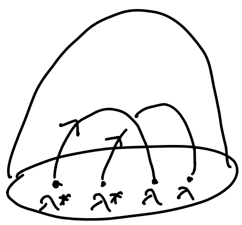
Recall: $q^{1/4} J_{L_+} - q^{-1/4} J_{L_-} = (q^{1/2} - q^{-1/2}) J_{L_0}$ (*)

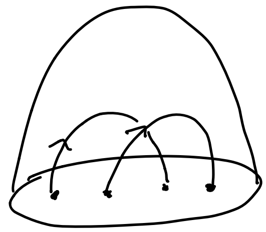
Interpretation:

Consider the space of conformal blocks $\mathcal{H}$ on $\mathbb{CP}^1$ with four points and highest weights $\lambda_1, \lambda_2, \lambda_3, \lambda_4$.

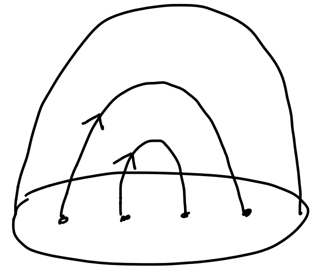
In our case: $\lambda, \lambda, \lambda^*, \lambda^*$



L_+ $\to v_+ \in \mathcal{H}$



L_- $\to v_- \in \mathcal{H}$



L_0 $\to v_0 \in \mathcal{H}$

There exists a vector $\omega \in \mathcal{H}^*$ such that

$J_{L_+} = \langle \omega, v_+ \rangle$, $J_{L_-} = \langle \omega, v_- \rangle$, $J_{L_0} = \langle \omega, v_0 \rangle$

where $\langle\ ,\ \rangle : \mathcal{H}^* \times \mathcal{H} \to \mathbb{C}$ is natural pairing

$\dim \mathcal{H} \leq 2 \Rightarrow \alpha v_+ + \beta v_- + \gamma v_0 = 0$

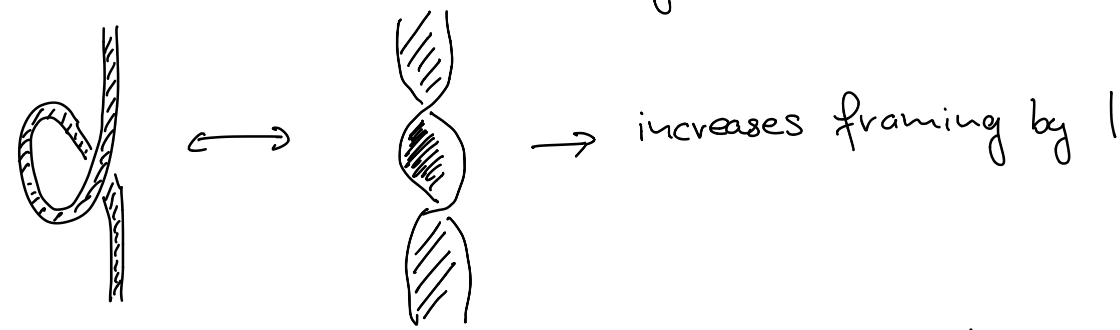
$$\to \alpha J_+ + \beta J_{L_-} + \gamma J_{L_0} = 0$$

α, β and γ are determined by (*).

Note: Our invariant J_L depends on the framing. To cure this, denote by $\omega(L)$ the "writhe" of L (# positive crossings − # negative crossings), and set

$$P_L = d(1)^{-1} \exp(-2\pi\sqrt{-1}\,\Delta_1 \omega(L)) J_L$$

we use "blackboard" framing to compute $w(L)$:



Proposition 1 $\longrightarrow$ P_L stays invariant

($\mathcal{J}_L \to e^{2\pi\sqrt{-1}\Delta_\lambda}\mathcal{J}_L$ by increasing framing)

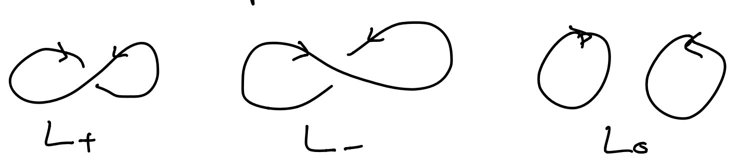
Comparing the writhes $w(L_+), w(L_-), w(L_0)$, we obtain the skein relation

$$qP_{L_+} - q^{-1}P_{L_-} = (q^{1/2} - q^{-1/2})P_{L_0} \qquad (**)$$

also we have : $P_O = 1$ (where O is trivial knot)

P_L is a version of the "Jones polynomial".

Let us compute $d(\lambda)$ in the case $\lambda = 1$.



For the links L_+, L_- and L_0 we have

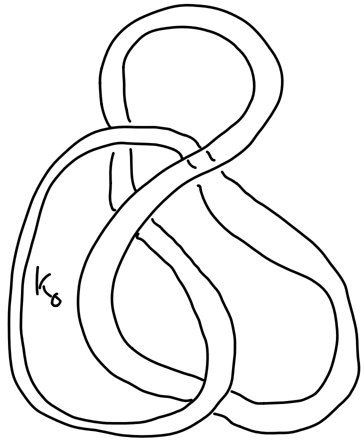
$$P_{L_+} = 1, \quad P_{L_-} = 1 \quad \text{and} \quad P_{L_0} = d(1)$$

Applying skein relation $(**)$ we get:

$$d(1) = \frac{q - q^{-1}}{q^{1/2} - q^{-1/2}}$$

We can compute $\mathcal{J}(L; \lambda_1, \dots, \lambda_m)$ from $\mathcal{J}_L$.

In order to show this, we need the concept of "cabling": Let K_0 be an oriented framed knot with a framing represented by t. Take K_1 to be the companion knot on tubular boundary of K_0 giving rise to framing t. $\longrightarrow$ two-component link $K_0 \cup K_1$ ("cabling")



We first compute $d(\lambda)$ for $\lambda > 1$:

<u>Lemma 2:</u>

$$d(\lambda) = \frac{q^{(\lambda+1)/2} - q^{-(\lambda+1)/2}}{q^{1/2} - q^{-1/2}}$$

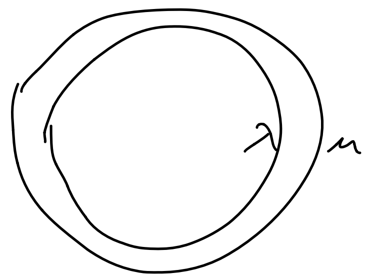
where $q^{1/2} = \exp\left(\frac{2\pi\sqrt{-1}}{2(K+2)}\right)$

<u>Proof:</u>

By construction $Z(K;\lambda) = 1$ for a trivial knot K with 0-framing. $\Rightarrow J(K;\lambda) = d(\lambda)$

Consider now the cabling for the trivial knot with

0 framing



$$\longrightarrow d(\lambda)\, d(\mu) = \sum_{\nu} N_{\lambda\mu}^{\nu}\, d(\nu)$$

observe that $\dfrac{q^{(\lambda+1)/2} - q^{-(\lambda+1)/2}}{q^{1/2} - q^{-1/2}} = \dfrac{S_{0\lambda}}{S_{00}}$

where $S_{\lambda\mu}$ is Verlinde's S-matrix.

enough to show $d(\lambda) = \dfrac{S_{0\lambda}}{S_{00}}$

But this follows from Verlinde formula (Prop. 6, §6):

$$\begin{aligned} N_{\lambda\mu\nu} &= \dim H(p_1, p_2, p_3; \lambda, \mu, \nu) \\ &= \sum_{\alpha} \frac{S_{\lambda\alpha} S_{\mu\alpha} S_{\nu\alpha}}{S_{0\alpha}} \end{aligned}$$

$\square$

Let K_0 be an oriented framed knot and let $K_0 \cup K_1$ be a link obtained by cabling of K_0. We have

<u>Lemma 3:</u>

The invariant $J(K_0, K_1; \lambda, \mu)$ of the link $K_0 \cup K_1$ obtained as a cabling of K_0 satisfies

$$J(K_0, K_1; \lambda, \mu) = \sum_{\nu} N_{\lambda\mu}^{\nu} J(K; \nu)$$

where $N_{\lambda\mu}^{\nu}$ are the structure constants of the fusion algebra R_K.

Define generalized notion of J-polynomial by considering invariant $J(L; x_1, \dots, x_m)$ with $x_1, \dots, x_m \in R_K$. For $x_j = v_{\lambda_j}$ for $j = 1, \dots, m$,

$J(L; x_1, \dots, x_m) = J(L; \lambda_1, \dots, \lambda_m)$.

Then for $x_j = v_\lambda \cdot v_\mu$ take

$$J(L; \dots, v_\lambda \cdot v_\mu, \dots) = \sum_\nu N_{\lambda\mu}^{\nu} J(K; \dots v_\nu, \dots).$$

$\rightarrow$ obtain multi-linear map

$$J(L) : R_K^{\otimes m} \rightarrow \mathbb{C}$$

<u>Proposition 3:</u>

For links L_1 and L_2 contained in disjoint 3-balls B_1 and B_2 respectively, we have

$$J(L_1 \cup L_2; \mu_1 \cup \mu_2) = J(L_1; \mu_1) J(L_2; \mu_2)$$

<u>Proof:</u>

In the construction of $Z(L_1 \cup L_2; \mu_1 \cup \mu_2)$ put B_1 and B_2 in such a way that

$$Z(L_1 \cup L_2; \mu_1 \cup \mu_2) = Z(L_1; \mu_1) \circ Z(L_2; \mu_2)$$
$$= Z(L_1; \mu), \ Z(L_2; \mu_2)$$

$\rightarrow$ correct by factors of $d(\mu_i)$ to obtain result $\square$

<u>Definition:</u>

We denote by $\bar{L}$ the mirror imag of L.
("look from the other side of the blackboard")

Proposition 4:

Let L be an oriented framed link. For the mirror image $\bar{L}$ we have

$$J(\bar{L};\lambda) = \overline{J(L;\lambda)}$$

where the right hand side stands for the complex conjugate of $J(L;\lambda)$.

Proof:

The monodromy matrix $\rho(\sigma^{-1})$ is obtained from $\rho(\sigma)$ by replacing q with q^{-1}. The entries of connection matrix F and $d(\lambda)$ are real

$\rightarrow J_{\bar{L}}(q) = J_L(q^{-1})$.

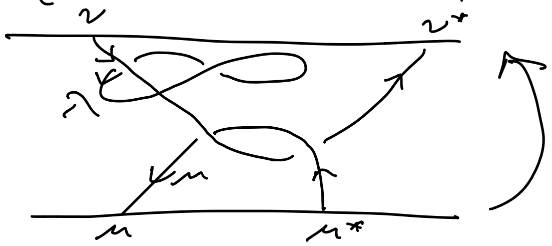
Since q is root of unity $\rightarrow J(\bar{L},\lambda) = \overline{J(L;\lambda)}$ $\square$

Oriented framed tangles:

set $X = \mathbb{C} \times [0,1]$

Let $p_1, \dots, p_m$ be m distinct points on the real line of $X_0 = \mathbb{C} \times \{0\}$ and let $q_1, \dots, q_n$ be n distinct points on real line of $X_1 = \mathbb{C} \times \{1\}$

A compact 1-manifold T in X with boundary $\{p_1, \dots, p_m, q_1, \dots, q_n\}$ is called an (m,n)-"tangle"



$Z(T;\lambda) : V_{\mu_1 \dots \mu_m} \rightarrow V_{\nu_1 \dots \nu_n}$

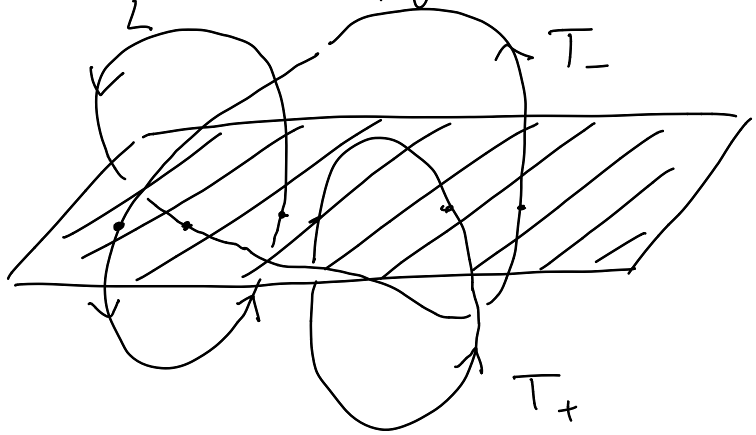
similarly, we get a linear map

$$\mathcal{J}(T;\lambda) : V_{\mu_1 \cdots \mu_m} \longrightarrow V_{\nu_1 \cdots \nu_n}$$

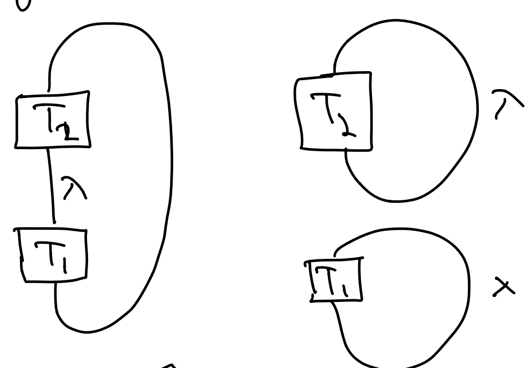
"tangle operator"

<u>Proposition 5:</u>

Let T be an oriented framed $(1,1)$ tangle. We denote by $\hat{T}$ a link obtained by closing T as in the figure below



For the composition of tangles $T_1 \circ T_2$ with a color λ given as shown



we have $\mathcal{J}(\widehat{T_1 \circ T_2}) = \mathcal{J}(\hat{T}_1)\, \mathcal{J}(\hat{T}_2) \dfrac{S_{00}}{S_{0\lambda}}$

For the link L associated to the above tangle we have

$\mathcal{J}(L;\lambda) = \langle \mathcal{J}(L_+), \mathcal{J}(L_-)\rangle$, where $\mathcal{J}(L_+) \in V(f)$ and $\mathcal{J}(L_-) \in V(f)^*$

space of conformal blocks

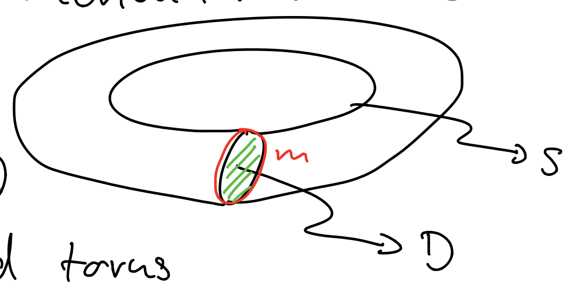
§ 9. Witten's invariants for 3-manifolds

Let L be a framed link in S^3.

Dehn surgery:

Take for simplicity L to be the unknot. Consider tubular neighborhood $N(L)$ of L. $N(L)$ is homeomorphic to $D \times S^1$. Take closed curve γ on $\partial N(L)$ giving the framing of L. Let m be the meridian on the boundary of $H \cong D \times S^1$

Put $E(L) = \overline{S^3 \setminus N(L)}$



$E(L)$ is a solid torus itself! We glue back H into $E(L)$ by identifying m with γ