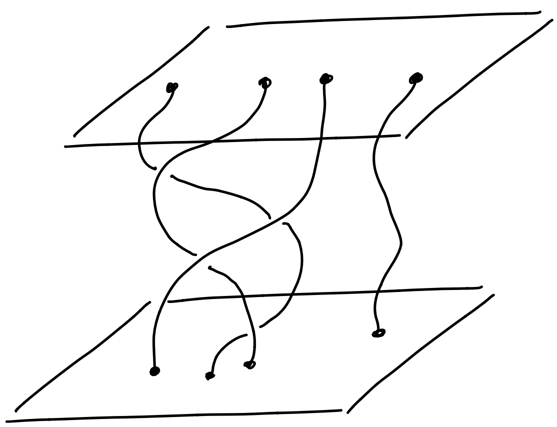
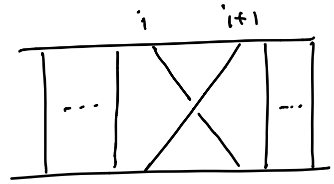
§7. KZ equation and representations of braid groups

Recall:

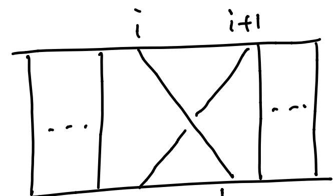
Braid group B_n on n strands has generators:

σ_i , $i = 1, \dots, n-1$





σ_i



σ_i^{-1}

satisfying the relations

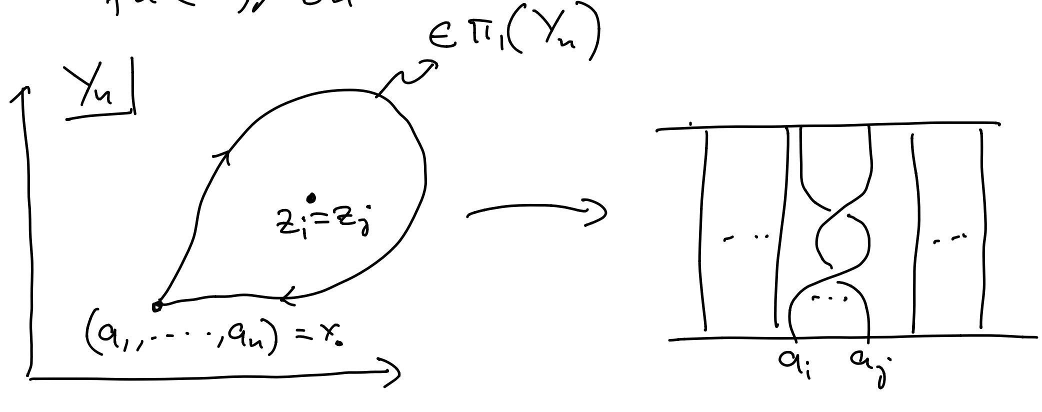
- $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$, $i = 1, \dots, n-2$
- $\sigma_i \sigma_j = \sigma_j \sigma_i$, $|i-j| > 1$

Want to relate this braid group to

$$\mathrm{Conf}_n(\mathbb{C}) = \mathbb{C}^n \setminus \bigcup_{i<j} H_{ij}$$

where H_{ij} denotes the hyperplane $z_i = z_j$ in $\mathbb{C}^n$. Note that symmetric group S_n acts on $\mathrm{Conf}_n(\mathbb{C})$ by permution of coordinates.

Then we have $B_n = \pi_1(Y_n)$ where
$Y_n = \mathrm{Conf}_n(\mathbb{C})/S_n$.



We have the exact sequence

$$1 \longrightarrow \underset{\underset{\displaystyle \pi_1(\mathrm{Conf}_n(\mathbb{C}), x_0)}{\shortparallel}}{P_n} \longrightarrow B_n \longrightarrow S_n \longrightarrow 1$$

P_n is also called "pure braid group".
We consider logarithmic differential 1-forms

$$\omega_{ij} = d\log(z_i - z_j) = \frac{dz_i - dz_j}{z_i - z_j}, \quad i \neq j$$

defined on $\mathrm{Conf}_n(\mathbb{C})$. They satisfy

$$\omega_{ij} \wedge \omega_{jk} + \omega_{jk} \wedge \omega_{ik} + \omega_{ik} \wedge \omega_{ij} = 0, \quad i < j < k,$$

called "Arnold relations", generators of cohomology ring $H^*(\mathrm{Conf}_n(\mathbb{C}), \mathbb{Z})$.

Recall KZ equation:

Fix finite dimensional complex semisimple Lie algebra $\mathfrak{g}$ together with representations

$$\rho_j : \mathfrak{g} \longrightarrow \mathrm{End}(V_j), \quad 1 \leq j \leq n.$$

Denote by $\{I_\mu\}$ orthonormal basis of $\mathfrak{g}$ with respect to Cartan-Killing form and set

$$\Omega = \sum_\mu I_\mu \otimes I_\mu$$

For example, for $\mathfrak{g} = sl_2(\mathbb{C})$,

$$\Omega = \frac{1}{2} H \otimes H + E \otimes F + F \otimes E$$

The element $C = \sum_\mu I_\mu I_\mu$ in the universal enveloping algebra $U(\mathfrak{g})$ is called "Casimir elem.".

We have

$$\Omega = \frac{1}{2}(\Delta C - C \otimes 1 - 1 \otimes C)$$

where $\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ is comultiplication (e.g. $\Delta(I_\mu I_\mu) = 2 I_\mu \otimes I_\mu + I_\mu I_\mu \otimes 1 + 1 \otimes I_\mu I_\mu$)

Let $\phi : V_1 \otimes V_2 \otimes \cdots \otimes V_n \to \mathbb{C}$ be a multilinear form. We denote by $\Omega^{(ij)}\phi$ the multi-linear form

$$(\Omega^{(ij)}\phi)(v_1 \otimes \cdots \otimes v_n)$$

$$= \sum_\mu \phi(v_1 \otimes \cdots \otimes \rho_i(I_\mu) v_i \otimes \cdots \otimes \rho_j(I_\mu) v_j \otimes \cdots \otimes v_n)$$

for $v_1 \otimes \cdots \otimes v_n \in V_1 \otimes V_2 \otimes \cdots \otimes V_n$. Then the KZ equation is given by

$$\frac{\partial \Phi}{\partial z_i} = \frac{1}{K} \sum_{j, j \neq i} \frac{\Omega^{(ij)} \Phi}{z_i - z_j} \qquad (*)$$

where K is a non-zero complex parameter and $\Phi(z_1, \cdots, z_n)$ is defined over $\mathrm{Conf}_n(\mathbb{C})$ with values in $\mathrm{Hom}_{\mathbb{C}}(V_1 \otimes V_2 \otimes \cdots \otimes V_n, \mathbb{C})$.

Now, we put

$$\omega = \frac{1}{K} \sum_{1 \leq i \leq j \leq n} \Omega^{(ij)} \omega_{ij}$$

$\longrightarrow$ (*) becomes $d\Phi = \omega \Phi$.

<u>Lemma 1:</u>

The above $\Omega^{(ij)}$, $1 \leq i \neq j \leq n$, satisfy the following relations:

1. $\Omega^{(ij)} = \Omega^{(ji)}$,
2. $[\Omega^{(ij)} + \Omega^{(jk)}, \Omega^{(ik)}] = 0$, i, j, k distinct.
3. $[\Omega^{(ij)}, \Omega^{(kl)}] = 0$, i, j, k, l distinct.

(exercise)

Combining with Arnold relation, we arrive at:

<u>Lemma 2:</u> $\omega \wedge \omega = 0$

<u>Proof:</u>

$$\omega \wedge \omega = \frac{1}{K^2} \sum_{i<j,\, k<l} [\Omega^{(ij)}, \Omega^{(kl)}] \omega_{ij} \wedge \omega_{kl}$$

$$\xrightarrow{\text{Arnold}} \sum_{i<j,\, k<l} [\Omega^{(ij)}, \Omega^{(kl)}] \omega_{ij} \wedge \omega_{kl}$$

$$= \sum_{i<j<k} \Big([\Omega^{(ij)} + \Omega^{(jk)}, \Omega^{(ik)}] \omega_{ij} \wedge \omega_{ik} + [\Omega^{(ij)} + \Omega^{(ik)}, \Omega^{(jk)}] \omega_{ij} \wedge \omega_{jk} \Big)$$

$$+ \sum_{\{i,j\} \cap \{k,l\} = 0} [\Omega^{(ij)}, \Omega^{(kl)}] \omega_{ij} \wedge \omega_{kl}$$

$= 0$ by Lemma 1 $\square$

Let E be a trivial vector bundle over $\mathrm{Conf}_n(\mathbb{C})$ with fiber $(V_1 \otimes V_2 \otimes \cdots \otimes V_n^*)$

$$= \mathrm{Hom}_{\mathbb{C}}(V_1 \otimes V_2 \otimes \cdots \otimes V_n, \mathbb{C})$$

Regard ω as a 1-form on $\mathrm{Conf}_n(\mathbb{C})$ with values in $\mathrm{End}(V_1^* \otimes V_2^* \otimes \cdots \otimes V_n^*)$. Define connection on E by $\nabla = d - \omega$.

$\longrightarrow$ flat since $d\omega + \omega \wedge \omega = 0$

$\longrightarrow$ horizontal sections are solutions of KZ eq.

Holonomy:

Let γ be a loop in $\mathrm{Conf}_n(\mathbb{C})$ with base point x_0. Then system of solutions $(\Phi_1, \dots, \Phi_m)$ transforms along γ as

$$(\Phi_1, \dots, \Phi_m)\,\theta(\gamma), \quad m = \dim V_1 \times \cdots \times \dim V_n$$

by a matrix $\theta(\gamma)$ only depending on the homotopy class of γ since ∇ is flat connection.

$\longrightarrow \theta : P_n \longrightarrow GL(V_1^* \otimes V_2^* \otimes \cdots \otimes V_n^*)$

with parameter k: "monodromy representation" of KZ equation. We have

$$\theta(\sigma\tau) = \theta(\sigma)\,\theta(\tau) \quad \forall\, \sigma, \tau \in P_n.$$

Thus we have arrived at the following.

Proposition 1:

For any complex semisimple Lie algebra $\mathfrak{g}$ and its representations $\rho_j : \mathfrak{g} \to \mathrm{End}(V_j), 1 \leq j \leq n$, the holonomy of the KZ connection ∇ gives linear representation of the pure braid group with a parameter k.

In the case $V_1 = \cdots = V_n = V$, the symmetric group S_n acts diagonally on the total space $\mathrm{Conf}_n(\mathbb{C}) \times (V^{\otimes n})^*$, where action of S_n on $(V^{\otimes n})^*$ is given by

$$(\phi \cdot \sigma)(v_1 \otimes \cdots \otimes v_n) = \phi(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)})$$

for $\phi \in (V^{\otimes n})^*$, $\sigma \in S_n$ and $v_j \in V_j$, $1 \leq j \leq n$.

$\longrightarrow$ ∇ descends to a connection on quotient space $F = \mathrm{Conf}_n(\mathbb{C}) \times_{S_n} (V^*)^{\otimes n}$ with holonomy given by braid group:

$$\theta : B_n \longrightarrow GL((V^*)^{\otimes n})$$

Let us return to the situation of the space of conformal blocks for the Riemann sphere for $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ at level k (see § 4).

Take four distinct points on $\mathbb{CP}^1$: p_1, p_2, p_3 and $p_4 = \infty$. Fix a global coordinate function z and set $z(p_j) = z_j$, $j = 1, 2, 3$.

Consider now the space of conformal blocks
$\mathcal{H}(p_1, p_2, p_3, p_4 ; \lambda_1, \lambda_2, \lambda_3, \lambda_4^*)$

$$\hookrightarrow \mathrm{Hom}_{\mathfrak{g}}(V_{\lambda_1} \otimes V_{\lambda_2} \otimes V_{\lambda_3} \otimes V_{\lambda_4}^*, \mathbb{C})$$

(see Lemma in §4)

$\longrightarrow$ conformal block bundle $\mathcal{E}$ over $\mathrm{Conf}_3(\mathbb{C})$ admits flat KZ-connection ∇ with $\kappa = K+2$

introduce coordinates $\zeta_1 = z_2 - z_1$, $\zeta_2 = z_3 - z_1$ and perform coordinate transformation

$$\zeta_1 = u_1 u_2 , \quad \zeta_2 = u_2$$

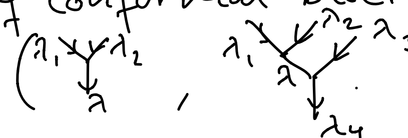
$$\longrightarrow \omega = \frac{1}{\kappa}\left(\frac{\Omega^{(12)}}{u_1} du_1 + \frac{\Omega^{(12)} + \Omega^{(13)} + \Omega^{(23)}}{u_2} du_2 + \omega_1 \right) \qquad (**)$$

where ω_1 is hol. 1-form around $u_1 = u_2 = 0$.

We have

(a) $\mathrm{Res}_{u_1=0}\, \omega = \frac{1}{\kappa} \Omega^{(12)}$,

(b) $\mathrm{Res}_{u_2=0}\, \omega = \frac{1}{\kappa}\left(\Omega^{(12)} + \Omega^{(13)} + \Omega^{(23)} \right)$

Basis of conformal blocks is given by labelled trees ( ...) denoted by $\{p_\lambda\}$

recall: $\sum_{1 \le i < j \le n} \Omega^{(ij)} \psi_0 = -(k+2) \sum_{j=1}^{n} \Delta_{\lambda_j} \psi_0 \quad (\Delta_{\lambda_j^*} = -\Delta_{\lambda_j})$

$\Rightarrow$ (a) and (b) are simultaneously diagonalized for the basis $\{p_\lambda\}$ with eigenvalues $\Delta_\lambda - \Delta_{\lambda_1} - \Delta_{\lambda_2}$ and $\Delta_{\lambda_4} - \Delta_{\lambda_1} - \Delta_{\lambda_2} - \Delta_{\lambda_3}$ respectively.

Proposition 2:

A basis of the space of horizontal sections of the conformal block bundle $\mathcal{E}$ is written around $u_1 = u_2 = 0$ as

$$u_1^{\Delta_\lambda - \Delta_{\lambda_1} - \Delta_{\lambda_2}} u_2^{\Delta_{\lambda_4} - \Delta_{\lambda_1} - \Delta_{\lambda_2} - \Delta_{\lambda_3}} h_\lambda(u_1, u_2) p_\lambda$$

for any λ such that each triple $(\lambda_1, \lambda_2, \lambda)$ and $(\lambda, \lambda_3, \lambda_4)$ satisfies the quantum Clebsch-Gordan condition at level K. Here $h_\lambda(u_1, u_2)$ is a single valued hol. function around $u_1 = u_2 = 0$.

Proof:

Consider vertex operators

$$\Psi_{\lambda_1 \lambda_2}^{\lambda}(\zeta_1) : H_{\lambda_1} \otimes H_{\lambda_2} \longrightarrow \overline{H_\lambda},$$

$$\Psi_{\lambda \lambda_3}^{\lambda_4}(\zeta_2) : H_\lambda \otimes H_{\lambda_3} \longrightarrow \overline{H_{\lambda_4}},$$

The composition $\Psi_{\lambda \lambda_3}^{\lambda_4}(\Psi_{\lambda_1 \lambda_2}^{\lambda}(\zeta_1) \otimes id_{H_{\lambda_3}})$ defined in the region $|\zeta_2| > |\zeta_1| > 0$ is written as

$$u_1^{\Delta_\lambda - \Delta_{\lambda_1} - \Delta_{\lambda_2}} u_2^{\Delta_{\lambda_4} - \Delta_{\lambda_1} - \Delta_{\lambda_2} - \Delta_{\lambda_3}} h_\lambda(u_1, u_2) p_\lambda$$

Since a horizontal section of $\mathcal{E}$ satisfies the KZ equation $\nabla \Psi_0 = (d - \omega)\Psi_0 = 0$, it follows from the form of ω given in $(**)$ that $h_\lambda(u_1, u_2)$ is holomorphic.

□