§3 6d Anomaly polynomials

§3.1 (2,0) theories

charge pairing in 4d:

$$q = (e, m), \qquad q' = (e', m')$$

$$\rightarrow \quad \langle q, q' \rangle_{4d} = em' - e'm \in \mathbb{Z}$$
anti-sym.

in 6d:

For n self-dual tensor fields, there are self-dual strings (M-strings) with charges in n-dim lattice Λ

symmetric pairing: $\langle q, q' \rangle_{6d} = \langle q', q \rangle_{6d}$

$$\downarrow T^2$$

4d charge : $q(\underbrace{mA + nB})$
$\qquad\qquad\qquad$ winding along A- and B-cycle
$\qquad\qquad\qquad$ of T^2

$$\langle q A, q' B \rangle_{4d} = \langle q, q' \rangle_{6d} \underbrace{\langle A, B \rangle_{T^2}}$$
$\qquad\qquad\qquad\qquad\qquad$ intersection number in T^2

Introduce $q = (q_i)_{i=1,\cdots,n} \in \Lambda$

$$\rightarrow \langle q, q' \rangle_{6d} = \underset{\uparrow}{\Omega^{ij}} q_i q_j'$$
$\qquad\qquad\qquad\qquad$ symmetric matrix

Introduce self-dual 3-form field strength H_i:

$$dH_i = q_i \prod_{a=2,3,4,5} \delta(x_a) dx_a$$

for M-string of charge q at $x_{a=2,3,4,5} = 0$

$\longrightarrow$ modification of Biandi identity:

$$\boxed{dH_i = \frac{X_i}{q}} \qquad (*)$$

4-form constructed of
metric and gauge fields

gives contribution to anomaly polynomial:

review:

- descent formalism:

$$I_8 = dI_7, \quad \delta_\Lambda I_7 = dI_6'(\Lambda)$$

$$\text{and} \quad \delta_\Lambda S = \int I_6'(\Lambda)$$

- $\overset{1}{I_8} = \frac{10 - \overset{\text{number of tensors}}{4}}{8} (tr R^2)^2 + \frac{1}{6} tr R^2 \sum_a X_a^{(2)}$

$$- \frac{2}{3} \sum_a X_a^{(4)} + 4 \sum_{a < b} Y_{ab}$$

where $X_a^{(n)} = Tr F_a^n - \sum_R n_{R_a} tr_{R_a} F_a^n$

$$Y_{ab} = \sum_{R_a, R_b'} n_{R_a R_b'} tr_{R_a} F_a^2 tr_{R_b'} F_b^2$$

notation: $\overline{\text{Tr}}$: trace in adjoint rep.

tr_{R_a} : trace in rep. R_a

n_{R_a} : number of hypermultiplets

$n_{R_a R_b'}$: " " " in

rep (R_a, R_b') of $G_a \times G_b$

* anomaly cancellation:

I_8 should be representable as

$$I_8 = \frac{1}{2} \Omega_{ij} X^i X^j$$

with $X^i = \frac{1}{2} a^i \text{tr} R^2 + 2 b_a^i \text{tr} F_a^2$

(closed and gauge invariant)

cancellation due to "Green-Schwarz"

mechanism:

$$A = \int_{\mathbb{R}^{5,1}} \Omega_{ij} B^i X^j$$

where $H^i = dB^i + \frac{1}{2} a^i \omega_{3L} + 2 b_a^i \omega_{3Y}^a$

gravitational ↑   ↑ Yang-Mills
Chern-Simons   CS 3-form
3-form

$$\to \delta_\Lambda (A + S) = \int \Omega_{ij} (\delta_\Lambda B^i) X^j + \int I_6^1(\Lambda) = 0$$

Since $dH^i = X^i$, equation (*) gives:

$$I^{GS} = \frac{1}{2} \Omega^{ij} X_i X_j$$

6d Green-Schwarz and 5d Chern-Simons:

- S^1 reduction of $H_i \longrightarrow$ n Abelian gauge fields A_i

$$\to F_{\mu\nu} = 2\pi R \cdot H_{\mu\nu 5}$$

$$\to 5d \text{ Kinetic term: } \frac{1}{2R} \Omega^{ij} F_i \wedge *F_j$$

and reduction of $dH_i = X_i$ gives

$$d\left(\frac{1}{2\pi R} *F_i\right) = X_i$$

$$\to \text{ Chern-Simons term in } 5d:$$

$$\frac{1}{2\pi} S^{CS} = \Omega^{ij} A_i X_j = A_i I^i$$

- consider a 5d fermion ψ with mass term $m \bar\psi \psi$ and charge q under a $U(1)$, coupling to non-Abelian background gauge field F_G in rep. ρ, coupling to metric

$\longrightarrow$ triangle diagrams give induced CS-term:

$$\frac{1}{2} (\text{sign } m) \, q \, A \left(\frac{1}{2} \text{tr}_\rho F_6^2 + \frac{1}{24} \, q_6 \, p_1(T) \right)$$

- Since reduction of 6d $(2,0)$ theory of ADE type on T^2 gives 4d $\mathcal{N}=4$ with gauge group G of ADE
  $\longrightarrow$ 6d charge lattice of M-strings is root lattice of G
  $\longrightarrow$ Ω^{ij} is Cartan matrix η^{ij} of G

  R-sym. of $(2,0)$ theory is $SO(5)_R$
  $\longrightarrow$ going to tensor branch gives $SO(4)_R$
  $$SO(4)_R \simeq SU(2)_R \times SU(2)_L$$
  by introducing veo $\phi \in$ Cartan$(\mathfrak{g})$
  $\longrightarrow$ reduction on S^1 gives massive charged $\mathcal{N}=2$ vector multiplets of mass $|\phi \cdot \alpha|$
  $\forall \, \alpha$ roots of $\mathfrak{g}$
  $\longrightarrow$ pair of massive $\mathcal{N}=1$ VM and $\mathcal{N}=1$ HM
  $\longrightarrow$ fermion masses $\psi T^{iL} \phi^I \psi$

VM has mass $-\phi\cdot\alpha$, HM has mass $+\phi\cdot\alpha$

The induced CS terms are then:

$$\frac{1}{2}\sum_{\alpha>0}(\alpha\cdot A)\left[\left(c_2(L)+\frac{2}{24}p_1(T)\right)-\left(c_2(R)+\frac{2}{24}p_1(T)\right)\right]$$

$$=\rho\cdot A\left(c_2(L)-c_2(R)\right)$$

$\uparrow$
Weyl vector

Lifting back to 6d gives:

$$dH_i = \rho_i\left(c_2(L)-c_2(R)\right)$$

$\rightarrow$ GS contribution to anomaly of 6d theory:

$$\frac{1}{2}\langle\rho,\rho\rangle\left(c_2(L)-c_2(R)\right)^2 = \frac{h_G^\vee d_G}{24}\left(c_2(L)-c_2(R)\right)^2$$

Using $p_2(N) = \left(c_2(L)-c_2(R)\right)^2$ gives then

$$I_G^{\mathcal{N}=(2,0)} = \frac{h_G^\vee d_G}{24}p_2(N) + r_G\, I^{\mathcal{N}=(2,0)\text{ tensor}}$$