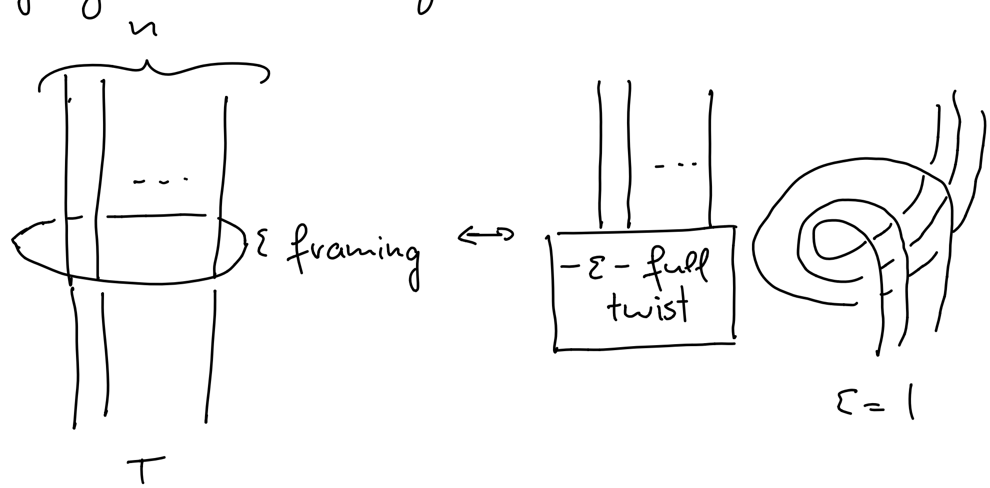
Let L and L' be framed links in S^3. Denote by M_L and M_L' the 3-manifolds obtained by Dehn surgery on L and L' respectively as in previous lecture. Then we have the following

Theorem 1 (Kirby moves):

There is an orientation preserving homeomorphism $M_L \cong M_L'$ if and only if L' is obtained from L by applying the following local move



finitely many times, where $\varepsilon = \pm 1$ and n stands for the number of strands passing through the trivial Knot with ε-framing.

For $n=0 \Rightarrow$ deleting/adding trivial Knot with ε framing

Next: Define Witten's invariants for arbitrary 3-manifolds obtained by Dehn surgery from S^3.

Let $\mathfrak{g}$ be the Lie algebra $sl_2(\mathbb{C})$. Fix a positive integer K and denote by $P_+(K)$ the set of level K highest weights of affine Lie algebra $\hat{\mathfrak{g}}$.

$\longrightarrow$ $P_+(K) = \{0, 1, \dots, K\}$. For each $\lambda \in P_+(K) \to H_\lambda$ on which Virasoro algebra acts with central charge $c = \frac{3K}{K+2}$. Set $C = \exp\left(2\pi\sqrt{-1}\,\frac{c}{24}\right)^3 = \exp\left(-\pi\sqrt{-1}\,\frac{c}{4}\right)$

Level K characters $\chi_\lambda(\tau)$, $\operatorname{Im}\tau > 0$, $\lambda \in P_+(K)$ satisfy:

$$\chi_\lambda\left(-\frac{1}{\tau}\right) = \sum_\mu S_{\lambda\mu}\chi_\mu(\tau),$$

$$\chi_\lambda(\tau+1) = \exp\left(2\pi\sqrt{-1}\left(\Delta_\lambda - \frac{c}{24}\right)\right)\chi_\lambda(\tau),$$

where we have

$$S_{\lambda\mu} = \sqrt{\frac{2}{K+2}}\,\sin\frac{(\lambda+1)(\mu+1)}{K+2}$$

$$\Delta_\lambda = \frac{\lambda(\lambda+2)}{4(K+2)}.$$

Modular transformations S and T satisfy

$$S^2 = (ST)^3 = \mathbb{1}$$

As a consequence of the above we have

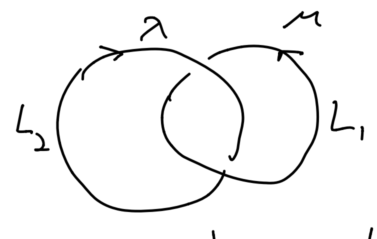
<u>Lemma 1:</u>

The above $S_{\lambda\mu}$, $0 \le \lambda, \mu \le K$, satisfy

$$C\sum_\mu S_{\lambda\mu}S_{\mu\nu}\exp\left(2\pi\sqrt{-1}\left(\Delta_\lambda + \Delta_\mu + \Delta_\nu\right)\right) = S_{\lambda\nu}$$

Let now L be an oriented framed link in S^3 with components $L_1, \dots, L_m$. Given a coloring $\lambda : \{1, \dots, n\} \to P_+(K)$ with highest weights of level $K \longrightarrow$ invariants $J(L; \lambda_1, \dots, \lambda_m)$

Let L be the Hopf link with two components L_1, L_2:



$$lk(L_1, L_2) = 1$$

Then we have the following

<u>Proposition 1:</u>

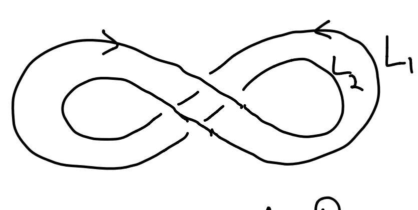
Let H be a Hopf link colored with $\lambda, \mu \in P_+(K)$

Then, we have

$$J(H; \lambda, \mu) = \frac{S_{\lambda\mu}}{S_{00}}$$

<u>Proof:</u>

Represent the Hopf link as a cabling of a trivial knot with -1 framing:



$$f(L_1) = -1$$
$$f(L_2) = -1$$
$$lk(L_1, L_2) = -1$$

By Lemma 3 and Proposition 1 of §8, we have

$$\exp 2\pi\sqrt{-1}(-\Delta_\lambda - \Delta_\mu) J(H; \lambda, \mu) = \sum_\nu N_{\lambda\mu}^{\nu} \exp(2\pi\sqrt{-1}(-\Delta_\nu)) \frac{S_{0\nu}}{S_{00}}$$

where we have used that $J(O;\nu) = \frac{S_{0\nu}}{S_{00}}$.

Then, using Verlinde's formula and Lemma 1, we compute:

$$N_{\lambda\mu\nu} = \sum_\alpha \frac{S_{\lambda\alpha} S_{\mu\alpha} S_{\nu\alpha}}{S_{0\alpha}} \quad \text{(Verlinde formula)}$$

$$\Rightarrow \sum_\nu N_{\lambda\mu}^{\nu} \exp\left(2\pi\sqrt{-1}\,(-\Delta_\nu)\right) \frac{S_{0\nu}}{S_{00}}$$

$$= \sum_{\nu,\alpha} \frac{S_{\lambda\alpha} S_{\mu\alpha} S_{\nu\alpha}}{S_{0\alpha}} \frac{S_{0\nu}}{S_{00}} \exp\left(2\pi\sqrt{-1}\,(-\Delta_\nu)\right)$$

$$e^{2\pi i(-\Delta_\lambda-\Delta_\mu)} J(H;\lambda,\mu) = \sum_{\nu,\alpha} \frac{S_{\lambda\alpha} S_{\mu\alpha} S_{\nu\alpha} S_{0\nu}}{S_{0\alpha} S_{00}} e^{-2\pi i\Delta_\nu}$$

$$\Leftrightarrow J(H;\lambda,\mu) \underset{|}{=} \sum_{\nu,\alpha} \frac{S_{\lambda\alpha} S_{\mu\alpha} S_{\nu\alpha} S_{0\nu}}{S_{0\alpha} S_{00}} e^{-2\pi i(\Delta_\nu-\Delta_\lambda-\Delta_\mu)}$$

$$\left[\ \sum_\nu S_{0\nu} S_{\nu\alpha} e^{-2\pi i\Delta_\nu} = C\, S_{0\alpha} e^{2\pi i\Delta_\alpha}\right.$$

(rearranging Lemma 1)

$$\overset{|}{\underset{|}{=}} C \sum_\alpha \frac{S_{\lambda\alpha} S_{\mu\alpha} \cancel{S_{0\alpha}}}{\cancel{S_{0\alpha}} S_{00}} e^{2\pi i(\Delta_\alpha+\Delta_\lambda+\Delta_\mu)}$$

Lemma 1 again

$$\overset{!}{=} \frac{S_{\lambda\mu}}{S_{00}}$$

$\square$

<u>Notation:</u> For link components L_i and L_j we write $L_i \cdot L_j = lk(L_i, L_j)$. In the case $i=j$, $L_i \cdot L_i$ denotes the integer representing the framing of L_i.

$\to$ obtain matrix A ($A_{ij} = (L_i \cdot L_j)$)
Let n_+ (resp. n_-) be the number of positive (resp neg.) eigenvalues of A Then we write

$$\sigma(L) = n_+ - n_-$$

for the signature of the link L.

Theorem 2:

Let M be a compact oriented 3-manifold without boundary. Suppose that M is obtained as Dehn surgery on a framed link L with m components L_j, $1 \leq j \leq m$, in S^3. Then,

$$Z_K(M) = S_{00} \, C^{\sigma(L)} \sum_{\{\lambda\}} S_{0\lambda_1} \cdots S_{0\lambda_m} J(L; \lambda_1, \ldots, \lambda_m)$$

is a topological invariant of M and does not depend on the choice of L which yields M. More precisely, if there is an orientation preserving homeomorphism $M_1 \cong M_2$, then $Z_K(M_1) = Z_K(M_2)$.

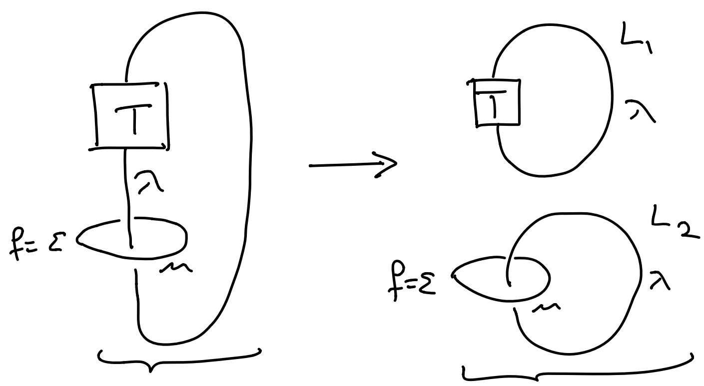
Proof:
By Theorem 1, we have to show that $Z_K(M)$ is invariant under Kirby moves. First, we deal with the case $\underline{n=0}$. The case $\lambda = \nu = 0$ in Lemma 1 implies

$$C \sum_{\mu \in P_+(K)} S_{0\mu} \underbrace{\frac{S_{0\mu}}{S_{00}}}_{= J(\bigcirc)} \underbrace{\exp(2\pi\sqrt{-1}\,\Delta_\mu)}_{= \text{framing shift}} = 1$$

Combining with $Z_K(S^3) = S_{00}$ we obtain the invariance of $Z_K(M)$ under Kirby moves with $n=0$.

<u>$n=1$:</u>

It is enough to consider case of Hopf link by factorization property (Proposition 5)

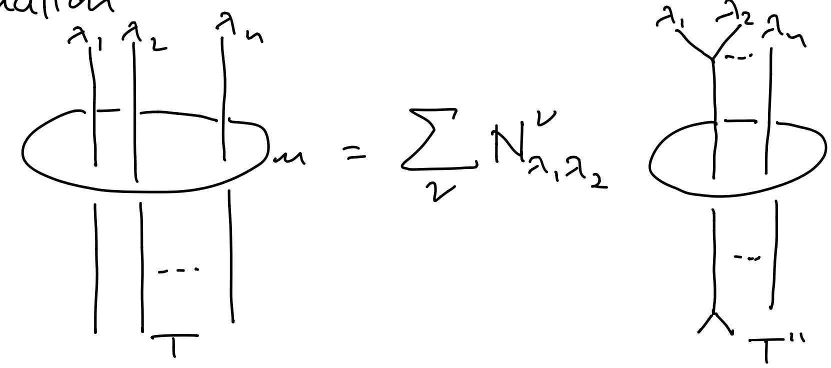


$$= J(L;\lambda_1,\dots) \qquad = J(L_1;\lambda_1,\dots)\, J(L_2;\lambda,\mu)\, \frac{S_{00}}{S_{0\lambda}}$$

Proposition 1 + Lemma 1 for $\nu=0$ give:

$$C \sum_{\mu\in P_+(K)} S_{0\mu} \underbrace{\frac{S_{\lambda\mu}}{S_{00}}}_{=J(\text{Hopf})} \underbrace{\exp\left(2\pi\sqrt{-1}\,\Delta_\mu\right)}_{\substack{\text{framing shift}\\ \text{of } O_\mu}} = \underbrace{\exp\left(-2\pi\sqrt{-1}\,\Delta_\lambda\right)}_{\substack{\text{framing}\\ =-1}} \underbrace{\frac{S_{0\lambda}}{S_{00}}}_{=J(O;\lambda)}$$

Next, we show invariance of $Z_K(M)$ under Kirby moves by induction on n. Consider the local situation



It will be sufficient to show

$$C^{\sigma(L)} \sum_{\mu} S_{0\mu}\, \mathcal{J}(T; \mu, \lambda_1, \dots, \lambda_n) = C^{\sigma(L')}\, \mathcal{J}(T'; \lambda_1, \dots, \lambda_n)$$

where L' is obtained from L by a Kirby move of deleting $\bigcirc^{\mu}$ and twisting by ε.

→ To achieve this, fuse two strands λ_1 and λ_2 as in above picture and write

→ The tangle operator $\mathcal{J}(T; \mu, \lambda_1, \dots, \lambda_n)$ is expressed as a linear combination

$$\sum_{\nu} N^{\nu}_{\lambda_1 \lambda_2}\, F_1\, \mathcal{J}(T''; \mu, \nu, \lambda_3, \dots, \lambda_n)\, F_2$$

where F_1 and F_2 are the above elementary connection matrices and T'' is an $(n-1, n-1)$-tangle

→ have reduced situation to $n-1$ strands

By induction hypothesis and change of $\sigma(L)$ under Kirby moves, we obtain the desired statement. □

$Z_K(M)$ is called "Witten's invariant" for M associated with the Lie algebra $sl_2(\mathbb{C})$ at level K.

Examples:

For $M = S^3$: $Z_K(S^3) = S_{00}$ $\quad = \mathcal{J}(\bigcirc; \mu)$, 0 framing

For $M = S^1 \times S^2$: $Z_K(S^1 \times S^2) = S_{00} \sum_{\mu} S_{0\mu} \frac{S_{0\mu}}{S_{00}} = 1$

Proposition 2:

For a connected sum $M_1 \# M_2$ of closed oriented 3-manifolds M_1 and M_2

$$Z_k(M_1 \# M_2) = \frac{1}{S_{00}} Z_k(M_1) Z_k(M_2)$$

holds.

Proposition 3:

We denote by $-M$ the 3-manifold M with the orientation reversed. Then we have

$$Z_k(-M) = \overline{Z_k(M)}$$

Proof:

If M is obtained as Dehn surgery on a framed link L, then surgery on its mirror image $-L$ yields $-M$. Result follows from Prop. 3, §8 $\square$

Extend the above construction to case where 3-manifold M contains a link L:

Let $L_1, \dots, L_n$ be components of L with coloring $\lambda_1, \dots, \lambda_n \in P_+(K)$. Suppose $(S^3, L') \longrightarrow (M, L)$ is obtained by Dehn surgery on framed link $N \subset S^3$. assume : $N \cap L' = \emptyset$. Let $N_1, \dots, N_m$ be the components of N. Then define

$$Z_k(M, L; \lambda_1, \dots, \lambda_n)$$
$$= S_{00} C^{\sigma(N)} \sum_{\mu} S_{0\mu_1} \cdots S_{0\mu_m} J(L' \cup N; \lambda_1, \dots, \lambda_m, \mu_1, \dots, \mu_m)$$