

## Lagrangians and Hamiltonians in classical mechanics

Let  $N$  be an  $n$ -dimensional smooth manifold and denote by  $TN$  its tangent bundle.

Definition:

Let  $\mathcal{L}(t, x, \dot{x})$  be a "Lagrangian" defined over  $TN$  with  $x = (x_1, \dots, x_n) \in N$ ,  $\dot{x} = (\dot{x}_1, \dots, \dot{x}_n) \in TN$ , and  $\gamma : [a, b] \rightarrow N$  a smooth curve on  $N$ .

Then

$$S = \int_a^b \mathcal{L}(t, \gamma(t), \gamma'(t)) dt$$

is called the "action integral".

A critical point of the action satisfies

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}_j} = \frac{\partial \mathcal{L}}{\partial x_j}, \quad j = 1, \dots, n$$

"Euler-Lagrange equations"

Example:

$\mathcal{L} = \frac{1}{2} m \sum_{j=1}^n \dot{x}_j^2 - V$ , where  $V$  is a potential, gives back Newton's equations for a particle.

Next, let  $M$  be a smooth manifold of dimension  $2n$ . A "symplectic form"  $\omega$  is a non-degenerate, closed 2-form on  $M$ . ( $\omega \neq 0$  on  $M$ )

Definition:

A smooth manifold equipped with a symplectic form is called a "symplectic manifold".

Given a smooth vector field  $X$  on  $M$ , the correspondence  $X \mapsto \iota(X)\omega$  gives isomorphism between 1-forms on  $M \longleftrightarrow$  smooth vector fields on  $M$

Definition (Hamiltonian vector field):

Let  $f$  be a smooth function on  $M$ .

Define vector field  $X_f$  by

$$\iota(X_f)\omega = df$$

$X_f$  is "Hamiltonian vector field" for  $f$ .

Definition (Poisson bracket):

For smooth functions  $f$  and  $g$  on  $M$ , define

$$\{f, g\} = -\omega(X_f, X_g)$$

Properties:

- bilinear and anti-symmetric

- satisfy Jacobi identity:

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$$

- $\{f, gh\} = \{f, g\}h + g\{f, h\}$  (derivation)

We have

$$\omega(X_f, X_g) = \iota(X_f)\omega(X_g) = df(X_g) = X_g f = -X_f g$$

Denote by  $\mathcal{X}(M)$  the space of smooth vector fields on  $M$ . Then

$$[X_f, X_g] = X(Yf) - Y(Xf), \quad f, g \in C^\infty(M),$$

→  $\mathcal{X}(M)$  is equipped with a Lie algebra structure.

$C^\infty(M)$  is a Lie algebra by the Poisson bracket and we have the following relation:

$$[X_f, X_g] = X_{\{f, g\}}$$

→  $f \mapsto X_f$  defines a Lie-algebra homomorphism

Consider a smooth function  $H$  on  $M$  called the "Hamiltonian" and the associated vector field  $X_H$ . Then

$$\frac{df}{dt} = X_H f = \{H, f\} \Rightarrow \frac{dH}{dt} = \{H, H\} = 0$$

## Complex line bundles and quantization

Let  $M$  be a smooth manifold and  $E$  a smooth vector bundle on  $M$ .

Denote

- $T^*M_{\mathbb{C}}$  : complexification of  $T^*M$
- $\Gamma(E)$  : space of smooth sections of  $E$
- $\Gamma(T^*M_{\mathbb{C}} \otimes E)$  : space of smooth sections of  $T^*M_{\mathbb{C}} \otimes E$

Definition (connection) :

A "connection" on  $E$  is a  $\mathbb{C}$ -linear map

$$\nabla : \Gamma(E) \rightarrow \Gamma(T^*M_{\mathbb{C}} \otimes E)$$

such that the Leibniz rule

$$\nabla(fs) = df \otimes s + f \nabla(s)$$

holds for  $f \in C^\infty(M)$  and  $s \in \Gamma(E)$ .

Definition (covariant derivative) :

For a vector field  $X \in \Gamma(T^*M_{\mathbb{C}})$  define a linear map  $\nabla_X : \Gamma(E) \rightarrow \Gamma(E)$  by

$$(\nabla_X s)(x) = (\nabla s)(X_x).$$

Definition (complex line bundle):

A "complex line bundle" is a complex vector bundle of rank 1.

Let  $L$  be a complex line bundle with Hermitian metric over  $M$ .

open covering:  $M = \bigcup_j U_j \rightarrow L = \bigcup_j U_j \times \mathbb{C}$  over  $U_j$   
(trivialization of  $L$ )

Let  $\nabla$  be a connection on  $L$  given by

$$\nabla = d - 2\pi\sqrt{-1}\alpha_j \text{ on } U_j,$$

where  $\alpha_j$  is 1-form on  $U_j$ .

Definition (first Chern class):

$d\alpha_j$  defines a global 2-form on  $M$ , the "first Chern-form" of  $\nabla$ , denoted by  $c_1(\nabla)$ .

Its de Rham cohomology class  $[c_1(\nabla)] \in H^2(M, \mathbb{R})$  is called "first Chern class" of  $L$ .

We have

$$[c_1(\nabla)] \in \text{Im } \iota,$$

where  $\iota: H^2(M, \mathbb{Z}) \rightarrow H^2(M, \mathbb{R})$  is the inclusion map

- classical Hamiltonian mechanics :

$$\frac{df}{dt} = \{H, f\}$$

- quantum mechanics :

$$\frac{d\hat{f}}{dt} = [\hat{H}, \hat{f}], \text{ where } \hat{f} = \sqrt{-1} \tilde{f}/\hbar$$

and  $\tilde{f}$  is the operator version of  $f$

"Heisenberg equation"

Let us apply this program of quantization to a symplectic manifold  $(M, \omega)$  together with a complex line bundle  $L$ :

- choose  $L$  and  $\nabla$  such that

$$c_1(\nabla) = \omega$$

- Denote by  $T(M, L)$  the space of smooth sections of  $L$  with inner product

$$\langle s_1, s_2 \rangle = \int_M (s_1(x), s_2(x)) \frac{\omega^n}{n!} \quad (*)$$

where  $(s_1(x), s_2(x))$  is hermitian metric on  $L$

- Denote by  $\mathcal{H}$  the space of  $L^2$  sections of  $L$  with respect to  $(*)$ .  
→ Hilbert space

- quantization map: for  $f \in C^\infty(M)$  define  
 $f \mapsto \hat{f}$  by  $\hat{f}s = \nabla_{X_f} s - \frac{i}{\hbar} \sqrt{-1} fs$ ,  $s \in \mathcal{H}$

Using  $\omega(X_f, X_g) = -\{f, g\}$ , one can verify

$$[\hat{f}, \hat{g}] = \widehat{\{f, g\}}$$

$\rightarrow f \mapsto \hat{f}$  determines a representation of  $C^\infty(M)$  on  $\mathcal{H}$  as a Lie algebra.

Example:

$N = \mathbb{R}^n \rightarrow M = T^*N$  with symplectic form

$$\omega = \sum_{i=1}^n dp_i \wedge dq_i$$

↑                           ↑  
coordinates               coordinates on  $N$   
on  $TN$

Define a connection on the trivial line bundle  $L = M \times \mathbb{C}$  by:

$$\nabla = d - \sqrt{-1} \tau^{-1} \theta, \quad \theta = \sum_{i=1}^n p_i dq_i$$

For a section  $s: M \rightarrow L$  define

$$\begin{aligned} \tilde{f} s &= \sqrt{-1} \tau \nabla_{X_f} s + fs \\ \rightarrow \tilde{p}_j &= -\sqrt{-1} \tau \frac{\partial}{\partial q_j}, \quad \tilde{q}_j = \sqrt{-1} \tau \frac{\partial}{\partial p_j} + q_j. \end{aligned}$$

Let  $\mathcal{H}_0$  be the subspace of  $T(M, L)$  consisting of  $L^2$  sections depending only on  $q_1, \dots, q_n$ . For  $s \in \mathcal{H}_0$ :  $\tilde{p}_j s = -\sqrt{-1} \tau \frac{\partial}{\partial q_j} s$ ,  $\tilde{q}_j s = q_j s$

→ recovered canonical quantization in quantum mechanics

Definition (Polarization):

Let  $(M, \omega)$  be a symplectic manifold of dimension  $2n$  and  $TM_{\mathbb{C}}$  the complexified tangent bundle.

$V_p \subset TM_{\mathbb{C}}$  subbundle is "integrable" if: for  $X, Y : M \rightarrow V_p \Rightarrow [X, Y] : M \rightarrow V_p$

$V_p$  is "Lagrangian" if

$\forall x \in M : \dim(V_p)_x = n$  and  
 $\omega|_{(V_p)_x} = 0$

A Lagrangian  $V_p$  is called "polarization", if it is integrable.

Define  $\mathcal{H}_p = \left\{ s \in \mathcal{H} \mid \nabla_X s = 0, X \in \Gamma(M, V_p) \right\}$   
↑  
quantum Hilbert space

Definition (Kähler polarization):

Let  $(M, \omega)$  be a Kähler manifold, set  $V_p = TM^{(0,1)}$   
→  $\mathcal{H}_p = H^0(M, L)$  space of holomorphic sections