Sensitivity and counterfactual analysis of structural models

Tamara Broderick, Ryan Giordano,
Jan-Christian Huetter, Yaroslav Mukhin

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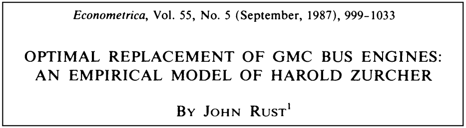
Econometrica, Vol. 55, No. 5 (September, 1987), 999-1033

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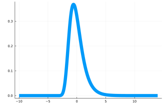

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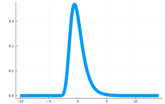

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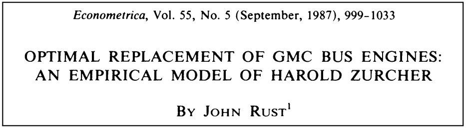
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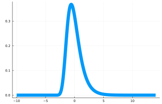

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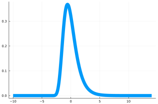

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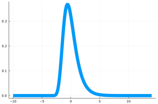

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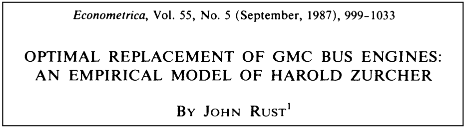
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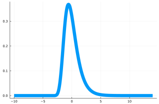

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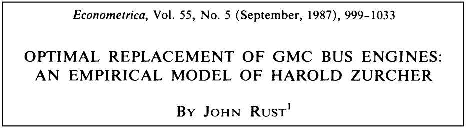

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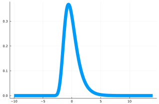

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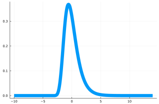

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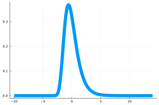

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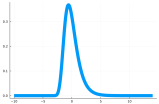

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Example: Forecast demand for a new product, an electric bus, with Rust87.

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Observation: forecast depends on the distributional assumption F.

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$$\pi_{d,x}(\theta) = \begin{cases} -\theta_0 \cdot x & \text{if } d = 0 \\ -\theta_1 & \text{if } d = 1 \end{cases}$$

parameter $\theta = (\theta_0, \theta_1)$, mileage x, decision $d \in \{0\text{:maintain}, 1\text{:replace}\}$.

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Assumption: latent state $U_d \overset{\text{iid}}{\sim} F$; solve for surplus:

$$V_x(\theta, F) = \mathrm{E}^F \max_{d \in \{0,1\}} \left\{ \pi_{d,x}(\theta) + U_d + \beta \mathrm{E}\left[V_{x'} | x, d \right] \right\}$$

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Counterfactual change in surplus:

$$k = w^\top \left(\tilde{V}\left(\hat{\theta}(F), F\right) - V\left(\hat{\theta}(F), F\right) \right)$$

depends on distributional assumption F.

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(iii) update model solution e.g. $\theta(F_\delta)$, $V(F_\delta)$ and $\mathrm{P}_{d|x}(F_\delta)$ without resolving the model;
(iv) repeat.

How to pose and solve the local problem?
How to integrate the local solutions?
How to update F_δ?
How to update $\theta(F), V(F)$, etc?

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- Influence function solves the local problem:
$$\text{const} \cdot \mathcal{I}_k(F) = \arg\max_{\|g\|_\rho\leq\delta} Dk_F[g]. \tag{1*}$$

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- Can impose normalizations on (F_δ), e.g. mean is 0, variance is 1, via normalizations on the right-hand side of equation (2).

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is the linear update in the manifold parameterization of the space.

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Solution: use influence functions $\mathcal{I}_k$ and $\mathcal{I}_\theta$ to form linear Taylor approximations to changes along the least favorable path:

$$\begin{aligned} \theta(F_{h+\epsilon}) &\approx \theta(F_h) + \epsilon \cdot D\theta_{F_h}\left[\mathcal{I}_k(F_h)\right] \\ &\approx \theta(F_h) + \epsilon \cdot \left\langle \mathcal{I}_k(F_h), \mathcal{I}_\theta(F_h) \right\rangle_{F_h}. \end{aligned}$$

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Compute exact change on $(F_t)_{0\le t\le h}$ via fundamental theorem of calculus:

$$\begin{aligned}\theta(F_h) - \theta(F_0) &= \int_0^h \Big\langle \mathcal{I}_k(F_t), \mathcal{I}_\theta(F_t) \Big\rangle_{F_t} \, \mathrm{d}t \\ &\approx \sum_j \Big\langle \mathcal{I}_\theta(F_{j\cdot\epsilon}), \mathcal{I}_\theta(F_{j\cdot\epsilon}) \Big\rangle_{F_{j\cdot\epsilon}}.\end{aligned}$$

Greedy algorithm III: influence function calculation

Question: how to compute the influence function?

Recall that score of a path F_h is the derivative of log-density of the path:

$$g(u) = \frac{d}{dh}_{|h=0} \log f_h(u).$$

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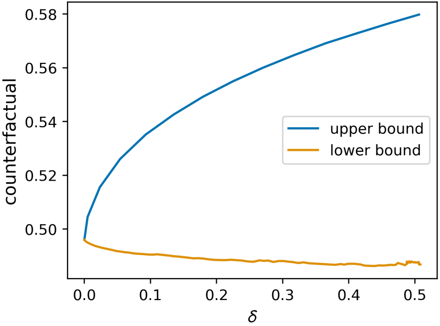
Rust87 example: numerical experiment I

Sensitivity analysis to the Gumbel assumption for counterfactual change in surplus (cost) of a switch from GMC diesel bus to electric bus.

Bounds on the counterfactual change in surplus over a δ ball of probability distributions around the Gumbel distribution of cost shocks in Rust model.



Note: preliminary numerical results using partial influence functions that hold θ fixed and allow $F \mapsto V(\theta, F)$ to vary.

Rust87 example: numerical experiment II

Least favorable distributions:

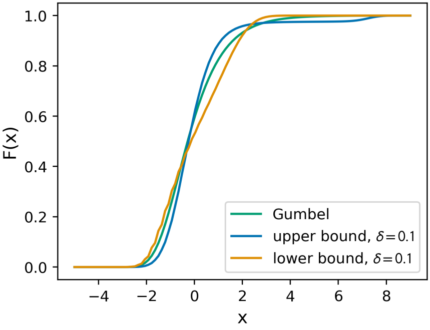


Figure: $\delta = 0.1$

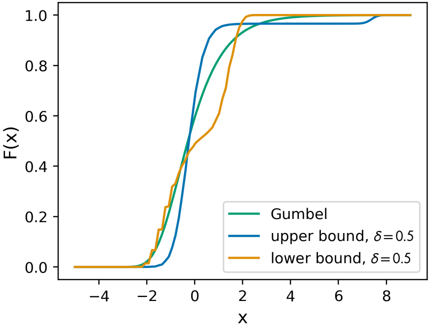


Figure: $\delta = 0.5$

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(i) structural models with general smooth dependence on F

(ii) perturbations over neighborhoods of a general metric, e.g. Wasserstein

(iii) not solving model under any F other than the one you have assumed

(iv) automatable†

(v) Limitation: no guarantees to find the global max (as with any greedy). Probably a bigger problem for estimation than sensitivity analysis.

Questions, comments, feedback are very welcome and highly appreciated!

This presentation is based on:

Y. Mukhin. "Sensitivity of regular estimators". In: *arXiv:1805.08883* (2018)
Y. Mukhin. "Counterfactual analysis of differentiable functionals". In: *MIT Thesis* (2019)

Joint work in progress with Tamara Broderick, Ryan Giordano and Jan-Christian Huetter.