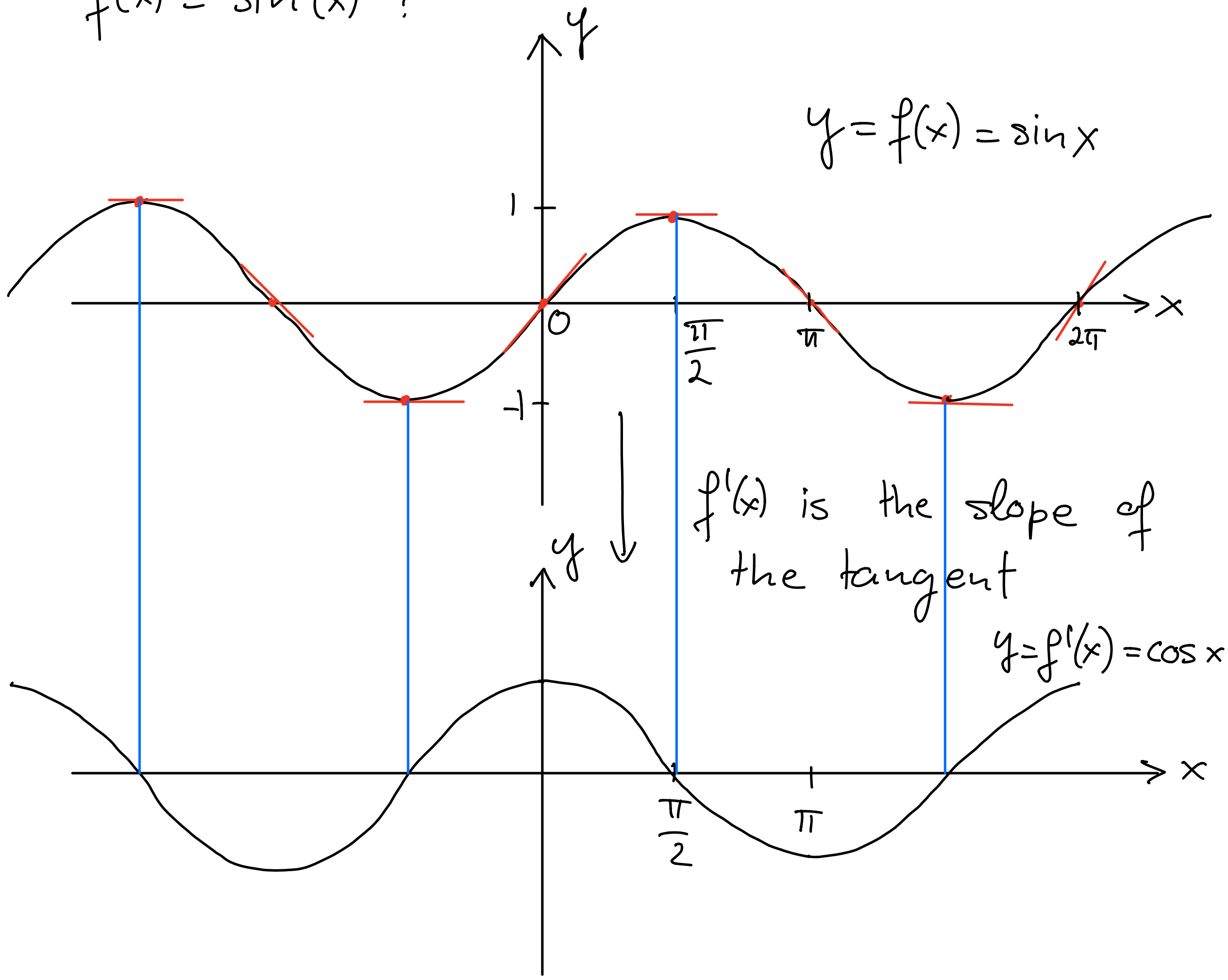
Last time we saw

$$\sin'(x) = \cos x\,, \quad \cos'(x) = -\sin(x)$$

Let us sketch the graph of the function $f(x) = \sin(x)$:



We see that the graph of the derivative nicely fits our interpretation of it being the slope of the tangent.

Example 5.7:

i) Differentiate $y = x^2 \sin x$

solution:

Using the product law, we obtain

$$\frac{dy}{dx} = x^2 \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(x^2)$$

$$= x^2 \cos x + 2x \sin x$$

ii) Differentiate $\cos = \sqrt{1 - \sin^2 x}$

solution:

$$\frac{d}{dx} \cos x = \frac{d}{dx} g(f(x))$$

where $g(y) = \sqrt{y}$ and $f(x) = 1 - \sin^2 x$

$$= g'(f(x))\, f'(x)$$

$$= \frac{1}{2\sqrt{1-\sin^2 x}} \frac{d}{dx}(1 - \sin^2 x)$$

$$= \frac{1}{2\cos x} \frac{d}{dx}\left(1 - \sin^2 x\right)$$

$$= \frac{1}{2\cos x}\left(-2 \sin x \cos x\right)$$

$$= -\sin x$$

iii) Similarly, we can compute the derivative of the tan-function:

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$$

$$= \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x}$$

$$= \frac{\cos x \cdot \cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x$$

iv) The derivatives of the remaining trigonometric functions, $\csc = \frac{1}{\sin}$, $\sec = \frac{1}{\cos}$, $\cot = \frac{1}{\tan}$, can also be calculated easily.

Summarizing, we get

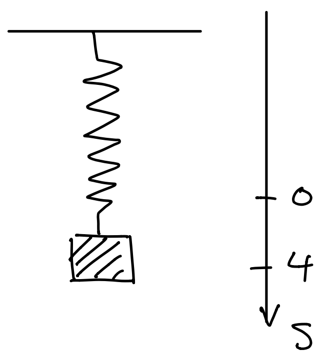
$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

Example 5.8:

Trigonometric functions are often used in modeling real-world phenomena:
An object at the end of a vertical spring is stretched 4cm beyond its rest position and released at time $t=0$



Its position at time t is $s = f(t) = 4\cos t$
Find velocity and acceleration
solution:

$$v = \frac{ds}{dt} = \frac{d}{dt}(4\cos t) = 4\frac{d}{dt}(\cos t) = -4\sin t$$

$$a = \frac{dv}{dt} = \frac{d}{dt}(-4\sin t) = -4\frac{d}{dt}(\sin t) = -4\cos t$$

The object oscillates from the lowest point $(s = 4\text{cm})$ to the highest point $(s = -4\text{cm})$
The period of oscillation is 2π.
The speed $|v| = 4|\sin t|$ is greatest when $|\sin t| = 1 \Rightarrow$ at $s = 0$.

Example 5.9:

i) $\arcsin : [-1,1] \longrightarrow \mathbb{R}$ is the inverse function to $\sin : [-\frac{\pi}{2}, \frac{\pi}{2}] \longrightarrow \mathbb{R}$. For $x \in (-1,1)$ we have:

$$\arcsin'(x) = \frac{1}{\sin'(\arcsin x)} = \frac{1}{\cos(\arcsin x)}$$

Let $y := \arcsin x$. Then $\sin y = x$ and $\cos y = \sqrt{1-x^2}$, as $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Therefore, we get

$$\frac{d\arcsin x}{dx} = \frac{1}{\sqrt{1-x^2}} \quad \text{for } -1 < x < 1.$$

ii) $\arctan : \mathbb{R} \longrightarrow \mathbb{R}$ is the inverse function to $\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \longrightarrow \mathbb{R}$. Therefore

$$\arctan'(x) = \frac{1}{\tan'(\arctan x)} = \cos^2(\arctan x)$$

Setting $y := \arctan x$, we have

$$x^2 = \tan^2 y = \frac{\sin^2 y}{\cos^2 y} = \frac{1-\cos^2 y}{\cos^2 y} = \frac{1}{\cos^2 y} - 1,$$

so

$$\cos^2 y = \frac{1}{1+x^2}$$

Thus

$$\frac{d\arctan x}{dx} = \frac{1}{1+x^2}.$$

§5.3 Implicit Differentiation

The functions we have met so far can be described explicitly, for example

$$y = \sqrt{x^3+1} \quad \text{or} \quad y = x \sin x$$

or, in general, $y = f(x)$. Some functions, however, are defined implicitly, such as

$$x^2 + y^2 = 25 \qquad (1)$$

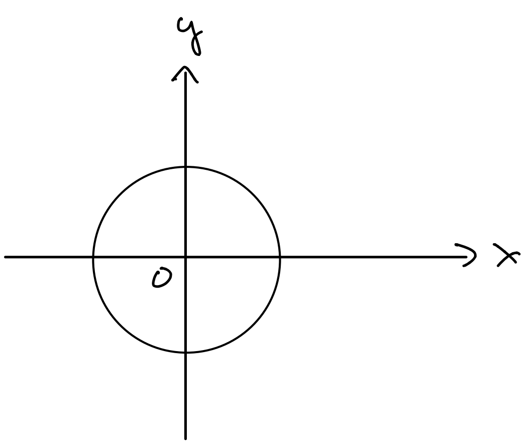
or

$$x^3 + y^3 = 6xy \qquad (2)$$

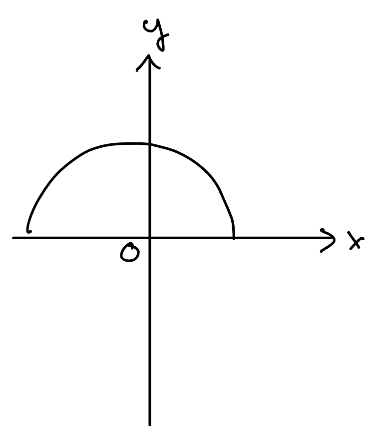
Solving (1) for y we get

$$y = \pm\sqrt{25 - x^2}$$

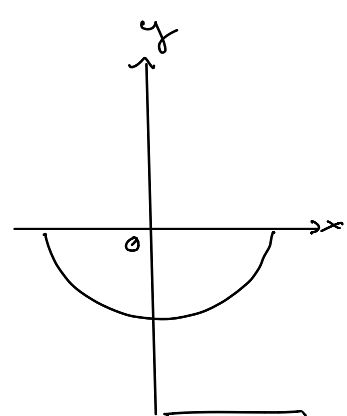
$\Rightarrow$ $f(x) = \sqrt{25-x^2}$ and $g(x) = -\sqrt{25-x^2}$ are both solutions.



$x^2 + y^2 = 25$

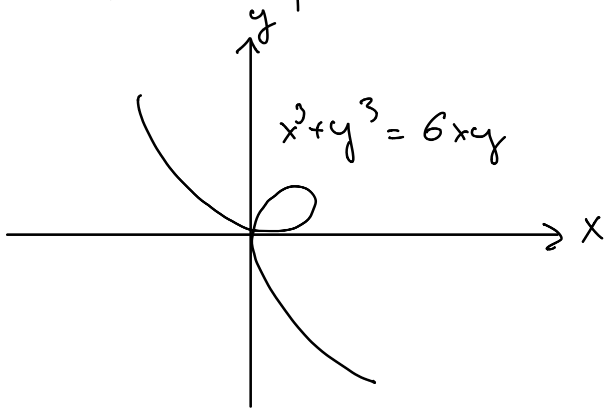


$f(x) = \sqrt{25-x^2}$

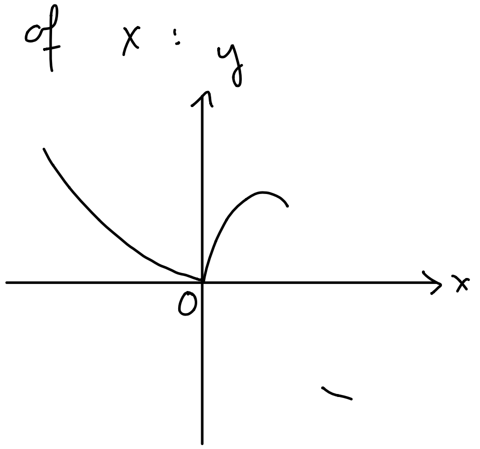


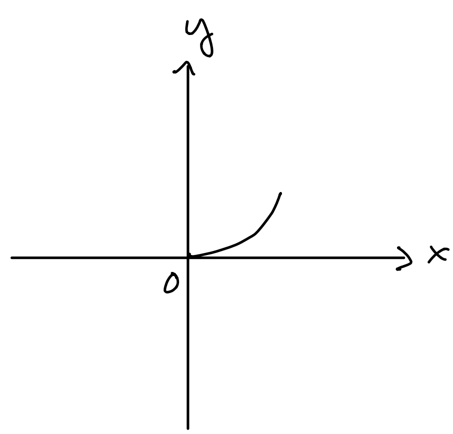
$g(x) = -\sqrt{25-x^2}$

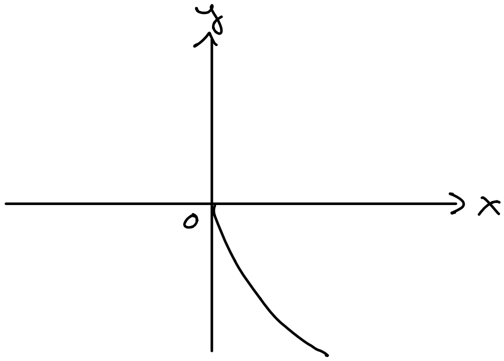
It's not easy to solve equation (2) for y explicitly as a function of x.
Nevertheless, (2) is the equation for the "folium of Descartes"



It implicitly defines y as several functions of x:







For all these graphs we have
$$x^3 + [f(x)]^3 = 6x\,f(x)$$
where only the domains of definition vary.

We can use "implicit differentiation" to differentiate such functions:

Example 5.10:

i) If $x^2 + y^2 = 25$, find $\frac{dy}{dx}$

Solution:

Differentiate both sides:

$$\frac{d}{dx}\left(x^2 + y^2\right) = \frac{d}{dx}(25)$$

$$\frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(y^2\right) = 0$$

Using the chain rule, we get

$$\frac{d}{dx}\left(y^2\right) = \frac{d}{dy}\left(y^2\right)\frac{d}{dx}y = 2y\frac{dy}{dx}$$

Thus $$2x + 2y\frac{dy}{dx} = 0$$

Solving this equation for $\frac{dy}{dx}$ we get

$$\frac{dy}{dx} = -\frac{x}{y} \qquad (*)$$

ii) Find an equation for the tangent to the circle $x^2 + y^2 = 25$ at the point $(3,4)$

Solution:

At the point $(3,4)$ we have $x = 3$, $y = 4$, so

$$\frac{dy}{dx} = -\frac{3}{4}$$

Thus the tangent at $(3,4)$ is given by

$$y - 4 = -\frac{3}{4}(x-3)$$

or $\quad 3x + 4y = 25$

<u>Note:</u> Equation $(*)$ is correct, no matter what function y is used:

- for $y = \sqrt{25 - x^2}$ we get

$$\frac{dy}{dx} = -\frac{x}{y} = -\frac{x}{\sqrt{25-x^2}}$$

- for $y = -\sqrt{25 - x^2}$ we get

$$\frac{dy}{dx} = -\frac{x}{y} = -\frac{x}{-\sqrt{25-x^2}} = \frac{x}{\sqrt{25-x^2}}$$

iii) Find y' if $x^3 + y^3 = 6xy$

solution:

Differentiating both sides, we get

$$3x^2 + 3y^2 y' = 6xy' + 6y$$

or $$x^2 + y^2 y' = 2xy' + 2y$$

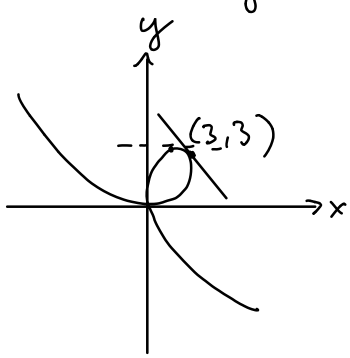
solving for y' gives: $\quad y' = \dfrac{2y - x^2}{y^2 - 2x}$

iv) Find the tangent to the folium of Descartes $x^3 + y^3 = 6xy$ at the point $(3,3)$.

solution:

When $x = y = 3$,

$$y' = \frac{2\cdot 3 - 3^2}{3^2 - 2\cdot 3} = -1$$



equation of the tangent at $(3,3)$ is:

$y - 3 = -1(x - 3)$ or $x + y = 6$

v) At what point in the first quadrant is the tangent line horizontal?

solution:

The tangent line is horizontal if $y' = 0$

$\Leftrightarrow 2y - x^2 = 0$

Substituing $y = \frac{1}{2}x^2$ in the curve equation, we get

$$x^3 + \left(\frac{1}{2}x^2\right)^3 = 6x\left(\frac{1}{2}x^2\right)$$

$\Rightarrow \quad x^6 = 16x^3 \quad \Rightarrow \quad x^3 = 16$

Thus $x = 16^{1/3} = 2^{4/3}$, $y = \frac{1}{2}(2^{8/3}) = 2^{5/3}$

$\Rightarrow$ the tangent is horizontal at $(2^{4/3}, 2^{5/3})$

Remark 5.3:

Finding explicit roots of the equation

$$y^3 + x^3 = 6xy$$

using a computer algebra system, we get

$$y = f(x) = \sqrt[3]{-\frac{1}{2}x^3 + \sqrt{\frac{1}{4}x^6 - 8x^3}} + \sqrt[3]{-\frac{1}{2}x^3 - \sqrt{\frac{1}{4}x^6 - 8x^3}}$$

and

$$y = \frac{1}{2}\left[-f(x) \pm \sqrt{-3}\left(\sqrt[3]{-\frac{1}{2}x^3 + \sqrt{\frac{1}{4}x^6 - 8x^3}} - \sqrt[3]{-\frac{1}{2}x^3 - \sqrt{\frac{1}{4}x^6 - 8x^3}}\right)\right]$$

These are the 3 functions from 3 graphs

$\rightarrow$ implicit differentiation saves enormeous amount of time

Moreover, for $y^5 + 3x^2y^2 + 5x^4 = 12$ it is <u>impossible</u> to solve for y!

(as shown fameously by Galois)

Example 5.11:

Find y' if $\sin(x+y) = y^2 \cos x$

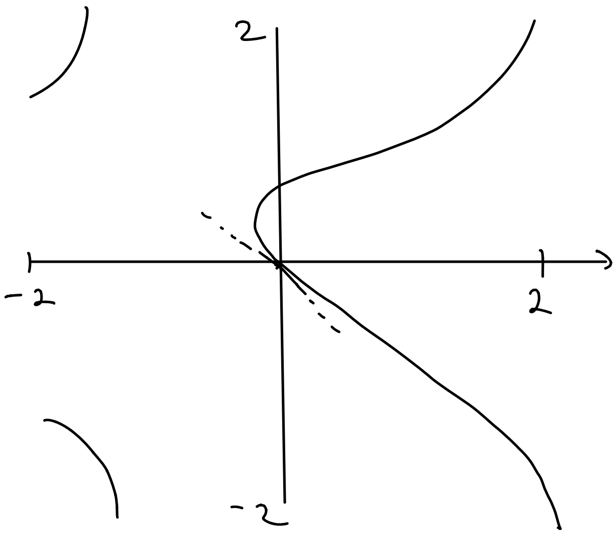
solution:

Differentiating both sides, we get

$$\cos(x+y) \cdot (1+y') = y^2(-\sin x) + \cos x (2yy')$$

$$\Leftrightarrow \cos(x+y) + y^2 \sin x = (2y\cos x) y' - \cos(x+y) \cdot y'$$

$$\Leftrightarrow \quad y' = \frac{y^2 \sin x + \cos(x+y)}{2y\cos x - \cos(x+y)}$$



Example 5.12 :

Find y'' if $x^4 + y^4 = 16$.

Solution :

Differentiation gives

$$4x^3 + 4y^3 y' = 0$$

Solving for y' gives

$$y' = -\frac{x^3}{y^3} \qquad (3)$$

Differentiating once again gives

$$y'' = \frac{d}{dx}\left(-\frac{x^3}{y^3}\right) = -\frac{y^3 (d/dx)(x^3) - x^3 (d/dx)(y^3)}{(y^3)^2}$$

$$= -\frac{y^3 \cdot 3x^2 - x^3 (3y^2 \cdot y')}{y^6}$$

If we substitute (3) into this expression, we get

$$y'' = -\frac{3x^2y^3 - 3x^3y^2\left(-\frac{x^3}{y^3}\right)}{y^6}$$

$$= -\frac{3(x^2y^4 + x^6)}{y^7} = -\frac{3x^2(y^4+x^4)}{y^7}$$

Since the values of x and y satisfy

$$x^4 + y^4 = 16,$$

we get

$$y'' = -\frac{3x^2(16)}{y^7} = -48\frac{x^2}{y^7}$$