Some dimensional analysis

Consider a bosonic field theory with kinetic terms

$$\partial_\mu \phi \, \partial^\mu \phi$$

$$\rightarrow \quad [\phi] = \Lambda^{\frac{1}{2}d - 1} \qquad (1)$$

where Λ denotes the unit of inverse length (energy)

(1) follows from $[\mathcal{L}] = \Lambda^d$

coupling constants will have dimensions

$$[\lambda_r \phi^r] = \Lambda^d \longrightarrow [\lambda_r] = \Lambda^{r + d - \frac{1}{2} rd} \equiv \Lambda^{\delta_r} \qquad (2)$$

For example,

$$[\mu^2] = \Lambda^2,$$

$$[\lambda_3] = \Lambda^{\frac{1}{2}(6-d)}$$

$$[\lambda_4] = \Lambda^{4-d}$$

etc.

$\rightarrow$ to each coupling constant there corresponds a number of space dimensions where $\delta_r = 0$

$\rightarrow$ theory is renormalizable in that dimension!

Let's see what this means

Let us look at the dimensions of Green's functions:

$$\left[G^{(N)}(x_1, \ldots, x_N) \right] = [\phi]^N = \Lambda^{N(\frac{1}{2}d-1)}$$

(recall $G^{(N)} \equiv \langle T\{\phi(x_1)\,\phi(x_2) \cdots \phi(x_N)\}\rangle$)

$\longrightarrow$ Fourier transform has dimension

$$\left[G^{(N)}(k_i) \right] = \underset{\prod_{i=1}^{N} \int d^4x_i \; e^{2\pi i k_i \cdot x_i}}{\underset{\uparrow}{\Lambda^{-Nd}}} \left[G^{(N)}(x_i) \right] = \Lambda^{-N(\frac{1}{2}d+1)} \qquad (3)$$

Using $G^{(N)}(k_i) = \underset{\text{dim} = \Lambda^{-d}}{\underset{\uparrow}{\delta^4\left(\sum_i k_i\right)}} \, \bar{G}^{(N)}(k_i)$,

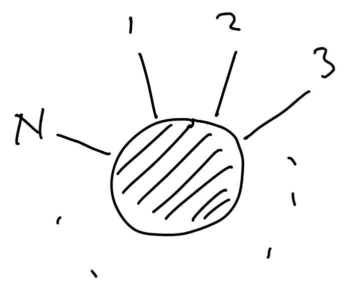
we get $\left[\bar{G}^{(N)}(k_i) \right] = \Lambda^{d - N(\frac{1}{2}d+1)}$ (4)

What is the dimension of the effective action?

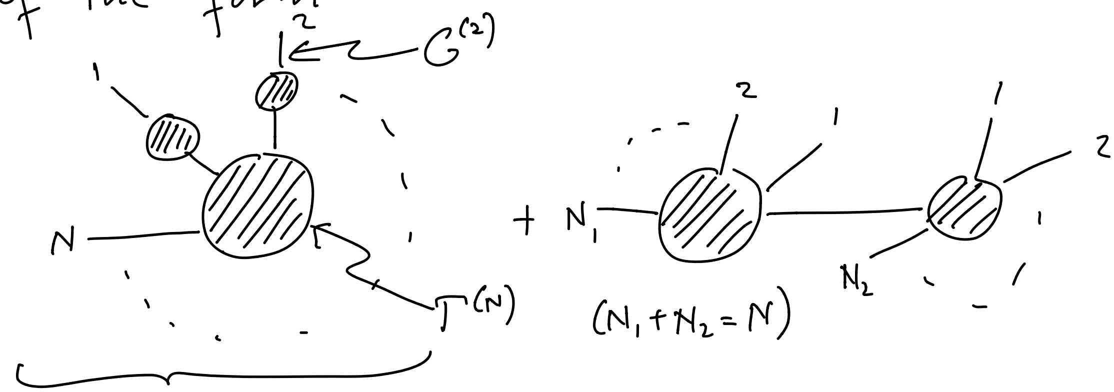
Note

$$\Gamma[\phi] = \sum_{N=1}^{\infty} \frac{1}{N!} \int d^4x_1 \cdots d^4x_N \, \Gamma^{(N)}(x_1, \ldots, x_N)\, \phi(x_1) \cdots \phi(x_N)$$

where

$\Gamma^{(N)} =$ 

Moreover, $G^{(N)}$ will consist of diagrams of the form



$+ \cdots$

$$\sim \Gamma^{(N)}(1',\dots,N')\, G_c^{(2)}(1,1')\, G_c^{(2)}(2,2') \cdots G_c^{(2)}(N,N')$$

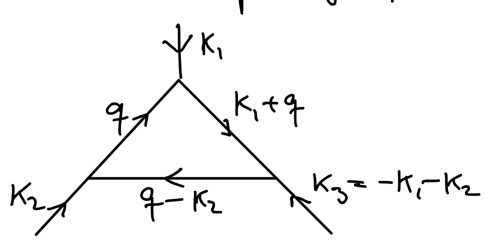
Thus
$$[\Gamma^{(N)}(x_i)] = [G^{(N)}(x_i)]\,[V]^{-N}\,[G^{(2)}(x_i)]^{-N} = \Lambda^{N(\frac{1}{2}d+1)} \qquad (5)$$

The Fourier transforms have the dims.:

$$\begin{aligned} [\Gamma^{(N)}(k_i)] &= \Lambda^{-N(\frac{1}{2}d-1)} \\ [\overline{\Gamma}^{(N)}(k_i)] &= \Lambda^{N+d-\frac{1}{2}Nd} \end{aligned} \qquad (6)$$

<u>Power counting and primitive divergences</u>

Consider the one-loop graph of $\overline{\Gamma}^{(3)}$ in ϕ^3-th.

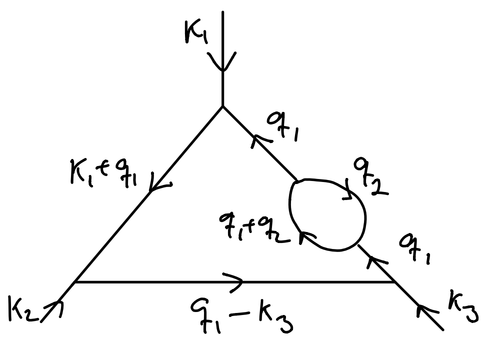


$\rightarrow$ gives rise to the integral

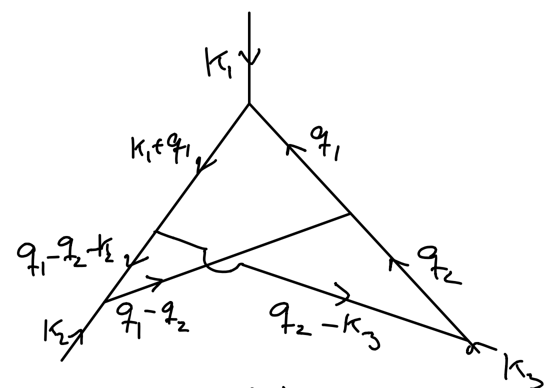
$$\int \frac{d^d q}{(q^2+\mu^2)\left[(q+k_1)^2+\mu^2\right]\left[(q-k_2)^2+\mu^2\right]}$$

Counting powers of q we see that this integral behaves as Λ^{d-6} as $\Lambda \rightarrow \infty$ (convergent for $d=6$)

At two-loop we obtain the following graphs



(a)



(b)

with integrals:

(a) $$\int dq_1\, d q_2 \frac{1}{(q_1^2+\mu^2)^2(q_2^2+\mu^2)\left[(q_2+q_1)^2+\mu^2\right]\left[(q-k_3)^2+\mu^2\right]\left[(k_1+q_1)^2+\mu^2\right]}$$

(b) $$\int dq_1\, dq_2 \frac{1}{(q_1^2+\mu^2)(q_2^2+\mu^2)\left[(k_1+q_2)^2+\mu^2\right]\left[(q_1-q_2)^2+\mu^2\right]\left[(q_2-k_3)^2+\mu^2\right] \times \left[(q_1-q_2-k_2)^2+\mu^2\right]}$$

Powercounting gives Λ^{2d-12} for both graphs

This can also be obtained from eqs. (2) and (6):

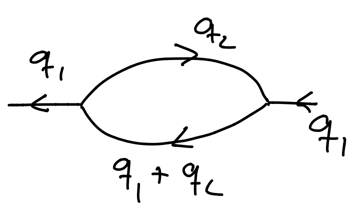
$$\Lambda^{[\text{integral}]} = \frac{\Lambda^{[\bar{\Gamma}_3]}}{\Lambda^{[(q_3)^5]}} = \Lambda^{3+d-\frac{1}{2}3d\, - 5\left(3+d-\frac{3}{2}d\right)}$$

$$= \Lambda^{2d-12}$$

However, the situation of graph (a) is more complicated:

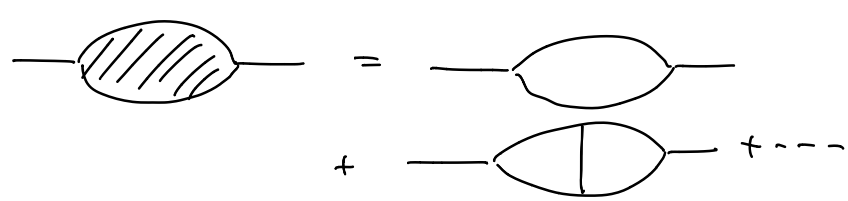
- integration over q_2 produces divergence $\sim \Lambda^{d-4} = \Lambda^2$ (for $d=6$)
- integration over q_1 is convergent $\sim \Lambda^0$ for $d=6$

This contradiction is due to the insertion of the "bubble"



This graph is part of the 2-vertex $\Gamma^{(2)}$:

$\Gamma^{(2)} =$



So if we take care of this divergence in $\Gamma^{(2)}$, this problem will <u>not</u> occur at the level of $\Gamma^{(3)}$.

(6) has no such problem

$\longrightarrow$ "primitive divergence"

(not a result of an insertion of another $\Gamma^{(N)}$ for $N < 3$)

$\longrightarrow$ General result:

In any theory focus on primitive divergences

Consider theory with pure ϕ^r interaction in d dimensions

$\longrightarrow$ nth order term of $\Gamma^{(E)}$ scales as

$$\Gamma^{(E),n} \sim \Lambda^{\delta(r,d,E,n)}$$

where $\delta(r,d,E,n) = \underset{\#\text{ loops}}{L}\,d - 2\underset{\#\text{ internal lines}}{I}$

above formula assumes bosonic propagators $\frac{1}{K^2+m^2}$ (can be generalized to other cases)

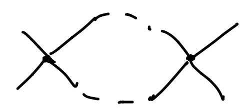
Furthermore,

- $L = I - (n-1)$

  ↑ internal momentum conservation conditions

- $I = \frac{1}{2}(nr - E)$

  ↑ # lines emitted by vertices



$\longrightarrow \delta = n\left(\frac{1}{2}rd - d - r\right) + \left(d + E - \frac{1}{2}Ed\right) = -n\delta_r + \left(d + E - \frac{1}{2}Ed\right)$

$= [\overline{\Gamma}^{(E)}] - [(\lambda_r)^n]$ (using (2), (6))

↑ independent of order in perturbation th.

$\longrightarrow$ iff $[\lambda_r] = 0$, δ is independent of n

this happens for $d_c = \frac{2r}{r-2}$

or $d_c = 4$ for ϕ^4 theory, 6 for ϕ^3 theory, and 3 for ϕ^6 theory, ...

$\longrightarrow$ • at $d = d_c$ only finitely many counterterms needed "renormalizable"

• at $d > d_c$ "non-renormalizable"

• at $d < d_c$ "super-renormalizable"