6d (2,0)-theories and ADE singularities

$$6d\ (2,0) \xrightarrow{S^1_R} 5d\ \mathcal{N}=2 \text{ SYM with gauge group } G$$

VM contains scalars ϕ^I, $I=1,\dots,5$

Coulomb branch $\longrightarrow$

$$\langle \phi^I \rangle = \sum_{i=1}^{r_{\mathfrak{g}}} h_i \varphi_i^I , \quad F = \sum_{i=1}^{r_{\mathfrak{g}}} h_i f_i$$

Recall:

- commutation relations:

$$[h_i, h_j] = 0, \quad [e_{+j}, e_{-j}] = \delta_{ij} h_j , \quad [h_i, e_{\pm j}] = \pm C_{ij} e_{\pm j}$$

Cartan matrix

- effective action:

$$\mathcal{L}^{(5)}_{\text{Coulomb}} = -\frac{1}{2g^2} \Omega_{ij} \left(f_i \wedge * f_j + \sum_{I=1}^{5} \partial_\mu \varphi_i^I \partial^\mu \varphi_j^I \right) + (\text{Fermions}) + \cdots$$

where $\Omega_{ij} = \mathrm{Tr}_{\mathfrak{g}}(h_i h_j) = \langle h_i, h_j \rangle_{\mathfrak{g}}$

6d abelian tensor multiplet:

- Φ^I $(I=1,\dots,5)$ in $\mathbf{5}$ of $SO(5)_R$
- self-dual 3-form $H = *H$
- Weyl fermions

$6d \to 5d$

Take $\Phi_i^I \to \frac{1}{2\pi R}\varphi_i^I$,

$$H_i \to \frac{1}{2\pi R}\left(f_i \wedge dx^5 + *^{(5)} f_i\right)$$

where $x^5 \sim x^5 + 2\pi R \to$ circle S^1_R

$*^{(5)}$ is 5d Hodge

$\to H_i = * H_i$

$\to$ obtain 6d action from 5d uplift:

$$-\frac{\pi R}{g^2}\Omega_{ij}\left(H_i \wedge * H_j + \sum_{I=1}^{5} \partial_\mu \Phi_i^I \partial^\mu \Phi_i^I\right)$$

$$+ \text{(Fermions)} \subset \mathcal{L}_{\text{tensor}}$$

Kinetic terms determine 6d Dirac pairing

$$dH_i = q_i \, \delta_{\Sigma_2} \iff q_i = \int_{\Sigma_3} H_i,$$

q_i: string charge; Σ_2: string world-sheet

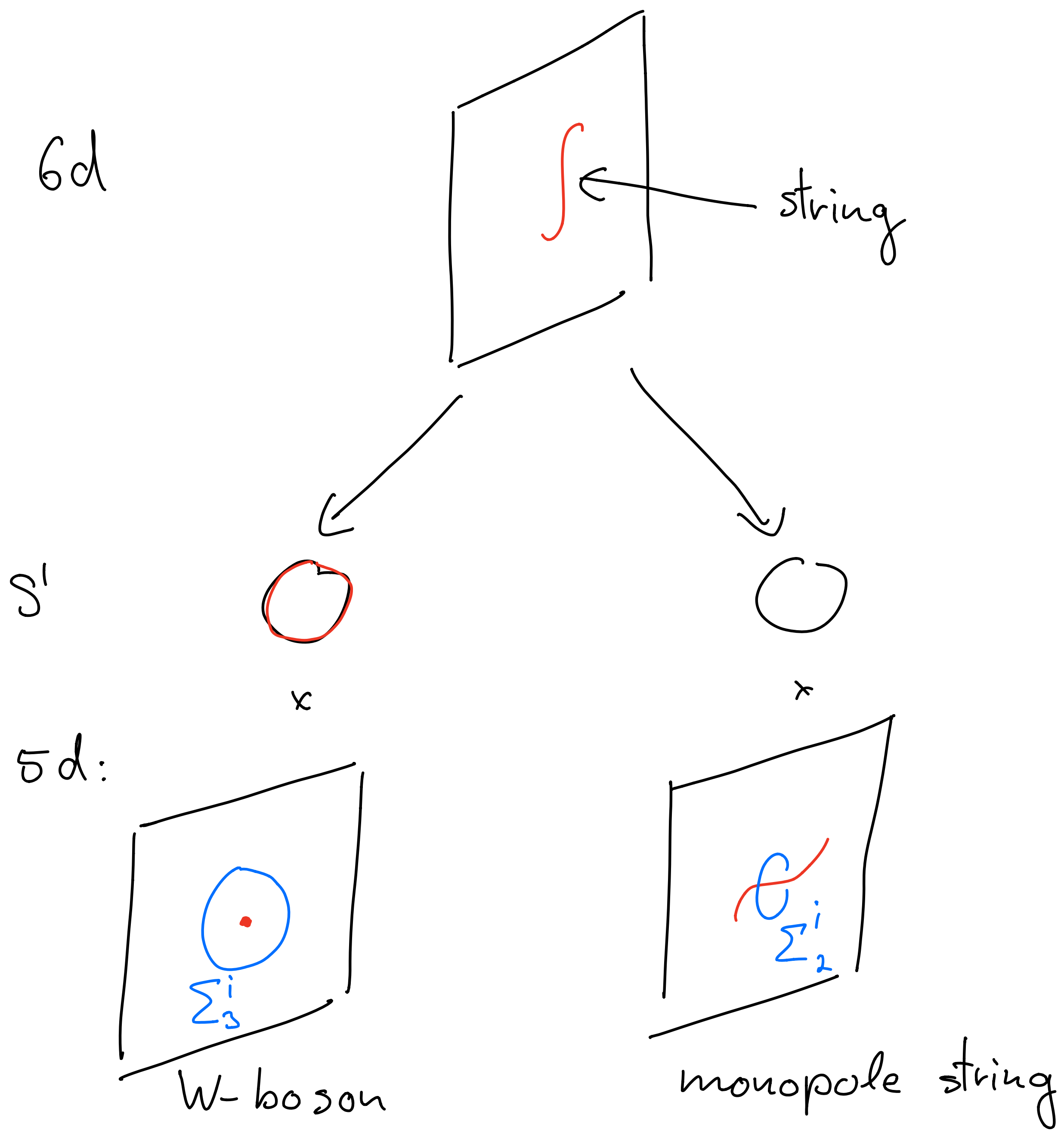
where Σ_2 is linked by Σ_3

$\to$ integer-valued Dirac-pairing between 2 strings:

$$\frac{R}{g^2}\Omega_{ij}\, q_i q_j' \in \mathbb{Z}$$

$\xrightarrow{S^1_R}$ 5d: correspond to BPS states on Coulomb branch

- electrically charged W-bosons correspond to roots $\alpha \in \Delta_{\mathfrak{g}}$
- magnetically charged monopole strings correspond to coroots h_α



$\rightarrow$ constraints on 5d theory:

Consider W-boson corresponding to α_i

Then $[h_j, e_{\pm i}] = \pm C_{ij} e_{\pm i}$

$\rightarrow$ have charge C_{ij} with respect to A_j:

(a) $$(e_i)_j = C_{ij} = \frac{\Omega_{jk}}{g^2} \int_{\Sigma_3^i} * f_k$$

similarly, magnetic charges are given by

(b) $$(m_i)_j = \delta_{ij} = \frac{1}{2\pi} \int_{\Sigma_2^i} f_j$$

Obtain (a) and (b) by integrating 3-form flux H_i over Σ_3 and $\Sigma_2^i \times S_R^1$ respectively

$$\rightarrow (q_i)_j = 2\pi \delta_{ij}, \quad C_{ij} = \frac{4\pi^2 R}{g^2} \Omega_{ij}$$

$$\int_{\Sigma_2^i \times S_R^1} \frac{1}{2\pi R} \left(f_j \wedge dx^5 + *^{(5)} f_j \right) = \int_{\Sigma_2^i} f_j$$

$$\int_{\Sigma_3^i} \frac{1}{2\pi R} \left(f_j \wedge dx^5 + *^{(5)} f_j \right) = \frac{1}{2\pi R} \int_{\Sigma_3^i} *^{(5)} f_j$$

$= (q_i)_j = 2\pi \delta_{ij}$

$$\Rightarrow \int_{\Sigma_3^i} *^{(5)} f_j = 4\pi^2 R \, \delta_{ij}$$

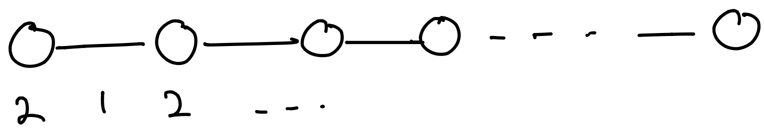
Since $\Omega_{ij} = \mathrm{Tr}_{\mathfrak{g}}(h_i h_j)$ is symmetric

$\longrightarrow$ C_{ij} is symmetric as well

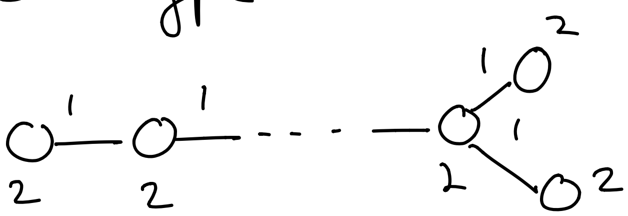
$\longrightarrow$ $\mathfrak{g}$ is simply laced

ADE classification:

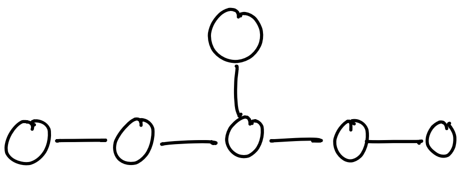
- A-type



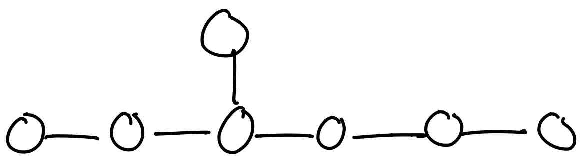
- D-type



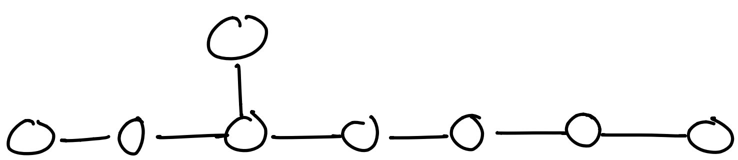
- E-type



(E_6)



(E_7)



(E_8)

Relation to ADE singularities

Type IIB string theory

$$\Big\downarrow \ \text{ADE singularity } \mathbb{C}^2/\Gamma_{\mathfrak{g}}$$

6d (2,0) theory of type $\mathfrak{g}$

What is $\Gamma_{\mathfrak{g}}$?

1) need to preserve supersymmetry

$\to$ Γ leaves $dz_1 \wedge dz_2$ (hol. 2-form) on $\mathbb{C}^2$ invariant

$\to$ Γ discrete subgroup of $SU(2)$

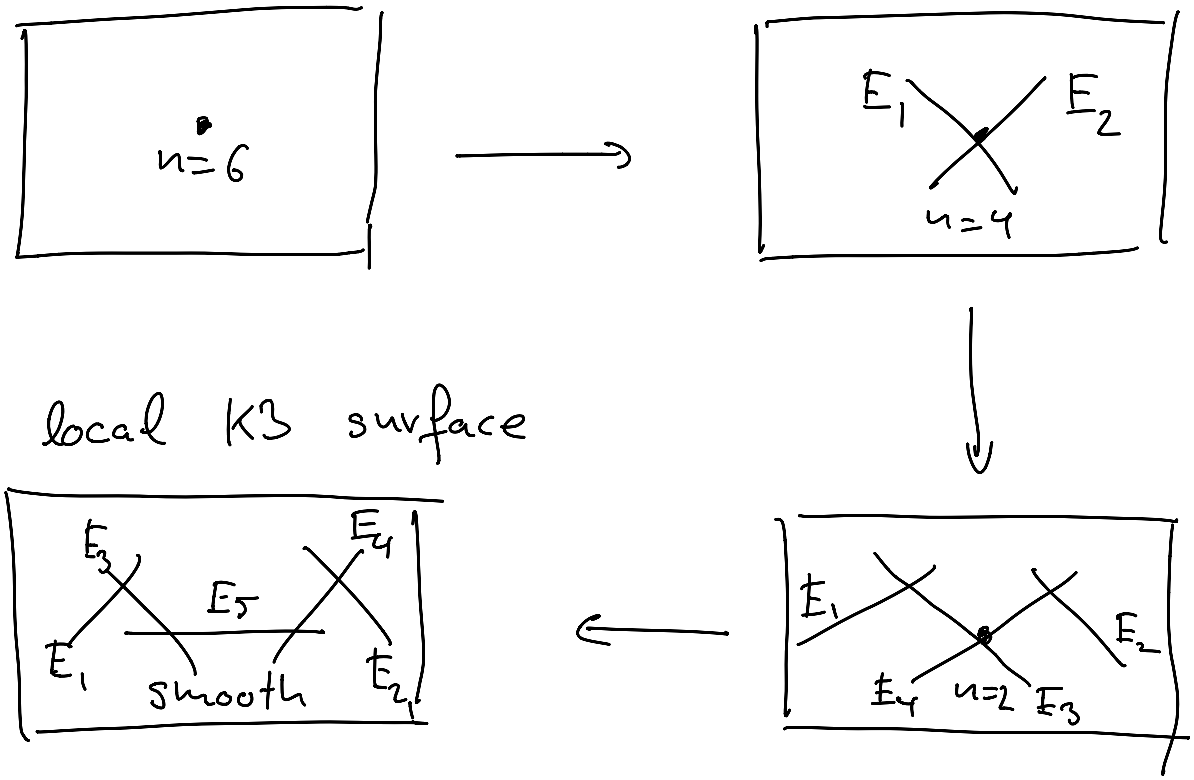
2) $1 \longrightarrow \mathbb{Z}_2 \longrightarrow SU(2) \longrightarrow SO(3) \longrightarrow 1$

projection to $SO(3)$ gives discrete subgroups of regular polygons.

Group	Generators	Resolution
Cyclic	$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$, $\alpha = e^{\frac{2\pi i}{n}}$	A_{n-1}
Binary Dihedral	$\begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\beta = e^{\frac{\pi i}{n}}$	D_{n+2}
Binary Tetrahedral	D_4, $\frac{1}{\sqrt{2}}\begin{pmatrix} \varepsilon^7 & \varepsilon^7 \\ \varepsilon^5 & \varepsilon \end{pmatrix}$, $\varepsilon = e^{\frac{2\pi i}{8}}$	E_6
Binary Octahedral	E_6, $\begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^7 \end{pmatrix}$, $\varepsilon = e^{\frac{2\pi i}{8}}$	E_7
Binary Isocahedral	$-\begin{pmatrix} \eta^3 & 0 \\ 0 & \eta^2 \end{pmatrix}$, $\frac{1}{\eta^2-\eta^3}\begin{pmatrix} \eta+\eta^4 & 1 \\ 1 & -\eta-\eta^4 \end{pmatrix}$, $\eta = e^{\frac{2\pi i}{5}}$	E_8

Resolution or "blow-up":

Consider for example the $\mathbb{C}^2/\mathbb{Z}_6$ singularity



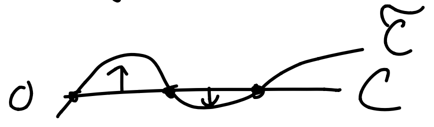
each of the lines is a rational curve, i.e. a $\mathbb{P}^1$.

<u>A little bit of algebraic geometry:</u>

consider a complex surface with $K = 0$ ← canonical class

consider $C \subset K$
C ← algebraic curve, genus $= g$

self-intersection of C:



with $[C] = [\tilde{C}]$

i.e. $\tilde{C}$ is a deformation of C in the same homology class deformation δ is section of normal bundle N.

$\longrightarrow$ self-intersection is number of zeros.

$$\Rightarrow \quad C \cdot C = \int_C c_1(N), \text{ where } c_1 \text{ is first Chern-class}$$

Now we have $N_C + T_C = K|_C = 0$

$$\Rightarrow T_C = - N_C$$

$$\Rightarrow c_1(N) = - c_1(T)$$

and $\int_C c_1(T) = 2 - 2g$

$$\Rightarrow C \cdot C = 2(g-1)$$

$\Rightarrow$ any $\mathbb{P}^1$ (i.e. $g=0$) in a K3 surface must be a (-2)-curve.

"Intersection matrices of resolutions of $\mathbb{C}^2/\Gamma_{\mathfrak{g}}$ are Cartan matrices of $\mathfrak{g}$"