\boldsymbol{k}\lambda} u^*_{\boldsymbol{k}\lambda}(t) + \widehat{c}^\dagger_{\boldsymbol{k}\lambda} u_{\boldsymbol{k}\lambda}(t)\right)\right\}, \tag{3}$$

where we used the following designation:

$$u^*_{\boldsymbol{k}\lambda}(t) = -\frac{i}{\hbar} \left(\frac{2\pi\hbar c^2}{\mathcal{V}\omega_k}\right)^{\frac{1}{2}} e^{-i\omega_k t} \boldsymbol{e}_{\boldsymbol{k}\lambda} \int \boldsymbol{j}(\boldsymbol{r}, t) e^{i\boldsymbol{k}\boldsymbol{r}} \mathrm{d}^3 r .$$

Relation (3) for the evolution operator can be written as a product:

$$S(t + \delta t, t) = \prod_{\boldsymbol{k}\lambda} \exp\left\{\delta t \left(-\widehat{c}_{\boldsymbol{k}\lambda} u^*_{\boldsymbol{k}\lambda}(t) + \widehat{c}^\dagger_{\boldsymbol{k}\lambda} u_{\boldsymbol{k}\lambda}(t)\right)\right\} = \prod_{\boldsymbol{k}\lambda} \widehat{D}_{\boldsymbol{k}\lambda}(\delta t u_{\boldsymbol{k}\lambda}(t)), \tag{4}$$

where on the right side the displacement operator for each mode was defined. This operator displaces the argument of the mode of small values.

Using the following property of displacement operator $\widehat{D}(\alpha_1)\widehat{D}(\alpha_2) = \widehat{D}(\alpha_1 + \alpha_2)$, if $\arg(\alpha_1) = \arg(\alpha_2)$, the density operator of the field in an arbitrary moment of time can be written as

$$\rho(t) = \prod_{\boldsymbol{k}\lambda} \widehat{D}_{\boldsymbol{k}\lambda}(\nu_{\boldsymbol{k}\lambda}(t)) \rho(0) \prod_{\boldsymbol{k}\lambda} \widehat{D}^\dagger_{\boldsymbol{k}\lambda}(\nu_{\boldsymbol{k}\lambda}(t)), \tag{5}$$

where $\nu_{\boldsymbol{k}\lambda}(t) = \int_0^t u_{\boldsymbol{k}\lambda}(\tau)\mathrm{d}\tau$. When the initial state is a vacuum $\rho(0) = |\mathrm{vac}\rangle\langle\mathrm{vac}|$, then state $\rho(t)$ is multimode coherent (see Problem 6.32).

6.48 The electric dipole transitions from the level with $l = 1$ to the level with $l = 0$ are unforbidden transitions in accordance with the selection rules (6.85). Therefore, the corresponding probabilities can be calculated with the help of formula (6.76). For these transitions the spin state does not change.

The initial and final wave functions are given in Appendix C, (C74)–(C76).
Let us write the radius in terms of cyclical components (see Problem 1.17):

$$\boldsymbol{r} = \sum_{\mu=-1}^{+1} \boldsymbol{e}_\mu x_\mu^* , \qquad x_\mu = r\sqrt{\frac{4\pi}{3}}(-1)^\mu Y_{1\mu}(\vartheta,\varphi) ,$$

$$\boldsymbol{e}_\mu = \pm\frac{1}{\sqrt{2}}(\boldsymbol{e}_x \pm i\boldsymbol{e}_y) , \qquad \boldsymbol{e}_0 = \boldsymbol{e}_z . \tag{1}$$

Taking the Ox axis as the plane determined by the quantization axis (Oz) and the wave vector of the emitted photon and calculating the transition dipole moment (6.75), we have

$$\frac{\mathrm{d}w^{\mathrm{sp}}}{\mathrm{d}\Omega_k} = \frac{2^{15}}{3^{10}}\frac{\omega^3(ea_\mathrm{B})^2}{2\pi\hbar c^3}|\boldsymbol{e}_\sigma^*\cdot\boldsymbol{e}_{m_l}|^2 . \tag{2}$$

We take the real basis vectors of linear polarization of the photon to be basis vector $\boldsymbol{e}_1$ placed in the xz plane and to have projections $\cos\theta, 0, -\sin\theta$ on the Cartesian coordinates; at the same time $\boldsymbol{e}_2 = \boldsymbol{e}_y$. In these designations, the angular dependence of radiation is described by the function $F_{\sigma,m_l}(\theta) = |\boldsymbol{e}_\sigma^*\cdot\boldsymbol{e}_{m_l}|^2$, which is presented in table form below.

m_l	$+1$	0	-1
$\sigma = 1$	$\frac{1}{2}\cos^2\theta$	$\sin^2\theta$	$\frac{1}{2}\cos^2\theta$
$\sigma = 2$	$\frac{1}{2}$	0	$\frac{1}{2}$
$\sum_\sigma F_{\sigma,m_l}(\theta)$	$\frac{1}{2}(1+\cos^2\theta)$	$\sin^2\theta$	$\frac{1}{2}(1+\cos^2\theta)$

In last row we wrote the angular distribution of the radiation summed over all polarizations. The angular function for a nonpolarized (nonoriented) atom is isotropic $\sum_{m_l} F_{\sigma,m_l}(\theta) = 1$ because of the absence of a preferential direction in the radiation source.

6.49 The intensity of radiation is calculated as the product of the photon energy $\hbar\omega = \mathcal{E}_i - \mathcal{E}_f$ and the transition probability per unit time summed over all directions. The expression for the effective charge density obtained in Problem 5.23 implies that we consider a transition from state 2p, $m_l = 0$, to the ground state 1s, $m_l = 0$. With the help of the results of previous problem we obtain a formula coinciding with the result obtained in Problem 5.23. This result can be written in another form:

$$I = \left(\frac{2}{3}\right)^8 \frac{\alpha^3 m_\mathrm{e} e^4}{\hbar^3}\hbar\omega ,$$

where α is the fine structure constant (6.51).

6.50• Independently of the magnetic quantum number m_l the lifetime of the excited atomic state 2p is

$$\tau = \frac{1}{w^{\text{sp}}} = \left(\frac{3}{2}\right)^8 \alpha^{-3} \frac{\hbar^3}{m_e e^4} \approx 1.6 \times 10^{-9}\,\text{s}\,.$$

6.51 In accordance with the results obtained in Problem 6.20*, a photon with right circular polarization has the following projection of spin $+\hbar$ on wave vector direction $\boldsymbol{k}$. When the photon is absorbed by an electron, the angular momentum of the photon is transferred to the electron (angular momentum conservation law) which transits to a state with $m_l = +1$. Similarly, a photon with left circular polarization results in an electron in a state with $m_l = -1$, and a linearly polarized photon results in a state with $m_l = 0$.

6.52 The intensity I_α of the Ly_α line was calculated in Problem 6.48. Similarly, we can calculate I_β. The result is $I_\alpha / I_\beta \approx 3.2$.

6.53* After integration of the differential expression (6.76) for the probability of electric dipole radiation over all photon emission directions, we obtain

$$w^{\text{sp}} = \frac{4\omega^3}{3\hbar c^3} |\boldsymbol{p}_{fi}|^2\,. \tag{1}$$

The matrix element of the dipole moment should be calculated with the following wave functions:

$$\Psi_i = R_{20}(r)\,Y_{00}\chi_\mu\,, \quad \Psi_f = R_{21}(r)\,\Phi_{1/2 m_j}\,. \tag{2}$$

Writing the radius in terms of spherical harmonics (see (3) from the solution to Problem 6.47•) and using orthonormal condition for spinors $(\chi_{\mu'}, \chi_\mu) = \delta_{\mu\mu'}$, we obtain

$$\boldsymbol{p}_{fi} = e\sqrt{\frac{4\pi}{3}}\, C^{1/2 m_j}_{1, m_j - \mu 1/2 \mu} \boldsymbol{e}_{\mu - m_j} \int\limits_0^\infty R_{20}(r) R_{21}(r) r^3 \mathrm{d}r\,. \tag{3}$$

The total transition probability should be summed over the final states of the angular momentum $m_j = \pm 1/2$ and averaged over the initial spin states $\mu = \pm 1/2$: $w = (1/2)\sum_{\mu, m_j} w^{\text{sp}}$. If we extract factors which depend on the magnetic quantum numbers, we obtain

$$\begin{aligned}
&\frac{1}{2}\sum_{m_j}\sum_{\mu} C^{1/2 m_j}_{1, m_j - \mu 1/2 \mu} C^{1/2 m_j}_{1, m_j - \mu 1/2 \mu} \left(\boldsymbol{e}^*_{\mu - m_j} \cdot \boldsymbol{e}_{\mu - m_j}\right) \\
&= \frac{1}{2}\sum_{m_j}\sum_{\mu} C^{1/2 m_j}_{1, m_j - \mu 1/2 \mu} C^{1/2 m_j}_{1, m_j - \mu 1/2 \mu} = 1\,.
\end{aligned} \tag{4}$$

The last equality is a result of spin-angle function $\Phi_{j m_j}$ orthonormality. With the help of its definition given in the statement of the problem we find

$$(\Phi_{j' m'_j}, \Phi_{j m_j}) = \sum_{m' \mu'} \sum_{m \mu} C^{j' m'_j}_{l' m' 1/2 \mu'} C^{j m_j}_{l m 1/2 \mu} (\chi_{\mu'}, \chi_\mu) \times \int Y^*_{l' m'}(\vartheta, \varphi) Y_{l m}(\vartheta, \varphi) \mathrm{d}\Omega = \sum_{m \mu} C^{j' m'_j}_{l m 1/2 \mu} C^{j m_j}_{l m 1/2 \mu} = \delta_{j j'} \delta_{m_j m'_j} . \tag{5}$$

This value of the sum of the Clebsch–Gordan coefficients was used in (4). Finally, by taking the simple integral in (3) and collecting all the values found, we obtain

$$w = \frac{12 e^2 a_\mathrm{B}^2 \omega^3}{\hbar c^3} \approx 0.8 \times 10^{-9}\,\mathrm{s}^{-1} .$$

6.54* In the assumption of an immobile proton the wave functions of the initial and final states are

$$\psi_i = R_{10} Y_{00} \Phi_{1M} , \quad \psi_f = R_{10} Y_{00} \Phi_{00} , \tag{1}$$

where R_{10} and Y_{00} are radial and angular functions of the electron, and Φ_{SM} are the spin wave functions of the electron (e) and proton (p) corresponding to singlet and triplet states; see (C86) and (C87).

A radiative transition is accompanied by a change in the spin state of two particles and is caused by interaction between the electron spin and the radiation field. Interaction between a proton and the field is $m_\mathrm{p}/m_\mathrm{e}$ times smaller and can be disregarded. Because of orthogonality between the wave functions of singlet and triplet states, it is reasonable to take into account only the spin part in the interaction Hamiltonian (6.57):

$$\widehat{V} = -\mu_\mathrm{B} \widehat{\boldsymbol{\sigma}} \cdot \widehat{\boldsymbol{H}} , \tag{2}$$

where $\widehat{\boldsymbol{\sigma}}$ is the vector Pauli operator and μ_B is the Bohr magneton. Using the wave functions Φ_{SM}, we calculate the matrix element of operator $\widehat{\boldsymbol{\sigma}}$: $(\Phi_{00}, \widehat{\boldsymbol{\sigma}} \Phi_{10}) = \boldsymbol{e}_z$. When $M = \pm 1$, we obtain

$$(\Phi_{00}, \widehat{\boldsymbol{\sigma}} \Phi_{1,\pm 1}) = \mp \boldsymbol{e}_{\pm 1} = \mp \frac{\boldsymbol{e}_x \pm \boldsymbol{e}_y}{\sqrt{2}} .$$

To calculate the transition probability per unit time we use formulas (6.10) and (6.59). The total probability averaged over initial states $M = 0, \pm 1$ and summed over photon polarizations has the form

$$w = \frac{1}{3} \frac{e^2}{\hbar c} \frac{(\Delta \mathcal{E})^3}{\hbar m_\mathrm{e}^2 c^4} \approx 2 \times 10^{-15}\,\mathrm{s}^{-1} . \tag{3}$$

The transition between hydrogen hyperfine sublevels considered corresponds to a wavelength of 21 cm. This fact is used extensively in radioastronomy for investigation of the distribution of neutral hydrogen (not ionized) in the universe. It is interesting to compare the lifetime of the triplet sublevel with the lifetime of the 2p level for hydrogen (see Problem 6.49). In the first case the lifetime is $\tau = 1/w \approx 10^7$ years. For the 2p $\to$ 1s transition in ultraviolet range we obtain $\tau \approx 1.6 \times 10^{-9}$ s. Generally, such a large difference is the result of strong photon frequency dependence: $\tau \propto \omega^{-3}$.

6.55* Let us use general formula (6.67) from perturbation theory

$$\mathrm{d}w^{\mathrm{abs}} = \frac{2\pi}{\hbar}|\langle f|\widehat{V}|i\rangle|^2\delta(\mathcal{E}_i - \mathcal{E}_f + \hbar\omega)\mathrm{d}\nu \ , \tag{1}$$

where

$$\widehat{V} = -\left(\frac{e}{mc}\right)\widehat{\boldsymbol{A}}\cdot\widehat{\boldsymbol{p}} \ , \tag{2}$$

and the initial and final state vectors are

$$|i\rangle = \psi_{100}(r)|N_s\rangle \ , \quad |f\rangle = \psi_{\boldsymbol{p}}(\boldsymbol{r})|N_s - 1\rangle \ , \quad \psi_{\boldsymbol{p}} = \frac{1}{(2\pi\hbar)^{3/2}}e^{i\boldsymbol{p}\cdot\boldsymbol{r}/\hbar} \ , \tag{3}$$

where $\psi_{\boldsymbol{p}}$ is the wave function of the free electron in the final state. For such a normalization condition the number of states in the continuous spectrum should be written in the following form:

$$\mathrm{d}\nu = p^2\mathrm{d}p\mathrm{d}\Omega_p = 2^{1/2}m^{3/2}\mathcal{E}_p^{1/2}\mathrm{d}\mathcal{E}_p\mathrm{d}\Omega_p \ . \tag{4}$$

Operator (2) does not change the spin state of the electron; therefore, the spin part of the wave function can be disregarded.

To obtain the scattering cross-section we need to divide (1) by the flux density $j_0 = N_s c/\mathcal{V}$ of the incident photons. In integration of the matrix element over space we need to take into account that the photon's wave vector is smaller than the wave vector of the electron, $k \ll p\hbar$. After calculations we obtain the differential cross-section of photoionization of the atom by polarized photons:

$$\mathrm{d}\sigma_{\mathrm{ph}} = 64\frac{e^2}{\hbar c}\left(\frac{I_0}{\hbar\omega}\right)^{7/2} a_{\mathrm{B}}^2|\boldsymbol{e}_s\cdot\boldsymbol{n}|^2\mathrm{d}\Omega_p \ , \tag{5}$$

where $\boldsymbol{n} = \boldsymbol{p}/p$ is the direction of electron departure and I_0 is the binding energy (ionization potential) of the electron in the ground state of hydrogen. The total photoionization cross-section is obtained after averaging over photon polarization and integrating over all directions of electron departure

$$\sigma_{\mathrm{ph}} = \frac{256\pi}{3}\frac{e^2}{\hbar c}\left(\frac{I_0}{\hbar\omega}\right)^{7/2} a_{\mathrm{B}}^2 \ . \tag{6}$$

The result is valid in the range $I_0 \ll \hbar\omega \ll mc^2$.

6.56 In comparison with the previous problem the wave functions of the initial and final states change over. This implies that the squared absolute value of the matrix element does not change. Instead of averaging over photon polarization, we should sum over photon polarizations, which gives the factor 2. The number of final states $d\nu = \mathcal{V}\omega^2 d\omega d\Omega_k/(2\pi c)^3$ relates now to the photon. The photon flux density should be replaced by flux density of recombining electrons $j_0 = v/(2\pi\hbar)^3$. Finally, the total recombination cross-section is

$$\sigma_{\text{rec}} = \frac{128\pi}{3}\left(\frac{e^2}{\hbar c}\right)^3\left(\frac{I_0}{\mathcal{E}_p}\right)^{5/2} a_{\text{B}}^2 \,.$$

In range in which the result is valid, $I_0 \ll \mathcal{E}_p \ll mc^2$, the cross-section obtained is small compared with the photoionization cross-section.

6.58• In the case of a free field, when interaction with atomic systems is absent the interaction representation coincides with the Heisenberg representation. Therefore, the desired representation can be obtained if we replace the photon annihilation and creation operators in (6.13)–(6.15) with the time-dependent (Heisenberg) operators according to (6.23). In particular,

$$\widehat{\boldsymbol{E}}_I(\boldsymbol{r}, t) = i\sum_j \left(\frac{2\pi\hbar\omega_j}{\mathcal{V}}\right)^{1/2}\left[\boldsymbol{e}_j a_j e^{i(\boldsymbol{k}_j\cdot\boldsymbol{r}-\omega_j t)} - \boldsymbol{e}_j^* a_j^\dagger e^{-i(\boldsymbol{k}_j\cdot\boldsymbol{r}-\omega_j t)}\right] .$$

6.59•

$$\widehat{V}_I(t) = -\widehat{\boldsymbol{d}}_I(t)\cdot\widehat{\boldsymbol{E}}_I(t) \,,$$

where

$$\widehat{\boldsymbol{d}}_I(t) = \sum_{mn}(\boldsymbol{d}_{nm} e^{i\omega_{nm}t}|n\rangle\langle m| + \boldsymbol{d}_{mn} e^{i\omega_{mn}t}|m\rangle\langle n|) \,,$$

and $\boldsymbol{d}_{mn}$ are prior matrix elements which do not depend on time.

6.60•

$$\Psi_I(t) = \exp\left(\frac{i}{\hbar}\widehat{\mathcal{H}}_0 t\right)\Psi_S(t) \,; \quad i\hbar\frac{\partial\Psi_I}{\partial t} = \widehat{V}_I(t)\Psi_I \,,$$

where $\widehat{V}_I(t)$ is the interaction Hamiltonian in the interaction representation.

6.61 Let us transform the matrix element considered in the following way:

$$\omega_{ln}\langle l|d_z|n\rangle = \frac{e}{\hbar}(\mathcal{E}_l - \mathcal{E}_n)\sum_{a=1}^N \langle l\,|z_a|\,n\rangle = \frac{e}{\hbar}\sum_{a=1}^N \langle l|\left[\widehat{\mathcal{H}}, \widehat{z}_a\right]|n\rangle$$

$$= -\frac{ie}{m}\sum_{a=1}^N \langle l\,|\widehat{p}_{za}|\,n\rangle \,, \quad (1)$$

where N is the total number of electrons and $\widehat{\mathcal{H}}$ is the exact atomic system Hamiltonian. Then we use the relation $\langle l|z|n\rangle = \langle n|z|l\rangle^*$, $\omega_{ln} = -\omega_{nl}$, which implies

$$\sum_l \omega_{ln}|\langle l|d_z|n\rangle|^2 = \frac{ie^2}{2m}\sum_{a=1}^{N}\sum_l \left\{\langle n|\widehat{p}_{za}|l\rangle\langle l|z_a|n\rangle - \langle n|z_a|l\rangle\langle l|\widehat{p}_{za}|n\rangle\right\}$$

$$= \frac{ie^2}{2m}\sum_{a=1}^{N}\langle n|[\widehat{p}_{za},\widehat{z}_a]|n\rangle = \frac{\hbar e^2}{2m}N\,. \qquad (2)$$

Here we used the completeness of atomic states: $\sum_l |l\rangle\langle l| = 1$.

6.62 Interaction between the system and the applied electrostatic field will be considered as a small perturbation. The interaction Hamiltonian is $\widehat{V} = -\widehat{\boldsymbol{d}}\cdot\boldsymbol{E}$. In accordance with perturbation theory (Landau and Lifshitz, 1977) the wave function in the first order is

$$|\psi\rangle = |0\rangle + \sum_n{}' \frac{\langle n|\widehat{V}|0\rangle}{\mathcal{E}_0 - \mathcal{E}_n}|n\rangle\,. \qquad (1)$$

The prime in the sum indicates the absence of a term with $n = 0$; $\mathcal{E}_n$ are the unperturbed energy levels of the system. The symbol $|0\rangle$ indicates the ground state wave function. Then, using $|\psi\rangle$, we calculate the mean value of the system's dipole moment:

$$\langle\psi|\widehat{\boldsymbol{d}}|\psi\rangle = \langle 0|\widehat{\boldsymbol{d}}|0\rangle + \sum_n{}' \frac{1}{\mathcal{E}_0 - \mathcal{E}_n}\left\{\langle n|\widehat{V}|0\rangle\langle 0|\widehat{\boldsymbol{d}}|n\rangle + \langle 0|\widehat{V}|n\rangle\langle n|\widehat{\boldsymbol{d}}|0\rangle\right\}\,. \qquad (2)$$

Here we took into account only the terms of first order in $\widehat{V}$. The first term on the right side is the electric dipole moment of the system in the unperturbed ground state (if it is not equal to zero).[22] The induced dipole moment is described by the sum over n, which is proportional to the applied field. By extraction of the proportionality factor, we obtain

$$\alpha_{\mu\nu} = \sum_n{}' \frac{1}{\mathcal{E}_n - \mathcal{E}_0}\left\{\langle 0|\widehat{d}_\mu|n\rangle\langle n|\widehat{d}_\nu|0\rangle + \langle 0|\widehat{d}_\nu|n\rangle\langle n|\widehat{d}_\mu|0\rangle\right\}\,. \qquad (3)$$

6.63 Let us use the interaction representation. In this case the atom-field wave function obeys the equation (see Problem 6.60$^\bullet$)

$$i\hbar\frac{\partial\psi}{\partial t} = \widehat{V}(t)\psi\,, \qquad (1)$$

where index I is omitted and the interaction Hamiltonian in the dipole approximation is

$$\widehat{V}(t) = -\widehat{\boldsymbol{d}}_{21}(t)\cdot\widehat{\boldsymbol{E}}^{(+)}(\boldsymbol{r},t)|2\rangle\langle 1| - \widehat{\boldsymbol{d}}_{12}(t)\cdot\widehat{\boldsymbol{E}}^{(-)}(\boldsymbol{r},t)|1\rangle\langle 2|\,. \qquad (2)$$

22) It is necessary for this that the ground state does not have any parity.

The dipole moment operator depends on time, $d_{21}^{\alpha}(t) = d_{21}^{\alpha}\exp(i\omega_{21}t)$, where d_{21}^{α} is the time-independent matrix element and ω_{21} is the transition frequency between atomic states. The time-dependent field operators in the interaction representation are given in Problem 6.58•. Below for simplicity we make all calculations in the resonant approximation (6.105) (or rotating wave approximation): only the positive frequency part which corresponds to photon absorption remains in the first term (6.63).

If we assume that initially the atom is in the ground state and the field is in the Fock state, we can write the initial wave function as a product: $\psi_0 = |1\rangle|\{N_s\}\rangle$.

In the first order of perturbation theory, the wave function at moment t is

$$\psi(t) = -\frac{i}{\hbar}\int_0^t \widehat{V}(\tau)\mathrm{d}\tau|1\rangle|\{N_s\}\rangle . \tag{3}$$

The probability $\tilde{W}(t)$ that the atom at moment t can be found in excited state $|2\rangle$ is equal to the squared absolute value $|\langle 2|\psi(t)\rangle|^2$ summed over all final states $|f\rangle$ of the field:

$$\tilde{W}(t) = \frac{1}{\hbar^2}\sum_f d_{21}^{\alpha} d_{12}^{\beta}\int_0^t\int_0^t \mathrm{d}\tau_1\mathrm{d}\tau_2 \\ \times \langle f|E_{\alpha}^{(+)}(r,\tau_1)|\{N_s\}\rangle\langle\{N_s\}|E_{\beta}^{(-)}(r,\tau_2)|f\rangle \exp(i\omega_{21}(\tau_1-\tau_2)) . \tag{4}$$

Using the completeness condition of field states $\sum_f |f\rangle\langle f| = 1$, we have

$$\tilde{W}(t) = \frac{1}{\hbar^2} d_{21}^{\alpha} d_{12}^{\beta}\int_0^t\int_0^t \mathrm{d}\tau_1\mathrm{d}\tau_2 \\ \times \langle\{N_s\}|E_{\beta}^{(-)}(r,\tau_2)E_{\alpha}^{(+)}(r,\tau_1)|\{N_s\}\rangle \exp(i\omega_{21}(\tau_1-\tau_2)) . \tag{5}$$

To obtain the final result we need to average the probability found over the initial field states using the density matrix given. Let us define the correlation function of the field:

$$\mathcal{G}_{\alpha\beta} = \mathrm{Sp}\left(\widehat{\rho}\widehat{E}_{\beta}^{(-)}(\boldsymbol{r},\tau_2)\widehat{E}_{\alpha}^{(+)}(\boldsymbol{r},\tau_1)\right) \\ = \sum_{\{N_j\}}\rho_{\{N_j\}}\langle\{N_j\}|E_{\beta}^{(-)}(\boldsymbol{r},\tau_2)E_{\alpha}^{(+)}(\boldsymbol{r},\tau_1)|\{N_j\}\rangle . \tag{6}$$

Here the density operator is written in terms of Fock states in which it has diagonal form. Using the explicit form of field operators (Problem 6.58•), we find

$$\mathcal{G}_{\alpha\beta}(\tau_1-\tau_2) = \frac{2\pi\hbar}{\mathcal{V}}\sum_s e_s^{\alpha} e_s^{*\beta}\omega_s\langle N_s\rangle e^{-i\omega_s(\tau_1-\tau_2)} \\ = \int G_{\alpha\beta}(\omega)e^{-i\omega(\tau_1-\tau_2)}\frac{\mathrm{d}\omega}{2\pi} . \tag{7}$$

The spectral density of the field $G_{\alpha\beta}(\omega)$ is defined through the mean values of the occupation numbers $\langle N_s \rangle$.

We designate the probability averaged by means of the density operator as $W(t)$, and its expression written via the field spectral density is

$$W(t) = \frac{1}{\hbar^2} d_{21}^{\alpha} d_{12}^{\beta} \int_0^t \int_0^t \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} d\tau_1 d\tau_2 G_{\beta\alpha}(\omega) \exp(i(\omega_{21} - \omega)(\tau_1 - \tau_2)) . \quad (8)$$

Time integrals give

$$\frac{|\exp(i(\omega_{21} - \omega)t - 1|^2}{(\omega_{21} - \omega)^2} = \frac{4\sin^2((\omega_{21} - \omega)t/2)}{(\omega_{21} - \omega)^2} . \quad (9)$$

For large times t the asymptotic expression can be used:

$$\frac{\sin^2(\omega_{21} - \omega)t/2}{(\omega_{21} - \omega)^2 t/2} \longrightarrow \pi\delta(\omega_{21} - \omega) . \quad (10)$$

Finally, we obtain

$$\frac{W(t)}{t} = \frac{1}{\hbar^2} d_{21}^{\alpha} d_{12}^{\beta} G_{\beta\alpha}(\omega_{21}) , \quad (11)$$

which is transition probability per unit time for a chaotic field defined by the spectral density of this field at the transition frequency.

6.64* From the perturbation theory point of view, the scattering process is associated with transition of the system from initial state $|i\rangle = \psi_i \Phi_{\{N_s\}}$ to final $|f\rangle = \psi_f \Phi_{\{N_s'\}}$, where ψ_i and ψ_f are the initial and final state wave functions of the atom, and $\Phi_{\{N_s\}}$ and $\Phi_{\{N_s'\}}$ are the field wave functions. The differential scattering cross-section is defined as the ratio between the scattered intensity in solid angle $d\Omega$ and the flux density of incident radiation:

$$d\sigma = \frac{1}{J_{\text{inc}}} \int_f dw_{fi} . \quad (1)$$

The final states required to be summed (integrated) in this expression are defined by the conditions of light scattering observation. In the case of a polarization analyzer in the detection scheme, the summation over photon polarization is not required. When the field spectrum is analyzed, the integration over frequencies is applied only in the filter transmission band.

Let us designate by index 1 the mode in which the photon number is reduced and by index 2 the mode in which the photon number is increased because of scattering. The light scattering is a two-photon process; therefore, we need to use the first term in (6.57) (second order of perturbation theory) and also the second term, which describes two-photon processes starting from the first order. The last term describes interaction between the spin magnetic moment of the atom and the magnetic field. This term can be disregarded if the size of the system is much smaller than the wavelength of the radiation (electric dipole approximation). Thus, the interaction Hamiltonian can be written in the following form:

$$\widehat{V} = \widehat{V}_1 + \widehat{V}_2 \,; \quad \widehat{V}_1 = -\frac{e}{mc}\widehat{\boldsymbol{p}}\cdot\widehat{\boldsymbol{A}} \,; \quad \widehat{V}_2 = \frac{e^2}{2mc^2}\widehat{\boldsymbol{A}}^2 \,. \tag{2}$$

In the first nonvanishing order, the scattering probability per unit time can be calculated by means of perturbation theory with summation over intermediate states:

$$\mathrm{d}w_{fi} = \frac{2\pi}{\hbar}\left|\langle f|\widehat{V}_2|i\rangle + \sum_l \frac{\langle f|\widehat{V}_1|l\rangle\langle l|\widehat{V}_1|i\rangle}{\epsilon_i - \epsilon_l}\right|^2 \delta(\epsilon_i - \epsilon_f)\frac{\mathcal{V}k_2^2\mathrm{d}k_2\mathrm{d}\Omega_2}{(2\pi)^3}\,, \tag{3}$$

where ϵ_l is the total energy of the system atom and field in state $|l\rangle$. Index 2 in the wave vector denotes that the corresponding quantities relate to the scattered photon. Note that we consider nonresonant scattering; therefore, we can completely disregard the finite width of atomic levels.

Let us consider separately the matrix element of two operators $\widehat{V}_1$ and $\widehat{V}_2$:

$$\begin{aligned}\langle f|\widehat{V}_2|i\rangle &= \frac{e^2}{2mc^2}\langle f|\widehat{\boldsymbol{A}}^2|i\rangle \\ &= \frac{e^2}{2mc^2}\sqrt{N_1(N_2+1)} \\ &\quad\times\left(\int \psi_f^*(\boldsymbol{A}_1(\boldsymbol{r})\cdot\boldsymbol{A}_2^*(\boldsymbol{r}))\psi_i\mathrm{d}^3r + \int \psi_f^*(\boldsymbol{A}_2^*(\boldsymbol{r})\cdot\boldsymbol{A}_1(\boldsymbol{r}))\psi_i\mathrm{d}^3r\right) \\ &= \frac{e^2}{2m}\sqrt{N_1(N_2+1)}\frac{4\pi\hbar}{\mathcal{V}\sqrt{\omega_1\omega_2}} \\ &\quad\times\int \psi_f^*\boldsymbol{e}_1\cdot\boldsymbol{e}_2^*\exp(i(\boldsymbol{k}_1-\boldsymbol{k}_2)\boldsymbol{r})\psi_i\mathrm{d}^3r\,.\end{aligned}$$

Here, N_1 and N_2 are the initial photon numbers for modes 1 and 2, respectively.

In the dipole approximation $\exp(i(\boldsymbol{k}_1-\boldsymbol{k}_2)\boldsymbol{r}) \approx 1$ and the expression given above can be simplified:

$$\langle f|\widehat{V}_2|i\rangle = \frac{2e^2\pi\hbar\sqrt{N_1(N_2+1)}}{\mathcal{V}m\sqrt{\omega_1\omega_2}}\boldsymbol{e}_1\cdot\boldsymbol{e}_2^*\delta_{if}\,. \tag{4}$$

Thus, the quadratic field term in the interaction Hamiltonian corresponds to a scattering processes that occurs without changing the atomic state (coherent or elastic scattering process).

The matrix element of operator $\widehat{V}_1$ was calculated in Problem 6.9:

$$\begin{aligned}\sum_l \frac{\langle f|\widehat{V}_1|l\rangle\langle l|\widehat{V}_1|i\rangle}{\epsilon_i-\epsilon_l} &= \frac{e^2}{m^2}\frac{2\pi\hbar\sqrt{N_1(N_2+1)}}{\mathcal{V}\sqrt{\omega_1\omega_2}} \\ \times\sum_l\left(\frac{\langle f|\widehat{\boldsymbol{p}}\cdot\boldsymbol{e}_2^*|l\rangle\langle l|\widehat{\boldsymbol{p}}\cdot\boldsymbol{e}_1|i\rangle}{\mathcal{E}_i-\mathcal{E}_l+\hbar\omega_1}\right. &+ \left.\frac{\langle f|\widehat{\boldsymbol{p}}\cdot\boldsymbol{e}_1|l\rangle\langle l|\widehat{\boldsymbol{p}}\cdot\boldsymbol{e}_2^*|i\rangle}{\mathcal{E}_i-\mathcal{E}_l-\hbar\omega_2}\right)\,.\end{aligned} \tag{5}$$

Note the important case of nonresonant scattering. The sum over l in (3) has two different types of terms. The first type describes a process when the incident photon is annihilated before photon creation in the scattered wave. The corresponding denominators of these terms have the following structure: $\mathcal{E}_i - \mathcal{E}_l + \hbar\omega_1$. The denominators of terms of the second type are $\mathcal{E}_i - \mathcal{E}_l - \hbar\omega_2$ and relate to processes with intermediate state transitions in which the scattered photon appears before incident photon absorption.

Substituting (4) and (5) into the cross-section definition (1) and taking into account that the photon density in the incident flux is $N_1 c/\mathcal{V}$, we obtain

$$\frac{d\sigma}{d\Omega} = \frac{e^4(N_2+1)\omega_2}{m^2c^4\omega_1}\Big|\boldsymbol{e}_1\boldsymbol{e}_2^*\delta_{if} + \frac{1}{m}\sum_l\left(\frac{\langle f|\widehat{\boldsymbol{p}}\cdot\boldsymbol{e}_2^*|l\rangle\langle l|\widehat{\boldsymbol{p}}\cdot\boldsymbol{e}_1|i\rangle}{\mathcal{E}_i-\mathcal{E}_l+\hbar\omega_1} + \frac{\langle f|\widehat{\boldsymbol{p}}\cdot\boldsymbol{e}_1|l\rangle\langle l|\widehat{\boldsymbol{p}}\cdot\boldsymbol{e}_2^*|i\rangle}{\mathcal{E}_i-\mathcal{E}_l-\hbar\omega_2}\right)\Big|^2 . \qquad (6)$$

Here, $d\Omega$ is the solid angle in which the scattered photon propagates, and ω_1 and ω_2 are frequencies which obey the energy conservation law: $\hbar\omega_1 + \mathcal{E}_i = \hbar\omega_2 + \mathcal{E}_f$ (see the Dirac delta function in (3)). The factor $(N_2 + 1)$ in (6) allows us to extract two types of contributions to the differential cross-section. The first of these is proportional to the photon number N_2 in the scattered wave and corresponds to a stimulated scattering process. The second contribution does not depend on N_2 and corresponds to spontaneous emission. Note that if we want to observe stimulated scattering we need to "illuminate" the scattering atom with radiation with a fixed frequency and propagation direction (and probably polarized) and all of these properties should be different from those of the light observed and its flux density should be contained in cross-section definition.

It is important to note that expression (6) essentially differs from the result obtained in classical electrodynamics where electrons were considered as charged classical oscillators (Problem 5.127*). The final atomic state can be different from the initial one. In this case the frequency of the scattered photon differs from the frequency of the incident photon. Such scattering is known as *Raman scattering* or *combinational scattering*. Raman scattering is accompanied by a reduction of the frequency (Stokes line) and by an increase of the frequency (anti-Stokes line). The scattering process when $\mathcal{E}_i = \mathcal{E}_f$ and $\omega_1 = \omega_2$ is also possible. This is the case of Rayleigh or coherent scattering (classical electrodynamics can describe only this case).

We note one more important two-photon process to which two-photon absorption and emission are also related. This process includes any intermediate state transitions. The contributions of different intermediate states can differ from each other by sign, which results in mutual reduction or enhancement. Such mutual influence of different transition channels is known as quantum interference of states.

6.65* In the dipole approximation the interaction Hamiltonian is

$$\widehat{V} = -\widehat{\boldsymbol{d}}\cdot\widehat{\boldsymbol{E}}. \qquad (1)$$

In the second order of perturbation theory, the result can be obtained as in the previous problem:

$$\frac{d\sigma}{d\Omega} = \frac{(N_2+1)\omega_1\omega_2^3}{c^4} \times \left|\sum_l \left(\frac{\langle f|\widehat{\boldsymbol{d}}\cdot\boldsymbol{e}_2^*|l\rangle\langle l|\widehat{\boldsymbol{d}}\cdot\boldsymbol{e}_1|i\rangle}{\mathcal{E}_i - \mathcal{E}_l + \hbar\omega_1} + \frac{\langle f|\widehat{\boldsymbol{d}}\cdot\boldsymbol{e}_1|l\rangle\langle l|\widehat{\boldsymbol{d}}\cdot\boldsymbol{e}_2^*|i\rangle}{\mathcal{E}_i - \mathcal{E}_l - \hbar\omega_2} \right)\right|^2 . \tag{2}$$

Let us show that the expression obtained coincides with result obtained in the previous problem. For that we will use the known relation between the matrix elements of the momentum and position operators:

$$\langle n'|\widehat{\boldsymbol{p}}|n\rangle = -i\frac{m\omega_{n'n}}{e}\langle n'|\widehat{\boldsymbol{d}}|n\rangle , \tag{3}$$

where $\omega_{n'n}$ is the transition frequency. First we will consider Rayleigh scattering. The second term in formula (6) in the solution of the previous problem can be transformed in the following way:

$$\begin{aligned} &\frac{1}{m}\sum_l \left(\frac{\langle i|\widehat{\boldsymbol{p}}\cdot\boldsymbol{e}_2^*|l\rangle\langle l|\widehat{\boldsymbol{p}}\cdot\boldsymbol{e}_1|i\rangle}{\mathcal{E}_i - \mathcal{E}_l + \hbar\omega_1} + \frac{\langle i|\widehat{\boldsymbol{p}}\cdot\boldsymbol{e}_1|l\rangle\langle l|\widehat{\boldsymbol{p}}\cdot\boldsymbol{e}_2^*|i\rangle}{\mathcal{E}_i - \mathcal{E}_l - \hbar\omega_1} \right) \\ &= \frac{(\boldsymbol{e}_1)_\alpha(\boldsymbol{e}_2)^*_\beta m}{e^2\hbar} \\ &\quad \times \sum_l \left((\omega_{il} - \omega_1)\langle i|\widehat{d}_\beta|l\rangle\langle l|\widehat{d}_\alpha|i\rangle + (\omega_{il} + \omega_1)\langle i|\widehat{d}_\alpha|l\rangle\langle l|\widehat{d}_\beta|i\rangle \right) \\ &\quad + \frac{m}{e^2}\sum_l \omega_1^2 \left(\frac{\langle i|\widehat{\boldsymbol{d}}\cdot\boldsymbol{e}_2^*|l\rangle\langle l|\widehat{\boldsymbol{d}}\cdot\boldsymbol{e}_1|i\rangle}{\mathcal{E}_i - \mathcal{E}_l + \hbar\omega_1} + \frac{\langle i|\widehat{\boldsymbol{d}}\cdot\boldsymbol{e}_1|l\rangle\langle l|\widehat{\boldsymbol{d}}\cdot\boldsymbol{e}_2^*|i\rangle}{\mathcal{E}_i - \mathcal{E}_l - \hbar\omega_1} \right) . \end{aligned} \tag{4}$$

The sum in the last expression, $\sum_l((\omega_{il} - \omega_1)\langle i|\widehat{d}_\beta|l\rangle\langle l|\widehat{d}_\alpha|i\rangle + (\omega_{il} + \omega_1)\langle i|\widehat{d}_\alpha|l\rangle \langle l|\widehat{d}_\beta|i\rangle)$, is equal to the first term in formula (6) in the solution of the previous problem. Indeed, according to completeness of the atomic states, we have $\sum_l\langle i|\widehat{d}_\alpha|l\rangle\langle l|\widehat{d}_\beta|i\rangle = \langle i|\widehat{d}_\alpha\widehat{d}_\beta|i\rangle = \langle i|\widehat{d}_\beta\widehat{d}_\alpha|i\rangle$. This implies that all terms under the sum which contains the factor ω_1 cancel each other. The remaining terms can be written as

$$\begin{aligned} &\sum_l \omega_{il}\langle i|\widehat{d}_\beta|l\rangle\langle l|\widehat{d}_\alpha|i\rangle + \omega_{il}\langle i|\widehat{d}_\alpha|l\rangle\langle l|\widehat{d}_\beta|i\rangle) \\ &= \frac{e^2 i}{m}\sum_l\langle i|\widehat{r}_\beta|l\rangle\langle l|\widehat{p}_\alpha|i\rangle - \langle i|\widehat{p}_\alpha|l\rangle\langle l|\widehat{r}_\beta|i\rangle) \\ &= \frac{e^2 i}{m}\left(\langle i|\widehat{r}_\beta\widehat{p}_\alpha - \widehat{p}_\alpha\widehat{r}_\beta|i\rangle\right) = -\frac{e^2\hbar}{m}\delta_{\alpha\beta} . \end{aligned} \tag{5}$$

If we substitute this expression into (4), we find that the Rayleigh scattering cross-section obtained with the help of the interaction Hamiltonian in the dipole approximation coincides with formula (6) (of course, when $i = f$).

In the case of Raman scattering we will be use the following relations: $\omega_2 = \omega_1 - \omega_{fl}$ and $\omega_{fl} = \omega_{il} + \omega_{fl}$, where ω_{21} is the transition frequency between initial and final atomic states. In this case we obtain

$$\frac{1}{m}\sum_l\left(\frac{\langle f|\widehat{\boldsymbol{p}}\cdot\boldsymbol{e}_2^*|l\rangle\langle l|\widehat{\boldsymbol{p}}\cdot\boldsymbol{e}_1|i\rangle}{\mathcal{E}_i-\mathcal{E}_l+\hbar\omega_1}+\frac{\langle f|\widehat{\boldsymbol{p}}\cdot\boldsymbol{e}_1|l\rangle\langle l|\widehat{\boldsymbol{p}}\cdot\boldsymbol{e}_2^*|i\rangle}{\mathcal{E}_i-\mathcal{E}_l-\hbar\omega_2}\right)$$
$$=\frac{(\boldsymbol{e}_1)_\alpha(\boldsymbol{e}_2^*)_\beta m\hbar}{e^2}\sum_l\Big((\omega_{il}-\omega_1+\omega_{fl})\langle f|\widehat{d}_\beta|l\rangle\langle l|\widehat{d}_\alpha|i\rangle$$
$$+(\omega_{il}+\omega_2+\omega_{fl})\langle f|\widehat{d}_\alpha|l\rangle\langle l|\widehat{d}_\beta|i\rangle\Big)$$
$$+\sum_l\omega_1\omega_2\left(\frac{\langle f|\widehat{\boldsymbol{d}}\cdot\boldsymbol{e}_2^*|l\rangle\langle l|\widehat{\boldsymbol{d}}\cdot\boldsymbol{e}_1|i\rangle}{\mathcal{E}_i-\mathcal{E}_l+\hbar\omega_1}+\frac{\langle f|\widehat{\boldsymbol{d}}\cdot\boldsymbol{e}_1|l\rangle\langle l|\widehat{\boldsymbol{d}}\cdot\boldsymbol{e}_2^*|i\rangle}{\mathcal{E}_i-\mathcal{E}_l-\hbar\omega_2}\right). \tag{6}$$

The first term on the right side equals zero, which gives the result obtained in previous problem.

6.66

1. In the case when the inequality $\omega_1, \omega_2 \ll \omega_{li}$ is satisfied for all intermediate states l, the scattering atom returns to its initial state (note that classical electrodynamics does not describe Raman scattering). Let us assume that the initial state is a ground atomic state designated by index 0. Using the trivial equality $\omega_1 = \omega_2 = \omega$, we can simplify the formula (2) from the solution of the previous problem:

$$\frac{d\sigma}{d\Omega}=\frac{(N_2+1)\omega^4}{\hbar c^4}\left|\sum_l\frac{(\boldsymbol{d}_{0l}\cdot\boldsymbol{e}_2^*)(\boldsymbol{d}_{l0}\cdot\boldsymbol{e}_1)+(\boldsymbol{d}_{0l}\cdot\boldsymbol{e}_1)(\boldsymbol{d}_{l0}\cdot\boldsymbol{e}_2^*)}{\omega_{l0}}\right|^2. \tag{1}$$

Let us define the static polarizability tensor (see Problem 6.62):

$$\alpha_{ik}=\sum_l\frac{(d_i)_{0l}(d_k)_{l0}+(d_k)_{0l}(d_i)_{l0}}{\omega_{l0}}. \tag{2}$$

If we take the real vector of polarization, the cross-section can be easily expressed in terms of the polarizability tensor:

$$\frac{d\sigma}{d\Omega}=\frac{(N_2+1)\omega^4}{\hbar^2c^4}\left|\sum_{i,k}\alpha_{ik}(e_{2i}e_{1k})\right|^2. \tag{3}$$

The properties of the polarizability tensor depend on the properties of the scattering system (atom or molecule). For an isotropic scatterer, such as an atom in state ${}^1S_0$, we have $\alpha_{ik} = \alpha_0\delta_{ik}$ and

$$\frac{d\sigma}{d\Omega}=\frac{(N_2+1)\omega^4}{\hbar^2c^4}\alpha_0(\boldsymbol{e}_2\cdot\boldsymbol{e}_1)^2. \tag{4}$$

The relation obtained is valid when the polarizations of incident and scattered light are linear and fixed. If we are not interested in the polarization of the light, we need to sum over all values of $\boldsymbol{e}_2$:

$$\sum_{\alpha=1,2} \left(\boldsymbol{e}_2{}^{(\alpha)} \cdot \boldsymbol{e}_1\right)^2 = \sin^2\vartheta = 1 - \cos^2\vartheta, \tag{5}$$

where ϑ is the angle between the escape direction of the scattered photon and the polarization vector of the incident photon.
To reduce the expression for the cross-section to an expression similar to that obtained in classical electrodynamics, we perform the following transformation for α_0

$$\alpha_0 = \sum_l \frac{2|(d_z)_{0l}|^2}{\omega_{l0}} \equiv \frac{\hbar e^2}{m} \sum_l \frac{2m}{\hbar e^2} \frac{f_{l0}}{\omega_{l0}^2}. \tag{6}$$

Here, f_{l0} are the oscillator strengths for transitions $l \leftrightarrow 0$:

$$f_{l0} = \frac{2m}{\hbar e^2} \omega_{n0} |(d_z)_{l0}|^2. \tag{7}$$

Thus,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = r_0^2 (N_2 + 1) \left(\sum_l f_{l0} \frac{\omega^2}{\omega_{l0}^2} \right)^2 (1 - \cos^2\vartheta), \tag{8}$$

where $r_0 = e^2/mc^2$ is the classical electron radius.
Integration over all scattering angles when $N_2 = 0$ gives

$$\sigma = \frac{8\pi}{3} r_0^2 \left(\sum_l f_{l0} \frac{\omega^2}{\omega_{l0}^2} \right)^2 . \tag{9}$$

If we take into account the sum rule for oscillator strengths, the expression obtained will be in very good agreement with the classical analogue. Indeed, the sum rule is

$$\sum_l f_{l0} = 1 , \tag{10}$$

which implies that

$$\sum_l \frac{f_{l0}}{\omega_{l0}^2} \simeq \frac{1}{\omega_0^2}, \tag{11}$$

where ω_0 is the averaged frequency of the atomic oscillator.
Using these relations, we obtain

$$\sigma \simeq \frac{8\pi}{3} r_0^2 \left(\frac{\omega}{\omega_0} \right)^4 . \tag{12}$$

If we want to compare the result obtained with the formulas in Problem 5.127*, we need to equate γ with zero and use $\omega \ll \omega_0$.

2. In the case when the energy of the photon is much greater than than electron binding energy, it is better to use expression (6) from Problem 6.64*. The second term in the modulus in (6) can be disregarded. In the assumption of linear polarization, we have

$$\frac{d\sigma}{d\Omega} = r_0^2(N_2 + 1)\cos^2\theta\,. \tag{13}$$

Here, θ is the angle between the polarization vectors of incident and scattered light. The result obtained corresponds to the classical formula of Thomson which describes the scattering off an unbound electron (see Problems 5.127* and 5.133). However, here the stimulated scattering is taken into account (term with N_2).

6.67* The first nonvanishing probability of the two-photon process can be calculated in the second order of perturbation theory. The corresponding transition probability per unit time is

$$dw_{fi} = \frac{2\pi}{\hbar}\left|\sum_l \frac{\langle f|\widehat{V}|l\rangle\langle l|\widehat{V}|i\rangle}{\epsilon_i - \epsilon_l}\right|^2 \delta(\epsilon_i - \epsilon_f)d\nu\,, \tag{1}$$

where $\widehat{V} = -\widehat{\boldsymbol{d}}\cdot\widehat{\boldsymbol{E}}$ is the interaction Hamiltonian in the dipole approximation, ϵ_l is the energy of field and atom state l, and $d\nu$ is the density of the final states (emission process) or initial states (absorption process).

As in a scattering problem, the intermediate states for two-photon emission (absorption) have a dual nature depending to which photon is emitted (absorbed) in the first step. For simplicity we consider the case of linear polarization. In this case we obtain

$$dw_{fi} = \frac{(2\pi)^2\omega_1\omega_2}{\mathcal{V}^2} f(N_1, N_2)\left|M_{\alpha\beta}e_{1\alpha}e_{2\beta}\right|^2 \delta(\mathcal{E}_i - \mathcal{E}_f \pm \hbar\omega_1 \pm \hbar\omega_2)d\nu\,, \tag{2}$$

$$M_{\alpha\beta} = \sum_l \left(\frac{\langle f|d_\beta|l\rangle\langle l|d_\alpha|i\rangle}{\mathcal{E}_i - \mathcal{E}_l \pm \hbar\omega_1} + \frac{\langle f|d_\alpha|l\rangle\langle l|d_\beta|i\rangle}{\mathcal{E}_i - \mathcal{E}_l \pm \hbar\omega_2}\right). \tag{3}$$

In these formulas the upper sign corresponds to absorption and the lower sign corresponds to emission. The function $f(N_1, N_2)$ depending on photon numbers is equal to $N_1 N_2$ for two-photon absorption and to $(N_1 + 1)(N_2 + 1)$ for two-photon emission. Thus, three type of two-photon emission are possible: spontaneous emission (the probability does not depend on photon numbers), spontaneously stimulated emission (proportional to the sum $N_1 + N_2$), and stimulated emission (proportional to the product $N_1 N_2$).

The selection rules for two-photon transitions can be obtained with the help of analysis of matrix $M_{\alpha\beta}$. In accordance with (3), the two-photon transition is possible if the one-photon transitions $i \to l$ and $l \to f$ can occur. Two-photon transitions between states which differ from each other in orbital momentum by values 0 and ± 2 ($\Delta l = 0, \pm 2$) are possible because of the validity of the relations $\Delta l = \pm 1$ and $\Delta m = 0, \pm 1$ for one-photon transitions in the dipole approximation. The magnetic quantum number can change at most by two units: $\Delta m = 0, \pm 1, \pm 2$. For one-photon transitions the parities of the initial and final states are opposite. This implies that two-photon transitions are possible only between states with the same parity. Thus, two-photon spectroscopy allows one to investigate atomic states which are inaccessible with one-photon or liner spectroscopy.

6.68* The first nonvanishing probability of parametric light generation as a three-photon process should be calculated in the third order of perturbation theory. In this case the transition probability per unit time is

$$\mathrm{d}w_{fi} = \frac{2\pi}{\hbar} \left| \sum_l \sum_n \frac{\langle f|\widehat{V}|n\rangle\langle n|\widehat{V}|l\rangle\langle l|\widehat{V}|i\rangle}{(\epsilon_i - \epsilon_l)(\epsilon_i - \epsilon_n)} \right|^2 \delta(\epsilon_i - \epsilon_f)\mathrm{d}\nu \,. \tag{1}$$

In expression (1) the six terms can be extracted according to the sequence of one-photon processes:

$$\begin{aligned}
\mathrm{d}w_{fi} = {} & \frac{(2\pi)^4\hbar}{\mathcal{V}^3}\omega_1\omega_2\omega_3 N_1(N_2+1)(N_3+1)\delta(\omega_1-\omega_2-\omega_3)\mathrm{d}\nu \\
& \times \sum_l \sum_n \left\{ \frac{\langle f|\boldsymbol{d}\cdot\boldsymbol{e}_3^*|n\rangle\langle n|\boldsymbol{d}\cdot\boldsymbol{e}_2^*|l\rangle\langle l|\boldsymbol{d}\cdot\boldsymbol{e}_1|i\rangle}{(\mathcal{E}_i-\mathcal{E}_l+\hbar\omega_1)(\mathcal{E}_i-\mathcal{E}_n+\hbar\omega_1-\hbar\omega_2)} \right. \\
& + \frac{\langle f|\boldsymbol{d}\cdot\boldsymbol{e}_2^*|n\rangle\langle n|\boldsymbol{d}\cdot\boldsymbol{e}_3^*|l\rangle\langle l|\boldsymbol{d}\cdot\boldsymbol{e}_1|i\rangle}{(\mathcal{E}_i-\mathcal{E}_l+\hbar\omega_1)(\mathcal{E}_i-\mathcal{E}_n+\hbar\omega_1-\hbar\omega_3)} \\
& + \frac{\langle f|\boldsymbol{d}\cdot\boldsymbol{e}_3^*|n\rangle\langle n|\boldsymbol{d}\cdot\boldsymbol{e}_1|l\rangle\langle l|\boldsymbol{d}\cdot\boldsymbol{e}_2^*|i\rangle}{(\mathcal{E}_i-\mathcal{E}_l-\hbar\omega_2)(\mathcal{E}_i-\mathcal{E}_n+\hbar\omega_1-\hbar\omega_2)} \\
& + \frac{\langle f|\boldsymbol{d}\cdot\boldsymbol{e}_1|n\rangle\langle n|\boldsymbol{d}\cdot\boldsymbol{e}_3^*|l\rangle\langle l|\boldsymbol{d}\cdot\boldsymbol{e}_2^*|i\rangle}{(\mathcal{E}_i-\mathcal{E}_l-\hbar\omega_2)(\mathcal{E}_i-\mathcal{E}_n-\hbar\omega_2-\hbar\omega_3)} \\
& + \frac{\langle f|\boldsymbol{d}\cdot\boldsymbol{e}_2^*|n\rangle\langle n|\boldsymbol{d}\cdot\boldsymbol{e}_1|l\rangle\langle l|\boldsymbol{d}\cdot\boldsymbol{e}_3^*|i\rangle}{(\mathcal{E}_i-\mathcal{E}_l-\hbar\omega_3)(\mathcal{E}_i-\mathcal{E}_n+\hbar\omega_1-\hbar\omega_3)} \\
& \left. + \frac{\langle f|\boldsymbol{d}\cdot\boldsymbol{e}_1|n\rangle\langle n|\boldsymbol{d}\cdot\boldsymbol{e}_2^*|l\rangle\langle l|\boldsymbol{d}\cdot\boldsymbol{e}_3^*|i\rangle}{(\mathcal{E}_i-\mathcal{E}_l-\hbar\omega_3)(\mathcal{E}_i-\mathcal{E}_n-\hbar\omega_2-\hbar\omega_3)} \right\} .
\end{aligned}$$

The initial and final states in the problem considered relate to a continuous spectrum; therefore, we have

$$\mathrm{d}\nu = \frac{\mathcal{V}k_1^2\mathrm{d}k_1\mathrm{d}\Omega_1}{(2\pi)^3}\frac{\mathcal{V}k_2^2\mathrm{d}k_2\mathrm{d}\Omega_2}{(2\pi)^3}\frac{\mathcal{V}k_3^2\mathrm{d}k_3\mathrm{d}\Omega_3}{(2\pi)^3}. \tag{2}$$

If we are not interested in the polarization of the detected parametric light, we should sum over all values of vectors $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$.

Similarly to the two-photon process, the parametric generation can be spontaneous, spontaneously stimulated, or stimulated depending on the presence of scattered photons in the incident light.

In conclusion, we note that in a similar way the probability of other three-photon processes (not only coherent) can be calculated.

6.69 Let us consider spontaneous decay in the frame of the dipole resonance approximation. The transition probability is

$$\mathrm{d}w_{fi} = \frac{2\pi}{\hbar}\left|\langle f|\widehat{V}|i\rangle\right|^2 \delta(\epsilon_i - \epsilon_f)\frac{\mathcal{V}k^2\mathrm{d}k\mathrm{d}\Omega}{(2\pi)^3}\ . \tag{1}$$

Here, $|i\rangle = \Psi|\mathrm{vac}\rangle$ and $\widehat{V} = -\boldsymbol{d}_{21}(t)\cdot\widehat{\boldsymbol{E}}^{(+)}(t)|2\rangle\langle 1| + \mathrm{h.c.} \equiv \widehat{V}^{(+)}(t) + \widehat{V}^{(-)}(t)$; the final state is $|f\rangle = |1\rangle_a|1\rangle_b|s\rangle$, where $|s\rangle$ is the field state with one photon in mode s. We assume that the distance between atoms is much smaller than the representative wavelength of radiation. In this assumption we can disregard the space argument in the wave functions and the field operators.

The calculation of the matrix element is similar to the single-atom case. Finally, for the single-photon emission probability per unit time summed over all polarizations, frequencies, and escape directions we obtain

$$W = \frac{2}{3}\frac{|d_{21}|^2\omega_{21}^3}{c^3\hbar} \tag{2}$$

for the initial state Ψ_1 and $W = 0$ for state Ψ_2.

Thus, the system in the first state radiates twice as fast as a single atom, and in the second state there is no radiation. Note that in both cases the total initial excitation of the systems is equal (the initial state energy is $\hbar\omega_{21}$ times larger than the ground state energy of the system).

6.70• After summation of both sides of the equation given in the statement of the problem over all oscillators, we find the equation for collective variable $\boldsymbol{R}(t) = \sum_{a=1}^{N} \boldsymbol{r}_a(t)$:

$$\ddot{\boldsymbol{R}} + 2\beta\dot{\boldsymbol{R}} + \omega_0^2\boldsymbol{R} = 0\ , \quad t > 0\ , \quad \beta = N\gamma\ .$$

The solution which obeys the initial condition $\boldsymbol{R}(0) = N\boldsymbol{r}_0$ has the following form: $\boldsymbol{R}(t) = N\boldsymbol{r}_0\Theta(t)e^{-\beta t}\cos\Omega t$, where $\Omega = \sqrt{\omega_0^2 - N^2\gamma^2}$ and Θ is a step function. The radiation spectrum is (see Problem 5.122*)

$$\frac{\mathrm{d}I_\omega}{\mathrm{d}\omega} = \frac{I_N\beta}{2\pi}\frac{1}{(\omega-\Omega)^2 + \beta^2/4}\ .$$

The total radiation intensity $I_N = N^2 I_0$ is N^2 times larger than the radiation intensity I_0 of a single oscillator and is N times larger than the radiation intensity of N independent oscillators. The decay constant is also N times larger than the decay constant of a single independent oscillator.

The phenomena considered are the classical analogue of the Dicke *superradiance* effect (Andreev *et al.*, 1988; Mandel and Wolf, 1995).

6.71 In classical electrodynamics the magnetic moment $\boldsymbol{\mu}_p$ produces the vector potential $\boldsymbol{A} = \boldsymbol{\mu}_p \times \boldsymbol{r}/r^3$ and magnetic field $\boldsymbol{H} = \nabla\times\boldsymbol{A}$. According to the correspondence principle, we treat these values,

$$\widehat{\boldsymbol{A}} = \nabla\times\frac{\boldsymbol{\mu}_p}{r}\ , \quad \widehat{\boldsymbol{H}} = \nabla\left(\nabla\cdot\frac{\widehat{\boldsymbol{\mu}}_p}{r}\right) - \nabla^2\frac{\boldsymbol{\mu}_p}{r} = -\nabla\frac{(\widehat{\boldsymbol{\mu}}_p\cdot\boldsymbol{r})}{r^3} + 4\pi\boldsymbol{\mu}_p\delta(\boldsymbol{r})\ , \tag{1}$$

as operators acting on the spin variables of a proton and on the electron coordinates. Here

$$\widehat{\boldsymbol{\mu}}_p = \mu_p \widehat{\boldsymbol{\sigma}}_p \,, \quad \mu_p = g_p \frac{e\hbar}{2m_p c} \,, \quad g_p \approx 2.79 \,. \tag{2}$$

The magnetic moment of a proton is approximately 1000 times less than that of an electron. The magnetic field produced by a proton for an electron is a small perturbation. The perturbation operator $\widehat{V}$ is derived from the total Pauli Hamiltonian by separating terms linear in $\widehat{\boldsymbol{A}}$:

$$\widehat{\mathcal{H}} = \widehat{\mathcal{H}}_0 + \widehat{V} \,, \quad \widehat{V} = -\widehat{\boldsymbol{\mu}}_e \cdot \widehat{\boldsymbol{H}} + \frac{e}{m_e c} \widehat{\boldsymbol{A}} \cdot \widehat{\boldsymbol{p}} \,, \quad \widehat{\boldsymbol{\mu}}_e = -\frac{e\hbar}{2m_e c} \widehat{\boldsymbol{\sigma}}_e \,, \quad e > 0 \,. \tag{3}$$

Corrections to the energy of the ground state are expressed through the matrix elements of the perturbation operator, calculated with nonperturbed wave functions. If the perturbation matrix is diagonal, its matrix elements are the required corrections to the energy. Nonperturbed wave functions of the ground state have the form $\Phi_{SM} = \varphi \chi_{SM}(p, e)$, where $\varphi = (\pi a_{\mathrm{B}}^3)^{-1/2} e^{-r/a_{\mathrm{B}}}$, and the spin wave functions $\chi_{SM}(p, e)$ of the singlet ($S = M = 0$) and triplet ($S = 1,\ M = 0, \pm 1$) states are given in Appendix C (see formulas (C86) and (C87)).

First we calculate the integrals over coordinates: $\int \varphi^* \widehat{\boldsymbol{A}} \cdot \widehat{\boldsymbol{p}} \varphi \mathrm{d}^3 r = 0$,

$$\begin{aligned}
\int \varphi^* \widehat{\boldsymbol{H}} \varphi \mathrm{d}^3 r &= 4\pi \widehat{\boldsymbol{\mu}}_p \varphi^2(0) - \int \varphi^2(r) \nabla \frac{(\widehat{\boldsymbol{\mu}}_p \cdot \boldsymbol{r})}{r^3} \mathrm{d}^3 r \\
&= 4\pi \boldsymbol{\mu}_p \varphi^2(0) + \int \frac{\boldsymbol{r}(\widehat{\boldsymbol{\mu}}_p \cdot \boldsymbol{r})}{r^4} \frac{\mathrm{d}\varphi^2}{\mathrm{d}r} r^2 \, \mathrm{d}r \, \mathrm{d}\Omega \\
&= \frac{8\pi}{3} \widehat{\boldsymbol{\mu}}_p \varphi^2(0) \,.
\end{aligned} \tag{4}$$

Let us express the product $\widehat{\boldsymbol{\sigma}}_p \cdot \widehat{\boldsymbol{\sigma}}_e$ through the operator of the atom's total spin $\widehat{\boldsymbol{S}} = (\boldsymbol{\sigma}_p + \boldsymbol{\sigma}_e)/2$:

$$\widehat{\boldsymbol{\sigma}}_p \cdot \widehat{\boldsymbol{\sigma}}_e = \frac{1}{2}[4\widehat{\boldsymbol{S}}^2 - \widehat{\boldsymbol{\sigma}}_p^2 - \widehat{\boldsymbol{\sigma}}_e^2] = \frac{1}{2}[4\widehat{\boldsymbol{S}}^2 - 6] \,. \tag{5}$$

The latter operator has only diagonal matrix elements

$$\langle S' M' | \widehat{\boldsymbol{\sigma}}_p \cdot \widehat{\boldsymbol{\sigma}}_e | S M \rangle = [2S(S+1) - 3] \delta_{SS'} \delta_{MM'} \,. \tag{6}$$

They depend only on the atom's total spin and are independent of its orientation. For the triplet t and singlet s states we have, correspondingly, the additives to the energy

$$\Delta E_t = \frac{8 g_p \mu_p \mu_{\mathrm{B}}}{3 a_{\mathrm{B}}^3} \,, \quad \Delta E_s = -\frac{8 g_p \mu_p \mu_{\mathrm{B}}}{a_{\mathrm{B}}^3} \,. \tag{7}$$

The energy of the transition between these levels is

$$\Delta E = \Delta E_t - \Delta E_s = \frac{32 g_p \mu_p \mu_{\mathrm{B}}}{3 a_{\mathrm{B}}^3} \approx 5.9 \times 10^{-6} \, \mathrm{eV} \,. \tag{8}$$

The wavelength of the superfine transition for hydrogen $\lambda \approx 21$ cm.

6.72

$$\Delta E_{1,1/2} = -\frac{e^2}{a}\frac{\alpha^2 Z^4}{8}\,, \quad \Delta E_{2,1/2} = -\frac{e^2}{a}\frac{5\alpha^2 Z^4}{128}\,,$$
$$\Delta E_{2,3/2} = -\frac{e^2}{a}\frac{\alpha^2 Z^4}{128}\,.$$

The numerical value $\Delta E_{2,3/2} - \Delta E_{2,1/2} \approx 3.45 \times 10^{-5}$ eV.

The general formula describing the fine level structure has the form

$$\Delta E_{nj} = -\frac{e^2}{a}\frac{\alpha^2 Z^4}{2n^3}\left(\frac{1}{j+1/2} - \frac{3}{4n}\right)\,.$$

The degeneracy on l is conserved, and at the stated value j the orbital moment may have values $l = j \pm 1/2$. In particular, the energies of the $2s_{1/2}$ and $2p_{1/2}$ sublevels are equal. This degeneracy is not connected with the assumptions made above. It occurs in the case of the accurate solution of the Dirac equation as well. The degeneracy is removed only if the vacuum corrections are accounted for (Lamb shift; see Problem 6.18$^{\bullet *}$). If an external field is not present, the degeneracy is conserved in the directions of the total moment, $-j \le m_j \le j$.

6.73 Let us write the vector potential of the external field as $\boldsymbol{A} = \boldsymbol{H} \times \boldsymbol{r}/2$. The terms of first order in $\boldsymbol{A}$, that is, the perturbation operator $\widehat{W}$, can be separated from the Pauli equation (6.122):

$$\widehat{W} = \frac{e}{mc}\boldsymbol{A}\cdot\widehat{\boldsymbol{p}} + \frac{e\hbar}{mc}\widehat{\boldsymbol{s}}\cdot\boldsymbol{H} = \mu_B\widehat{\boldsymbol{l}}\cdot\boldsymbol{H} + 2\mu_B\widehat{\boldsymbol{s}}\cdot\boldsymbol{H} = \mu_B(\widehat{\boldsymbol{j}} + \widehat{\boldsymbol{s}})\cdot\boldsymbol{H}\,, \quad e > 0\,. \quad (1)$$

The total Hamiltonian of an atom in this assumption has the form $\widehat{\mathcal{H}} + \widehat{W}$, where $\widehat{\mathcal{H}}$, given by formula (10) from the solution of Example 6.17, accounts for the relativistic corrections. Since the relativistic effect was taken into account in the previous problem, we have to account only for additional interaction with an external field. When averaging the operator $\widehat{W}$, we can use the wave functions of a nonrelativistic atom $|n l j m_j\rangle = R_{nl}(r)\Phi_{l j m_j}(\vartheta, \varphi)$ without relativistic corrections, since these corrections give the terms of higher order of smallness. The operator $\widehat{W}$ is independent of the coordinates; therefore, the integral over dr is equal to unity owing to the normalization of the radial wave functions $\int_0^\infty R_{nl}^2(r) r^2 \mathrm{d}r = 1$, and the matrix elements are determined by spin–angular functions $\Phi_{l j m_j}$.

Let us first average the operator $\widehat{\boldsymbol{s}}$ over nonperturbed states, using considerations of symmetry and the correspondence principle. Denoting the averaging with angle brackets, we can write for classical vectors

$$\langle \boldsymbol{j} + \boldsymbol{s} \rangle = g\boldsymbol{j}\,, \quad (1)$$

since the averaged vector can be directed along the single conserving vector that is the total moment $\boldsymbol{j} = \langle \boldsymbol{j} \rangle$. Here, g is an unknown constant. Multiplying both parts of (1) by the constant vector $\boldsymbol{j}$, we obtain $g j^2 = g\langle j^2 \rangle = \langle j^2 + \boldsymbol{j}\cdot\boldsymbol{s} \rangle$. Further we will consider $\boldsymbol{j}$ and $\boldsymbol{s}$ as the quantum operators under the averaging over the state $|l\,j\,m_j\rangle$:

$$\langle l\,j\,m_j|\widehat{\boldsymbol{j}}^2|l\,j\,m_j\rangle = j(j+1)\,, \quad \widehat{\boldsymbol{j}}\cdot\widehat{\boldsymbol{s}} = \frac{1}{2}[\widehat{\boldsymbol{j}}^2 + \widehat{\boldsymbol{s}}^2 - \widehat{\boldsymbol{l}}^2]\,,$$

$$\langle l\,j\,m_j|\widehat{\boldsymbol{j}}\cdot\widehat{\boldsymbol{s}}|l\,j\,m_j\rangle = \frac{1}{2}[j(j+1) + s(s+1) - l(l+1)]\,. \tag{2}$$

This permits to obtain the constant (*Landé factor*)[23]

$$g = 1 + \frac{1}{2j(j+1)}[j(j+1) + s(s+1) - l(l+1)] \tag{3}$$

and the correction to the energy, which now depends on the magnetic quantum number m_j:

$$\Delta E_{l\,j\,m_j} = \langle l\,j\,m_j|\widehat{W}|l\,j\,m_j\rangle = \frac{e\hbar H}{2mc} g m_j\,, \quad m_j = -j, -j+1, \ldots, j\,. \tag{4}$$

The degeneracy is totally removed, and the energy levels depend on the projection of the total momentum on the direction of the external magnetic field. The assumed conservation of the total moment of the electron will be approximately fulfilled if the Zeeman splitting is small in comparison with the distance between the levels in the multiplets, formed by spin–orbital interaction. Invoking the result obtained in the previous problem, we find for hydrogen the usability condition of formula (3): $H < 10^3$ G.

6.74 In this case one can disregard the spin–orbital coupling and use the wave functions of zero approximation in the form $|l\,m_l m_s\rangle = Y_{l m_l}(\vartheta, \varphi)\chi_{m_s}$. So we obtain

$$\Delta E_{n l m_l m_s} = \frac{e\hbar H}{2mc}(m_l + 2m_s)\,.$$

The level $E_{nl}^{(0)}$ is split into the $2l + 3$ sublevels since $-(l+1) \le m_l + 2m_s \le l + 1$. In this case the degeneracy is not totally removed. The two upper and two lower sublevels become non degenerate. The other terms are twofold degenerate.

6.75 The perturbation operator has the form $\widehat{V} = e\mathcal{E} r \cos\vartheta\,(e > 0)$. The presence of electron spin in the given case is insignificant since the spin moment does not interact directly with the magnetic field. Therefore, the electron can be considered as a zero-spin particle. The energy level considered is fourfold degenerate (over coordinate degrees of freedom). A priori considerations on the choice of the basis of nonperturbed functions, which permits us to obtain immediately the diagonal matrix of the operator $\widehat{V}$, are absent in this case. Therefore, the correct nonperturbed basis should be obtained in the course of solving the problem. Denoting the wave

23) Alfred Landé (1888–1976) was a German theoretical physicist.

functions $|nlm_l\rangle$, related to the level considered, as $\psi_1 = |200\rangle$, $\psi_2 = |210\rangle$, $\psi_3 = |21, +1\rangle$, and $\psi_4 = |21, -1\rangle$, we construct their linear superposition:

$$\varphi(r, \vartheta, \alpha) = \sum_{i=1}^{4} c_i \psi_i \, , \tag{1}$$

where all ψ_i satisfy the stationary nonperturbed Srödinger equation

$$\widehat{\mathcal{H}}_0 \psi_i = E \psi_i \, , \tag{2}$$

in which E is the nonperturbed energy of the level considered.

We find the correction to the energy ΔE and superposition coefficients c_i from the Schrödinger equation with regard for the perturbation:

$$(\widehat{\mathcal{H}}_0 + \widehat{V})\varphi = (E + \Delta E)\varphi \, . \tag{3}$$

With the help of (1) and (2) we find

$$\sum_i c_i \widehat{V} \psi_i = \Delta E \sum_i c_i \psi_i \tag{4}$$

and, using the orthonormality of functions ψ_i, obtain the system of algebraic equations

$$\sum_i V_{ki} c_i = \Delta E c_k \, , \quad k = 1, 2, 3, 4 \, ; \quad V_{ki} = \int \psi_k^* \widehat{V} \psi_i \mathrm{d}^3 r \, ,$$

in which the values of ΔE and c_i are unknown. The corrections to the energy ΔE are calculated on the basis of the fact that the determinant of the system is equal to zero:

$$|V_{ki} - \Delta E \delta_{ki}| = 0 \, . \tag{5}$$

Only two matrix elements differ from zero; these relate to the states with different parity of the wave functions: $V_{12} = V_{21} = -3e\mathcal{E}a$. This gives the corrections to the energy $\Delta E_1 = 3e\mathcal{E}a$, $\Delta E_2 = \Delta E_3 = 0$, and $\Delta E_4 = -3e\mathcal{E}a$ as the linear (with respect to the intensity of the electric field) Stark effect. The linear effect is possible only with degeneracy over l. If all states have the same parity, the matrix elements become zero, and only quadratic corrections to the energy level are possible. The spin–orbital interaction may be disregarded only at $\mathcal{E} > 10^3$ V/cm .

6.76 The wave function of the stationary state is sought in the form $\Psi(\boldsymbol{r}, t) = u(\boldsymbol{r}) \exp(-i/\hbar \mathcal{E} t)$, and the Dirac equation is written as

$$c\widehat{\boldsymbol{\alpha}} \cdot \left(\widehat{\boldsymbol{p}} - \frac{e}{c}\boldsymbol{A}\right) u + mc^2 \widehat{\beta} u = \mathcal{E} u \, , \tag{1}$$

where $\mathcal{E}$ is the energy of relativistic electron and $u(\boldsymbol{r}) = \binom{\varphi}{\chi}$ is the bispinor depending on the coordinates. The two-component spinors satisfy the system of equations

$$(\mathcal{E} - mc^2)\varphi = c\widehat{\boldsymbol{\sigma}} \cdot \left(\widehat{\boldsymbol{p}} - \frac{e}{c}\boldsymbol{A}\right)\chi \, , \quad (\mathcal{E} + mc^2)\chi = c\widehat{\boldsymbol{\sigma}} \cdot \left(\widehat{\boldsymbol{p}} - \frac{e}{c}\boldsymbol{A}\right)\varphi \, . \tag{2}$$

If we substitute in the first equation spinor χ, taken from the second equation, and produce transformations analogous to that which we made in Example 6.14, the first equation becomes

$$\left\{\frac{1}{2m}\left(\widehat{\boldsymbol{p}}-\frac{e}{c}\boldsymbol{A}\right)^2-\frac{e\hbar}{2mc}\widehat{\boldsymbol{\sigma}}\cdot\boldsymbol{H}\right\}\varphi=\frac{\mathcal{E}^2-m^2c^4}{2mc^2}\varphi\ . \tag{3}$$

In the case of stationary states it coincides with the Pauli equation (6.122) if in the latter the nonrelativistic particle energy is changed by $(\mathcal{E}^2-m^2c^4)/2mc^2$.

From equation (3) the spin dependence of the wave function is easily determined. The single spin operator is in the term containing $\widehat{\boldsymbol{\sigma}}\cdot\boldsymbol{H}=\widehat{\sigma}_z H$; therefore, there exist states with projection ν of spin on the direction of the magnetic field. The wave function can be written as $\varphi=\psi(\boldsymbol{r})w_\nu$, where $\psi(\boldsymbol{r})$ is a scalar function, and $w_\nu(\nu=\pm 1/2)$ is a two-component spin factor, $\widehat{\sigma}_z w_\nu=\pm w_\nu$.

The coordinate part of the wave function satisfies the equation

$$\frac{1}{2m}\left\{\widehat{p^2}_x+\left(\widehat{p}_y-\frac{e}{c}Hx\right)^2+\widehat{p^2}_z\right\}\psi=E_\pm\psi\ , \tag{4}$$

where

$$E_\pm=\frac{\mathcal{E}^2-m^2c^4}{2mc^2}\pm\mu_B H\ , \tag{5}$$

where $\mu_B>0$ is the Bohr magneton and the plus/minus signs for an electron denote the states with spin projection $\mp 1/2$ on the field direction.

The effective Hamiltonian on the left side of equation (4) commutes with operators of the components of the generalized momentum $\widehat{p}_y$ and $\widehat{p}_z$. This permits us to seek the solution in the form $\psi(x,y,z)=F(x)\exp\left\{i/\hbar(p_y y+p_z z)\right\}$, where p_y and p_z are real values. On substitution of this solution in (4), we obtain the equation determining the function $F(x)$:

$$\frac{1}{2m}\left\{\widehat{p^2}_x+\left(p_y-\frac{e}{c}Hx\right)^2\right\}F(x)=\left(E_\pm-\frac{p_z^2}{2m}\right)F(x)\ . \tag{6}$$

From this equation it follows that $p_z^2/2m$ is the energy of longitudinal motion.

The equality derived coincides with the equation for stationary states of the linear harmonic oscillator with frequency $\omega=\omega_c=eH/mc$ and equilibrium point shifted by $x_0=cp_y/eH$ from the origin of the coordinates. This shift is removed by introducing the coordinate $x'=x-x_0$. The frequency of the oscillator is equal to the cyclotron rotation frequency of a classical particle around the direction of the magnetic field (see Problem 4.52*).

The solution of the problem for the oscillator is considered in detail in Appendix C. The oscillator energy is discrete:

$$E_{n\pm}-\frac{p_z^2}{2m}=\hbar\omega_c\left(n+\frac{1}{2}\right)\ ,\qquad n=0,1,2,\ldots \tag{7}$$

It represents the energy of transverse motion of a particle *(Landau levels)*. The feature of the problem is the absence of the dependence of energy on the transverse generalized momentum p_y. Therefore, the energy levels of transverse motion are degenerate with infinite multiplicity. The total particle energy has a continuum spectrum owing to the continuous energy of longitudinal motion. The electron quantum states are determined by four quantum numbers: n, p_y, p_z, and ν.

6.77 The Lagrangian is the generalization of expression (4.100):

$$\mathcal{L} = \frac{i\hbar}{2}\left(\Psi^\dagger \frac{\partial \Psi}{\partial t} - \frac{\partial \Psi^\dagger}{\partial t}\Psi\right) + \frac{i\hbar}{2}(\Psi^\dagger c\widehat{\boldsymbol{\alpha}}\cdot\nabla\Psi - (\nabla\Psi^\dagger)\cdot c\widehat{\boldsymbol{\alpha}}\Psi) + \Psi^\dagger[e\widehat{\boldsymbol{\alpha}}\cdot\widehat{\mathbf{A}} - \widehat{\beta}mc^2 - U]\Psi\,. \quad (1)$$

The density of the Hamiltonian function is set in accord with formula (4.97), and for the Dirac field has the form

$$\mathcal{H} = \frac{1}{2}(\Psi^\dagger c\widehat{\boldsymbol{\alpha}}\cdot\widehat{\boldsymbol{p}}\Psi - (\widehat{\boldsymbol{p}}\Psi^\dagger)\cdot c\widehat{\boldsymbol{\alpha}}\Psi) + \Psi^\dagger[-e\widehat{\boldsymbol{\alpha}}\cdot\widehat{\mathbf{A}} + \widehat{\beta}mc^2 + U]\Psi\,. \quad (2)$$

The quantum mechanical Hamiltonian operator can be derived from (2), if, using integration by parts, the total field energy $\mathcal{E}$ is represented as

$$\mathcal{E} \equiv \int \mathcal{H}\mathrm{d}^3 r = \int \Psi^\dagger\widehat{\mathcal{H}}\Psi\mathrm{d}^3 r\,. \quad (3)$$

From (2) and (3) we obtain (6.117):

$$\widehat{\mathcal{H}} = c\widehat{\boldsymbol{\alpha}}\cdot\left(\widehat{\boldsymbol{p}} - \frac{e}{c}\widehat{\mathbf{A}}\right) + \widehat{\beta}mc^2 + e\varphi(\boldsymbol{r})\,. \quad (4)$$

6.78 Once the operator $\widehat{\boldsymbol{v}} = \widehat{\dot{\boldsymbol{r}}}$ has been calculated by formula (C34) with the Dirac Hamiltonian (6.111), we have $\widehat{\boldsymbol{v}} = c\widehat{\boldsymbol{\alpha}}$. Finding the eigenvalues of the operator $\widehat{v_x} = c\widehat{\alpha}_x$ (matrices), we obtain $v_x = \pm c$. The same eigenvalues will be obtained for any other spatial axis. Thus, the direct interpretation of $c\widehat{\boldsymbol{\alpha}}$ as the velocity operator $\widehat{\boldsymbol{v}}$ leads to the incorrect conclusion that any Dirac particle moves with the limiting velocity c. The appearance of the limiting velocity c may be understood as the result of the unsuccessful attempt to determine the coordinate in close moments of time with high accuracy. The minimal error of the change of the electron coordinate is limited by its Compton wavelength $\Lambda_\mathrm{C} = \hbar/mc$. When there is localization in the region of such dimensions, the electrons becomes relativistic and can create an uncontrollable number of real and virtual electron–positron pairs.

The second reason why the result of setting up the velocity operator has no physical meaning is the presence of states with negative energy. The wave functions of these states do not allow for the common quantum mechanical interpretation, since particles with negative energy do not exist in nature. Let us try to determine the velocity operator of a free Dirac particle, excluding the states with negative energy. This can be done with the help of the projection operator on the states with positive energy, which was used in Example 6.19:

$$\widehat{P} = \frac{\mathcal{E} + \widehat{\mathcal{H}}}{2\mathcal{E}}\,, \quad \widehat{P} = \widehat{P}^{\dagger}\,, \quad \text{where}$$
$$\mathcal{E} = \sqrt{(cp)^2 + m^2c^4} > 0\,, \quad \widehat{\mathcal{H}} = c(\widehat{\boldsymbol{\alpha}} \cdot \widehat{\boldsymbol{p}}) + \widehat{\beta} mc^2\,. \tag{1}$$

Let us use the momentum representation. Then the bispinor, describing the particle state, will depend on the momentum: $\Phi(\boldsymbol{p}) = \Phi_{(+)}(\boldsymbol{p}) + \Phi_{-}(\boldsymbol{p})$, where the lower index indicates the sign of the energy, and the Hamiltonian operator $\widehat{\mathcal{H}} = c(\widehat{\boldsymbol{\alpha}} \cdot \boldsymbol{p}) + \widehat{\beta} mc^2$ does not contain the differentiation. But the coordinate operators become differential: $\widehat{\boldsymbol{r}} = i\hbar\partial/\partial\boldsymbol{r}$. The projection operator, acting on the state vector, leaves only the states with positive energy: $\widehat{P}\Phi(\boldsymbol{p}) = \Phi_{(+)}(\boldsymbol{p})$, $\widehat{P}\Phi_{(-)}(\boldsymbol{p}) = 0$.

Now we write the quantum mechanical value of the velocity operator:

$$\left\langle \frac{\mathrm{d}\widehat{\boldsymbol{r}}}{\mathrm{d}t} \right\rangle = \int (\Phi^{\dagger}_{(+)}, c\widehat{\boldsymbol{\alpha}}\Phi_{(+)})\mathrm{d}^3 p = \int (\Phi^{\dagger}, \widehat{P}^{\dagger} c\widehat{\boldsymbol{\alpha}}\widehat{P}\Phi)\mathrm{d}^3 p\,. \tag{2}$$

Using the relations of anticommutation of matrices $\widehat{\boldsymbol{\alpha}}$ and $\widehat{\beta}$, we perform the transformation

$$\widehat{P}^{\dagger} c\widehat{\boldsymbol{\alpha}}\widehat{P} = \frac{c^2\boldsymbol{p}}{\mathcal{E}}\widehat{P} \tag{3}$$

and obtain

$$\left\langle \frac{\mathrm{d}\widehat{\boldsymbol{r}}}{\mathrm{d}t} \right\rangle = \int \left(\Phi^{\dagger}, \frac{c^2\boldsymbol{p}}{\mathcal{E}}\widehat{P}\Phi\right)\mathrm{d}^3 p\,. \tag{4}$$

The last equality is valid because $(\Phi^{\dagger}_{(-)}, \Phi_{(+)}) = 0$, and the operator $c^2\boldsymbol{p}/\mathcal{E}$ does not act on the bispinor indices. The consideration performed led to the correct dependence of the velocity on the momentum $\boldsymbol{v} = c^2\boldsymbol{p}/\mathcal{E}(p)$. In the coordinate representation one should change the momentum $\boldsymbol{p}$ for the operator $\widehat{\boldsymbol{p}} = -i\hbar\nabla$.

6.79

$$\omega = \frac{\omega_0}{1 + \hbar\omega_0/(mc^2)(1 - \cos\theta)}\,, \quad \lambda - \lambda_0 = 2\pi\Lambda_{\mathrm{C}}(1 - \cos\theta)\,.$$

6.83•

$$\mathrm{Sp}\left\{\widehat{B}\widehat{P}_0\widehat{B}^{\dagger}\widehat{P}\right\} = \frac{8mc^2}{\mathcal{E}}(\boldsymbol{e}_0\cdot\boldsymbol{e})^2 + \frac{2\hbar(\omega_0 - \omega)}{\mathcal{E}}(1 - \boldsymbol{n}\cdot\boldsymbol{n}_0)\,.$$

6.84•

1.

$$\mathrm{d}\sigma_{\parallel} = \frac{1}{4}r_0^2\frac{\omega^2}{\omega_0^2}\left[\frac{\omega_0}{\omega} + \frac{\omega}{\omega_0} + 2 - 4\sin^2\theta\cos^2\phi\right]\mathrm{d}\Omega\,,$$

$$d\sigma_\perp = \frac{1}{4} r_0^2 \frac{\omega^2}{\omega_0^2} \left[\frac{\omega_0}{\omega} + \frac{\omega}{\omega_0} - 2 \right] d\Omega ,$$

$$d\sigma = d\sigma_\parallel + d\sigma_\perp$$
$$= \frac{1}{2} r_0^2 \frac{\omega^2}{\omega_0^2} \left[\frac{\omega_0}{\omega} + \frac{\omega}{\omega_0} - 2 \sin^2\theta \cos^2\phi \right] d\Omega ,$$

where r_0 is the classical electron radius, and θ and ϕ are the angles of the vector $\boldsymbol{k}$ in the Cartesian coordinate system, in which the Oz axis is directed along $\boldsymbol{k}_0$ and the Ox axis is directed along $\boldsymbol{e}_0$.

3.

$$d\sigma_\parallel = r_0^2 (1 - \sin^2\theta \cos^2\phi) d\Omega , \quad d\sigma_\perp = 0 ,$$
$$d\bar{\sigma} = \frac{1}{2} r_0^2 (1 + \cos^2\theta) d\Omega .$$

The last cross-section coincides with the Thomson one, which was derived in the classical calculation (see Problem 5.129).

4. In the ultrarelativistic case, $\hbar\omega_0 \gg mc^2$, at large scattering angles, $\hbar\omega_0(1 - \cos\theta) \gg mc^2$, we have

$$d\sigma_\parallel = d\sigma_\perp = \frac{1}{4} r_0^2 \frac{mc^2 d\Omega}{\hbar\omega_0(1 - \cos\theta)} .$$

At small scattering angles, $\hbar\omega_0(1 - \cos\theta) \approx \hbar\omega_0\theta^2/2 \ll mc^2$, we obtain

$$d\sigma_\perp = 0 , \quad d\sigma_\parallel = r_0^2 d\Omega .$$

6.85

$$\sigma = \pi r_0^2 \left\{ \frac{1}{x} \left(1 - \frac{2(1+x)}{x^2} \right) \ln(1+2x) + \frac{2(1+x)}{(1+2x)^2} + \frac{4}{x^2} \right\} ,$$
$$x = \frac{\hbar\omega_0}{mc^2} .$$

In the nonrelativistic case, $x \ll 1$, we have the Thomson cross-section $\sigma = \sigma_T = 8\pi r_0^2/3$. In the ultrarelativistic case

$$\sigma = \pi r_0^2 \frac{mc^2}{\hbar\omega_0} \ln\left(\frac{2\hbar\omega_0}{mc^2} \right) .$$

When the photon energy increases, the scattering cross-section decreases.

6.86

$$\frac{d\sigma}{d(\hbar\omega)} = \pi r_0^2 \frac{mc^2}{(\hbar\omega_0)^2}$$
$$\times \left[\frac{\omega_0}{\omega} + \frac{\omega}{\omega_0} + \left(\frac{mc^2}{\hbar\omega} - \frac{mc^2}{\hbar\omega_0} \right)^2 - 2\left(\frac{mc^2}{\hbar\omega} - \frac{mc^2}{\hbar\omega_0} \right) \right] ,$$

where

$$\hbar\omega_0 \geq \hbar\omega \geq \frac{\hbar\omega_0}{1 + 2\hbar\omega_0/mc^2} \,.$$

6.87 The radiation of photons by an electron under the action of the incoming ultrarelativistic particle may be considered as the result of scattering of pseudophotons by an electron with the creation of real scattered photons. The emission of photons by a heavy particle is strongly suppressed owing to the M^{-2} dependence of the emission cross-section on the mass M. The radiation cross-section $d\sigma_r(\omega)$ can be calculated by the formula

$$d\sigma_r(\omega) = \int_{\omega_{0\min}}^{\omega_{0\max}} d\omega_0 n(\omega_0) d\sigma_C(\omega_0, \omega) \,, \tag{1}$$

where $n(\omega_0)$ is the spectrum of equivalent photons (6.129) and $d\sigma_C$ is the Compton scattering cross-section, which was derived in Problem 6.86. The integration limits are found from the conservation laws (Problem 6.79):

$$\hbar\omega_0 = \frac{\hbar\omega}{1 - (\hbar\omega/mc^2)(1 - \cos\theta)} \,. \tag{2}$$

It is necessary to recognize two intervals of the frequency change of the scattered quantum:

$$1.\quad 0 < \hbar\omega \leq \frac{1}{2}mc^2 \,, \quad \omega_{0\min} = \omega \,, \quad \omega_{0\max} = \frac{\omega}{1 - 2\hbar\omega/mc^2} \,. \tag{3}$$

$$2.\quad \frac{1}{2}mc^2 \leq \hbar\omega < \mathcal{E} - Mc^2 \,, \quad \omega_{0\min} = \omega \,, \quad \omega_{0\max} \to \infty \,. \tag{4}$$

When integrating the frequencies, we remove the slowly changing factor $\ln(\mathcal{E}/\hbar\omega_0)$ from the integral at the point $\omega_0 = \omega$. Having reduced similar terms, we obtain

$$d\sigma_r\left(\omega < \frac{mc^2}{2\hbar}\right) = \frac{16}{3}\frac{q^2}{\hbar c} r_0^2 \frac{d\omega}{\omega} \ln\left(\frac{\mathcal{E}}{\hbar\omega}\right) \left\{1 - \frac{\hbar\omega}{mc^2} + \left(\frac{\hbar\omega}{mc^2}\right)^2\right\} \,. \tag{5}$$

This cross-section diverges at $\omega \to 0$ ("infrared catastrophe"), but the total emitted energy is finite:

$$d\sigma_r\left(\omega > \frac{mc^2}{2\hbar}\right) = \frac{2}{3}\frac{q^2}{\hbar c} r_0^2 \frac{mc^2 d\omega}{\omega^2} \ln\left(\frac{\mathcal{E}}{\hbar\omega}\right) \left\{4 - \frac{mc^2}{\hbar\omega} + \left(\frac{mc^2}{2\hbar\omega}\right)^2\right\} \,. \tag{6}$$

The last formula can be used only at frequencies $\ln(\mathcal{E}/\hbar\omega) \gg 1$. The method of equivalent photons is not applicable in the case of the radiation of ultra-high-energy quanta, $\hbar\omega \approx \mathcal{E}$.

6.88* In the rest system of an electron S' the emission could be considered as the Compton scattering of pseudophotons. In this system of reference the radiation cross-section $d\sigma'_r(\omega')$ can be computed by formula (1) obtained in the preceding problem. However, the values entering this formula should be transformed to the laboratory system S. For this purpose it is necessary to take into account that the cross-sections $d\sigma_r$ in the system of a nucleus and $d\sigma'_r$ in the system of an electron are equal, and only the frequencies should be transformed. Denoting by θ' the scattering angle in system S' (it is counted from the nucleus velocity direction in the system of an electron), we obtain with the help of the Lorenz transformation the relation between the frequencies of the scattered quantum in both systems:

$$\omega = \omega'\gamma\left(1 - \frac{v}{c}\cos\theta'\right) \approx \omega'\gamma\left(1 - \cos\theta'\right) . \tag{1}$$

In the latter approximate equality we omitted terms of the order γ^{-2} compared with unity. Further, we have from the conservation laws the coupling between frequencies ω' and ω'_0:

$$1 - \cos\theta' = \frac{mc^2}{\hbar\omega'} - \frac{mc^2}{\hbar\omega'_0} . \tag{2}$$

Using the energy conservation law in the laboratory coordinate system $\hbar\omega = \mathcal{E} - \mathcal{E}'$, where $\mathcal{E} = \gamma mc^2$ is the initial energy of the electron and $\mathcal{E}'$ is its energy after the emission of a quantum, we find from equalities (1) and (2) the coupling between the energies of quanta in the system of an electron and between the energies of an electron in the laboratory system:

$$\hbar\omega' = \hbar\omega'_0\frac{\mathcal{E}'}{\mathcal{E}} . \tag{3}$$

From these equalities we also obtain the integration limits over frequency ω'_0 at fixed values of the electron's initial energy $\mathcal{E}$ and the energy of the emitted quantum $\hbar\omega$:

$$\omega'_{0\min} = \omega\frac{mc^2}{2\mathcal{E}'} , \quad \omega'_{0\max} = \omega\frac{2\gamma\mathcal{E}}{\mathcal{E}'} . \tag{4}$$

On substitution of these values into formula (1) obtained in the preceding problem, the bremsstrahlung cross-section of an ultrarelativistic electron takes the form

$$\begin{aligned} d\sigma_r(\omega) = {}& 2r_0^2\frac{Z^2e^2}{\hbar c}\frac{mc^2}{\mathcal{E}}\int\limits_{\omega'_{0\min}}^{\omega'_{0\max}}\frac{d\omega'_0}{\omega'^2_0} \\ & \times\ln\left(\frac{\mathcal{E}}{\hbar\omega'_0}\right)\left\{\frac{\mathcal{E}}{\mathcal{E}'} + \frac{\mathcal{E}'}{\mathcal{E}} + \left(\frac{mc^2}{\mathcal{E}'}\right)^2\frac{\omega^2}{\omega'^2_0} - \frac{2mc^2\omega}{\mathcal{E}\omega'_0}\right\}d\omega . \end{aligned} \tag{5}$$

The integration should be performed with the same accuracy with which the spectrum of equivalent photons was determined. For that it is sufficient to remove the logarithm in (5) from the integral, taking the integral at $\omega'_0 = \omega'_{0\min}$, and changing the upper limit of integration to infinity. On integration, we obtain

$$d\sigma_r(\omega) = 4r_0^2 \frac{Z^2 e^2}{\hbar c} \frac{\mathcal{E}'}{\mathcal{E}} \frac{d\omega}{\omega} \ln\left(\frac{2\mathcal{E}\mathcal{E}'}{mc^2\hbar\omega}\right) \left\{\frac{\mathcal{E}}{\mathcal{E}'} + \frac{\mathcal{E}'}{\mathcal{E}} - \frac{2}{3}\right\} . \tag{6}$$

The result is applicable under the following conditions:

$$\gamma = \frac{\mathcal{E}}{mc^2} \gg 1 , \quad \ln\left(\frac{2\mathcal{E}\mathcal{E}'}{mc^2\hbar\omega}\right) \gg 1 . \tag{7}$$

The calculated result given by perturbation theory in the first Born approximation differs from (6) by replacement of the logarithmic factor by

$$\ln\left(\frac{2\mathcal{E}\mathcal{E}'}{mc^2\hbar\omega}\right) - \frac{1}{2}$$

(see Berestetskii *et al.*, 1982, Section 93)

6.89 The total radiation of photons is produced by emission of a projectile ultra-relativistic electron and by emission of a recoil electron. The first process is characterized by the cross-section determined in Problem 6.82 (at $Z = 1$). The second cross-section was calculated for two intervals of frequencies in Problem 6.81 (at $q = e$).

7
Fundamentals of Quantum Theory of the Electron–Positron Field

7.1
Covariant Form of the Dirac Equation. Relativistic Bispinor Transformation

In the preceding chapter, we used the nonrelativistic form (6.109) of the Dirac equation. However, many problems require the use of covariant quantities. For this purpose, it is convenient to change the writing of the initial equation.

Let us turn to the relativistic system of units in which

$$c = \hbar = 1 \quad \text{and} \quad \widehat{p}_k = -i\frac{\partial}{\partial x^k} \equiv -i\partial_k\,, \quad k = 0, 1, 2, 3\,, \tag{7.1}$$

is the four-dimensional momentum operator. Let us further introduce γ Dirac matrices:

$$\widehat{\gamma^0} = \widehat{\beta}\,, \quad \widehat{\gamma^\lambda} = \widehat{\beta}\widehat{\alpha^\lambda}\,, \quad \lambda = 1, 2, 3\,, \quad \widehat{\boldsymbol{\gamma}} = \begin{pmatrix} 0 & \widehat{\boldsymbol{\sigma}} \\ -\widehat{\boldsymbol{\sigma}} & 0 \end{pmatrix}\,. \tag{7.2}$$

Matrices $\widehat{\gamma}$ are non-Hermitian; they are "compromised" – "tainted" by the pseudo-Euclidean character of the four-dimensional space–time – and possess the following properties:

$$\widehat{\gamma^i}\widehat{\gamma^k} + \widehat{\gamma^k}\widehat{\gamma^i} = 2g^{ik}\,, \quad (\widehat{\gamma^0})^2 = -(\widehat{\gamma^1})^2 = -(\widehat{\gamma^2})^2 = -(\widehat{\gamma^3})^2 = 1\,,$$
$$(\widehat{\gamma^0})^\dagger = \widehat{\gamma}^0\,, \quad (\widehat{\gamma^\lambda})^\dagger = -\widehat{\gamma^\lambda}\,, \quad \lambda = 1, 2, 3\,, \tag{7.3}$$

where g^{ik} is a metric tensor.

Substituting Hamiltonian (6.116) into (6.109) and multiplying both sides of resulting equation by $\widehat{\beta} = \widehat{\gamma^0}$ yields the following form:

$$(i\partial_k - eA_k(x))\widehat{\gamma^k}\psi(x) - m\psi(x) = 0\,, \tag{7.4}$$

where $A_k = (A_0, -\boldsymbol{A})$ and x denotes the totality of four Cartesian coordinates.

Foundations of Classical and Quantum Electrodynamics, First Edition. Igor N. Toptygin.

Let us derive the equation for the Dirac conjugate function $\overline{\psi}(x) = \psi^{\dagger}(x)\widehat{\gamma^0}$ (denoted by a bar). Applying Hermitian conjugation to equality (7.4) and multiplying both sides of resulting equation by $\widehat{\gamma^0}$ leads to

$$-(i\partial_0 + eA_0)\psi^+\widehat{\gamma^0}\widehat{\gamma^0} - (i\partial_\lambda + eA_\lambda)\psi^+(\widehat{\gamma^\lambda})^+\widehat{\gamma^0} - m\psi^+\widehat{\gamma^0} = 0\,.$$

Using (7.3) and the definition $\overline{\psi}(x)$ gives

$$(i\partial_k + eA_k(x))\overline{\psi}(x)\widehat{\gamma^k} + m\overline{\psi}(x) = 0\,. \tag{7.5}$$

Equations (7.4) and (7.5) must retain their form on transition into a different inertial frame of reference even though the four-dimensional vectors ∂_k and A_k undergo transformation in accordance with (4.4) and numerical matrices (7.2) remain unaltered. This means that the bispinor must transform too. Let us find this transformation.

Let

$$\psi'(x') = \widehat{U}\psi(x)\,, \quad \psi(x) = \widehat{U}^{-1}\psi'(x')\,, \tag{7.6}$$

where $\widehat{U}$ is a four-row matrix independent of x owing to the uniformity of the four-dimensional space but dependent on Lorentz transformation parameters. Let us use formulas (4.4) and (7.6) in (7.4) to transform it to the form

$$\left(i\partial'_k - eA'_k\right)\Lambda^k{}_i\widehat{\gamma^i}\widehat{U}^{-1}\psi' - m\widehat{U}^{-1}\psi' = 0$$

so that this equation retains its form in the new system. We multiply its left-hand side by $\widehat{U}$ and require that the transformation matrix $\widehat{U}$ satisfy the condition

$$\Lambda^k{}_i\widehat{U}\widehat{\gamma^i}\widehat{U}^{-1} = \widehat{\gamma^k}\,. \tag{7.7}$$

This relation can be written in different forms:

$$\widehat{U}^{-1}\widehat{\gamma^k}\widehat{U} = \Lambda^k{}_i\widehat{\gamma^i}\,, \quad \Lambda^k{}_i\widehat{U}\widehat{\gamma^i} = \widehat{\gamma^k}\widehat{U}\,. \tag{7.8}$$

Relativistic covariance of the Dirac equation is preserved owing to the transformation of bispinors performed above. Equation (7.6) allows the $\widehat{U}$ matrix to be expressed through matrix $\widehat{\Lambda}$ of the Lorentz transformation.

We present here without derivation the Lorentz transformation matrix for the case in which the axes of two inertial systems are parallel to each other and their relative velocity $\boldsymbol{V}$ has an arbitrary direction:

$$\widehat{U} = \exp\left\{-\left(\frac{1}{2}\right)\omega\boldsymbol{n}\cdot\widehat{\boldsymbol{\alpha}}\right\}\,, \tag{7.9}$$

where $\boldsymbol{n} = \boldsymbol{V}/V$, $\tanh\omega = V$, and $\widehat{\boldsymbol{\alpha}}$ is the Dirac matrix. The reader is referred to the textbooks by Berestetskii *et al.* (1982), Davydov (1973), and Moskaleov (2006) for a more detailed discussion of this issue (see also Problem 7.7).

Let us construct the transformation matrix of the conjugate bispinor. By definition,

$$\overline{\psi}'(x') = \psi'^{\dagger}(x')\widehat{\gamma^0} = \psi^{\dagger}(x)\widehat{U}^{\dagger}\widehat{\gamma^0} = \overline{\psi}(x)\widehat{\overline{U}} ,$$

where $\widehat{\overline{U}} = \widehat{\gamma^0}\widehat{U}^{\dagger}\widehat{\gamma^0}$ and $\overline{\psi}(x) = \overline{\psi}'(x')\widehat{\overline{U}}^{-1}$.

On the other hand, transition into (7.5) and primed coordinates with the help of Lorentz transformation (4.4) leads to

$$\left(i\partial'_k + eA'_k(x')\right)\overline{\psi}'(x')\Lambda^k{}_i\widehat{\overline{U}}^{-1}\widehat{\gamma^i} + m\overline{\psi}'(x')\widehat{\overline{U}}^{-1} = 0 .$$

Multiplying the right-hand side of the last equation by $\widehat{\overline{U}}$ and using the relativistic covariance principle, we require that the condition

$$\Lambda^k{}_i\widehat{\overline{U}}^{-1}\widehat{\gamma^i}\widehat{\overline{U}} = \widehat{\gamma^k} . \tag{7.10}$$

be fulfilled. Then, we find, from comparison with expressions (7.7) and (7.9), the identity that must be fulfilled for any Lorentz transformation:

$$\Lambda^k{}_i\widehat{\overline{U}}^{-1}\widehat{\gamma^i}\widehat{\overline{U}} = \widehat{\gamma^k} = \Lambda^k{}_i\widehat{U}\widehat{\gamma^i}\widehat{U}^{-1} = \widehat{\gamma^k} . \tag{7.11}$$

This identity leads to the expression for the matrix of inverse bispinor transformation:

$$\widehat{U}^{-1} = \widehat{\overline{U}} = \widehat{\gamma^0}\widehat{U}^{\dagger}\widehat{\gamma^0} , \quad \overline{\psi}'(x') = \overline{\psi}(x)\widehat{U}^{-1} . \tag{7.12}$$

The matrix $\widehat{U}$ is nonunitary and does not satisfy the condition $\widehat{U}^{\dagger} = \widehat{U}^{-1}$.

Example 7.1

Using the Dirac equation for a free particle, write down the equations satisfied by bispinors $u^{(\pm)}$ describing the states with a given momentum p and different signs of the energy.

Solution. For the state with positive energy, $\psi^{(+)} = u^{(+)}(p)e^{-ipx}$, where $px = p_0 t - \boldsymbol{p}\cdot\boldsymbol{r}$. Substituting this function into (7.4) and (7.5) in which $A_k = 0$ yields

$$\left(\widehat{\gamma^k}p_k - m\right)u^{(+)}(p) = 0 , \quad \overline{u}^{(+)}(p)\left(\widehat{\gamma^k}p_k - m\right) = 0 . \tag{7.13}$$

For the states with negative energy where the sign of the component $p_0 = \mathcal{E}$ changes, it is convenient to change the sign of the remaining components of the momentum, that is, $\boldsymbol{p} \to -\boldsymbol{p}$. Then, $\psi^{(-)} = u^{(-)}(-p)e^{ipx}$ and (7.4) and (7.5) give

$$\left(\widehat{\gamma^k}p_k + m\right)u^{(-)}(-p) = 0 , \quad \overline{u}^{(-)}(-p)\left(\widehat{\gamma^k}p_k + m\right) = 0 . \tag{7.14}$$

Systems (7.13) and (7.14) are derived from each other by the substitution $m \to -m$. □

Example 7.2

Calculate the sum $\sum_\mu u_\mu^{(+)}(p)\overline{u}_\mu^{(+)}(p)$ over spin states with positive energy and momentum $\boldsymbol{p}$ from the product of bispinors. This sum denoted by $\widehat{\Pi}_+(p)$ should be understood as a 4×4 matrix whose elements have the form of the products of spinor $u^{(+)}(p)$ components and spinor $\overline{u}^{(+)}(p)$ components:

$$\sum_{\mu=-1/2}^{1/2} \left(u_\mu^{(+)}(p)\right)_\alpha \left(\overline{u}_\mu^{(+)}(p)\right)_\beta = \left(\widehat{\Pi}_+(p)\right)_{\alpha\beta} . \tag{7.15}$$

Do the same for the sum of bispinor products related to the states with negative energy.

Solution. Find the condition defining the matrix being sought. Its trace

$$\operatorname{Tr}\widehat{\Pi}_+(p) = \sum_\mu \overline{u}_\mu^{(+)}(p)u_\mu^{(+)}(p) = \begin{cases} 4m , & \text{normalization (a)} , \\ 2 , & \text{normalization (b)} , \end{cases} \tag{1}$$

where normalization (a) means $\overline{u}^{(+)}(p)u^{(+)}(p) = 2m$ and normalization (b) means $\overline{u}^{(+)}(p)u^{(+)}(p) = 1$. Moreover, the definition and (7.13) lead to two equations which must be satisfied by the matrix being sought,

$$\left(\widehat{\gamma^k} p_k - m\right)\widehat{\Pi}_+(p) = 0 , \quad \widehat{\Pi}_+(p)\left(\widehat{\gamma^k} p_k - m\right) = 0 . \tag{2}$$

These equations become identities after the substitution $\widehat{\Pi}_+(p) = C(m + p_l\widehat{\gamma^l})$, where C is a normalization constant. It follows from condition (1) that

$$\widehat{\Pi}_+(p) = \begin{cases} m + p_l\widehat{\gamma^l} , & \text{normalization (a)} , \\ \frac{m+p_l\widehat{\gamma^l}}{2m} , & \text{normalization (b)} . \end{cases} \tag{7.16}$$

An analogous line of reasoning and the use of equation (7.14) leads to

$$\begin{aligned} \widehat{\Pi}_-(p) &= \sum_{\mu=-1/2}^{1/2} u_\mu^{(-)}(-p))\overline{u}_\mu^{(-)}(-p) \\ &= \begin{cases} m - p_l\widehat{\gamma^l} , & \text{normalization (a)} , \\ \frac{m-p_l\widehat{\gamma^l}}{2m} , & \text{normalization (b)} . \end{cases} \end{aligned} \tag{7.17}$$

Normalization constant C has the former meaning, but the signs of the normalization need to be changed: $\overline{u}^{(-)}(-p)u^{(-)}(-p) = -2m$ for normalization (a) and $\overline{u}^{(-)}(-p)u^{(-)}(-p) = -1$ for normalization (b). □

Example 7.3

Show with the help of equations for bispinors with a given momentum and different signs of energy that the operators

$$\widehat{\Pi}_{+}(p) = \frac{\left(m + \widehat{\gamma^k} p_k\right)}{2m}\,, \quad \widehat{\Pi}_{-}(p) = \frac{\left(m - \widehat{\gamma^k} p_k\right)}{2m} \tag{7.18}$$

are the projection operators onto the states with positive and negative energies. Obtain the following relations for this purpose:

$$\widehat{\Pi}_{+} u^{(+)} = u^{(+)}\,, \quad \widehat{\Pi}_{+} u^{(-)} = 0\,; \quad \widehat{\Pi}_{-} u^{(-)} = u^{(-)}\,, \quad \widehat{\Pi}_{-} u^{(+)} = 0\,. \tag{7.19}$$

$$\widehat{\Pi}_{+}\widehat{\Pi}_{-} = \widehat{\Pi}_{-}\widehat{\Pi}_{+} = 0\,, \quad \widehat{\Pi}_{+}^2 = \widehat{\Pi}_{+}\,, \quad \widehat{\Pi}_{-}^2 = \widehat{\Pi}_{-}\,. \tag{7.20}$$

What is the result of the action of each operator on an arbitrary bispinor $u(p)$?

Solution. The arbitrary bispinor describes the superposition of the states with positive and negative energies: $u = u^{(+)} + u^{(-)}$. Each of the operators acts on u and makes up part of the bispinor with a definite energy sign. We have already used the projection operator in Example 6.19 and Problem 6.72, where it was written in the noncovariant form. □

Problems

7.1. Write matrices $\widehat{\gamma^i}$, $i = 0, 1, 2, 3$, in the explicit four-row form.

7.2. Calculate the products of the $\widehat{\gamma}$ matrices presented below with the help of anticommutation relations (7.3). For the matrices with subscripts, we use the notation $\widehat{\gamma}_i = g_{ik}\widehat{\gamma^k}$. The quantities a_k, b_k, and c_k are the usual vectors (not matrices).

1. $\widehat{\gamma}_k \widehat{\gamma^k} = 4\,.$
2. $\widehat{\gamma}_k \widehat{\gamma^i}\widehat{\gamma^k} = -2\widehat{\gamma^i}\,.$
3. $\widehat{\gamma}_k \widehat{\gamma^i}\widehat{\gamma^l}\widehat{\gamma^k} = 4g^{ik}\,.$
4. $\widehat{\gamma}_k \widehat{\gamma^i}\widehat{\gamma^l}\widehat{\gamma^n}\widehat{\gamma^k} = -2\widehat{\gamma}_n\widehat{\gamma^l}\widehat{\gamma^i}\,.$
5. $\left(a_i\widehat{\gamma^i}\right)\left(b_k\widehat{\gamma^k}\right) + \left(b_k\widehat{\gamma^k}\right)\left(a_i\widehat{\gamma^i}\right) = 2a_i b^i\,.$
6. $\widehat{\gamma}_k \left(a_i\widehat{\gamma^i}\right)\widehat{\gamma^k} = -2\left(a_i\widehat{\gamma^i}\right)\,.$

7. $\widehat{\gamma}_k \left(a_i \widehat{\gamma}^i\right) \left(b_l \widehat{\gamma}^l\right) \widehat{\gamma}^k = 4 a_i b^i$.

8. $\widehat{\gamma}_k \left(a_i \widehat{\gamma}^i\right) \left(b_l \widehat{\gamma}^l\right) \left(c_n \widehat{\gamma}^n\right) \widehat{\gamma}^k = -2 \left(c_n \widehat{\gamma}^n\right) \left(b_l \widehat{\gamma}^l\right) \left(a_i \widehat{\gamma}^i\right)$.

7.3. Calculate the traces (the sums of diagonal elements) of matrix products.

1. Make sure that the cyclic interchange of cofactors (without changing their order) does not affect the trace value:
$$\mathrm{Tr}\left(\widehat{\gamma}^i \widehat{\gamma}^k \widehat{\gamma}^l \ldots \widehat{\gamma}^m\right) = \mathrm{Tr}\left(\widehat{\gamma}^k \widehat{\gamma}^l \ldots \widehat{\gamma}^m \widehat{\gamma}^i\right) = \ldots$$
2. Make sure that the trace of odd number $\widehat{\gamma}$ matrices equals zero.
3. Calculate
$$\mathrm{Tr}\left(\widehat{\gamma}^i \widehat{\gamma}^k\right) = 4 g^{ik} , \quad \mathrm{Tr}\left(\widehat{\gamma}^i \widehat{\gamma}^k \widehat{\gamma}^l \widehat{\gamma}^n\right) = 4\left(g^{ik} g^{ln} + g^{in} g^{lk} - g^{il} g^{kn}\right) .$$

7.2 Covariant Quadratic Forms

The bispinors can be used to make four-dimensional tensors of different ranks. Such quantities may be necessary when considering interactions between different fields. In this book, we deal only with interactions between electron–positron and electromagnetic fields.

1. The simplest bilinear form is the product of conjugate and initial bispinors. Using relations (7.6) and (7.11), we find
$$S = \overline{\psi}'(x')\psi'(x') = \overline{\psi}(x)\widehat{U}^{-1}\widehat{U}\psi(x) = \overline{\psi}(x)\psi(x) = \mathrm{inv} . \qquad (7.21)$$
This quantity is the invariant of the Lorentz transformation, that is, a 4-scalar.
2. The fundamental role in the process of field interaction is played by the four-dimensional density of the electric current $j^k = (e\rho, e\boldsymbol{j})$, where e is the particle's charge, ρ and $\boldsymbol{j}$ are the probability density and the probability current density, respectively, for which the expressions (6.119) and (6.120) were obtained. Combine them and write down with the use of the $\widehat{\gamma}^k$ matrix
$$j^k(x) = e\overline{\psi}(x)\widehat{\gamma}^k\psi(x) , \quad k = 0, 1, 2, 3 . \qquad (7.22)$$
Find the Lorentz transformation law of the current density:
$$\begin{aligned} j'^k &= e\overline{\psi}'(x')\widehat{\gamma}^k\psi'(x') = e\overline{\psi}(x)\widehat{U}^{-1}\widehat{\gamma}^k\widehat{U}\psi(x) = e\Lambda^k{}_i\overline{\psi}(x)\widehat{\gamma}^i\psi(x) \\ &= \Lambda_i^k j^i . \end{aligned} \qquad (7.23)$$
Here, relation (7.8) is used. Formula (7.23) coincides with (4.14) (at $a = 0$) and expresses the transformation of a 4-vector.

3. The product of two $\widehat{\gamma}$ matrices leads to the rank 2 antisymmetric tensor

$$\widehat{\gamma^i}\widehat{\gamma^k} = \frac{1}{2}\left(\widehat{\gamma^i}\widehat{\gamma^k} + \widehat{\gamma^k}\widehat{\gamma^i}\right) + \frac{1}{2}\left(\widehat{\gamma^i}\widehat{\gamma^k} - \widehat{\gamma^k}\widehat{\gamma^i}\right) = g^{ik} + \widehat{S}^{ik} ,$$

where the first term derived from the anticommutator is proportional to a four-row unit matrix that is not written like the usual unit matrix. The quantity

$$S^{ik} = \overline{\psi}(x)\widehat{S}^{ik}\psi(x) , \quad \text{where} \quad \widehat{S}^{ik} = \frac{1}{2}\left(\widehat{\gamma^i}\widehat{\gamma^k} - \widehat{\gamma^k}\widehat{\gamma^i}\right) , \tag{7.24}$$

is a rank 2 4-tensor, which is easy to check in the same way as for the j^k vector.

4. Further increase of the number of $\widehat{\gamma}$ matrices between $\overline{\psi}(x)$ and $\psi(x)$ does not increase the rank of the tensor because it will contain identical matrices, the squares of which give units. Moreover it is possible to introduce dual tensors of lower ranks. However, qualitatively new quantities emerge. The commonest of them is the product of all four matrices denoted by $\widehat{\gamma^5}$:

$$\widehat{\gamma^5} = -i\widehat{\gamma^0}\widehat{\gamma^1}\widehat{\gamma^2}\widehat{\gamma^3} = -\begin{pmatrix} 0 & \widehat{1} \\ \widehat{1} & 0 \end{pmatrix} . \tag{7.25}$$

This matrix can be written in the invariant form

$$\widehat{\gamma^5} = \frac{i}{4!} e_{iklm}\widehat{\gamma^i}\widehat{\gamma^k}\widehat{\gamma^l}\widehat{\gamma^m} . \tag{7.26}$$

Let us find the transformation of quantity $P' = \overline{\psi}'(x')\widehat{\gamma^5}\psi'(x') = \overline{\psi}(x)$ $\widehat{U}^{-1}\widehat{\gamma^5}\widehat{U}\psi(x)$. Using (7.17) and (7.8), we arrive at

$$\widehat{U}^{-1}\widehat{\gamma^5}\widehat{U} = \frac{i}{4!} e_{iklm}\Lambda^i_{i'}\Lambda^k_{k'}\Lambda^l_{l'}\Lambda^m_{m'}\widehat{\gamma^{i'}}\widehat{\gamma^{k'}}\widehat{\gamma^{l'}}\widehat{\gamma^{m'}} .$$

Formula (4.34′) from the theory of determinants leads to

$$\widehat{U}^{-1}\widehat{\gamma^5}\widehat{U} = \frac{i}{4!}\widehat{\gamma^5}\det|\widehat{\Lambda}| .$$

Thus, the transformation above gives rise to the factor $\det|\widehat{\Lambda}|$. For the proper Lorentz transformation $\det|\widehat{\Lambda}| = 1$, but for the inverted spatial axes $\det|\widehat{\Lambda}| = -1$. The quantity

$$P = \overline{\psi}(x)\widehat{\gamma^5}\psi(x) \tag{7.27}$$

is a pseudoscalar. The addition of factor $\widehat{\gamma^5}$ to the definition of vector (7.22) and tensor (7.24) makes the bilinear forms $\overline{\psi}(x)\widehat{\gamma^5}\widehat{\gamma^i}\psi(x)$ and $\overline{\psi}(x)\widehat{\gamma^5}\widehat{S}^{ik}\psi(x)$ a pseudovector and a pseudotensor, respectively.

Problems

7.4. Prove the following properties of matrix $\widehat{\gamma^5}$:

$$\widehat{\gamma^5}\widehat{\gamma^k} + \widehat{\gamma^k}\widehat{\gamma^5} = 0\,, \quad (\widehat{\gamma^5})^\dagger = \widehat{\gamma^5}\,, \quad (\widehat{\gamma^5})^2 = 1\,, \quad \mathrm{Tr}\,\widehat{\gamma^5} = 0\,.$$

7.5. Consider different bispinor normalizations describing motion with a given momentum $\boldsymbol{p}$. The normalization used in Example 6.18 is used along with other invariant conditions: (i) $\overline{u}u = \pm 2m$ and (ii) $\overline{u}u = \pm 1$ (the signs correspond to the signs of the energy of states; such normalizations have already been used in Examples 7.2 and 7.3). In the latter two cases, the invariant normalization is used for the exponential factor $(2\mathcal{E}\mathcal{V})^{-1/2}\exp(i\boldsymbol{p}\cdot\boldsymbol{r} - i\mathcal{E}t)$ too. Calculate the normalization factor N for the bispinor as well as the probability densities and currents (three dimensional) for different normalizations.

7.6.

1. Write down bispinors u at $\mathcal{E} > 0$ in the form of four-component columns for the states in which the spin in the particle's rest system is oriented along and opposite to the Oz axis and its momentum has an arbitrary direction.
2. Do the same for the case when a particle is in states characterized by definite values of spirality, with the momentum direction specified by the polar angles θ and ϕ in the Cartesian system chosen.

7.7. On the basis of the result obtained in Problem 4.20, construct the $\widehat{U}$ matrix that performs a small (taking account of only the first-order terms) Lagrange transformation for the bispinor.

7.8. Show that the current 4-vector (7.13) undergoes transformation in accordance with formulas (3.9) using the Lorentz transformation matrix (7.9) for the bispinor and directing the relative velocity along the Ox axis.

7.9. Use the Lorentz transformation matrix (7.9) and the explicit expression for the bispinor of a motionless particle $u_0 = \binom{\varphi}{0}$, where φ is a two-component spinor, to construct bispinor $u(\boldsymbol{p})$ of a particle having momentum $\boldsymbol{p}$.

7.10. An electron in its rest system has the average value of the polarization vector (double spin) $\boldsymbol{\zeta}_0 = (\varphi^\dagger\widehat{\boldsymbol{\sigma}}\varphi)$, where φ is the spinor describing the particle's state (i.e., the two upper components of the bispinor with $(\varphi^\dagger\varphi) = 1$). Calculate the average polarization vector $\boldsymbol{\zeta} = (u^\dagger\widehat{\boldsymbol{\Sigma}}u)/(u^\dagger u)$ in the frame of reference in which the electron propagates with momentum $\boldsymbol{p}$. Here, $\boldsymbol{\Sigma}$ is the spin operator found in Example 6.16. Analyze the nonrelativistic and ultrarelativistic cases.

7.3
Charge Conjugation and Wave Functions of Antiparticles

The solutions of the Dirac equation with negative energies $\psi^{(-)}$ do not allow an interpretation as the solutions with positive energies do. There are no free particles with negative energy in nature. An antiparticle (e.g., a positron) having positive energy differs from an electron in the sign of both the charge and the magnetic moment. To correlate in a similar fashion the wave functions with particles and antiparticles, the wave functions of states with negative energy need to be transformed. These transformations must change the sign of the particle's energy and charge.

Let us denote the charge-conjugate function by ψ^c, and the energy of particles and antiparticles by $\mathcal{E} > 0$. Then, for the stationary state $\psi^{(-)} \sim e^{i\mathcal{E}t}$, and the sign of the energy can be changed by passing to the complex-conjugate function $\psi^{*(-)}$. Let us write it down in the form

$$\psi^c = \widehat{C}\widehat{\gamma^0}\psi^{*(-)} . \tag{7.28}$$

Here, $\widehat{C}$ denotes the four-row matrix chosen in such a way that ψ^c satisfies (7.4) with inverse sign of charge:

$$(i\partial_k + eA_k(x))\widehat{\gamma^k}\psi(x) - m\psi(x) = 0 . \tag{7.29}$$

To this effect, we use (7.5) for the conjugate bispinor, but perform transposition in all matrices:

$$(i\partial_k + eA_k(x))\widehat{\tilde{\gamma}^k}\tilde{\overline{\psi}} + m\tilde{\overline{\psi}} . \tag{7.30}$$

This means that the columns in each square matrix are substituted with lines and vice versa, whereas bispinor columns are substituted with bispinor lines (and vice versa). It follows from the explicit form of the matrices that

$$\widehat{\tilde{\gamma}^0} = \widehat{\gamma^0} , \quad \widehat{\tilde{\gamma}^1} = -\widehat{\gamma^1} , \quad \widehat{\tilde{\gamma}^2} = \widehat{\gamma^2} , \quad \widehat{\tilde{\gamma}^3} = -\widehat{\gamma^3} . \tag{7.31}$$

The definition of a conjugate bispinor implies that $\tilde{\overline{\psi}} = \widehat{\gamma^0}\psi^*$, where ψ^* is the column with complex-conjugate components. Substitution of these components into (7.30) yields the equality

$$\left(D_0\widehat{\gamma^0} - D_1\widehat{\gamma^1} + D_2\widehat{\gamma^2} - D_3\widehat{\gamma^3}\right)\widehat{\gamma^0}\psi^* + m\widehat{\gamma^0}\psi^* = 0 , \tag{7.32}$$

where the form is shortened by denoting $D_k = i\partial_k + eA_k$. Multiplying both parts of the left-hand side of equality (7.32) by the product $\widehat{\gamma^2}\widehat{\gamma^0}$ and making use of anticommutation of $\widehat{\gamma}$ matrices gives (7.29), in which $\psi(x)$ is replaced by

$$\psi^c(x) = \eta\widehat{\gamma^2}\psi^*(x) . \tag{7.33}$$

Here, η is an arbitrary phase factor, $|\eta| = 1$. In other words, the charge conjugation matrix is

$$\widehat{C} = \widehat{\gamma^2}\widehat{\gamma^0} = -\widehat{\alpha}_2 \,. \tag{7.34}$$

Example 7.4

Perform charge conjugation of the wave function describing the state with energy $\mathcal{E} < 0$, three-dimensional momentum $-\boldsymbol{p}$, and spin projection $-\mu$ in the particle's rest system. Do the same for the wave function of the state with energy $\mathcal{E} > 0$, momentum $\boldsymbol{p}$, and spin projection μ.

Solution. The initial wave function was constructed in Example 6.18 and has the form

$$\begin{aligned}\psi_{-p,-\mu} &= N \begin{pmatrix} -\frac{\mathbf{p}\cdot\widehat{\sigma}}{|\mathcal{E}|+m} w_{-\mu} \\ w_{-\mu} \end{pmatrix} e^{ipx} = N \begin{pmatrix} -\frac{2\mu|\mathbf{p}|}{|\mathcal{E}|+m} w_{-\mu} \\ w_{-\mu} \end{pmatrix} e^{ipx} \,, \\ px &= |\mathcal{E}|t - \boldsymbol{p}\cdot\boldsymbol{r} \,. \end{aligned} \tag{1}$$

Find using formula (7.34) and the two-row form of matrix $\widehat{\gamma^2}$ that

$$\psi^{(c)}_{-p,-\mu} = \eta N \begin{pmatrix} \widehat{\sigma}_2 w_{-\mu} \\ -\frac{2\mu|\mathbf{p}|}{|\mathcal{E}|+m}\widehat{\sigma}_2 w_{-\mu} \end{pmatrix} e^{-ipx} \,. \tag{2}$$

Now, use the explicit form of matrix $\widehat{\sigma}_2$ and spinor $w_{-\mu}$ to obtain $\widehat{\sigma}_2 w_{-\mu} = -iw_\mu$ at $\mu = 1/2$ and $\widehat{\sigma}_2 w_{-\mu} = iw_\mu$ at $\mu = -1/2$. The phase factors $\pm i$ can be included in η. As a result, the charge conjugation (7.33) of the wave function (1) leads to the state with positive energy: $\psi^{(c)}_{-p,-\mu} = \eta\psi_{p\mu}(x)$.

The application of charge conjugation to $\psi_{p\mu}(x)$ yields the wave function of the negative-energy state: $\psi^{(c)}_{p\mu}(x) = \eta\psi_{-p,-\mu}(x)$.

Charge conjugation reverses the direction of both the spin and the momentum; because of this, the sign of the spin projection onto the momentum (spirality) remains unaltered. □

7.4 Secondary Quantization of the Dirac Field. Creation and Annihilation Operators for Field Quanta

Up to now, we have considered the Dirac equation as an equation describing the state of a single particle with a spin (electron, muon, taon, and their antiparticles). However, such an approach is highly restricted in the relativistic theory because high energies are able to generate a large number of analogous or different particles. For this reason, relativistic systems usually have a variable number of particles. A description is therefore needed that permits us to take into account the processes of creation and annihilation of different particles. For this purpose, the treatment of the Dirac equation needs to be extended; it should be regarded as an equation describing as a whole the electron–positron (or another) field, the number of quanta (particles and antiparticles) in which may vary. We have already applied such an approach to the electromagnetic field in Section 6.1 when the vector potential (6.13) was considered to be an operator capable of changing the number of photons.

Let us write down the general solution $\psi(x)$ of the Dirac equation for a free particle in the form of superposition of the states with a given momentum $\boldsymbol{p}$, spirality $\sigma = \pm 1/2$, and sign of the energy $\pm\mathcal{E}$, $\mathcal{E} = \sqrt{m^2 + p^2}$:

$$\psi(x) = \sum_{\mathbf{p}\sigma} \left[a^{(+)}_{\mathbf{p}\sigma} \psi^{(+)}_{\mathbf{p}\sigma}(x) + a^{(-)}_{\mathbf{p}\sigma} \psi^{(-)}_{\mathbf{p}\sigma}(x) \right] , \tag{7.35}$$

where $a^{(\pm)}_{\mathbf{p}\sigma}$ are the complex coefficients of expansion. The superscripts denote the sign of the energy.

$$\psi^{(+)} \sim \exp(i\boldsymbol{p}\cdot\boldsymbol{r} - i\mathcal{E}t) , \quad \psi^{(-)} \sim \exp(i\boldsymbol{p}\cdot\boldsymbol{r} + i\mathcal{E}t) .$$

To unify and simplify the writing, we substitute summation indices $\boldsymbol{p} \to -\boldsymbol{p}, \sigma \to -\sigma$ in the second sum of (7.35), introduce a single index for the three quantities $s = (\boldsymbol{p}, \mathcal{E}, \sigma)$, and redesignate the complex coefficients of expansion as $a^{(+)}_{\mathbf{p}\sigma} = a_s$ and $a^{(-)}_{-\mathbf{p},-\sigma} = b^*_s$. Then, expansion (7.35) will take the form

$$\psi(x) = \sum_s \left[a_s \psi_s(x) + b^*_s \psi_{-s}(x) \right] . \tag{7.36}$$

We obtained the analogue of the expansion of the vector potential (6.8), but in this case we expanded the complex bispinor field.

Let us further perform secondary quantization and assign to constants a_s and b^*_s the sense of annihilation operators of fermions with quantum numbers $s = (\boldsymbol{p}, \mathcal{E} > 0, \sigma)$ and $-s = (-\boldsymbol{p}, -\mathcal{E} < 0, -\sigma)$, respectively. However, the annihilation of a particle with negative energy $-\mathcal{E}$, momentum $-\boldsymbol{p}$, and spirality $-\sigma$ is equivalent to the creation of a fermion with energy $\mathcal{E} > 0$, momentum $\boldsymbol{p}$, and spirality σ. For this reason, a_s should be identified as the annihilation operator of a fermion with a set of quantum numbers s, and b^*_s should be assigned the sense of the creation operator $\widehat{b}^\dagger_s$ of an antifermion with the same set of quantum numbers. As a result, the bispinor (7.36) becomes the absorption operator of fermions and the creation operator of antifermions:

$$\widehat{\psi}(x) = \sum_s \left[\widehat{a}_s \psi_s(x) + \widehat{b}^\dagger_s \psi_{-s}(x) \right] . \tag{7.37}$$

In the present case, the replacement of complex numbers by operators is referred to as secondary quantization because primary quantization was performed earlier during the transition from the equations of classical relativistic mechanics to the quantum single-particle Dirac equation. The Dirac conjugate operator is defined like the conjugate bispinor: $\widehat{\overline{\psi}}(x) = \widehat{\psi}^\dagger(x)\widehat{\gamma^0}$. It is expressed through the sum

$$\widehat{\overline{\psi}}(x) = \sum_s \left[\widehat{a}_s^\dagger \overline{\psi}_s(x) + \widehat{b}_s \overline{\psi}_{-s}(x)\right] , \tag{7.38}$$

which contains the creation operators for fermions and the annihilation operators for antifermions in different states.

The operators $\widehat{\psi}(x)$ and $\widehat{\overline{\psi}}(x)$ are written in the Heisenberg representation for free particles that ensures relativistic covariance. They depend on time and undergo transformation after transition into a different inertial frame of reference in accordance with the rules formulated in Section 7.1. Functions $\psi_s(x)$ and $\overline{\psi}_s(x)$ satisfy the Dirac equations for free particles:

$$i\partial_l \widehat{\gamma^l} \psi_s - m\psi_s = 0 , \quad i\partial_l \overline{\psi}_s \widehat{\gamma^l} + m\overline{\psi}_s = 0 . \tag{7.39}$$

They constitute a complete orthonormalized system of bispinors. They are eigenfunctions of the complete set of Hermitian operators. Let us consider them to be normalized to unity:

$$\begin{aligned} \int \psi_{s'}^\dagger(x)\psi_s(x)\mathrm{d}^3x &= \int \overline{\psi}_{s'}(x)\widehat{\gamma^0}\psi_s(x)\mathrm{d}^3x \\ &= \int \overline{\psi}_{-s'}(x)\widehat{\gamma^0}\psi_{-s}(x)\mathrm{d}^3x = \delta_{ss'} , \end{aligned} \tag{7.40}$$

$$\int \overline{\psi}_{-s'}(x)\widehat{\gamma^0}\psi_s(x)\mathrm{d}^3x = \int \overline{\psi}_{s'}(x)\widehat{\gamma^0}\psi_{-s}(x)\mathrm{d}^3x = 0 . \tag{7.41}$$

The last equalities hold at $s = s'$ too because bispinors are the eigenfunctions of the Hamiltonian related to the states with different signs of energy. Equations (7.39) are satisfied by operators $\widehat{\psi}(x)$ and $\widehat{\overline{\psi}}(x)$ themselves:

$$i\partial_l \widehat{\gamma^l}\widehat{\psi} - m\widehat{\psi} = 0 , \quad i\partial_l \widehat{\overline{\psi}}\widehat{\gamma^l} + m\widehat{\overline{\psi}} = 0 . \tag{7.42}$$

The next step in the construction of the scheme for secondary quantization is the establishment of commutation relations for the creation and annihilation operators. By analogy with the case of the electromagnetic field (6.1), we shall characterize the field state by occupation numbers N_s and $\overline{N}_s$ of the Dirac particles and antiparticles (the latter are denoted by a bar). The vector of the state of an electron–positron field is given by the product

$$\Phi_{N_s,\overline{N}_{s'}} = \prod_{s,s'} |N_s\rangle|\overline{N}_{s'}\rangle \tag{7.43}$$

of the state vectors of particles $|N_s\rangle$ and antiparticles $|\overline{N}_s\rangle$.

The fundamental difference of the electromagnetic field consists in that the number of photons $N_s = 0, 1, 2$ can be arbitrary in each quantum state, whereas fermions obey the Pauli exclusion principle ensuing from experience and their occupation numbers are limited:

$$N_s = 0, 1\ ; \quad \overline{N}_s = 0, 1\ . \tag{7.44}$$

Make sure that if the operators of the occupation numbers are defined as the products

$$\widehat{N}_s = \widehat{a}_s^\dagger \widehat{a}_s\ , \quad \widehat{\overline{N}}_s = \widehat{b}_s^\dagger \widehat{b}_s\ , \tag{7.45}$$

then the Pauli principle is fulfilled for given anticommutators $\widehat{a}$ and $\widehat{b}$ in the following form:

$$\begin{gathered}\left\{\widehat{a}_s, \widehat{a}_{s'}^\dagger\right\} = \delta_{ss'}\ , \quad \left\{\widehat{b}_s, \widehat{b}_{s'}^\dagger\right\} = \delta_{ss'}\ , \\ \{\widehat{a}_s, \widehat{a}_{s'}\} = \left\{\widehat{a}_s, \widehat{b}_{s'}^\dagger\right\} = \left\{\widehat{b}_s^\dagger, \widehat{b}_{s'}^\dagger\right\} = \cdots = 0\ .\end{gathered} \tag{7.46}$$

Verify the above inference. It follows from (7.46) that $\widehat{a}_s^2 = (\widehat{a}_s^\dagger)^2 = 0$. The construction of

$$\widehat{N}_s^2 = \widehat{a}_s^\dagger \widehat{a}_s \widehat{a}_s^\dagger \widehat{a}_s = \widehat{a}_s^\dagger \widehat{a}_s (1 - \widehat{a}_s \widehat{a}_s^\dagger) = \widehat{a}_s^\dagger \widehat{a}_s = \widehat{N}_s$$

leads to the idempotent operator $\widehat{N}_s$, that is, $\widehat{N}_s^n = \widehat{N}_s$ for any integer n. Its eigenvalue has an analogous property: if $\widehat{N}_s \Phi = N_s \Phi$, then $\widehat{N}_s^2 \Phi = N_s^2 \Phi = \widehat{N}_s \Phi = N_s \Phi$, that is, $N_s^2 = N_s$, $N_s = 0, 1$. The same is obtained for $\overline{N}_s = 0, 1$. Quantization with the use of anticommutators (7.46) gives an adequate picture of the behavior of Dirac field quanta – particles and antiparticles with half-integer spin.

7.5
Energy and Current Density Operators for Dirac Particles

To find operators for the above quantities in the secondary quantization representation, we make use of the conformity principle, The energy $\mathcal{E}$ of a free particle whose state is described by bispinor $\psi(x)$ is given by integral (7.47),

$$\overline{\mathcal{E}} = \int \psi^\dagger(x) \widehat{\mathcal{H}}_{\mathrm{D}} \psi(x) \mathrm{d}^3 x = i \int \overline{\psi}(x) \widehat{\gamma^0} \frac{\partial \psi(x)}{\partial t} \mathrm{d}^3 x\ , \tag{7.47}$$

where $\widehat{\mathcal{H}}_{\mathrm{D}}$ is the Dirac Hamiltonian (6.111) of a free particle. This formula holds for both the stationary state and the general case. If ψ and $\overline{\psi}$ are regarded as the operators describing an electron–positron field by and large, quantity (7.47) becomes the energy operator of this field

$$\overline{\mathcal{E}} \to \widehat{\mathcal{H}} = i \int \widehat{\overline{\psi}}(x) \widehat{\gamma^0} \frac{\partial \widehat{\psi}(x)}{\partial t} \mathrm{d}^3 x\ , \tag{7.48}$$

where, according to (7.37),

$$i\frac{\partial\widehat{\psi}(x)}{\partial t} = \sum_s \mathcal{E}_s \left[\widehat{a}_s \psi_s(x) - \widehat{b}_s^\dagger \psi_{-s}(x)\right] . \tag{7.49}$$

Substituting expansions (7.38) and (7.49) into (7.48) and integrating with the help of formulas (7.40) and (7.41) yields the Hamiltonian operator $\widehat{\mathcal{H}}$ (energy operator) of the free Dirac field:

$$\widehat{\mathcal{H}} = \sum_s \mathcal{E}_s \left(\widehat{a}_s^\dagger \widehat{a}_s - \widehat{b}_s \widehat{b}_s^\dagger\right) = \sum_s \mathcal{E}_s \left(\widehat{N}_s + \widehat{\overline{N}}_s\right) + \mathcal{E}_0 . \tag{7.50}$$

Here, the commutation relation $\widehat{b}_s\widehat{b}_s^\dagger = -\widehat{b}_s^\dagger\widehat{b}_s + 1$ is used and the notation $\mathcal{E}_0 = -\sum_s \mathcal{E}_s$ is used for the infinite constant. We have already had consider the appearance of infinite energy (6.7) in quantization of the electromagnetic field. It should be omitted to predetermine the point of reference for the energy to arrive at the explicit expression for the Hamiltonian of the free Dirac field as the sum of Hamiltonians of individual modes related to particles and antiparticles:

$$\widehat{\mathcal{H}} = \sum_s \mathcal{E}_s \left(\widehat{N}_s + \widehat{\overline{N}}_s\right) . \tag{7.51}$$

The eigenvalue of this Hamiltonian

$$E_{N_s,\overline{N}_s} = \sum_s \mathcal{E}_s \left(N_s + \overline{N}_s\right) \tag{7.52}$$

is the sum of the energies of all particles and antiparticles, with the energies of both being positive as is expected from experience.

Now, let us create the electric current density operator in the representation of secondary quantization. Consideration of bispinors in expression (7.13) as the operators leads to

$$\widehat{j^k}(x) = e\widehat{\overline{\psi}}(x)\widehat{\gamma^k}\widehat{\psi}(x) , \quad k = 0, 1, 2, 3 . \tag{7.53}$$

We use this expression to calculate the total electric charge operator $\widehat{Q}$ by integration of the charge density $\widehat{j^0}(x)$ over the three-dimensional space:

$$\begin{aligned} \widehat{Q} &= e\int \widehat{\overline{\psi}}(x)\widehat{\gamma^0}\widehat{\psi}(x)\mathrm{d}^3x = e\sum_s \left(\widehat{a}_s^\dagger\widehat{a}_s + \widehat{b}_s\widehat{b}_s^\dagger\right) \\ &= \sum_s \left(e\widehat{N}_s - e\widehat{\overline{N}}_s\right) + Q_0 , \\ Q_0 &= e\sum_s 1 \to \infty . \end{aligned} \tag{7.54}$$

Its eigenvalue is

$$Q = \sum_s \left(eN_s - e\overline{N}_s\right) + Q_0 . \tag{7.55}$$

Q_0 is a constant, that is, an infinite electric charge of the field, emerging as in the calculation of the quantum field energy. The appearance of such charge is consistent with the original Dirac hypothesis about filling an infinite number of negative-energy states by the particles; however, it is in conflict with experiment. The "infinite vacuum charge" does not manifest itself in any way.

There are two ways to resolve this discrepancy:

1. To recognize constant Q_0 as being "nonphysical," that is, having no physical sense, and omit it.
2. To modify the operator expression (7.53) for electric current in such a way that it does not lead to the appearance of infinite charge but at the same time satisfies the continuity equation (i.e., the law of conservation of electric charge) and corresponds to expression (7.22) of the single-particle theory.

The latter way is more consistent and is realized by using the following equation for the current:

$$\widehat{j^k}(x) = \frac{e}{2}\left[\widehat{\overline{\psi}}(x), \widehat{\gamma^k \widehat{\psi}}(x)\right] = \frac{e}{2}\left(\widehat{\gamma^k}\right)_{\lambda'\lambda}\left[\widehat{\overline{\psi}}_{\lambda'}(x), \widehat{\psi}_\lambda(x)\right] . \tag{7.56}$$

Here, the square brackets denote the commutator of relevant operators. If $\widehat{\psi}(x)$ and $\widehat{\overline{\psi}}(x)$ did not contain operators $\widehat{a}$ and $\widehat{b}$, the commutator would vanish. However, the operator $\widehat{j^k}(x)$ differs from zero by virtue of noncommutativity of $\widehat{\psi}(x)$ and $\widehat{\overline{\psi}}(x)$. It is easy to see that it leads to the correct value of the field charge, $Q = \sum_s (eN_s - e\overline{N}_s)$.

Similarly to the case of the energy operator, expressions (7.53) and (7.56) for the electric current operator lead to identical and experimentally confirmed probabilities for different electrodynamic processes. For this reason, a simpler expression (7.53) is usually used for practical purposes (see, e.g., Berestetskii *et al.*, 1982).

Not surprisingly, certain relations of the quantum Dirac field theory differ from the results of the single-particle theory. A more general and deeper world view in which particles and antiparticles behave as the quanta (excitations) of a unified field based on the already elaborated notions of nonrelativistic and relativistic single-particle theory must specify these notions and interpret them differently. It is in this way that the nonrelativistic quantum mechanics based on the notions and mathematical apparatus of classical Newtonian/Galilean mechanics was changed into the form into which it was shaped by Euler, Lagrange, Hamilton, and other outstanding scientists.

7.6
Interaction between Electron–Positron and Electromagnetic Fields

In the classical field theory (Chapter 4), the part of the Lagrangian describing the interaction between an electromagnetic field A_i and the current j^i created by charged particles is determined by the constituent of action (4.108):

$$S_{\text{int}} = -\int j^i(x) A_i(x) \mathrm{d}^4 x \,. \tag{7.57}$$

The subintegral function is a part of the Lagrangian density function (Lagrangian):

$$\mathcal{L}_{\text{int}}(x) = -j^i(x) A_i(x) \,. \tag{7.58}$$

The contribution of this term to the Lagrangian function L_{int} is given by the integral over three-dimensional space:

$$L_{\text{int}} = \int \mathcal{L}_{\text{int}}(x) \mathrm{d}^3 x \,. \tag{7.59}$$

To construct the interaction operator, one should pass from the Lagrangian to the Hamiltonian using formula (4.97). The Lagrangian (7.58) contains no derivatives of the $q^A_{,0}$ type nor does it change during transition to the Hamiltonian (4.97), barring the sign. Therefore, the classical interaction Hamiltonian has the form

$$\mathcal{H}_{\text{int}}(x) = -\mathcal{L}_{\text{int}}(x) = j^i(x) A_i(x) \,. \tag{7.60}$$

In the quantum theory of interacting fields, the quantities $j^i(x)$ and $A_i(x)$ should be replaced by the respective operators. To pass to the interaction operator $\widehat{V}$ analogous to the Hamiltonian operator (7.51) of noninteracting particles, (7.60) needs to be integrated over three-dimensional space:

$$\widehat{V}(t) = \int \widehat{j}^i(x) \widehat{A}_i(x) \mathrm{d}^3 x \,. \tag{7.61}$$

This operator depends on time and is written in the covariant form in the Heisenberg representation. It differs from the interaction operators (6.126) and (6.127) used earlier not only in the presence of bispinors in the current and in the presence of the four-component vector potential $\widehat{A}_i$, $i = 0, 1, 2, 3$. Formula (6.126) contains only transverse (with respect to photon momentum $\boldsymbol{k}$) components owing to the absence of longitudinal and scalar photons in nature, whereas the operator $\widehat{\boldsymbol{A}}$ is a three-dimensional vector. For the conservation of covariance in formula (7.61), the 4-potential $\widehat{A}_i$ containing the four-dimensional polarization vectors must be present.

The space-like polarization 4-vector e_i satisfying the normalization condition $e^*_i \cdot e^i = -1$ and the four-dimensional transversality condition $e_i k^i = 0$ should be introduced into formula (6.16) instead of the three-dimensional vector $\boldsymbol{e}_s$, where $s = (\boldsymbol{k}, \sigma)$, $\sigma = 1, 2$, are two independent polarizations of the photon. The aforementioned conditions define the polarization vector with an accuracy up to the gauge transformation $e'_i = e_i + \kappa k_i$, where k_i is the 4-momentum of the photon and κ is an arbitrary scalar. This transformation of the polarization vector follows from the gauge transformation of the 4-potential: $A'_i = A_i + \partial_i \chi$.

The calibration leading to the value of the temporal component $e'_0 = e_0 + \kappa\omega = 0$ is always possible. Thus, the polarization 4-vector takes the form

$$e_i = (0, -\boldsymbol{e}) \ , \quad \boldsymbol{e} \cdot \boldsymbol{k} = 0 \ . \tag{7.62}$$

Two polarization 4-vectors containing different three-dimensional vectors $\boldsymbol{e}^{(1)}$ and $\boldsymbol{e}^{(2)}$ remain physically different at a given value of k^i. As a result, the wave functions of photons in formula (7.61) instead of (6.16) take the form

$$A_l^s(\boldsymbol{r}) = e_l^s \sqrt{\frac{2\pi}{\mathcal{V}\omega}} e^{i\mathbf{k}\cdot\mathbf{r}} \ , \tag{7.63}$$

where index $s = (\boldsymbol{k}, \sigma)$ at a given $\boldsymbol{k}$ takes two values, indicating the type of polarization.

An essential fact is the photon wave functions (6.16) and (7.63) are written without time dependence. If these functions are substituted into (6.13), the $\widehat{A}$ operator does not depend on time either, which corresponds to the Schrödinger representation. To pass to the Heisenberg representation, either the time-dependent operators (6.23) should be used or their exponents $e^{\pm i\omega t}$ to refer to the wave function and to write them in the form

$$A_l^s(x) = e_l^s \sqrt{\frac{2\pi}{\mathcal{V}\omega}} e^{-ikx} \ , \quad kx = \omega t - \boldsymbol{k} \cdot \boldsymbol{r} \ . \tag{7.64}$$

The same is true of the fermion operators $\widehat{\psi}(\boldsymbol{r})$, $\widehat{\overline{\psi}}(\boldsymbol{r})$: they lose time dependence in the Schrödinger representation. Therefore, the interaction operator $\widehat{V}$ in the Schrodinger representation is time independent too:

$$\widehat{V} = \int \widehat{j}^{\,i}(\boldsymbol{r}) \widehat{A}_i(\boldsymbol{r}) \mathrm{d}^3 x \ . \tag{7.65}$$

If an external nonquantized field $A_i^{\text{ext}}(x)$ is present in the system, the interaction operator must contain the summated potential $A_i^{\text{ext}}(x) + \widehat{A}_i(x)$.

7.7
Schrödinger Equation for Interacting Fields and the Evolution Operator

Let us consider a physical system of interacting electron–positron and electromagnetic fields. We write down the Schrödinger equation for the state vector Φ_S of this complicated system depending on the occupation number of the quantum states of both fields:

$$i \frac{\partial \Phi_S(t)}{\partial t} = (\widehat{\mathcal{H}}_0 + \widehat{V}) \Phi_S(t) \ . \tag{7.66}$$

Here, the operator $\widehat{\mathcal{H}}_0$ refers to the free fields

$$\widehat{\mathcal{H}}_0 = \widehat{\mathcal{H}}_p + \widehat{\mathcal{H}}_{\text{em}} \ , \quad \widehat{\mathcal{H}} = \sum_s \mathcal{E}_s \left(\widehat{a}_s^\dagger \widehat{a}_s + \widehat{b}_s^\dagger \widehat{b}_s \right) \ , \quad \widehat{\mathcal{H}}_{\text{em}} = \sum_s \omega_s \widehat{c}_s^\dagger \widehat{c}_s \ . \tag{7.67}$$

Operator $\widehat{V}$ is given by expression (7.61); it describes the interaction between the fields and contains operators $\widehat{a}$, $\widehat{b}$, and $\widehat{c}$, related to both fields. The Schrödinger equation is written in the Schrödinger representation (marked by symbol S): the state vector is time dependent, unlike the operators that are independent of time.

Let us now pass to a different time representation called the *interaction representation*. We transform the state vector with the help of the free field Hamiltonian:

$$\Phi_I(t) = \exp\left\{i\widehat{\mathcal{H}}_0(t-t_0)\right\}\Phi_S(t)\,, \quad \Phi_S(t) = \exp\left\{-i\widehat{\mathcal{H}}_0(t-t_0)\right\}\Phi_I(t)\,. \tag{7.68}$$

At $t = t_0$ the two vectors coincide: $\Phi_I(t_0) = \Phi_S(t_0)$. The difference from the Heisenberg representation (C30) consists in only part of the Hamiltonian participating in the transformation, that is, the free field operator $\widehat{\mathcal{H}}_0$.

Let us find the Schrödinger equation in the interaction representation:

$$i\frac{\partial\Phi_S(t)}{\partial t} = \exp\left\{-i\widehat{\mathcal{H}}_0(t-t_0)\right\}\widehat{\mathcal{H}}_0\Phi_I + \exp\left\{-i\widehat{\mathcal{H}}_0(t-t_0)\right\}i\frac{\partial\Phi_I(t)}{\partial t}\,.$$

Substituting this derivative into (7.66) yields

$$i\frac{\partial\Phi_I(t)}{\partial t} = \widehat{V}_I(t)\Phi_I(t)\,, \quad \widehat{V}_I(t) = \exp\left\{i\widehat{\mathcal{H}}_0(t-t_0)\right\}\widehat{V}\exp\left\{-i\widehat{\mathcal{H}}_0(t-t_0)\right\}\,. \tag{7.69}$$

The interaction representation is possible for any operator:

$$\widehat{F}_I(t) = \exp\left\{i\widehat{\mathcal{H}}_0(t-t_0)\right\}\widehat{F}_S\exp\left\{-i\widehat{\mathcal{H}}_0(t-t_0)\right\}\,. \tag{7.70}$$

Finally, let us relate the state vector $\Phi_I(t)$ in the interaction representation to the state vector $\Phi_H(t_0)$ in the Heisenberg representation, which is time independent (in accordance with (C30)). We further introduce the unitary evolution operator $\widehat{S}(t, t_0)$ satisfying the initial condition $\widehat{S}(t_0, t_0) = 1$ and write the relationship between the state vectors in the form

$$\Phi_I(t) = \widehat{S}(t, t_0)\Phi_H(t_0)\,. \tag{7.71}$$

Substituting the last expression into (7.69) yields

$$i\frac{\partial\widehat{S}}{\partial t}\Phi_H = \widehat{V}_I(t)\widehat{S}(t, t_0)\Phi_H\,. \tag{7.72}$$

The state vector Φ_H being arbitrary, the following equation for the evolution operator ensues from (7.69):

$$i\frac{\partial\widehat{S}}{\partial t} = \widehat{V}_I(t)\widehat{S}(t, t_0)\,, \quad \widehat{S}(t, t_0)\Big|_{t=t_0} = 1\,. \tag{7.73}$$

It should be emphasized that operators $\widehat{V}_I$, $\widehat{A}_I$, $\widehat{\psi}$, and $\widehat{\overline{\psi}}$ for the interacting systems in the interaction representation coincide with the same operators in the Heisenberg representation for free (noninteracting) systems. Because of this, operators (7.61), (7.35), and (7.36) obtained earlier can be used in (7.73) written in the interaction representation. This representation proves most convenient for the approximate solution of electrodynamic problems. Therefore, it will be used below without special mention, omitting index I.

7.8 Scattering Matrix and Its Calculation

The evolution operator $\widehat{S}(t, t_0)$ permits us in principle to retrace the development of interacting fields in time. However, any exact measurement of characteristic parameters of individual particles during their interaction is impracticable owing to quantum factors, such as indeterminancy relations for pairs of mutually complementary observables and the influence of the measuring device on the system of interest. As a rule, it is possible to fix (i.e., to measure accurately enough) the momenta, energy, and polarizations of the particles either before their interaction while they are far apart from one another and can be regarded as being free or after the interaction when the particles in the finite state become free as well. Indeed, a real interaction process is rather short and confined in space.

For these reasons, the following statement of the problem is typical: the state of a system of noninteracting fields at the starting moment ($t_0 \to -\infty$) is specified as

$$\Phi(-\infty) \equiv \Phi_i = \Phi_H = \Phi_S = \Phi_I \tag{7.74}$$

(at $t = t_0$ the state vectors in all representations coincide and describe one of the states of the free fields). It is necessary to determine in which states the system may happen to be after the interaction process ($t \to \infty$) has terminated. According to definition (7.71) of the evolution operator, the finite state vector of the system is written in the form

$$\Phi(\infty) = \widehat{S}(\infty, -\infty)\Phi_i \equiv \widehat{S}\Phi_i \ . \tag{7.75}$$

The evolution operator in infinite limits is denoted by $\widehat{S}$ and is referred to as the scattering operator (matrix). Here, the term "scattering" is understood in the generalized sense; it implies the possibility of creation and absorption of bosons and fermions.

Let us expand $\Phi(\infty)$ in states of noninteracting fields, that is, in eigenfunctions Φ_f of the $\mathcal{H}_0$ operator. Then, the expansion coefficients (7.76)

$$\Phi(\infty) = \sum_f C_f \Phi_f \ , \quad C_f = \left(\Phi_f, \widehat{S}\Phi_i\right) \equiv \langle f|\widehat{S}|i\rangle \tag{7.76}$$

are the amplitudes of transition between states $i \to f$ as a result of interaction and the probabilities are proportional to the squares of the modulus of the amplitude:

$$\mathrm{d}W_{i\to f} = |\langle f|\widehat{S}|i\rangle|^2 \mathrm{d}\nu_f \,. \tag{7.77}$$

Free particles are in continuous spectrum states; therefore, by $\mathrm{d}\nu_f$ is meant a certain range of observables determined by the characteristics of the instruments used to register particles.

The scattering operator can be calculated using (7.73). We write it in the form of an integral equation including the initial condition:

$$\widehat{S}(t, t_0) = 1 - i \int\limits_{t_0}^{t} \widehat{V}(t_1)\widehat{S}(t_1, t_0)\mathrm{d}t_1 \,. \tag{7.78}$$

In the absence of interaction, at $\widehat{V} = 0$, matrix S becomes the unit operator $\widehat{S}^{(0)} = 1$. The integral term in expression (7.78) contains a small factor $e \approx 1/\sqrt{137}$ – the field coupling constant which makes possible the expansion in this small constant. In fact, because the probability is proportional to the square of the transition amplitude, the expansion is performed in the smaller quantity $e^2 \approx 1/137$.

Equation (7.78) is solved by the iteration method:

$$\widehat{S}^{(1)} = -i \int\limits_{t_0}^{t} \widehat{V}(t_1)\mathrm{d}t_1 \,, \quad \widehat{S}^{(2)} = (-i)^2 \int\limits_{t_0}^{t} \mathrm{d}t_1 \int\limits_{t_0}^{t_1} \mathrm{d}t_2 \widehat{V}(t_1)\widehat{V}(t_2) \,,$$

$$\widehat{S}^{(n)} = (-i)^n \int\limits_{t_0}^{t} \mathrm{d}t_1 \int\limits_{t_0}^{t_1} \mathrm{d}t_2 \ldots \int\limits_{t_0}^{t_{n-1}} \mathrm{d}t_n \widehat{V}(t_1)\widehat{V}(t_2)\ldots\widehat{V}(t_n) \,. \tag{7.79}$$

All terms of this expansion have a structure of the same type. It can be simplified by means of a certain transformation of the integrand expressions. Let us start from the second approximation, $\widehat{S}^{(2)}$. The region of integration over $\mathrm{d}t_1\mathrm{d}t_2$ at the plane (t_1, t_2) is the triangle at the bottom of Figure 7.1.

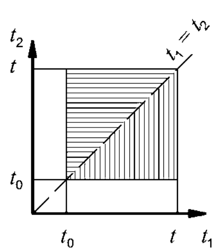

Figure 7.1 Regions of integration over time.

In the case of integration of a symmetric algebraic function $V(t_1)V(t_2) = V(t_2)V(t_1)$, the integral can be written in the form

$$\frac{(-i)^2}{2!} \int\limits_{t_0}^{t} \mathrm{d}t_1 \int\limits_{t_0}^{t} \mathrm{d}t_2 \, V(t_1)V(t_2) \,.$$

The integration region becomes simplified and becomes a square. But the operators under the sign of the integral in (7.79) do not commute with each other at different moments of time. For this reason, the substitution $t_1 \rightleftharpoons t_2$ and the transition to integration over the upper triangle results in a change of the order of the operators under the integral sign:

$$\widehat{S}^{(2)} = (-i)^2 \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \widehat{V}(t_1)\widehat{V}(t_2) = (-i)^2 \int_{t_0}^{t} dt_2 \int_{t_0}^{t_2} dt_1 \widehat{V}(t_2)\widehat{V}(t_1) ,$$

with

$$\widehat{V}(t_1)\widehat{V}(t_2) \neq \widehat{V}(t_2)\widehat{V}(t_1) .$$

Therefore, the *chronological operator* $\widehat{T}$ is introduced and by definition rearranges operators in the correct order:

$$\widehat{T}(\widehat{V}(t_1)\widehat{V}(t_2)) = \begin{bmatrix} \widehat{V}(t_1)\widehat{V}(t_2), & t_2 < t_1 \\ \widehat{V}(t_2)\widehat{V}(t_1), & t_1 < t_2 \end{bmatrix}$$
$$= \Theta(t_1 - t_2)\widehat{V}(t_1)\widehat{V}(t_2) + \Theta(t_2 - t_1)\widehat{V}(t_2)\widehat{V}(t_1) . \tag{7.80}$$

The chronological operator acts in a similar manner on the product of a large number of operators: it arranges them in such an order that the operator with the largest time is on the left side and is followed by operators with decreasing times (chronological product of the operators).

The use of the chronological operator makes it possible to write

$$\widehat{S}^{(2)} = \frac{(-i)^2}{2!} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \widehat{T}(\widehat{V}(t_1)\widehat{V}(t_2)) . \tag{7.81}$$

Now, it is pertinent to recall that a scattering operator, that is, the evolution operator in infinite limits, $t_0 \to -\infty$, $t \to \infty$, is needed to calculate probabilities. Each $\widehat{V}(t)$ operator is expressed in the form of the integral over the entire three-dimensional space (see (7.61)). On the basis of these results, the quantity being sought $\widehat{V}(t)$ is written in the form of the double integral over the entire 4-space:

$$\widehat{S}^{(2)}(\infty, -\infty) \equiv \widehat{S}^{(2)} = \frac{(-i)^2}{2!} \int \widehat{T}(\widehat{V}(x_1)\widehat{V}(x_2)) d^4x_1 d^4x_2 , \tag{7.82}$$

where $\widehat{V}(x) = \widehat{j^k(x)}\widehat{A}_k(x)$. An analogous expression can be written for the common expansion term:

$$\widehat{S}^{(n)} = \frac{(-i)^n}{n!} \int \widehat{T}(\widehat{V}(x_1)\widehat{V}(x_2) \dots \widehat{V}(x_n)) d^4x_1 d^4x_2 \dots d^4x_n . \tag{7.83}$$

The coefficients $(-i)^n/n!$ in front of the integral coincide with the exponent expansion coefficients, which makes it possible to write the entire series for the scattering operators in a symbolic form through the exponent:

$$\widehat{S} = \widehat{T} \exp\left\{-i \int \widehat{j^k}(x)\widehat{A}_k(x)\mathrm{d}^4x\right\} . \tag{7.84}$$

This gives the unitary operator $\widehat{S}\widehat{S}^\dagger = 1$. This property is due to hermiticity of the interaction operator: $\widehat{V} = \widehat{V}^\dagger$. An important property of the $\widehat{S}$ operator is its relativistic invariance due to the extension of the integration region over the entire four-dimensional space. Representation (7.84) is only a short form in which the terms in (7.83) are written because the closed expression for the exponent (7.84) is unknown and separate terms of the expansion serve as the working tool allowing calculations to be performed in different orders of perturbation theory.

7.9
Calculations of Probabilities and Effective Differential Cross-Sections

To calculate the probabilities of quantum transitions, it is convenient to distinguish in matrix elements $\langle f|\widehat{S}|i\rangle$ some factors characterizing the common properties of the system of interacting fields being considered. To begin with, if $f \neq i$, then the zero-order item is absent in the matrix element that contains only items with the interaction operator. Moreover, any interaction process in the absence of external fields (i.e., in a free space) proceeds with the conservation of the entire four-dimensional momentum P of the physical system (i.e., with the conservation of the energy E and the three-dimensional momentum $\boldsymbol{P}$). Because of this, the matrix element is proportional to the four-dimensional delta function:

$$\delta^{(4)}(P^f - P^i) \equiv \delta^{(3)}(\boldsymbol{P}^f - \boldsymbol{P}^i)\delta(E_f - E_i) , \tag{7.85}$$

which ensures the fulfillment of the conservation law,

$$\langle f|\widehat{S}|i\rangle = i(2\pi)^4 M_{fi}\delta^{(4)}(P^f - P^i) \tag{7.86}$$

Here, the factor $i(4\pi)^4$ is written traditionally and taking into consideration further transformations. The matrix element M_{fi} does not contain singularities; actually, it is a nonsingular part of the transition amplitude. In what follows, particular examples will be presented where the $\delta^{(4)}$ function is obtained by integration of matrix elements over the 4-space.

Squaring of the matrix element according to (7.77) yields the square of the $\delta^{(4)}$ function, which can be transformed in the following way:

$$[\delta^{(4)}(P^f - P^i)]^2 = \delta^{(4)}(P^f - P^i)\delta^{(4)}(0) .$$

Thereafter, we consider the integral through which it is possible to express the $\delta^{(4)}$ function and perform integration over a large but finite three-dimensional volume $\mathcal{V}$ and a large time interval Δt:

$$(2\pi)^4\delta^{(4)}(0) = \int \exp\left(iP_k x^k\right)|_{P\to 0}\mathrm{d}^4x = \mathcal{V}\Delta t . \tag{7.87}$$

Here, the $\mathcal{V}$ volume should be identified as the volume of the field periodicity region that enters the normalization of photons (7.64) and Dirac particles (see Problem 7.5). The probability $\mathrm{d}W_{i\to f}$ should be divided by the time interval Δt in order to obtain the probability of transition per unit time:

$$\mathrm{d}w_{i\to f} = (2\pi)^4\mathcal{V}|M_{fi}|^2\delta^{(4)}(P^f - P^i)\mathrm{d}\nu_f\ . \tag{7.88}$$

The formula for the probability of transition changes if the system experiences the action of an external static field. In this case, part of the system's momentum can be transferred to the sources of the external field and the law of conservation of the three-dimensional momentum fails to be fulfilled even though the total energy of the particles is preserved. For this reason only the energy-dependent delta function remains in formula (7.88) and the constant factors change:

$$\mathrm{d}w_{i\to f} = 2\pi|M_{fi}|^2\delta(E_f - E_i)\mathrm{d}\nu_f\ . \tag{7.89}$$

The expression for the number of quantum states $\mathrm{d}\nu_f$ after the transition depends on the number of particles in the finite state. In Chapter 2 (see formula (2.164)), we obtained $\mathrm{d}\nu_f = \mathcal{V}\mathrm{d}^3k/(2\pi)^3$ for photons with given polarization. The number of quantum states of Dirac particles is expressed in exactly the same way through the momentum. Therefore, the total number of states will be expressed in the form of the product of the factors related to each individual particle in the finite state:

$$\mathrm{d}\nu_f = \prod_a \frac{\mathcal{V}\mathrm{d}^3p_a}{(2\pi)^3}\ . \tag{7.90}$$

The process of collision of two particles is usually characterized by the differential cross-section (see Problem 3.67). If the collision gives rise to other particles, such a process is called a reaction. The effective differential cross-section $\mathrm{d}\sigma_{i\to f}$ of scattering (or reaction) is the ratio of the number of transitions $\mathrm{d}w_{i\to f}$ per unit time to the flux density j of the colliding particles:

$$\mathrm{d}\sigma_{i\to f} = \frac{\mathrm{d}w_{i\to f}}{j} = \frac{(2\pi)^4\mathcal{V}}{j}|M_{fi}|^2\delta^{(4)}(P^f - P^i)\mathrm{d}\nu_f\ . \tag{7.91}$$

The scattering cross-section has the dimension of squared length and is a characteristic of the process of particle–particle interaction in the given initial and finite states. Unlike the probability of transitions, it cannot depend on the normalization condition of the wave function, for example, on the normalization volume.

7.10
Scattering of a Relativistic Particle with a Spin in the Coulomb Field

Let us apply the scheme for the solution of electrodynamic problems developed in this chapter to the process of elastic scattering of a spin-1/2 fermion (e.g., an electron) in the Coulomb field of the nucleus. We shall solve the problem in the first approximation of perturbation theory on the assumption that the nucleus is a motionless point-like object with charge Ze creating the external field. Photons do not participate in this process, which allows us to take into consideration only the vector potential of the nonquantized external electric field:

$$A_k = (A_0, \boldsymbol{A} = 0)\,, \quad A_0 = \frac{Ze}{r}\,. \tag{7.92}$$

The quantum transition occurs between two electron states, the initial one i with momentum p and polarization μ, and the final state f with momentum p' and polarization μ'.

The two states have positive energy. Such a process in the first vanishing approximation of perturbation theory is described by the first-order scattering matrix $\widehat{S}^{(1)}$:

$$\widehat{S}^{(1)} = -i\int \widehat{j^k}(x)A_k(x)\mathrm{d}^4x = ie\int \widehat{\overline{\psi}}(x)\widehat{\gamma^0}\widehat{\psi}(x)A_0(r)\mathrm{d}^3r\mathrm{d}t\,. \tag{7.93}$$

Let us write down vectors Φ_i and Φ_f of the initial and final states of the system through the vacuum state vector $|0\rangle$ with the help of electron creation operators $|0\rangle$:

$$\Phi_i = \widehat{a}^\dagger_{p\mu}|0\rangle\,, \quad \Phi_f = \widehat{a}^\dagger_{p'\mu'}|0\rangle\,, \quad \Phi_f^\dagger = \langle 0|\widehat{a}_{p'\mu'}\,. \tag{7.94}$$

The matrix element of the transition has the form

$$\langle f|\widehat{S}^{(1)}|i\rangle = ie\int \mathrm{d}^3r\mathrm{d}tA_0(r)\langle 0|\widehat{a}_{p'\mu'}\widehat{\overline{\psi}}(x)\widehat{\gamma^0}\widehat{\psi}(x)\widehat{a}^\dagger_{p\mu}|0\rangle\,, \tag{7.95}$$

where $\widehat{\psi}$ and $\widehat{\overline{\psi}}$ are operators (7.37) and (7.38) containing infinite series. Averaging over the vacuum yields a nonzero result in the case of the sole combination

$$\langle 0|\widehat{a}_{p'\mu'}\widehat{a}^\dagger_{s'}\widehat{a}_s\widehat{a}^\dagger_{p\mu}|0\rangle = 1$$

if $s = p, \mu; s' = p', \mu'$. All combinations with an odd number of operators $\widehat{a}$ or $\widehat{b}$ necessarily lead to the zero result. The same is true of the quantity $\langle 0|\widehat{a}_{p'\,\mu'}\widehat{b}_{s'}\widehat{b}^\dagger_s\widehat{a}^\dagger_p, \mu|0\rangle$ at $p', \mu' \neq p, \mu$. This means that a single term should be chosen from each $\widehat{\psi}, \widehat{\overline{\psi}}$ series.

Now, let us define concretely the wave functions of the initial and final states of an electron in the frame of reference where the nucleus is motionless:

$$\psi_{p\mu} = \frac{1}{\sqrt{2\mathcal{E}\mathcal{V}}}u_\mu(p)e^{-ipx}\,, \quad \overline{\psi}_{p'\mu'} = \frac{1}{\sqrt{2\mathcal{E}\mathcal{V}}}\overline{u}_{\mu'}(p')e^{ip'x}\,,$$
$$\overline{u}_\mu(p)u_\mu(p) = 2m\,. \tag{7.96}$$

Such normalization corresponds to the probability density $\rho = 1/\mathcal{V}$ (one particle in volume $\mathcal{V}$) and the probability current density in the initial state $\boldsymbol{j} = \overline{\psi}_{p,\mu}\widehat{\boldsymbol{\gamma}}\psi_{p\mu} = \boldsymbol{v}/\mathcal{V}$, where $\boldsymbol{v} = \boldsymbol{p}/\mathcal{E}$ is the particle's velocity.

Let us calculate the integral over time in equality (7.93) and obtain $\int_{-\infty}^{\infty}\exp[i(\mathcal{E}'-\mathcal{E})t]\mathrm{d}t = 2\pi\delta(\mathcal{E}'-\mathcal{E})$. We further calculate the integral over three-dimensional space: $\int A_0(r)\exp(-i\boldsymbol{q}\cdot\boldsymbol{r}) = 4\pi Ze/q^2$, where $\boldsymbol{q} = \boldsymbol{p} - \boldsymbol{p}'$ is the momentum transferred onto the nucleus. As a result,

$$\langle f|\widehat{S}^{(1)}|i\rangle = i\frac{4\pi Ze/q^2}{q^2}\frac{1}{2\mathcal{E}\mathcal{V}}\overline{u}_{\mu'}(p')\widehat{\gamma^0}u_\mu(p)2\pi\delta(\mathcal{E}'-\mathcal{E})\,. \tag{7.97}$$

The structure of the matrix element thus obtained and written in the momentum representation can be depicted graphically as a Feynman diagram (Figure 7.2); a similar picture was used in Example 6.19.

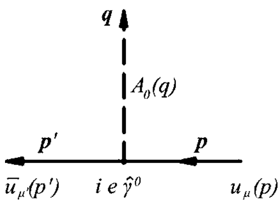

Figure 7.2 Feynman diagram for electron scattering by an external field.

Straight lines with arrows show the bispinor of the initial and final states of the electron. The vertex corresponds to the product of the e coupling constant, The vertex corresponds to the product of the e coupling constant, $\widehat{\gamma^0}$ matrix, and the Fourier transform $A_0(q)$ of the external field potential. The initial and final states are related by the laws of conservation of energy $\mathcal{E}' = \mathcal{E}$ and the three-dimensional momentum $\boldsymbol{p}' + \nu q = \nu p$. These factors are supplemented by some others depending on the concrete normalization of the particles participating in the process.

The squaring of this matrix element is associated with the substitution as in the preceding section: $[2\pi\delta(\mathcal{E}'-\mathcal{E})]^2 \to 2\pi\delta(\mathcal{E}'-\mathcal{E})\Delta t$, where Δt is a large time interval; the probability of transition is divided by this interval. This yields the formula for the probability per unit time and cross-section,

$$\mathrm{d}w_{i\to f} = \frac{|\langle f|\widehat{S}^{(1)}|i\rangle|^2}{\Delta t}\mathrm{d}\nu\,, \quad \mathrm{d}\sigma = \frac{\mathrm{d}w_{i\to f}}{j}\,, \tag{7.98}$$

which were written in the preceding section from general considerations. Taking together all the factors including

$$\mathrm{d}\nu = \frac{\mathcal{V}\mathrm{d}^3 p}{(2\pi)^3} = \frac{\mathcal{V}p\mathcal{E}}{(2\pi)^3}\mathrm{d}\Omega\,\mathrm{d}\mathcal{E}'\,,$$

where $\mathrm{d}\Omega$ is the solid angle into which scattering occurs, and integrating over $\mathrm{d}\mathcal{E}'$ with the help of the delta function, yields the cross-section of scattering into a given solid angle accompanied in the general case by a change of the polarization state of the electron:

$$\mathrm{d}\sigma_{\mu\to\mu'} = \left(\frac{Ze^2}{q^2}\right)^2\left|\overline{u}_{\mu'}(p')\widehat{\gamma^0}u_\mu(p)\right|^2\mathrm{d}\Omega\,. \tag{7.99}$$

Let us first consider the nonrelativistic limit: the two lower components of the bispinor being vanishingly small, $\overline{u}_{\mu'}\widehat{\gamma}^0 u_\mu \approx 2mw_{\mu'}^\dagger w_\mu = 2m\delta_{\mu'\mu}$. The polarization state of the electron does not change. The differential cross-section (7.99) takes the form of the Rutherford formula:

$$\frac{d\sigma}{d\Omega} = \left(\frac{Ze^2}{2mv^2}\right)^2 \frac{1}{\sin^4 \theta/2} \tag{7.100}$$

(see Problem 4.67*, where this formula was derived from the equations of classical mechanics, that is, in the absence of the notion of spin).

In the relativistic case, the particle's spin plays an important role. If the particle incident on the scatterer has random spin directions and the scattering spin is not measured, then the cross-section $d\sigma_{\mu\to\mu'}$ should be averaged over the initial states μ and summed over the final states μ':

$$d\overline{\sigma} = \frac{1}{2}\sum_{\mu'\mu} d\sigma_{\mu\to\mu'} \ . \tag{7.101}$$

These operations can be performed by making use of the projection operator onto the states with positive energy $\widehat{\Pi}_+(p)$, as considered in Examples 7.2 and 7.3:

$$\begin{aligned}
&\sum_{\mu'\mu}\left(\overline{u}_{\mu'}(p')\widehat{\gamma^0}u_\mu(p)\right)\left(\overline{u}_{\mu'}(p')\widehat{\gamma^0}u_\mu(p)\right)^* \\
&= \sum_{\mu'\mu}\left(\overline{u}_{\mu'}(p')\widehat{\gamma^0}u_\mu(p)\right)\left(u^\dagger_\mu(p)\widehat{\gamma^0}\widehat{\gamma^0}u_{\mu'}(p)\right) \\
&= \sum_{\mu'\mu}\left(\overline{u}_{\mu'}(p')\widehat{\gamma^0}u_\mu(p)\right)\left(\overline{u}_\mu(p)\widehat{\gamma^0}u_{\mu'}(p')\right) \\
&= \mathrm{Tr}\left[\widehat{\gamma^0}\widehat{\Pi}_+(p)\widehat{\gamma^0}\widehat{\Pi}_+(p')\right] .
\end{aligned}$$

Here, the properties of the $\widehat{\gamma^0}$, $\widehat{\gamma^0} = \widehat{\tilde{\gamma}}^0 = (\widehat{\gamma^0})^\dagger$, and the definition (7.16) of the $\widehat{\Pi}_+(p)$ operator are used. The trace is calculated with the help of the formulas derived in Problems (7.2) and (7.3). We first transpose the $\widehat{\gamma^0}$ matrix through the $\widehat{\Pi}_+(p)$ operator, which leads to the relation

$$\mathrm{Tr}\left[\widehat{\gamma^0}\widehat{\Pi}_+(p)\widehat{\gamma^0}\widehat{\Pi}_+(p')\right] = \mathrm{Tr}\left[\widehat{\Pi}_+(\overline{p})\widehat{\Pi}_+(p')\right] ,$$

where $\overline{p}_k = (p_0, \boldsymbol{p})$ and $\widehat{\Pi}_+(p) = (m + p_l\widehat{\gamma^l})$. Then, we find

$$\mathrm{Tr}\left[\widehat{\Pi}_+(\overline{p})\widehat{\Pi}_+(p')\right] = 8\mathcal{E}^2\left(1 - \frac{q^2}{4\mathcal{E}^2}\right) ,$$

where the relation $\boldsymbol{p}\cdot\boldsymbol{p}' = p^2 - q^2/2$ is used. The substitution of these results into (7.99) and (7.101) yields the Mott[1] formula:

$$\frac{d\overline{\sigma}}{d\Omega} = \left(\frac{2Ze^2\mathcal{E}}{q^2}\right)^2\left(1 - \frac{q^2}{4\mathcal{E}^2}\right) . \tag{7.102}$$

1) Nevill Francis Mott (1905–1996) was an English theoretical physicist known for his extensive research on quantum mechanics, the theory of atomic collisions, nuclear physics, and the physics of metals and semiconductors. He was a Nobel Prize recipient.

In usual units,

$$\frac{d\overline{\sigma}}{d\Omega} = \left(\frac{Ze^2c^2}{2\mathcal{E}v^2}\right)^2 \frac{1}{\sin^4\theta/2}\left(1 - \frac{v^2}{c^2}\sin^2\theta/2\right) . \tag{7.103}$$

The last factor on the right-hand side is due to the spin. It can appreciably reduce the cross-section at large scattering angles of an ultrarelativistic particle with $v \approx c$.

Problems

7.11. Calculate in the first order of perturbation theory the cross-section of scattering of a relativistic positron by a point-like Coulomb center. The positron beam thus scattered is not polarized and the spin states of the scattered particles are not analyzed.

7.12. Calculate the cross-section of elastic scattering of a relativistic electron by a neutral atom. The electron shell should be taken into consideration in the exponential screening approximation with effective radius $a = a_B Z^{-1/3}$, Z being the charge number. Moreover, account must be taken of the finite size of the nucleus as a spherically symmetric object with radius r_0 ($r_0 \approx 10^{-12}$cm for heavy nuclei), in which charged particles (protons) are distributed with a particle number density of $n(r)$, $\int n(r)d^3r = Z$. Express the scattering cross-section of an electron as the nuclear form factor $F(\boldsymbol{q}) = \int n(r)e^{i\mathbf{q}\cdot\mathbf{r}}d^3r$.

7.11
Green's Functions of Electron–Positron and Electromagnetic Fields

Example 7.5

Determine the four-row matrix $\widehat{G}^{(F)}(x - x')$ as the average over the vacuum from the chronological product of the field operators,

$$\widehat{G}^{(F)}(x - x') = -i\langle 0|\widehat{T}\widehat{\psi}(x)\widehat{\overline{\psi}}(x')|0\rangle \tag{7.104}$$

or, in a more explicit form, with the indication of bispinor indices (Greek ones assuming four values each):

$$\widehat{G}^{(F)}_{\alpha\beta}(x - x') = -i\langle 0|\widehat{T}\widehat{\psi}_\alpha(x)\widehat{\overline{\psi}}_\beta(x')|0\rangle . \tag{1}$$

Show that $\widehat{G}^{(F)}(x - x')$ is the Green's function of the Dirac equation (7.4) for a free particle ("electron propagator"), that is, it satisfies the equation

$$\left(\widehat{\gamma^k \widehat{p}_k} - m\right)\widehat{G}^{(F)}(x - x') = \delta^{(4)}(x - x') . \tag{7.105}$$

Analyze the physical sense of the Green's function for times $t > t'$ and $t < t'$.

Solution. At $t \neq t'$, the Green's function satisfies (7.105) as follows from the fact that this equation is also satisfied by the $\widehat{\psi}(x)$ operator (see Section 7.4). However, at $t = t'$, the Green's function undergoes a jump giving rise to the delta-like term when it is differentiated with respect to time. Let us find this term.

For this purpose we use the definition (7.80) of the chronological operator:

$$\widehat{T}\left(\widehat{\psi}_\alpha(x)\widehat{\overline{\psi}}_\beta(x')\right) = \begin{bmatrix} \widehat{\psi}(x)\widehat{\overline{\psi}}_\beta(x'), & t' < t \\ -\widehat{\overline{\psi}}_\beta(x')\widehat{\psi}(x), & t < t' \end{bmatrix}$$
$$= \Theta(t-t')\widehat{\psi}(x)\widehat{\overline{\psi}}_\beta(x') - \Theta(t'-t)\widehat{\overline{\psi}}_\beta(x')\widehat{\psi}(x) \ . \quad (1)$$

The minus sign in the second line arises as a result of anticommutation of the Dirac operators. In (7.105), the $i(\widehat{\gamma^0})_{\lambda\alpha}\partial/\partial t$ term contains a time derivative. We perform differentiation and use the expansions of operators (7.37) and (7.38) to find in the small vicinity of point $t = t'$

$$i\left(\widehat{\gamma^0}\right)_{\lambda\alpha}\frac{\partial \widehat{G}^{(F)}_{\alpha\beta}}{\partial t} = \left(\widehat{\gamma^0}\right)_{\lambda\alpha}\sum_{s,s'}\Big\{\langle 0|\widehat{a}_s\widehat{a}^\dagger_{s'}|0\rangle\psi_{s\alpha}(\boldsymbol{r},t)\overline{\psi}_{s'\beta}(\boldsymbol{r}',t)$$
$$+\langle 0|\widehat{b}_s\widehat{b}^\dagger_{s'}|0\rangle\overline{\psi}_{-s'\beta}(\boldsymbol{r}',t)\psi_{-s\alpha}(\boldsymbol{r},t)\Big\}\,\delta(t-t')$$
$$= (\widehat{\gamma^0})_{\lambda\alpha}\sum_s\Big\{\overline{\psi}_{s\beta}(\boldsymbol{r}',t)\psi_{s\alpha}(\boldsymbol{r},t)$$
$$+\,\overline{\psi}_{-s\beta}(\boldsymbol{r}',t)\psi_{-s\alpha}(\boldsymbol{r},t)\Big\}\,\delta(t-t') \ . \quad (2)$$

Then, we write $\overline{\psi}_{s\beta} = (\widehat{\gamma^0})_{\beta\sigma}\psi^*_{s\sigma}$, and take advantage of the completeness of the system of functions of the particle's free movements:

$$\sum_s\left\{\psi^*_{s\sigma}(\boldsymbol{r}',t)\psi_{s\alpha}(\boldsymbol{r},t) + \psi^*_{-s\sigma}(\boldsymbol{r}',t)\psi_{-s\alpha}(\boldsymbol{r},t)\right\} = \delta_{\sigma\alpha}\delta(\boldsymbol{r}-\boldsymbol{r}') \ . \quad (3)$$

We finally obtain

$$i(\widehat{\gamma^0})_{\lambda\alpha}\frac{\partial \widehat{G}^{(F)}_{\alpha\beta}}{\partial t} = \delta_{\lambda\beta}\delta^{(4)}(x-x') \ , \quad (4)$$

with the help of the relation $(\widehat{\gamma^0})^2 = 1$; this confirms the validity of (7.105).

Referring to the explicit form,

$$\widehat{G}^{(F)}(x-x') = \begin{cases} \sum_s \psi_s(x)\overline{\psi}_s(x') \ , & t > t' \ , \\ -\sum_s \psi_{-s}(x)\overline{\psi}_{-s}(x') \ , & t < t' \ , \end{cases} \quad (7.106)$$

of the Green's function, we arrive at the conclusion that its expansion at $t > t'$ contains only positive-energy states, that is, all possible states of the particles propagating forward in time. At times $t < t'$, the expansion contains only the wave function of the states with negative energy. These states describe antiparticles propagating backward in time ("retrogressively") since they encompass the $-\infty < t < t'$ time interval. The Green's function with such properties is called the *causal* or *Feynman* function. It should be noted that this Green's function is not the sole one possible. Both advanced and retarded Green's functions considered in Chapter 5 are solutions of (7.105). But there they described the propagation of electromagnetic waves. The Feynman Green's function of the electron–positron field is needed to calculate elements of the scattering matrix. □

Example 7.6

Find the momentum representation for the Green's function and elucidate the rule for a detour around the poles in the complex plane of momentum variable p_0.

Solution. The momentum representation for the Green's function can be obtained from (7.105) by means of the four-dimensional Fourier transformation:

$$\widehat{G}^{(F)}(p) = \int G^{(F)}(x) e^{ipx} \mathrm{d}^4 x \,,$$
$$G^{(F)}(x - x') = \int G^{(F)}(p) e^{-ip(x-x')} \frac{\mathrm{d}^4 p}{(2\pi)^4} \,. \tag{1}$$

Here, p should be regarded as the totality $(p_0, -\boldsymbol{p})$ of four Fourier variables not related by the dispersion condition $p_0^2 - \boldsymbol{p}^2 = m^2$. In this case, (7.105) becomes an algebraic equation,

$$\left(\widehat{\gamma^k p_k} - m\right) \widehat{G}^{(F)}(p) = 1 \,, \tag{2}$$

and has the solution

$$\widehat{G}^{(F)}(p) = \frac{1}{\widehat{\gamma^k p_k} - m} = -\frac{m + \widehat{\gamma^k p_k}}{m^2 - p^2} \,. \tag{3}$$

But this solution is ambiguous and allows us to obtain a large family of Green's functions with different properties because the denominator in (3) has peculiarities ; the rules for a detour around singular points need to be specified (see the analogous discussion in Section 5.1, Example 5.2). Transition to the coordinate representation according to (1) yields

$$\widehat{G}^{(F)}(x - x') = \int \frac{m + \widehat{\gamma^k p_k}}{p^2 - m^2} e^{-ip(x-x')} \frac{\mathrm{d}^4 p}{(2\pi)^4}$$
$$= \left(\widehat{\gamma^k \widehat{p}_k} + m\right) G^{(F)}(x - x') \,, \tag{4}$$

where

$$G^{(F)}(x - x') = \int \frac{e^{-ip(x-x')}}{p^2 - m^2} \frac{\mathrm{d}^4 p}{(2\pi)^4} \tag{5}$$

is the scalar Green's function of the relativistic Klein–Gordon–Fock equation (see Problem 4.125* and the beginning of Section 6.3).

We select integration over $\mathrm{d}p_0$

$$G^{(F)}(x-x') = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \exp\{i\boldsymbol{p}\cdot(\boldsymbol{r}-\boldsymbol{r}')\} \int \frac{\mathrm{d}p_0}{2\pi} \frac{e^{-ip_0(t-t')}}{p_0^2 - m^2 - \mathbf{p}^2} \tag{6}$$

and determine the positions of the poles $p_0 = \pm\mathcal{E}(p) = \pm\sqrt{\boldsymbol{p}^2 + m^2}$ to chose the detours around them such that the time factor at $t > t'$ corresponds to the positive energies in accordance with (7.106), that is, it has the form $e^{-i\mathcal{E}(t-t')}$. At $t < t'$, only antiparticle states remain in (7.106); Therefore, the time factor must have the opposite sign in the exponent $e^{i\mathcal{E}(t-t')}$. This rule (*Feynman rule*) is realized by passing along the contour drawn in Figure 7.3a. At $t > t'$, we close the contour with an arc of large radius in the lower semiplane to obtain the contribution from the pole $p_0 = \mathcal{E}$ with the desired sign in the exponent index. At $t < t'$, the contour should be closed in the upper semiplane.

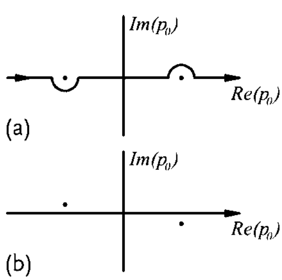

Figure 7.3 The Feynman rule for bypassing the poles in a complex plane p_0 for the calculation of the causative Green's function. (a) the detour of poles on small semicircles. (b) the shift of poles from an integration contour.

Instead of bypassing the poles along small-radius semicircles, we can displace the poles themselves over a small distance onto the upper and lower semiplanes, respectively (Figure 7.3b). This approach was used in Chapter 1 when discussing the properties of the delta function and allows the momentum representation of the Green's function to be written in a compact form:

$$\widehat{G}^{(F)}(p) = \left.\frac{m + \widehat{\gamma^k p_k}}{p^2 - m^2 + i\epsilon}\right|_{\epsilon\to+0} \tag{7.107}$$

The limiting transition $\epsilon \to 0$ is performed at the very last stage of the calculations.

To be certain that that expression (7.106) and the result ensuing from (7.107) after the transition to the coordinate representation are identical, one should make use of the relations found in Example 7.2 and the transition from summating over the quasi-discrete value of p to integration. □

The *photon Green's function* analogous to the Feynman propagator (7.104) is defined as the average over the vacuum;

$$D_{ik}^{(F)}(x-x') = i\langle 0|\widehat{T}\widehat{A}_i(x)\widehat{A}_k(x')|0\rangle \; . \tag{7.108}$$

It is a second rank 4-tensor. In the general case, the calculation of (7.108) cannot be confined to the use of operator (6.13) with the corrected (7.63) functions because it takes account only of the transverse electromagnetic field, that is, the real photons. However, the interaction between charged particles does not reduce to the exchange of photons alone. It also involves the nontransverse Coulomb field with $A_3 \neq 0$, $A_0 \neq 0$.

We should make sure that this quantity is the Green's function of the d'Alembert equation for the vector potential (i.e., the photon propagator even if this term is inexact as encompassing the longitudinal and scalar fields too):

$$\Box D_{il}^{(F)}(x-x') = -4\pi g_{il}\delta^{(4)}(x-x') \tag{7.109}$$

Similarly to Example 7.5, the photon propagator at $x \neq x'$ satisfies the homogeneous d'Alembert equation because $\Box A_i = 0$. It is necessary to calculate the second time derivative near the point $t = t'$, at which the propagator (7.108) loses continuity.

Let us distinguish the point of discontinuity

$$\widehat{T}\widehat{A}_i(x)\widehat{A}_l(x') = \Theta(t-t')\widehat{A}_i(x)\widehat{A}_l(x') + \Theta(t'-t)\widehat{A}_l(x')\widehat{A}_i(x)$$

and calculate the first time derivative:

$$\begin{aligned}\frac{\partial}{\partial t}D_{il}^{(F)}(x-x') &= \delta(t-t')\langle 0|\left(\widehat{A}_i(x)\widehat{A}_l(x') - \widehat{A}_l(x')\widehat{A}_i(x)\right)|0\rangle \\ &\quad + \Theta(t-t')\langle 0|\frac{\partial}{\partial t}\widehat{A}_i(x)\widehat{A}_l(x')|0\rangle \\ &\quad + \Theta(t'-t)\langle 0|\widehat{A}_l(x')\frac{\partial}{\partial t}\widehat{A}_i(x)|0\rangle \; .\end{aligned}$$

On the basis of the expansion of potentials in flat waves and the rules of commutation for photon operators $\widehat{c}_s$, we make sure that at $t = t'$ the term proportional to $\delta(t-t')$ becomes zero on averaging over the vacuum. To calculate the second derivative, only singular terms are left,

$$-\frac{\partial^2}{\partial t^2}D_{il}^{(F)}(x-x') = \langle 0|\frac{\partial}{\partial t}\widehat{A}_i(x)\widehat{A}_l(x') - \widehat{A}_l(x')\frac{\partial}{\partial t}\widehat{A}_i(x)|0\rangle\delta(t-t') \; ,$$

and the average over the vacuum becomes a convolution in the delta function of spatial coordinates with the coefficient -4π . As a result,

$$-\frac{\partial^2}{\partial t^2}D_{il}^{(F)}(x-x') = -4\pi g_{il}\delta^{(4)}(x-x') \; ,$$

which confirms the validity of (7.109).

Transition to the momentum representation raises the problem of the rule for a detour around singular points. It is resolved in the same way as in the case of the electron propagator, that is, with the use of Feynman's detour rule:

$$D_{il}^{(F)}(k) = \frac{4\pi}{k^2 + i\epsilon} g_{il} \quad \text{(the Feynman gauge)} . \tag{7.110}$$

Potential A_k, unlike Dirac state vectors ψ, is defined ambiguously and admits a broad class of gauge transformations. It accounts for a variety of representations of photon propagator $D_{il}(x - x')$, which give similar results in the calculation of electrodynamic processes. We present one more relativistically covariant momentum representation of the propagator without a derivation:

$$D_{il}^{(F)}(k) = \frac{4\pi}{k^2 + i\epsilon} \left(g_{il} - \frac{k_i k_l}{k^2} \right) \quad \text{(the Landau gauge)} . \tag{7.111}$$

Other gauge variants and substantiation of the above formulas differing from that presented in the preceding paragraphs can be found in textbooks (Berestetskii *et al.*, 1982; Peskin and Schroeder, 1995; Akhiezer and Berestetskii, 1981; Bogolubov and Shirkov, 1980; Moskaleov, 2006).

7.12
Interaction between Electrons and Muons

7.12.1
Electron–Muon Collisions

Let us move to a problem that is more complicated than that considered above, that is, elastic collision of two Dirac particles, with the second object being a muon. Electron–electron scattering (Möller scattering) is a somewhat more complicated process than electron scattering from a muon because it includes the effect of exchange of two identical particles. In accordance with the scheme applied in Section 7.10, in the first order of perturbation theory,

$$\widehat{S} = -i \int \widehat{j}^k(x) \widehat{A}_k(x) \mathrm{d}^4 x . \tag{7.112}$$

However, the $\widehat{A}_k(x)$ potential is created in this case by a muon rather than a motionless nucleus; it is also described by the Dirac equation. The $\widehat{A}_k(x)$ operator should be calculated from the equation for the electromagnetic potential written in the Heisenberg representation:

$$\Box \widehat{A}_k(x) = -4\pi \widehat{j}_k = -4\pi e \widehat{\overline{\varphi}}(x) \widehat{\gamma}_k \widehat{\varphi}(x) , \tag{7.113}$$

where $\widehat{\gamma}_k = g_{kl} \widehat{\gamma^l}$, and $\widehat{\overline{\varphi}}(x)$ and $\widehat{\varphi}(x)$ are the operators of a muon–antimuon field. The electrons and muons have similar electric charges. We use the photon Green's function (7.110) in the Feynman gauge to write down the potential created by muon in the form

$$\widehat{A}_i(x) = -\int D_{il}^{(F)}(x-x')\widehat{j}^l(x')\mathrm{d}^4x'$$
$$= -e\int D_{il}^{(F)}(x-x')\widehat{\overline{\varphi}}(x')\widehat{\gamma^l}\widehat{\varphi}(x')\mathrm{d}^4x' . \tag{7.114}$$

After the substitution of this result into (7.112), the scattering operator assumes the form

$$\widehat{S} = -ie^2\int\left(\widehat{\overline{\psi}}(x)\widehat{\gamma^i}\widehat{\psi}(x)\right)D_{il}^{(F)}(x-x')\left(\widehat{\overline{\varphi}}(x')\widehat{\gamma^l}\widehat{\varphi}(x')\right)\mathrm{d}^4x\mathrm{d}^4x' . \tag{7.115}$$

Electron and muon operators naturally enter in a symmetric manner: each particle plays the role of the source of the "external field" with respect to its partner.

Let us now specify more concretely the particle's initial and final states: p, ν and q, μ are the initial momenta and polarization indices of the electron and the muon, and p', ν' and q', μ' are the same for the final states. We shall use normalization $\psi_{p\nu}(x) = \frac{u_{p\nu}}{\sqrt{2\mathcal{E}_e\mathcal{V}}}e^{-ipx}$, where $\overline{u}_{p\nu}u_{p\nu} = 2m$, and analogous normalization for muonic states: $\varphi_{q\mu}(x) = \frac{u_{q\mu}}{\sqrt{2\mathcal{E}_\mu\mathcal{V}}}e^{-iqx}$, where $\mathcal{E}_e = \sqrt{m_e^2 + \boldsymbol{p}^2}$ and $\mathcal{E}_\mu = \sqrt{m_\mu^2 + \boldsymbol{q}^2}$ are the electron and muon energies, respectively.

Transition to the momentum representation was done earlier, which allows to immediately write down the matrix element (7.76):

$$\langle f|\widehat{S}|i\rangle = i(2\pi)^4 M_{fi}\delta^{(4)}(P^f - P^i)$$
$$= i(2\pi)^4\delta^{(4)}(p+q-q'-p')\frac{1}{4\mathcal{V}^2\sqrt{\mathcal{E}_e\mathcal{E}_\mu\mathcal{E}'_e\mathcal{E}'_\mu}}\left(\overline{u}_{p'\nu'}\widehat{\gamma}^n u_{p\nu}\right)$$
$$\times\frac{-4\pi e^2 g_{nl}}{k^2 + i\epsilon}\left(\overline{u}_{q'\mu'}\widehat{\gamma^l}u_{q\mu}\right) . \tag{7.116}$$

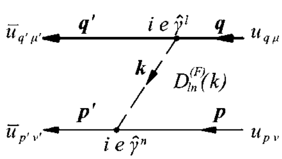

Figure 7.4 Feynman diagram for electron scattering by a muon.

The quantity M_{fi} can be depicted using a Feynman diagram (Figure 7.4). The diagram shows multipliers that should be matched with diagram elements (such as inner and outer lines and vertices) with the summation of repeated indices. The matrix element M_{fi} contains the normalizing factors of the wave functions, $(4\mathcal{V}^2\sqrt{\mathcal{E}_e\mathcal{E}_\mu\mathcal{E}'_e\mathcal{E}'_\mu})^{-1}$, besides the multipliers above. Each vertex retains the three-dimensional momentum, but not the energy. The arrows are placed so that time increases from right to left.

Further calculations are made using formulas (7.86)–(7.91) and lead to the following expression:

$$\mathrm{d}\sigma_{i\to f} = \frac{(2\pi)^4\mathcal{V}}{j}|M_{fi}|^2\delta^{(4)}(p+q-q'-p')\frac{\mathcal{V}\mathrm{d}^3p'}{(2\pi)^3}\frac{\mathcal{V}\mathrm{d}^3q'}{(2\pi)^3}\,, \tag{7.117}$$

where the last two multipliers represent the number of final quantum states $\mathrm{d}\nu_f$. The singular $\delta^{(4)}$ function can be removed by integration over one of the finite three-dimensional momenta, for example, d^3q', and the absolute value of the second momentum $\mathrm{d}p'^2$.

At this stage, the frame of reference is specified more concretely and the center-of-inertia system is chosen in which the complete three-dimensional momentum becomes zero, for example, $\boldsymbol{q} = -\boldsymbol{p}$, $\boldsymbol{q}' = -\boldsymbol{p}'$. In this system, $\mathrm{d}(\mathcal{E}'_\mathrm{e} + \mathcal{E}'_\mu) = \frac{\mathcal{E}'_\mathrm{e}+\mathcal{E}'_\mu}{2\mathcal{E}'_\mathrm{e}\mathcal{E}'_\mu}\mathrm{d}p'^2$, and integration of the delta function from energies gives

$$\int \delta(\mathcal{E}_\mathrm{e} + \mathcal{E}_\mu - \mathcal{E}'_\mathrm{e} - \mathcal{E}'_\mu)\mathrm{d}p'^2 = \frac{2\mathcal{E}_\mathrm{e}\mathcal{E}_\mu}{\mathcal{E}_\mathrm{e}+\mathcal{E}_\mu}\,.$$

In all other multipliers, $\mathcal{E}'_\mathrm{e} = \mathcal{E}_\mathrm{e}$, $\mathcal{E}'_\mu = \mathcal{E}_\mu$. The particle flow density in the center-of-inertia system

$$j = \frac{1}{\mathcal{V}}(v_e + v_\mu) = \frac{|\mathbf{p}|}{\mathcal{V}}\frac{\mathcal{E}_\mathrm{e}+\mathcal{E}_\mu}{\mathcal{E}_\mathrm{e}\mathcal{E}_\mu}\,, \tag{7.118}$$

where v_e and v_μ are particles velocities. Substitution into (7.117) leads to

$$\mathrm{d}\sigma_{i\to f} = \frac{\mathcal{V}^3|\mathbf{p}|\mathcal{E}_\mathrm{e}^2\mathcal{E}_\mu^2}{(2\pi)^2\mathcal{E}^2}|M_{fi}|^2\mathrm{d}\Omega\,, \quad \mathcal{E} = \mathcal{E}_\mathrm{e} + \mathcal{E}_\mu\,, \tag{7.119}$$

where $\mathrm{d}\Omega$ is the solid angle into which electrons are scattered, $\mathcal{E}$ is the total energy of two particles retained after elastic scattering, and M_{fi} is given by (7.116).

Let us regard the particles as being initially unpolarized and perform summation over polarizations in the final state. As shown in Section 7.10, this operation leaves traces of the product of Dirac matrices $\widehat{\gamma}$ and projection operators $\widehat{\Pi}_+$. To this effect, we use the relation $(\overline{u}_{q'\mu'}\widehat{\gamma^l}u_{q\mu})^* = (u^\dagger_{q\mu}(\widehat{\gamma^l})^\dagger(\widehat{\gamma^0})^\dagger(u^\dagger_{q'\mu'})^\dagger) = (\overline{u}_{q\mu}\widehat{\gamma^l}u_{q'\mu'})$ and an analogous one for electron bispinors. This reduces the cross-section averaged over spin states to the form

$$\begin{aligned}
\mathrm{d}\overline{\sigma}_{i\to f} &= \frac{1}{4}\sum_{\mu\mu'\nu\nu'}\mathrm{d}\sigma_{i\to f}\\
&= \frac{e^4}{4k^4\mathcal{E}^2}\frac{1}{2}\mathrm{Tr}\left[\widehat{\gamma^l}\widehat{\Pi}_+(q)\widehat{\gamma}^n\widehat{\Pi}_+(q')\right]\\
&\quad\times\frac{1}{2}\mathrm{Tr}\left[\widehat{\gamma}_l\widehat{\Pi}_+(p)\widehat{\gamma}_n\widehat{\Pi}_+(p')\right]\mathrm{d}\Omega\,,
\end{aligned} \tag{7.120}$$

where $k^2 = (p-p')^2 = (q-q')^2$.

Calculation of the traces and their multiplication is rather simple mechanical work requiring, however, diligence and carefulness. The results are as follows:

$$\tfrac{1}{2}\mathrm{Tr}\,[\widehat{\gamma^l}\widehat{\Pi}_+(q)\widehat{\gamma}^n\widehat{\Pi}_+(q')] = 2(q^l q'^n + q^n q'^l) + 2g^{ln}(m_\mu^2 - q^k q'_k)\,;$$

$$\tfrac{1}{2}\mathrm{Tr}\,[\widehat{\gamma}_l\widehat{\Pi}_+(p)\widehat{\gamma}_n\widehat{\Pi}_+(p')] = 2(p_l p'_n + p_n p'_l) + 2g_{ln}(m_e^2 - p^k p'_k)\,.$$

The result of trace multiplication can be conveniently expressed through kinematic invariants s, t, and u defined in Section 3.2. In the present case, they are written in the following form:

$$\begin{aligned} s &=(p+q)^2 = (p'+q')^2 = (\mathcal{E}_e + \mathcal{E}_\mu)^2 = \mathcal{E}^2\,; \\ t &=(p-p')^2 = (q-q')^2 = -2\boldsymbol{p}^2(1-\cos\theta) = k^2\,; \\ u &=(p-q')^2 = (p'-q)^2 = (\mathcal{E}_e - \mathcal{E}_\mu)^2 - 2\boldsymbol{p}^2(1-\cos\theta) = (\mathcal{E}_e - \mathcal{E}_\mu)^2 + t\,. \end{aligned} \tag{7.121}$$

In these chains of equalities, the last one expresses the respective invariant as the quantities referring to the center-of-inertia system. The solid angle is also expressed through the invariant:

$$\mathrm{d}\Omega = \frac{\pi\mathcal{E}^2}{I^2}d(-t)\,, \quad I^2 = (pq)^2 - m_\mu^2 m_e^2 = \boldsymbol{p}^2\mathcal{E}^2\,.$$

The substitution of these quantities into (7.120) results in the differential scattering cross-section in the invariant form:

$$\frac{\mathrm{d}\overline{\sigma}_{i\to f}}{\mathrm{d}\Omega} = \frac{\alpha^2}{t^2 s}\left[st + (s-m_\mu^2)^2 + \frac{1}{2}t^2\right]\,, \quad \alpha = e^2 \approx \frac{1}{137}\,. \tag{7.122}$$

Here, the quantity $(m_e/m_\mu)^2 \ll 1$ is disregarded throughout. Using formulas (7.121), we can easily write the differential scattering cross-section through the total energy $\mathcal{E}$, the scattering angle θ, and the electron three-dimensional momentum $\boldsymbol{p}^2$ in the center-of-inertia system:

$$\begin{aligned} \frac{\mathrm{d}\overline{\sigma}_{i\to f}}{\mathrm{d}\Omega} &= \frac{\alpha^2}{4\mathcal{E}^2\boldsymbol{p}^4(1-\cos\theta)^2} \\ &\quad \times \left\{\left(\mathcal{E}^2 - m_\mu^2\right)^2 - 2\mathcal{E}^2\boldsymbol{p}^2(1-\cos\theta) + 2\boldsymbol{p}^4(1-\cos\theta)^2\right\}\,. \end{aligned} \tag{7.123}$$

A characteristic feature of the cross-section is its strong dependence on the scattering angle $\mathrm{d}\sigma/\mathrm{d}\Omega \sim \theta^{-4}$ conditioned by long-range Coulomb forces. If the total energy of an electron $\mathcal{E}_e$ is low compared with the muon rest energy m_μ, the last formula transforms into the Mott formula (7.102) (see Problem 7.13).

7.12.2
Conversion of an Electron–Positron Pair into a Muon Pair

A diagram of the process being considered is presented in Figure 7.5, where antiparticles are depicted as objects moving backward in time, that is, from the future to the past. Muons are shown by solid lines. Creation of a muon pair is possible only at $\mathcal{E}_\mathrm{e} \gg m_\mathrm{e}$; therefore, the electron mass is disregarded in what follows.

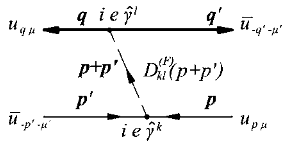

Figure 7.5 Conversion of an electron–positron par into a muon pair.

We write down the matrix element M_{fi} by making use of the diagram and the notation therein:

$$M_{fi} = \left(\overline{u}_{-p'\nu'} ie\widehat{\gamma^k} u_{p\nu}\right) \times D_{kl}^{(F)}(p+p')\left(\overline{u}_{-q'\mu'} ie\widehat{\gamma^l} u_{q\mu}\right)\left(4\mathcal{V}^2\sqrt{\mathcal{E}_\mathrm{e}\mathcal{E}'_\mathrm{e}\mathcal{E}_\mu\mathcal{E}'_\mu}\right)^{-1} . \tag{7.124}$$

Then, we proceed in accordance with the scheme presented in Section 7.9. Squaring the matrix element should be followed by averaging electron and positron bispinors over polarizations:

$$\begin{aligned}&\frac{1}{4}\sum_{\nu'\nu}\left(\overline{u}_{-p'\nu'}\widehat{\gamma}_l u_{p\nu}\right)^*\left(\overline{u}_{-p'\nu'}\widehat{\gamma}_k u_{p\nu}\right)\\&=\frac{1}{4}\sum_{\nu'\nu}\left(\overline{u}_{p\nu}\widehat{\gamma}_l u_{-p'\nu'}\right)\left(\overline{u}_{-p'\nu'}\widehat{\gamma}_k u_{p\nu}\right)\\&=\frac{1}{4}\mathrm{Tr}\left[\widehat{\gamma}_k\widehat{\Pi}_+(p)\widehat{\gamma}_l\widehat{\Pi}_-(p')\right]=g_{kl}\left(m_\mathrm{e}^2+p^ip'_i\right)-\left(p_kp'_l+p_lp'_k\right) .\end{aligned} \tag{7.125}$$

Muon bispinors are summed over polarizations:

$$\begin{aligned}&\sum_{\mu'\mu}\left(\overline{u}_{-q'\mu'}\widehat{\gamma^l}u_{q\mu}\right)^*\left(\overline{u}_{-q'\mu'}\widehat{\gamma^k}u_{q\mu}\right)\\&=\sum_{\mu'\mu}\left(\overline{u}_{q\mu}\widehat{\gamma^l}u_{-q'\mu'}\right)\left(\overline{u}_{-q'\mu'}\widehat{\gamma^k}u_{q\mu}\right)\\&=\mathrm{Tr}\left[\widehat{\gamma^k}\widehat{\Pi}_+(q)\widehat{\gamma^l}\widehat{\Pi}_-(q')\right]\\&=4g^{kl}\left(m_\mu^2+q^iq'_i\right)-4\left(q^kq'^l+q^lq'^k\right) .\end{aligned} \tag{7.126}$$

The $\delta^{(3)}$ function of three-dimensional momenta is removed by means of integration over d^3q'. Then, $\mathrm{d}^3q = q^2\mathrm{d}q\mathrm{d}\Omega = (1/2)|\boldsymbol{q}|\mathrm{d}\boldsymbol{q}^2\mathrm{d}\Omega = |\boldsymbol{q}|\mathcal{E}_\mu\mathrm{d}\mathcal{E}_\mu\mathrm{d}\Omega$ is written down and the integration is performed with the help of the last delta function of the energy by making use of all conservation laws and choosing the center-of-inertia system to arrive at relations $p + p' = q + q'$, that is, $\mathcal{E}_e + \mathcal{E}'_e = \mathcal{E}_\mu + \mathcal{E}'_\mu$, $\boldsymbol{p} + \boldsymbol{p}' = \boldsymbol{q} + \boldsymbol{q}'$, and also $\mathcal{E}'_e = \mathcal{E}_e$, $\mathcal{E}'_\mu = \mathcal{E}_\psi = \mathcal{E}_e$, and $\delta(2\mathcal{E}_e - 2\mathcal{E}_\mu)|\boldsymbol{q}|\mathcal{E}_\mu d\mathcal{E}_\mu = (1/2)|\boldsymbol{q}|\mathcal{E}_\mu = (1/2)\mathcal{E}^2\sqrt{1 - m_\mu^2/\mathcal{E}_e^2}$. The following equalities are used in multiplying the traces of electron–positron and muon operators: $\mathcal{E}'_e = \mathcal{E}_e$, $\mathcal{E}'_\mu = \mathcal{E}_\psi = \mathcal{E}_e$, and $\delta(2\mathcal{E}_e - 2\mathcal{E}_\mu)|\boldsymbol{q}|\mathcal{E}_\mu d\mathcal{E}_\mu = (1/2)|\boldsymbol{q}|\mathcal{E}_\mu = (1/2)\mathcal{E}^2\sqrt{1 - m_\mu^2/\mathcal{E}_e^2}$. After all the results have been obtained, we find the differential cross-section expressed as the initial energy $\mathcal{E} = 2\mathcal{E}_e$ of an electron–positron pair and the angle determining the line of divergence of a muon pair:

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4\mathcal{E}^2}\sqrt{1 - \frac{4m_\mu^2}{\mathcal{E}^2}}\left\{\left(1 + \frac{4m_\mu^2}{\mathcal{E}^2}\right) + \left(1 - \frac{4m_\mu^2}{\mathcal{E}^2}\right)\cos^2\theta\right\} . \tag{7.127}$$

Integration over the solid angle gives the full cross-section:

$$\sigma = \frac{4\pi\alpha^2}{3\mathcal{E}^2}\sqrt{1 - \frac{4m_\mu^2}{\mathcal{E}^2}}\left(1 + \frac{2m_\mu^2}{\mathcal{E}^2}\right) . \tag{7.128}$$

Problems

7.13. Specify the applicability conditions and obtain the limiting expression for the cross-section (7.123) at which it coincides with the Mott formula (7.102), which describes scattering of a Dirac particle by the Coulomb field of a given external spinless source.

7.14*. Depict the process of annihilation of an electron–positron pair with the creation of two hard photons (gamma quanta) on the Feynman diagram. Find the differential and total cross-sections of this process averaged over polarization states of the initial particles and summed over their final states. Express the differential cross-section through the starting energy of the electron and the scattering angle in the center-of-inertia system.

7.13 Higher-Order Corrections

We have considered a few quantum electrodynamics problems in the first, nonvanishing approximation in the preceding sections. Had we tried to calculate corrections in the next order of smallness, we would have come to divergent integrals leading to infinite values of these corrections. At first sight, this difficulty renders the entire scheme of perturbation theory inconsistent. However, the founders of modern quantum electrodynamics, such as Kramers, Dyson[2)], Tomonaga, Feyn-

2) American physicist Freeman Dyson (*1925) investigated the renormalization problem in quantum electrodynamics and other problems of the quantum theory of a field. It one of the first

man[3], Schwinger, and their successors, developed a set of rules for "subtraction of infinities" resulting in final corrections in excellent agreement with experiment.

This set of rules formulated in the mathematical language gives rise to the *renormalization theory*, the consideration of which is beyond the scope of this book. The essence of this method is in assigning certain prescribed values (obtained in experiment) to divergent quantities. In quantum electrodynamics, such quantities are zero mass and photon charge as well as mass and electron charge known exactly from experiment. Some idea of the renormalization method is given by the renormalization of mass described in Chapter 5 in the framework of classical electrodynamics (see Section 5.4.2).

The study of higher approximations ("radiative corrections") in quantum electrodynamics made it possible not only to determine more exactly the cross-sections of the processes of interest but also to discover essentially new physical phenomena. The radiative corrections for the energy of atomic electrons led to the shift of atomic levels ("Lamb shift") (see the estimates in Problem 6.18$^{\bullet}$*). They indicate that electromagnetic and electron–positron fields in a vacuum, that is, in the absence of quanta of these fields, may act on other physical systems and cause the observed effects.

Radiative corrections lead to a change of the electron magnetic moment compared with its value following from the Dirac equation and equaling the Bohr magneton. Calculations yield the following value of the "anomalous magnetic moment":

$$\mu_{\text{th}} = \mu_{\text{B}} \left(1 + \frac{\alpha}{2\pi} - 0.328 \frac{\alpha^2}{\pi^2} + 1.184 \frac{\alpha^3}{\pi^3} \right) ,$$

which is in good agreement with the measured value

$$\mu_{\text{exp}} = 1.001\,159\,652\,41(20) \mu_{\text{B}} .$$

Here, α is the fine structure constant.

The effects described above and other subtle effects leave no doubt that modern electrodynamics for all its limitations and shortcomings (e.g., the appearance of infinite values) adequately describes a wide range of phenomena of the microworld. Indeed, it provides a basis for the formulation of up-to-date and more general quantum field theories.

has joined in working out of the theory of neutron stars and has stated some ideas directed on search of extraterrestrial civilisations.

3) American physicists Richard Feynman (1918–1988), Julian Schwinger (1918–1994) and Japan physicist Sinyatiro Tomonaga (1906–1979) have formulated quantum electrodynamics in the modern form and have been awarded by the Nobel Prize in 1965.

7.14 Answers and Solutions

7.1

$$\widehat{\gamma^0} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \widehat{\gamma^1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\widehat{\gamma^2} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \quad \widehat{\gamma^3} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

7.5 Normalization $u^\dagger u = 1$ from Example 6.18 gives the value of $\overline{u}u = m/\mathcal{E}$. But with regard for the normalized exponential factor, we have $\overline{\psi}(x)\psi(x) = m/\mathcal{E}\mathcal{V} =$ inv, $\rho = 1/\mathcal{V}$, and $\boldsymbol{j} = \boldsymbol{p}/\mathcal{E}\mathcal{V} = \rho\boldsymbol{v}$.

Case (i): $N = \sqrt{|\mathcal{E}| + m}$, $\rho = 1/\mathcal{V}$, $\boldsymbol{j} = \pm\boldsymbol{p}/\mathcal{E}\mathcal{V} = \pm\rho\boldsymbol{v}$.

Case (ii): $N = \sqrt{(|\mathcal{E}| + m)/2m}$, $\rho = 1/2m\mathcal{V}$, $\boldsymbol{j} = \pm\boldsymbol{p}/2m\mathcal{E}\mathcal{V} = \pm\rho\boldsymbol{v}$.

In all three cases, $\overline{\psi}\psi =$ inv.

7.6

1. $$u_{p,1/2} = N \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{(\mathcal{E}+m)} \\ \frac{(p_x+ip_y)}{(\mathcal{E}+m)} \end{pmatrix}; \tag{7.129}$$

$$u_{p,-1/2} = N \begin{pmatrix} 0 \\ 1 \\ \frac{(p_x-ip_y)}{(\mathcal{E}+m)} \\ \frac{-p_z}{(\mathcal{E}+m)} \end{pmatrix}. \tag{7.130}$$

2. $$u_{p,+} = N \begin{pmatrix} e^{-i\phi/2}\cos\left(\frac{\theta}{2}\right) \\ e^{i\phi/2}\sin\left(\frac{\theta}{2}\right) \\ \frac{p\,e^{-i\phi/2}\cos(\theta/2)}{(\mathcal{E}+m)} \\ \frac{p\,e^{i\phi/2}\sin(\theta/2)}{(\mathcal{E}+m)} \end{pmatrix}; \tag{7.131}$$

$$u_{p,-} = N \begin{pmatrix} -e^{-i\phi/2}\sin\left(\frac{\theta}{2}\right) \\ e^{i\phi/2}\cos\left(\frac{\theta}{2}\right) \\ \frac{p\,e^{-i\phi/2}\sin(\theta/2)}{(\mathcal{E}+m)} \\ \frac{-p\,e^{i\phi/2}\cos(\theta/2)}{(\mathcal{E}+m)} \end{pmatrix}. \tag{7.132}$$

7.7 The solution of Problem 4.20 implies that the infinitely small proper Lorentz transformation is defined by antisymmetric tensor $\delta\Omega_{ik}$ having six independent components (small angles of rotation in planes i, k). Taking into account the terms of the first order of smallness, we find that the spinor transformation matrix must have the form

$$\widehat{U} = 1 + \eta\delta\Omega_{ik}\widehat{S}^{ik}\,, \quad \widehat{U} = 1 + \eta\delta\Omega_{ik}\widehat{S}^{ik}\,, \tag{1}$$

where $\widehat{S}^{ik}$ is a four-row matrix antisymmetric with respect to indices i and k, and η is an unknown numerical coefficient. It should be found from (7.8):

$$\widehat{U}^{-1}\widehat{\gamma^k}\widehat{U} = \Lambda_i{}^k\widehat{\gamma^i}\,. \tag{2}$$

Bearing in mind the covariant properties of the $\widehat{\gamma}^k$ matrices which they acquire when localized between the bispinors $\overline{u}$ and u, we would naturally seek $\widehat{S}^{lm}$ as a combination of these matrices. But we already have the antisymmetric tensor structure (7.24). Substituting quantities (1) and the small Lorentz transformation matrix $\Lambda_i^k = \delta_i^k + \delta\Omega_{ik}$ yields with an accuracy up to the terms of the first order of smallness

$$\eta\delta\Omega_{lm}\left(\widehat{\gamma^k}\widehat{S}^{lm} - \widehat{S}^{lm}\widehat{\gamma^k}\right) = g^{kl}\delta\Omega_{lm}\widehat{\gamma^m}\,. \tag{3}$$

Using matrix anticommutation $\widehat{\gamma}$ and the explicit form of $\widehat{S}^{lm}$, we find from (3) $\eta = 1/4$.

7.8 Expand the transformation matrix $\widehat{U} = e^{\omega\widehat{\alpha}_x/2}$ as a power series using readily controllable relations $\widehat{\alpha}_x^{2n+1} = \widehat{\alpha}_x$, $\widehat{\alpha}_x^{2n} = 1$ to obtain

$$\widehat{U} = \cosh\frac{\omega}{2} - \widehat{\alpha}_x\sinh\frac{\omega}{2}\,, \quad \widehat{U}^{-1} = \cosh\frac{\omega}{2} + \widehat{\alpha}_x\sinh\frac{\omega}{2}\,. \tag{1}$$

This permits us to find

$$\begin{aligned}\widehat{U}^{-1}\widehat{\gamma^1}\widehat{U} &= \left(\cosh^2\frac{\omega}{2} + \sinh^2\frac{\omega}{2}\right)\widehat{\gamma^1} - 2\cosh\frac{\omega}{2}\sinh\frac{\omega}{2}\widehat{\gamma^0}\\ &= \frac{\widehat{\gamma^1} - V\widehat{\gamma^0}}{\sqrt{1-V^2}}\,.\end{aligned} \tag{2}$$

The last equation is used to obtain the transformed current component,

$$j'^1 = \frac{j^1 - Vj^0}{\sqrt{1-V^2}}\,, \tag{3}$$

and, in an analogous way, other formulas of 4-vector transformation.

7.9 The frame of reference in which the bispinor is sought moves with respect to the particle's rest system with velocity $\boldsymbol{V} = -\boldsymbol{v} = -\boldsymbol{p}/\mathcal{E}$, where $\boldsymbol{v}$ is the particle's velocity. Therefore,

$$\widehat{U} = \cosh\frac{\omega}{2} + \left(\widehat{\boldsymbol{\alpha}}\cdot\frac{\boldsymbol{v}}{v}\right)\sinh\frac{\omega}{2}\ ;$$

$$u(p) = \begin{pmatrix} \varphi\cosh\frac{\omega}{2} \\ \frac{\boldsymbol{v}\cdot\widehat{\boldsymbol{\sigma}}}{v}\varphi\sinh\frac{\omega}{2} \end{pmatrix} = \sqrt{\frac{\mathcal{E}+m}{2m}}\begin{pmatrix} \varphi \\ \frac{\boldsymbol{p}\cdot\widehat{\boldsymbol{\sigma}}}{\mathcal{E}+m}\varphi \end{pmatrix} .$$

7.10

$$\zeta = \frac{m}{\mathcal{E}}\zeta_0 + \frac{\mathbf{p}}{\mathcal{E}(\mathcal{E}+m)}(\boldsymbol{p}\cdot\zeta_0)\ .$$

In the nonrelativistic case ($\mathcal{E} \approx m$, $p \ll m$), $\zeta = \zeta_0$. In the relativistic case, $\zeta_\perp = m\zeta_{0\perp}/\mathcal{E}$ and $\zeta_\parallel = \zeta_{0\parallel}$, where the symbols $\perp$ and $\parallel$ pertain to direction $\boldsymbol{p}$. In the ultrarelativistic limit, $\mathcal{E} \gg m$ and the mean polarization vector is arranged along the momentum.

7.11 The Feynman diagram is presented in Figure 7.6. In the first approximation of perturbation theory, the electron and positron cross-sections coincide; they are given by the Mott formula (7.102).

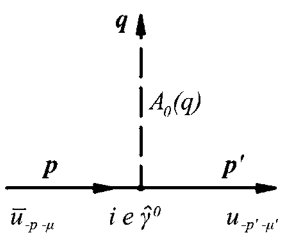

Figure 7.6 Positron scattering by an external field.

7.12 The difference from the problem considered in Section 7.10 reduces to a change of the external field potential and its Fourier transform that depends on the momentum $q = 2p\sin(\theta/2)$ transferred to the scatterer. The characteristic values of the transferred momentum at which the corrections being considered become essential are determined by the reciprocal values of characteristic radii a and r_0. Their role can be considered separately because they differ by approximately four to five orders of magnitude. For high-energy particles with $p \gg a^{-1} = \alpha$, the effect of screening is appreciable under the condition $q \lesssim \alpha$ or under the condition $p\theta \lesssim \alpha$, that is, at small scattering angles. In the case of exponential screening,

$$A_0(q) = \int \frac{Ze}{r} e^{-\alpha r + \mathrm{i}\mathbf{q}\cdot\mathbf{r}} \mathrm{d}^3 r = \frac{4\pi Z e^2}{q^2+\alpha^2}\ . \tag{1}$$

The screening in this approximation does not influence the spin factor; therefore, expression (7.102) is replaced by

$$\frac{\mathrm{d}\overline{\sigma}}{\mathrm{d}\Omega} = \left(\frac{2Ze^2\mathcal{E}}{q^2+\alpha^2}\right)^2\left(1-\frac{q^2}{4\mathcal{E}^2}\right)\ . \tag{2}$$

Coulomb divergence of the cross-section at $\theta \to 0$ disappears because the field outside the screening radius no longer deflects the particle.

Taking account of the finite nuclear size alters the form of the potential in the region with $r < r_0$. We use the Poisson equation in the Fourier representation:

$$q^2 A_0(q) = 4\pi e \int n(r) e^{i\mathbf{k}\cdot\mathbf{r}} \mathrm{d}^3 r = 4\pi e F(q) \,. \tag{3}$$

In the spherically symmetric case,

$$F(q) = \frac{4\pi}{q} \int_0^{r_0} r n(r) \sin(qr) \mathrm{d}r \,. \tag{4}$$

These results are used to find the modified Mott formula:

$$\frac{\mathrm{d}\overline{\sigma}}{\mathrm{d}\Omega} = \left(\frac{2e^2\mathcal{E}}{q^2}\right)^2 \left(1 - \frac{q^2}{4\mathcal{E}^2}\right) F^2(q) \,. \tag{5}$$

Finiteness of the nuclear size weakly affects small-angle scattering because $F(q) \approx Z$ at $qr_0 \ll 1$. However, at large scattering angles, $qr_0 \gg 1$, the form factor decreases as $F(q) \sim (qr)^{-2}$, which reduces the cross-section. This is quite natural since the field inside a nucleus of finite size is significantly smaller than the point-charge field.

7.13 The energy of an oncoming particle, in contrast to its momentum, is not transferred to the external source of the static field. The source remains motionless and has a large mass. For this reason, the total electron energy must be small compared with the rest energy of the field source (in our case, a muon). Such conditions can be roughly fulfilled because the mass of a muon mass is approximately 207 times the mass of an electron. Under the conditions being considered, the electron can be even moderately ultrarelativistic. The spin magnetic moment of the muon is much smaller than that of the electron and plays but a minor role in scattering. At $m_\mu \gg \mathcal{E}_\mathrm{e} \gg m_\mathrm{e}$, the Mott formula and (7.123) assume the simple form

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2(1+\cos\theta)}{2\boldsymbol{p}^2(1-\cos\theta)^2} \,.$$

7.14* Two Feynman diagrams depicting the process of annihilation of the pair are presented in Figure 7.7.

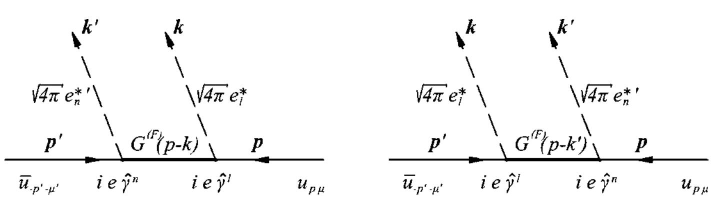

Figure 7.7 Annihilation of an electron–positron pair into two gamma quanta.

They differ in quantum rearrangement. The diagrams show the factors to be matched with such elements as inner lines, outer lines, and vertices in the analytical expression of the matrix element. Apart from the expressions presented in the diagram, the matrix element includes the factor $(2\mathcal{V}\mathcal{E})^{-1/2}$ per particle in the initial and final states; that is, $(16\mathcal{V}^4\mathcal{E}\mathcal{E}'\omega\omega')^{-1/2}$. As a result, the matrix element taking account of both diagrams is written in the form

$$M_{fi} = -\frac{\pi e^2}{\mathcal{V}^2\sqrt{\mathcal{E}\mathcal{E}'\omega\omega'}} e_l^* e_n'^* \left(\overline{u}_{-p,-\mu}\widehat{U}^{nl} u_{p\mu}\right) , \tag{1}$$

where

$$\widehat{U}^{nl} = \widehat{\gamma^n}\widehat{G}^{(F)}(p-k)\widehat{\gamma^l} + \widehat{\gamma^l}\widehat{G}^{(F)}(p-k')\widehat{\gamma^n} , \tag{2}$$

and $\widehat{G}^{(F)}$ are the electron propagators constructed in Section 7.11.

According to the relations in Section 7.9, the cross-section of the process is expressed through the squared modulus of the matrix element as follows:

$$\begin{aligned} d\sigma_{i\to f} = {} & \frac{(2\pi)^4\mathcal{V}}{j}\overline{|M_{fi}|^2}\delta^{(3)}(\boldsymbol{p}+\boldsymbol{p}'-\boldsymbol{k}-\boldsymbol{k}') \\ & \times \delta(\mathcal{E}+\mathcal{E}'-\omega-\omega')\frac{\mathcal{V}d^3k'}{(2\pi)^3}\frac{\mathcal{V}d^3k}{(2\pi)^3} . \end{aligned} \tag{3}$$

Find the relationship between the particles' three-dimensional momenta $\boldsymbol{p}' = -\boldsymbol{p}$ and $\boldsymbol{k}' = -\boldsymbol{k}$ and energies $\mathcal{E}' = \mathcal{E} = \omega = \omega'$ from the conservation laws supported by delta functions in the center-of-inertia system. The delta functions are eliminated by integrating over d^3k' and $dk = d\omega$. As a result, the right-hand side of equation (3) contains only the differential of the solid angle $d\Omega_k = 2\pi \sin\theta\, d\theta$, indicating the direction of escape of one of the quanta relative to the initial electron momentum. Retain the former notation for the cross-section integrated as above to obtain

$$d\sigma_{i\to f} = \frac{\mathcal{V}^3\mathcal{E}^2}{2(2\pi)^2 j}\overline{|M_{fi}|^2} d\Omega_k . \tag{4}$$

Kinematic invariants defined by formulas (3.49) in the center-of-inertia system have the form

$$\begin{aligned} & s = (p+p')^2 = 4\mathcal{E}^2 , \quad t = (p-k)^2 = m^2 - 2\mathcal{E}(\mathcal{E} - |\boldsymbol{p}|\cos\theta) , \\ & u = (p-k')^2 = m^2 - 2\mathcal{E}(\mathcal{E} + |\boldsymbol{p}|\cos\theta) . \end{aligned} \tag{5}$$

The flow density j of colliding particles is given by

$$j = \frac{|\boldsymbol{p}|}{\mathcal{V}}\left(\frac{1}{\mathcal{E}} + \frac{1}{\mathcal{E}}\right) = \frac{2|\boldsymbol{p}|}{\mathcal{V}\mathcal{E}} . \tag{6}$$

In the product $|M_{fi}|^2 = M_{fi}M_{fi}^*$ the complex-conjugate matrix element M_{fi}^* emerges from the initial one after a change in the sequence order of bispinors, transition to Hermitian conjugate quantities, and the use of γ matrices (7.2) and (7.3):

$$(\overline{u}_{-p,-\mu}\widehat{U}^{is}u_{p\mu})^* = \left(u^\dagger_{p\mu}(\widehat{U}^{is})^\dagger(\widehat{\gamma^0})^\dagger u_{-p',-\mu'}\right)$$
$$= \left(\overline{u}_{p\mu}\widehat{\gamma^0}(\widehat{U}^{is})^\dagger\widehat{\gamma^0}u_{-p',-\mu'}\right) . \tag{7}$$

Further, take advantage of the explicit form (2) of the $\widehat{U}^{nl}$ operator and find with the help of (7.2) and Feynman propagator (7.107)

$$\widehat{\gamma^0}(\widehat{U}^{is})^\dagger\widehat{\gamma^0} = \widehat{U}^{si} = \widehat{\gamma^s}\frac{\widehat{\gamma}(p-k)+m}{(p-k)^2-m^2}\widehat{\gamma^i} + \widehat{\gamma^i}\frac{\widehat{\gamma}(p-k')+m}{(p-k')^2-m^2}\widehat{\gamma^s} . \tag{8}$$

Similar to formulas (5), summation indices are omitted in both the numerator and the denominator of fractions of the scalar products in order to simplify the writing.

The sign of averaging above the square of the matrix element modulus in (3) means averaging over electron and positron spin states and summation over photon polarizations. Averaging over the spin states of Dirac particles was performed in previous calculations; it implies the use of projection operators (7.6) and (7.17) and leads to the traces of the products of Dirac matrices. Summation of the products of photon polarization vectors yields the components of the metric tensor:

$$\sum_{\lambda=1}^{2} e_l^{*\lambda}e_n^{\lambda} = g_{ln} , \quad \sum_{\lambda=1}^{2} e_l^{*\prime\lambda}e_n^{\prime\lambda} = g_{ln} . \tag{9}$$

Accomplishment of all these laborious calculations results in a relatively simple expression for the angular distribution of annihilation quanta in the center-of-inertia system:

$$\frac{\mathrm{d}\sigma_{i\to f}}{\mathrm{d}\Omega_k} = r_0^2\frac{m^2}{4\mathcal{E}|\boldsymbol{p}|}\left\{\frac{\mathcal{E}^2+\boldsymbol{p}^2(1+\sin^2\theta)}{\mathcal{E}^2-\boldsymbol{p}^2\cos^2\theta} - \frac{2\boldsymbol{p}^4\sin^4\theta}{(\mathcal{E}^2-\boldsymbol{p}^2\cos^2\theta)^2}\right\} . \tag{10}$$

The quantity $r_0 \approx 2.8 \times 10^{-13}$ cm here is the classical electron radius (see Problem 3.120$^\bullet$). Integration over the angle gives the overall cross-section of the annihilation pair:

$$\sigma_{i\to f} = \pi r_0^2\frac{1-v^2}{4v}\left\{\frac{3-v^4}{v}\ln\frac{1+v}{1-v} - 2(2-v^2)\right\} , \tag{11}$$

where $v = |\boldsymbol{p}|/\mathcal{E}$ is the dimensionless velocity. The cross-section decreases in the ultrarelativistic limit ($v \to 1$) and increases on transition to nonrelativistic energies, $v \ll 1$:

$$\sigma_{i\to f} \approx \frac{\pi r_0^2}{2v} . \tag{12}$$

At the same time, Coulomb attraction of the electron and positron, which was disregarded by the formulas above, becomes essential at $v \lesssim \alpha \approx 1/137$.

Appendix A
Conversion of Electric and Magnetic Quantities between the International System of Units and the Gaussian System

Formulas and equations

Name	CGS	SI
Speed of light	c	$(\epsilon_0\mu_0)^{-1/2}$
Strength of electric field, scalar potential	E, φ	$\sqrt{4\pi\epsilon_0}(E, \varphi)$
Electric induction	$\mathbf{D}, \mathcal{D}$	$\sqrt{4\pi/\epsilon_0}(\mathbf{D}, \mathcal{D})$
Electric charge, density of charge, electric current, density of electric current, electric polarization, electric dipole moment	$q, \rho, \mathcal{J}, j, \mathbf{P}, \mathbf{p}$	$\frac{1}{\sqrt{4\pi\epsilon_0}}(q, \rho, \mathcal{J}, j, \mathbf{P}, \mathbf{p})$
Magnetic induction, magnetic flux, vector potential	$\mathbf{B}, \Phi, \mathbf{A}$	$\sqrt{\frac{4\pi}{\mu_0}}(\mathbf{B}, \Phi, \mathbf{A})$
Strength of magnetic field	$\mathbf{H}$	$\sqrt{4\pi\mu_0}\mathbf{H}$
Magnetic moment, magnetic polarization	$\mathbf{m}, \mathbf{M}$	$\sqrt{\frac{\mu_0}{4\pi}}(\mathbf{m}, \mathbf{M})$
Relative electric and magnetic permeabilities	ϵ, μ	ϵ, μ
Electric and magnetic susceptibilities	α, χ	$\frac{1}{4\pi}(\alpha, \chi)$
Specific electric conductivity	κ	$\frac{\kappa}{4\pi\epsilon_0}$
Resistance	R	$4\pi\epsilon_0 R$
Electric capacitance	C	$\frac{1}{4\pi\epsilon_0} C$
Inductance	L	$\frac{4\pi}{\mu_0} L$
Lorentz force per unit charge	$\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}$	$\mathbf{E} + \mathbf{v} \times \mathbf{B}$
Density of electromagnetic energy in dispersionless matter	$\frac{1}{8\pi}(\epsilon E^2 + \mu H^2)$	$\frac{1}{2}(\epsilon\epsilon_0 E^2 + \mu\mu_0 H^2)$
Poynting vector	$\frac{c}{4\pi}\mathbf{E} \times \mathbf{H}$	$\mathbf{E} \times \mathbf{H}$
Maxwell equations	$\mathrm{curl}\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$	$\mathrm{curl}\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$
	$\mathrm{curl}\mathbf{H} = \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}$	$\mathrm{curl}\mathbf{H} = \mathbf{j} + \frac{\partial \mathbf{D}}{\partial t}$
	$\mathrm{div}\mathbf{D} = 4\pi\rho$	$\mathrm{div}\mathbf{D} = \rho$
	$\mathrm{div}\mathbf{B} = 0$	$\mathrm{div}\mathbf{B} = 0$
	$\mathbf{D} = \epsilon\mathbf{E} = \mathbf{E} + 4\pi\mathbf{P}$	$\mathbf{D} = \epsilon\epsilon_0\mathbf{E} = \epsilon_0\mathbf{E} + \mathbf{P}$
Material equations	$\mathbf{B} = \mu\mathbf{H} = \mathbf{H} + 4\pi\mathbf{M}$	$\mathbf{B} = \mu\mu_0\mathbf{H} = \mu_0(\mathbf{H} + \mathbf{M})$
	$\mathbf{j} = \kappa\mathbf{E}$	$\mathbf{j} = \kappa\mathbf{E}$

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Numerical values

Name	Notation	SI	CGS
Length	l	1 m (meter)	10^2 cm
Mass	m	1 kg (kilogram)	10^3 g
Time	t	1 s (second)	1 s
Frequency	$\nu = \omega/2\pi$	1 Hz (hertz)	1 Hz
Force	F	1 N (newton)	10^5 dyn
Energy	$\mathcal{E}$, W	1 J (joule)	10^7 erg
Intensity of radiation (power)	I	1 W (watt)	10^7 erg/s
Pressure	P	1 Pa (pascal)	10 dyn/cm^2
Electric current	J	1 A (ampere)	3×10^9 cm$^{3/2}$ g$^{1/2}$/s^2 (statampere)
Electric charge	e, q	1 C (coulomb)	3×10^9 cm$^{3/2}$ g$^{1/2}$/s (statcoulomb)
Strength of electric field	$\mathbf{E}$	1 V/m (volt per meter)	$(1/3) \times 10^{-4}$ g$^{1/2}$/cm$^{1/2}$ s (statvolt per centimeter)
Scalar potential	φ	1 V	$(1/3) \times 10^{-2}$ cm$^{1/2}$ g$^{1/2}$/s (statvolt)
Electric polarization	$\mathbf{P}$	1 C/m^2 (coulomb per square meter)	3×10^5 g$^{1/2}$/cm$^{1/2}$ s (dipole moment per cubic centimeter)
Electric induction (displacement)	$\mathbf{D}$	1 C/m^2 (coulomb per square meter)	$12\pi \times 10^5$ g$^{1/2}$/cm$^{1/2}$ s
Electric capacitance	C	1 F (farad)	9×10^{11} cm
Electric resistance	R	1Ω (ohm)	$(1/9) \times 10^{-11}$ s/cm
Specific electric conductivity	κ	1 Sm/m (siemens per meter)	9×10^9 s^{-1}
Magnetic flux	Φ	1 Wb (weber)	10^8 Mx (maxwell; or gauss square centimeter)
Magnetic induction	$\boldsymbol{B}$	1 T (tesla)	10^4 G (gauss)
Strength of magnetic field	$\boldsymbol{H}$	1 A/m (ampere per meter)	4×10^{-3} Oe (oersted)
Magnetic polarization (magnetization)	$\boldsymbol{M}$	1 A/m (ampere per meter)	0.25×10^4 G
Inductance	L	1 H (henry)	10^9 cm
Electric constant	ϵ_0	8.85×10^{-12} F/m (farad per meter)	
Magnetic constant	μ_0	1.26×10^{-6} H/m (henry per meter)	

Appendix B
Variation Principle for Continuous Systems

Classical mechanics, whose fundamentals were formulated by Newton, became the first physical scientific theory in the modern understanding. It provided a basis for developments in other fields of theoretical physics, such as electrodynamics, quantum mechanics, statistical physics, and the theory of elementary particles. The mathematical methods and many notions worked out in classical mechanics are widely used in other areas of theoretical physics. The variation principle is one of the most profound ideas introduced by classical mechanics into classical and quantum field theory. Therefore, the limiting transition from mechanics of discrete point masses to continuous system mechanics appears to be a natural way to introduce this principle in the classical field theory.

B.1
Vibrations of an Elastic Medium as the Vibration Limit of Discrete Point Masses

Let us consider a linear chain of point masses m connected by springs of similar stiffness k (Figure B.1). Recall the description of the motion of such a system in classical mechanics where generalized coordinates $q_n(t)$ or deviations of point masses from the equilibrium position are introduced. The kinetic T and potential U energies of the system are given by the following expressions:

$$T = \frac{1}{2}\sum_{n=1}^{N} m\dot{q}_n^2 \,, \quad U = \frac{1}{2}\sum_{n=1}^{N} k(q_{n+1} - q_n)^2 \,. \tag{B1}$$

Let us assume the extreme masses are fixed at equilibrium positions so that $q_1 = q_N = 0$ at any t.

The equations of motion of the system can be derived from the stationary action condition $\delta S = 0$, where

$$S = \int_{t_1}^{t_2} L(q_n, \dot{q}_n)\mathrm{d}t \,,$$

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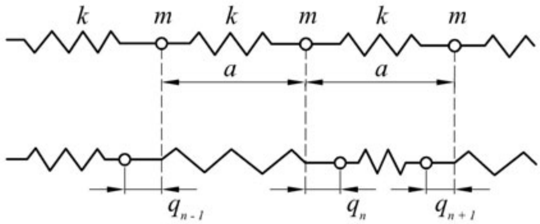

Figure B.1 The vibrations of a chain of point masses.

and

$$L(q_n, \dot{q}_n) = T - U = \frac{1}{2}\sum_{n=1}^{N}\left[m\dot{q}_n^2 - k(q_{n+1} - q_n)^2\right] \tag{B2}$$

is the Lagrangian of the system of interest. The condition $\delta S = 0$ leads to the equations of motion in the Lagrangian form,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_n} - \frac{\partial L}{\partial q_n} = 0\,, \tag{B3}$$

which assume the following form after the substitution of (B2):

$$m\ddot{q}_n - k(q_{n+1} - q_n) + k(q_n - q_{n-1}) = 0\,, \quad n = 2, 3, \ldots, N-1\,. \tag{B4}$$

Let us consider our mechanical system as a discrete model of an elastic rod. For the model to represent adequately what is called the elastic rod in continuous medium mechanics, it is necessary to perform the limiting transition to a continuous system (with the mass number tending toward infinity and the equilibrium distance a between the point masses tending toward zero). We are especially interested in how the Lagrangian function and the variation principle become modified at such a limiting transition. But we shall start the consideration from the limiting transition in the equations of motion.

Let us divide both parts of (B2) by the equilibrium distance of a single link a and rewrite the equality in the form

$$\frac{m}{a}\ddot{q}_n - ka\left(\frac{q_{n+1} - q_n}{a^2}\right) + ka\left(\frac{q_n - q_{n-1}}{a^2}\right) = 0\,. \tag{B5}$$

The ratio $m/a = \mu$ is the mass per unit length of the rod. This mass should be regarded as constant in the case of a limiting transition. The product ka enters the expression for the force acting between the neighboring masses: $F = k(q_{n+1} - q_n) = ka(q_{n+1} - q_n)/a$. The ratio $\xi = (q_{n+1} - q_n)/a$ is the relative elongation of a single link. To recall, Hooke's[1] law states that $F = E\xi$, where ξ is the relative elongation

1) Robert Hooke (1635–1703) was an English scientist, mathematician, physicist, and astronomer, and one of the founders of the Royal Society of London.

and E is Young's[2] modulus (one of the elastic constants of matter). Thus, the product $ka = E$ is Young's modulus and must be considered as constant in the case of a limiting transition.

Let us further introduce the equilibrium coordinate of the nth node: $x_n = na$. For a discrete system, this quantity changes jumpwise; however, $x_n = na$ becomes a continuous variable (Cartesian coordinate) in the limiting transition $a \to 0$, $n \to \infty$. Consideration of a as a small increment of the coordinate x leads to

$$\frac{q_{n+1} - q_n}{a} = \frac{q(x+a) - q(x)}{a} \approx \frac{\partial q(x)}{\partial x} ;$$

similarly

$$\frac{q_n - q_{n-1}}{a} \approx \frac{\partial q(x-a)}{\partial x} .$$

It should be remembered that q also depends on t. Bearing in mind the notation introduced, we find (B5) assumes the form

$$\mu \frac{\partial^2 q}{\partial t^2} = \frac{E}{a} \left[\left(\frac{\partial q}{\partial x} \right)_x - \left(\frac{\partial q}{\partial x} \right)_{x-a} \right] = 0 .$$

Finally, the difference between the first derivatives at adjacent points divided by a is replaced by the second derivative with respect to coordinates, which yields the equation for elastic rod fluctuations:

$$\mu \frac{\partial^2 q}{\partial t^2} - E \frac{\partial^2 q}{\partial x^2} = 0 . \tag{B6}$$

It is a wave equation that can be rewritten in the form

$$\frac{1}{c_l^2} \frac{\partial^2 q}{\partial t^2} - \frac{\partial^2 q}{\partial x^2} = 0 , \tag{B7}$$

where $c_l = \sqrt{E/\mu}$ is the velocity of longitudinal waves.

Let us turn now to Lagrangian (B2) and perform a limiting transition in this function:

$$\begin{aligned} L &= \frac{1}{2} \sum_n \left[\frac{m}{a} \dot{q}_n^2 - ka \left(\frac{q_n - q_{n-1}}{a} \right)^2 \right] \\ &\to \frac{1}{2} \int \left[\mu \left(\frac{\partial q}{\partial t} \right)^2 - E \left(\frac{\partial q}{\partial x} \right)^2 \right] \mathrm{d}x . \end{aligned} \tag{B8}$$

In such a limiting transition, the sum over all particles turns into the integral over the coordinate. The function under the integral sign is called the Lagrangian density or simply the Lagrangian:

$$\mathcal{L} = \frac{1}{2} \left[\mu \left(\frac{\partial q}{\partial t} \right)^2 - E \left(\frac{\partial q}{\partial x} \right)^2 \right] . \tag{B9}$$

2) Thomas Young (1773–1829) was an outstanding English scientist and polymath.

To summarize, we have obtained the following rules for transition from mechanics of material points to continuous system mechanics:

1. The particle number n transforms into coordinate x;
2. The generalized coordinate $q_n(t)$ becomes a function of the coordinate and time: $q(x,t)$ or (in the case of three-dimensional motion) $q(\boldsymbol{r},t)$. The number of generalized coordinates may be arbitrary. By way of example, in the case of three-dimensional motion of a medium, the strain vector has three constituent components, $q_\alpha(\boldsymbol{r},t)$, $\alpha = 1,2,3$. The electromagnetic field is described at each point by four generalized coordinates, the components of 4-potential $A_i(\boldsymbol{r},t)$, $i = 0,1,2,3$, and so on.
3. The Lagrangian function is expressed in the form of the integral over coordinates of the Lagrangian that depends in the example above on the first derivatives of the generalized coordinates (field functions). In the general case, it may depend also on the generalized coordinates $q^i(\boldsymbol{r},t)$ themselves, spatial coordinates $\boldsymbol{r}$, and time t:

$$\mathcal{L} = \mathcal{L}\left(q^i, \frac{\partial q^i}{\partial x_\alpha}, \frac{\partial q^i}{\partial t}, \boldsymbol{r}, t\right) . \tag{B10}$$

   In this case, the equations of motion for the field functions q contain the second-order and lower derivatives with respect to the coordinates and time.
4. Because the Lagrangian function L becomes an integral over the spatial coordinates, action S is expressed in the form of the integral over a four-dimensional manifold:

$$S = \int \mathcal{L}\left(q^i, \frac{\partial q^i}{\partial x_\alpha}, \frac{\partial q^i}{\partial t}, \boldsymbol{r}, t\right) \mathrm{d}^3x\mathrm{d}t . \tag{B11}$$

   We use superscripts in (B10) and (B11). However, in the present case, they do not necessarily label the contravariant components of any 4-vector.

B.2
The Lagrangian Form of Equations of Motion for a Continuous Medium

Let us make clear how the variation principle should be formulated for continuous systems in such a way that the equations of motion could follow from it. We shall resort to the analogy with material point mechanics considered in detail in the previous section. The role of generalized coordinates for a continuous system is played by the field functions $q^i(r,t)$. Index i and spatial coordinates $\boldsymbol{r}$ now label the degrees of freedom of the system that make up a continuum set. When the action is varied, the generalized coordinates $q^i(r,t)$ must get small independent increments $\delta q^i(r,t)$ unrelated to the changes of time and coordinates, similar to what occurs in material point mechanics where variation of $\delta q_n(t)$ is introduced at a fixed moment of time and for a fixed degree of freedom n. The generalized coordinates must have a specified value at the boundary of the integration domain in (B11), that is, at the three-dimensional hypersurface Σ, and their variations become zero:

$$\delta q^i(\boldsymbol{r}, t)\big|_\Sigma = 0\,. \tag{B12}$$

Bearing in mind this observation intended to emphasize the analogy with material point mechanics, we may turn to the formulation of the variation principle for a continuous system acting as described below.

The real evolution of a continuous system in space and time at given values of its generalized coordinates at the boundary of the four-dimensional region proceeds in such a manner that action S has a stationary value, that is, its first variation vanishes:

$$\delta S = 0\,. \tag{B13}$$

Passing to the calculation of the first action variation, we use abridged notation for the derivatives:

$$\frac{\partial q^i}{\partial x_\alpha} \equiv q^i_{,\alpha} \quad (\alpha = 1, 2, 3)\,; \qquad \frac{\partial q^i}{\partial t} \equiv q^i_{,t}\,.$$

This allows the form of the resultant relations to be simplified.

Consider the action as a functional of the generalized coordinates and compute in the first approximation in δq^i the difference

$$\begin{aligned}\delta S &= S[q^i(\boldsymbol{r}, t) + \delta q^i(\boldsymbol{r}, t)] - S[q^i(\boldsymbol{r}, t)] \\ &= \sum_i \int \left\{ \frac{\partial \mathcal{L}}{\partial q^i}\delta q^i + \frac{\partial \mathcal{L}}{\partial q^i_{,\alpha}}\delta q^i_{,\alpha} + \frac{\partial \mathcal{L}}{\partial q^i_{,t}}\delta q^i_{,t} \right\} \mathrm{d}^3x\mathrm{d}t\,.\end{aligned} \tag{B14}$$

This expression contains the explicitly written sum over all values of index i (i.e., over all the components of the field functions) and (in the second term under the integral sign) the sum from 1 to 3 over the Cartesian coordinates x_α, which is not written explicitly.

Taking advantage of the fact that the operation of varying δ is performed at constant $\boldsymbol{r}$ and t and is therefore permutational with respect to differentiation, we write

$$\delta q^i_{,t} = \frac{\partial}{\partial t}\delta q^i\,, \quad \delta q^i_{,\alpha} = \frac{\partial}{\partial x_\alpha}\delta q^i$$

and integrate the respective terms in (B14) by parts:

$$\begin{aligned}\int \frac{\partial \mathcal{L}}{\partial q^i_{,t}}\delta q^i_{,t}\mathrm{d}^3x\mathrm{d}t &= \int \mathrm{d}^3x \frac{\partial \mathcal{L}}{\partial q^i_{,t}}\delta q^i(\boldsymbol{r}, t)\bigg|_\Sigma \\ &\quad - \int \left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial q^i_{,t}}\right)\delta q^i(\boldsymbol{r}, t)\mathrm{d}^3x\mathrm{d}t\,.\end{aligned} \tag{B15}$$

Similarly,

$$\begin{aligned}\int \frac{\partial \mathcal{L}}{\partial q^i_{,\alpha}}\delta q^i_{,\alpha}\mathrm{d}^3x\mathrm{d}t &= \int \mathrm{d}^2S_\alpha\mathrm{d}t \frac{\partial \mathcal{L}}{\partial q^i_{,\alpha}}\delta q^i(\boldsymbol{r}, t)\bigg|_\Sigma \\ &\quad - \int \left(\frac{\mathrm{d}}{\mathrm{d}x_\alpha}\frac{\partial \mathcal{L}}{\partial q^i_{,\alpha}}\right)\delta q^i(\boldsymbol{r}, t)\mathrm{d}^3x\mathrm{d}t\,.\end{aligned} \tag{B16}$$

Here, $d^2 S_\alpha$ is the projection of an element of the two-dimensional surface onto the x_α axis.

The terms must be integrated over the three-dimensional surface and they become zero by virtue of (B12). Taking into account (B14)–(B16), we find that (B13) takes the form

$$\delta S = \int \left\{ \frac{\partial \mathcal{L}}{\partial q^i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial q^i_{,t}} - \frac{d}{dx_\alpha}\frac{\partial \mathcal{L}}{\partial q^i_{,\alpha}} \right\} \delta q^i(\boldsymbol{r}, t) d^3x dt = 0 \,. \tag{B17}$$

Because both the integration domain and variation δq^i are arbitrary, (B17) yields the following equations of motion for a continuous medium in the Lagrangian form:

$$\frac{\partial \mathcal{L}}{\partial q^i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial q^i_{,t}} - \frac{d}{dx_\alpha}\frac{\partial \mathcal{L}}{\partial q^i_{,\alpha}} = 0 \,, \tag{B18}$$

where i assumes the entire set of possible values (summation over α!).[3]

We consider now two examples.

1. Let us take Lagrangian (B9). We find

$$\frac{\partial \mathcal{L}}{\partial q} = 0 \,, \quad \frac{\partial \mathcal{L}}{\partial q_{,t}} = \mu \frac{\partial q(\boldsymbol{r}, t)}{\partial t} \,, \quad \frac{\partial \mathcal{L}}{\partial q_{,x}} = -E\frac{\partial q}{\partial x}$$

and obtain from (B18) the equation of motion that coincides (as could be expected) with (B6).

2. Let us consider a complex scalar field $\psi(\boldsymbol{r}, t) = q^1(\boldsymbol{r}, t) + i q^2(\boldsymbol{r}, t)$ and represent the Lagrangian in the form

$$\mathcal{L} = \frac{i\hbar}{2}\left(\psi \frac{\partial \psi^*}{\partial t} - \psi^* \frac{\partial \psi}{\partial t}\right) + \frac{\hbar^2}{2m}\frac{\partial \psi^*}{\partial x_\alpha}\frac{\partial \psi}{\partial x_\alpha} + U\psi\psi^* \,, \tag{B19}$$

where m and $\hbar$ are constants and $U(\boldsymbol{r}, t)$ is a real-valued function of the coordinates and time. The Lagrangian (B19) and action S are real even though ψ is a complex quantity. This complex quantity is equivalent to two real fields, q^1 and q^2. Instead of regarding δq^1 and δq^2 as independent variations, we more conveniently choose their linear combinations $\delta q^1 + i\delta q^2$ and $\delta q^1 - i\delta q^2$. When the action is varied with respect to ψ^*, (B18) assumes the form

$$\frac{\partial \mathcal{L}}{\partial \psi^*} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \psi^*_{,t}} - \frac{d}{dx_\alpha}\frac{\partial \mathcal{L}}{\partial \psi^*_{,\alpha}} = 0 \,. \tag{B20}$$

3) In (B15)–(B18), we write (as generally accepted) the direct derivatives d/dt and d/dx_α. This means that both the explicit dependence on x_α and t and the dependence on these variables mediated through q^i and their derivatives must be taken into account in the Lagrangian $\mathcal{L}$. However, x_α and t for a continuous system, unlike the situation in material point mechanics, are always independent variables; therefore, for example,

$$\frac{d}{dt}q^i \equiv \frac{\partial q^i(\boldsymbol{r}, t)}{\partial t} \,.$$

Calculating

$$\frac{\partial \mathcal{L}}{\partial \psi^*} = -\frac{i\hbar}{2}\frac{\partial \psi}{\partial t} + U\psi , \quad \frac{\partial \mathcal{L}}{\partial \psi^*_{,t}} = \frac{i\hbar}{2}\psi , \quad \frac{\partial \mathcal{L}}{\partial \psi^*_{,\alpha}} = \frac{\hbar^2}{2m}\frac{\partial \psi}{\partial x_\alpha} ,$$

we find the field equation

$$i\hbar\frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\triangle\psi + U\psi . \qquad \text{(B21)}$$

This is the Schrödinger equation for the wave function of a nonrelativistic particle in the potential field $U(\boldsymbol{r}, t)$, and $\hbar$ and m are the reduced Planck constant and the particle's mass, respectively.

When the Lagrangian method is applied in field theory, the form of the Lagrangian is not derived by means of a limiting transition from the Lagrangian function of a mechanical system as in the case of Lagrangian (B9) because neither the electromagnetic field nor other fields can be reduced to mechanical models. The form of the Lagrangian is usually postulated on the basis of certain general considerations. The reader is referred to Section 4.3 for a detailed discussion of the choice of the Lagrangian of the electromagnetic field.

Appendix C
General Outline of Quantum Theory

This appendix is intended for readers who are already familiar with the basis of the quantum theory. It cannot replace detailed textbooks on quantum mechanics of different levels and styles (e.g., Dirac, 1958; Landau and Lifshitz, 1977; Fock, 1978; Davydov, 1976; Blokhintsev, 1964; Schiff, 1955; Zelevinsky, 2011a,b). Problems related to measurements in quantum physics and a number of questions regarding principles of quantum theory are discussed in books and reviews (Mandelstam, 1972; Blokhintsev, 2010, 1973; Lipkin, 2001; Menskii, 2001, 2007).

C.1
Spectrum of Physical Values and the Wave Function

From experiments it is known that physical quantities q_i ("observables", on Dirac terminology) characterizing a quantum system can have eigenvalues that are defined for a given system. The set of all possible eigenvalues is called the spectrum of the given value q_i. We will use the same symbols for both physical quantity in an abstract sense and its concrete value. We will write the operator with a "wide hat" $\widehat{q}_i$ label. Spectra can be discrete, continuous, and mixed. Discrete values of given quantity can be numbered: $q_i = q_n^i$, where n is an integer number or another discrete *quantum number*.

If the system has s degrees of freedom, it is possible to have *s simultaneously measured* (with the commutative operators) quantities $q = q_1, \dots, q_n$, which can be considered as the *generalized coordinates* of the system. The completeness of eigenvalues (one for each quantity q_i) forms a point of an s-dimensional spectrum. The great number of all such spectral points forms the *configuration space* of the quantum system. It is possible to describe generalized coordinates in different ways. Any pure[1] state of a system is described by the wave function $\psi(q, t)$, generally complex, which depends on the point in the configuration space and on time. The wave function has the sense of probability, that is,

$$w_q(t) = |\psi(q, t)|^2 \tag{C1}$$

1) A description of general (mixed) states is given later.

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represents the probability to observe values $q = (q_1, \dots, q_n)$ (a concrete point of a discrete spectrum) on simultaneous measurement of the set of observables. In this context $\psi(q, t)$ is called the probability amplitude. If a spectrum is continuous, it is possible to calculate the probability of observing the system in a volume element $d^s q$ of configuration space about point q:

$$dw_q(t) = \rho(q, t) d^s q = |\psi(q, t)|^2 d^s q \,, \tag{C2}$$

where $\rho(q, t) = |\psi(q, t)|^2$ is the *probability density* for the spectral points. The distribution of probabilities in intermediate cases is evidently obtained from (C1) and (C2). For calculation of the distribution of one of the coordinates it is necessary to sum (C1) or (C2) for all values of all superfluous coordinates. On summation of (C1) or (C2) over all configuration space, the total probability becomes

$$\sum_q w_q(t) = \sum_q |\psi(q, t)|^2 = \sum_q \psi^*(q, t)\psi(q, t) \,. \tag{C3}$$

The sum over q is understood here in the generalized sense, that is, as the finite sum or a converging series if the spectrum is discrete, and as an integral if the spectrum is continuous. In the case of a converging series or an integral, probability is normalized to unity. If the integral diverges, the normalization is performed on a delta function.

C.2
State Vector

The totality $\{\psi(q, t)\}$ of values $\psi(q, t)$ in all points of configuration space can be considered as a vector $\psi(t) \equiv |\psi(t)\rangle = \{\psi(q, t)\}$ in the functional Hilbert space of states for the quantum system considered ("*state vector*"). In relation to vector $|\psi(t)\rangle$, the wave function at point q can be considered as a component of a state vector "along the q axis" of a Hilbert space. In this case the abstract vector $|\psi(t)\rangle$ of a pure state is given in the q representation as $\{\psi(q, t)\}$.

It is also convenient to consider the totality $\{\psi^*(q, t)\}$ as a vector $\langle\psi(t)| \equiv \psi^\dagger(t)$, conjugate to vector $|\psi(t)\rangle$. In the terminology of Dirac, vectors $|\psi(t)\rangle$ are called as ket vectors and $\langle\psi(t)|$ are called bra vectors. For vectors $|\psi(t)\rangle$ and $\langle\varphi(t)|$ the action of scalar multiplication is defined. The parameter S (complex in the general case) is related to these to vectors by the rule

$$S = (\varphi, \psi) \equiv \varphi^\dagger \psi \equiv \langle\varphi|\psi\rangle = \sum_q \varphi^*(q)\psi(q) \,. \tag{C4}$$

Common designations for the scalar product result. In the first term in (C4), complex conjugation, the component of the first factor is meant, but it is not marked obviously. The scalar product possesses obvious properties:

$$\begin{gathered} \langle\varphi|\psi\rangle = \langle\psi|\varphi\rangle^* \,, \\ \langle\varphi|c_1\psi_1 + c_2\psi_2\rangle = c_1\langle\varphi|\psi_1\rangle + c_2\langle\varphi|\psi_2\rangle, \quad \langle\varphi\varphi_1|\psi\rangle + c_2^*\langle\varphi_2|\psi\rangle \,. \end{gathered} \tag{C5}$$

The concept of a state vector is especially evident in the case of a discrete spectrum. For example, for a particle with spin 1/2, the spin coordinate $s_z = \pm 1/2$ can have only two values: the column contains the spin state vectors

$$|\psi\rangle = \begin{pmatrix} \psi\left(\frac{1}{2}\right) \\ \psi\left(-\frac{1}{2}\right) \end{pmatrix} \equiv \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} ,$$

and the conjugate vector is given by the rows $\langle\psi| = (\psi^*(1/2), \psi^*(-1/2)) \equiv (c_1^*, c_2^*)$. The scalar product which is expressed by the normalization condition is formed by the rule of matrix multiplication of a row line and a column: $\langle\psi|\psi\rangle = c_1^* c_1 + c_2^* c_2 = 1$.

The wave functions (state vectors) allow us to calculate the physical values measured in experiments (probabilities, currents, observable values and mean values). Therefore, the wave functions should possess mathematical properties which provide consistency to quantum theory. Such general properties are the *limitation* of wave functions at all points of configuration space, their *unambiguity* (for each point there corresponds one value of the function), and their *continuity* together with their first derivatives. Here it is a question of continuous dynamic variables, for example, about Cartesian or spherical coordinates. The requirements formulated need to be considered in the solution of the differential equations defining eigenvectors and eigenvalues of physical quantities, or the equations which describe the evolution of states in time.

C.3
Indistinguishability of Identical Particles

Identical particles have identical internal quantum numbers (charges, masses, spins, etc.). Such particles are completely indiscernible in view of the wave character of their movement and probabilistic nature of the wave function (as a probability amplitude). From these physical properties – indiscernibility of identical particles – important mathematical properties follow for state vectors of systems of identical particles. All particles, elementary and compound, are of two classes – fermions[2)] (particles with half-integer spins) and bosons[3)] (particles with integer spins). The state vector of a fermion system is antisymmetric, that is, there is a change of sign on a change of the arguments concerning two identical particles, i and k:

$$\psi(q_1, \ldots, q_i, \ldots, q_k, \ldots, t) = -\psi(q_1, \ldots, q_k, \ldots, q_i, \ldots, t) . \tag{C6}$$

Here, q_i means a set of four dynamic variables (coordinates of configuration space) concerning particle i.

The state vectors of the boson system are symmetric, that is, do not change on a change of the arguments of identical particles:

$$\psi(q_1, \ldots, q_i, \ldots, q_k, \ldots, t) = \psi(q_1, \ldots, q_k, \ldots, q_i, \ldots, t) . \tag{C7}$$

2) Named after the outstanding Italian physicist and Nobel Prize recipient Enrico Fermi (1901–1954).
3) Named after the Indian physicist Satyendra Nath Bose (1894–1974).

The physical sense of transposition of arguments consists in a change in the position of two particles (in the corresponding configuration space). In view of the indistinguishability of particles, such an operation does not involve a change of a state of the system, and leads to the necessity of permutable symmetry of wave functions (C6) and (C7).

If interaction between particles can be disregarded, each particle can be represented by a certain set of four quantum numbers (a quantum state). In this case, the property of antisymmetry (C6) of the wave function leads to the principle that there can be no more than one fermion in a given quantum state (the *Pauli exclusion principle*[4]). The number of bosons in a given quantum state is unlimited.

C.4
Operators and Their Properties

The primary goals of quantum theory are the calculation of spectra observed for concrete quantum systems, their probabilities in various states, their average values, and dispersions. For the solution of these problems, the observable g is connected with some linear operators $\widehat{g}$ acting in the Hilbert space of the system. If the given observable is used, in the classical description of the system its operator is expressed through the operators of canonical variables (coordinates and momenta). The relation between the observable's operator and the operators of canonical variables is the same as that between the corresponding classical quantities (*correspondence principle*)[5].

For spectrum $\{g\}$, g can be found by the solution of the problem in which the eigenvalues and corresponding eigenfunctions (eigenvectors) ψ_k are found:

$$\widehat{g}\psi_k = g_k\psi_k \ , \tag{C8}$$

The role of boundary conditions is played by the requirements of a limitation, a continuity, and an unambiguity, as already discussed. An important property of eigenstate ψ_k is that measurement of g in this state (by means of a real device) will result in the exact value g_k being recorded. For this value, the uncertainty $\Delta g = 0$.

If it is possible to measure the physical values $q_1, q_2, \dots, q_s$ simultaneously with great accuracy, their operators $\widehat{q}_i$ in pairs commute among themselves:

$$\left[\widehat{q}_i, \widehat{q}_j\right] \equiv \widehat{q}_i\widehat{q}_j - \widehat{q}_j\widehat{q}_i = 0 \ . \tag{C9}$$

The physical values and their operators form a *full set* if all of them are independent, and any another value and its operator are expressed through the values forming a full set. The full set of physical values defines the state of the system with the greatest possible completeness in quantum mechanics.

4) Wolfgang Pauli (1900–1958) was an outstanding German physicist-theorist, Nobel Prize winner, working in Germany, Switzerland and USA.

5) Generally, operators do not commute; therefore, in their construction except for correspondence principle addition reasons, for example, self-conjugation is required, as discussed further later.

The common eigenvectors of a system of commuting operators

$$\widehat{q}_j \psi_q = q_j \psi_q \,, \quad j = 1, \ldots, s \,, \quad q = (q_1, \ldots, s) \tag{C10}$$

form the total orthogonal and normalized basis. They satisfy the condition

$$\left(\psi'_{q}, \psi_q\right) \equiv \langle q'_1 \ldots q'_s | q_1 \ldots q_s \rangle = \delta_{q_1 q'_1} \ldots \delta_{q_s q'_s} \,. \tag{C11}$$

In (C11), the quantity $\delta_{qq'}$ is the Kronecker symbol $\delta_{q_n q'_n}$ if the spectrum is discrete and the Dirac delta function if the spectrum is continuous. Any state vector Φ of a system can be expanded in a generalized sum of the kind

$$\Phi = \sum_q C_q \psi_q \equiv \sum_q C_q |q\rangle \tag{C12}$$

(*the principle of superposition of states* in quantum mechanics). Coefficients $C_q \equiv \varphi(q_1 \ldots q_s)$ in superposition (C12) represent the wave function of state Φ in the q representation. They define the distribution of probabilities over q according to (C1) and (C2) and can be found by means of (C11) as scalar products:

$$C_q \equiv \varphi(q) = (\psi_q, \Phi) = \langle q | \Phi \rangle \,. \tag{C13}$$

The mean value of quantity F in state Φ is

$$\overline{F}_\Phi \equiv \langle F \rangle_\Phi = (\Phi, \widehat{F}\Phi) \equiv \langle \Phi | \widehat{F} | \Phi \rangle \,, \tag{C14}$$

where $\widehat{F}$ is the operator of observable F. If there is a q representation with a continuous spectrum,

$$\overline{F}_\Phi = \int \varphi^*(q) \widehat{F}_q \varphi(q) \mathrm{d}q \,, \tag{C15}$$

where $\widehat{F}_q$ and $\varphi(q)$ are the operator and the wave function in the q representation. In any arbitrary g representation we can write the expectation value in the form of the generalized sum:

$$\overline{F}_\Phi = \sum_{gg'} C^*_{g'} F_{g'g} C_g \,, \tag{C16}$$

where $C_g = \langle g | \Phi \rangle$ is a coefficient in the type (C12) superposition. Here set $\{C_g\} = |\Phi\rangle$ should be understood as a vector column, and set $\{C^*_{g'}\} = \langle \Phi |$ should be understood as a vector row. Operator $\widehat{F}$ in the g representation is the square matrix $\widehat{F} = \{F_{g'g}\}$ with matrix elements

$$F_{g'g} = \langle g' | \widehat{F} | g \rangle \,. \tag{C17}$$

Formula (C17) is written in the general (abstract) form. On transition to the continuous q representation it will become

$$F_{g'g} = \int \varphi^*_{g'}(q) \widehat{F}_q \varphi_g(q) \mathrm{d}q \,. \tag{C18}$$

The linear operator $\widehat{F}^\dagger$ is called a Hermitian[6] conjugate to the operator $\widehat{F}$ if for any state vectors φ and ψ

$$(\varphi, \widehat{F}\psi) = (\widehat{F}^\dagger\varphi, \psi) \quad \text{or} \quad \langle\varphi|\widehat{F}|\psi\rangle = \langle\widehat{F}^\dagger\varphi|\psi\rangle \,. \tag{C19}$$

is valid. The operator is called *self-conjugate (Hermitian)* if $\widehat{F}^\dagger = \widehat{F}$. This condition has the forms

$$F_{gg'} = F^*_{g'g} \quad \text{and} \quad \int \varphi^*(q)\widehat{F}_q\psi(q)\mathrm{d}q = \int (\widehat{F}_q\varphi(q))^*\psi(q)\mathrm{d}q \tag{C20}$$

for operators in matrix form and in the continuous q representation, respectively. The major property of Hermitian operators is that their eigenvalues, and also the average values of any physical quantity in any quantum state, are real. Therefore all physical values (observed) should be represented by Hermitian operators.

The measure of the uncertainty of the physical value in any state is defined as

$$\Delta F = \sqrt{\overline{(F - \overline{F})^2}} = \sqrt{\overline{(F^2)} - (\overline{F})^2} \,, \tag{C21}$$

where averaging is done with the wave function of state considered. If Hermitian operators $\widehat{F}$ and $\widehat{G}$ of two physical values do not commute, $[\widehat{F}, \widehat{G}] = i\widehat{M}$, where $\widehat{M}$ is also a Hermitian operator, then in any quantum state the following *uncertainty relation* holds for these values:

$$\Delta F \Delta G \geq \frac{1}{2}|\overline{M}| \,. \tag{C22}$$

Here averaging was done with the wave function of the state considered. In particular, for Cartesian coordinates and canonical momenta conjugate to them in any representation Heisenberg's[7] permutable relations and uncertainty relations

$$\begin{aligned} &\left[\widehat{x}_\alpha, \widehat{p}_\beta\right] = i\hbar\delta_{\alpha\beta} \,, \qquad &&\left[\widehat{x}_\alpha, \widehat{x}_\beta\right] = 0 \,, \\ &\left[\widehat{p}_\alpha, \widehat{p}_\beta\right] = 0 \,, &&\Delta x_\alpha \Delta p_\beta \geq \frac{\hbar}{2}\delta_{\alpha\beta} \,, \end{aligned} \tag{C23}$$

hold, where $\hbar$ is the reduced Planck constant. From (C23) it follows that the component of the momentum and the associated coordinate have no simultaneously certain values and cannot be measured precisely. However, it is possible to measure three coordinates or three components of the momentum simultaneously and precisely.

The projection operator onto state $\psi_{q_0} \equiv |q_0\rangle$ can be written in the designation of Dirac as

$$\widehat{P}_{q_0} = |q_0\rangle\langle q_0| \,. \tag{C24}$$

6) Named after the French mathematician Charles Hermite (1822–1901).
7) Werner Heisenberg (1901–1976) was a German theoretical physicist, one of founders of quantum mechanics, and a Nobel Prize recipient.

Operating on superposition (C12), this operator transforms a full state vector to a component, corresponding to ort $|q_0\rangle$:

$$\widehat{P}_{q_0}\Phi = \sum_q C_q|q_0\rangle\langle q_0|q\rangle = C_{q_0}|q_0\rangle$$

The condition of completeness for a system of eigenfunctions can be written through the projection operator in the form

$$\sum_q \widehat{P}_q \equiv \sum_q |q\rangle\langle q| = \widehat{1}\,, \tag{C25}$$

where $\widehat{1}$ is the operator of identical transformation (Kronecker's delta symbol or the Dirac delta function). Special cases of condition (C25) are obtained with equalities (1.239) and (1.242). Equality (C25) leads to a convenient representation of any operator

$$\widehat{F} = \widehat{1}\cdot\widehat{F}\cdot|\widehat{1} = \sum_{qq'} |q\rangle F_{qq'}\langle q'| \tag{C26}$$

and allows us to pass easily from one representation to another.

Let some quantum state be described by wave function $\Phi(x)$ in the x representation. Having made the decomposition $\Phi(x) = \sum_q \varphi(q)\psi_q(x)$, we obtain the wave function $\varphi(q) = \int \psi_q^*(x)\Phi(x)\mathrm{d}x \equiv \langle q|\Phi\rangle$ of the initial state in the q representation. Finally, having written

$$\chi(Q) = \langle Q|\widehat{1}\cdot\Phi\rangle = \sum_q \langle Q|q\rangle\langle q|\Phi\rangle = \widehat{U}\varphi(q)\,,$$

we obtain the relation between the wave functions in the q and Q representations. Here $\widehat{U} = \{\langle Q|q\rangle\}$ is the operator of *unitary* transformation from the q representation to the Q representation. The operator of the inverse transformation can be written as $\widehat{U}^{-1} = \{\langle q|Q\rangle\}$, so $\varphi(q) = \widehat{U}^{-1}\chi(Q)$. The operator $\widehat{U}$ can be considered as a matrix whose rows are numbered points of the observable's eigenvalues in Q space, and whose columns are points in q space. In operator $\widehat{U}^{-1}$ rows and columns interchanged their position, and $\langle q|Q\rangle = \langle Q|q\rangle^*$. Therefore, the operator of unitary transformation possesses important properties,

$$\widehat{U}^+ = \widehat{U}^{-1}\,, \quad \widehat{U}\widehat{U}^+ = \widehat{U}^+\widehat{U} = \widehat{1}\,, \tag{C27}$$

which preserve of the normalization of wave functions. Operators of physical values will similarly be transformed on the transition $q \to Q$, for example,

$$\widehat{F}_Q = \widehat{U}\widehat{F}_q\widehat{U}^{-1} \tag{C28}$$

The *development of a quantum system in time* is described in one of three equivalent representations (not mixed up by a choice of the generalized coordinates!).

1. In the *Schrödinger representation* operators of the observable do not depend on time t (except for possible dependence on the time of the operator $\widehat{V}(t)$ interaction of the system with an external field). The dependence on time is contained in the wave function and is defined in the dynamic *Schrödinger equation*[8)]

$$\widehat{\mathcal{H}}\Psi(t) = i\hbar\frac{\partial\Psi(t)}{\partial t} \tag{C29}$$

with the initial condition $\Psi(t_0) = \Psi_0$. Here, Ψ_0 is an initial state vector and $\widehat{\mathcal{H}}$ is the Hamiltonian operator of the system, whose form is in most cases defined by the correspondence principle.
2. In the *Heisenberg representation* dependence on time is transferred completely to operators, and the state vector does not depend on time:

$$\Phi(t) = \exp\left(\frac{i}{\hbar}\widehat{\mathcal{H}}t\right)\Psi(t) = \Psi_0 \tag{C30}$$

Here and afterward it is supposed that the Hamiltonian does not depend obviously on time and $t_0 = 0$. Operators of value F in the Schrödinger (S) and Heisenberg (H) representations are connected by the relation

$$\widehat{F}_H(t) = \exp\left(\frac{i}{\hbar}\widehat{\mathcal{H}}t\right)\widehat{F}_S\exp\left(-\frac{i}{\hbar}\widehat{\mathcal{H}}t\right) = \widehat{U}^+(t)\widehat{F}_S\widehat{U}(t)\,. \tag{C31}$$

Here

$$\widehat{U}(t) = \exp\left(-\frac{i}{\hbar}\widehat{\mathcal{H}}t\right)\,,\quad \widehat{U}^+(t) = \widehat{U}^{-1}(t)\,, \tag{C32}$$

is a unitary transformation operator. The Hamiltonian of the system is the same in the Schrödinger and Heisenberg representations: $\widehat{\mathcal{H}}_H = \widehat{\mathcal{H}}_S = \widehat{\mathcal{H}}$.
3. There is also the interaction representation (see Example 6.10 and Section 7.7).

All previous formulas are true in any representation.

The operator $\widehat{\dot{F}}$ of a derivative with regard to time of physical value F is an operator which in any state satisfies the relation for averages:

$$\frac{\mathrm{d}\overline{F}}{\mathrm{d}t} = \langle\psi|\widehat{\dot{F}}|\psi\rangle\,. \tag{C33}$$

In the Schrödinger representation

$$\widehat{\dot{F}}_S = \frac{\partial\widehat{F}_S}{\partial t} + \frac{i}{\hbar}\left[\widehat{\mathcal{H}}, \widehat{F}_S\right]\,, \tag{C34}$$

The first term on the right is a derivative with regard to time, on which there is an obvious dependence. In the Heisenberg representation

$$\widehat{\dot{F}}_H = \widehat{U}^+(t)\frac{\partial\widehat{F}_S}{\partial t}\widehat{U}(t) + \frac{i}{\hbar}\left[\widehat{\mathcal{H}}, \widehat{F}_H\right]\,. \tag{C35}$$

8) Erwin Schrödinger (1887–1961) was an outstanding Austrian theoretical physicist, one of founders of quantum mechanics, and a Nobel Prize recipient.

Movement integrals in quantum mechanics are observables whose average values in any state of the system do not depend on time, that is, operators of derivatives with regard to time are equal to zero. If the operator of the observable does not depend on time, a condition for its conservation according to (C34) and (C35) is commutation of its operator with the Hamiltonian: $[\widehat{F}_{H,S}, \widehat{\mathcal{H}}] = 0$.

A special role among all states of the system is played by *stationary states*. They are solutions of the *stationary Schrödinger equation*,

$$\widehat{\mathcal{H}}\psi_E = E\psi_E \,, \tag{C36}$$

and exist only if the Hamiltonian system does not depend on time. Vectors of stationary states in the Schrödinger representation depend on time according to the law

$$\Psi_E(x,t) = \psi_E(x)\exp\left(-\frac{i}{\hbar}Et\right) \,, \tag{C37}$$

where x is a set of the generalized coordinates. In the Heisenberg representation state vectors do not depend on time. Average values and probability distributions in stationary states do not depend on time for any values whose operators $\widehat{F}_S$ do not depend on time.

Mixed states of a quantum system cannot be described by wave functions (unlike pure states, which are connected by state vectors and which were considered above). Mixed states arise if the quantum system interacted in the past (or continues to interact) with another system having degrees of freedom. In this case the system cannot be described by a wave function, which depends only on the generalized coordinates of the system considered. The mixed state can be understood as a statistical (not coherent) mix of pure states, and information on the relative phases of such states is absent. Mixed states are described by means of *density operators (matrices)*

$$\widehat{\rho}(t) = \sum_{qq'} |q\rangle\rho_{qq'}\langle q'| \,, \tag{C38}$$

where q, as before, designates a full set of the observed values. The diagonal elements of a density matrix represent the probabilities of observing eigenvalues q,

$$w_q = \rho_{qq} = \langle q|\widehat{\rho}|q\rangle \,, \tag{C39}$$

and satisfy the normalization condition

$$\sum_q \rho_{qq} \equiv \mathrm{Tr}\widehat{\rho} = 1 \,. \tag{C40}$$

The average value of the observable is calculated as

$$\overline{F} = \mathrm{Tr}(\widehat{\rho}\widehat{F}) = \mathrm{Tr}(\widehat{F}\widehat{\rho}) = \sum_{qq'} F_{qq'}\rho_{q'q} \,. \tag{C41}$$

With use of the trace it is possible to perform a cyclic shift of factors. The density operator can be applied and to the description of pure states $|\psi>$. In this case the density operator $\widehat{\rho} = |\psi><\psi|$ possesses the characteristic property $\widehat{\rho}^2 = \widehat{\rho}$.

The dynamics of a system which is in a mixed state and interacts with an external field, but not with other dynamic systems, is described by the equation of Liouville[9] and Schrödinger (it is also called the Landau–von Neumann[10] equation):

$$i\hbar\frac{\partial\widehat{\rho}}{\partial t} = [\widehat{\mathcal{H}}, \widehat{\rho}] \,. \tag{C42}$$

Here the derivative depends on time and we have a Hamiltonian system.

An important example of the mixed state is given by the conditions of thermodynamic balance. The density operator for the fixed number of particles in the system represents the canonical *distribution of Gibbs*:[11]

$$\widehat{\rho} = \frac{1}{Z}\exp\left(-\frac{\widehat{\mathcal{H}}}{T}\right) \,. \tag{C43}$$

The probability of observing a steady state ψ_n with energy E_n is given by the diagonal elements of this operator:

$$w_n = \rho_{nn} = \frac{1}{Z}\exp\left(-\frac{E_n}{T}\right) \,, \tag{C44}$$

where T is the absolute temperature in energy units, and

$$Z = \mathrm{Tr}\exp\left(-\frac{\widehat{\mathcal{H}}}{T}\right) = \sum_n e^{-E_n/T} \tag{C45}$$

is the statistical sum (over all quantum states of the system). Through the statistical sum, the thermodynamic characteristics of system, in particular, the Helmholtz free energy F and the internal energy $\mathcal{E}$ are expressed:

$$\mathcal{E} : F = -T\ln Z \,, \quad \mathcal{E} = T^2\frac{\partial \ln Z}{\partial T} \,. \tag{C46}$$

In turn through them it is possible to calculate other thermodynamic potentials, including entropy S, pressure $\mathcal{P}$, thermal capacity C_V, and the Gibbs free energy Φ:

$$\begin{aligned} S &= -\left(\frac{\partial F}{\partial T}\right)_V \,, & \mathcal{P} &= \left(\frac{\partial F}{\partial V}\right)_T \,, \\ C_V &= T\left(\frac{\partial S}{\partial T}\right)_V = \left(\frac{\partial \mathcal{E}}{\partial T}\right)_V \,, & \Phi &= F + \mathcal{P}\mathcal{V} \,. \end{aligned} \tag{C46$'$}$$

9) Joseph Liouville (1809–1882) was a French mathematician.

10) John (Janos) von Neumann (1903–1957) was an American mathematician and physicist who developed the mathematical basis of quantum mechanics and many other things.

11) Josiah Willard Gibbs (1839–1903) was an outstanding American physicist and chemist, and a founder of classical statistical mechanics and thermodynamics.

If in the system there are particles of different kinds and the numbers of the particles change, the equilibrium state of such a system is described by the density operator (the *big canonical distribution of Gibbs*):

$$\hat{\rho} = \frac{1}{\mathcal{Z}} \exp\left(\frac{\sum_a \mu_a \hat{N}_a - \hat{\mathcal{H}}}{T}\right) , \tag{C47}$$

where $\hat{N}_a$ is the operator of the particle number of a given kind of particle, and μ_a is the chemical potential of the corresponding components. The probabilities of states are

$$w_{\{N_a\}n} = \frac{1}{\mathcal{Z}} \exp\left(\frac{\sum_a \mu_a N_a - E_{\{N_a\}n}}{T}\right) , \tag{C48}$$

where the generalized sum over states is

$$\begin{aligned} \mathcal{Z} &= \operatorname{Tr} \exp\left(\frac{\sum_a \mu_a \hat{N}_a - \hat{\mathcal{H}}}{T}\right) \\ &= \sum_{\{N_a\}} \exp\left(\frac{\sum_a \mu_a N_a}{T}\right) \sum_n \exp\left(-\frac{E_{\{N_a\}n}}{T}\right) . \end{aligned} \tag{C49}$$

Through this the generalized free energy is expressed,

$$\Omega = -T \ln \mathcal{Z} , \tag{C50}$$

with the help of which, in particular, it is possible to calculate the average number of particles:

$$\overline{N_a} = -\frac{\partial \Omega}{\partial \mu_a} . \tag{C51}$$

With this we will finish our general review of the basic concepts of quantum theory.

Example C.1

Find the eigenvectors of the stationary states and the energy spectrum of a linear harmonic oscillator represented by Hamiltonian (2.160). Construct the non-Hermitian operators $\hat{c}$ and $\hat{c}^\dagger$ corresponding to complex amplitudes q and q^*. Find the rules of action for these operators on vectors of stationary states and find out their physical sense.

Solution. By the correspondence principle we build from the classical Hamiltonian function (2.160) the quantum Hamiltonian (operator)

$$\hat{\mathcal{H}} = \frac{1}{2}\left(\hat{P}^2 + \omega^2 \hat{Q}^2\right) . \tag{C52}$$

(indices $\boldsymbol{k}$ and σ are omitted). We choose the coordinate representation:

$$\widehat{Q} = Q = \sqrt{\frac{\hbar}{\omega}}\,\xi\,, \quad \widehat{P} = -i\hbar\frac{\mathrm{d}}{\mathrm{d}Q} = -i\sqrt{\hbar\omega}\frac{\mathrm{d}}{\mathrm{d}\xi}\,,$$

where $\xi = \sqrt{\omega/\hbar}\,Q$ is the dimensionless coordinate. These operators of the coordinate and momentum satisfy the permutable Heisenberg relations (C23):

$$[\widehat{Q}, \widehat{P}] = i\hbar\,. \tag{C53}$$

We will define also the operators corresponding to dimensionless complex amplitudes. According to (2.157) and (6.12),

$$\begin{aligned}\widehat{c} &= \sqrt{\frac{\omega}{2\hbar}}\left(\widehat{Q} + \frac{i}{\omega}\widehat{P}\right) = \frac{1}{\sqrt{2}}\left(\xi + \frac{\mathrm{d}}{\mathrm{d}\xi}\right)\,,\\ \widehat{c}^{\dagger} &= \sqrt{\frac{\omega}{2\hbar}}\left(\widehat{Q} - \frac{i}{\omega}\widehat{P}\right) = \frac{1}{\sqrt{2}}\left(\xi - \frac{\mathrm{d}}{\mathrm{d}\xi}\right)\,.\end{aligned} \tag{C54}$$

These operators are non-Hermitian, $\widehat{c} \neq \widehat{c}^{\dagger}$. But through them Hamiltonian (C52) is simply expressed. Having found the product

$$\widehat{c}^{\dagger}\widehat{c} = \frac{1}{2\hbar\omega}\left(\widehat{P}^2 + \omega^2\widehat{Q}^2\right) + \frac{i}{2\hbar}(\widehat{Q}\widehat{P} - \widehat{P}\widehat{Q})$$

and using (C53), we obtain from (C52)

$$\widehat{\mathcal{H}} = \hbar\omega\left(\widehat{c}^{\dagger}\widehat{c} + \frac{1}{2}\right)\,. \tag{C55}$$

Multiplying

$$\widehat{c}\widehat{c}^{\dagger} = \frac{1}{2\hbar\omega}\left(\widehat{P}^2 + \omega^2\widehat{Q}^2\right) - \frac{i}{2\hbar}(\widehat{Q}\widehat{P} - \widehat{P}\widehat{Q})\,,$$

we find the commutation relation for operators $\widehat{c}$:

$$[\widehat{c}, \widehat{c}^{\dagger}] = 1\,, \quad [\widehat{c}, \widehat{c}] = 0\,, \quad [\widehat{c}^{\dagger}, \widehat{c}^{\dagger}] = 0\,. \tag{C56}$$

We note that $\widehat{c}$ and $\widehat{c}^{\dagger}$ can be write in a form equivalent to (C54):

$$\widehat{c} = \frac{1}{\sqrt{2}}e^{-\xi^2/2}\frac{\mathrm{d}}{\mathrm{d}\xi}e^{\xi^2/2}\,, \quad \widehat{c}^{\dagger} = -\frac{1}{\sqrt{2}}e^{\xi^2/2}\frac{\mathrm{d}}{\mathrm{d}\xi}e^{-\xi^2/2}\,. \tag{C57}$$

Now we will move directly to the solution of the stationary Schrödinger equation. We will write it down through the dimensionless Hamiltonian, having used $\widehat{\mathcal{E}} = \widehat{\mathcal{H}}/\hbar\omega - 1/2 = \widehat{c}^{\dagger}\widehat{c}$:

$$\widehat{\mathcal{E}}\Phi_{\mathcal{E}} = \mathcal{E}\Phi_{\mathcal{E}}\,. \tag{C58}$$

Let us consider also the other eigenvector, $\widehat{c}\Phi_{\mathcal{E}}$. We have with the help (C56) $\widehat{\mathcal{E}}\widehat{c}\Phi_{\mathcal{E}} = (\widehat{c}\widehat{c}^{\dagger}\widehat{c} - \widehat{c})\Phi_{\mathcal{E}} = (\mathcal{E} - 1)\widehat{c}\Phi_{\mathcal{E}}$. From this equality it follows that $\widehat{c}\Phi_{\mathcal{E}} =$

$\alpha\Phi_{\mathcal{E}-1}$ is also an eigenvector corresponding to eigenvalue $\mathcal{E}-1$, and α is a normalization constant. Similarly, we find $\widehat{\mathcal{E}}\widehat{c}^\dagger\Phi_{\mathcal{E}} = (\mathcal{E}+1)\widehat{a}^\dagger\Phi_{\mathcal{E}}$, that is, $\widehat{c}^\dagger\Phi_{\mathcal{E}} = \beta\Phi_{\mathcal{E}+1}$ is the eigenvector corresponding to eigenvalue $\mathcal{E}+1$. Thus, the action of operators $\widehat{c}$ and $\widehat{c}^\dagger$ on the wave function of some state leads to a state with energy one unit lower or higher than the energy of the initial state (in the dimensionless scale). Therefore, these operators can be called lowering and raising operators.

As the average value of initial Hamiltonian (C52)

$$\overline{\mathcal{H}} = \langle\Phi|\widehat{\mathcal{H}}|\Phi\rangle = \int\limits_{-\infty}^{\infty}\left(\frac{\hbar^2}{2}\left|\frac{\mathrm{d}\Phi}{\mathrm{d}Q}\right| + \omega^2 Q^2|\Phi|^2\right)\mathrm{d}Q > 0$$

is positive in any state, the spectrum of the operator $\widehat{\mathcal{E}}$ is limited from below. We will designate its least eigenvalue through $\mathcal{E}_0$. The state with the least energy is called the ground state. As states with wave function $\Phi_{\mathcal{E}_0-1}$ do not exist, the action of the lowering operator $\widehat{c}$ on the wave function of the ground state should give zero:

$$\widehat{c}\Phi_{\mathcal{E}_0} = \frac{1}{\sqrt{2}}\left(\xi + \frac{\mathrm{d}}{\mathrm{d}\xi}\right)\Phi_{\mathcal{E}_0} = 0\,. \tag{C59}$$

This differential equation of the first order has a unique solution defined to within a normalization constant

$$\Phi_0(\xi) = A_0 e^{-\xi^2/2}\,. \tag{C60}$$

We have fiven the wave function of the ground state the index 0.

Applying the raising operator $\widehat{c}^\dagger$ to the wave function Φ_0 consistently once, twice, and so on, we obtain the wave functions of the exited states of the oscillator which correspond to dimensionless energies $\mathcal{E}_1 = 1,\ \mathcal{E}_2 = 2, \ldots, \mathcal{E}_n = n \ldots$ Their number is infinite. We will be convinced by a method of contradiction that other than the sequence found, the system has no other levels. Let there be some value $\mathcal{E}'$ which does not enter into the sequence found. Applying the lowering operator to the corresponding wave function a number of times, we will eventually obtain for the wave function of the ground state (C59) again, from which the former sequence of states again follows. As a result, coming back to dimensional energy, we can write down the spectrum of a harmonic oscillator in the form

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right)\,, \quad \text{where} \quad n = 0, 1, \ldots \tag{C61}$$

We note that the minimum energy of quantum oscillator has a nonzero value, $E_0 = \hbar\omega/2 > 0$, whereas the classical oscillator can be immobile at the bottom of a potential well, having $E_0 = 0$. Comparing spectrum (C61) to the Hamiltonian $\widehat{\mathcal{E}}$ representation (C55), we come to the conclusion that the Hermitian operator

$$\widehat{n} = \widehat{c}^\dagger\widehat{c} \tag{C62}$$

has integer eigenvalues $n = 0, 1, \dots$ which are the number of energy levels of the ground and the excited states of the oscillator.

Let us address now how to calculate wave functions. From the normalization condition $\int_{-\infty}^{\infty} |\Phi_0(Q)|^2 \mathrm{d}Q = 1$, substituting in the integral the function (C60) and using thePoisson integral $\int_{-\infty}^{\infty} e^{-\alpha x^2} \mathrm{d}x = \sqrt{\pi/\alpha}$, we find the normalization constant $|A_0| = (\omega/\pi\hbar)^{1/4}$. It is defined to within a phase factor as the modulus equal to unity. Further, we define the constants α_n and β_n given above for lowering and raising operators. Having written down the scalar product, $\langle \widehat{a}\Phi_n | \widehat{a}\Phi_n \rangle = \langle \alpha_n \Phi_n | \alpha_n \Phi_n \rangle = |\alpha_n|^2$, we transform it, using the definition of the conjugated operator and (C62):

$$\langle \widehat{a}\Phi_n | \widehat{a}\Phi_n \rangle = \langle \Phi_n | \widehat{a}^\dagger \widehat{a}\Phi \rangle = \langle \Phi_n | \widehat{n}\Phi_n \rangle = n \, .$$

From comparison of two representations of a scalar square it is found that $|\alpha_n| = \sqrt{n}$ and choosing a phase factor in the simplest form, we accept $\alpha_n = \sqrt{n}$. In the same way it is found that $\beta_n = \sqrt{n+1}$. Now it is possible to write down the important relations

$$\widehat{a}^\dagger \Phi_n = \sqrt{n+1}\,\Phi_{n+1} \, , \quad \widehat{a}\Phi_n = \sqrt{n}\,\Phi_{n-1} \, , \tag{C63}$$

which have many applications, and to express Φ_n through Φ_0:

$$\Phi_n(\xi) = \frac{1}{\sqrt{n!}} \left(\widehat{a}^\dagger\right)^n \Phi_0(\xi) \, . \tag{C64}$$

We can obtain another representation of the wave function similar to the Rodrigues formula for Legendre polynomials (see Section 1.3), having applied expression (C57) for $\widehat{a}^\dagger$:

$$\Phi_n(\xi) = A_n e^{-\xi^2/2} H_n(\xi) \, , \quad \text{where} \quad H_n(\xi) = (-1)^n e^{\xi^2} \frac{\mathrm{d}^n e^{-\xi^2}}{\mathrm{d}\xi^n} \tag{C65}$$

is the Hermite polynomial of order n, and $A_n = \omega^{1/4}/\sqrt{2^n n! \sqrt{\pi}}$ is a normalization constant. □

C.5
Some Useful Formulas of Operator Algebra

$$[\widehat{f}\widehat{g}, \widehat{h}] = [\widehat{f}, \widehat{h}]\widehat{g} + \widehat{f}[\widehat{g}, \widehat{h}] \, ; \quad [\widehat{f}, \widehat{g}\widehat{h}] = [\widehat{f}, \widehat{g}]\widehat{h} + \widehat{g}[\widehat{f}, \widehat{h}] \, . \tag{C66}$$

$$e^{\alpha\widehat{A}} \widehat{B} e^{-\alpha\widehat{A}} = \widehat{B} + \alpha[\widehat{A}, \widehat{B}] + \frac{\alpha^2}{2!}[\widehat{A}, [\widehat{A}, \widehat{B}]] + \dots \tag{C67}$$

In particular, if $[\widehat{A}, \widehat{B}] = C$ (nonoperator value), we have

$$e^{\alpha\widehat{A}} \widehat{B} e^{-\alpha\widehat{A}} = \widehat{B} + C\alpha \, . \tag{C67}$$

For lowering and raising operators $\widehat{c}$ and $\widehat{c}^\dagger$, and also for $\widehat{n} = \widehat{c}\widehat{c}^\dagger$ we obtain

$$e^{\alpha\widehat{n}}\begin{bmatrix}\widehat{c}\\ \widehat{c}^\dagger\end{bmatrix}e^{-\alpha\widehat{n}} = \begin{bmatrix}\widehat{c}e^{-\alpha}\\ \widehat{c}^\dagger e^{\alpha}\end{bmatrix}; \tag{C68}$$

$$e^{-z\widehat{c}^\dagger + z^*\widehat{c}}\begin{bmatrix}\widehat{c}\\ \widehat{c}^\dagger\end{bmatrix}e^{z\widehat{c}^\dagger - z^*\widehat{c}} = \begin{bmatrix}\widehat{c}+z\\ \widehat{c}^\dagger + z^*\end{bmatrix}. \tag{C69}$$

$$\left(e^{\alpha\widehat{A}}\widehat{B}e^{-\alpha\widehat{A}}\right)^n = e^{\alpha\widehat{A}}\widehat{B}^n e^{-\alpha\widehat{A}}; \tag{C70}$$

$$e^{\alpha\widehat{A}}f(\widehat{B})e^{-\alpha\widehat{A}} = f\left(e^{\alpha\widehat{A}}\widehat{B}e^{-\alpha\widehat{A}}\right), \tag{C71}$$

where the function $f(\widehat{B})$ may be expanded in a power series over $\widehat{B}$.

If $f(\widehat{c}, \widehat{c}^\dagger)$ can be expanded in a power series over $\widehat{c}$ and $\widehat{c}^\dagger$, the derivatives have the form

$$\frac{\partial f}{\partial \widehat{c}} = -[\widehat{c}^\dagger, f(\widehat{c}, \widehat{c}^\dagger)], \quad \frac{\partial f}{\partial \widehat{c}^\dagger} = [\widehat{c}, f(\widehat{c}, \widehat{c}^\dagger)]. \tag{C72}$$

If $[\widehat{A}, [\widehat{A}, \widehat{B}]] = [\widehat{B}, [\widehat{A}, \widehat{B}]] = 0$, the equality

$$e^{\alpha(\widehat{A}+\widehat{B})} = e^{\alpha\widehat{A}}e^{\alpha\widehat{B}}e^{-\alpha^2[\widehat{A},\widehat{B}]/2} = e^{\alpha\widehat{B}}e^{\alpha\widehat{A}}e^{\alpha^2[\widehat{A},\widehat{B}]/2} \tag{C73}$$

is valid.

C.6
Wave Functions of the Hydrogen-Like Atom (the Lowest Levels)

$$\begin{aligned}\varphi_{nlm_l} &= R_{nl}(r)Y_{lm_l}(\vartheta, \alpha);\\ \varphi_{100} &= \frac{1}{\sqrt{\pi a^3}}e^{-r/a_{\mathrm{B}}};\\ R_{10} &= \frac{2}{a^{3/2}}e^{-r/a};\end{aligned} \tag{C74}$$

$$\begin{aligned}R_{21} &= \frac{r}{2\sqrt{6}a^{5/2}}e^{-r/2a};\\ R_{20} &= \frac{1}{(2a)^{3/2}}\left(2-\frac{r}{a}\right)e^{-r/2a};\\ R_{31} &= \frac{8r}{27\sqrt{6}a^{5/2}}\left(1-\frac{r}{6a}\right)e^{-r/3a};\end{aligned} \tag{C75}$$

Here, $a = a_{\mathrm{B}}/Z$, where $a_{\mathrm{B}} = \hbar^2/me^2$ is the Bohr radius.

$$\begin{aligned}&Y_{00}(\vartheta, \alpha) = \frac{1}{\sqrt{4\pi}}; \quad Y_{10}(\vartheta, \alpha) = \sqrt{\frac{3}{4\pi}}\cos\vartheta;\\ &Y_{1,\pm1}(\vartheta, \alpha) = \pm\sqrt{\frac{3}{8\pi}}\sin\vartheta\, e^{\pm i\alpha}.\end{aligned} \tag{C76}$$

C.6.1
Addition of Angular Moments

$$Y_{l_1 m_1}(\vartheta,\varphi)\, Y_{l_2 m_2}(\vartheta,\varphi) = \sum_{L,M} \sqrt{\frac{(2l_1+1)(2l_2+1)}{4\pi(2L+1)}}\, C^{L0}_{l_1 0 l_2 0}\, C^{LM}_{l_1 m_1 l_2 m_2}\, Y_{LM}(\vartheta,\varphi)\,. \tag{C77}$$

The wave function of the system consisting of two subsystems with fixed moments j_1 and j_2 (decomposition of Clebsch–Gordan) is

$$\psi_{jm}(1,2) = \sum_{m_1 m_2} C^{jm}_{j_1 \nu_1 j_2 m_2}\, \psi_{j_1 m_1}(1)\, \psi_{j_2 m_2}(2)\,. \tag{C78}$$

The Clebsch–Gordan coefficients $C^{jm}_{j_1 \nu_1 j_2 m_2}$ are distinct from zero only when there is an inequality of a triangle, $j_1 + j_2 \geq j \geq |j_1 - j_2|$, and the rules of addition of projections hold, $m_1 + m_2 = m$. They submit to a variety of symmetry relations, in particular, $C^{jm}_{j_1 \nu_1 j_2 m_2} = (-1)^{j_1+j_2+j}\, C^{j,-m}_{j_1 - m_1 j_2 - m_2}$

C.6.2
Spin Operators and Wave Functions of Fermions ($s = 1/2$)

The dimensionless operator of the spin mechanical moment is expressed through two-column Pauli's matrices

$$\widehat{\sigma}_x = \widehat{\sigma}_x^\dagger = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \widehat{\sigma}_y = \widehat{\sigma}_y^\dagger = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$
$$\widehat{\sigma}_z = \widehat{\sigma}_z^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \widehat{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{C79}$$

as

$$\widehat{\boldsymbol{s}} = \frac{1}{2}\widehat{\boldsymbol{\sigma}}\,, \quad \widehat{\boldsymbol{s}}^2 = \frac{1}{4}\widehat{\boldsymbol{\sigma}}^2 = \frac{3}{4}\widehat{1}\,. \tag{C80}$$

Pauli's matrices together with a unit matrix $\widehat{1}$ form a full set of two-column and two-row matrices. They submit to the relations

$$\widehat{\sigma}_\mu \widehat{\sigma}_\nu + \widehat{\sigma}_\nu \widehat{\sigma}_\mu = 2\delta_{\mu\nu}\,, \quad [\widehat{\boldsymbol{\sigma}} \times \widehat{\boldsymbol{\sigma}}] = 2i\widehat{\boldsymbol{\sigma}}\,, \quad \widehat{\sigma}_\mu \widehat{\sigma}_\nu = \delta_{\mu\nu} + i e_{\mu\nu\kappa} \widehat{\sigma}_\kappa\,. \tag{C81}$$

In these equalities $\delta_{\mu\nu}$ is multiplied by the unit matrix not written in an explicit form. Operators $\widehat{\boldsymbol{s}}^2$ and $\widehat{s}_z$ have a diagonal form, and their matrix elements distinct from zero represent eigenvalues $\boldsymbol{s}^2 = 3/4$ and $s_z \equiv m = \pm 1/2$.

Spin wave functions (spinors) have two components and are normalized to unity:

$$\varphi = \begin{pmatrix} a \\ b \end{pmatrix}, \quad \varphi^\dagger \varphi = |a|^2 + |b|^2 = 1\,. \tag{C82}$$

The eigenfunctions of operators $\widehat{\boldsymbol{s}}^2$ and $\widehat{s}_z$ have the form

$$w_{1/2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \quad w_{-1/2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} , \tag{C83}$$

The operator of spin projection in the direction of a unit vector $\boldsymbol{n}$ is constructed under the laws of Euclidean geometry:

$$\widehat{s}_n = \frac{1}{2}(\boldsymbol{n}\cdot\widehat{\boldsymbol{\sigma}}) = \frac{1}{2}\begin{pmatrix} \cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta \end{pmatrix} , \tag{C84}$$

where θ and ϕ are angles defining the direction of $\boldsymbol{n}$. The eigenvalues of $\widehat{s}_n$ operators are $\pm 1/2$ and the eigenspinors are

$$w_{1/2} = \begin{pmatrix} e^{-i\phi/2}\cos(\theta/2) \\ e^{i\phi/2}\sin(\theta/2) \end{pmatrix} , \quad w_{-1/2} = \begin{pmatrix} -e^{-i\phi/2}\sin(\theta/2) \\ e^{i\phi/2}\cos(\theta/2) \end{pmatrix} . \tag{C85}$$

Let us remind ourselves that any wave function in quantum mechanics is defined to within a phase factor $e^{i\alpha}$, where α is a real constant. If the Oz axis is superposed with the direction $\boldsymbol{n}$, then $\theta = 0$ and spinors (C85) become (C83).

The operator $\widehat{\boldsymbol{S}}$ with the spin of two fermions is defined as the vector sum $\widehat{\boldsymbol{S}} = \widehat{\boldsymbol{s}}_1 + \widehat{\boldsymbol{s}}_2$ of their spin operators. By the rule of addition of the moments, $s_1 = s_2 = 1/2$, two values of the spin quantum number are possible: $S = 0$ and $S = 1$. The wave functions Φ_{SM} for a spin triplet are

$$\begin{aligned} \Phi_{11} &= w_{1/2}(1)w_{1/2}(2) , \\ \Phi_{10} &= \frac{1}{\sqrt{2}}[w_{1/2}(1)w_{-1/2}(2) + w_{-1/2}(1)w_{1/2}(2)] , \\ \Phi_{1-1} &= w_{-1/2}(1)w_{-1/2}(2) . \end{aligned} \tag{C86}$$

The wave function for a spin singlet is

$$\Phi_{00} = \frac{1}{\sqrt{2}}[w_{1/2}(1)w_{-1/2}(2) - w_{-1/2}(1)w_{1/2}(2)] . \tag{C87}$$

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