# §5. 2d Landau–Ginzburg models from 3d theory $T[M_3]$

Circle reduction of 3d $\mathcal{N}=2$ leads to 2d $\mathcal{N}=(2,2)$ theory with twisted superpotential $\widetilde{W}(\Sigma_a, M_i)$

gauge fields

background fields

Contributions:

1) Integrating out chiral multiplets

$\longrightarrow$ $\widetilde{W}$ receives one-loop corrections:

$$\delta\widetilde{W}(M_\phi) = \sum_{n\in\mathbb{Z}} \left(M_\phi + \frac{2\pi i n}{R}\right)\left[\log\left(RM_\phi + 2\pi i n\right) - 1\right]$$

KK-modes

$$\simeq \frac{R}{4} M_\phi^2 + \frac{1}{R} \mathrm{Li}_2\left(-e^{-RM_\phi}\right)$$

where $\mathrm{Li}_2(z) := \sum_{k=1}^{\infty} \frac{z^k}{k^2}$

2) Chern–Simons terms contribute as

$$\frac{1}{R}\widetilde{W}_{CS}(\Sigma_a, M_i) = \frac{1}{2} k_{ab}\Sigma_a\Sigma_b + k_{ia}\Sigma_a M_i + \frac{1}{2} k_{ij} M_i M_j$$

where $K := \begin{pmatrix} k_{ab} & k_{aj} \\ k_{ia} & k_{ij} \end{pmatrix}$ is level-matrix

# Vacua and boundary conditions at infinity

vacua are given by:

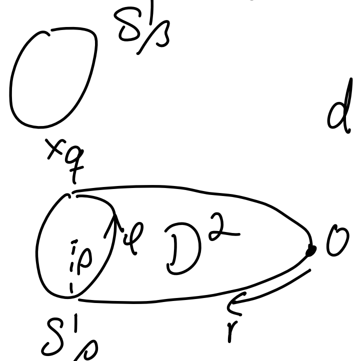
$$\frac{\partial \widetilde{W}}{\partial \sigma_a} = 2\pi i n_a, \qquad n_a \in \mathbb{Z}$$

Using $\quad s_a = e^{\sigma_a}, \quad x_i = e^{m_i},$

this becomes

$$\exp\left( s_a \frac{\partial \widetilde{W}}{\partial s_a} \right) = 1$$

Putting 3d geometry on




$$ds^2 = dr^2 + f(r)^2 (d\varphi + \varepsilon\beta d\theta)^2 + \beta^2 d\theta^2$$

$$q = e^{2\pi i \varepsilon\beta} = e^{t_\hbar}$$

we can further compactify on $S^1_\rho$ to obtain $\mathcal{N}=4$ QM in 1d with SP:

$$W_{t_\hbar}(s_a, m_i; t_\hbar) = \frac{1}{2} k_{ab} \sigma_a \sigma_b + k_{ai} \sigma_a m_i + \frac{1}{2} k_{ij} m_i m_j + \sum_\phi \left[ \frac{1}{4} m_\phi^2 + Li_2\left(-e^{-m_\phi - \frac{t_\hbar}{2}}; t_\hbar\right) \right]$$

where $Li_2(x;\hbar) := \sum_{n=0}^{\infty} \frac{B_n \hbar^n}{n!} Li_{2-n}(x)$

with $B_n$ the Bernoulli numbers $(1, \frac{1}{2}, \frac{1}{6}, 0, \dots)$

## Localization of the bulk path integral

The bosonic part of the action is given by

$$I_{\vec{n}} = \int dt\, d^4\theta\, g_{a\bar{b}}\, \Sigma^a \bar{\Sigma}^b$$

$$+ \int dt\, d\theta\, d\bar{\theta}\; W^{QM}_{\vec{n}}(\Sigma_a, M_i) + c.c.$$

Wave-functions are given by

$$\psi^\alpha \simeq \int_{\Gamma^\alpha} \Omega \, \exp\left(\frac{1}{\hbar} \widetilde{W}_\hbar(s_a, m_i; \hbar)\right)$$

Lefschetz-cycles (mid-dimensional)

where $\Omega = d\sigma_1 \wedge \dots \wedge d\sigma_r$

## § 5.1 BPS solitons in $\mathcal{N}=2$ Landau-Ginzburg theories

The action for a Landau Ginzburg model of $n$ chiral superfields $\Phi_i$ $(i=1,\dots,n)$ with superpotential $W(\Phi)$ is given by

$$S = \int d^2x \left[ \int d^4\theta\, K(\Phi_i, \bar{\Phi}_i) + \frac{1}{2}\left( \int d^2\theta\, W(\Phi_i) + \int d^2\bar{\theta}\, \bar{W}(\bar{\Phi}_i) \right) \right]$$

where $K(\Phi_i, \bar{\Phi}_i)$ is the Kähler potential which defines the Kähler metric $g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K(\Phi_i, \bar{\Phi}_i)$

Vacua are labeled by critical points of $W$:

$$\phi^i(x) = \phi^i_*\,, \qquad \partial_i W\big|_{\phi^i_*} = 0$$

theory is purely "massive" if all the critical points are isolated and non-degenerate

$\to W$ is quadratic near $\phi^i_*$

→ label non-degenerate critical points as

$$\{\phi_a \mid a=1,\dots,N\}$$

→ # vacua is equal to dimension of local ring of $W(\Phi)$, $\mathcal{R}=\frac{\mathbb{C}[\Phi]}{\partial_{\phi_i}W}$

When we have more than one vacuum, we can have "solitonic" states in which the boundary conditions of the fields at the left spacial infinity $x^1=-\infty$ is at one vacuum and is different from the one at right infinity $x^1=+\infty$

→ energy of interpolating field config:

$$E_{ab}=\int_{-\infty}^{+\infty}dx^1\left\{g_{i\bar{j}}\frac{d\phi^i}{dx^1}\frac{d\bar{\Phi}^{\bar{j}}}{dx^1}+\frac{1}{4}g^{i\bar{j}}\partial_iW\partial_{\bar{j}}\bar{W}\right\}$$

$$=\int_{-\infty}^{\infty}dx^1\left|\frac{d\phi^i}{dx^1}-\frac{\overset{\text{arbitrary phase}}{\alpha}}{2}g^{i\bar{j}}\partial_{\bar{j}}\bar{W}\right|^2$$

$$+\ \mathrm{Re}\left(\bar{\alpha}\left(W(b)-W(a)\right)\right)$$

BPS solitons are given by

$$\frac{d\phi^i}{dx^1} = \frac{\alpha}{2} g^{i\bar{j}} \partial_{\bar{j}} \overline{W}, \qquad \alpha = \frac{W(b) - W(b)}{|W(b) - W(a)|}$$

→ energy saturates bound in

$$E_{ab} \geq |W(b) - W(a)|$$

→ superpotential satisfies equation

$$\partial_{x^1} W = \frac{\alpha}{2} g^{i\bar{j}} \partial_i W \partial_{\bar{j}} \overline{W} \qquad (*)$$

Since $g^{i\bar{j}}$ is positive definite, $g^{i\bar{j}} \partial_i W \partial_{\bar{j}} \overline{W}$ is real

→ image of BPS soliton in W-plane is straight line connecting $W(a)$ and $W(b)$.

<u>Vanishing cycles</u>

With no loss of generality assume $\alpha = 1$

→ near critical point $\phi^i_a$, can choose coordinates $u^i_\alpha$ s.th.

$$W(\phi) = W(\phi) + \sum_{i=1}^{n} (u^i_a)^2$$

→ solutions of (*) will have straight line image on W-plane

→ real (n-1)-dim sphere emanates from $u_a^i = 0$

$$\sum_{i=1}^{n} (\mathrm{Re}(u_a^i))^2 = w - w_a , \quad \mathrm{Im}(u_a^i) = 0$$

where $w_a = W(\phi_a)$

Note that as we take $w \mapsto w_a$ the sphere vanishes "vanishing cycle"

(n-1)-dim homology cycle $\Delta_a$ in the (n-1)-dim complex manifold defined by $W^{-1}(w)$

→ solitons originating from $\phi_a$ and traveling all the way to $\phi_b$ correspond to intersection points $\Delta_a \cap \Delta_b$