Recall from last lecture:

Have defined set of equivalence classes

$$\mathrm{Map}_0(D_\infty, G_{\mathbb{C}}) \times \mathbb{C} / \sim$$

where $(f_\infty, u) \sim (g_\infty, v)$ iff:

a) $f_\infty(z) = g_\infty(z)$ on ∂D

b) For $g_\infty = f_\infty h_\infty$ one has

$$v = u \exp\left(-S_{\mathbb{C}P^1}(h) + T_{D_\infty}(f_\infty, h_\infty)\right)$$

with h being extension of h_∞ s.t. $h(z) = e$ on D.

$\longrightarrow$ line bundle $\mathcal{L}$

Now: Define projection map $\pi: \mathcal{L} \longrightarrow LG_{\mathbb{C}}$

by $\pi([f_\infty, u]) = f_\infty \circ \iota$

where $\iota: \partial D \to D$ is inclusion map

In the above, f_∞ and g_∞ correspond to different paths γ_1 and γ_2 s.t.

$$\gamma_i : [0,1] \longrightarrow LG_{\mathbb{C}}, \quad i = 1,2$$

$$\gamma_1(0) = \gamma_2(0) = e,$$

a) $\Leftrightarrow$ $\gamma_1(1) = \gamma_2(1)$, b) $\Leftrightarrow$ holonomy along $\gamma_1 \cdot \gamma_2^{-1}$

$\rightarrow$ analogous to line bundle of Prop. 3 in §1

one can show: $\mathcal{L}$ is isomorphic to k-fold tensor product of fundamental line bundle

Suppose $f : \mathbb{C}P^1 \to G_\mathbb{C}$ is extension of $f_0 : D \to G_\mathbb{C}$. Define $\exp(-S_D(f_0)) \equiv [f_\infty, \exp(-S_{\mathbb{C}P^1}(f))]$

<u>Lemma:</u>
For $f_0 : D \to G_\mathbb{C}$, $[f_\infty, \exp(-S_{\mathbb{C}P^1}(f))]$ does not depend on choice of extension of f_0
$\to$ element in fibre of $\mathcal{L}$ over $f_0 \circ \iota$.

<u>Proof:</u>
Take another extension $f' : \mathbb{C}P^1 \to G_\mathbb{C}$ of $f_0 : D \to G_\mathbb{C}$ with $f' = fh$. Then

$$(f_\infty, \exp(-S_{\mathbb{C}P^1}(f))) \sim (f_\infty h_\infty, \exp(-S_{\mathbb{C}P^1}(fh)))$$

follows from Polyakov–Wiegmann formula. $\square$

<u>Dual line bundle $\mathcal{L}^{-1}$:</u>
Denote by $\mathrm{Map}_0(D, G_\mathbb{C})$ set of smooth maps $\varphi : D \to G_\mathbb{C}$ with $\varphi(0) = e$ and define equiv. relation $(f_0, u) \sim (g_0, v)$ by

a) $f_0(z) = g_0(z)$ for $z \in \partial D$

b) $g_0 = f_0 h_0 \to v = u \exp(-S_{\mathbb{C}P^1}(h) + \Gamma_D(f_0, h_0))$

$\to$ Define $\mathcal{L}^{-1}$ as $\mathrm{Map}_0(D, G_\mathbb{C}) \times \mathbb{C} / \sim$

$\to$ $\exp(-S_{D_\infty}(f_\infty))$ is well-defined as element of fibre of $\mathcal{L}^{-1}$ over $f_\infty \circ \iota$.

Denote by $\gamma : S^1 \to G_\mathbb{C}$ the loop defined by $f_0 \iota$.

Then we have pairing

$$\mathcal{L}_\gamma \times \mathcal{L}_\gamma^{-1} \longrightarrow \mathbb{C}$$

given by

$$\langle [f_\infty, u], [f_0, v] \rangle = uv \exp(S_{\mathbb{C}P^1}(f))$$

where $\mathcal{L}_\gamma$ is the fibre of $\mathcal{L}$ over γ.

$\rightarrow$ pairing well-defined as right-hand side is independent of representatives of equivalence relations

<u>Lemma:</u>

The following operation

$$\exp(-S_D(g_1)) \cdot \exp(-S_D(g_2))$$
$$= \exp(-\Gamma_D(g_1, g_2)) \exp(-S_D(g_1 g_2)),$$

for $g_i : D \longrightarrow G_{\mathbb{C}}$, $i = 1, 2$, defines a product

$$\mathcal{L}_{\gamma_1} \times \mathcal{L}_{\gamma_2} \longrightarrow \mathcal{L}_{\gamma_1 \cdot \gamma_2}$$

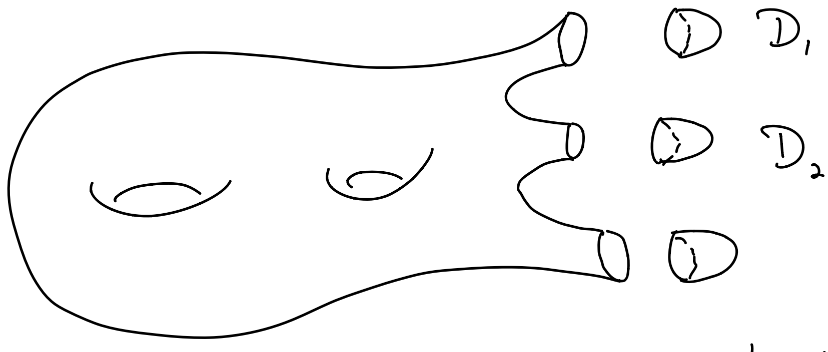
where $\gamma_i = g \circ \iota$. This product equippes $\widehat{LG_{\mathbb{C}}} = \mathcal{L} \setminus s(LG_{\mathbb{C}})$ with a group structure.

$\uparrow$ zero section

Next, let Σ be a compact Riemann surface with boundary. $\partial\Sigma$ is homeomorphic to a disjoint union of circles and we have diffs.

$$p_i : S^1 \longrightarrow \partial\Sigma, \quad 1 \leq i \leq m$$

for each connected component of $\partial \Sigma$.



Glue the boundary of unit discs D_i, $1 \le i \le m$, with $p_i(S^1)$, $1 \le i \le m$, to obtain a closed Riemann surface $\tilde{\Sigma}$.

For a smooth map $g : \Sigma \longrightarrow G_{\mathbb{C}}$ define the extension to $\tilde{\Sigma}$ as $\tilde{g} : \tilde{\Sigma} \longrightarrow G_{\mathbb{C}}$ and the restriction on D_i by g_i.

$\longrightarrow$ $\exp(-S_{D_i}(g_i))$ defines an element of the fibre of $\mathcal{L}^{-1}_{g \circ p_i}$

$\longrightarrow$ Define $\exp(-S_{\Sigma}(g))$ as element of $\bigotimes_{i=1}^{m} \mathcal{L}^{-1}_{g \circ p_i}$ specified by

$$\left\langle \exp\left(-S_{\Sigma}(g)\right), \bigotimes_{i=1}^{m} \exp\left(-S_{D_i}(g_i)\right) \right\rangle = \exp\left(-S_{\tilde{\Sigma}}(\tilde{g})\right)$$

By the Polyakov-Wiegmann formula this definition does not depend on choice of extension $\tilde{g}$.

Let us specify to the case $m=1$ for simplicity.

For a smooth map $f: \Sigma \to G_{\mathbb{C}}$ define the "left action" $l(f)$ on $\mathcal{L}$ by

$$l(f)\exp\left(-S_\Sigma(g)\right) = \exp\left(-S_\Sigma(f)\right)\bullet\exp\left(-S_\Sigma(g)\right)$$
$$= \exp\left(-S_\Sigma(fg) - T_\Sigma(f,g)\right)$$

Similarly, we define "right action" by

$$r(f)\exp\left(-S_\Sigma(g)\right) = \exp\left(-S_\Sigma(g)\right)\bullet\exp\left(-S_\Sigma(f)\right)$$

<u>Proposition 2:</u>

Let $g: \Sigma \to G_{\mathbb{C}}$ be a smooth map and let $h: \Sigma \to G_{\mathbb{C}}$ be a smooth map which is hol. on $\mathrm{Int}\,\Sigma$. Then

$$l(h)\exp\left(-S_\Sigma(g)\right) = \exp\left(-S_\Sigma(hg)\right)$$

and for anti-hol. $h^*: \Sigma \to G_{\mathbb{C}}$ we have

$$r(h^*)\exp\left(-S_\Sigma(g)\right) = \exp\left(-S_\Sigma(gh^*)\right)$$

<u>Proof:</u> Use $\bar\partial h = 0$ $\square$

<u>Definition:</u>

A representation $\rho: \mathrm{Map}(\Sigma, G_{\mathbb{C}}) \to \mathrm{Aut}(\Gamma(\mathcal{L}))$ is given by

space of sections of $\mathcal{L}$

$$[\rho(f)s](\gamma) = l(f)\,s\left(\left(f|_{\partial\Sigma}\right)^{-1}\cdot\gamma\right),$$
$$s \in \Gamma(\mathcal{L}),\ \gamma \in LG_{\mathbb{C}},\ \text{for } f \in \mathrm{Map}(\Sigma, G_{\mathbb{C}})$$

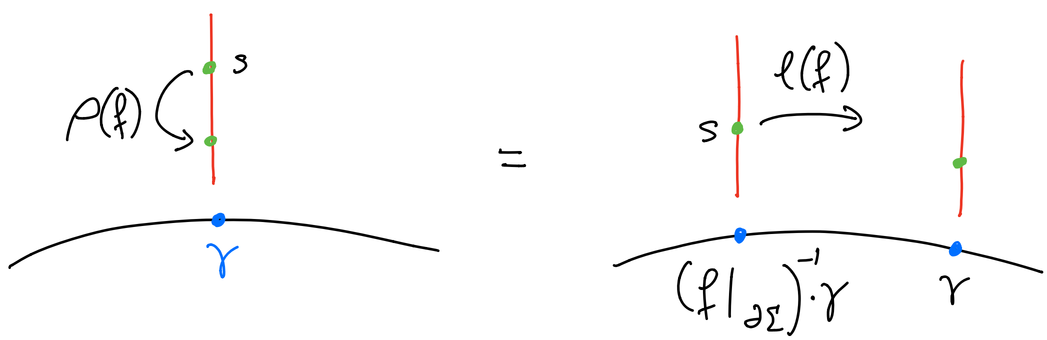
Similarly, define a representation

$\rho^*: \mathrm{Map}(\Sigma, G_{\mathbb{C}}) \longrightarrow \mathrm{Aut}(\Gamma(\mathcal{L}))$ by

$$[\rho^*(f)s](\gamma) = r(f^*)\, s\left(\gamma\cdot (f^*|_{\partial\Sigma})^{-1}\right),$$

$$s \in \Gamma(\mathcal{L}),\ \gamma \in LG_{\mathbb{C}},$$

where $f^*(z) = {}^t\overline{f(z)}$



<u>Infinitesimal action of $\mathrm{Map}(\Sigma, G_{\mathbb{C}})$</u>:

set for non-negative integer n and $X \in \mathfrak{g}$

$$X_{n,\varepsilon}(z) = e^{\varepsilon X z^n},\ z \in D,\ \varepsilon \in \mathbb{R}$$

and for negative n

$$X_{n,\varepsilon}(z) = e^{\varepsilon X z^{-n}},\ z \in D,\ \varepsilon \in \mathbb{R}$$

Infinitesimal action of $X_{n,\varepsilon}$ by ρ is defined by

$$X_n s = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \rho(X_{n,\varepsilon})\, s,\quad s \in \Gamma(\mathcal{L})$$

$\rightarrow$ The map defined by $X \otimes t^n \longmapsto X_n$ gives representation of affine Lie algebra

$\hat{g}$ on space of sections $\Gamma(\mathcal{L})$:

Lemma:

The operators X_m and Y_n, $m, n \in \mathbb{Z}$, satisfy the relation

$$[X_m, Y_n] = [X, Y]_{m+n} + mK\,\delta_{m+n,0}\langle X, Y\rangle$$

Proof:

Put $f = X_{m,\varepsilon_1}$ and $g = Y_{n,\varepsilon_2}$ for $\varepsilon_1, \varepsilon_2 \in \mathbb{R}$.

In the case $m, n \geq 0$ or $m, n \leq 0$ the relation $[X_m, Y_n] = [X, Y]_{m+n}$ follows from

$$\Gamma_D(f, g) = \Gamma_D(g, f) = 0.$$

Let us suppose $m \geq 0$ and $n \leq 0$. Then

$$\Gamma_D(f, g) = 0, \text{ but}$$

$$\lim_{\varepsilon_1, \varepsilon_2 \to 0} \frac{1}{\varepsilon_1 \varepsilon_2} \Gamma_D(g, f) = \frac{K}{2\pi\sqrt{-1}} \int_D \mathrm{Tr}\left(m z^{m-1} X dz \wedge n \bar{z}^{-n-1} Y d\bar{z}\right)$$

$= mK\langle X, Y\rangle$ for $m = -n$ and zero otherwise

$\square$

Define operators $\overline{X}_n s = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \rho^*(X_{n,\varepsilon}) s$, $s \in \Gamma(\mathcal{L})$

Then $X \otimes t^n \mapsto \overline{X}_n$ also defines a repr. of $\hat{g}$

$\rightarrow [\overline{X}_m, \overline{Y}_n] = \overline{[X, Y]_{m+n}} + mK\,\delta_{m+n,0}\langle X, Y\rangle$

but: $[X_m, \overline{Y}_n] = 0$

Definition:

A smooth section $\varphi \in \Gamma(\mathcal{L})$ is called "primary" if and only if

$$X_n \varphi = \overline{X}_n \varphi = 0 \qquad \forall X \in \mathfrak{g},\ n > 0$$

One can show that the space of sections of $\mathcal{L}$ contains a subspace in the representation

$$\bigoplus_{0 \leq \lambda \leq K} H_\lambda \otimes H_\lambda^*$$

Physics interpretation:

For a closed Riemann surface $\tilde{\Sigma}$ consider

$$\int_{f: \tilde{\Sigma} \to G_{\mathbb{C}}} \exp(-S_{\tilde{\Sigma}}(f))\, \mathcal{D}f$$

Decompose $\tilde{\Sigma} = \Sigma \cup \bigoplus_{i=1}^{m} D_i$. Take $m = 1$.

$\exp(-S_\Sigma(f)) \in \pi^{-1}(f \circ c)$ where $\pi: \mathcal{L} \to LG_{\mathbb{C}}$

This section will be an element of

$$\bigoplus_{0 \leq \lambda \leq K} H_\lambda \otimes H_\lambda^*$$

Then the path integral over $\tilde{\Sigma}$ will be obtained through the pairing $\mathcal{L} \times \mathcal{L}^{-1} \to \mathbb{C}$