Elliptic CY 3-folds

We focus on elliptic fibrations

$$\pi : X \longrightarrow \Theta,$$

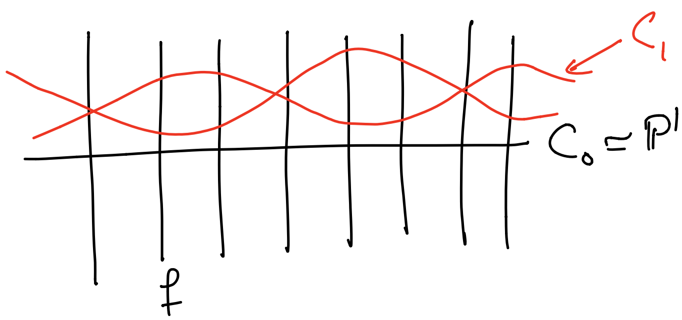
such that Θ is $\mathbb{P}^1$ fibration over $\mathbb{P}^1$.

$\rightarrow$ simplest possibility: $\Theta = \mathbb{F}_n$ "Hirzebruch surface"

consider line bundle $\pi : L \longrightarrow \mathbb{P}^1$

with $c_1(L) = -n$

$\rightarrow \mathbb{F}_n := \mathbb{P}(L)$



L is normal bundle to C_0

$\Rightarrow C_0 \cdot C_0 = \int_{C_0} c_1(L) = -n$

$f \cdot f = 0 \;,\quad f \cdot C_0 = 1$

Define $C_1 = C_0 + nf \Rightarrow C_1 \cdot C_1 = +n$

For $n=0$ we get $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$

elliptic fibration over $\mathbb{F}_n$:

$$\begin{array}{ccc} T^2 \hookrightarrow X & & T^2 \hookrightarrow K3 \longrightarrow X \\ \quad\ \downarrow & \cong & \qquad\ \downarrow \qquad\quad \downarrow \\ \quad\ \mathbb{F}_n & & \qquad \mathbb{P}^1 \qquad \mathbb{P}^1 = W \end{array}$$

Let $[t_0, t_1]$ be homogeneous coordinates on W

$\rightarrow$ affine coordinate $t = t_1/t_0$

$[s_0, s_1]$ homogeneous coordinates of $\mathbb{P}^1$ fiber

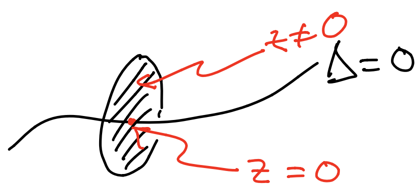
$\rightarrow$ affine coordinate $s = \frac{s_1}{s_0}$

$T^2: \quad y^2 = x^3 + a(s,t)x + b(s,t)$

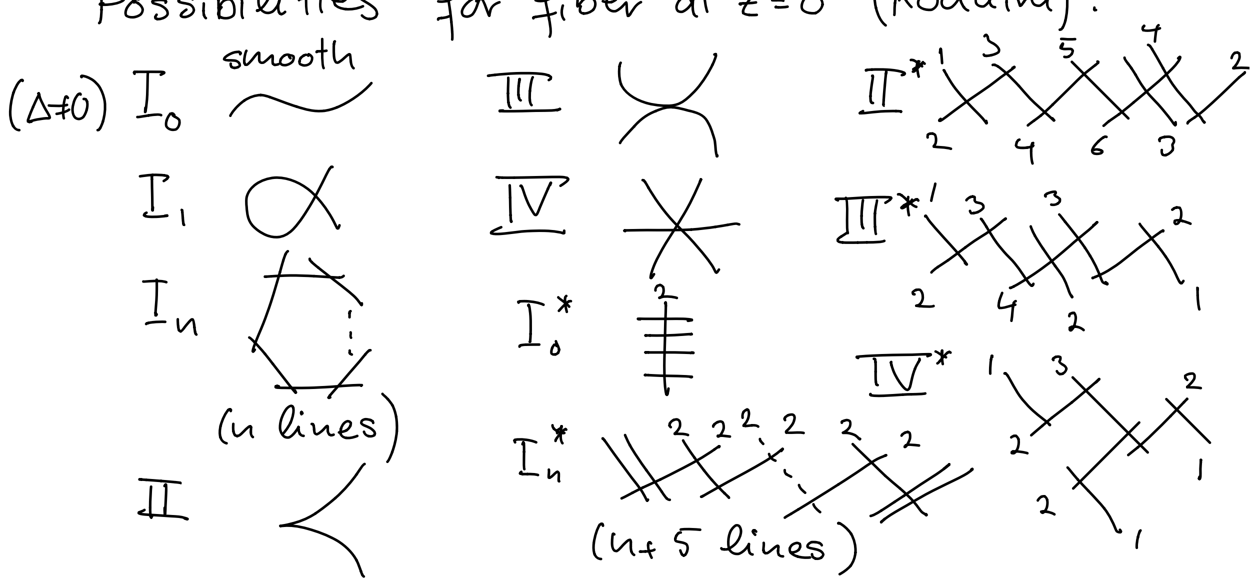
$\rightarrow \Delta = 4a^3 + 27b$

$\rightarrow \Delta(s,t) = 0$ gives locus of singular fibers

consider small disc $D \subset \mathbb{F}_n$ with



Possibilities for fiber at $z=0$ (Kodaira):



each line corresponds to a $\mathbb{P}^1$
small numbers denote multiplicity
Weierstrass form:

$$a(z) = z^L a_0(z)$$

$$b(z) = z^K b_0(z)$$

$$\Delta(z) = z^N \Delta_0(z)$$

and $a_0, b_0, \Delta_0 \neq 0$ at $z=0$. Then we have

L	K	N	Fiber	Λ'
≥ 0	≥ 0	0	I_0	
0	0	>0	I_N	A_{N-1}
≥ 1	1	2	II	
1	≥ 2	3	III	A_1
≥ 2	2	4	IV	A_2
≥ 2	≥ 3	6	I_0^*	D_4
2	3	≥ 7	I_{N-6}^*	D_{N-2}
≥ 3	4	8	IV^*	E_6
3	≥ 5	9	III^*	E_7
≥ 4	5	10	II^*	E_8

homogeneous coordinates:

$$(*) \quad x_0 x_2^2 = x_1^3 + a x_0^2 x_1 + b x_0^3 \longrightarrow \text{cubic in } \mathbb{P}^2$$

$\mathbb{P}^2$ is $\mathbb{P}(L_1 \oplus L_2 \oplus L_3)$

sum of line bundles over $\mathbb{F}_n$

→ $L_1 \simeq \mathcal{O}$, $L_2 \simeq \mathcal{L}^2$, $L_3 \simeq \mathcal{L}^3$

a is section of $\mathcal{L}^4$, b is section of $\mathcal{L}^6$

$[x_0, x_1, x_2] = [0,0,1]$ always solves $(*)$

→ section σ of elliptic fibration

$\sigma : \mathbb{F}_n \longrightarrow T^2$

affine coordinates $\xi_1 = x_1/x_2$, $\xi_2 = x_0/x_2$

→ $(\xi_1, \xi_2) = (0,0)$ defines σ

ξ_1 is section of $\mathcal{L}^{-1}$, ξ_2 is section of $\mathcal{L}^{-3}$

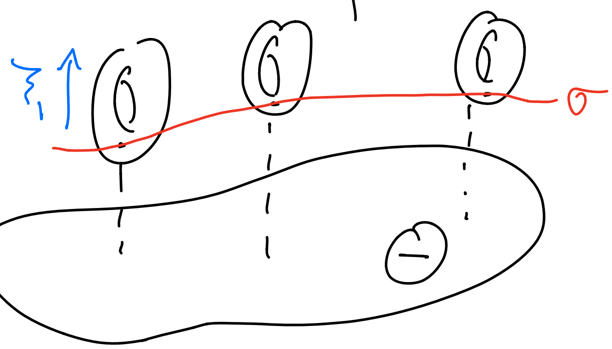
$$\frac{x_0}{x_2} = \left(\frac{x_1}{x_2}\right)^3 + a\left(\frac{x_0}{x_2}\right)^2 \frac{x_1}{x_2} + b\left(\frac{x_0}{x_2}\right)^3$$

$$\xi_2 = \xi_1^3 + a\,\xi_2^2\,\xi_1 + b\,\xi_2^3$$

$$= \xi_1^3 + \mathcal{O}(\xi_2^2)$$

→ ξ_1 is a good coordinate on normal bundle of σ

$\mathcal{N}_\sigma = \mathcal{L}^{-1}$



$\Rightarrow K_X|_\sigma = K_\sigma + \mathcal{L}$

$K_X = \pi^*(K_\Theta + \mathcal{L})$

as $K_X = 0$

→ $\mathcal{L} = -K_\Theta$

→ at $z=0$ fiber degenerates according to simply laced Lie algebra lattice Λ'

→ D7-branes located at $z=0$ with gauge group $G_{\Lambda'}$ on world volume!

<u>Question:</u> Gauge group for $\Theta = \mathbb{F}_n$, $n=1,2,3,\ldots$?

curve $C \in \mathbb{F}_n$

→ adjunction formula gives:

$$\chi(C) = -C\cdot\underbrace{(C+K_\Theta)}_{=\mathcal{N}} \qquad (1)$$

Take $C = C_0 \Rightarrow$ (1) gives

$$\begin{aligned} 2 &= -C_0\cdot(C_0+K_\Theta) \\ &= -C_0\cdot(C_0 + aC_0 + bf) \\ &= n + an - b = -n - b \Rightarrow b = -n-2 \end{aligned}$$

$C = f$ and (1) give

$$\begin{aligned} 2 &= -f\cdot(f + aC_0 + bf) \\ &= 0 - a \Rightarrow a = -2 \end{aligned}$$

$$\Rightarrow K_{\mathbb{F}_n} = -2C_0 - (2+n)f$$

Divisors in $\mathbb{F}_n$:

- A : $a=0$, • B : $b=0$, Δ: $\Delta=0$

From $K_X = \pi^*(K_\Theta + \mathcal{L})$ we get $\mathcal{L} = 2C_0 + (2+n)f$

- $\mathcal{N}_A \simeq \mathcal{L}^4$

  $\rightarrow A = 8C_0 + (8+4n)f$

- $\mathcal{N}_B \simeq \mathcal{L}^6$

  $\rightarrow B = 12C_0 + (12+6n)f$

- $\mathcal{N}_\Delta \simeq \mathcal{L}^{12}$

  $\rightarrow \Delta = 24C_0 + (24+12n)f$

split off C_0 from Δ:

$$\Delta = NC_0 + \underset{\uparrow}{\Delta'} \text{ not containing } C_0$$

and $\Delta' \cdot C_0 \geq 0$

$$\Rightarrow (24-N)\underbrace{C_0 \cdot C_0}_{=-n} + (24+12n)\underbrace{f \cdot C_0}_{=1} \geq 0$$

$$= -n24 + nN + 24 + 12n$$

$$= -12n + nN + 24 \geq 0$$

$$\Leftrightarrow N \geq 12 - \frac{24}{n}$$

Similarly, we get:

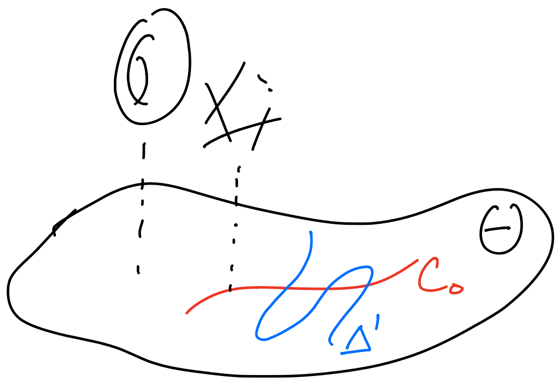
$$L \geq 4 - \frac{8}{n}, \quad K \geq 6 - \frac{12}{n}$$

$\underline{n=1,2:}$ no singularity since $N, L, K \leq 0$

$\rightarrow$ no gauge group

$\underline{n>2:}$ singular fibers on C_0

$\rightarrow$ gauge group



choose $N = 12 - \frac{24}{2} \rightarrow \Delta' \cdot C_0 = 0$

$\underline{n=3:}$

$$L \geq 4 - \frac{8}{3} = 1\tfrac{1}{3} \Rightarrow L = 2, 3, \ldots$$

$$K \geq 6 - \frac{12}{3} = 2$$

$$N = 4$$

$\rightarrow$ Fiber $\underline{\text{IV}}$ $\rightarrow$ $SU(3)$ gauge group

$\underline{n=4:}$

$L \geq 2$, $K \geq 3$, $N = 6$

$\rightarrow$ I_0^* Fiber $\rightarrow$ $SO(8)$ gauge group

$\underline{n=5:}$

F_4

$\underline{n=6:}$

E_6

$\underline{n=8:}$ E_7

$\underline{n=12:}$ E_8