Last time:

Saw that linear end-point configurations are possible:

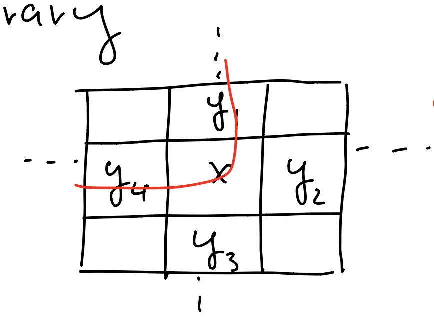
$$\mathcal{C}_{end} = x_1 \, x_2 \cdots x_r$$

blow-up leads to valid F-th. configuration

blow-down leads to $\mathbb{C}^2/\Gamma$-sing of type $A(x_1 \cdots x_r)$

No quartic vertices

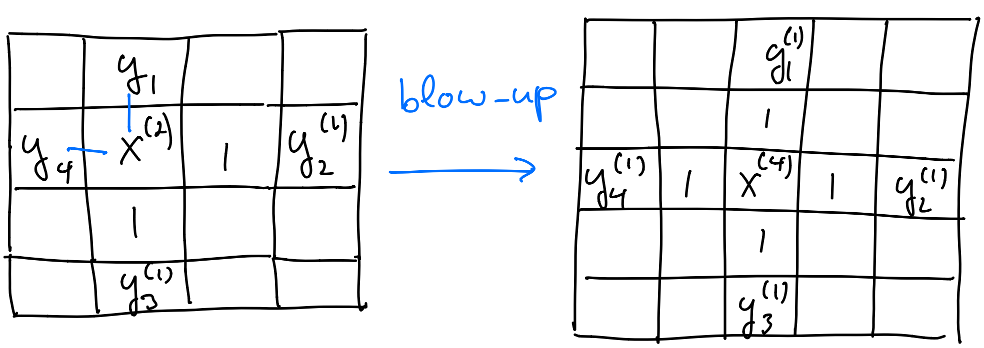
Assume the contrary

$\mathcal{C}_{end} =$ 

one NHC

with at least one $-n$-curve with $n \geq 3$

$\longrightarrow$ need at least 2 blow-ups



$\rightarrow \exists\, i$: gauge algebra at $(x^{(4)}, y_i^{(1)})$ is at least $e_6 \oplus so(8)$ or $e_7 \oplus su(3)$

$\rightarrow$ blow-up 4 more times

			$y_1^{(1)}$			
			2			
			1			
$y_4^{(1)}$	2	1	$x^{(8)}$	1	2	$y_2^{(1)}$
			1			
			2			
			$y_3^{(1)}$			

now gauge algebra at $x^{(8)}$ is e_8 and at $(2, y_i^{(1)})$ at least

$su(2) \oplus g_2$

$\rightarrow$ need to blow-up 4 more times

$\rightarrow$ $x^{(12)} > 12$ not possible $\lightning$

$\rightarrow$ no quartic vertices !

<u>Restriction to trivalent vertices</u>

$\mathcal{C}_{sub} =$

	y_1	
y_3	x	y_2

claim $\mathcal{C}_{sub} =$

	2	
2	x	y

suppose to the contrary:

$\mathcal{C}_{sub}$ =

	≥ 3	
≥ 3	x	y

blow-up

		$y_1^{(1)}$		
		1		
$y_3^{(1)}$	1	$x^{(3)}$	1	$y_2^{(1)}$

$\longrightarrow$

				$y_1^{(2)}$			
				1			
				3			
				1			
$y_3^{(2)}$	1	3	1	$x^{(6)}$	1	2	$y_2^{(1)}$

$x^{(6)} \geq 8 \longrightarrow$ gauge algebra is at least e_7

but $e_7 \oplus su(3) \not\subset e_8$

$\longrightarrow$ blow-up more

$\mathcal{C}_{sub} \longrightarrow$

						$y_1^{(2)}$				
						1				
						3				
						2				
						2				
						1				
$y_3^{(2)}$	1	3	2	2	1	$x^{(11)}$	1	2	2	$y_2^{(1)}$

but $x^{(11)} > 12$ ↯

only possible trivalent vertices:

$$\tau_{sub} = \begin{array}{|c|c|c|} \hline & 2 & \\ \hline 2 & 3 & 2 \\ \hline \end{array} \quad \text{or} \quad \begin{array}{|c|c|c|} \hline & 2 & \\ \hline 2 & 2 & y \\ \hline \end{array} \; \cdots$$

can be proved using similar argument

<u>Endpoint classification</u>

Suppose we have a subgraph of the form

$$\tau_{sub} = x_1\, x_2\, x_3\, x_4\, x_5\, x_6\, x_7\, x_8\, x_9\, x_{10}\, x_{11}$$

Then $x_6 = 2$

suppose to the contrary

$$\tau_{sub} \longrightarrow x_1 x_2 x_3^{(1)} \mid x_4^{(2)} \mid x_5^{(2)} \mid x_6^{(2)} \mid x_7^{(2)} \mid x_8^{(2)} \mid x_9^{(1)} x_{10} x_{11}$$

$$\longrightarrow \vdots$$

$$\longrightarrow x_1 x_2 x_3^{(1)} \mid x_4^{(3)} \mid 3 \mid x_5^{(5)} \mid 3 \mid 5 \mid 3\, 2\, 2 \mid x_6^{(10)} \mid 2\, 2\, 3 \mid 5 \mid 3 \mid x_7^{(5)} \mid 3 \mid x_8^{(3)} \mid x_9^{(1)} x_{10} x_{11}$$

but $x_6^{(10)} > 12$ for $x_6 > 2$ ↯

→ starting at eleven or more curves:

$$\tau_{sub} = x_1\, x_2\, x_3\, x_4\, x_5\, \underbrace{2 \cdots 2}_{N}\, y_5\, y_4\, y_3\, y_2\, y_1$$

Generalized D-type theories

at 4 curves:

$$\mathcal{C}_{rigid} = \begin{array}{|c|c|c|}\hline & 2 & \\ \hline 2 & 3 & 2 \\ \hline\end{array} \;, \quad \begin{array}{|c|c|c|}\hline & 2 & \\ \hline 2 & 2 & 4 \\ \hline\end{array}$$

with minimal resolutions:

$$\begin{array}{|c|c|c|}\hline & 2 & \\ \hline 2 & 3 & 2 \\ \hline\end{array} \longrightarrow \begin{array}{|c|c|c|c|c|}\hline & & 3 & & \\ \hline & & 1 & & \\ \hline 3 & 1 & 6 & 1 & 3 \\ \hline\end{array}$$

$$\begin{array}{|c|c|c|}\hline & 2 & \\ \hline 2 & 2 & 4 \\ \hline\end{array} \longrightarrow \begin{array}{|c|c|c|c|}\hline & 2 & & \\ \hline 2 & 3 & 1 & 5 \\ \hline\end{array}$$

NHC: $\mathfrak{f}_4 \oplus \mathfrak{so}(7) \not\subset \mathfrak{e}_8$

$$\big\downarrow$$

$$\begin{array}{|c|c|c|c|c|c|c|}\hline & & 3 & & & & \\ \hline & & 1 & & & & \\ \hline 3 & 1 & 6 & 1 & 3 & 1 & 6 \\ \hline\end{array}$$

At 5 curves and above:

$$\mathcal{C}_{end} = \begin{array}{|c|c|c|}\hline & 2 & \\ \hline 2 & 2 & x_1 \cdots x_r \\ \hline\end{array}$$

For rigid D-type theories, one finds

$$\mathcal{C}^{(D)}_{\text{rigid}} = D_N \gamma \quad \text{for } N \geq 2$$

with $\quad \gamma \in \{32, 24\}$

Orbifold examples: A-type theories

$A(x_1, \ldots, x_r)$:

$$(z_1, z_2) \mapsto (\omega z_1, \omega^q z_2) \quad \text{where } \omega = e^{2\pi i/p}$$

and $\displaystyle \frac{p}{q} = x_1 - \frac{1}{x_2 - \cdots - \frac{1}{x_r}}$

Consider now the cases $x A_N y$ for $2 \leq x, y \leq 7$

$$\rightarrow \quad \frac{1}{p} = \frac{1}{N(x-1)(y-1) + xy - 1}, \quad \frac{q}{p} = \frac{N(y-1) + y}{N(x-1)(y-1) + xy - 1} \quad (*)$$

For $x = y = 7$ there is F-th orbifold realization:

$$(\mathbb{C}^2 \times T^2)/\Gamma, \quad \Gamma: (z_1, z_2, \lambda) \mapsto (\eta \zeta^{-1} z_1, \eta \zeta z_2, \eta^{-2} \lambda)$$

with

$$\eta = \exp\left(2\pi i \cdot \frac{1}{12}\right), \quad \zeta = \exp\left(2\pi i \cdot \frac{k}{12k+1}\right)$$

$$\rightarrow \quad \frac{1}{p} = \frac{1}{12(12k+1)}, \quad \frac{q}{p} = \frac{24k+1}{12(12k+1)}$$

matches with $(*)$ for $N = 4k-1$, $x = y = 7$.

Orbifold examples: D-type theories

$$\mathcal{C}_{end} = D_N\, 24 \longrightarrow \frac{p}{q} = \frac{18N-12}{6N-5}$$

$$D_N\, 32 \longrightarrow \frac{p}{q} = \frac{18N-24}{12N-17}$$

supplemented with $\Lambda: (z_1, z_2) \mapsto (-z_2, z_1)$