# Supplementing a minimal model

1) Start from an end-point configuration $\mathcal{C}_{end}$
$\longrightarrow$ make fiber over each $\mathbb{P}^1$ more singular

- ADE cases:

$$\mathcal{C}_{A\text{-type}} = \underbrace{2 \cdots 2}_{N} \qquad \mathcal{C}_{E_6} = \begin{array}{ccccc} & & 2 & & \\ 2 & 2 & 2 & 2 & 2 \end{array}$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow su(2m)$$

$$\mathcal{C}_{tuned} = \underbrace{\overset{su(m)}{2} \cdots \overset{su(m)}{2}}_{N} \qquad \mathcal{C}_{tuned} = \begin{array}{ccccc} & & \overset{su(2m)}{2} & & \\ \underset{su(m)}{2} & \underset{su(2m)}{2} & \underset{su(3m)}{2} & \underset{su(2m)}{2} & \underset{su(m)}{2} \end{array}$$

2) Do more blow-ups

- $\mathcal{C}_{end} = 33$, $\mathcal{C}_{min} = 414 = \overset{so(8)}{4} \; 1 \; \overset{so(8)}{4}$

$$\downarrow \text{blow-up}$$

$$4151 \quad \text{not consistent } (so(8) \oplus f_4 \not\subset e_8)$$

$$\downarrow \text{blow-up}$$

$$\overset{f_4}{5} \; 1 \; \overset{su(3)}{3} \; 1 \; \overset{e_6}{6} \; 1 = 513161$$

$$\downarrow \text{tuning}$$

$$\overset{e_6}{5} \; 1 \; \overset{su(3)}{3} \; 1 \; \overset{e_6}{6} \; 1 \quad \checkmark$$

consider next

$\mathcal{T}_{E_7} =$

| | | | | | |
|---|---|---|---|---|---|
| | | 2 | | | |
| 2 | 2 | 2 | 2 | 2 | 2 |

$\downarrow$ blow-up

$\vdots$

$\downarrow$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | | | 3 | | | | |
| | | | 1 | | | | |
| 2 | 3 | 1 | 5 | 1 | 3 | 2 | 2 |

with gauge algebra :

| | | | | | | |
|---|---|---|---|---|---|---|
| | | | $su(3)$ | | | |
| | | | $\oplus$ | | | |
| $su(2)$ | $g_2$ | $\oplus$ | $f_4$ | $\oplus$ | $g_2$ | $sp(1)$ |

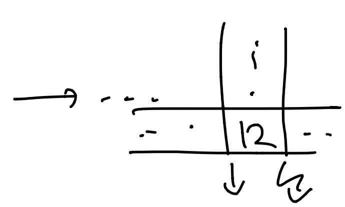
can be viewed as gauging $e_8^{\oplus 3}$ flavor sym.
for 3 E-string theories $\leftarrow$ (-1) - curves

- rigid theories

$\mathcal{T}_{end} = (12)$ single -12 curve

$\mathcal{T}_{E_8} =$

| | | | | | | |
|---|---|---|---|---|---|---|
| | | 2 | | | | |
| 2 | 2 | 2 | 2 | 2 | 2 | 2 |



## Duality Moves

We seek "dual" descriptions of the same SCFT fixed point.

1) Example:

$$\overset{\mathfrak{su}(3)}{2\ \ 1\ \ 3} \xrightarrow{RG} \overset{\mathfrak{su}(3)}{1\ \ 2} \xleftarrow{RG} 1\ \ \overset{\mathfrak{su}(3)}{3}\ \ 1$$

but different from $\mathcal{C}_{generic} = 1\ \ 2$

2) Conformal matter

Consider the following end-point configurations

a) $3,3 \xrightarrow{up} 4,1,4$ , $\mathfrak{so}(8) \oplus \mathfrak{so}(8) \subset \mathfrak{e}_8$ ✓

b) $4,4 \xrightarrow{up} 5,1,5$ $\quad \mathfrak{f}_4 \oplus \mathfrak{f}_4 \not\subset \mathfrak{e}_8$

$$5,1,5 \xrightarrow{up} 6,1,2,5 \xrightarrow{up} \underset{\mathfrak{e}_6}{6},1,\underset{\mathfrak{su}(3)}{3},1,\underset{\mathfrak{e}_6}{6}$$ ✓

c) $5,5 \xrightarrow{up} 6,1,6 \xrightarrow{up} 7,1,3,1,7$

$\xrightarrow{up}$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | $\frac{1}{2}(2,8)$ | | $\frac{1}{2}(8,2)$ | | | |
| 8 | 1 | 2 | | 3 | | 2 | 1 | 8 |
| $e_7$ | | $su(2)$ | | $so(7)$ | | $su(2)$ | | $e_7$ |

d) $7,7 \xrightarrow{up} 8,1,8 \xrightarrow{up} 9,1,3,1,9$

$\xrightarrow{up}$

$(10),1,2,3,2,1,(10) \xrightarrow{up} 11,1,2,2,3,2,2,1,11$

$\xrightarrow{up}$

$(12),1,2,2,2,3,2,2,1,(12)$

$\downarrow up$

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $e_8$ | | | $sp(1)$ | $g_2$ | | $f_4$ | | $g_2$ | $sp(1)$ | | | $e_8$ |
| 12 | 1 | 2 | 2 | 3 | 1 | 5 | 1 | 3 | 2 | 2 | 1 | 12 |
| | | | $\frac{1}{2}(2,7+1)$ | | | | | $\frac{1}{2}(7+1,2)$ | | | | |

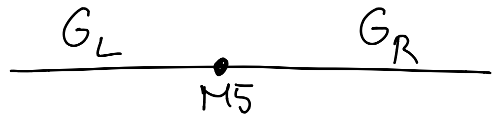
"conformal matter SCFT's"

M5-branes probing ADE singularities:

consider M5-branes probing $\mathbb{C}^2/\Gamma_{ADE}$ sing.

$\rightarrow$ M-th. on $\mathbb{R}^{6,1} \times \mathbb{C}^2/\Gamma_G$

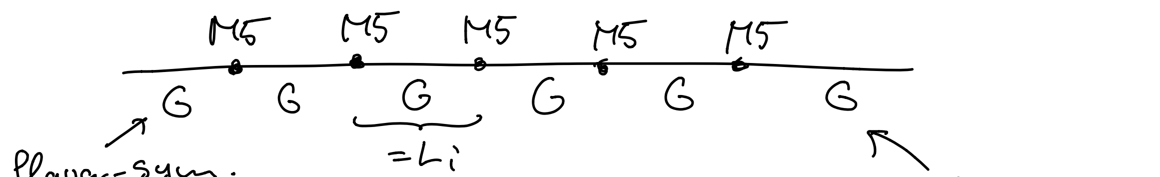
$$\Big\downarrow \; M5 \subset \mathbb{R}^{6,1}$$




$\mathbb{R} \subset \mathbb{R} \times \mathbb{C}^2/\Gamma_G$

"domain-wall" solution in M-th.
with (1,0) SUSY (M5-brane is $\frac{1}{2}$BPS)

Similarly, we can introduce multiple domain-walls:




$\longrightarrow \; G_{quiver} = G_1 \times \cdots \times G_{N-1}$

$$\frac{1}{g_i^2} \sim L_i$$

$\longrightarrow$ SCFT fixed point by taking M5's on top of each other (strong coupling point)

F-theory realization: $[G_L] \; \overset{\mathfrak{g}}{2} \; \overset{\mathfrak{g}}{2} \; \cdots \; \overset{\mathfrak{g}}{2} \; \overset{\mathfrak{g}}{2} \; [G_R]$

each $-2$ curve is wrapped by 7-brane with gauge sym. $\mathfrak{g}$.

$\rightarrow$ have to blow-up intersection points

example: $[E_8], \overset{\mathfrak{e}_8}{2}, [E_8]$

non-compact curve — non-compact curve

$\downarrow$ blow-up

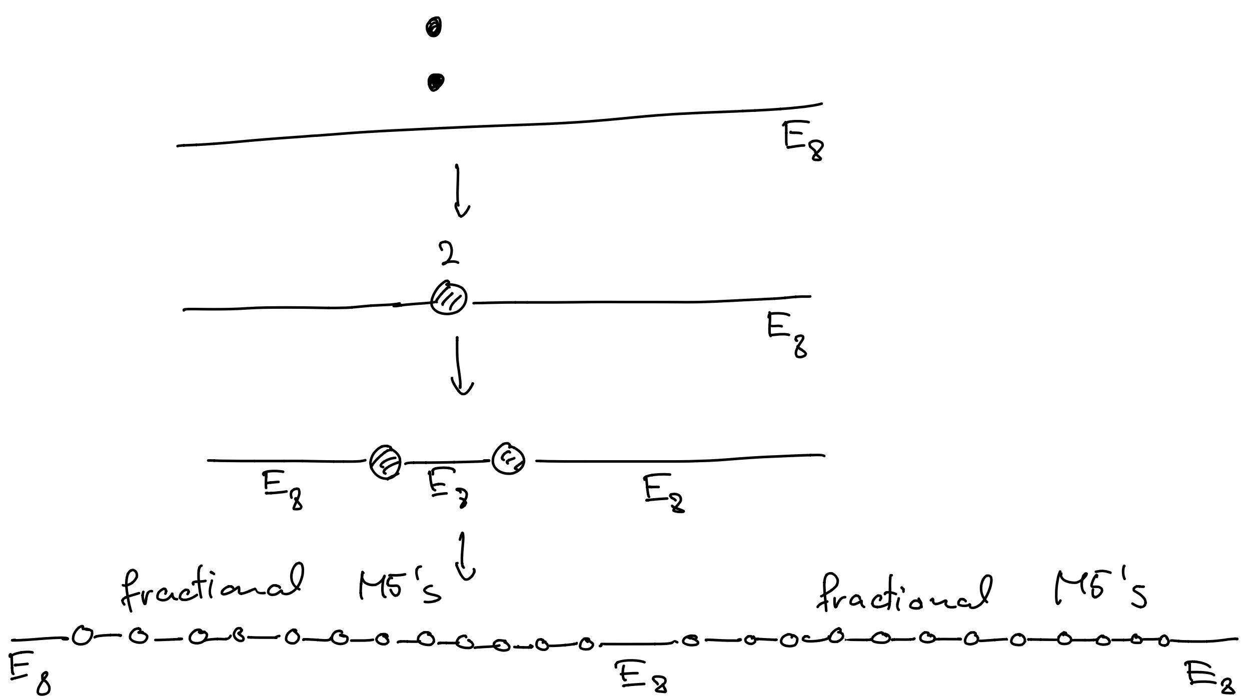
$$\overset{\mathfrak{e}_8}{12} \;\; 1 \;\; 2 \;\; \overset{\mathfrak{sp}(1)}{2} \;\; \overset{\mathfrak{g}_2}{3} \;\; 1 \;\; \overset{\mathfrak{f}_4}{5} \;\; 1 \;\; \overset{\mathfrak{g}_2}{3} \;\; \overset{\mathfrak{sp}(1)}{2} \;\; 2 \;\; 1 \;\; \overset{\mathfrak{e}_8}{12}$$

$\frac{1}{2}(2, 7+1)$ $\qquad\qquad$ $\frac{1}{2}(7+1, 2)$

"$\mathfrak{e}_8$ conformal matter"

similarly, for $D_4$, $E_6$, $E_7$

"M5-branes split"



## → 5d dualities:

consider N M5-branes probing $A_K$-sing.

$\rightarrow \quad G^{6d}_{quiver} = SU(K+1)^N \quad$ in 6d

$\downarrow$ compactify on $S^1$

N D4-branes probing $A_K$-sing

$\rightarrow \quad G^{5d}_{quiver} = SU(N)^{K+1}$

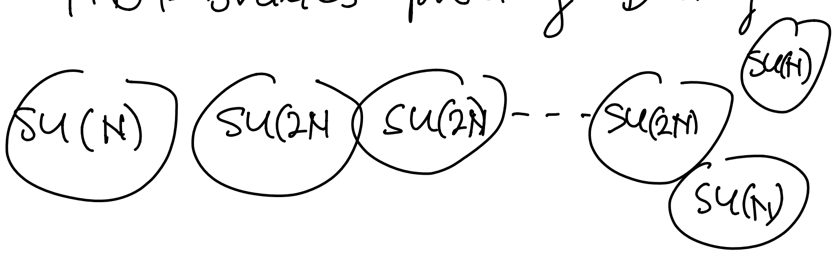
"Douglas- Moore" construction,

D-sing.:

6d: $\quad SO \times Sp \times SO \cdots \quad$ chain

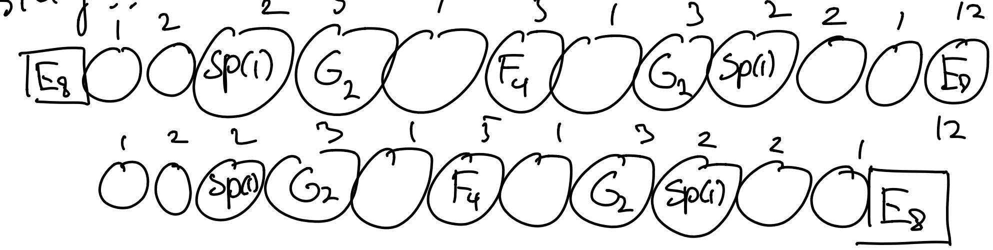
$\downarrow S^1$

5d: $\quad$ N D4-branes probing D-sing.

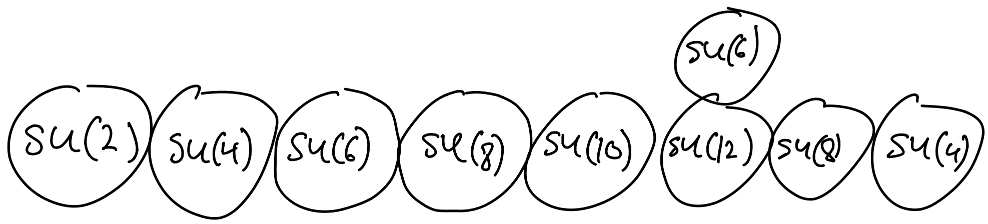



E-sing.:

6d:



5d:




Counting parameters:

$$\dim_{6d\ \text{tensor}}(G,N) = n_{\text{interval}} + n_{\text{cmatter}}$$

$$n_{\text{interval}} = (N-1)\, r_G + (N-1)$$

$A_k$: $n_{\text{cmatter}} = 0$

$D_p$: $n_{\text{cmatter}} = (p-3)N$

$E_6$: $n_{\text{cmatter}} = (2+3)N$

$E_7$: $n_{\text{cmatter}} = (5+5)N$

$E_8$: $n_{\text{cmatter}} = (10+11)N$

$$\dim_{5d\ \text{Coul.}}(G) = \sum_i \Big(N \underset{\substack{\uparrow \\ \text{affine} \\ \text{Dynkin number}}}{d_i^{(\hat{G})}} - 1\Big) = N \underset{\substack{\uparrow \\ \text{dual} \\ \text{Coxeter}}}{h_G} - \underset{\substack{\uparrow \\ \text{rank}}}{r_{\hat{G}}}$$