§8. Conformal field theory and the Jones polynomial

A "link" L is an embedding

$$f: S^1 \sqcup \cdots \sqcup S^1 \to S^3$$

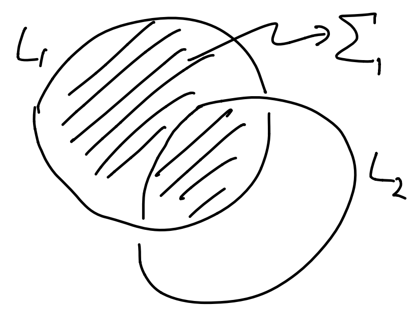
The image of each S^1 is called "link component".

$\to L = L_1 \cup L_2 \cdots \cup L_m$

A link L with one component is a "Knot".

Let $L = L_1 \cup L_2$ be a link with two components

The "linking number" $\ell k(L_1, L_2)$ is the intersection number of an oriented surface Σ_1 in S^3 s.t. $\partial \Sigma_1 = L_1$ with L_2



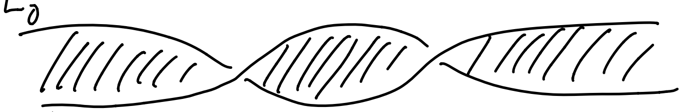
$$\ell k(L_1, L_2) = \frac{1}{2}(\# \text{positive crossings} - \# \text{negative crossings})$$

A "framing" of a link L is an integer n_j for each component L_j given by

$$n_j = \ell k(L_j, L_j')$$

where L_j' is a simple closed curve on the boundary of a tubular neighborhood of L_j

L_j'

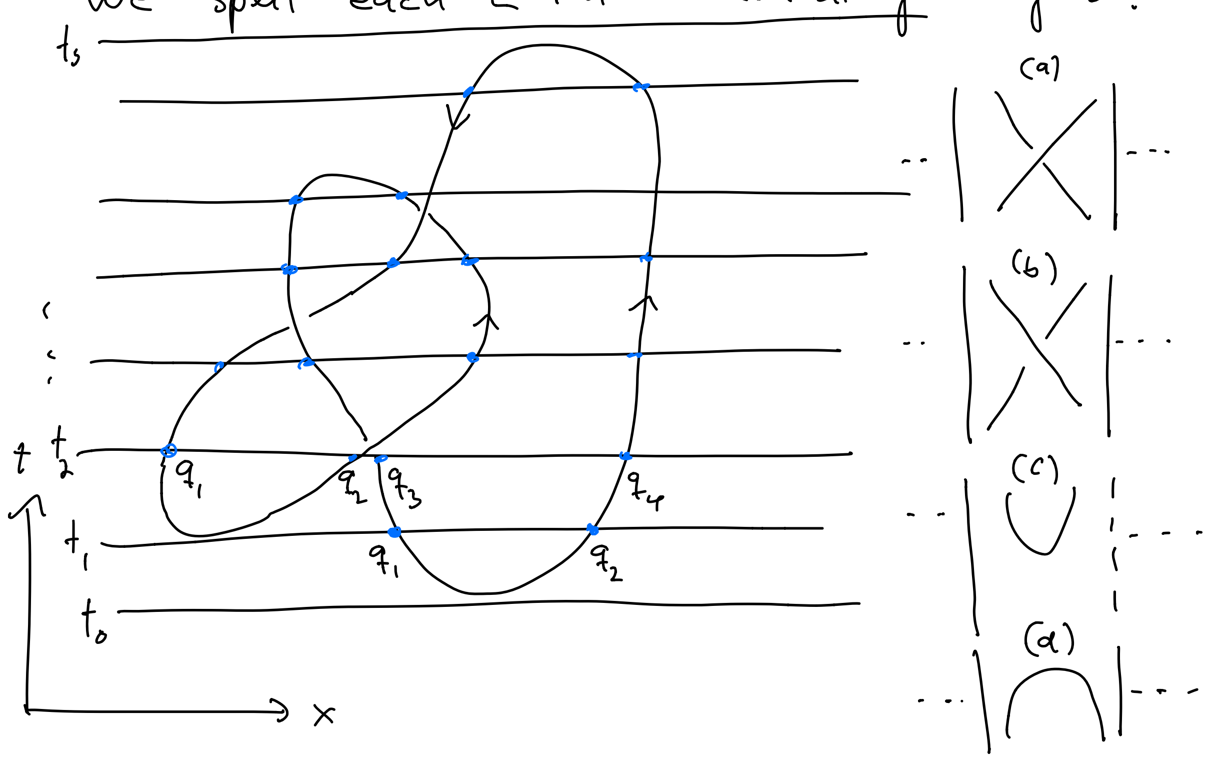


L_j $\longrightarrow n_j = 1$

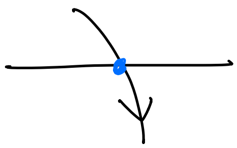
Let L be an oriented framed link in $\mathbb{R}^3$.

$\longrightarrow$ associate level K highest weights $\lambda_1, \dots, \lambda_m$ to $L_1, \dots, L_m$

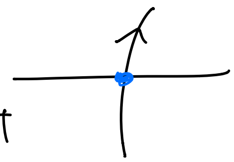
We split each L into "elementary tangles":



To each q_i we associate a level K highest weight:



$\to \lambda$ highest weight



$\to \lambda^*$ dual highest weight

at t_j

$\longrightarrow$ consider space of conformal blocks for Riemann sphere with points $q_1, \dots, q_n$ and highest weights as defined above

$\longrightarrow V(t_j)$

in particular: $V(t_0) = V(t_s) = \mathbb{C}$

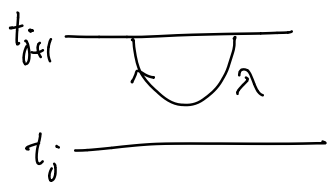
Associate a linear map

$$Z_j : V(t_j) \longrightarrow V(t_{j+1}), \quad 0 \leq j \leq s-1$$

to each elementary tangle as follows:

(1) $\sigma_i, \sigma_i^{-1} \longrightarrow$ holonomy of KZ equation

(2) for maximal/minimal points define



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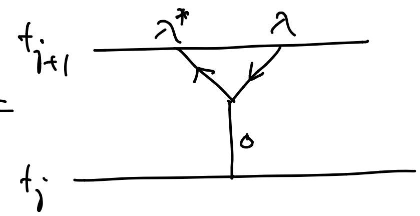
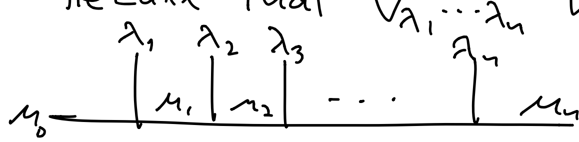


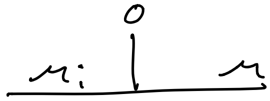
and set $V(t_j) = V_{\lambda_1 \cdots \lambda_i \lambda_{i+1} \cdots \lambda_n}$

$V(t_{j+1}) = V_{\lambda_1 \cdots \lambda_i \lambda \lambda^* \lambda_{i+1} \cdots \lambda_n}$

Recall that $V_{\lambda_1 \cdots \lambda_n}$ has basis

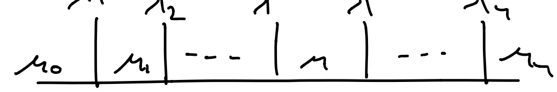


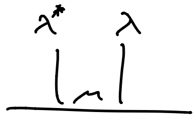
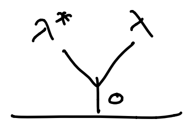
$v_{\mu_0 \cdots \mu_n}$

inserting  gives natural identification $V_{\lambda_1 \cdots \lambda_i \lambda_{i+1} \cdots \lambda_n} \cong V_{\lambda_1 \cdots \lambda_i 0 \lambda_{i+1} \cdots \lambda_n}$ defined by $v_{\mu_0 \cdots \mu_i \cdots \mu_n} \mapsto v_{\mu_0 \cdots \mu_i \mu_i \cdots \mu_n}$

Now define $Z_j : V(t_j) \to V(t_{j+1})$ by

$$v_{\mu_0 \cdots \mu_i \mu_i \cdots \mu_n} \mapsto \sum_{\mu} F_{\mu 0}\, v_{\mu_0 \cdots \mu_i \mu \mu_i \cdots \mu_n}$$

where $v_{\mu_0 \cdots \mu_i \mu \mu_i \cdots \mu_n} =$ 

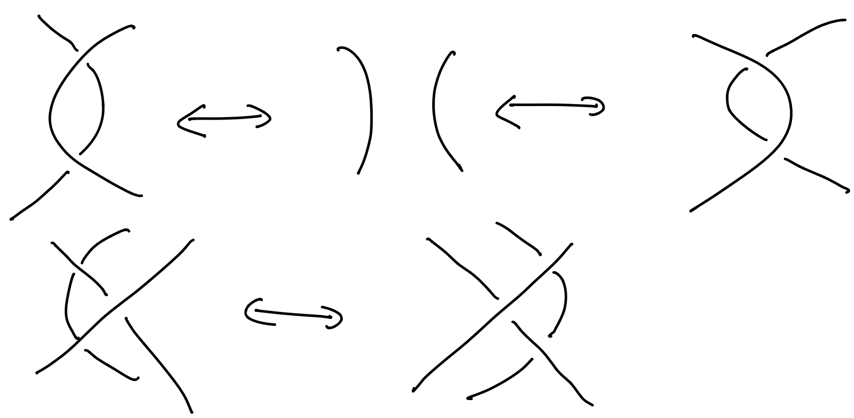
and $\sum_{\mu} F_{\mu 0}$  $=$ 

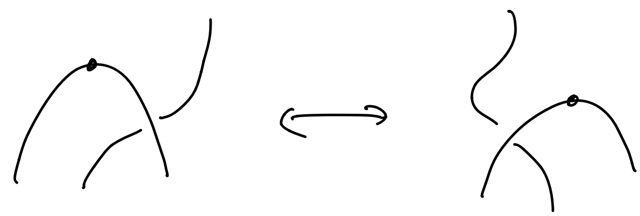
Composing the above linear maps $Z_j, 0 \leq j \leq s-1$ we get $Z(L; \lambda_1, \ldots, \lambda_m) = Z_{s-1} \circ \cdots \circ Z_1 \circ Z_0(1)$

Flatness of KZ connection gives:

__Lemma 1:__

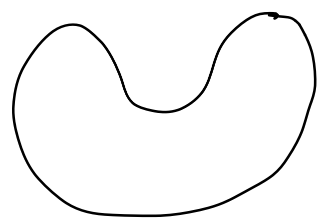
The above $Z(L; \lambda_1, \cdots, \lambda_m)$ is invariant under the local horizontal moves below





These are also called "Reidemeister moves"

Next define

$K_0 =$ 

as the unique unknot with two minimal and two maximal points and set

$$d(\lambda) = Z(K_0;\lambda)^{-1}$$

Define $\mu(j)$ = # maximal points in L_j

$$\to J(L;\lambda_1,\dots,\lambda_m) = d(\lambda_1)^{\mu(1)}\cdots d(\lambda_m)^{\mu(m)}\, Z(L;\lambda_1,\dots,\lambda_m)$$

Theorem 1:

$J(L;\lambda_1,\dots,\lambda_m)$ is an invariant of a colored oriented framed link.

Proof:

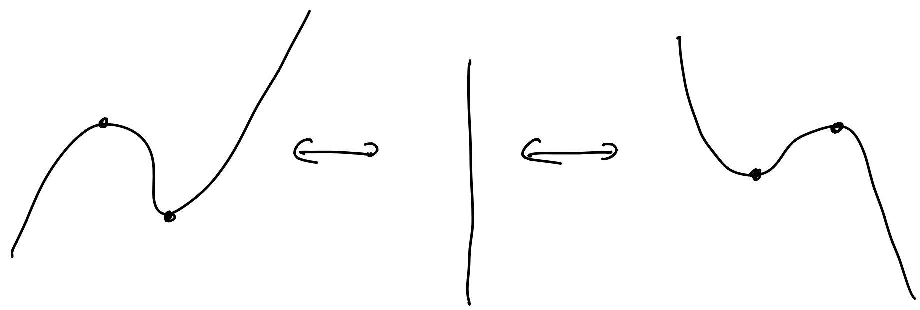
Suppose $L' = L_1' \cup \dots \cup L_m'$ is an equivalent link, i.e. $\exists$ orientation perserving homeomorphism h of S^3 s.t.

$$h(L_j) = L_j', \quad 1 \le j \le m$$

Have to show: $J(L) = J(L')$

$L' \Longleftrightarrow L$ if and only if L' is obtained from L through a sequence of Reidemeister moves and a cancellation of two critical points

(*)



Z is <u>not</u> invariant under operation (*) (exercise) but invariant under the Reidemeister moves (Lemma 1)

$\rightarrow$ have to show invariance under (*)

$$L' = L + K_0$$

$$\Rightarrow Z(L'; \lambda_1, \dots, \lambda_m) = Z(K_0; \lambda_j)\, Z(L; \lambda_1, \dots, \lambda_m)$$

$\longrightarrow$ $J(L; \lambda_1, \dots, \lambda_m)$ is invariant under a cancellation of critical points.

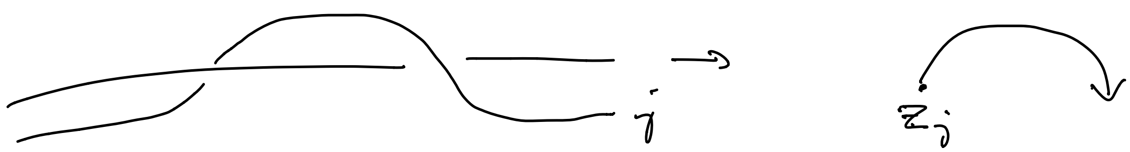
<u>Proposition 1:</u>

Let L' be a link obtained from $L = \bigcup_{j=1}^{m} L_j$ by increasing the framing of the component L_j by 1. Then

$$J(L'; \lambda_1, \dots, \lambda_m) = \exp 2\pi\sqrt{-1}\,\Delta_{\lambda_j}\, J(L; \lambda_1, \dots, \lambda_m)$$

Proof:

Increase of framing by 1 means: cross-section:



or $z_j \mapsto e^{\pi i} z_j$

We know that under $w_j = \dfrac{a z_j + b}{c z_j + d}$

conformal blocks $\Psi_0(z_1, \dots, z_n)$ transform as

$$\Psi_0(z_1, \dots, z_n) = \prod_{j=1}^{n} (c z_j + d)^{-2\Delta_{\lambda_j}} \Psi_0(w_1, \dots, w_n)$$

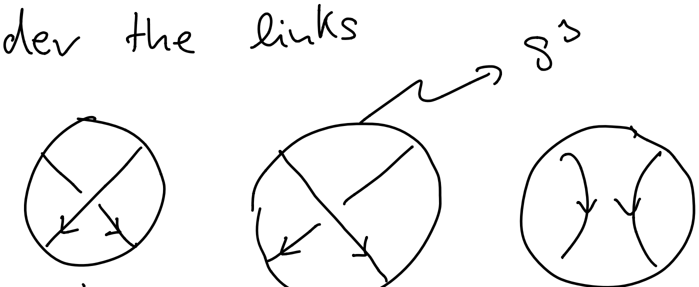
or for $z_j \mapsto \alpha z_j$: $\Psi_0(z_1, \dots, z_n) = \alpha^{2\Delta_{\lambda_j}} \Psi_0(w_1, \dots, w_n)$

set $\alpha = e^{\pi\sqrt{-1}}$

$\square$

Notation: In the case $\lambda_1 = \lambda_2 = \dots \lambda_m = 1$, we write J_L for $J(L, \lambda, \dots, \lambda)$

Consider the links



L_+ L_- L_0

and identical outside S^3.

Proposition 2:

set $q^{1/m} = \exp\left(\frac{2\pi\sqrt{-1}}{m(K+2)}\right)$

The link invariant J_L satisfies the skein relation

$$q^{1/4} J_{L_+} - q^{-1/4} J_{L_-} = \left(q^{1/2} - q^{-1/2}\right) J_{L_0}$$

Proof:

We have seen that the monodromy matrix $\rho(\sigma_1)$ acts on conformal blocks as

$$\rho(\sigma_1) = P_{12} \exp\left(\pi\sqrt{-1}\,\Omega_{12}/\kappa\right), \quad \kappa = K+2$$

The matrix Ω_{12}/κ is diagonalized with eigenval.

$$\Delta_\lambda - \Delta_{\lambda_1} - \Delta_{\lambda_2}$$

since $\lambda_1 = \lambda_2 = 1$ we get from Clebsch-Gordan rule $\lambda = 0, 2$

Recall $\Delta_{2j} = \frac{j(j+1)}{K+2} \Rightarrow \Delta_0 = 0$, $\Delta_1 = \frac{3/4}{K+2}$, $\Delta_2 = \frac{2}{K+2}$

set $G_i = \rho(\sigma_i)$. Then we have

$P_{12} = \mathbb{1}$ as $\lambda_1 = \lambda_2$

$$\left(G_i + q^{-3/4}\right)\left(G_i - q^{1/4}\right)$$

$$= G_i^2 - G_i q^{1/4} + q^{-3/4} G_i - q^{-1/2} = 0$$

This is equivalent to

$$q^{1/4} G_i - q^{-1/4} G_i^{-1} = \left(q^{1/2} - q^{-1/2}\right)\mathbb{1}$$

$\square$