Last time:

Consider blowing-down a -1 curve C_A in $\mathcal{B}$ which intersects with C_{A-1} and C_{A+1}

$\longrightarrow \hat{\mathcal{B}}$ with $p : \mathcal{B} \to \hat{\mathcal{B}}$ the blow-down map

homology cycles $\hat{C}$ in $\hat{\mathcal{B}}$ and C in $\mathcal{B}$ are related by

$$p^* [\hat{C}_i] = \begin{cases} [C_i] + [C_A] & i = A-1, A+1 \\ [C_i] & i \neq A-1, A, A-1 \end{cases}$$

$$\Rightarrow \hat{\eta}^{ij} = -\hat{C}^i \cdot \hat{C}^j = \begin{cases} \eta^{ii} - 1 & i = j = A \pm 1 \\ -1 & (i,j) = (A-1, A+1), (A+1, A-1) \\ \eta^{ij} & \text{otherwise} \end{cases}$$

$$d\hat{H}_i = \hat{I}_i,$$

$$\hat{I}_i = \begin{cases} I^i + I^A & i = A+1, A-1 \\ I^i & \text{otherwise} \end{cases}$$

$$(I^i = \eta^{ij} I_j)$$

$$\Rightarrow \hat{I}^{GS} = \frac{1}{2} \hat{\eta}^{ij} \hat{I}_i \hat{I}_j = \frac{1}{2} (\hat{\eta}^{-1})_{ij} \hat{I}^i \hat{I}^j,$$

$$I^{tot} = I^{\text{one-loop}} + I^{GS} = \hat{I}^{\text{one-loop}} + \hat{I}^{GS}$$

required by anomaly matching

We also have

$$I^{GS} = \hat{I}^{GS} + \frac{1}{2}(I^A)^2$$

$$\Rightarrow \quad \hat{I}^{\text{one-loop}} = I^{\text{one-loop}} + \frac{1}{2}(I^A)^2$$

The difference is

$$\hat{I}^{\text{one-loop}} - I^{\text{one-loop}} = I^{\text{rank1}}_{\text{E-string}} - I^{\text{tensor}}$$

$$(1) \quad = \frac{1}{2}\left(c_2(R) - \frac{1}{4}p_1(T) - \frac{1}{4}\mathrm{Tr}\,F_{E_8}^2\right)^2$$

On the other hand:

$$(2) \quad \frac{1}{2}(I^A)^2 = \frac{1}{2}\left(y^A c_2(R) - \frac{1}{4}p_1(T) - \frac{1}{4}\mathrm{Tr}\,F_{A-1}^2 - \frac{1}{4}\mathrm{Tr}\,F_{A+1}^2\right)^2$$

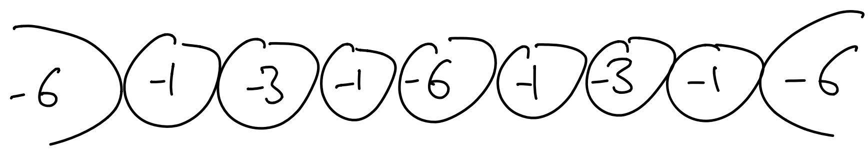
Comparing (1) and (2) we get

- $y_A = 1$
- $\mathrm{Tr}\,F_{E_8}^2 = \mathrm{Tr}\,F_{A-1}^2 + \mathrm{Tr}\,F_{A+1}^2$

$\rightarrow$ gauge groups $G_{A-1} \times G_{A+1}$ are embedded into E_8 of E-string theory

Example:

2 M5 branes probing E_6 singularity



$$\downarrow$$

$$\boxed{E_6} \,\Big|\, \emptyset \,\Big|\, su(3) \,\Big|\, \emptyset \,\Big|\, E_6 \,\Big|\, \emptyset \,\Big|\, su(3) \,\Big|\, \emptyset \,\Big|\, \boxed{E_6}$$

η^{ii}	1	3	1	6	1	3	1
y^i	1	3	1	12	1	3	1

$\downarrow$ shrink all -1 curves

$$\boxed{E_6} \,\|\, su(3) \,\|\, E_6 \,\|\, su(3) \,\|\, \boxed{E_6}$$

η^{ii}	1	4	1
y^i	5	14	5

$\downarrow$ shrink -3 curves

$$\boxed{E_6} \,\|\|\, E_6 \,\|\|\, E_6$$

η^{ii}	2
y^i	24

Computation of anomaly polynomial for single M5 brane probing E_6 sing.:

$$E_6 \,\|\, SU(3) \,\|\, E_6$$

$\longrightarrow$
- 1 $SU(3)$ VM
- 1 $(1,0)$ TM
- bifundamental matter: rank 1 E-string theory via embedding $E_6 \times SU(3) \subset E_8$

$$\Rightarrow \quad I^{\text{one-loop}} = \sum_A I^{\text{vec}}_{F_A} + \sum_{A,B} I^{\text{matter}}_{F_A, F_B} + I^{\text{tensor}}$$

$$= I^{\text{rank 1}}_{\text{E-string}}\left(\operatorname{Tr} F_L^2 + \operatorname{Tr} F^2_{SU(3)}\right) + I^{\text{vec}}_{SU(3)}\left(\operatorname{Tr} F^2_{SU(3)}\right)$$

$$+ I^{\text{tensor}} + I^{\text{rank 1}}_{\text{E-string}}\left(\operatorname{Tr} F^2_{SU(3)} + \operatorname{Tr} F_R^2\right)$$

$$= \frac{1}{32}\left(\operatorname{Tr} F_L^2\right)^2 + \frac{1}{32}\left(\operatorname{Tr} F_R^2\right)^2$$

$$+ \left(\operatorname{Tr} F_L^2 + \operatorname{Tr} F_R^2\right)\left(\frac{1}{16} p_1(T) - \frac{1}{4} c_2(R)\right) + \frac{19}{24} c_2^2(R)$$

$$- \frac{29}{48} c_2(R) p_1(T) + \frac{373}{5760} p_1^2(T) - \frac{79}{1440} p_2(T)$$

$$- \frac{1}{32} \left(\mathrm{Tr}\, F^2_{su(3)}\right)^2$$

$$+ \mathrm{Tr}\, F^2_{su(3)} \left(-\frac{5}{4} c_2(R) - \frac{1}{16} p_1(T) + \frac{1}{16} \mathrm{Tr}\, F_L^2 + \frac{1}{16} \mathrm{Tr}\, F_R^2 \right)$$

Green - Schwarz term:

recall that

- $I^{vec} = -\frac{1}{24} \left(\frac{3}{4} \omega_G (\mathrm{Tr}\, F^2)^2 + 6 h^\vee_G \mathrm{Tr}\, F^2 c_2(R) + d_G c_2(R)^2 \right)$

  and $\omega_{su(3)} = 3$, $h^\vee_{su(3)} = 3$, $d_{su(3)} = 8$

- $I^{GS} = \frac{1}{2} \eta^{ij} I_i I_j$

  with $\eta^{ij} I_j = \frac{1}{4} \left(\eta^{ia} \mathrm{Tr}\, F_a^2 - K^i p_1(T) \right) + y^i c_2(R)$

Now
$$\eta^{ij} = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{pmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix} \end{array} \xrightarrow{\text{blow-down}} \hat{\eta}^{11} = (1)$$

$$\hat{I}^1 = I^1 + \underbrace{I^0 + I^2}$$
$$= I^{E\text{-str}} \left(\mathrm{Tr}\, F^2_{su(3)} + \mathrm{Tr}\, F_L^2\right) + I^{E\text{-str}} \left(\mathrm{Tr}\, F^2_{su(3)} + \mathrm{Tr}\, F_R^2\right)$$
$$= 2 c_2(R) - \frac{1}{2} p_1(T) - \frac{1}{4} \mathrm{Tr}\, F_L^2 - \frac{1}{4} F_R^2 - \frac{1}{2} F^2_{su(3)}$$

Using $-K^1 = \frac{1}{4}$ we get for $I^{GS} = \frac{1}{2} \underbrace{(\hat{\eta}^{-1})_{11}}_{=1} \hat{I}^1 \hat{I}^1$:

$$I^{GS} = \frac{1}{2} \left(\frac{1}{4} \mathrm{Tr}\, F^2_{su(3)} + 5 c_2(R) - \frac{1}{4} p_1(T) - \frac{1}{4} F_L^2 - \frac{1}{4} F_R^2 \right)^2$$

$\Rightarrow$ total anomaly:

$$I^{bif}_{E_6, E_6}(F_L, F_R) = I^{one-loop} + I^{GS}$$

$$= \frac{1}{16}(\mathrm{Tr}\,F_L^2)^2 + \frac{1}{16}\mathrm{Tr}\,F_L^2\,\mathrm{Tr}\,F_R^2 + \frac{1}{16}(\mathrm{Tr}\,F_R^2)^2$$

$$+ \left(\mathrm{Tr}\,F_L^2 + \mathrm{Tr}\,F_R^2\right)\left(\frac{1}{8}p_1(T) - \frac{3}{2}c_2(R)\right)$$

$$+ \frac{319}{24}c_2^2(R) - \frac{89}{48}c_2(R)p_1(T) + \frac{553}{5760}p_1^2(T) - \frac{29}{1440}p_2(T)$$

Example 2: Q M5-branes on $\mathbb{C}^2/\mathbb{Z}_k$

$\rightarrow [SU(k)_0] \times SU(k)_1 \times \cdots \times SU(k)_{Q-1} \times [SU(k)_Q]$

with $(Q-1)$ TM's and bifundamentals under $SU(k)_i \times SU(k)_{i+1}$

$$I_i^{vec} = -\frac{1}{24}\Big(2k\,\mathrm{tr}_{fund}F_i^4 + (3/2)(\mathrm{Tr}\,F_i^2)^2 + 6k\,\mathrm{Tr}\,F_i^2 c_2(R) + (k^2-1)c_2(R)^2\Big)$$

$$I^{bif}_{i,i+1} = \frac{1}{24}\left(k\,\mathrm{tr}_{fund}F_i^4 + k\,\mathrm{tr}_{fund}F_{i+1}^4 + \frac{3}{2}\mathrm{Tr}\,F_i^2\,\mathrm{Tr}\,F_{i+1}^2\right)$$

and $I^{tensor} = \frac{1}{24}c_2(R)^2$

$$\Rightarrow I^{one-loop} = -\frac{1}{32}\eta^{ij} - \frac{k}{4}\mathrm{Tr}\,F_i^2\rho^i c_2(R) - \frac{1}{24}(Q-1)(k^2-2)c_2(R)^2$$

where η^{ij} for $i, j = 1, \cdots, Q-1$ is given by

$$\eta^{ij} = \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & 0 & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & 0 & & & -1 & 2 \end{pmatrix}$$

and $\rho^i = (1, 1, \cdots, 1)$ and have set $F_0 = F_Q = 0$ (for simplicity)

$$I^{GS} = \frac{1}{2} \eta^{ij} I_i I_j \,, \quad I_i = \frac{1}{4} \operatorname{Tr} F_i^2 + K (\eta^{-1})_{ij} \rho^j c_2(R)$$

$$\Rightarrow I^{tot} = I^{one\text{-}loop} + I^{GS}$$

$$= \left(\frac{K^2}{2} \rho^i (\eta^{-1})_{ij} \rho^j - \frac{1}{24} (Q-1)(K^2-2) \right) c_2(R)^2$$

$$= \frac{1}{24} \left((Q^3 - Q) K^2 - (Q-1)(K^2-2) \right) c_2(R)^2$$

where we used $\rho^i (\eta^{-1})_{ij} \rho^j = (Q^3 - Q)/12$