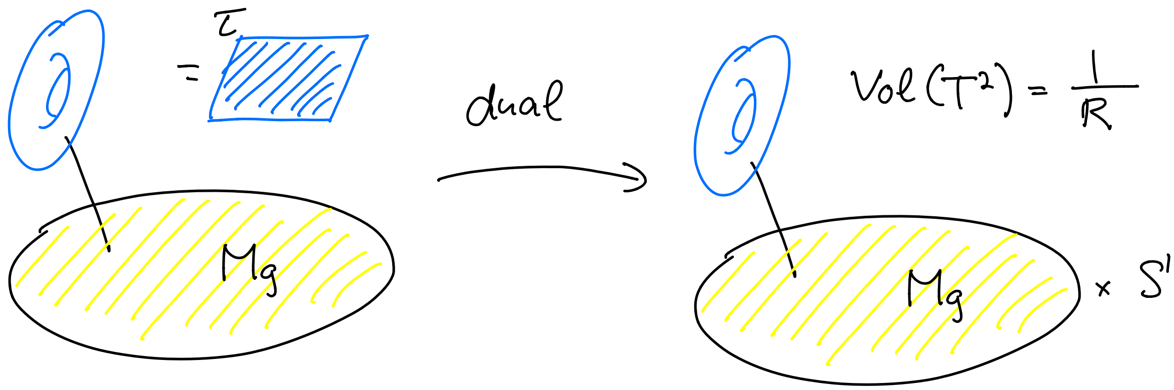
Last time we saw

<u>M-th</u> on $\qquad\qquad\qquad\qquad$ <u>F-th</u> on



Take M_g to be $\mathbb{R}^5 \times B_4$

$\longrightarrow$ to preserve SUSY, need
$$\begin{tikzcd} T^2 \arrow[r, hook] & X \arrow[d] \\ & B_4 \end{tikzcd}$$
to be CY_3

B_4 ← Kähler manifold

<u>Example:</u> K3

$B_4 = \mathbb{P}^1 \times \mathbb{R}^2$, i.e. F-theory on $\mathbb{R}^{1,7} \times K3$

K3 eq. : $\quad y^2 = x^3 + f(u,v)\,x z^4 + g(u,v)\, z^6 \qquad (*)$

where $\quad x, y, z, u, v \in \mathbb{C}/\sim$

$$\begin{aligned}(u, v, x, y, z) &\sim (\lambda u, \lambda v, \lambda^4 x, \lambda^6 y, z) \\ &\sim (u, v, \mu^2 x, \mu^3 y, \mu z)\end{aligned}$$

where $\mu, \lambda \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, and $(u,v) \neq (0,0)$,

$(x,y,z) \neq (0,0,0)$

$$\left.\begin{aligned} f(\lambda u, \lambda v) &= \lambda^8 f(u,v) \\ g(\lambda u, \lambda v) &= \lambda^{12} g(u,v) \end{aligned}\right\} \text{ consistent with } (*)$$

<u>rule:</u> deg of $(*)$ = sum of weights

$$\begin{aligned} \lambda: \quad & 12 &&= 1+1+4+6+0 \\ \mu: \quad & 6 &&= 0+0+2+3+1 \end{aligned}$$

$\longrightarrow$ total space is Calabi-Yau

projection $\pi: K3 \longrightarrow \mathbb{P}^1 : (x,y,z,u,v) \mapsto (u,v)$

$$\mathbb{P}^1 = \{(u,v) \neq (0,0) \mid (u,v) \simeq (\lambda u, \lambda v)\}$$

at fixed $z=1$, $v=1$, we get

$$(*) \longrightarrow y^2 = x^3 + f(u)x + g(u) \qquad (**)$$

$\longrightarrow$ equation of elliptic curve in (x,y,z) space

deg $= 6$ $\quad = \quad$ sum of weights $= 2+3+1 = 6$

<u>Relation to $T^2 = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$</u>

holomorphic coordinate: $z = x + \tau y$

for $P \in T^2$: $\quad z(P) = \int_0^P \Omega_1$, $\quad \Omega_1 = dz$

For T^2 described by $(**)$, $\Omega_1 = \frac{c\,dx}{y}$, $c = $ const.

then $\tau = \dfrac{\oint_B \Omega_1}{\oint_A \Omega_1}$

$\longrightarrow$ ambiguity in basis choice is $SL(2,\mathbb{Z})$

<u>How to compute τ ?</u>

Use j-invariant !

$$j(\tau) = \frac{4 \cdot (24f)^3}{\Delta}, \qquad \Delta = 27g^2 + 4f^3$$

and $j(\tau)$ is $SL(2,\mathbb{Z})$ modular invariant

$$j(\tau) = e^{-2\pi i \tau} + 744 + \mathcal{O}(e^{2\pi i \tau})$$

$\Delta = 0$ is "discriminant locus"

$\rightarrow \deg(\Delta) = 24 \rightarrow 24$ zero's on $\mathbb{P}^1$

$\rightarrow$ denote by u_i, $i = 1, \dots, 24$

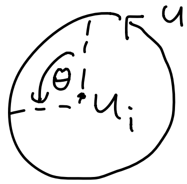
We have: $j(\tau(u)) \sim \dfrac{1}{u - u_i}$ near u_i

$$\longrightarrow \tau(u) \simeq \frac{1}{2\pi i} \ln(u - u_i)$$

for $u \longrightarrow u_i$: $\tau \longrightarrow i\infty$ (ratio of A- and B-cycle of T^2 vanishes)

Since $\tau = C_0 + \dfrac{i}{g_{IIB}}$ this is "weak coupling" limit : $g_{IIB} \to 0$

monodromy:



$u(\theta) = u_i + \varepsilon e^{2\pi i \theta}$

$\to T: \tau \longmapsto \tau + 1$

equivalently, $C_0 \longmapsto C_0 + 1$, or $\oint_{u_i} F_1 = \oint_{u_i} dC_0 = 1$

$\longrightarrow$ D7-brane $\quad u = u_i$

altogether, we have 24 D7-branes

since $\mathbb{P}^1$ is compact, we expect $\sum_{i=1}^{24} \oint_{u_i} F_1 = 0$

How can this be ?

solution: monodromies at different u_i are related through $SL(2,\mathbb{Z})$ transformations: $T_i = M T M^{-1}$

$M \in SL(2,\mathbb{Z})$

$\longrightarrow$ (p,q) 7-brane related to $(1,0)$ 7-brane by "S-duality" transformation M.

vanishing of $pA + qB$ 1-cycle of T^2

$\begin{pmatrix} p \\ q \end{pmatrix} = M \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

The weak coupling limit:

Want to find region with $\tau \sim$ const

$$\Rightarrow \quad \frac{f^3}{g^2} = \text{const.}$$

$$\text{solved by} \quad g = p^3, \quad f = \overset{\text{const.}}{\alpha} p^2$$

with p a homogeneous polynomial of degree 4

$\rightarrow$ go to coordinate patch $v=1$:

$$p = \prod_{i=1}^{4} (u - u_i)$$

$$\rightarrow \Delta = (4\alpha^3 + 27) \prod_{i=1}^{4} (u-u_i)^6, \quad j(\tau) = \frac{4(24\alpha)^3}{27+4\alpha^3}$$

For $\alpha \sim -3/4^{1/3} \rightarrow j(\tau) \sim \infty$

weak coupling everywhere on base $\mathbb{P}^1$

Only non-trivial $SL(2,\mathbb{Z})$ element with

$$M\tau = \frac{a\tau + b}{c\tau + d} = \tau \quad \text{is} \quad M = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

define $x = p\tilde{x}$, $y = p^{3/2}\tilde{y}$

$$\rightarrow \tilde{y}^2 = \tilde{x}^3 + \alpha \tilde{x} + 1$$

$$\rightarrow \Omega_1 = p^{-1/2} \frac{d\tilde{x}}{\tilde{y}}$$

gives $\Omega_1 \longrightarrow -\Omega_1$ and $\oint_A \Omega_1 = -\oint_A \Omega_1$

(1) $\oint_B \Omega_1 = -\oint_B \Omega_1$

and thus $z \mapsto -z$

leads to double-cover construction of $\mathbb{P}^1$:

$$X: \xi^2 = p(u,v), \quad (u,v,\xi) \simeq (\lambda u, \lambda v, \lambda^2 \xi)$$

with $(u,v,\xi) \neq (0,0,0)$

$\rightarrow$ equation of 2nd elliptic curve

original $\mathbb{P}^1$ is recovered from X as quotient

X/σ_1 where $\sigma: \xi \longrightarrow -\xi$ (2)

when circling around u_i

$\mathbb{Z}_2$ transformations (1) and (2) give together

$$K3 = (T^2 \times T^2)/\mathbb{Z}_2$$

Fixed loci of $\mathbb{Z}_2$-involution σ, i.e. the u_i, are positions of O7-planes with D7-charge $-4 \longrightarrow$ 4 D7-branes on top of O7-branes