§5. KZ equation

Definition:

The "configuration space" of n ordered distinct points on $\mathbb{CP}^1$ is

$$\mathrm{Conf}_n(\mathbb{CP}^1) = \{(p_1, \dots, p_n) \in (\mathbb{CP}^1)^n \mid p_i \neq p_j \text{ if } i \neq j\}$$

Now choose level K highest weights $\lambda_1, \dots, \lambda_n$ and consider the space

$$\mathcal{E}_{\lambda_1, \dots, \lambda_n} = \bigcup_{(p_1, \dots, p_n) \in \mathrm{Conf}_n(\mathbb{CP}^1)} \mathcal{H}(p_1, \dots, p_n; \lambda_1, \dots, \lambda_n)$$

as a vector bundle over $\mathrm{Conf}_n(\mathbb{CP}^1)$.

Want to put structure of vector bundle on $\mathcal{E}_{\lambda_1, \dots, \lambda_n}$

$\to$ consider trivial vector bundle

$$E = \mathrm{Conf}_n(\mathbb{CP}^1) \times \mathrm{Hom}_{\mathbb{C}}\left(\bigotimes_{j=1}^{n} H_{\lambda_j}, \mathbb{C}\right)$$

and $\mathcal{E}_{\lambda_1, \dots, \lambda_n} \subset E$ as subset.

For $j = 1, \dots, n$ consider hyperplanes D_j of $(\mathbb{CP}^1)^{n+1}$ defined by $z_{n+1} = z_j$ wher $(z_1, \dots, z_{n+1}) \in (\mathbb{CP}^1)^{n+1}$.

Let U be open set of $\mathrm{Conf}_n(\mathbb{CP}^1)$ and denote by $\mathcal{M}_{D_1 \dots D_n}(U)$ set of meromorphic functions with poles of any order at most at $D_1, \dots, D_n$

$\longrightarrow \mathfrak{g} \otimes \mathcal{M}_{D_1, \dots, D_n}(U)$ has structure of Lie algebra

$f \in \mathfrak{g} \otimes \mathcal{M}_{D_1 \cdots D_n}(U)$ has Laurent expansion along D_j, $1 \le j \le n$, of the form

$$f_{D_j}(t_j) = \sum_{m=-N}^{\infty} a_m(z_1, \dots, z_n) t_j^m \qquad (*)$$

where $t_j = z_{n+1} - z_j$ and $a_m(z_1, \dots, z_n)$ is a holomorphic function in $z_1, \dots, z_n$ with values in $\mathfrak{g}$. Denote by $\mathcal{O}(U)$ set of hol. functions on U. Then $(*)$ gives a map

$$\tau_j : \mathfrak{g} \otimes \mathcal{M}_{D_1 \cdots D_n}(U) \to \mathfrak{g} \otimes \mathcal{O}(U) \otimes \mathbb{C}((t_j))$$

determined by $\tau_j(f) = f_{D_j}(t_j)$.

$\Rightarrow$ get action of Lie algebra $\mathfrak{g} \otimes \mathcal{M}_{D_1 \cdots D_n}(U)$ on $\mathcal{H}_{\lambda_1} \otimes \cdots \otimes \mathcal{H}_{\lambda_n}$ depending on $(p_1, \dots, p_n) \in U$.

Define $\mathcal{E}_{\lambda_1 \cdots \lambda_n}(U)$ to be set of smooth sections

$$\Psi : U \to E$$

satisfying

$$\sum_{j=1}^{n} [\Psi(p_1, \dots, p_n)](\xi_1, \dots, \tau_j(f)\xi_j, \dots, \xi_n) = 0$$

for any $f \in \mathfrak{g} \otimes \mathcal{M}_{D_1 \cdots D_n}(U)$ and $\xi_j \in \mathcal{H}_{\lambda_j}$, $1 \le j \le n$ at any $(p_1, \dots, p_n) \in U$. Note $\Psi(p_1, \dots, p_n) \in \mathcal{H}(\vec{p}; \vec{\lambda})$

Have to show: $\mathcal{E}_{\lambda_1 \cdots \lambda_n}$ has structure of vector bundle with flat connection on open subset

$$\mathrm{Conf}_n(\mathbb{C}) = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j \text{ if } i \neq j\}$$

of $\mathrm{Conf}_n(\mathbb{CP}^1)$. To this end, define multi-linear map $X^{(j)}\Psi : H_{\lambda_1} \otimes \dots \otimes H_{\lambda_n} \to \mathbb{C}$ by

$$[X^{(j)}\Psi](\xi_1, \dots, \xi_n) = \Psi(\xi_1, \dots, X\xi_j, \dots, \xi_n),$$

where $\xi_1 \in H_{\lambda_1}, \dots, \xi_n \in H_{\lambda_n}$ and $X \in \mathfrak{g}$ acts on jth component.

<u>Proposition 1:</u>

If Ψ is a smooth section of $\mathcal{E}_{\lambda_1 \dots \lambda_n}$ over open subset $U \subset \mathrm{Conf}_n(\mathbb{C})$, then

$$\frac{\partial \Psi}{\partial z_j} - L_{-1}^{(j)} \Psi$$

is smooth section of $\mathcal{E}_{\lambda_1 \dots \lambda_n}$ over U for $\forall\, 1 \leq j \leq n$.

<u>Proof:</u>

For $f \in \mathfrak{g} \otimes \mathcal{M}_{D_1 \dots D_n}(U)$ get Laurent expansion

$$\tau_j(f) = f_{D_j}(t_j) = \sum_{m=-N}^{\infty} a_m(z_1, \dots, z_n) t_j^n$$

along D_j, where $t_j = z_{n+1} - z_j$. $\Rightarrow f_{z_j} = \frac{\partial f}{\partial z_j} \in \mathfrak{g} \otimes \mathcal{M}_{D_1 \dots D_n}(U)$ with Laurent expansion

$$\tau_j(f_{z_j}) = \sum_{m=-N}^{\infty} \left(\frac{\partial a_m}{\partial z_j} t_j^m - a_m m t_j^{m-1} \right)$$

define operator $\partial_j : \mathfrak{g} \otimes \mathcal{O}(U) \to \mathfrak{g} \otimes \mathcal{O}(U)$, $1 \leq j \leq n$

by $\partial_j(h) = h_{z_j}$ and extend to action on $\mathfrak{g} \otimes \mathcal{O}(U) \otimes \mathbb{C}((t_j))$ with trivial action on $\mathbb{C}((t_j))$.

$$\longrightarrow \quad \tau_j(f_{z_j}) = \partial_j \tau_j(f) - \frac{\partial}{\partial t_j} \tau_j(f)$$

By Prop. 6 of §2 we have

$$\frac{\partial}{\partial t_j} \tau_j(f) = -[L_{-1}, \tau_j(f)]$$

$$\Rightarrow \tau_j(f_{z_j}) = \partial_j \tau_j(f) + [L_{-1}, \tau_j(f)]$$

Have to show that:

$$\sum_{i=1}^{n} \left(\frac{\partial \Psi}{\partial z_j} - L_{-1}^{(j)} \Psi \right)(\xi_1, \ldots, \tau_i(f)\xi_i, \ldots, \xi_n) = 0$$

We have

$$\frac{\partial}{\partial z_j} \left[\Psi(\xi_1, \ldots, \tau_i(f)\xi_i, \ldots, \xi_n) \right]$$

$$= \left(\frac{\partial \Psi}{\partial z_j} \right)(\xi_1, \ldots, \tau_i(f)\xi_i, \ldots, \xi_n) + \Psi(\xi_1, \ldots, \partial_j \tau_i(f)\xi_i, \ldots \xi_n)$$

Altogether we thus obtain:

$$\sum_{i=1}^{n} \left(\frac{\partial \Psi}{\partial z_j} - L_{-1}^{(j)} \Psi \right)(\xi_1, \ldots, \tau_i(f)\xi_i, \ldots, \xi_n)$$

$$= \sum_{i=1}^{n} \left(\frac{\partial}{\partial z_j} \left[\Psi(\xi_1, \ldots, \tau_i(f)\xi_i, \ldots, \xi_n) \right] - \Psi(\xi_1, \ldots, \tau_i(f_{z_j})\xi_i, \ldots, \xi_n) \right)$$

$= 0$ by definition of Ψ

□

Intrduce the following linear operator:

$$\nabla_{\frac{\partial}{\partial z_j}} : \mathcal{E}_{\lambda_1 \cdots \lambda_n}(U) \longrightarrow \mathcal{E}_{\lambda_1 \cdots \lambda_n}(U)$$

defined by

$$\nabla_{\frac{\partial}{\partial z_j}} \Psi = \frac{\partial \Psi}{\partial z_j} - L_{-1}^{(j)} \Psi$$

Theorem 1:

The family of conformal blocks $\mathcal{E}_{\lambda_1 \cdots \lambda_n}$ over the configuration space $\mathrm{Conf}_n(\mathbb{C})$ has the structure of a vector bundle with a flat connection.

Proof:

We set $M = \mathrm{Conf}_n(\mathbb{C})$ and consider $E = M \times F$ with $F = \mathrm{Hom}_{\mathbb{C}}\left(\bigotimes_{j=1}^{n} H_{\lambda_j}, \mathbb{C}\right)$

Introduce connection

$$\nabla : \Gamma(E) \longrightarrow \Gamma(T^* M_{\mathbb{C}} \otimes E)$$

given by

$$\nabla \Psi = d\Psi - \underbrace{\sum_{i=1}^{n} L_{-1}^{(i)} \Psi \, dz_i}_{=: \omega}$$

$\longrightarrow d\omega = 0$ since $L_{-1}^{(i)}$ is independent of $z \in M$.

$\omega \wedge \omega = 0$ since $[L_{-1}^{(i)}, L_{-1}^{(j)}] = 0, \ 1 \leq i, j \leq n$

$\longrightarrow \nabla$ is flat connection $\square$

We call $\mathcal{E}_{\lambda_1 \cdots \lambda_n}$ the "conformal block bundle".

Set $\Psi_0 : V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n} \to \mathbb{C}$ restriction of Ψ

Set $\Omega = \sum_\mu I_\mu \otimes I_\mu$ and

$$[\Omega^{(ij)}\Psi_0](\xi_1, \ldots, \xi_n)$$
$$= \sum_\mu \Psi_0(\xi_1, \ldots, I_\mu \xi_i, \ldots, I_\mu \xi_j, \ldots, \xi_n)$$

Recall: $[(X \otimes t^{-1})^{(i)}\Psi](\xi_1, \ldots, \xi_n)$ $(**)$

$$= \sum_{j, j \neq i} (z_i - z_j)^{-1} \Psi(\xi_1, \ldots, X\xi_j, \ldots, \xi_n)$$

Proposition 2:

If a multilinear form $\Psi : \mathcal{H}_{\lambda_1} \otimes \cdots \otimes \mathcal{H}_{\lambda_n} \to \mathbb{C}$ belongs to the space of conformal blocks $\mathcal{H}(p_1, \ldots, p_n ; \lambda_1, \ldots, \lambda_n)$, then the restriction $(L_{-1}^{(i)}\Psi)_0$ of $L_{-1}^{(i)}\Psi : \mathcal{H}_{\lambda_1} \otimes \cdots \otimes \mathcal{H}_{\lambda_n} \to \mathbb{C}$ on $V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}$ is given by

$$(L_{-1}^{(i)}\Psi)_0 = \sum_{j, j \neq i} \frac{\Omega^{(ij)}\Psi_0}{z_i - z_j} .$$

Furthermore, we have

(1) $(L_n^{(i)}\Psi)_0 = 0$, $n > 0$

(2) $(L_0^{(i)}\Psi)_0 = \Delta_{\lambda_i}\Psi_0$

where Δ_{λ_i} is eigenvalue of L_0 on V_{λ_i} .

Proof:

For $v \in V_{\lambda_j} \subset H_{\lambda_j}$ we have

$$L_{-1} v = \frac{1}{k+2} \left(\sum_{\mu} I_\mu \otimes t^{-1} \cdot I_\mu \right) v$$

for $v \in V_{\lambda_j} \subset H_\lambda$. Combining with (**) we get

$$\sum_{\mu} \psi(\xi_1, \ldots, (I_\mu \otimes t^{-1} \cdot I_\mu)\xi_i, \ldots, \xi_n)$$

$$= \sum_{j, j \neq i} (z_i - z_j)^{-1} \psi_0(\xi_1, \ldots, I_\mu \xi_i, \ldots, I_\mu \xi_j, \ldots, \xi_n)$$

$$= \sum_{j, j \neq i} \frac{\Omega^{(ij)} \psi_0}{z_i - z_j}$$

Equations (1) and (2) follow directly from definition of Sugawara operators. $\square$

Combining Theorem 1 and Proposition 2 we obtain

Theorem 2:

Let ψ be a horizontal section of the conformal block bundle $\mathcal{E}_{\lambda_1 \ldots \lambda_n}$. Then the restriction ψ_0 satisfies

$$\frac{\partial \psi_0}{\partial z_i} = \frac{1}{k+2} \sum_{j, j \neq i} \frac{\Omega^{(ij)} \psi_0}{z_i - z_j}, \quad 1 \le i \le n$$

"Knizhnik-Zamolodchikov equation"

∇ is call "KZ connection"

Proposition 3:
Let $\mathcal{Y}$ be a horizontal section of the conformal block bundle $\mathcal{E}_{\lambda_1 \cdots \lambda_n}$. Then $\mathcal{Y}_0$ satisfies

$$\sum_{i=1}^{n} z_i^r \left(z_i \frac{\partial}{\partial z_i} + (r+1) \Delta_{\lambda_i} \right) \mathcal{Y}_0 = 0$$

for $r = -1, 0, 1$

Proof:
Invariance of $\mathcal{Y}_0$ under diagonal action of $\mathfrak{g}$ gives

$$\sum_{j=1}^{n} \Omega^{(ij)} \mathcal{Y}_0 = 0, \quad 1 \le i \le n.$$

→ Taking sum over i gives:

$$\sum_{j \neq i} \Omega^{(ij)} \mathcal{Y}_0 = - \sum_{j=1}^{n} \Omega^{(jj)} \mathcal{Y}_0$$

$$\Rightarrow \sum_{1 \le i \le j \le n} \Omega^{(ij)} \mathcal{Y}_0 = -(k+2) \sum_{j=1}^{n} \Delta_{\lambda_j} \mathcal{Y}_0$$

by using that $\Omega^{(ij)} = \Omega^{(ji)}$ and $\Omega^{(jj)}$ Casimir

$$\Rightarrow \sum_{i=1}^{n} z_i^{r+1} \frac{\partial \mathcal{Y}_0}{\partial z_i} = \frac{1}{k+2} \sum_{i \neq j} \frac{z_i^{r+1} \Omega^{(ij)} \mathcal{Y}_0}{z_i - z_j}$$

One can now check that Prop. follows for $r = -1, 0, 1$ □

Proposition 4:
Under a Möbius transformation $w_j = \frac{a z_j + b}{c z_j + d}$, $1 \le j \le n$, $a, b, c, d \in \mathbb{C}$, $ad - bc = 1$, $\mathcal{Y}_0$ behaves as

$$\mathcal{Y}_0(z_1, \ldots, z_n) = \prod_{j=1}^{n} (c z_j + d)^{-2\Delta_{\lambda_j}} \mathcal{Y}_0(w_1, \ldots, w_n)$$