3D topological phases from six dimensions

Plan of lecture

1) Introduction to superconformal alg.
   $\longrightarrow$ 6d (2,0) SCFTs
2) Compactification on M_3 and topological twist
3) Defects and the theory $T[M_3]$
4) $T[M_3]$ as modular tensor category

§1. The superconformal algebra

Before attempting to understand the superconformal algebra, we first need to look at some preliminaries of the conformal algebra...

§1.1 Conformal field theories

What is a conformal transformation?

A transformation $x^\mu \mapsto \tilde{x}^\mu$ such that

$$g_{\mu\nu}(x) \longmapsto \lambda\, g_{\mu\nu}(\tilde{x})$$

"angle preserving" transformation

The group of such transformations is called the "conformal symmetry group" and is an extension of the Poincaré group

Its generators are given by:

(Greek indices run from 0 to $d-1$)

$M_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$ Lorentz generators

$P_\mu = -i\,\partial_\mu$ translations

$D = (-i)\,[-x\cdot\partial]$ dilatations

$K_\mu = (-i)\left[-2x_\mu(x\cdot\partial) + x^2\partial_\mu\right]$ special conf. transformations

D is a scalar and K_μ is a covariant vector under Lorentz transformations.

Intuitively, D is obvious because it induces ordinary scale transformations. K_μ are less obvious, they induce so called "Möbius transformations"

$$x^\mu \longmapsto \frac{x^\mu - a^\mu x^2}{1 - 2a \cdot x + a^2 x^2}$$

"inversion + translation + inversion"

inversion: $x^\mu \longmapsto \frac{x^\mu}{x^2}$

The above generators obey the commutation relations:

$$[M_{\mu\nu}, M_{\alpha\beta}] = (-i)[\eta_{\mu\beta} M_{\nu\alpha} + \eta_{\nu\alpha} M_{\mu\beta} - \eta_{\mu\alpha} M_{\nu\beta} - \eta_{\nu\beta} M_{\mu\alpha}]$$

$$[M_{\mu\nu}, P_\alpha] = (-i)[\eta_{\nu\alpha} P_\mu - \eta_{\mu\alpha} P_\nu]$$

$$[D, M_{\mu\nu}] = 0$$

$$[M_{\mu\nu}, K_\alpha] = (-i)[\eta_{\nu\alpha} K_\mu - \eta_{\mu\alpha} K_\nu]$$

$$[D, P_\mu] = -i P_\mu, \quad [D, K_\mu] = -i(-K_\mu)$$

$$[D, D] = 0 \,, \quad [P_\mu, P_\nu] = 0$$

$$[P_\mu, K_\nu] = (-i)[2\eta_{\mu\nu} D + 2M_{\mu\nu}]$$

$$[K_\mu, K_\nu] = 0$$

The conformal group is locally isomorphic to $SO(d,2)$. Denote $SO(d,2)$ generators by S_{ab} where latin indices run from -1 to d. Then we have:

$$S_{\mu\nu} = M_{\mu\nu}$$

$$S_{-1d} = D$$

$$S_{\mu -1} = \frac{1}{2}[P_\mu + K_\mu]$$

$$S_{\mu d} = \frac{1}{2}[P_\mu - K_\mu]$$

We can also define a Euclidean conformal algebra:

$$\left.\begin{aligned} M'_{pq} &= S_{pq} \\ D' &= i S_{-10} \\ P'_p &= [S_{p-1} + i S_{p0}] \\ K'_p &= [S_{p-1} - i S_{p0}] \end{aligned}\right\} \text{generators of Euclidean conf. group } SO(d+1,1)$$

$$\begin{aligned} M'^{\dagger} &= M' \\ D'^{\dagger} &= -D' \\ P'^{\dagger} &= K' \\ K'^{\dagger} &= P' \end{aligned}$$

§1.2 The superconformal algebra

In conformal field theory the $(d-1,1)$ Lorentzian spinor $Q_\alpha \longrightarrow (d,2)$ conformal spinor

Let us choose Γ matrices for $SO(d,2)$:

$$\Gamma_\mu = \begin{pmatrix} \sigma_\mu & 0 \\ 0 & -\sigma_\mu \end{pmatrix}$$

$$\Gamma_{-1} = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

$$\Gamma_d = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

where σ_μ are $SO(d-1,1)$ Γ matrices.

Γ_μ are constructed iteratively from the $d=2$ expressions :

$$\sigma_0' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Then the above Γ matrices satisfy:

$$\{\Gamma_a, \Gamma_b\} = 2\eta_{ab}$$

where $\eta_{ab} = \mathrm{diag}(\underset{a=-1}{-1}, \underset{a=0}{-1}, 1, \ldots, 1)$

For odd d we have to add

$$\Gamma_{d+1} = \begin{pmatrix} \sigma_{d+1} & 0 \\ 0 & -\sigma_{d+1} \end{pmatrix}$$

Q is completed to full conformal spinor V through new Lorentz spinor S

$$V = \begin{pmatrix} Q_\alpha \\ C_{\theta\phi} \bar{S}^\phi \end{pmatrix}$$

where C is the charge conjugation matrix

$$C \Gamma_\mu C^{-1} = -\Gamma_\mu^T$$

$C = B\sigma_0$ where B is the matrix used to impose the Majorana condition

$$Q^\dagger = BQ$$

We set

$$[S_{ab}, V_\alpha] = R(M_{ab})_\alpha{}^\beta V_\beta$$

with $R(M_{ab}) = \frac{i}{4}[\Gamma_a, \Gamma_b]$. Specifically

$$R(P_\mu) = (-i)\begin{pmatrix} 0 & 0 \\ \sigma_\mu & 0 \end{pmatrix}, \quad R(K_\mu) = (-i)\begin{pmatrix} 0 & \bar{\sigma}_\mu \\ 0 & 0 \end{pmatrix}$$

$$R(D) = \left(-\frac{i}{2}\right)\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix},$$

$$R(M_{\mu\nu}) = \begin{pmatrix} R(m_{\mu\nu}) & 0 \\ 0 & R(m_{\mu\nu}) \end{pmatrix}$$

where $R(m_{\mu\nu}) = \frac{i}{4}\left[\sigma_\mu, \sigma_\nu\right]$

Euclidean spinors:

$$Q' = \frac{1}{\sqrt{2}}\left(Q - i\sigma_0 S\right), \quad S' = \frac{1}{\sqrt{2}}\left(Q + i\sigma_0 S\right)$$

This gives

$$[M'_{pq}, Q'_\alpha] = \left(\frac{i}{4}\right)\left[\Gamma_p, \Gamma_q\right]_\alpha{}^\beta Q'_\beta$$

$$[M'_{pq}, S'_{\tilde{\alpha}}] = \left(\frac{i}{4}\right)\left[\tilde{\Gamma}_p, \tilde{\Gamma}_q\right]_{\tilde{\alpha}}{}^{\tilde{\beta}} Q'_{\tilde{\beta}}$$

$$[D', Q'_\alpha] = \left(-\frac{i}{2}\right) Q'_\alpha$$

$$[D', S'_{\alpha'}] = \left(-\frac{i}{2}\right) S'_{\tilde{\alpha}}$$

$$[P'_p, Q'_\alpha] = 0, \quad [K'_p, S'_{\tilde{\alpha}}] = 0$$

$$[P'_p, S'_{\tilde{\alpha}}] = -\left(\tilde{\Gamma}_p \sigma_0\right)_{\tilde{\alpha}}{}^\beta Q_\beta$$

$$[K'_p, Q'_\alpha] = \left(\Gamma_p \sigma_0\right)_\alpha{}^{\tilde{\beta}} S_{\tilde{\beta}}$$

where $\Gamma_i = \sigma_i \,, \quad \Gamma_d = -i\sigma_0$

$\tilde{\Gamma}_i = \sigma_i \,, \quad \tilde{\Gamma}_d = i\sigma_0$

Note: σ_0 interpolates between the two representations

<u>Next</u>: need to specify the R-sym of the superconformal alg.

→ Jacobi-identities <u>only</u> consistent for dimensions $d = 3, 4, 5, 6$ (will give proof in next lecture)

<u>$d=4$</u>:

Choose Majorana spinors Q and S

$C = \sigma_0$

$$\longrightarrow \quad Q^\dagger = Q \quad ; \quad S^\dagger = S$$

$$Q'^\dagger = S' \quad ; \quad S'^\dagger = Q'$$

R-symmetry: $U(n)$

Define $P_\pm = (I \pm \sigma_5)/2$

$$\longrightarrow P_+^T = P_- \; ; \; P_+^* = P_- \; ; \; (P_+)^\dagger = P_+$$

generators T_{ij} of $U(n)$ obey:

$$[T_{ij}, Q_m] = [P_+ Q_i \delta_{jm} - P_- Q_j \delta_{im}]$$

$$[T_{ij}, Q'_m] = [P_+ Q'_i \delta_{jm} - P_- Q'_j \delta_{im}]$$

$$[T_{ij}, S_m] = [P_- S_i \delta_{jm} - P_+ S_j \delta_{im}]$$

$$[T_{ij}, S'_m] = [P_+ S'_i \delta_{jm} - P_- S'_j \delta_{im}]$$

$$[T_{ij}, M_{pq}] = 0$$

anti-commutation relations:

$$\{Q'_{i\alpha}, Q'_{j\beta}\} = (\not{P}'C)_{\alpha\beta}\,\delta_{ij}$$

$$\{S'_{i\tilde{\alpha}}, S_{j\tilde{\beta}}\} = (\tilde{\not{K}}'C)_{\tilde{\alpha}\tilde{\beta}}\,\delta_{ij}$$

$$\{Q'_{i\alpha}, S'_{j\tilde{\beta}}\} = ((i)\,\delta_{ij}/2)\left[(M'_{\mu\nu}\Gamma_\mu\Gamma_\nu)_{\alpha\tilde{\beta}} + 2\delta_{\alpha\tilde{\beta}}D'\right]$$

$$- 2(P_+)_{\alpha\tilde{\beta}}\,T_{ij} + 2(P_-)_{\alpha\tilde{\beta}}\,T_{ji} + \tfrac{1}{2}(\sigma_5)_{\alpha\tilde{\beta}}\,R$$