§ 6. Vertex operators and OPE

Consider the space of conformal blocks for three points: $0 \in \mathbb{C}$, p and ∞

→ associate level K highest weights λ_0, λ and λ_∞

→ obtain space of conformal blocks

$\mathcal{H}(0, p, \infty; \lambda_0, \lambda, \lambda_\infty^*)$ as the space of multi-linear maps

$$\Psi : H_{\lambda_0} \times H_\lambda \times H^{\infty}_{\lambda_\infty} \longrightarrow \mathbb{C}$$

invariant under diagonal action of meromorphic functions with values in $\mathfrak{g}$ and poles at $0, p, \infty$.

Consider conformal block bundle

$$\mathcal{E} = \bigcup_{p \in \mathbb{C} \setminus \{0\}} \mathcal{H}(0, p, \infty; \lambda_0, \lambda, \lambda_\infty^*)$$

and let Ψ be a section of $\mathcal{E}$. Introduce bilinear map

$$\phi(v, z) : H_{\lambda_0} \otimes H^*_{\lambda_\infty} \longrightarrow \mathbb{C}$$

given by

$$\phi(v, z)(u \otimes w) = \Psi(z)(u, v, w)$$

for $u \in H_{\lambda_0}$, $v \in H_\lambda$ and $w \in H^*_{\lambda_\infty}$.

Regard $\phi(v, z)$ as linear operator from H_{λ_0} to H_{λ_∞}.

Note $H_\lambda = \prod_{d \geq 0} H_\lambda(d)$, with $H_\lambda(0) = V_\lambda$ (recall: $L_0 H_\lambda(d) = (\Delta_\lambda + d) H_\lambda(d)$)

Then we have the following

Proposition 1:

Let Ψ be a section of the above conformal block bundle $\mathcal{E}$. Then the linear operator $\phi(v,z)$, $z \in V_\lambda$, $v \in V_\lambda$, $z \in \mathbb{C} \setminus \{0\}$, defined by $\phi(v,z)(u \otimes w) = \Psi(z)(u,v,w)$ satisfies the commutation relation

$$[X \otimes t^n, \phi(v,z)] = z^n \phi(Xv, z) \qquad (*)$$

for $X \otimes t^n \in \hat{\mathfrak{g}}$.

Proof:

Consider the meromorphic function $f(z) = X \otimes z^n$, $X \in \mathfrak{g}$, $n \in \mathbb{Z}$. The action of $f(z)$ on H_{λ_0}, V_λ and $H^*_{\lambda_\infty}$ are given by

$$f(z)\,u = (X \otimes t^n)\,u, \quad u \in H_{\lambda_0},$$

$$f(z)\,v = z^n X v, \quad v \in V_\lambda,$$

$$f(z)\,w = -w(X \otimes t^n), \quad w \in H^*_{\lambda_\infty}$$

$\rightarrow$ invariance of Ψ under action of f implies:

$$\Psi((X \otimes t^n)u, v, w) + z^n \Psi(u, Xv, w) - \Psi(u, v, w(X \otimes t^n)) = 0$$

$$\Leftrightarrow \quad z^n \langle w, \phi(Xv, z)\, u \rangle$$

$$= \underbrace{\langle w(X \otimes t^n), \phi u \rangle}_{= \langle w, (X \otimes t^n)\phi u \rangle} - \langle w, \phi\,(X \otimes t^n)\, u \rangle$$

$$= \langle w, [X \otimes t^n, \phi(v,z)]\, u \rangle$$

$\square$

Relation (*) is called "gauge invariance".
Ψ is uniquely determined by its restriction Ψ_0 on $V_{\lambda_0} \otimes V_\lambda \otimes V_{\lambda_\infty}^*$.

Definition:

Suppose that Ψ is horizontal section of $\mathcal{E}$ with respect to the connection ∇. Such an operator

$$\Psi(z): H_{\lambda_0} \otimes H_\lambda \otimes H_{\lambda_\infty}^* \longrightarrow \mathbb{C}$$

is called a "chiral vertex operator". $\phi(v,z)$, $v \in V_\lambda$, is called "primary field". The operators $\phi(v,z)$, $v \in \bigoplus_{d>0} H_\lambda(d)$, are called "descendents".

Proposition 2:

A non-trivial chiral vertex operator

$$\Psi(z): H_{\lambda_0} \otimes H_\lambda \otimes H_{\lambda_\infty}^* \longrightarrow \mathbb{C}$$

exists if and only if the highest weights λ_0, λ and λ_∞ satisfy the quantum Clebsch-Gordan condition at level K.

Conformal invariance (Prop. 4 §5)

$\longrightarrow$ restriction $\Psi_0 : V_{\lambda_0} \otimes V_\lambda \longrightarrow V_{\lambda_\infty}$ is given by

(**) $z^{\Delta_{\lambda_\infty} - \Delta_{\lambda_0} - \Delta_\lambda} \vec{p}$, where $\vec{p}$ is a basis of

$$\mathrm{Hom}_{\mathfrak{g}}(V_{\lambda_0} \otimes V_\lambda, V_{\lambda_\infty}).$$

Ψ is uniquely determined by Ψ_0.

Decompose $\phi(v,z)$ into $\phi(v,z) = \sum_{n \in \mathbb{Z}} \phi_n(v,z)$, such that

Φ_n sends $H_\lambda(d)$ to $H_\lambda(d-n)$. By using $(*)$ and $(**)$ together with the definition of L_0 one can check the relation

$$[L_0, \Phi_0(v,z)] = \left(z\frac{d}{dz} + \Delta_\lambda\right)\Phi_0(v,z)$$

(exercise)

$\to$ $\phi(v,z)$, $v \in V_\lambda$ can be written in the form

$$\phi(v,z) = \sum_{n\in\mathbb{Z}} \phi_n(v)\, z^{-n-\Delta}$$

where $\Delta = -\Delta_{\lambda_\infty} + \Delta_{\lambda_0} + \Delta_\lambda$ and $\phi_n(v)$ is operator sending $H_{\lambda_0}(d)$ to $H_{\lambda_\infty}(d-n)$.

<u>Proposition 3</u>:

The primary field $\phi(v,z)$, $v \in V_\lambda$, satisfies the relation

$$(1) \qquad [L_n, \phi(v,z)] = z^n\left(z\frac{d}{dz} + (n+1)\Delta_\lambda\right)\phi(v,z)$$

for any integer n.

(exercise)

<u>Interpretation</u>: invariance of $\phi(v,z)(dz)^{\Delta_\lambda}$ under local hol. conformal transformations, namely we have $\quad \phi(v, f(z)) = \left(\dfrac{df}{dz}\right)^{-\Delta_\lambda} \phi(v,z)$

$\to$ for $f_\varepsilon(z) = z - \varepsilon(z)$ this gives:

$$(2) \qquad \delta_\varepsilon \phi(v,z) = \left(\Delta_\lambda \varepsilon'(z) + \varepsilon(z)\frac{d}{dz}\right)\phi(v,z)$$

In particular, in the case $\varepsilon(z) = \varepsilon z^{n+1}$ right-hand side of (2) coincides with (1). Left-hand sides also coincide. Next, define $X(z)$ and $T(z)$ by

$$X(z) = \sum_{n\in\mathbb{Z}} (X\otimes t^n)\, z^{-n-1},$$

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$$

where $X \in \mathfrak{g}$ and both $X(z)$ and $T(z)$ are formal power series in z. For $u \in H_\lambda$, $\eta \in H_\lambda^*$,

$$\langle \eta, X(z)u\rangle = \sum_{n\in\mathbb{Z}} \langle \eta, (X\otimes t^n) z^{-n-1} u\rangle$$

is expressed as a finite sum, similarly for $T(z)$ ("energy momentum tensor").

OPE:

$$X(w)\,\phi(v,z) = \sum_{k\in\mathbb{Z}} w^{-1} z^{-\Delta-k} \sum_{m\in\mathbb{Z}} \left(\frac{z}{w}\right)^m (X\otimes t^m)\,\phi_{k-m}(v)$$

Assume $|w|>|z|>0$. Using gauge invariance, the above expression can be written as

$$\sum_{k\in\mathbb{Z}} w^{-1} z^{-\Delta-k} \sum_{m\geq 0} \left(\frac{z}{w}\right)^m \phi_k(Xv) + R_1(w-z),$$

where $R_1(w-z)$ is regular in the sense that $\langle \eta, R_1(w-z)\xi\rangle$ is hol. $\forall\, \xi\in H_{\lambda_0}, \eta\in H_{\lambda_\infty}^*$

$$\rightarrow\ X(w)\,\phi(v,z) = \frac{1}{w-z}\,\phi(Xu, z) + R_1(w-z)$$

$X(w)\phi(v,z)$ in region $|w|>|z|>0$ is analytically continued to $\phi(v,z)X(w)$ defined in region $|z|>|w|>0$

"operator product expansion" of $X(\omega)$ and $\phi(v,z)$

Similarly, we have

$$(3)\quad T(\omega)\,\phi(v,z) = \left(\frac{\Delta_\lambda}{(\omega-z)^2} + \frac{1}{\omega-z}\frac{\partial}{\partial z}\right)\phi(v,z) + R_2(\omega-z)$$

in the region $|\omega| > |z| > 0$, where $R_2(\omega-z)$ is regular in $\omega-z$. $\rightsquigarrow$ analytically continued to $\phi(v,z)T(\omega)$ in region $|z| > |\omega| > 0$.

In the region $|\omega| > |z| > 0$ we have:

$$(4)\quad T(\omega)\,T(z) = \frac{c/2}{(\omega-z)^4} + \frac{2T(z)}{(\omega-z)^2} + \frac{1}{\omega-z}\frac{\partial}{\partial z}T(z) + R_3(\omega-z),$$

where $R_3(\omega-z)$ is regular in $\omega-z$.

$\longrightarrow$ analytically continued to region $|z|>|\omega|>0$: $T(z)\,T(\omega)$

<u>Lemma 1:</u>

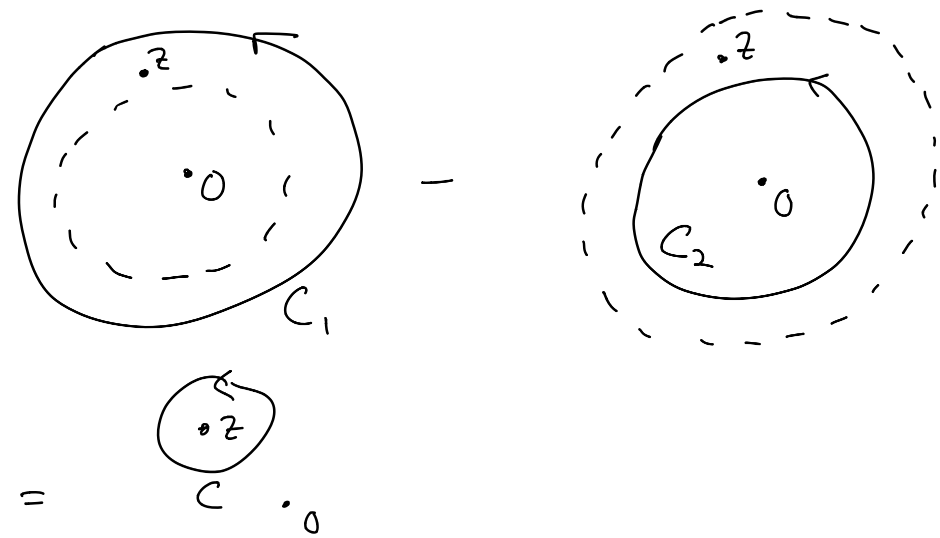
$$[X\otimes t^n, \phi(v,z)] = \frac{1}{2\pi\sqrt{-1}}\int_C \omega^n X(\omega)\,\phi(v,z)\,d\omega,$$

$$[L_n, \phi(v,z)] = \frac{1}{2\pi\sqrt{-1}}\int_C \omega^{n+1} T(\omega)\,\phi(v,z)\,d\omega,$$

where C is an oriented small circle in the ω-plane turning around z counterclockwise.

<u>Proof:</u>

We will show the second equality, the first one is analogous. Fix a point in the ω-plane with coordinate z. Consider the following contours



and the corresponding residues:

$$T(w) = \sum_{n \in \mathbb{Z}} L_n w^{-n-2}$$

$$\longrightarrow \quad L_n \phi(v, z) = \frac{1}{2\pi\sqrt{-1}} \int_{C_1} w^{n+1} T(w) \phi(v, z) dw .$$

$$\phi(v, z) L_n = \frac{1}{2\pi\sqrt{-1}} \int_{C_2} w^{n+1} \phi(v, z) T(w) dw$$

$$\Rightarrow \int_{C_1} w^{n+1} T(w) \phi(v, z) dw - \int_{C_2} w^{n+1} \phi(v, z) T(w) dw$$

$$= \int_{C} w^{n+1} T(w) \phi(v, z) dw$$

$\square$

Combining Lemma 1 and OPE (3) we obtain commutator (1).

Next, we explain how (4) gives rise to Virasoro Lie algebra. Let γ be a circle in the z-plane with parameter $z = re^{2\pi\sqrt{-1}\theta}, 0 \le \theta \le 1$.

$$\rightarrow L_n = \frac{1}{2\pi\sqrt{-1}} \int_\gamma z^{n+1} T(z)\, dz.$$

Take circles C_1 and C_2 in the w-plane and suppose $r_2 < r < r_1$. Then we have

$$L_m L_n = \left(\frac{1}{2\pi\sqrt{-1}}\right)^2 \int_\gamma \int_{C_1} w^{m+1} z^{n+1} T(w) T(z)\, dw\, dz$$

$$L_n L_m = \left(\frac{1}{2\pi\sqrt{-1}}\right)^2 \int_\gamma \int_{C_2} w^{m+1} z^{n+1} T(z) T(w)\, dw\, dz$$

Thus for a circle C as in the above picture, we get

$$[L_m, L_n] = \left(\frac{1}{2\pi\sqrt{-1}}\right)^2 \int_\gamma \left(\int_C w^{m+1} z^{n+1} T(w) T(z)\, dw \right) dz$$

Combining with OPE (4) we obtain the Virasoro Lie algebra. $\square$