§1 Loop groups and affine Lie algebras

Definition (Loop group)

Let G be a compact connected Lie group.

Define $LG \equiv \{ \gamma : S^1 \to G \mid \gamma \text{ smooth map} \}$

where $S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \}$

group structure on LG:

$$(\gamma_1 \cdot \gamma_2)(z) = \gamma_1(z)\gamma_2(z), \quad \gamma_1, \gamma_2 \in LG$$

$\rightarrow$ $(\gamma_1, \gamma_2) \mapsto \gamma_1 \cdot \gamma_2$ and $\gamma \mapsto \gamma^{-1}$ are smooth maps

$\Rightarrow$ LG is infinite dimensional Lie group "Loop group" of G

In these lectures: $G = SU(2)$

Let $\mathbb{C}((t))$ denote the $\mathbb{C}$ algebra of the Laurent series: $f(t) = \sum_{n=-m}^{\infty} a_n t^n$, $m \in \mathbb{Z}$

Let g be the complexified Lie algebra of G

$\rightarrow$ in our case: $g = sl_2(\mathbb{C})$. Set

$$Lg = g \otimes \mathbb{C}((t)) \quad (\text{compl. Lie algebra of } LG)$$

Lie bracket:

$$[X \otimes f, Y \otimes g] = [X, Y] \otimes fg$$

Now define $\hat{\mathfrak{g}}$ as the direct sum

$$L\mathfrak{g} \oplus \mathbb{C}c$$

$\uparrow$ one-dim complex vector space with basis c

Lie-bracket for $\hat{\mathfrak{g}}$:

$$[\xi + \alpha c, \eta + \beta c] = [\xi, \eta] + \omega(\xi, \eta)\, c, \qquad (*)$$

$$\xi, \eta \in L\mathfrak{g} \quad \alpha, \beta \in \mathbb{C}$$

where $\omega : L\mathfrak{g} \times L\mathfrak{g} \to \mathbb{C}$ bilinear form

$\to$ c belongs to center of $\hat{\mathfrak{g}}$:

$$[\xi, c] = [\xi, 0 + c] = [\xi, 0] + \underbrace{\omega(\xi, 0)}_{=0} c = 0$$

$\to$ $(*)$ defines a Lie algebra structure on $\hat{\mathfrak{g}}$ <u>iff</u> for $x, y, z \in L\mathfrak{g}$:

(a) $\omega(x, y) = -\omega(y, x)$ (anti-symmetry)

(b) $\omega([x,y], z) + \omega([y,z], x) + \omega([z,x], y) = 0$ (Jacobi identity)

$\to$ Lie bracket for $\hat{\mathfrak{g}}$:

$$[X \otimes f, Y \otimes g] = [X, Y] \otimes fg + \omega(X \otimes f, Y \otimes g) c$$

Condition (b) is called "2-cocycle condition"

ω and ω' are equiv. iff $\exists\ \mu : L\mathfrak{g} \to \mathbb{C}$ linear s.t. $\omega(x,y) = \omega'(x,y) + \mu([x,y]) \quad \forall\ x, y \in L\mathfrak{g}$

$\to$ $\hat{\mathfrak{g}} = L\mathfrak{g} \oplus \mathbb{C}c$ is Lie algebra with $c \in \mathrm{Center}(\hat{\mathfrak{g}})$

Definition:

The Lie algebra $\hat{g}$ is called "central extension" of Lg.

Definition (Lie algebra cohomology):

For a Lie algebra a and left a module M define

$$C^p(a, M) = \mathrm{Hom}_{\mathbb{C}}\left(\bigwedge^p a, M\right)$$

(p-th cochain group)

and differential $d_p : C^p(a, M) \to C^{p+1}(a, M)$:

$$(d\omega)(x_0, x_1, \dots, x_p)$$
$$= \sum_{i=0}^{p} (-1)^i x_i \, \omega(x_0, \dots, \hat{x}_i, \dots, x_p)$$
$$+ \sum_{0 \le i < j \le p} (-1)^{i+j} \omega([x_i, x_j], x_0, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_p)$$

for $\omega \in C^p(a, M)$. Then

$$H^p(a, M) = \mathrm{Ker}\, d_p / \mathrm{Im}\, d_{p-1}$$

is called p-th cohomology of a with coefficients in M.

Regarding $\mathbb{C}$ as a trivial g module ($g\mathbb{C} = 0$)

$\longrightarrow$ $\omega \in H^2(Lg, \mathbb{C})$ (condition (b) becomes $d\omega = 0$ and $\omega([x, y]) = d\sigma$ for $\sigma \in H^1(Lg, \mathbb{C}$))

The converse can also by shown

$$\rightarrow \left\{ \begin{array}{l} \text{isomorphism classes} \\ \text{of central extensions of } Lg \end{array} \right\} \longleftrightarrow H^2(Lg, \mathbb{C})$$

Definition (Cartan-Killing form):

A non-degenerate symmetric bilinear form

$$\langle \, , \, \rangle : g \times g \longrightarrow \mathbb{C}$$

satisfying

$$\langle [X, Y], Z \rangle = \langle X, [Y, Z] \rangle \qquad (**)$$

for $X, Y, Z \in g$ is called "Cartan-Killing form"

For $g = sl_2(\mathbb{C})$ we set $\langle X, Y \rangle = Tr(XY)$

Proposition 1:

For the loop algebra Lg we have

$H^2(Lg, \mathbb{C}) \cong \mathbb{C}$. The generator of $H^2(Lg, \mathbb{C})$ is given by

$$\omega(X \otimes f, Y \otimes g) = \langle X, Y \rangle \operatorname{Res}_{t=0}(df\, g)$$

where $\operatorname{Res}_{t=0}\left(\sum_n c_n t^n dt\right) = c_{-1}$

Proof:

$G \subset LG$ by choosing $\gamma : S^1 \rightarrow G$ constant

For $g \in G$ write $g = \exp t Z$, $Z \in g$. For $X \in Lg$ we have $g X g^{-1} = X + t[Z, X] + O(t^2)$

$\rightarrow$ For α 2-cocycle of Lg we have

$$\lim_{t\to 0} \frac{1}{t}\left[\alpha(gXg^{-1}, gYg^{-1}) - \alpha(X,Y)\right] = \alpha(Z,[X,Y])$$

for $X, Y \in Lg$ by 2-cocycle condition.

Define 1-cochain $\mu_Z : Lg : \to \mathbb{C}$ by

$$\mu_Z(u) = \alpha(Z, u) \quad \text{for } Z \in Lg$$

$$\Rightarrow \alpha(Z,[X,Y]) = \mu_Z([X,Y]) = d\mu_Z(X,Y)$$

trivial in $H^2(Lg, \mathbb{C})$

Denote $\alpha_g(X,Y) = \alpha(gXg^{-1}, gYg^{-1})$. Then

$$\int_G \alpha_g \, dg$$

in invariant under conjugation and is cohomologous to α (G is simply connected).

$\to$ suppose that α is invariant under conj.:

$$\alpha([Z,X],Y) + \underbrace{\alpha(X,[Z,Y])}_{= -\alpha(X,[Y,Z])} = 0 \to (**)$$

Set $\alpha_{m,n}(X,Y) = \alpha(X\otimes t^m, Y\otimes t^n)$ for $X, Y \in g$

$\to$ $\alpha_{m,n} : g\times g \to \mathbb{C}$ is bilinear and satisfies $(**)$

$\to$ $\alpha_{m,n}$ is symmetric (and therefore Killing-form) since g is simple.

Then $\alpha_{m,n} = -\alpha_{n,m}$ (α anti-sym.)

Cocycle condition for α becomes

$$\alpha_{m+n,p} + \alpha_{n+p,m} + \alpha_{p+m,n} = 0$$

$n = p = 0 \to \alpha_{m,0} = 0 \;\; \forall m$

$p = -m-n \rightarrow \alpha_{m+n,-m-n} = \alpha_{m,-m} + \alpha_{n,-n}$

$\Rightarrow \alpha_{m,-m} = m\,\alpha_{1,-1}$

$p = q-m-n \rightarrow \alpha_{q-m-n,m+n} = \alpha_{q-m,m} + \alpha_{q-n,n}$

$\Rightarrow \alpha_{q-k,k} = k\,\alpha_{q-1,1}$

$\Rightarrow \alpha_{m,n} = 0$ if $m+n \neq 0$ $\quad (q\alpha_{q-1,1} = \alpha_{0,q} = 0)$

$$\Rightarrow \alpha_{m,n} = m\,\delta_{m+n,0}\,\alpha_{1,-1}$$

$\alpha_{1,-1} : g \times g \rightarrow \mathbb{C}$ is g invariant sym. bilinear form $\rightarrow$ equal to Cartan-Killing form up to const.

set $\omega = \alpha_{1,-1}$

It can be easily shown that ω is not coboundary (exercise). $\square$

<u>Definition</u> (affine Lie algebra):
The central extension $\hat{g}$ of g with Lie bracket

$$[X\otimes t^m, Y\otimes t^n] = [X,Y]\otimes t^{m+n} + \langle X,Y\rangle m\,\delta_{m+n,0}\,c$$

for $X, Y \in g$ is called "affine Lie algebra" associated with g.

The group of diffeomorphisms of $S^1 = \{z\in\mathbb{C} \mid |z|=1\}$, denoted by $\mathrm{Diff}(S^1)$, is infinite dimensional Lie group. Corresponding Lie algebra:

$$A = \left\{ f(z)\frac{d}{dz} \,\middle|\, f(z) \in \mathbb{C}[z,z^{-1}] \right\}$$

where $\mathbb{C}[z,z^{-1}]$ is $\mathbb{C}$ algebra of Laurent pols.

Lie bracket:

$$\left[f(z)\frac{d}{dz}, g(z)\frac{d}{dz}\right] = \left\{f(z)g'(z) - g(z)f'(z)\right\}\frac{d}{dz}$$

Basis generators:

$$L_n = -z^{n+1}\frac{d}{dz}, \quad n \in \mathbb{Z}$$

with restriction to S^1 given by:

$$L_n = ie^{in\theta}\frac{\partial}{\partial\theta}, \quad z = e^{i\theta}$$

Commutation relation: $[L_m, L_n] = (m-n)L_{m+n}$

L_n generates infinitesimal transformations of $\mathbb{C}\setminus\{0\}$ given by $\varphi_t(z) = z - tz^{n+1}$

L_0, L_{-1}, L_1 extend to $\mathbb{CP}^1$:

$$[L_0, L_1] = -L_1, \quad [L_0, L_{-1}] = L_{-1}, \quad [L_1, L_{-1}] = 2L_0$$

$\longrightarrow$ generate $sl_2(\mathbb{C})$

<u>Proposition 2:</u>

$H^2(A, \mathbb{C}) \cong \mathbb{C}$. The cohomology $H^2(A, \mathbb{C})$ has a basis represented by the 2-cocycle

$$\alpha := \omega\left(f\frac{d}{dz}, g\frac{d}{dz}\right) = \frac{1}{12}\mathrm{Res}_{t=0}\left(f'''g\,dz\right)$$

<u>Proof:</u>

Let α be a 2-cocycle of A. Put $\alpha_{p,q} = \alpha(L_p, L_q)$

2-cocycle condition for (L_0, L_p, L_q)

$\xrightarrow[\text{Prop. 1}]{\text{similar to proof of}}$ α is invariant under rotation also $\alpha_{p,q} = 0$ if $p+q \neq 0$

set $\alpha_p := \alpha_{p,-p} \rightarrow$ 2-cycle condition:

$$(p+2q)\alpha_q - (2p+q)\alpha_q = (p-q)\alpha_{p+q}$$

(exercise)

$\Rightarrow \alpha_p = \lambda p^3 + \mu p$

We can get μp by

$$d\beta(L_p) = \beta([L_p, L_{-p}]) = 2p\,\beta(L_0)$$

by setting $\mu := \frac{1}{2}\beta(L_0)$

$\rightarrow \mu p$ is coboundary and does not change $H^2(A, \mathbb{C})$.

Set $\lambda = \frac{1}{12}$, $\mu = -\frac{1}{12}$ $\square$

Lie-bracket on $\mathcal{V} = A \oplus \mathbb{C}c$:

$$[f\tfrac{d}{dz} + \xi c, g\tfrac{d}{dz} + \eta c] = [f\tfrac{d}{dz}, g\tfrac{d}{dz}] + \alpha(f\tfrac{d}{dz}, g\tfrac{d}{dz})c,$$

$\xi, \eta \in \mathbb{C}$

<u>Definition:</u>

The Lie algebra $\mathcal{V}$ is called "Virasoro algebra":

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0}\,c$$