§10. Chern-Simons theory and connections on surfaces

Let M be a compact oriented 3-manifold, let G be the compact Lie group $SU(2)$ and let P be a principal G bundle over M.

Define $\mathcal{A}_M$ as the space of connections on P. Identify $\mathcal{A}_M \cong \Omega^1(M, \mathfrak{g})$ (space of 1-forms with values in $\mathfrak{g}$)

Recall:

For $A \in \mathcal{A}_M$ the "Chern-Simons functional" is given by

$$CS(A) = \frac{1}{8\pi^2} \int_M \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)$$

Definition (gauge trf.):

The gauge group of P is $\mathcal{G} \cong \mathrm{Map}(M, G)$

We have a right-action of $\mathcal{G}$ on $\mathcal{A}_M$ given by

$$g^* A = g^{-1} A g + g^{-1} dg, \quad A \in \mathcal{A}_M, g \in \mathcal{G}$$

Proposition 1:

Let M be a compact oriented 3-manifold with $\partial M \neq \emptyset$. Then we have

$$CS(g^* A) = CS(A) + \frac{1}{8\pi^2} \int_{\partial M} \mathrm{Tr}(A \wedge dg\, g^{-1}) - \int_M g^* \sigma$$

Proof:

$$g^*A = g^{-1}Ag + g^{-1}dg$$

$$g^*F_A = g^{-1}F_A g \,, \quad \text{where } F_A = dA + A\wedge A$$

$$\Rightarrow \quad CS(g^*A) = \frac{1}{8\pi^2}\int_M \mathrm{Tr}\left(g^*A \wedge g^*F_A - \frac{1}{3} g^*A \wedge g^*A \wedge g^*A\right)$$

$$= \frac{1}{8\pi^2}\int_M \mathrm{Tr}\Big(A\wedge F_A + \underbrace{g^{-1}dg \wedge g^{-1}F_A g}_{= -dA\wedge dg\, g^{-1} + \cancel{A\wedge A\wedge dg\, g^{-1}}} \Big)$$

$$- \frac{1}{24\pi^2}\int_M \Big(g^*\sigma + \cancel{g^{-1}dg \wedge g^{-1}Ag} \wedge g^{-1}Ag + \cancel{\text{perm.}}$$

$$+ g^{-1}\cancel{dg \wedge g^{-1}dg} \wedge g^{-1}Ag + \cancel{\text{perm.}}\Big)$$

$$-dA\wedge dg\, g^{-1} = d\left(A\wedge dg\, g^{-1}\right) - A\wedge \cancel{dg \wedge g^{-1}dg}\, g^{-1}$$

$$- \frac{1}{24\pi^2}\int_M \mathrm{Tr}(A\wedge A\wedge A)$$

$\square$

Set now $\partial M = \Sigma$ (Riemann surface)

Denote by Q a principal G bundle over Σ.

For $G = SU(2) \longrightarrow Q \cong \Sigma \times SU(2)$, since $SU(2)$ simply connected

Denote by $\mathcal{A}_\Sigma$ the space of connections on Q.

We have $\mathcal{A}_\Sigma \cong \Omega^1(\Sigma, \mathfrak{g})$

On $\mathcal{A}_\Sigma$ there is non-degenerate anti-symmetric bilinear form ω defined by

$$\omega(\alpha, \beta) = -\frac{1}{8\pi^2} \int_\Sigma \mathrm{Tr}(\alpha \wedge \beta), \quad \alpha, \beta \in \Omega^1(\Sigma, \mathfrak{g})$$

with $d\omega = 0$.

$\longrightarrow$ $\mathcal{A}_\Sigma$ has structure of infinite dimensional symplectic manifold.

By Prop. 3 §1 $\exists$ line bundle $\mathcal{L}$ over $\mathcal{A}_\Sigma$ and a connection ∇ on $\mathcal{L}$ s.t. $\omega = c_1(\nabla)$ (first Chern class)

$\longrightarrow$ Quantization of $\mathcal{A}_\Sigma$

In the following we shall construct $\mathcal{L}$.

Denote by $\mathcal{G}_\Sigma$ the gauge group of Q.

$\rightarrow \mathcal{G}_\Sigma \cong \mathrm{Map}(\Sigma, G)$

For $a \in \mathcal{A}_\Sigma$ and $g \in \mathcal{G}_\Sigma$ let A be an extension of a on M and $\tilde{g} : M \to G$ an extension of g as a smooth map on M to G.

Set

$$c(a, g) = \exp\left(2\pi\sqrt{-1}\left(CS(\tilde{g}^* A) - CS(A)\right)\right)$$

More explicitly,

$$c(a, g) = \exp 2\pi\sqrt{-1}\left(\int_\Sigma \frac{1}{8\pi^2} \mathrm{Tr}\left(g^{-1} a g \wedge g^{-1} dg\right) - \underbrace{\int_M \tilde{g}^* \sigma}_{\text{Wess-Zumino term}}\right)$$

Proposition 2:

Let M be a compact oriented 3-manifold with boundary Σ. For a connection A of a principal G bundle P over M and a gauge transformation $g \in Map(M, G)$ we have

$$\exp(2\pi\sqrt{-1}\, CS(g^* A)) = c(a, g|_\Sigma) \exp(2\pi\sqrt{-1}\, CS(A))$$

where $g|_\Sigma$ denotes the restriction of g on Σ.

Let $a \in \mathcal{A}_\Sigma$. Define $L_{\Sigma,a}$ as the set of maps $f : Map(\Sigma, G) \to \mathbb{C}$ satisfying

$$f(e \cdot g) = c(a,g)\, f(e), \quad g \in Map(\Sigma, G)$$

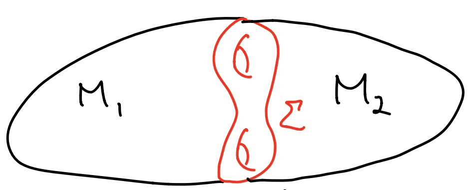
$\to$ $L_{\Sigma,a}$ is 1-dimensional complex vector space with Hermitian inner product.

Prop. 2 $\to$ $\exp(2\pi\sqrt{-1}\, CS(A)) \in L_{\Sigma,a}$

For $-\Sigma$ (Σ with <u>reversed</u> orientation) we have

$$L_{-\Sigma,a} \cong \overline{L_{\Sigma,a}}$$

Let $M = M_1 \cup M_2$ with $\partial M_1 = \Sigma$ and $\partial M_2 = -\Sigma$



Let A be a connection on M and A_1 and A_2 its restrictions on M_1 and M_2.
$\alpha =$ restriction of A on Σ

$\to \exp\left(2\pi\sqrt{-1}\, CS_{M_1}(A_1)\right) \in L_{\Sigma,\alpha}$,

$\exp\left(2\pi\sqrt{-1}\, CS_{M_2}(A_2)\right) \in \overline{L_{\Sigma,\alpha}}$

Using the canonical pairing $L_{\Sigma,\alpha} \times L_{-\Sigma\alpha} \to \mathbb{C}$ we have

$$\exp\left(2\pi\sqrt{-1}\, CS_M(A)\right) = \langle \exp\left(2\pi\sqrt{-1}\, CS_{M_1}(A_1)\right), \exp\left(2\pi\sqrt{-1}\, CS_{M_2}(A_2)\right)\rangle$$

Denote by η_t, $0 \leq t \leq 1$ a one-parameter family of connections of a G-bundle Q over Σ.

$\to$ regard η as connection over $\Sigma \times [0,1]$

$\to$ $CS_{\Sigma\times[0,1]}$ defines a map

$$\exp\left(2\pi\sqrt{-1}\, CS_{\Sigma\times[0,1]}\right) : L_{\eta_0} \to L_{\eta_1}$$

Let L_Σ be a topologically trivial line bundle over $\mathcal{A}_\Sigma$. For a path η_t, $0 \leq t \leq 1$, in $\mathcal{A}_\Sigma$

$\to$ lift on the total space of L_Σ

$\to$ connection ∇ on L_Σ with hor. sections given by above lift

$\to$ can verify: $c_1(\nabla) = \omega$

Lift action of gauge group $\mathcal{G}_\Sigma$ to L_Σ. Define

$$\mathcal{M}_\Sigma = \mathcal{A}_\Sigma /\!/ \mathcal{G}_\Sigma = \mu^{-1}(0)/\mathcal{G}_\Sigma$$

(Marsden-Weinstein quotient)

moduli space of flat G-connections on Σ

$\to$ complex line bundle $\mathcal{L}$ on $\mathcal{M}_\Sigma$

Witten's invariant for 3-manifolds is formally written as

$$Z_k(M) = \int_{\mathcal{A}_M/\mathcal{G}} \exp\left(2\pi\sqrt{-1}\, k\, CS(A)\right) \mathcal{D}A \quad (*)$$

Suppose that M is oriented 3-manifold with boundary Σ. Have shown

$$\exp\left(2\pi\sqrt{-1}\ k\, CS(A)\right) \in L_{\Sigma,\alpha}$$

$\mathcal{A}_{M,\alpha} :=$ space of G-connections on M whose restriction on Σ $= \alpha$

Restrict the path integral in $(*)$ on $\mathcal{A}_{M,\alpha}$

Since

$$\exp\left(2\pi\sqrt{-1}\ k\, CS(g^*A)\right) = c(g,\alpha)^k \exp\left(2\pi\sqrt{-1}\ k\, CS(A)\right)$$

$\to Z_k(M)$ is section of complex line bundle $\mathcal{L}^{\otimes k}$

Introduce a complex structure J on $\mathcal{M}_\Sigma$ and denote the space by $\mathcal{M}_\Sigma^{(J)}$

$\to$ Denote by $\mathcal{H}_\Sigma$ the space of hol. sections of $\mathcal{L}^{\otimes k}$ with respect to complex structure J

$\to \mathcal{H}_\Sigma$ is finite dimensional Hilbert space and is identified with space of conformal blocks for Σ !

In the case of a link L in M:

$$Z_k(M; C_1, \ldots, C_r) = \int \exp\left(2\pi\sqrt{-1}\, k\, CS(A)\right) \prod_{j=1}^{r} W_{C_j, R_j}(A)\, \mathcal{D}A$$

where $W_{C_j, R_j}(A) = \mathrm{Tr}_{R_j} \mathrm{Hol}_{C_j}(A)$