## Recall:

Let $M$ be a smooth manifold and let $L$ be a complex line bundle over $M$.
Fix connection $\nabla$ on $L$. Then

$$\nabla = d - 2\pi\sqrt{-1}\,\alpha_U \quad \text{locally on } U \subset M$$

$\rightarrow$ $d\alpha_{U_\lambda}$ defines a global 2-form for open covering $\{U_\lambda\}, \lambda \in \Lambda$ of $M$

$\rightarrow$ first Chern form of $\nabla$: $c_1(\nabla)$

$c_1(\nabla)$ defines a class in $H^2(M, \mathbb{R})$ in the image of $H^2(M, \mathbb{Z})$ $\rightarrow$ first Chern class of $L$.

## Now:

Suppose $M$ is simply connected and fix a base point $x_0 \in M$.
Let $\gamma : [0,1] \rightarrow M$ be a smooth loop with

$$\gamma(0) = \gamma(1) = x_0.$$

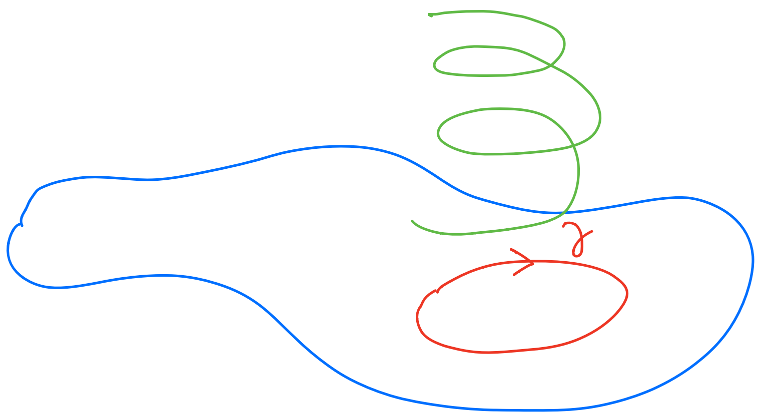
Then $\gamma^*(L)$ $\rightarrow$ complex line bundle on $[0,1]$ with connection $\gamma^*\nabla$.
"horizontal section" $s$: $(\gamma^*\nabla)s = 0$
For $u$ in fibre of $\gamma^*L$ at $0$ select section $s$ s.t. $s(0) = u$.
We have disc $D \subset M$ with $\partial D = \gamma$
($M$ simply connected)

→ Stokes theorem gives

$$(*) \qquad S(1) = u \exp\left(2\pi\sqrt{-1}\int_{D_t} c_1(\nabla)\right).$$

Denote by $L_{x_0}$ the fibre of $L$ over $x_0$.

→ $(*)$ gives linear transformation of $L_{x_0}$ denoted "holonomy" of $\nabla$ around $\gamma$.




<u>Proposition 3:</u>

Let $M$ be a simply connected smooth manifold and $\omega$ a closed 2-form on $M$ with $\omega \in \operatorname{Im} i$ where

$$i : H^2(M,\mathbb{Z}) \to H^2(M,\mathbb{R})$$

Then $\exists$ complex line bundle $L$ over $M$ and connection $\nabla$ on $L$ s.t.

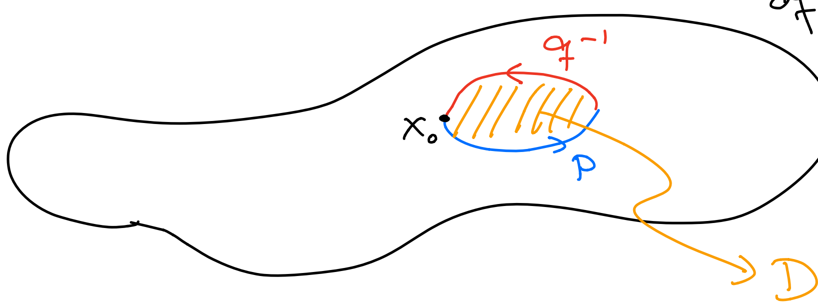
$$c_1(\nabla) = \omega$$

<u>Proof:</u>

Denote by $P_{x_0}(M)$ the set of smooth paths
$$p : [0,1] \longrightarrow M \quad \text{with} \quad p(0) = x_0$$

equivalence relation $\sim$ on $P_{x_0}(M) \times \mathbb{C}$ :
$$(p,u) \sim (q,v) \quad \underline{\text{iff}}$$
$$p(1) = q(1) \quad \text{and} \quad v = u \exp\left(2\pi\sqrt{-1}\int_D \omega\right)$$

independent of $D$ as $\omega \in H^2(M,\mathbb{Z})$




Define $L \equiv P_{x_0}(M) \times \mathbb{C}/\sim$ and proj. map
$\pi : L \longrightarrow M$ by $\pi(p,u) = p(1)$

$\longrightarrow$ connection $\nabla$ of $L$ has holonomy
$$\exp\left(2\pi\sqrt{-1}\int_D \omega\right) \quad \text{around loops } \gamma$$

$\longrightarrow \quad c_1(\nabla) = \omega$

$\square$

<u>Example:</u>

consider $M = LG_{\mathbb{C}}$ and $\omega \in H^2(LG_{\mathbb{C}}, \mathbb{Z})$

Take $G = SU(2)$ and $\mathfrak{g}_{\mathbb{C}} = sl_2(\mathbb{C})$

Define for $X \in SU(2)$ 1-form $\mu = X^{-1}dX$

("Maurer-Cartan form") of Lie group $SU(2)$.

Then $\sigma = \frac{1}{24\pi^2} \operatorname{Tr}(\mu \wedge \mu \wedge \mu)$ is generator of $H^3(SU(2), \mathbb{Z})$. Let $\phi : LG \times S^1 \to G$ be given by $\phi(\gamma, z) = \gamma(z)$. Define

$$\omega = -\int_{S^1} \phi^* \sigma$$

It can be shown that $H^2(LG, \mathbb{Z}) \cong \mathbb{Z}$ with generator $\omega$. Associated complex line bundle $L$ is called "fundamental line bundle" over $LG_{\mathbb{C}}$.

## §2 Representations of affine Lie algebras

Let $g$ be a complex Lie algebra and $V$ a complex vector space.

Definition:

We call a linear map $\rho : g \to \operatorname{End}(V)$ s.t.

$$\rho([X, Y]) = \rho(X)\rho(Y) - \rho(Y)\rho(X)$$

$\forall\ X, Y \in g$ a "linear representation" of $g$ on $V$.

We will also simply write $Xv$ for $\rho(X)v$.

Focus on $g = sl_2(\mathbb{C}) \longrightarrow \hat{g} = \hat{A}_1$

Lie-algebra: $[H, E] = 2E$, $[H, F] = -2F$, $[E, F] = H$

$$\langle H, H \rangle = 2, \ \langle E, F \rangle = 1, \ \langle H, E \rangle = 0 = \langle H, F \rangle$$

Denote by $\{I_n\}$ orthonormal basis with respect to $\langle , \rangle$. Then the "Casimir element" $C$ in $U(g)$ is given by $C = \sum_\mu I_\mu I_\mu$

universal enveloping alg

or $C = \frac{1}{2} H^2 + EF + FE$ (angular momentum operator)

<u>Definition</u> (highest weight representation):

$V$ is "highest weight rep" if:

(a) $\exists\, v \in V$ non-zero s.t. $Hv = \lambda v$ and $Ev = 0$

(b) $V$ is generated by $F^n v$, $n = 0, 1, \cdots$

If $F^n v$, $n = 0, 1, \cdots$, are linearly independent, $V$ is called "Verma module" $M_\lambda$.

For $\lambda \in \mathbb{C}$ generic, $M_\lambda$ is irreducible $g$-module. But for $\lambda = 2j$, $j \in \mathbb{Z}$ $M_\lambda$ becomes reducible and we have sub-rep $V_\lambda$ spanned by:

$$u_m, \quad m = j, j-1, \cdots, -j+1, -j$$

$$H u_m = 2m u_m,$$

$$E u_m = \sqrt{(j+m+1)(j-m)}\, u_{m+1}$$

$$F u_m = \sqrt{(j+m)(j-m+1)}\, u_{m-1}$$

In particular: $F^{\lambda+1} u_j = 0$. We have

$$V_\lambda = M_\lambda / F^{\lambda+1} v \quad \text{"spin } j \text{ rep."}$$

$C$ acts as scalar on $V_\lambda$ s.t.

$$C u_j = 2j(j+1)$$

<u>Next:</u> Representations of affine Lie algebras

Denote by

- $A_+ \subset \mathbb{C}((t))$ : the subalgebra $\sum_{n>0} a_n t^n$
- $A_-$ : sb. alg. $\sum_{n<0} a_n t^n$

Define

- $N_+ = [g \otimes A_+] \oplus \mathbb{C}E,$
- $N_0 = \mathbb{C}H \oplus \mathbb{C}c,$
- $N_- = [g \otimes A_-] \oplus \mathbb{C}F$

$\rightarrow \hat{g} = N_+ \oplus N_0 \oplus N_-$

<u>Definition:</u>

Let $k$ and $\lambda$ be complex numbers. A left $\hat{g}$-module $\hat{V}_{k,\lambda}$ is "highest weight rep." with level $k$ and highest weight $\lambda$ if:

(a) $\exists$ $v \in \hat{V}_{k,\lambda}$ non-zero with

$$N_+ v = 0, \quad cv = kv, \quad Hv = \lambda v$$

(b) $U(N_-)$ generated by $v$ coincides with $\hat{V}_{k,\lambda}$

(a) $\Rightarrow$ $cu = ku$ $\forall$ $u \in \hat{V}_{k,\lambda}$

$\hat{V}_{\kappa,\lambda}$ is generated by $a_1 \cdots a_r v$ with $a_1, \ldots, a_r \in N_-$, $r \geq 0$.

Again, we have $\hat{V}_{\kappa,\lambda} = M_{\kappa,\lambda} / \mathcal{J}$ where $M_{\kappa,\lambda}$ is Verma module.

<u>Proposition 5:</u>

Let $\kappa$ be a positive integer and $\lambda$ an integer s.t. $0 \leq \lambda \leq \kappa$. Define $x \in M_{\kappa,\lambda}$ by $x = (E \otimes t^{-1})^{\kappa - \lambda + 1} v$ where $v$ is highest weight vector in $M_{\kappa,\lambda}$. Then $N_+ x = 0$ and $\mathcal{U}(N_-) x$ is sub-module of $M_{\kappa,\lambda}$ and

$$H_{\kappa,\lambda} = M_{\kappa,\lambda} / \mathcal{U}(N_-) x$$

is irreducible $\hat{g}$ module.

$H_{\kappa,\lambda}$ is called "integrable highest weight" module and $x$ is called "null vector".

<u>Dual representation:</u>

The dual vector space $H_\lambda^*$ of $H_\lambda$ has the structure of a right $\hat{g}$ module:

$$\langle \xi \alpha, \eta \rangle = \langle \xi, \alpha \eta \rangle \quad \forall\, \xi \in H_\lambda^*, \eta \in H_\lambda, \alpha \in \hat{g}$$

$v^*$ dual to $v \Rightarrow v^* N_- = 0 \Rightarrow H_\lambda^*$ is generated by $v^*$ as right $\mathcal{U}(N_+)$ module.

Define left-representation:

$$\rho^*(X\otimes t^n)\xi = -\xi X\otimes t^{-n}, \quad X\in g,\ \xi\in H_\lambda^*$$

$\longrightarrow$ dual representation of $H_\lambda$.

Action of Virasoro Lie algebra on $H_\lambda$:

Define "Sugawara operators"

$$(n\neq 0):\quad L_n = \frac{1}{2(k+2)} \sum_{j\in\mathbb{Z}} \sum_\mu I_\mu\otimes t^{-j}\cdot I_\mu\otimes t^{n+j}$$

$$(n=0):\quad L_0 = \frac{1}{k+2} \sum_{j\geq 1} \sum_\mu I_\mu\otimes t^{-j}\cdot I_\mu\otimes t^{j}$$

$$+ \frac{1}{2(k+2)} \sum_\mu I_\mu\cdot I_\mu$$

$\longrightarrow$ defines action of Virasoro algebra on $H_\lambda$!

Proposition 6:

The Sugawara operators $L_m$, $m\in\mathbb{Z}$, acting on the integrable highest weight $\hat{g}$ module $H_{k,\lambda}$ satisfy

$$[L_m, X\otimes t^n] = -n X\otimes t^{m+n}, \quad X\in g$$

for any integer $n$.

Proposition 7:

As linear operators on $H_{k,\lambda}$, we have

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3-m}{12}\delta_{m+n,0}\frac{3k}{k+2}$$

$$\Longrightarrow \quad c = \frac{3k}{k+2} \qquad \text{"central charge"}$$

$H_{k,\lambda}$ contains a subspace $V_\lambda$ which is finite-dim. rep. of $\mathfrak{sl}_2(\mathbb{C})$. For $u \in V_\lambda$:

$$L_0 u = \frac{1}{2(k+2)} \left( \sum_\mu I_\mu \cdot I_\mu \right) u = \frac{j(j+1)}{k+2} u$$

Have set $\lambda = 2j$. Set $\Delta \equiv \frac{j(j+1)}{k+2}$ "conformal weight".