

§ 1. 6d superconformal field theories

Some general remarks about SCFT's :

- Consider affine Minkowski space $\mathbb{R}^{1, d-1}$
 - d-dimensional real affine space with Lorentz metric of signature $(-1, +1^{d-1})$
 - automorphism group is Poincaré group with Lie algebra $\text{iso}(1, d-1)$ with generators P ("translations") and M ("rotations").
 - Consider further the group which takes light-cones to light-cones
 - conformal group
 - Lie algebra is $\text{conf}(1, d-1)$ with generators D, P, K, M .
 - D denotes "dilatations"
 - K is special conformal transformation
 - We have : $\text{conf}(1, d-1) \cong \text{so}(2, d)$
- relation between generators:
- $|PI|^{-1} = K$, where I is the "inversion" operation $I: x^m \rightarrow x^m/x^2$

- Super-Poincaré algebras:
S-matrix of a Poincaré invariant theory must have Lie algebra:

$$\Sigma^\circ = \text{iso}(1, d-1) \oplus t \text{ "Coleman-Mandula"}$$

where t is a compact real Lie algebra.

→ some theories can be enriched by having a \mathbb{Z}_2 -graded Lie algebra as a symmetry of the S-matrix:

$$\Sigma = \Sigma^\circ \oplus \Sigma'$$

with a graded bracket $[,] : \Sigma \otimes \Sigma \rightarrow \Sigma$

such that $[X, Y] = (-1)^{|X||Y|} [Y, X]$

satisfying the graded Jacobi identity:

$$[X, [Y, Z]] = [[X, Y], Z] + (-1)^{|X||Y|} [Y, [X, Z]]$$

We subdivide 4 cases:

BBB : Σ° is ordinary Lie algebra

BBF : Σ' is a module for Σ°

FFB : There is an Σ° equivariant map
 $\text{Sym}^2 \Sigma' \rightarrow \Sigma^\circ$

FFF : $[F_1, [F_2, F_3]] + \text{cyclic} = 0$ (no signs)

→ generalization of Coleman-Mandula theorem
known as Haag-Lopuszanski-Sohnius
theorem:

If the theory contains mass-less particles,
then the graded Lie algebra symmetry
of the S-matrix must be a
"Super-Poincaré symmetries"

Definition:

A d-dimensional superPoincaré symmetry Σ
is a super Lie algebra such that

- 1) $\Sigma^0 = \text{iso}(1, d-1) \oplus t$ where t is as above
- 2) Σ^1 is a real spinorial representation
of $\text{iso}(1, d-1)$
- 3) $\text{Sym}^2 \Sigma^1 \rightarrow \Sigma^0$ is non-vanishing

Notation: $\Sigma(\mathbb{R}^{1, d-1 | s})$

Elements of Σ^1 are called supersymmetry
operators. The global symmetries in t
which do not commute with supersymmetries
are "R-symmetries".

Poincaré susy algebras exist in all dimensions:

- choose real spinorial rep. of S with pairing

$$S \otimes S \rightarrow V^*$$

$$\text{where } V = T_p \mathbb{R}^{1, d-1}$$

- take $\sum' = S$ to define odd bracket:

$$[Q, Q'] \sim P$$

- The FFF Jacobi identity is trivially satisfied.

Supercosmological algebras:

A theory with conformal symmetry gives

$$\text{iso}(1, d-1) \rightarrow \text{so}(2, d)$$

Then we have:

Definition:

A d -dimensional "superconformal Lie algebra" is a super Lie algebra $\sum = \sum^\circ \oplus \sum'$ over \mathbb{R} such that

- 1) $\sum^\circ = \text{so}(2, d) \oplus t$ for some $d \geq 1$ with t a compact real Lie algebra.
- 2) \sum' is a real spinorial rep. of $\text{so}(2, d)$
- 3) $\text{Sym}^2 \sum' \rightarrow \sum^\circ$ is non-vanishing

Nahm's Lemma:

Suppose Σ is a superconformal algebra.

Then, $\Sigma = \mathfrak{o}_S \oplus \{ \}$ where \mathfrak{o}_S is a "simple" super Lie algebra and $\{ \}$ is central
 \hookrightarrow : no non-trivial ideals

- use classification of Kac of simple super Lie algebras
- \mathfrak{o}_S' spinorial is very constraining:
 dimension grows linearly with rank
but spinor representations have a dimension which grows exponentially with rank.

Theorem:

The complete list of superconformal algebras

- 1) 7 types for $d=2$, one type has continuous deformations
- 2) $d=3$: $\mathfrak{osp}(N|4) \cong [\mathfrak{so}(N) \oplus \mathfrak{so}(2,3)]^0 \oplus (N,4)$,
 $N \geq 1$
- 3) $d=4$: $\mathfrak{u}(2,2|N) \cong [\mathfrak{so}(2,4) \oplus \mathfrak{u}(N)]^0 \oplus [(4,N) + (\bar{4},\bar{N})]_{\mathbb{R}}$
- 4) $d=4$: $\mathfrak{psu}(2,2|4) \cong [\mathfrak{so}(2,4) \oplus \mathfrak{su}(4)]^0 \oplus [(4,4) + (\bar{4},\bar{4})]_{\mathbb{R}}$

- 5) $d=5$: $\text{osp}(2,5|2) \cong [\text{so}(2,5) \oplus \text{usp}(2)]^\circ \oplus (8,2)_R$
- 6) $d=6$: $\text{osp}(2,6|2k) \cong [\text{so}(2,6) \oplus \text{usp}(2k)]^\circ \oplus (8,2k),$
 $k = 1, 2, 3, \dots$

Remarks:

1. The notation $\text{usp}(2k)$ means the real Lie algebra of the compact symplectic group of $k \times k$ unitary matrices over the quaternions.
2. In physics there is an important distinction between Q and S supersymmetries. The eigenspace $[D, Q] = -\frac{1}{2} Q$
 \nearrow
scaling operator
 are the Q supersymmetries and the eigenspace $[D, S] = +\frac{1}{2} S$ are the S supersymmetries.
 Since $[Q, Q] \sim P \rightarrow Q$: Poincaré SUSY
 Note: $[K, Q] \sim S$ and $[S, S] \sim K$

Six-dimensional superconformal algebras

Relevant Lie superalgebra: $\text{osp}(2,6|2k)$

→ particles of spin ≥ 2 for $k > 2$

→ ruled out

thus the relevant cases are:

- 1) $\text{osp}(2,6|2)$ " (1,0) algebra "
- 2) $\text{osp}(2,6|4)$ " (2,0) algebra "

some group theory:

- denote real Clifford algebras by $\text{Cl}(t_+, s_-)$
 - irreducible spinor rep. $\Delta(t_+, s_-)$ if non-chiral
 $\Delta(t_+, s_-)_\pm$ if chiral
 - ↑ chirality
- $\text{Cl}(2_+, 6_-) \cong \underbrace{\text{Cl}^{\{0\}}(2_+, 6_-)}_{\cong \mathbb{H}(4)} \oplus \underbrace{\text{Cl}^{\{1\}}(2_+, 6_-)}_{\cong \mathbb{H}(4)}$
 - two spinor representations $\Delta_\pm \cong \mathbb{H}^4$
- R-symmetry $\text{usp}(4) \cong \text{SO}(5)$

$$\begin{aligned}\text{Cl}(5_\pm) &\cong \underbrace{\text{Cl}^{\{0\}}}_{\cong \mathbb{H}(2)} \oplus \underbrace{\text{Cl}^{\{1\}}}_{\cong \mathbb{H}(2)} \\ &\cong \mathbb{H}(2)\end{aligned}$$

$$\rightarrow \Delta(5) \cong \mathbb{H}^2$$

- The superconformal algebra $\Sigma(\mathbb{R}^{1,5|32})$:

We have $\Sigma^0 = \text{so}(2,6) \oplus \text{usp}(4) = \text{so}(2,6) \oplus \text{sd}_{\mathbb{R}}$

$$\Sigma^1 = \begin{cases} (\Delta_+(2,6) \otimes_{\mathbb{C}} \Delta(5))_{\mathbb{R}} & (2,0) \text{ alg.} \\ (\Delta_-(2,6) \otimes_{\mathbb{C}} \Delta(5))_{\mathbb{R}} & (0,2) \text{ alg.} \end{cases}$$

$\Delta_{\pm}(2,6)$ are pseudo-real :

as complex rep. 8-dim but

\exists multiplication by quaternion j

as anti-linear operator j with $j^2 = -1$

$\Delta(5)$ is also pseudo-real

→ tensor product gives 32-dim compl. rep.

→ project to 32 real spinors by the use of $j_1 \otimes j_2$:

$$(Q_{\alpha i})^+ = j^{\alpha\beta} j^{ij} Q_{\beta j} \quad \alpha, \beta = 1, \dots, 8, \\ i, j = 1, \dots, 4$$