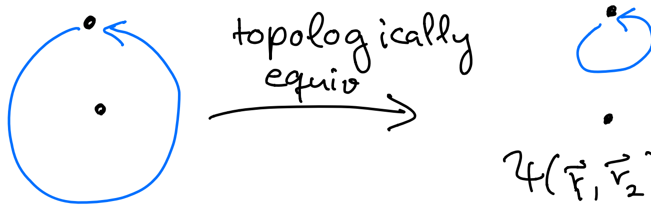
Digression: Non-abelian anyons and CS-th

- quantum statistics in 3+1D:
  2x interchange of two particles is equivalent to



(a) (b)

$\psi(\vec{r}_1, \vec{r}_2)$ is invariant

$\Rightarrow$ under 1x interchange we have:

$$\psi(\vec{r}_1, \vec{r}_2) = \pm \psi(\vec{r}_1, \vec{r}_2)$$

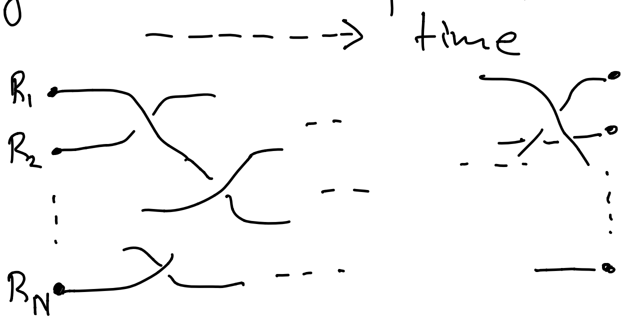
+ : bosons
− : fermions

- quantum statistics in 2+1D:

$$\psi(\vec{r}_1, \vec{r}_2) = e^{i\theta} \psi(\vec{r}_1, \vec{r}_2) \qquad \big((a) \not\Leftrightarrow (b)\big)$$

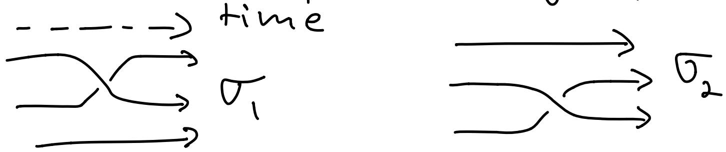
$\longrightarrow$ "anyons"

general case of N particles:



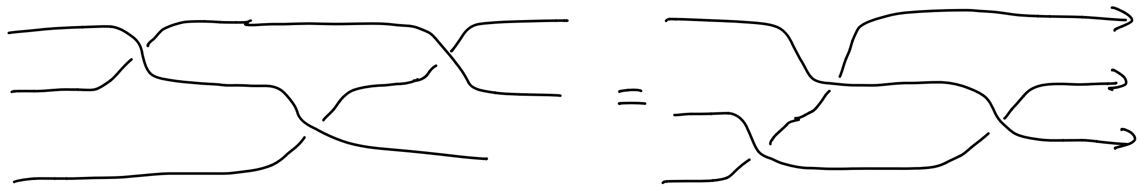
$\longrightarrow$ "braid group" B_N

generators of braid group:



for 3 particles. In general $\sigma_1, \dots, \sigma_{N-1}$

braid relation:



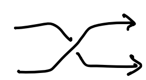
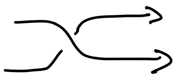
$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

also: $\sigma_i \sigma_j = \sigma_j \cdot \sigma_i$ for $|i-j| \geq 2$

$\sigma_i^2 \neq 1 \Rightarrow$ infinitely many elements

- abelian representations of B_N:

$$\psi(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N) \longrightarrow e^{im\theta} \psi(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N)$$

where $m = \#$  $- \#$ 

for identical particles. Non-identical case:

θ_{ab}, $a, b = 1, \dots, n_s$

$\uparrow$ number of particle species

- non-abelian repr. of B_N:

degenerate states

$\longrightarrow$ g states with particles at $R_1, \dots, R_n$

$$\psi_\alpha, \ \alpha = 1, 2, \dots, g$$

Then σ_i act as $g \times g$ unitary matrix $\rho(\sigma_i)$

$$\psi_\alpha \to [\rho(\sigma_1)]_{\alpha\beta}\,\psi_\beta$$

in particular: $\rho(\sigma_1)\rho(\sigma_2) \neq \rho(\sigma_2)\rho(\sigma_1)$

$\to$ "non-abelian braiding statistics"

<u>Example:</u>

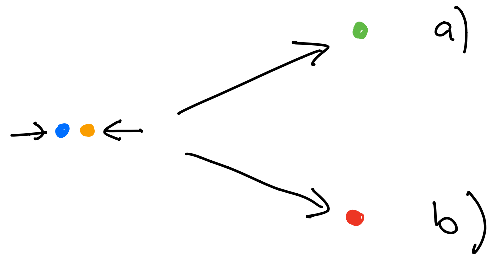
Consider a model with 3 anyon types:

$1, \sigma, \psi$

with "fusion rules":

$$\sigma \times \sigma = 1 + \psi, \quad \sigma \times \psi = \sigma, \quad \psi \times \psi = 1,$$

$$1 \times x = x \quad \text{for } x = 1, \sigma, \psi$$



fusion channels

Note the similarity to tensor products of $su(2)$ representations:

$$\underset{\sigma}{\tfrac{1}{2}} \times \underset{\sigma}{\tfrac{1}{2}} = \underset{1}{0} + \underset{\psi}{1}, \quad \underset{\sigma}{\tfrac{1}{2}} \times \underset{\psi}{1} = \underset{\sigma}{\tfrac{1}{2}}, \quad \underset{\psi}{1} \times \underset{\psi}{1} = \underset{1}{0}$$

with important <u>constraint</u>: maximum spin $= 1$

$\to$ realized in integrable highest weight modules $H_{k,\lambda}$ of $\widehat{su(2)}$ introduced in Prop. 5

here $k = 2$ ($\Rightarrow$ highest $\lambda = 2j = 2$)

Hilbert space of $4\,\sigma$s:

group according to $(1,2)$ and $(3,4)$

$$\begin{array}{ccc} & (1,2) & \\ \swarrow & & \searrow \\ 1 & & \psi \end{array} \qquad \begin{array}{ccc} & (3,4) & \\ \swarrow & & \searrow \\ & & \psi \end{array}$$

constraint: global topological charge $= 1$

$\Rightarrow$ σ_1 and σ_2 fuse to 1 (σ_3 and σ_4 too)

or σ_1 and σ_2 fuse to ψ (σ_3 and σ_4 too)

$\Rightarrow$ two-dim Hilbert space generated by Ψ_1 and Ψ_ψ

In general: for $2n$ quasi-particles Hilbert space is 2^{n-1} dimensional.

action of braid group generators: spinor representation of $SO(2n)$!

braiding particles i and j

$\rightarrow$ $\frac{\pi}{2}$ rotation in the i-j plane of $\mathbb{R}^{2n}$

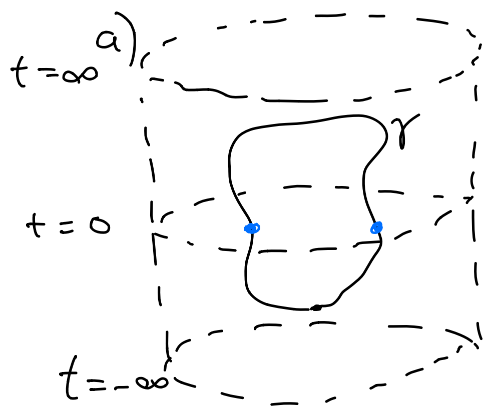
<u>Realization in physics</u>:

Fractional quantum hall effect with $\nu = \frac{5}{2}$

quasiparticle excitations and their corresponding wavefunctions (in $2+0$ D) are mapped to correlation functions in chiral WZW (Wess-Zumino-Witten) theory (1+1D) : "Moore-Read Pfaffian State"

<u>Relation to Chern-Simons theory :</u>

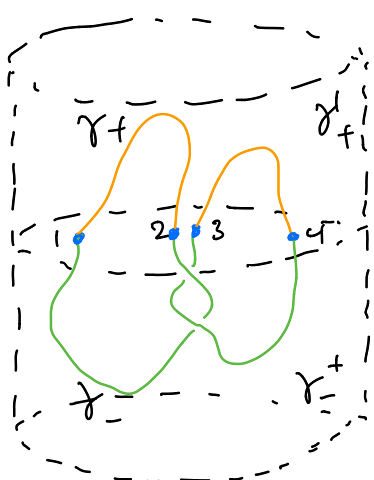
Restrict to quasi-particles of FQHE with $\nu = \frac{5}{2} \longrightarrow$ motion of quasiparticles in 2+1D is described by SU(2) CS-theory at level $k = 2$.



$$= \langle 0|0\rangle_{\gamma, j} = \int \mathcal{D}a\, e^{iS_{cs}[a]}\, W_{\gamma, j}[a]$$

$\uparrow$ type of q.p. $(= 1, \sigma, \psi)$

b) 2 pairs of quasi-particles



$$= \langle 0|0\rangle_{\gamma_1, j_1; \gamma_2, j_2}$$

$$= \int \mathcal{D}a\, e^{iS_{cs}[a]}\, W_{\gamma, j}[a]\, W_{\gamma', j'}[a]$$

$$= \langle \chi | \psi \rangle \quad \text{where}$$

$$\psi[A] = \int_{a(\vec{x},0)=A(\vec{x})} \mathcal{D}a(\vec{x},t)\, W_{\gamma_-, j}[a]\, W_{\gamma'_-, j'}[a] \times e^{\int_{-\infty}^{0} dt \int d^2x\, \mathcal{L}_{cs}}$$

$$\chi^*[A] = \int_{a(\vec{x},0)=A(\vec{x})} \mathcal{D}a(\vec{x},t)\, W_{\gamma_+, j}[a]\, W_{\gamma'_+, j'}[a] \times e^{\int_{0}^{\infty} dt \int d^2x\, \mathcal{L}_{cs}}$$

$$\boxed{|\psi\rangle = \rho(\sigma_2^{\,2})\, |\chi\rangle}$$

§3. Wess-Zumino-Witten model

Let Σ be a compact Riemann surface with $\partial\Sigma = \emptyset$. Consider

$$\sigma = \frac{1}{24\pi^2} \mathrm{Tr}(\mu \wedge \mu \wedge \mu)$$

$\rightarrow$ left-invariant volume form of $SU(2)$

$\sigma \in H^3(SU(2), \mathbb{Z})$

Let $f : \Sigma \rightarrow G$ be a smooth map

Define

$$E_\Sigma \equiv -\sqrt{-1} \int_\Sigma \mathrm{Tr}(f^{-1}\partial f \wedge f^{-1}\bar{\partial} f)$$

as the "energy" of f.

Definition:

The "Wess-Zumino-Witten action" $S_\Sigma(f)$ is defined by

$$S_\Sigma(f) = \frac{k}{4\pi} E_\Sigma(f) - \frac{\sqrt{-1}\,k}{12\pi} \int_B \mathrm{Tr}(\tilde{f}^{-1}d\tilde{f} \wedge \tilde{f}^{-1}d\tilde{f} \wedge \tilde{f}^{-1}d\tilde{f})$$

where B is compact oriented smooth 3-manifold with $\partial B = \Sigma$ and $\tilde{f} : B \rightarrow G$ s.t. $\tilde{f}\big|_\Sigma = f$.

$\rightarrow$ $\tilde{f}^{-1}d\tilde{f}$ is pull back of Maurer-Cartan form $\mu = X^{-1}dX$ by $\tilde{f}$.

Lemma:

$\exp(-S_\Sigma(f))$ does not depend on choice of B and extension $\tilde{f}$.

Proof:

Consider second 3-manifold B' with $\partial B' = \Sigma$ and $\tilde{f}' : B \to G$ s.t. $\tilde{f}'|_\Sigma = f$.

Defin 3-manifold $M = B \cup_\Sigma -B'$ where $-B'$ is B' with reverse orientation

Let $F : M \to G$ be a smooth map with $F|_B = \tilde{f}$ and $F|_{B'} = \tilde{f}'$. Then

$$\frac{K}{24\pi^2}\left(\int_B \mathrm{Tr}\left(\tilde{f}^{-1}d\tilde{f} \wedge \tilde{f}^{-1}d\tilde{f} \wedge \tilde{f}^{-1}d\tilde{f}\right) - \int_{B'} \mathrm{Tr}\left(\tilde{f}'^{-1}d\tilde{f}' \wedge \tilde{f}'^{-1}d\tilde{f}' \wedge \tilde{f}'^{-1}d\tilde{f}'\right)\right)$$

$$= K \int_M F^* \sigma$$

As $\sigma \in H^3(G, \mathbb{Z}) \to \int_M F^*\sigma \in \mathbb{Z}$

$\Rightarrow \exp(-S_\Sigma(f))$ does not depend on choice of B and extensions $\tilde{f}$ $\square$

The term

$$\frac{\sqrt{-1}\,K}{12\pi}\int_B \mathrm{Tr}\left(\tilde f^{-1}d\tilde f\wedge \tilde f^{-1}d\tilde f\wedge \tilde f^{-1}d\tilde f\right)$$

is called "Wess-Zumino" term.

<u>Proposition 1</u> (Polyakov-Wiegmann formula):
Let $f, g : \Sigma \to G$ be smooth maps.
Then

$$\exp\left(-S_\Sigma(fg)\right) = \exp\left(-S_\Sigma(f) - S_\Sigma(g) - \frac{\sqrt{-1}\,K}{2\pi}\int_\Sigma \mathrm{Tr}\left(f^{-1}\bar\partial f\wedge \partial g g^{-1}\right)\right)$$

holds.

<u>Proof:</u>
Note that $\mathrm{Tr}(\omega\wedge\eta) = (-1)^{pq}\,\mathrm{Tr}(\eta\wedge\omega)$ for differential forms ω and η of degrees p and q respectively. Then we compute

$$I = \int_\Sigma \mathrm{Tr}\left((fg)^{-1}\partial(fg)\wedge(fg)^{-1}\bar\partial(fg)\right)$$

$$= \int_\Sigma \mathrm{Tr}\left(f^{-1}\partial f\wedge f^{-1}\bar\partial f + g^{-1}\partial g\wedge g^{-1}\bar\partial g\right)$$

$$+ \int_\Sigma \mathrm{Tr}\left(f^{-1}\partial f\wedge\bar\partial g g^{-1} + \partial g g^{-1}\wedge f^{-1}\bar\partial f\right)$$

(exercise)

applying $df^{-1} = -f^{-1} df\, f^{-1}$ and Stokes' theorem gives the result (exercise) $\square$

set
$$T_\Sigma(f,g) = -\frac{\sqrt{-1}\,k}{2\pi}\left(\int_\Sigma \mathrm{Tr}\left(f^{-1}\bar{\partial} f \wedge \partial g\, g^{-1}\right)\right)$$

For the complexification $G_{\mathbb{C}} = SL(2,\mathbb{C})$ consider
$$f : D \longrightarrow G_{\mathbb{C}} \quad \text{where} \quad D = \{ z \in \mathbb{C} \mid |z| \leq 1 \}$$
Denote the complement in $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ by D_∞, i.e. $D_\infty = \{ z \in \mathbb{C} \mid |z| \geq 1 \} \cup \{\infty\}$

consider $f : \mathbb{CP}^1 \longrightarrow G_{\mathbb{C}}$ smooth s.t.
$$f|_D = f_0 \quad \text{and} \quad f|_D = f_\infty$$
for some f_∞. Then $\exp\left(-S_{\mathbb{CP}^1}(f)\right) \in \mathbb{C}$ depends on extension f.

Consider second extension $f' : \mathbb{CP}^1 \longrightarrow G_{\mathbb{C}}$. Then
$$f' = f h, \quad \text{where } h : \mathbb{CP}^1 \longrightarrow G_{\mathbb{C}} \text{ with } h|_D = e$$
unit of $G_{\mathbb{C}}$

and denote $h_\infty \equiv h|_{D_\infty}$.

Using the Polyakov-Wiegmann formula we then obtain
$$\exp\left(-S_{\mathbb{CP}^1}(fh)\right) = \exp\left(-S_{\mathbb{CP}^1}(f) - S_{\mathbb{CP}^1}(h) + T_{\mathbb{CP}^1}(f,h)\right)$$

and $\quad T_{\mathbb{CP}^1}(f,h) = T_{D_\infty}(f_\infty, h_\infty)$

$$\Rightarrow \exp\left(-S_{\mathbb{CP}^1}(fh)\right) = \exp\Big(-S_{\mathbb{CP}^1}(f) - S_{\mathbb{CP}^1}(h) + T_{D_\infty}(f_\infty, h_\infty)\Big) \qquad (*)$$

Denote by $\mathrm{Map}_0(D_\infty, G_{\mathbb{C}})$ the set of smooth maps $\varphi : D_\infty \longrightarrow G_{\mathbb{C}}$ with $\varphi(\infty) = e$

using $z^{-1} = re^{\sqrt{-1}\theta}$ we set $\varphi(re^{\sqrt{-1}\theta}) = p_r(e^{\sqrt{-1}\theta})$.

The map $p_r : S^1 \longrightarrow G_{\mathbb{C}}$, $0 \le r \le 1$

defines loop of $G_{\mathbb{C}}$ for fixed r, for $r = 0$

$\rightarrow p_0 : \theta \mapsto e$ for $\theta \in S^1$

$\rightarrow p_r$, $0 \le r \le 1$ corresponds to a path in $LG_{\mathbb{C}}$ starting at $e \in LG_{\mathbb{C}}$ (constant map from S^1 to $G_{\mathbb{C}}$)

Introduce equivalence relation $\sim$ on

$$\mathrm{Map}_0(D_\infty, G_{\mathbb{C}}) \times \mathbb{C}$$

by setting for $(f_\infty, u), (g_\infty, v) \in \mathrm{Map}_0(D_\infty, G_{\mathbb{C}})$:

$$(f_\infty, u) \sim (g_\infty, v)$$

<u>iff:</u>

a) $f_\infty(z) = g_\infty(z)$ holds for $z \in \partial D$.

b) for $g_\infty = f_\infty h_\infty$ one has

$$v = u \exp\left(-S_{\mathbb{CP}^1}(h) + T_{D_\infty}(f_\infty, h_\infty)\right)$$

$\rightarrow \mathrm{Map}_0(D_\infty, G_{\mathbb{C}}) \times \mathbb{C}/\sim$ gives line bundle L on $LG_{\mathbb{C}}$

L is K-fold tensor product of fundamental l.b.