Essay

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1 Abstract

Solving linear equations is a very important problem in many situations as we know[1]. A inear equations is

$$A\vec{x} = \vec{b}. \tag{1}$$

In this paper, we assume that we do not need to know $\vec{x}$ itself, but we want to find a value related to $\vec{x}$ such as $\vec{x}^\dagger M\vec{x}$, where M is a certain matrix. Consider the case where A is a matrix of $N \times N$ with up to s non-zero components per row and the condition number [1] is κ, the fastest known classical algorithm can calculate $\vec{x}, \vec{x}^\dagger M\vec{x}$ in a time scale of $N\sqrt{\kappa}$. On the other hand, in the quantum algorithm presented here, when the state $|b\rangle$ corresponding to $\vec{b}$ is available, the computational steps to find $|x\rangle$ can be computed on a polynomial time scale of $\log(N)$ and κ. This is an exponential speedup compared to classical computers.

2 Introduction

A quantum computer is a device that uses quantum mechanics to perform calculations, including calculations that are not possible with currently known algorithms from classical computers. For certain problems, such as Shor's algorithm, a quantum computer can perform calculations much faster than a classical computer. In this paper, we will consider the characteristics of the solution of a linear system of linear equations.

[1] condition number is an upper bound on the uncertainty of the solution of a linear equation and is defined as the maximum ratio of the relative error of x divided by the relative error of b.

Linear equations have applications in almost every field of science, technology, and engineering, and are very important[1]. On a classical computer, it generally takes at least N orders of time to solve a N-order linear equation.

In this paper, we use the same equations as in Abstract;

$$A\vec{x} = \vec{b}. \tag{2}$$

If A is a $N \times N$ matrix with up to s non-zero components per row, and the condition number is κ, then it shows that it takes a polynomial time scale of $\log(N), \kappa$ to compute the values associated with the solution $|x\rangle$ to any precision. This is an exponential speedup compared to a classical computer. Also, in typical cases, accuracy is not often required. However, the condition number can significantly increase the computational complexity. This is a strong constraint of the algorithm in this paper.

We will consider the same conditions as in the linear equation above. First, we consider $\vec{b}$ to be the matrix representation of the abstract state $|b\rangle$. Note that $\vec{b}$ and $|b\rangle$ are not necessarily equivalent.

Related work includes examples of linear operations with restrictions, and extensions to nonlinear differential equations.

3 Preliminary

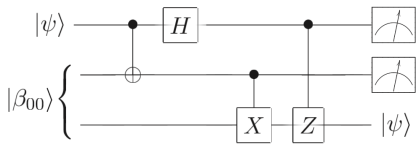
3.1 Basics of Quantum Computation

A qubit is the smallest unit of storage for quantum information. In a two-level basis, the general state $|\psi\rangle$ is often expressed as $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, with $|0\rangle, |1\rangle$ as the bases, where α, β are complex constants, $|\alpha|^2 + |\beta|^2 = 1$.

A quantum register is a sequence of qubits prepared in different Hilbert spaces, represented here using tensor products as $|0\rangle \otimes |1\rangle \otimes \cdots |1\rangle$.

A quantum gate is an operation on a quantum register, and in today's mainstream quantum computers, the desired state is achieved by a combination of several different unitary operators (which are also known to be unitary operators).

A quantum circuit is a circuit diagram in which gates are applied to a quantum register in sequence, as shown in the following figure.



.

Figure 1: example of a quautum circuit (taken from QCQI)

Measurement is the operation of reading information from the final state. It refers to knowing $p_k = |c_k|^2$ from the final state $\sum_{k=0}^{2^n-1} c_k|k\rangle$, repeating the execution and measurement of the circuit, and recording the number of times $0, 1, \ldots, 2^{n-1}$ is obtained. Estimate $\{p_k\}$ from the histogram.

4 Main Result

Here is the specific algorithm.

1. Convert the matrix A to e^{iAt} [2].
2. Prepare the state $|b\rangle$ corresponding to $\vec{b}$ by [3].
3. By [3], similarly prepares the state $|\Psi_0\rangle := \sqrt{\frac{2}{T}} \sum_{\tau=0}^{T-1} \sin \frac{\pi\left(\tau + \frac{1}{2}\right)}{T} |\tau\rangle$.
4. $\sum_{\tau=0}^{T-1} |\tau\rangle\langle\tau| \otimes e^{iA\tau t_0/T}$ (unitary) on $|\Psi_0\rangle \otimes |b\rangle$, where $t_0 = O(\kappa/\epsilon)$. The tensor products of the states $|b\rangle$ and $|\Psi_0\rangle$ are subject to conditional Hamiltonian evolution.
5. Perform a Fourier Transformation on the first register (register with initial state $|\Psi_0\rangle$).
6. Add the qubit and make the rotation depend on the value of $\tilde{\lambda}_k$.
7. Inverse Fourier Transformation.
8. Do the inverse transform of 4.

9. Measure the last qubit; a 1 means success and a 0 means failure.

With the above algorithm, when the state $|b\rangle$ corresponding to the quantity $\vec{x}$ can be prepared, the computation step to find the remaining $\vec{x}^\dagger M\vec{x}$.

5 Discussion

5.1 QFT

QFT(Quantum Fourier transformation) is a operation which convert $|j\rangle$;

$$
\begin{aligned}
|j\rangle &\to \frac{1}{2^{n/2}} \sum_{k=0}^{2^n-1} e^{2\pi ijk/2^n} |k\rangle \\
&= \frac{1}{2^{n/2}} \sum_{k_1=0}^{1} \cdots \sum_{k_n=0}^{1} e^{2\pi ij\left(\sum_{l=1}^{n} k_l 2^{-l}\right)} |k_1 \ldots k_n\rangle \\
&= \frac{1}{2^{n/2}} \sum_{k_1=0}^{1} \cdots \sum_{k_n=0}^{1} \bigotimes_{l=1}^{n} e^{2\pi ijk_l 2^{-l}} |k_l\rangle \\
&= \frac{1}{2^{n/2}} \bigotimes_{l=1}^{n} \left[\sum_{k_l=0}^{1} e^{2\pi ijk_l 2^{-l}} |k_l\rangle \right] \\
&= \frac{1}{2^{n/2}} \bigotimes_{l=1}^{n} \left[|0\rangle + e^{2\pi ij2^{-l}} |1\rangle \right] \\
&= \frac{\left(|0\rangle + e^{2\pi i 0.j_n}|1\rangle\right)\left(|0\rangle + e^{2\pi i 0.j_{n-1}j_n}|1\rangle\right)\cdots\left(|0\rangle + e^{2\pi i 0.j_1 j_2 \cdots j_n}|1\rangle\right)}{2^{n/2}}.
\end{aligned}
$$

This operation is complemented by

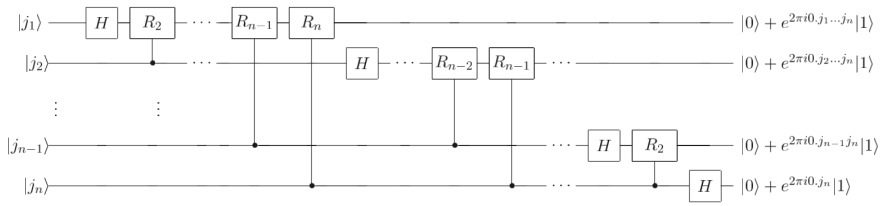


Figure 2: Example of QFT(taken from QCQI)

5.2 QPE

QPE (Quantum Phase Estimation) is an algorithm for estimating the phase, which is calculated by applying QFT. The specific algorithm is as follows (Quoted from QCQI).

> Inputs:
> (1) A black box wich performs a controlled- U^j operation, for integer j
> (2) an eigenstate $|u\rangle$ of U with eigenvalue $e^{2\pi i \varphi_u}$
> (3) $t = n + \left\lceil \log\left(2 + \frac{1}{2\epsilon}\right)\right\rceil$ qubits initialized to $|0\rangle$.
> Outputs: An n-bit approximation $\widetilde{\varphi_u}$ to φ_u.
> Runtime: $O\left(t^2\right)$ operations and one call to controlled- U^j black box. Succeeds with probability at least $1 - \epsilon$.
> Procedure:
> 1. $|0\rangle|u\rangle$
> 2. $\rightarrow \frac{1}{\sqrt{2^t}} \sum_{j=0}^{2^t-1} |j\rangle|u\rangle$ create superposition
> 3. $\rightarrow \frac{1}{\sqrt{2^t}} \sum_{j=0}^{2^t-1} |j\rangle U^j|u\rangle = \frac{1}{\sqrt{2^t}} \sum_{j=0}^{2^t-1} e^{2\pi i j \varphi_u}|j\rangle|u\rangle$ apply black box
> 4. $\rightarrow |\widetilde{\varphi_u}\rangle\,|u\rangle$ apply inverse Fourier transform
> 5. $\rightarrow \widetilde{\varphi_u}$ measure first register

If this is realized by a quantum circuit, it is as follows.

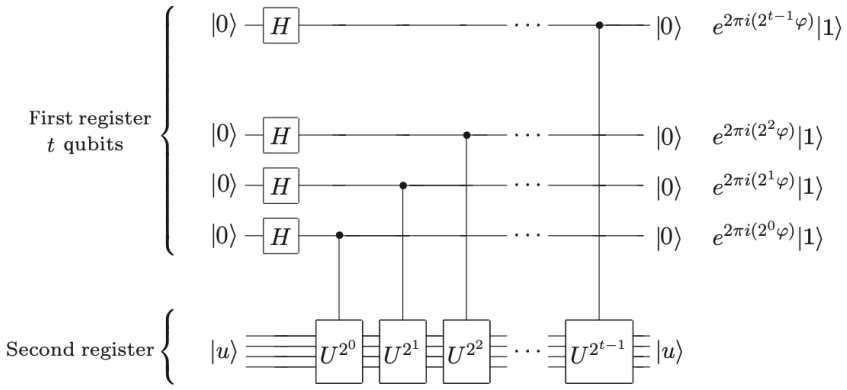


Figure 3: Examples of a part of QPE (quoted from QCQI)

5.3 Algorithm Description

In the following, we will explain why the algorithm introduced in the Main Result works well. First, in [2], the matrix A can be computed in $\exp(iAt)$ with $O(\log(N)s^2t)$ as the upper bound.

Next, we divide the cases according to whether the matrix A is Hermitian or not.

If A is not Hermitian, then

$$\tilde{A} := \begin{pmatrix} 0 & A \\ A^\dagger & 0 \end{pmatrix} \tag{3}$$

Since $\tilde{A}$ is Hermitian by definition,

$$.\tilde{A}\vec{y} = \begin{pmatrix})\vec{b} \\ \\ 0 \end{pmatrix}$$

. By solving above,

$$\vec{y} = \begin{pmatrix} 0 \\ \vec{x} \end{pmatrix}$$

. is obtained. Thus, when A is not Hermitian, the series of operations to define $\tilde{A}$ and make it solvable is called reduction.

In the following, we consider the case where A is Hermitian. By [3], if b_i and $\sum_{i=i_1}^{i_2} |b_i|^2$ are efficiently computable, then state $|b\rangle$ in an arbitrary basis for arbitrary probability amplitudes. If $|b\rangle$ is efficiently computable, then state $|b\rangle$ can be converted to a superposition for any probability amplitude in any basis. We will denote $|b\rangle$ by the superposition of the eigenvectors (choose a unit vector) of A. For the probability amplitude, we consider assigning a component of $\vec{b}$. Now we also define $|u_j\rangle$ as the eigenvectors of A , and λ_j as the corresponding eigenvalues. So far, $|b\rangle$ is ready.

$$. |\Psi_0\rangle := \sqrt{\frac{2}{T}} \sum_{\tau=0}^{T-1} \sin \frac{\pi \left(\tau + \frac{1}{2}\right)}{T} \tau\rangle$$

.

However, here T is assumed to be a very large value. This gives us $|\Psi_0\rangle$ is now ready. By applying conditional Hamiltonian evolution to the tensor products of $|b\rangle$ and $|\Psi_0\rangle$ prepared in this way

$\sum_{\tau=0}^{T-1} |\tau\rangle\langle\tau| \otimes e^{iA\tau t_0/T}$ (unitary) on $|\Psi_0\rangle \otimes |b\rangle$, where $t_0 = O(\kappa/\epsilon)$. By Fourier Transformation of the first register (the register whose initial state is $|\Psi_0\rangle$), we get The state in the first register is

$$. \sum_{j=1}^{N} \sum_{k=0}^{T-1} \alpha_{k|j} \beta_j |k\rangle |u_j\rangle$$

.

Replacing $|k\rangle$ with $\left|\tilde{\lambda}_k\right\rangle$.

$$. \sum_{j=1}^{N} \sum_{k=0}^{T-1} \alpha_{k|j} \beta_j \left|\tilde{\lambda}_k\right\rangle |u_j\rangle Youcandothatwith$$

. We can add a qubit and rotate $\left|\tilde{\lambda}_k\right\rangle$ to get

$$. \sum_{j=1}^{N} \sum_{k=0}^{T-1} \alpha_{k|j} \beta_j \left|\tilde{\lambda}_k\right\rangle |u_j\rangle \left(\sqrt{1 - \frac{C^2}{\tilde{\lambda}_k^2}}|0\rangle + \frac{C}{\tilde{\lambda}_k}|1\rangle\right)$$

.

However, C is $O(1/\kappa)$. Phase estimation allows us to calculate $\left|\tilde{\lambda}_k\right\rangle$ backwards. If phase estimation is perfect, then$\alpha_{k|j} = 1$ if $\tilde{\lambda}_k = \lambda_j$, and 0 otherwise.

Then,

$$\sum_{j=1}^{N} \beta_j |u_j\rangle \left(\sqrt{1 - \frac{C^2}{\lambda_j^2}} |0\rangle + \frac{C}{\lambda_j} |1\rangle \right)$$

If we measure the last qubit, we get last

$$\sqrt{\frac{1}{\sum_{j=1}^{N} C^2 |\beta_j|^2 / |\lambda_j|^2}} \sum_{j=1}^{N} \beta_j \frac{C}{\lambda_j} |u_j\rangle$$

This corresponds to $|x\rangle = \sum_{j=1}^{n} \beta_j \lambda_j^{-1} |u_j\rangle$. If we measure with M, we get $\langle x|M|x\rangle$.

6 Conclusion

Solving the linear system of equations $A\vec{x} = \vec{b}$ is an important problem in any field[1], and it is not necessary to know $\vec{x}$ itself, but if we want to find a quantity related to $\vec{x}$, such as $\vec{x}^{\dagger} M$, we can apply quantum phase estimation to find it probabilistically. The success probability depends on the condition number κ. When the state $|b\rangle$ corresponding to $\vec{b}$ is ready, the computational step to find $|x\rangle$ can be computed on a polynomial time scale of $\log(N), \kappa$. This is an exponential speedup compared to the fastest known algorithm on a classical computer.

7 Reference

References

[1] Aram W. Harrow, Avinatan Hassidim, and Seth Lloyd. Quantum algorithm for linear systems of equations. Physical Review Letters, 103(15):1–8, 2009.

[2] Dominic W. Berry, Graeme Ahokas, Richard Cleve, and Barry C. Sanders. Efficient quantum algorithms for simulating sparse hamiltonians. Communications in Mathematical Physics, 270(2):359–371, 2007.

[3] Lov Grover and Terry Rudolph. Creating superpositions that correspond to efficiently integrable probability distributions. pages 1–2, 2002.