Definition:

For a transformation f of the complex plane we introduce the "Schwarzian derivative"

$$S(f,z) = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2$$

Lemma 2:

For a Möbius transformation

$$f(z) = \frac{az+b}{cz+d}, \quad a,b,c,d \in \mathbb{C}, \ ad-bc=1,$$

the equality $S(f,z)=0$ holds for any $z \in \mathbb{C}$. Conversely, if $S(f,z)=0$ for any $z \in \mathbb{C}$, then f is a Möbius transformation.

From the OPE of the energy momentum tensor we get

$$(*) \quad \delta_\varepsilon T(z) = \varepsilon(z)\frac{\partial}{\partial z}T(z) + 2\varepsilon'(z)T(z) + \frac{c}{12}\varepsilon'''(z)$$

$\rightarrow T(z)$ is not a co-variant tensor of order 2

for $\varepsilon = z^{n+1}$, $n=-1,0,1$, f_ε generates a global Möbius trf. and $\frac{c}{12}\varepsilon''' = 0$.

Integral form of $(*)$:

Proposition 4:

For a hol. transformation $w = f(z)$, we have

$$T(z) = \left(\frac{\partial w}{\partial z}\right)^2 T(w) + \frac{c}{12} S(f,z)$$

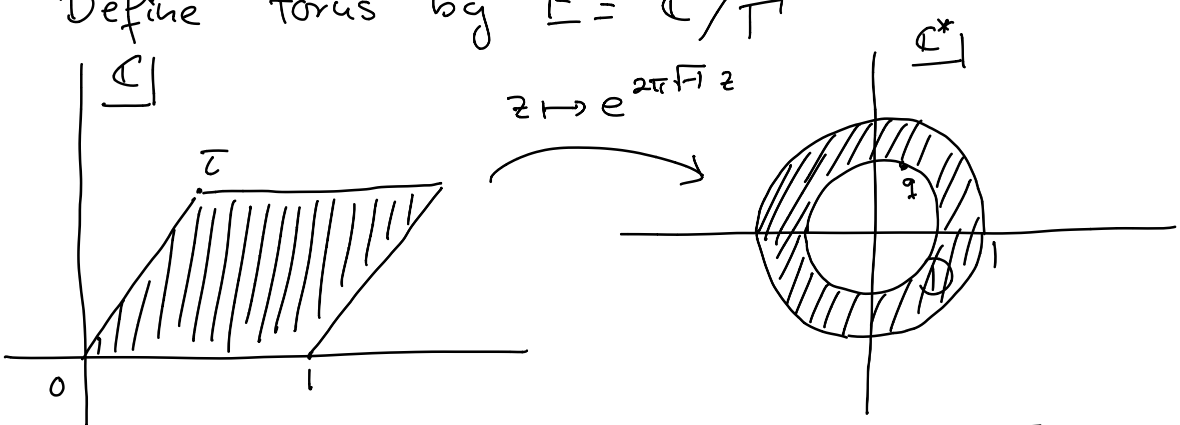
Conformal blocks on torus:

Let τ be an element of the upper half plane

$$H = \{ z \in \mathbb{C} \mid \operatorname{Im} z > 0 \}$$

Denote the lattice $\mathbb{Z} \oplus \mathbb{Z}\tau$ by Γ.

Define torus by $E = \mathbb{C}/\Gamma$



Suppose $0 \le \operatorname{Re} \tau < 1$ and set $q = e^{2\pi\sqrt{-1}\,\tau}$.

Denote by G the transformation group of $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ generated by $f(w) = qw$, $w \in \mathbb{C}^*$. "dilatation"

Then $\mathbb{C}^*/G \cong E$ $\quad \left(z \longmapsto w = e^{2\pi\sqrt{-1}\, z} \right)$

Now set $\zeta = 2\pi\sqrt{-1}\, z$ for the torus $\mathbb{C}^*/G$.

Since $w = e^{\zeta} \longrightarrow S(w, z) = -\frac{1}{2}$

Prop 4. $\longrightarrow T(\zeta) = \sum_{n \in \mathbb{Z}} \left(L_n - \frac{c}{24} \delta_{n,0} \right) e^{-n\zeta}$

$\Rightarrow L_0'$ of torus $\mathbb{C}^*/G$ is given by

$$L_0' = L_0 - \frac{1}{24} c.$$

action of dilatation operator:

$$q^{L_0 - \frac{1}{24} c} \xi = \exp 2\pi\sqrt{-1}\,\tau \left(\Delta_\lambda + d - \frac{c}{24} \right) \xi, \quad \xi \in \mathcal{H}_\lambda(d).$$

Take distinct points $p_1, \dots, p_n$ on the torus $\mathbb{C}^*/G$ and represent them as points in D.

$\to$ associate level k highest weights $\mu_1, \dots, \mu_n$ to $p_1, \dots, p_n$.

associate H_λ to origin of w-plane and H_λ^* to infinity

$\to$ space of conformal blocks

$$\mathcal{H}(0, p_1, \dots, p_n, \infty; \lambda, \mu_1, \dots, \mu_n, \lambda^*)$$

$\psi \in \mathcal{H}$ is linear operator

$$\psi : H_\lambda \otimes H_{\mu_1} \otimes \dots \otimes H_{\mu_n} \to \overline{H_\lambda} .$$

Consider

$$\mathrm{Tr}_{H_\lambda} q^{L_0 - \frac{c}{24}} : H_{\mu_1} \otimes \dots \otimes H_{\mu_n} \to \mathbb{C}$$

and denote by

$$\mathcal{H}_\lambda(D; p_1, \dots, p_n; \mu_1, \dots, \mu_n)$$

the vector space of linear operators $\mathrm{Tr}_{H_\lambda} q^{L_0 - \frac{c}{24}}$ for any $\psi \in \mathcal{H}(0, p_1, \dots, p_n, \infty; \lambda, \mu_1, \dots, \mu_n, \lambda^*)$

$\to$ Define the space of conformal blocks for the torus E by

$$\mathcal{H}(E; p_1, \dots, p_n; \mu_1, \dots, \mu_n) = \bigoplus_{0 \le \lambda \le k} \mathcal{H}_\lambda(D; p_1, \dots, p_n; \mu_1, \dots, \mu_n)$$

$\to$ for $\underline{n=0}$: basis of $\mathcal{H}(E)$ is given by

$$\chi_\lambda(\tau) = \mathrm{Tr}_{H_\lambda} q^{L_0 - \frac{c}{24}} , \lambda = 0, 1, \dots, k$$

"characters of affine Lie algebra $A_1^{(1)}$".

We have:

$$\chi_\lambda\left(-\frac{1}{\tau}\right) = \sum_\mu S_{\lambda\mu}\,\chi_\mu(\tau),$$

$$\chi_\lambda(\tau+1) = \exp 2\pi\sqrt{-1}\left(\Delta_\lambda - \frac{c}{24}\right)\chi_\lambda(\tau)$$

where $S_{\lambda\mu}$ and Δ_λ are given by

$$S_{\lambda\mu} = \sqrt{\frac{2}{K+2}}\ \sin\frac{(\lambda+1)(\mu+1)}{K+2}$$

$$\Delta_\lambda = \frac{\lambda(\lambda+2)}{4(K+2)}$$

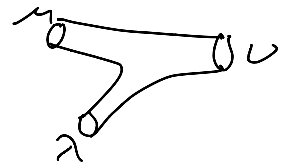
Put $S = (S_{\lambda\mu})$ and $\mathrm{Diag}\left(\exp 2\pi\sqrt{-1}\left(\Delta_\lambda - \frac{c}{24}\right)\right)$,
$0 \le \lambda \le K$.

$\longrightarrow S^2 = (ST)^3 = I$.

Let R_K be complex vector space with basis
v_λ, $0 \le \lambda \le K$.

Define $v_\lambda \cdot v_\mu = \sum_\nu N_{\lambda\mu}^\nu v_\nu$ with linear extension on R_K. Here

$N_{\lambda\mu}^\nu = \dim \mathcal{H}(p_1, p_2, p_3; \lambda, \mu, \nu^*)$



For $\mathfrak{g} = sl_2(\mathbb{C})$: $N_{\lambda\mu}^\nu = N_{\lambda\mu\nu}$
and $N_{\lambda\mu}^\nu = 0$ or 1.

Proposition 5:

The algebra R_K is commutative and associative.

Proof:

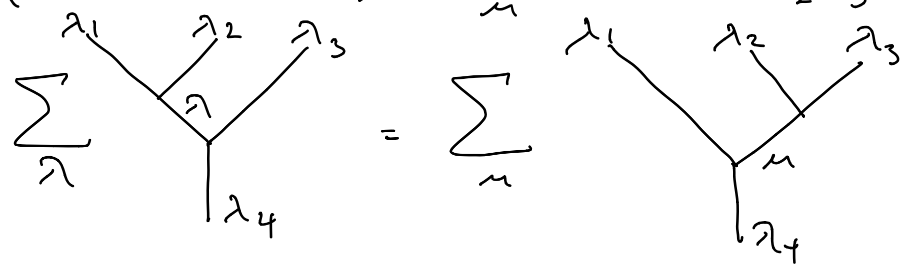
The commutativity follows from $N_{\lambda\mu}^{\nu} = N_{\mu\lambda}^{\nu}$. Associativity:

$$(v_{\lambda_1} \cdot v_{\lambda_2}) \cdot v_{\lambda_3} = \sum_{\lambda, \lambda_4} N_{\lambda_1 \lambda_2}^{\lambda} N_{\lambda \lambda_3}^{\lambda_4} v_{\lambda_4}$$

$$v_{\lambda_1} \cdot (v_{\lambda_2} \cdot v_{\lambda_3}) = \sum_{\mu, \lambda_4} N_{\lambda_1 \mu}^{\lambda_4} N_{\lambda_2 \lambda_3}^{\mu} v_{\lambda_4}$$

equality follows from

$$\sum_{\lambda} N_{\lambda_1 \lambda_2}^{\lambda} N_{\lambda \lambda_3}^{\lambda_4} = \sum_{\mu} N_{\lambda_1 \mu}^{\lambda_4} N_{\lambda_2 \lambda_3}^{\mu}$$



□

The algebra R_K is called "Verlinde algebra" or "fusion algebra" for the $SU(2)$ Wess-Zumino-Witten model at level K.

It can be shown that

$$\phi : \mathbb{C}[X]/(X^{K+1}) \longrightarrow R_K \qquad (**)$$

defined by $\phi(X) = v_1$ is isomorphism.

Proposition 6:

$$N_{\lambda\mu\nu} = \dim H(p_1, p_2, p_3; \lambda, \mu, \nu)$$
$$= \sum_{\alpha} \frac{S_{\lambda\alpha} S_{\mu\alpha} S_{\nu\alpha}}{S_{0\alpha}} \qquad \text{"Verlinde formula"}$$

<u>Proof:</u>

Denote $(N_\lambda) = N_{\lambda\mu\nu}$, $0 \le \mu, \nu \le k$ $(k+1)\times(k+1)$ matrix. For $\lambda = 1$: $N_{1\mu\nu} = 1$ if $|\mu - \nu| = 1$ and $N_{1\mu\nu} = 0$ else.

Check that matrix N_1 is diagonalised by matrix S with eigenvalue $S_{\lambda 1}/S_{00}$, $\lambda = 0, 1, \dots, k$.

$v_\lambda, \lambda \ge 1$ is polynomial in v_1 (see $(**)$)

$\longrightarrow$ N_λ, $\lambda \ge 1$ is polynomial in N_1.

$\Rightarrow$ N_λ is diagonalised by S as well, with eigenvalues $S_{\lambda\mu}/S_{0\mu}$, $\mu = 0, 1, \dots, k$ (exercise).

$\longrightarrow$ Verlinde formula

$\square$

<u>Next:</u> Basis of conformal blocks on sphere

$$H(p_1, \dots, p_n; \lambda_1, \dots, \lambda_n)$$

Take μ_j, $0 \le j \le n$ level k highest weights and $\mu_0 = \mu_n = 0$ s.t. $(\mu_{j-1}, \lambda_j, \mu_j)$ satisfies quantum Clebsch-Gordan rule at level k.

Consider chiral vertex operators

$$\Psi_j(z) : H_{\mu_{j-1}} \otimes H_{\lambda_j} \longrightarrow \overline{H}_{\mu_j}, \quad 1 \le j \le n,$$

and associated operators

$$\phi_j(z, \xi_j) : H_{\mu_{j-1}} \longrightarrow \overline{H}_{\mu_j}, \quad \xi_j \in H_{\lambda_j, 1}, \ 1 \le j \le n.$$

Then the composition

$$\phi_n(z_n, \xi_n) \cdots \phi_1(z, \xi_1) : H_0 \longrightarrow \overline{H}_0$$

together with the correspondence

$$\xi_1 \otimes \cdots \otimes \xi_n \longrightarrow \langle v_0^*, \phi_n(z_n, \xi_n) \cdots \phi_1(z_1, \xi_1) v_0 \rangle$$

$$(v_0 \in H_0, \ v_0^* \in H_0^*)$$

determines a multilinear map

$$\mathcal{V}_{\mu_0 \mu_1 \cdots \mu_n}(z_1, \cdots, z_n) : H_{\lambda_1} \otimes \cdots \otimes H_{\lambda_n} \to \mathbb{C}$$

$\longrightarrow$ restriction on $V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}$ satisfies KZ equation.

Proposition 7:

$$\dim \mathcal{H}(p_1, \cdots, p_n; \lambda_1, \cdots, \lambda_n)$$

$$= \sum_{0 \le \lambda \le k} \frac{S_{\lambda_1 \lambda} \cdots S_{\lambda_n \lambda}}{(S_{0\lambda})^{n-2}}$$

Proof:

Suppose $n \ge 3$. Then

$$\dim \mathcal{H}(p_1, \cdots, p_n; \lambda_1, \cdots, \lambda_n) = \sum_{\mu_1, \cdots, \mu_{n-1}} \prod_{j=1}^{n} N_{\mu_{j-1} \lambda_j \mu_j}$$

$$\xrightarrow{\text{Prop. 6}} \sum_{0 \le \lambda \le k} \frac{S_{\lambda_1 \lambda} \cdots S_{\lambda_n \lambda}}{(S_{0\lambda})^{n-2}}$$

$\square$

"pairs of pants decomposition":