Classification of F-theory bases

6d $\mathcal{N}=(1,0)$ theories arise from

F-th on $T^2 \hookrightarrow X$
$$\begin{array}{c} X \\ \downarrow \\ B \end{array} \quad \text{with } \dim_{\mathbb{C}} B = 2$$

$\longrightarrow \quad T = h^{1,1}(B) - 1$

T: number of tensor multiplets

last time: $B = \mathbb{F}_n$

today: general B

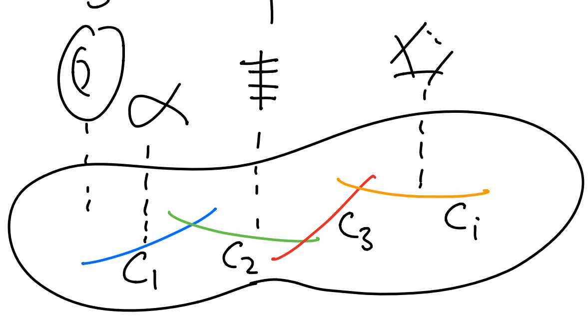
Weierstrass equation: $y^2 = x^3 + f x + g$

$$\longrightarrow \quad \Delta = 4f^3 + 27g^2$$

with $f \in -4K_B$, $g \in -6K_B$, $\Delta \in -12K_B$

K_B: canonical bundle of B

$K = K_B$ satisfies $K \cdot K = 9 - T$



Want to classify $\Delta = 0$ in terms of irreducible components $C_i \subset \Delta = 0$

Along irreducible components:

ord(f)	ord(g)	ord(Δ)	singularity	gauge sym.
≥ 0	≥ 0	0	none	none
0	0	$n \geq 2$	A_{n-1}	$su(n)$ or $sp(\lfloor n/2 \rfloor)$
≥ 1	1	2	none	none
1	≥ 2	3	A_1	$su(2)$
≥ 2	2	4	A_2	$su(3)$ or $su(2)$
≥ 2	≥ 3	6	D_4	$so(8)$ or $so(7)$ or g_2
2	3	$n \geq 7$	D_{n-2}	$so(2n-4)$ or $so(2n-5)$
≥ 3	4	8	E_6	E_6 or F_4
3	≥ 5	9	E_7	E_7
≥ 4	5	10	E_8	E_8
≥ 4	≥ 6	≥ 12	does not occur in F-th.	

$\Delta = -12K$ need not be irreducible

For irr. divisor $C \subset B$ with $C \cdot C < 0$,
and divisor $A \subset B$ with $A \cdot C < 0$
$\longrightarrow$ C is irreducible component of A:

$$C \cdot C < 0, \quad A \cdot C < 0 \Rightarrow A = C + X$$

Example: $C \cdot C = -8, \quad \chi(C) = 2 - 2g$
$\Rightarrow (K + C) \cdot C = 2g - 2$, for $C = \mathbb{P}^1 \ (g = 0) \rightarrow K \cdot C = 6$
$\Rightarrow -4K \cdot C = -24 \Rightarrow -4K = 3C + X_4, \quad X_4 \cdot C \geq 0$

Similarly, $-6K = 5C + X_6$, $-12K = 9C + X_{12}$

$\rightarrow$ f, g and Δ are vanishing on C with degrees $3, 5, 9$ $\rightarrow$ E_7 gauge algebra

$B = \mathbb{P}^2, \mathbb{F}_m$ with $m \leq 12$ $\rightarrow$ all cases with $T=0$ and $T=1$

$\mathbb{P}^2 \rightarrow T=0$, $\mathbb{F}_m \rightarrow T=1$

$\rightarrow$ all other F-theory bases are blow-ups of these.

degrees of f, g, Δ should not exceed $4, 6, 12$!

$\rightarrow$ singularity becomes too bad

Non-Higgsable clusters

minimal possible gauge group on curves C_i

$\rightarrow$ all possible matter fields Higgsed

Note: $C_i \cdot C_i \geq -2 \rightarrow -nK \cdot C_i \geq 0$

$\rightarrow$ C_i does not appear as component of $-nK$

$\rightarrow$ no non-abelian gauge group required

$\rightarrow$ focus on clusters with $\exists\, i : C_i \cdot C_i \leq -3$

- single irreducible divisors ($B = \mathbb{F}_m$):

$$-K = \gamma C + Y, \quad Y \cdot C = 0, \quad \gamma \in \mathbb{Q}$$

$$-K \cdot C = 2 - m, \quad C \cdot C = -m \rightarrow \gamma = (m-2)/m$$

write $-nK = cC + X$, $\quad X\cdot C \geq 0$, $c \in \mathbb{Z}$

$\rightarrow \quad c = \lceil n(m-2)/m \rceil$

$\rightarrow \quad [f] = \lceil 4(m-2)/m \rceil$, $[g] = \lceil 6(m-2)/m \rceil$,

$\quad [\Delta] = \lceil 12(m-2)/m \rceil$

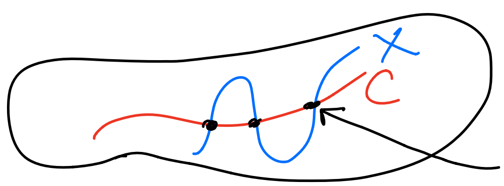
for $m = 9, 10, 11 \rightarrow [f] = 4$, $[g] = 5$, $[\Delta] = 10$

$\rightarrow$ E_8 singularity on C

writing $\Delta = -12K = \lceil 12(m-2)/m \rceil + X$,

we see that $X \cdot C \neq 0$ for $m = 9, 10, 11$

$\rightarrow$



$[f] = 4$, $[g] = 6$, $[\Delta] = 12$

$\rightarrow$ fiber too singular

"no fundamental matter field for E_8"

$\rightarrow$ only value of $m > 8$ possible in a good F-theory model: $m = 12$ ($\rightarrow E_8$)

further constraint: $g(C) = 0$!

assume opposite: $C\cdot C < 0$, $g > 0$

$\rightarrow K\cdot C \geq -C\cdot C \rightarrow -nK = cC + X$

$\Rightarrow X\cdot C = -nK\cdot C - cC\cdot C < 0$

unless $c \geq n$

$\rightarrow$ too singular

- Pairs of intersecting divisors

consider curves A, B with

$$A \cdot A = -x < 0$$
$$B \cdot B = -y < 0$$
$$A \cdot B = p > 0$$

example: $x = y = 3$, $p = 1$

$\rightarrow -4K = aA + bB + X$, $X \cdot A \geq 0$, $X \cdot B \geq 0$

from $(K+C) \cdot C = 2g-2 = -2$ we get

$$-4K \cdot A = 8 + 4 \underbrace{A \cdot A}_{=-3} = -4 = -4K \cdot B$$

$$\| $$

$$(aA + bB + X) \cdot A = -3a + b + \underbrace{X \cdot A}_{\geq 0} = -4$$

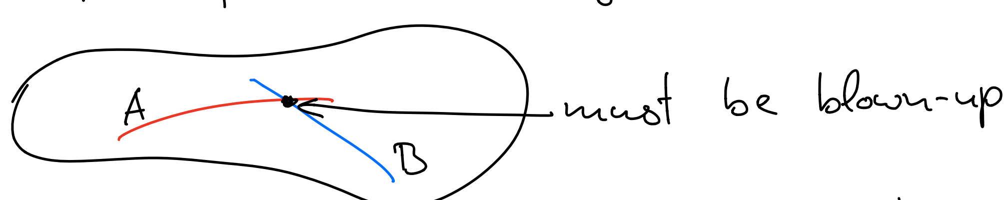
$\Rightarrow 3a - b \geq 4$, $3b - a \geq 4$

$\Rightarrow \quad a + b \geq 4$

$\rightarrow \quad [f] \geq 4$ on $A \cap B$

similarly, $[g] \geq 6$, $[\Delta] \geq 12$ on $A \cap B$

$\rightarrow$ elliptic fiber too singular



$(-3)(-3)$ curve configurations are not consistent in F-theory.

general situation:

$$-nK = aA + bB + X$$

$$\Rightarrow -nK \cdot A = n(2-x) = -ax + bp + X \cdot A,$$

$$-nK \cdot B = n(2-y) = ap - by + X \cdot B$$

$$\Rightarrow \quad ax - pb \geq n(x-2)$$

$$by - pa \geq n(y-2)$$

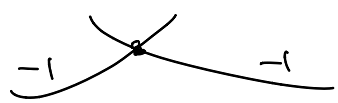
$p^2 < xy$

$$\Rightarrow \quad a \geq \frac{n}{xy-p^2}(xy + py - 2y - 2p)$$

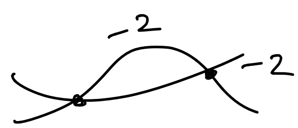
$$b \geq \frac{n}{xy-p^2}(xy + px - 2x - 2p)$$

no solution for $p^2 > xy$.

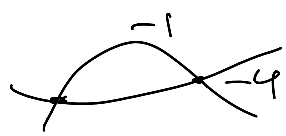
3 marginal solutions:

1) $x = y = p = 1$: 

$a = b = 0$

2) $x = y = p = 2$: 

$a = b = 0$

3) $x = 4$, $y = 1$, $p = 2$: 

$b = 0$, a as for isolated (-4)

For $p^2 < xy$ we get:

$$a + b \geq \frac{n}{xy - p^2}(2xy + p(x+y) - 4p - 2x - 2y)$$

need $a + b < n$

only possible if

$$(x+p-2)(y+p-2) < 4$$

$\longrightarrow$ only possible pairs with $p=1$ (single int.):

$(-x,-y) = (-3,-2), (-2,-2), (-m,-1)$ with $m \leq 12$

$p=2$:

$(-x,-y) = (-m,-1)$ with $m = 1,2,3,4$.