§4. The space of conformal blocks and fusion rules

Consider the Riemann sphere $\mathbb{CP}^1$ with homogeneous coordinates $[\zeta_0 : \zeta_1] \to z = \zeta_0/\zeta_1$

identify $\mathbb{CP}^1 \cong \mathbb{C} \cup \{\infty\}$

Let $p_1, \dots, p_n \in \mathbb{CP}^1$ be n distinct points

introduce coordinates $z_j = z(p_j)$ at p_j

$\to$ locally $t_j = z - z_j$ for $p_j \neq \infty$

for $p_j = \infty$ take $t_j = \frac{1}{z}$

Now suppose $p_j \neq \infty \;\forall j$

Denote by $\mathcal{M}_{p_1, \dots, p_n}$ the vector space of meromorphic functions on $\mathbb{CP}^1$ with poles of any order at most at $p_1, \dots, p_n$. Set

$$\mathfrak{g}(p_1, \dots, p_n) = \mathfrak{g} \otimes \mathcal{M}_{p_1, \dots, p_n}$$

where $\mathfrak{g}$ is $\mathfrak{sl}(2, \mathbb{C})$ Lie algebra.

$\to \mathfrak{g}(p_1, \dots, p_n)$ has structure of Lie algebra:

$$[X \otimes f, Y \otimes g] = [X, Y] \otimes fg, \quad X, Y \in \mathfrak{g},$$

$$f, g \in \mathcal{M}_{p_1, \dots, p_n}.$$

$\to$ Laurent expansion of elements of $\mathfrak{g}(p_1, \dots, p_n)$ at p_j with respect to t_j gives linear map

$$\tau_j : g(p_1, \dots, p_n) \longrightarrow g \otimes \mathbb{C}((t_j))$$

for each j, $1 \leq j \leq n$. Injection of $g \otimes \mathbb{C}((t_j))$ to affine Lie algebra $\hat{g}_j = g \otimes \mathbb{C}((t_j)) \oplus \mathbb{C}$ then gives:

$$\iota_j : g(p_1, \dots, p_n) \longrightarrow \hat{g}_j .$$

Fix level K. Associate $p_j \longrightarrow H_{\lambda_j}$ (integrable highest weight module)

<u>Definition:</u>

The diagonal action Δ of $g(p_1, \dots, p_n)$ on $H_{\lambda_1} \otimes \dots \otimes H_{\lambda_n}$ is given by

$$\Delta(\varphi)(\xi_1 \otimes \dots \otimes \xi_n)$$

$$= \sum_{j=1}^{n} \xi_1 \otimes \dots \otimes \iota_j(\varphi)\xi_j \otimes \dots \otimes \xi_n$$

for $\varphi \in g(p_1, \dots, p_n)$ and $\xi_j \in H_{\lambda_j}$, $1 \leq j \leq n$.

<u>Lemma:</u>

The above action

$$\Delta : g(p_1, \dots, p_n) \longrightarrow \mathrm{End}(H_{\lambda_1} \otimes \dots \otimes H_{\lambda_n})$$

is representation of the Lie algebra $g(p_1, \dots, p_n)$.

<u>Proof:</u>

For $f \in \mathcal{M}_{p_1, \dots, p_n}$ denote by f_{p_j} the Laurent series in t_j at p_j. The 2-cocycle ω introduced in §1, Proposition 1 satisfies:

$$\sum_{j=1}^{n} \omega(X \otimes f_{p_j}, Y \otimes g_{p_j}) = 0$$

for any $X \otimes f$, $Y \otimes g(p_1, \dots, p_n)$ since

$$\sum_{j=1}^{n} \omega(X \otimes f_{p_j}, Y \otimes g_{p_j}) = \langle X, Y \rangle \sum_{j=1}^{n} \operatorname{Res}_{t_j=0}(df g)$$

and sum of residues of a meromorphic 1-form is zero.

$$\longrightarrow \quad \Delta([X,Y] \otimes fg) = [\Delta(X \otimes f), \Delta(Y \otimes g)]$$

$\square$

<u>Definition:</u>

The "space of conformal blocks" $\mathcal{H}(p_1, \dots, p_n ; \lambda_1, \dots, \lambda_n)$ is defined as the space of linear maps

$$\Psi : H_{\lambda_1} \otimes \cdots \otimes H_{\lambda_n} \longrightarrow \mathbb{C}$$

which are invariant under diagonal action Δ of the Lie algebra $\mathfrak{g}(p_1, \dots, p_n)$, i.e

$$\sum_{j=1}^{n} \Psi(\xi_1, \dots, (X \otimes f_{p_j}) \xi_j, \dots, \xi_n) = 0$$

$\forall\ \xi_1 \in H_{\lambda_1}, \dots, \xi_n \in H_{\lambda_n}$ and $X \otimes f \in \mathfrak{g}(p_1, \dots, p_n)$.

Notation: $\operatorname{Hom}_{\mathfrak{g}(p_1, \dots, p_n)}(\bigotimes_{j=1}^{n} H_{\lambda_j}, \mathbb{C})$

Consider the meromorphic function

$$f(z) = (z - z_i)^r, \quad r < 0$$

defined on $\mathbb{CP}^1$. Taylor expansion at p_j:

$$f_{p_j}(t_j) = \sum_{m=0}^{\infty} a_m^{(j)} t_j^m$$

Then invariance property of Ψ gives:

$$\Psi(\xi_1, \dots, (X \otimes t_i^r)\xi_i, \dots, \xi_n) \qquad (*)$$

$$= - \sum_{j, j \neq i} \sum_{m \geq 0} a_m^{(j)} \Psi(\xi_1, \dots, (X \otimes t_j^m)\xi_j, \dots, \xi_n)$$

$\forall \xi_j \in H_j, 1 \leq j \leq n.$

Define the embedding map:

$$\iota : \bigotimes_{j=1}^{n} V_{\lambda_j} \longrightarrow \bigotimes_{j=1}^{n} H_{\lambda_j}$$

$\uparrow$ finite dim. irreducible rep. of $\mathfrak{g}$ with highest weight λ_j

$\rightarrow$ restriction map:

$$\iota^* \Psi : \bigotimes_{j=1}^{n} V_{\lambda_j} \longrightarrow \mathbb{C}$$

wher $\iota^* \Psi = \Psi \circ \iota$. Then we have the following

Lemma:

For $\Psi \in \mathcal{H}(p_1, \dots, p_n; \lambda_1, \dots, \lambda_n)$, its restriction Ψ_0 on $\bigotimes_{j=1}^{n} V_{\lambda_j}$ is invariant under $\Delta(\mathfrak{g})$.

Moreover,

$$\iota^*: \mathcal{H}(p_1, \dots, p_n; \lambda_1, \dots, \lambda_n) \to \mathrm{Hom}_{\mathfrak{g}}\left(\bigotimes_{j=1}^n V_{\lambda_j}, \mathbb{C}\right)$$

is injective. $\to$ Ψ is uniquely determined by Ψ_0.

<u>Proof:</u>

Ψ is in particular invariant under diagonal action of $X \otimes 1_{\mathbb{CP}^1}$

$\in \mathfrak{g}$ constant function on $\mathbb{CP}^1$

$\to$ for any $X \in \mathfrak{g}$, $\xi_j \in V_{\lambda_j}$, $1 \le j \le n$:

$$\sum_{j=1}^n \Psi(\xi_1 \otimes \dots \otimes X\xi_j \otimes \dots \otimes \xi_n) = 0$$

Now set

$$\mathcal{F}_d = \bigoplus_{d_1 + \dots + d_n \le d} \left(\bigotimes_{j=1}^n H_{\lambda_j}(d_j)\right)$$

using the direct sum decomposition $H_{\lambda_j} = \bigoplus_{d \ge 0} H_{\lambda_j}(d)$

We have $\mathcal{F}_0 = \bigotimes_{j=1}^n V_{\lambda_j}$ and

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots \subset \mathcal{F}_d \subset \dots$$

Suppose $\Psi|_{\mathcal{F}_0} = 0$. Have to show $\Psi(\xi) = 0$

where

$$\xi = \eta_1 \otimes \dots \otimes (X \otimes t_i^r)\eta_i \otimes \dots \otimes \eta_n,$$

$$X \in \mathfrak{g}, \quad r < 0, \quad \eta_1 \otimes \dots \otimes \eta_n \in \mathcal{F}_{d-1}$$

$\to$ proof follows by induction on d $\square$

Recall:

H_λ is quotient of M_λ by submodule generated by $x = (E \otimes t^{-1})^d v$, $d = k - \lambda + 1$, $v \in V_\lambda$

null-vector

Set $d_i = k - \lambda_i + 1$, $1 \leq i \leq n$. Then we have

Proposition 1:

For Ψ belonging to above space of conformal blocks, the restriction map $\Psi_0 : V_{\lambda_1} \times \cdots \times V_{\lambda_n} \to \mathbb{C}$ satisfies

$$\Psi_0 \left(E^{m_1} \xi_1, \ldots, E^{m_{i-1}} \xi_{i-1}, v_i, E^{m_{i+1}} \xi_{i+1}, \ldots, E^{m_n} \xi_n \right) = 0$$

for v_i highest weight vector, $\xi_j \in V_j$, $j \neq i$, $1 \leq j \leq n$, and $m_j \geq 0$, $1 \leq j \leq n$, $\sum_{j, j \neq i} m_j = d_i$

Proof:

We show the statement in the case $i = 1$.

$x_1 = (E \otimes t_1^{-1})^{d_1} v_1$ null-vector

$$\to \Psi \left((E \otimes t_1^{-1})^{d_1} v_1, \xi_2, \ldots, \xi_n \right) = 0$$

Applying equation $(*)$ then gives:

$$\sum_{m_2 + \cdots + m_n = d_1} \frac{d_1!}{m_2! \cdots m_n!} \prod_{2 \leq j \leq n} (z_j - z_1)^{-m_j} f_{m_2, \ldots, m_n} = 0$$

where

$$(**) \quad f_{m_2, \ldots, m_n} = \Psi_0 \left(v_1, E^{m_2} \xi_2, \ldots, E^{m_n} \xi_n \right)$$

consider $d_1 = 0$

$t_1^{-1} = (z - z_1)^{-1}$

$$\Rightarrow t_1^{-1} = (t_2 + z_2 - z_1)^{-1} = (z_2 - z_1)^{-1} - \frac{t_2}{(z_2 - z_1)^2} + \frac{t_2^2}{(z_2 - z_1)^3} + O(t_2^3)$$

$$\overset{(*)}{\Rightarrow} \Psi((E \otimes t_1^{-1}) v_1, \xi_2, \ldots, \xi_n)$$

$$= -\sum_{j, j \neq 1} \sum_{m \geq 0} (z_j - z_1)^{-m-1} (-1)^m \Psi(v_1, \ldots, (E \otimes t_j^m) \xi_j, \ldots, \xi_n)$$

$$= -\sum_{j, j \neq 1} (z_j - z_1)^{-1} \Psi(v_1, \ldots, E\xi_j, \ldots, \xi_n)$$

$\longrightarrow$ general case follows by induction

Since $(**)$ holds for any $z_1, \ldots, z_n$

$\longrightarrow f_{m_2, \ldots, m_n} = 0$ $\square$

Denote by $N_{\lambda_1 \lambda_2 \lambda_3}$ the dimension of the space of conformal blocks $\mathcal{H}(p_1, p_2, p_3 ; \lambda_1, \lambda_2, \lambda_3)$

Then we have

Proposition 2:

In the case $n = 3$, if the following holds

$$\lambda_1 + \lambda_2 + \lambda_3 \in 2\mathbb{Z},$$
$$|\lambda_1 - \lambda_2| \leq \lambda_3 \leq \lambda_1 + \lambda_2$$
$$\lambda_1 + \lambda_2 + \lambda_3 \leq 2k,$$

then $N_{\lambda_1 \lambda_2 \lambda_3} = 1$. Otherwise, $N_{\lambda_1 \lambda_2 \lambda_3} = 0$

$\longrightarrow$ "quantum Clebsch-Gordan rule" at level k.

Let us next introduce the "fusion rule" for counting dimensions of conformal blocks

Take $n+1$ distinct points $p_1, \cdots, p_n, p_{n+1}$ so that $p_{n+1} = \infty \longrightarrow$ associated integrable highest weight modules: $H_{\lambda_1}, \cdots, H_{\lambda_n}, H^*_{\lambda_{n+1}}$ of level K.

$\uparrow$ dual module

Define space of conformal blocks

$$\mathcal{H}(p_1, \cdots, p_n, p_{n+1}; \lambda_1, \cdots, \lambda_n, \lambda^*_{n+1})$$

$$\equiv \mathrm{Hom}_{g(p_1, \cdots, p_n, p_{n+1})}\left(\left(\bigotimes_{j=1}^{n} H_{\lambda_j}\right) \otimes H^*_{\lambda_{n+1}}, \mathbb{C}\right)$$

$$\Big\downarrow$$

$$\mathrm{Hom}_{g}\left(\left(\bigotimes_{j=1}^{n} V_{\lambda_j}\right) \otimes V^*_{\lambda_{n+1}}, \mathbb{C}\right)$$

with $\dim \mathcal{H} = N^{\lambda_{n+1}}_{\lambda_1 \cdots \lambda_n}$. In the case of $g = sl_2(\mathbb{C})$: $N^{\lambda_3}_{\lambda_1 \lambda_2} = N_{\lambda_1 \lambda_2 \lambda_3}$ since dual rep is equiv to original one.

In the case $\underline{n=3}$: $\mathcal{H} \hookrightarrow \mathrm{Hom}_g(V_{\lambda_1} \otimes V_{\lambda_2} \otimes V_{\lambda_3}, V_{\lambda_4})$

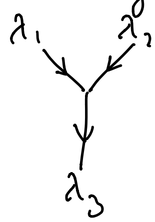
We have the following composition of projection maps

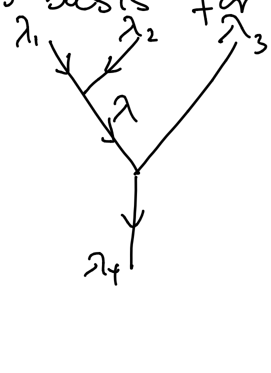
$$p^{\lambda}_{\lambda_1 \lambda_2} \otimes id_{V_{\lambda_3}} : (V_{\lambda_1} \otimes V_{\lambda_2}) \otimes V_{\lambda_3} \to V_\lambda \otimes V_{\lambda_3}$$

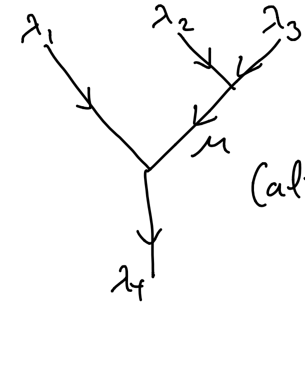
$$p^{\lambda_4}_{\lambda \lambda_3} : V_\lambda \otimes V_{\lambda_3} \to V_{\lambda_4}$$

"labeled trees"

$\longrightarrow$ gives basis for $\mathcal{H}$







(alternate basis)