Cosmological Perturbations, from Quantum Fluctuations to Large Scale Structure

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Abstract: Classical perturbation theory is developed from the 3+1 form of the Einstein equations. A somewhat unusual form of the perturbation equations in the synchronous gauge is recommended for carrying out computations, but interpretation is based on certain hypersurface-invariant combination of the variables. The formalism is used to analyze the origin of density perturbations from quantum fluctuations during inflation, with particular emphasis on dealing with "double inflation" and deviations from the Zel'dovich spectrum. The evolution of the density perturbations to the present gives the final density perturbation power spectrum, whose relationship to observed large scale structure in discussed in the context of simple cold-dark-matter biasing schemes.

Keywords: cosmology, inflation, perturbations, biasing, large scale structure.

Contents

1 Introduction

The concept of inflation, as originally proposed by Guth [1] and subsequently elaborated by Linde [2–6] and Albrecht and Steinhardt [7], among others, has had an overwhelming impact on theoretical approaches to cosmology, despite the fact that there is still no specific model which combines convincing particle physics with an acceptable cosmology. While inflation was invoked to explain the large scale homogeneity of the universe as well as to avoid embarrassingly large numbers of such relics of the early universe as magnetic monopoles, it was soon discovered to contain a mechanism for generating inhomogeneities on a wide range of length scales [8–11]. Quantum fluctuations in scalar fields present during inflation, and in particular in the inflaton field itself freeze out and become classical density perturbations.

The Fourier power spectrum of these perturbations naturally approximates the Harrison-Zel'dovich spectrum, which was proposed on general astrophysical grounds by Harrison [12] and Zel'dovich [13]. The *primordial* amplitude of the perturbations, in a sense to be defined precisely later, is roughly independent of wavenumber over all scales relevant to observations in the present universe. The value of this amplitude was too large in the original "new" inflation models, in which the inflaton field was a Higgs field in a grand unified theory (GUT), but other models with more weakly coupled scalar fields were soon devised which allowed adjusting the amplitude to the desired value. Steinhardt and Turner [14] give the general prescription for a successful inflation model. See Linde [15] and Turner turner:1985pr for overall reviews.

Inflation ends when the energy density stops being dominated by the potential energy of the inflaton field. The scalar field starts to oscillate in time, and the oscillations decay into particles and radiation which thermalize into the heat bath of the standard big bang. The scalar field perturbations become adiabatic density perturbations. On comoving scales appropriate to present large scale structure the perturbations are dynamically inert for many e-foldings of expansion. After they come back into the horizon in the more recent universe the perturbations evolve in a way which depends on the nature of the dark matter. Except possibly for decaying particle scenarios [16], the shape of the Fourier power spectrum of the density perturbations is largely fixed by a redshift of 1000, after the universe is matter-dominated and hydrogen has recombined. The amplitude of the density perturbation field grows until the perturbations become nonlinear and structures start to separate out from the Hubble flow.

In cold-dark-matter (CDM) models structure forms first on sub-galactic scales and then on successively larger scales, while in hot-dark-mater (HDM) models supercluster scales are the first to go nonlinear [17–19]. While numerical simulations (M. Davis, this volume) are necessary to follow the nonlinear gravitational clustering in detail, in CDM the sites of galaxy formation can plausibly be related to peaks in the linear density perturbation field. On sufficiently large scales, such as those being probed by observations of large scale streaming velocities (A. Dressler, this volume), linear perturbation theory should have some validity even today. The most direct information about the linear perturbation spectrum will come from observations of anisotropies in the cosmic microwave background (CMB), once these have been unambiguously detected.

My lectures begin with the classical theory of scalar cosmological perturbations [20, 21], presented from the point of view of the 3+1 formalism of general relativity. The proper role of gauge invariance [22] in physically interpreting the results is emphasiszed, but the synchronous gauge simplifies the dynamics of multi-component models of the matter. Potential numerical problems with the usual formulation of the synchronous gauge perturbation equations [23] can be avoided by using a different combination of the Einstein equations.

The evolution of zero-point quantum fluctuations of a scalar field into classical perturbations as the wavelengths of the Fourier modes become larger than the Hubble radius during inflation is examined in detail, with particular attention to computational techniques which are valid even when the effective mass of the scalar field is comparable to the expansion rate and/or the expansion rate is varying rapidly between phases of inflation, as in models for generating non-scale-invariant density perturbations from inflation [24, 25]. A gauge-invariant measure of the amplitude of adiabatic perturbations introduced by Bardeen, Steinhardt, and Turner [11] is used to extrapolate the amplitude of the perturbations until they come back within the horizon in the Friedmann epoch.

Transfer functions which relate present to primordial perturbation amplitudes are defined and accurate numerical formulas given for standard CDM and HDM models. Some of the basic concepts behind the quasi-analytic statistical theory of large scale structure in CDM models [26] are reviewed. Galaxies are identified with high peaks in the density perturbation field, treated as a Gaussian random field. Some newer results from applying the statistical techniques to current questions regarding the "Great Attractor" model of large scale streaming motions [27] and the origin of the peculiar acceleration of the Local Group [27, 28] are mentioned briefly.

2 Cosmological Perturbation Formalism

Gauge-invariant perturbation theory was introduced to Cosmology by Bardeen [22], and has since been rather widely applied (see [29] for an elaborate review). While a useful tool, gauge-invariance in itself does not remove all ambiguity in physical interpretation, and the form of the perturbation equations most often taken from [22] can cause numerical problems in some situations. The approach here is to insist on invariance of all variables under *spatial* gauge transformations, in accord with the spatial homogeneity and isotropy of the Robertson-Walker background, but to deal explicitly with the true physical ambiguity in the choice of constant-t spacelike hypersurfaces in the context of the 3+1 formalism of general relativity.

2.1 3+1 Formalism

This review follows [30] and [31]. Spacetime is considered an ordered sequence of 3-geometries (spacelike hypersurfaces). The hypersurfaces are labelled by the time coordinate t. Points within each hyper surface are distinguished by spatial coordinate labels x^i, $i = 1, 2, 3$. The *intrinsic* geometry of a hyper surface is described by a spatial 3-metric h_{ij}, whose inverse is

h^{ij}. Denote the covariant derivative in the 3-space by D_i. The spatial Ricci tensor is R_{ij}, and the spatial scalar curvature is $R \equiv h^{ij} R_{ij}$.

The lapse function N measures the proper time interval $\Delta\tau$ between hypersurfaces at t and $t + \Delta\tau$ along a normal worldline,

$$\Delta\tau = N \delta t. \tag{2.1}$$

Spatial coordinates may vary in any continuous way from one hyper surface to the next, as described by the shift vector N^i. The change in spatial coordinate labels of a normal worldline from a hyper surface at t to a hyper surface at $t + \Delta t$ is

$$\Delta x^i = N^i \Delta t. \tag{2.2}$$

This N_i is the coordinate 3-velocity of an observer "at rest" in the hyper surface and has the opposite sign from the N_i in [30].

The spacetime metric tensor $g_{\alpha\beta}$ in terms of h_{ij}, N, and N^i is

$$\begin{aligned} g_{00} &= -N^2 + N^i N_i, \\ g_{0i} &= -N_i, \\ g_{ij} &= h_{ij}. \end{aligned} \tag{2.3}$$

The inverse spacetime metric tensor $g_{\alpha\beta}$ is

$$\begin{aligned} g^{00} &= -\frac{1}{N^2}, \\ g^{0i} &= -\frac{N_i}{N^2}, \\ g^{ij} &= h^{ij} - \frac{N^i N^j}{N^2}. \end{aligned} \tag{2.4}$$

The unit normal 4-vector to the constant-t hyper surface has components

$$n^\alpha = N^{-1} \left(1, N^1, N^2, N^3\right), \quad n_\alpha = \left(-N, 0, 0, 0\right). \tag{2.5}$$

A spacetime covariant derivative is denoted by a semi-colon, an ordinary partial derivative by a comma.

The *extrinsic* curvature tensor K_{ij} describes the local bending of the hypersurfaces as they are stacked together to make up the 4-d spacetime, how the normal 4-vector changes in going from one point to another in the hyper surface. It is a tensor in the hyper surface, related to the spatial metric tensor by

$$h_{ij,0} + N^k h_{ij,k} + N^k{}_{,i} h_{kj} + N^k{}_{,j} h_{ik} = -2N K_{ij}. \tag{2.6}$$

The trace $K \equiv K^i{}_i$ is minus the rate of expansion of the normal worldlines.

The tensor which projects from the spacetime into the 3-space of the constant-t hypersurface is

$$P_{\alpha\beta} = g_{\alpha\beta} + n_\alpha n_\beta, \quad P^0{}_0 = 0, \quad P^i{}_j = \delta^i{}_j. \tag{2.7}$$

This tensor and n_α can be used to make a 3+1 split of any spacetime tensor. For instance, the energy-momentum tensor $T^\alpha\beta$ decomposes into the energy density 3-scalar

$$E = n_\alpha n_\beta T^{\alpha\beta} = N^2 T^{00}, \tag{2.8}$$

the momentum-density 3-vector

$$J_i = -n_\alpha T^{\alpha\beta} P_{\beta i} = N T^0{}_i, \tag{2.9}$$

and the stress tensor

$$S_{ij} = P_{i\alpha} T^{\alpha\beta} P_{\beta j} = T_{ij}. \tag{2.10}$$

Local conservation of energy-momentum $T^{\alpha\beta}{}_{;\beta} = 0$ becomes conservation of energy

$$E_{,0} + N^k E_{,k} = NKE + NK^{ij} S_{ij} - N^{-1} D_i (N^2 J^i) \tag{2.11}$$

and conservation of momentum

$$J_{i,0} + N^k J_{i,k} + N^k{}_{,i} J_k = NKJ_i - \left(E\delta_i^k + S_i{}^k \right) N_{,k} - ND_k S_i{}^k. \tag{2.12}$$

The Einstein equations split up into four (one scalar and one 3-vector) initial value constraint equations, the energy constraint

$$R - K_{ij} K^{ij} + K^2 = 16\pi G E \tag{2.13}$$

and the momentum constraint

$$D_j K^j{}_i - D_i K = 8\pi G J_i, \tag{2.14}$$

and six "dynamical" equations for the time derivatives of the K_{ij},

$$\begin{aligned} K^i{}_{j,0} + N^k K^i{}_{j,k} - N^i{}_{,k} K^k{}_j + N^k{}_{,j} K^i{}_k = -D^i D_j N \\ + N \left[R^i{}_j + K K^i{}_j - 8\pi G S^i{}_j + 4\pi G \delta^i_j (S^k{}_k - E) \right]. \end{aligned} \tag{2.15}$$

Eq. (2.6) evolves the h_{ij}, given the K_{ij}.

There are no evolution equations for N and N^i, since these only reflect the arbitrary choice of time coordinate and spatial coordinates, respectively.

The choice of a time coordinate is equivalent to *hyper surface condition* which specifies how the spacetime is to be sliced up into spacelike hypersurfaces and how these hypersurfaces are to be labelled. *Synchronous* hypersurfaces are separated by uniform intervals in proper time along the hyper surface normals, starting from some initial hypersurface, and have $N \equiv 1$.

If the hyper surface normals should start to converge ($K > 0$) somewhere a coordinate singularity develops in a finite proper time, but for small perturbations away from a homogeneous expanding universe there is no problem. The standard choice in numerical cosmology [32], when inhomogeneities are not assumed small, has been *uniform-expansion* hypersurfaces, for which $D_i K \equiv 0$. Then the trace of Eq. (2.15) becomes an elliptic equation for N,

$$\Delta N \equiv D^k D_k N = N \left[K^{ij} K_{ij} + 4\pi G(E + S^k{}_k) \right] - K_{,0}. \tag{2.16}$$

K can be any explicit function $K(t)$ such that t increases monotonically toward the future.

I now apply this formalism to the theory of small perturbation away from a homogeneous, isotropic cosmological background.

2.2 The Robertson-Walker Background

The spacelike hypersurfaces have maximal symmetry [33], and the spatial metric can be put in the form

$$h_{ij} = a(t)^2 f_{ij}, \quad f_{ij} dx^i dx^j = dr^2 + \left(\frac{1}{\sqrt{C}} \sin(\sqrt{C} r) \right)^2 (d\theta^2 + \sin^2\theta d\phi^2). \tag{2.17}$$

The curvature constant C is positive for a spatially closed universe and negative or zero for a spatially open universe. The time evolution is governed only by the scale factor a. The spatial curvature of the background is

$$R_{ij} = \frac{2C}{a^2} h_{ij} = 2C f_{ij}, \quad R = \frac{6C}{a^2}. \tag{2.18}$$

The scaling of the time coordinate is fixed by the choice of the background lapse function $N_0(t)$. If $N_0 = 1$, then t is proper time. If $N_0 = a$, then t is called conformal time. The spacetime line element is

$$ds^2 = -N_0^2 dt^2 + a^2 f_{ij} dx^i dx^j. \tag{2.19}$$

The background expansion rate, or “Hubble constant", is

$$H(t) \equiv -\frac{K(t)}{3} = \frac{\dot{a}}{N_0 a}, \tag{2.20}$$

and the corresponding proper distance H^{-1} will be called the proper Hubble radius. The ‘dot’ denotes a coordinate time derivative. The coordinate distance which light travels per unit e-folding of expansion,

$$r_{\rm H} \equiv \frac{1}{aH} = \frac{N_0}{\dot{a}}, \tag{2.21}$$

will be called the coordinate Hubble radius, or (*effective*) *horizon*. It is the largest comoving scale on which real dynamics, generated by causal interactions, can take place. The limit of causal communication from or to the origin, integrated over time from the initial singularity, is the *particle horizon*,

$$r_{\rm p} \equiv \int_0^t \frac{N_0}{a} dt'. \tag{2.22}$$

The background energy-momentum tensor is

$$E = E_0(t), \quad J_i = 0, \quad S_{ij} = P_0(t) h_{ij}. \tag{2.23}$$

The energy initial value equation determines the scale factor from the energy density,

$$H^2 = \left(\frac{\dot{a}}{N_0 a}\right)^2 = \left(\frac{8\pi G}{3}\right) E_0 - \frac{C}{a^2}, \tag{2.24}$$

local energy conservation determines the energy density from the pressure,

$$\frac{\dot{E}_0}{N_0} = -3H(E_0 + P_0), \tag{2.25}$$

and the evolution equation (Eq. (2.15) with $i = j$),

$$\frac{\dot{H}}{N_0} + H^2 = \frac{(aH)'}{N_0 a} = -\frac{4\pi G}{3}(E_0 + 3P_0), \tag{2.26}$$

is redundant. Eqs. (2.23) and (2.25) combine to give

$$\frac{\dot{H}}{N_0} = -4\pi G(E_0 + P_0) + \frac{c}{a^2}. \tag{2.27}$$

Note from Eq. (2.26) that the dominant energy condition $E_0 + P_0 \geq 0$ corresponds to $\dot{H} \leq 0$ if $C = 0$. From Eqs. (2.21) and (2.26), $P_0 > -E_0/3$ implies $\dot{r}_{\mathrm{H}} > 0$ and decelerated expansion, what I will call a Friedmann epoch, and $P_0 < -E_0/3$ implies the opposite, what I will call an inflation epoch.

2.3 The Perturbations

General perturbation on homogeneous, isotropic backgrounds decompose into three disjoint class: *scalar*, *vector*, and *tensor*.

Tensor perturbations are constructed from traceless, divergenceless 3-tensors. Physically they represent gravitational waves, and are a potential cause of microwave background anisotropy. Upper limits on their amplitude are important constraints on inflation models [34, 35].

Vector perturbations are constructed from divergenceless 3-vectors and their covariant derivatives. They correspond to vortical motions of the cosmological “fluid", but these tend to die out in an expanding universe. No one has come up with a plausible mechanism for generating large enough primordial amplitudes for them to have any dynamical significance by the time of galaxy formation.

Scalar perturbations are constructed from 3-scalars, their covariant derivatives, and the background spatial metric. These are the ones directly involved with structure formation in linear perturbation theory, since they encompass density perturbations and everything coupled to them. From now on only scalar perturbations will be considered.

The most general scalar perturbation to the geometry, *i.e.*, the spacetime metric tensor, involves four scalar functions of t and the x^i, which I will denote by α, β, φ, and γ. In the 3+1 formalism these describe the perturbations in the lapse function,

$$N = N_0(1+\alpha), \tag{2.28}$$

in the shift vector,

$$N_i = a^2 D_i \beta, \tag{2.29}$$

and in the spatial metric tensor,

$$h_{ij} = a^2 \left(f_{ij} + 2\varphi f_{ij} + 2D_i D_j \gamma\right) \tag{2.30}$$

The intrinsic Ricci curvature tensor of a constant-t hypersurface only depends on φ, with[1]

$$R^i{}_j = \frac{2C}{a^2}\delta^i{}_j - \left[\Delta\varphi + \frac{4C}{a^2}\varphi\right]\delta^i{}_j - D^i D_j \varphi. \tag{2.31}$$

The extrinsic curvature tensor, calculated from Eq. (2.6), is[2]

$$K^i{}_j = -H\delta^i{}_j - \left(\frac{\dot{\varphi}}{N_0} - H\alpha\right)\delta^i{}_j - \frac{a^2}{N_0} D^i D_j(\dot{\gamma} + \beta). \tag{2.32}$$

It is convenient to define two additional scalars in terms of the basic four,

$$\chi \equiv \frac{a^2}{N_0}(\dot{\gamma} + \beta) \tag{2.33}$$

and

$$\kappa \equiv -3\left(\frac{\dot{\varphi}}{N_0} - H\alpha\right) - \Delta\chi. \tag{2.34}$$

Note that κ is the perturbation in the trace of the extrinsic curvature, K, and χ generates the trackless part of $K^i{}_j$, *i.e.*, the shear in the normal worldlines.

The perturbations in the overall energy-momentum tensor are generated by four more independent scalars ε, ψ, π, and σ, with

$$E = E_0 + \varepsilon, \tag{2.35a}$$

$$J_i = D_i \psi, \tag{2.35b}$$

$$S^i{}_j = (P_0 + \pi)\delta^i{}_j + D^i D_j \sigma - \frac{1}{3}\Delta\sigma\delta^i{}_j. \tag{2.35c}$$

The description as absolute perturbations, as opposed to the relative ones of [22], more easily accommodates complex models of the matter.

[1] $\Delta \equiv D^i D_i$

[2] N.B., $H \equiv \dot{a}/(N_0 a)$. The first term is from background. The second and third terms are from linear perturbation.

2.4 Gauge Transformations and Gauge Invariance

A gauge transformation can be thought of as a coordinate transformation induced by a change in the correspondence between the physical perturbed spacetime and the fictitious background spacetime introduced to define the perturbations. A given point in the physical spacetime with coordinates t, x^i acquires in the gauge transformation new coordinates $t+T(t,x^i), x^i+L^i(t,x^i)$ as the point it is matched with in the background spacetime changes. If the displacement in the background spacetime is ϵ^α, the new metric tensor for the physical spacetime is

$$\tilde{g}_{\alpha\beta} = g_{\alpha\beta} - \epsilon_{\alpha;\beta} - \epsilon_{\beta;\alpha}. \tag{2.36}$$

In the context of scalar perturbations the *spatial* gauge transformation generator L^i must be the gradient of a scalar. If

$$L^i \equiv f^{ij} D_j \lambda, \tag{2.37}$$

then the only perturbation scalars to change are β and γ,

$$\tilde{\beta} = \beta + \dot{\lambda}, \quad \tilde{\gamma} = \gamma - \lambda. \tag{2.38}$$

Since the background 3-space is homogeneous and isotropic, the perturbations in all physical quantities must in fact be *gauge invariant* under purely spatial gauge transformations. Note that the perturbation in the intrinsic curvature is independent of λ, since φ is unchanged, and so is the perturbation in the extrinsic curvature tensor, since the changes in β and γ cancel each other in χ. The perturbations in the Einstein equations depend on β and γ only through the combination χ.

On the other hand, the background does evolve in time, so perturbations in physical quantities, even quantities which are spacetime scalars in the physical spacetime, will in general *not* be gauge invariant under time gauge transformation. For instance, the change in the background energy density is $\dot{E}_0 T$, so the perturbation ε changes according to

$$\tilde{\varepsilon} = \varepsilon - \dot{E}_0 T, \tag{2.39}$$

even though the total physical energy density is unchanged by the gauge transformation to first order. The time dependence of the scale factor a enters into the time gauge transformations of the metric tensor perturbations. The only physical quantities which are completely gauge invariant are those which are zero or time independent in the background.

The time gauge transformations of the physically significant (spatially gauge invariant) scalars defined above are straightforwardly found to be:

$$\tilde{\alpha} = \alpha - \frac{(\dot{N_0 T})}{N_0}; \tag{2.40a}$$

$$\tilde{\varphi} = \varphi - H(N_0 T); \tag{2.40b}$$

$$\tilde{\chi} = \chi - (N_0 T); \tag{2.40c}$$

$$\tilde{\kappa} = \kappa + 3\dot{H}T + N_0 \Delta T; \tag{2.40d}$$

$$\tilde{\varepsilon} = \varepsilon + 3H(E_0 + P_0)(N_0 T); \tag{2.40e}$$

$$\tilde{\psi} = \psi + (E_0 + P_0)(N_0 T); \tag{2.40f}$$

$$\tilde{\pi} = \pi - \dot{P}_0 T; \tag{2.40g}$$

$$\tilde{\sigma} = \sigma. \tag{2.40h}$$

Many gauge-invariant combinations of these scalars can be constructed, but for the most part they have no physical meaning independent of a particular time gauge, or hyper surface condition. For instance, Φ_H and Φ_A of [22] are $\varphi - H\chi$ and $\alpha - \dot{\chi}/N_0$, respectively, and are interpretable as a curvature perturbation and a lapse function perturbation only if the hypersurfaces are chosen to make $\chi = 0$. The value of ε on scales larger than the horizon r_H is physically ambiguous because causal measurements cannot compare values of E and determine a background level E_0.

2.5 Perturbation Equations

It is straightforward to substitute the expressions for the perturbed metric, intrinsic and extrinsic curvature, and energy-momentum tensor into the 3+1 field equations and linearize to find equations for the perturbation scalars. The results are:

1. From Eq. (2.6) for the evolution of the spatial metric,

$$\frac{\dot{\varphi}}{N_0} = H\alpha - \frac{\kappa}{3} - \frac{\Delta\chi}{3}; \tag{2.41}$$

2. From Eq. (2.13), the energy initial value constraint,

$$\left(\Delta + \frac{3C}{a^2}\right)\varphi + H\kappa = -4\pi G\varepsilon; \tag{2.42}$$

3. From Eq. (2.14), the momentum initial value constraint,

$$\left(\Delta + \frac{3C}{a^2}\right)\chi + \kappa = -12\pi G\psi; \tag{2.43}$$

4. From the trace of Eq. (2.15) for the evolution of the extrinsic curvature,

$$\frac{\dot{\kappa}}{N_0} + 2H\kappa = -\left(\Delta + \frac{3\dot{H}}{N_0}\right)\alpha + 4\pi G(\varepsilon + 3\pi); \tag{2.44}$$

5. From the traceless part of Eq. (2.15),

$$\left(D^i D_j - \frac{1}{3}\delta^i{}_j \Delta\right)\left(\frac{\dot{\chi}}{N_0} + H\chi - \alpha - \varphi - 8\pi G\sigma\right) = 0. \tag{2.45}$$

Actually, Eq. (2.44) is the result of using Eq. (2.13) to eliminate R from the trace of Eq. (2.15).

The Einstein equations must be supplemented by equations of motion for the matter. The matter dynamics is necessarily consistent with local conservation of energy and momentum, Eqs. (2.11) and (2.12), but in general involves too many degrees of freedom for Eqs. (2.11) and (2.12) to be sufficient. Whatever non-gravitational variables and evolution equations are used, as appropriate to the particular situation, the overall perturbation in the energy-momentum tensor as embodied in the scalar ε, ψ, *etc.*needs to be calculated and fed as input into the Einstein equations. The linearized versions of Eqs. (2.11) and (2.12)

$$\frac{\dot{\varepsilon}}{N_0} = -3H(\varepsilon + \pi) + (E_0 + P_0)(\kappa - 3H\alpha) - \Delta\psi, \tag{2.46}$$

$$\frac{\dot{\psi}}{N_0} = -3H\psi - (E_0 + P_0)\alpha - \pi - \frac{2}{3}\left(\Delta + \frac{3C}{a^2}\right)\sigma. \tag{2.47}$$

The standard procedure is to reduce the perturbation equations to ordinary differential equations in time by expanding the perturbations scalars in spatial harmonics, mode functions $Q_k(x^i)$ which satisfy the Helmholtz equation in the background 3-space,

$$\Delta Q_k + \frac{k^2}{a^2} Q_k = 0. \tag{2.48}$$

The scale factor a factors out of this equation, since the Laplacian operator Δ scales as $1/a^2$, ad the Q_k are time independent. If the background curvature constant $C = 0$, as will be assumed in the applications to be discussed latter, the Q_k are just plane waves $\exp(ik_jx^j)$, and $k^2 = k_jk^j$. The magnitude of the physical wave vector is k/a. For individual Fourier modes, all spatial derivatives can be eliminated from Eqs. (2.41)–(2.47) using Eq. (2.48). The initial value Eqs. (2.42) and (2.43) become algebraic relations.

2.6 Solution Strategies

The perturbed Einstein equations are five equations for the four unknown α, φ, κ, and χ. The gauge freedom in the choice of time coordinate, *i.e.*, the hypersurface condition which specifies the way the spacetime is sliced into space-like hypersurfaces, allows one of these scalars to be fixed arbitrarily. Then there are five equations for three functions, and two of the equations are redundant.

The traditional choice of gauge [21, 23, 33] has been the *synchronous* gauge, which sets $\alpha \equiv 0$. There is a residual gauge freedom in the choice of initial constant-t hypersurface which shows up as a "gauge mode" when solving the perturbations equations. Instead of being able to solve algebraically for the metric perturbations, as in some other gauges, one time integration with one free constant of integration must be performed. There is no one-to-one relation between the values of the perturbation scalars and the physical state of the perturbation. I criticized the synchronous gauge on these grounds in [22], but there are compensating advantages. In an expanding background the synchronous gauge is always mathematically well behaved, with no spurious coordinate singularities. Setting α to zero minimizes the presence of metric perturbations in the matter evolution equations, which for complicated models of the matter more than compensates for the slight additional complexity of the Einstein equations. I have now have come to believe that in dealing with such situations as the generation of scalar field perturbations during inflation and their conversion into fluid density perturbations during reheating the synchronous gauge is *the* preferred gauge.

The *uniform-expansion* gauge set $\kappa \equiv 0$. The metric perturbations are determined entirely by elliptic equations (algebraic equations for individual Fourier modes): the two initial value equations give φ and χ, while Eq. (2.44) becomes an elliptic equation for α. The gauge corresponds to a Newtonian gauge when the perturbation is well within the horizon ($k/aH \gg 1$), in that coordinate velocities become Newtonian peculiar velocities. Which κ does appear in the matter evolution equations, *e.g.*, Eq. (2.46), it is not as prominent as α , and not as much is gained by making it zero. The gauge is usually mathematically well behaved, but artificial blips are introduced into solutions if $E_0 + P_0 \to 0$, as can happen for a coherent scalar field, which $k/aH \ll 1$.

The *zero-shear* gauge is defined by $\chi \equiv 0$. This is also a newtonian gauge when the perturbation is within the horizon, and elliptic equations determine the remaining metric perturbations. However, there is no simplification of the matter evolution equations, and as discussed in [22], the metric perturbation amplitudes can be much larger than the true physical amplitude of the perturbations in some situations.

The *comoving* gauge sets $\psi \equiv 0$. When the model of the matter is sufficiently simple (a single scalar field or single perfect fluid), this gauge facilitates finding analytic solutions to the perturbation equations [11, 22]. Eqs. (2.44), (2.46) and (2.47) combine to give a single second-order equation for the time dependence of ε. However, this advantage disappears when the matter is more complex and π is not simply related to ε. Even more seriously, the equation for ε becomes highly singular when $E_0 + P_0 \to 0$, making accurate numerical solutions hard to obtain.

Much has been made by some authors over the last several years of the advantages of using gauge-invariant variables, specifically the variables $\Phi_H = \varphi - H\chi$ and $\Phi_A = \alpha - \dot{\chi}/N_0$ introduced in [22]. However, it is apparent from the gauge transformation Eqs. (2.40) that the perturbation scalars for any of the last three specific gauges mentioned above can be represented by gauge-invariant expressions. For instance, $\varphi + (\Delta + 3\dot{H}/N_0)^{-1}H\kappa$ is gauge-invariant and equals φ in the uniform-expansion gauge, which $\varphi - H\chi$ is φ in the zero-shear gauge, and $\varphi + H\psi/(E_0 + P_0)$ is φ in the comoving gauge. These gauge-invariant expressions are just the time gauge transformations in the gauge in question for an arbitrary initial gauge. The advantages of Φ_H and Φ_A as variables are the advantages of working in the zero-shear gauge, no more and no less, which as remarked above, are not overwhelming.

The moral is that one should work in the gauge that is mathematically most convenient for the problem at hand, which is often the synchronous gauge [36]. The real value f the gauge-invariant expressions is in interpreting the results of the calculations. For example, a “Newtonian" peculiar velocity can be found by taking the gradient of the gauge-invariant expression $[\psi/(E_0 + P_0) + \chi]$, using ψ and χ from a synchronous gauge calculations, since the momentum density of a fluid is $(E+P)$ times the velocity for small velocities and the gauge $\chi = 0$ is a Newtonian gauge for scales smaller than the horizon. Such gauge-invariant expressions are guaranteed to be independent of the gauge mode associated with the initial choice of hypersurface in the synchronous gauge.

Working in the synchronous gauge, one still has to choose which variables to take as the primary integration variables, and which of the redundant set of Einstein equations to solve. Traditionally, Eq. (2.44) is integrated to find κ. The matter evolution equations typically contain κ, but not φ or χ, so apparently this is the most efficient procedure. However, it is also dangerous. As I discuss in the next subsection, the overall amplitude of the perturbation in the geometry and the total energy density of the matter is characterized by a gauge-invariant linear combination of φ and $\varepsilon/(E_0 + P_0)$. If κ is the primary integration viable, then φ is found from Eq. (2.42). If $C = 0$, Eq. (2.42) can be written as

$$\varphi = \left(\frac{aH}{k}\right)^2 \left(\frac{\kappa}{H} + \frac{3}{2}\frac{\varepsilon}{E_0}\right). \tag{2.49}$$

If both κ/H and ε/E_0 are of order $(k/aH)^2\varphi$ when $k/aH \ll 1$, as for a pure "adiabatic" mode in a Friedmann epoch, there is no problem. However, evolution during the inflation epoch typically leaves φ and ε/E_0 roughly comparable in magnitude. Since $k/aH \sim 10^{-22}$ at the end of inflation for the comoving scales of present large scale structure, there must be an extraordinarily delicate cancellation in Eq. (2.49), far beyond what is feasible in a numerical calculation, in order to come even close to the correct value of φ. The "adiabatic mode" is swamped by a "gauge mode".

To keep track of the amplitude of the adiabatic mode in this kind of situation the scalar φ should be the primary integration variable in place of κ. With $C = 0$, as is normally an excellent approximation, Eqs. (2.41) and (2.43) combine to give

$$\frac{\dot{\varphi}}{N_0} = 4\pi G\psi \tag{2.50}$$

in the synchronous gauge. Then from Eq. (2.42),

$$\kappa = \frac{1}{H}\left[\left(\frac{k}{a}\right)^2 \varphi - 4\pi G\varepsilon\right], \tag{2.51}$$

and Eq. (2.43) gives

$$\Delta\chi = \frac{1}{H}\left[4\pi G\left(\varepsilon - 3H\psi\right) - \left(\frac{k}{a}\right)^2 \varphi\right]. \tag{2.52}$$

As long as H does not approach zero or become negative. Eqs. (2.50)–(2.52), when used along with the synchronous gauge matter evolution equations, are numerically well behaved.

The local energy and momentum conservation Eqs. (2.46) and (2.47) can also have problems with a gauge mode which make ε and $H\psi$ individually much larger than the gauge-invariant combination

$$\varepsilon_C \equiv \varepsilon - 3H\psi, \tag{2.53}$$

[36]. To use ε_C as an integration variable in place of ε, combine Eqs. (2.42), (2.46) and (2.47) to get

$$\begin{aligned}\frac{\dot{\varepsilon}_C}{N_0} =& - 3H\varepsilon_C - 4\pi G(E_0+P_0)\frac{\varepsilon_C}{H} + 2H\left(\Delta + \frac{3C}{a^2}\right)\sigma \\ & - (E_0+P_0)\left(\Delta + \frac{3C}{a^2}\right)\left(\varphi + \frac{H\psi}{E_0+P_0}\right).\end{aligned} \tag{2.54}$$

Eq. (2.54) is a comoving gauge equation, written in a gauge-invariant form so it can be used with the synchronous gauge φ and ψ.

2.7 Physical Interpretation Using Gauge-Invariant Variables

Gauge-invariant variables give mathematically unambiguous ways of comparing results obtained in different gauges, but their physical interpretation is not necessarily straightforward, in that it is usually tied to a particular way of slicing the spacetime into hypersurfaces. I know of no way to characterize completely the deviations from homogeneity and isotropy independent of the slicing into spacelike hypersurfaces. Such measures of the amplitude of the perturbation as the fractional energy density perturbation, ε/E_0, the warping of the spacelike hypersurface due to the perturbation in the intrinsic curvature over the distance a/k the curvature is coherent, $\sim (a/k)^2\Delta\phi \sim \phi$, the fractional perturbation in the expansion of they hypersurface normals, $\sim \kappa/H$, the ratio of the shear in the hypersurface normals to the expansion rate, $\sim \Delta\chi/H$, are hypersurface-dependent. The "velocity" of the matter, $\sim (k/a)\psi/(E_0+P_0)$, is relevant for a fluid, but perhaps not for a scalar field. A few physically hypersurface-independent measures of the rate of "shear" of the matter to the expansion rate, $\sim (\Delta\psi/(E_0+P_0)+\Delta\chi)/H$, and hypersurface-independent measures of the amplitude of the stress perturbations,

$$\frac{\pi - (dP_0/dE_0)\varepsilon}{E_0+P_0}, \quad \frac{(\Delta + 3C/a^2)\sigma}{E_0+P_0}, \tag{2.55}$$

do exist. The stress perturbations are not compared with P_0 because P_0 small or zero does not invalidate first-order perturbation theory. The comparison with E_0+P_0 is the dynamically relevant one.

The amplitude of the perturbation with respect to a particular hypersurface condition is the *largest* of the physically relevant relative amplitudes. However, the hypersurfaces associated with a particular hypersurface condition may become strongly warped, most likely when the scale of the perturbation is much larger than the horizon ($k/aH \ll 1$), which can make some of the relative amplitudes artificially large. To get a hypersurface-independent measure of the overall amplitude of the perturbation, take a linear combination of two of the relative amplitudes, one or the other of which is likely to be comparable to the largest face condition. Allowing only measures of the amplitude of the matter perturbations based on the *total* energy-momentum tensor, the essentially unique such quantity is

$$\zeta \equiv \varphi + \frac{1}{3}\frac{\varepsilon}{E_0+P_0}. \tag{2.56}$$

Note the need to compare ε with (E_0+P_0), rather than E_0, in order to make ζ gauge invariant.

Even for multi-component models of the matter, ζ is a good measure of the amplitude of the *adiabatic* mode. When the scale of the perturbation is very large compared with the horizon, each horizon-sized region evolves independently. The adiabatic mode is when the evolution of each of these regions is similar to the background, with the same relation between pressure and energy density. The perturbation is associated with different regions reaching the same stage of evolution (*e.g.*, same values of the physical energy density E) at different proper times. There is no local way of detecting the perturbation. It becomes apparent only when the horizon expands to be the order of the perturbation wavelength in size, $k/aH \sim 1$. For such an adiabatic mode, the synchronous gauge hypersurfaces can be chosen to make all quantities such as $\varepsilon/(E_0 + P_0)$, κ/H, *etc.*, very small, the order of $(k/aH)^2$, leaving only the scalar φ to indicate the large-scale warping of the hypersurfaces.

The quantity ζ is extremely useful, not only as a measure of the amplitude of the perturbation, but because when $k/aH \ll 1$ and at least approximate conditions of adiabaticity are satisfied (the relative stress perturbations of Eq. (2.55) averaged over an e-folding of expansion should be small compared to ζ), ζ becomes independent of time. This feature of ζ was first exploited by [11] to extrapolate the amplitude of scalar field perturbations produced by quantum fluctuations during inflation to the amplitude of density perturbations reentering the horizon during the Friedmann epoch.

Other "conserved" quantities similar to ζ are $\varphi_C = \varphi + H\psi/(E_0 + P_0)$, the comoving curvature perturbation amplitude put forward by [37], and a more *ad hoc* quantity advocated by [38]. Note that the difference between φ_C and ζ is $(1/3)\varepsilon_C/(E_0+P_0)$, which we have already noted tends to be very small when $k/aH \ll 1$. However, neither of these other quantities are reliable as measures of the amplitude of the adiabatic perturbation when a decaying mode is present.

When there are two or more components of the matter contributing to the energy-momentum tensor, additional degrees of freedom are present in the form of *isocurvature* modes, which are invariantly characterized by $\zeta = 0$ initially. As the relative contribution of the different components changes with time, the stresses evolve non-adiabatically, and a non-zero ζ may be generated.

Examples of the use of ζ to define initial conditions and extrapolate amplitude will be given later in these lectures.

3 Classical Perturbations from quantum fluctuations

3.1 Standard Inflation Models

To establish the framework in which I will discuss the generation of perturbation during inflation, standard inflation models will be briefly reviewed. For a more thorough discussion, see [39] (this volume). The units in this section will be $\hbar = c = 1$. Energies will be scaled explicitly by the Planck mass, $M_p \equiv G^{-1/2} = 1.2 \times 10^{19}$ GeV.

The basic ingredient of standard inflation models is a scalar field Φ, minimally coupled to gravity, with a combined Lagrangian for the scalar field and gravity

$$L = \int d^4x\sqrt{-g}\left[\frac{M_p^2}{16\pi}\,{}^{(4)}R - \frac{1}{2}g^{\mu\nu}\Phi_{,\mu}\Phi_{,\nu} - V(\Phi)\right]. \tag{3.1}$$

The spatially uniform background scalar field is treated as a classical field, with the equation of motion

$$\Box\,\Phi = \frac{dV}{d\Phi}, \tag{3.2}$$

which in the Robertson-Walker spacetime becomes

$$\ddot{\Phi} + 3H\dot{\Phi} + \frac{dV}{d\Phi} = 0 \tag{3.3}$$

with $N_0 = 1$ and t the proper time. The scalar field starts on a potential "hill" and rolls down toward a "valley " with "frictional drag" from the expansion of the universe.

The energy density of the scalar field is

$$E_0 = \frac{1}{2}\dot{\Phi}^2 + V(\Phi), \tag{3.4}$$

while

$$P_0 = \frac{1}{2}\dot{\Phi}^2 - V(\Phi). \tag{3.5}$$

Inflation requires $\dot{\Phi} \ll V$, so $P_0 \simeq -E_0$. In this limit the *slow roll down approximation* is valid. In Eq. (3.3) the $\ddot{\Phi}$ term is small compared with the other two terms, and

$$\dot{\Phi} \simeq -\frac{1}{3H}\frac{dV}{d\Phi}. \tag{3.6}$$

The potential must be flat enough that the slow roll down lasts for more than about 60 Hubble times, and the requirement that the density perturbations generated from quantum fluctuations in Φ not be too large is much more stringent. The general prescription for inflation is given by [14].

The precise form of V varies with the type of inflation, and is of no fundamental importance. In a simple example of "chaotic" inflation [4] the potential is a polynomial in Φ,

$$V(\Phi) = \frac{1}{2}\mu^2\Phi^2 + \frac{1}{4}\lambda\Phi^4, \tag{3.7}$$

and Φ must start rolling down from an initial value much larger than M_p, toward $\Phi = 0$. In the slow roll down approximation, with the $\lambda\Phi^4$ term dominant,

$$\frac{\dot{\Phi}^2}{V} \simeq \frac{M_p^2}{24\pi}\left(\frac{1}{V}\frac{dV}{d\Phi}\right)^2 \simeq \frac{2}{3\pi}\left(\frac{M_p}{\Phi}\right)^2, \tag{3.8}$$

and inflation lasts as long as $\Phi \gtrsim M_p/2$. Once $H < \mu$, Φ oscillates in time. The energy-momentum tensor averaged over these oscillations is like that of an incoherent fluid, with energy density falling as a^{-3}.

Eventually the oscillation energy must be thermalized. An ad hoc way of accomplishing this is to add a dissipation term $\Gamma\dot{\Phi}$ to Eq. (3.3), so

$$\ddot{\Phi} + (3H + \Gamma)\dot{\Phi} + \frac{dV}{d\Phi} = 0, \tag{3.9}$$

and thermal energy density and pressure term E_r and $P_r = E_r/3$ to Eqs. (3.4) and (3.5). Overall conservation of energy then requires that

$$\dot{E_r} = \Gamma\dot{\Phi}^2 - 4HE_r. \tag{3.10}$$

The universe becomes radiation dominated once H becomes less than Γ.

3.2 Quick and Dirty Look at Quantum Fluctuations

To illustrate the basic principles behind the generation of cosmological perturbations from quantum fluctuations in the inflation field Φ, I first give a hand-waving derivation, ignoring factors of two. For a more thorough consideration of the basic issues see, for example, [40]. The first step is to make a kind of mean field approximation to Phi as a quantum field. This is necessary to avoid dealing with all the complexity of quantum gravity, since Φ is coupled to the gravitational field through the Einstein equations. Φ is split up into a spatially uniform classical background field $\Phi_0(t)$ and a Heisenberg quantum operator $\phi(\vec{x}, t)$, which is treated as a first-order fluctuation about Φ_0. The quantum operator ϕ satisfies equal-time commutation relations,

$$\left[\phi(\vec{x}, t), \dot{\phi}(\vec{x}', t)\right] = \frac{i}{a^3}\delta^3\left(\vec{x} - \vec{x}'\right). \tag{3.11}$$

Assume that the slow roll down approximation is well satisfied, with $d^2V/d\Phi^2 \ll H^2$, so H is constant and the background spacetime is approximately deSitter space. The spatial coordinates x^i are comoving coordinates, and t is proper time.

Consider the part of ϕ with coordinate wave numbers $\approx k$, physical wave numbers $\approx k/a$, and corresponding frequencies $\omega(k, t)$. Then $\dot{\phi} \sim \omega\phi$. Integrate the commutation relation over a coordinate volume $\sim k^{-3}$ to estimate the minimum quantum uncertainty in $\langle\phi^2\rangle$,

$$\frac{\omega\left\langle\phi^2\right\rangle}{k^3} \gtrsim \frac{1}{a^3}, \quad \left\langle\phi^2\right\rangle \gtrsim \frac{1}{\omega}\left(\frac{k}{a}\right)^3. \tag{3.12}$$

As long as these modes are well within the horizon, $k/a \gg H$, $\omega \simeq k/a$, and

$$\left\langle\phi^2\right\rangle \gtrsim \left(\frac{k}{a}\right)^2. \tag{3.13}$$

In the same limit there is a WKB solution for the time dependence of the individual Fourier modes in ϕ,

$$\phi \simeq \phi_0\left(\frac{a_0}{a}\right)\exp\left(-i\int dt\,\frac{k}{a}\right). \tag{3.14}$$

If the initial quantum state is the ground state, with only zero-point excitation of ϕ, then Eq. (3.13) is an equality initially, and by Eq. (3.14) it remains an equality as long as the WKB approximation is valid.

The WKB approximation breaks down once $k/a \leq H$, at which point ϕ approaches a constant the order of H. More precisely, the calculations of [8], [9] and others show that, summing over all modes with $k/aH \ll 1$, the power spectrum for the mean square fluctuation in ϕ defined by

$$\langle\phi\rangle^2 = \int d(\ln k) P_\phi(k) \tag{3.15}$$

is

$$P_\phi(k) \simeq \left(\frac{H}{2\pi}\right)^2. \tag{3.16}$$

For modes with $k/aH \ll 1$, ϕ can be treated as a *classical perturbation* of the classical field Φ. Since the slow roll down approximation is assumed valid, the energy density is dominated by the potential energy, and the perturbation in the energy density is

$$\varepsilon \simeq \left(\frac{dV}{d\phi}\right)\phi. \tag{3.17}$$

In the background $E_0 + P_0 = \dot{\Phi}^2$, so the scalar ζ characterizing the amplitude of the classical perturbation is

$$\zeta \simeq \frac{1}{3}\frac{\varepsilon}{E_0 + P_0} \simeq -H\frac{\phi}{\dot{\Phi}}, \tag{3.18}$$

assuming the metric perturbation scalar φ is negligible and using Eq. (3.6) to simplify. The power spectrum for ζ, defined similarly to the power spectrum for ϕ, is

$$P_\zeta(k) \simeq \left(\frac{H}{\dot{\Phi}}\right)^2 P_\phi(k) \simeq \left(\frac{H^2}{2\pi\dot{\Phi}}\right)^2, \tag{3.19}$$

with $\dot{\Phi}$ given by Eq. (3.6). Within differences in notation, this is the standard result of [8, 9, 11] and others.

During the roll down process H and $\dot{\Phi}$ will change slowly over many Hubble times. For a given value of k, the correct values to use in Eq. (3.19) are the value when $k/aH \approx 1$. As H and $\dot{\Phi}$ evolves, ζ remains constant for each k because the metric perturbation scalar φ compensates for the change in $\varepsilon/(E_0+P_0)$. Thus [41] made a mistake in applying something like Eq. (3.19) to all k just before reheating, and got the wrong answer for the density perturbation power spectrum. Also, Eq. (3.19) does not apply if H and/or $\dot{\Phi}$ change appreciably in a few Hubble times around the time $k/aH = 1$, as they can in models of double inflation [24, 25].

As an example, consider chaotic inflation with a $\lambda\Phi^4$ potential. In the slow roll down approximation,

$$H \simeq \left(\frac{2\pi\lambda}{3}\right)^{1/2}\frac{\Phi^2}{M_p}, \quad \dot{\Phi} \simeq -\left(\frac{\lambda}{6\pi}\right)^{1/2}\Phi, \quad P_\zeta(k)^{1/2} \simeq \left(\frac{2\pi\lambda}{3}\right)^{1/2}\left(\frac{\Phi}{M_p}\right)^3. \tag{3.20}$$

The comoving scales corresponding to present large scale structure leave the horizon about 50 e-foldings of expansion before the end of inflation, when $\Phi \sim 4M_p$. To get $P_\zeta(k)^{1/2}$ as small as 10^{-5}, as required for the timing of large scale structure formation in CDM models, requires $\lambda \sim 10^{-14}$.

The scale invariance of the perturbation spectrum produced by standard inflation is only approximate. It comes about because the astrophysical relevant range of wave numbers leaves the horizon over just a few Hubble times, during which period the percentage changes in Φ, $\dot{\Phi}$, and H are rather small.

3.3 Elaborate Treatment of Quantum Fluctuations

The observational indications of excess power in density perturbation on supercluster scales, beyond what is expected from a scale-invariant spectrum normalized on scales on several Mpc [42], have motivated attempts to find inflation moels with "ramps" or "mountains" in the perturbation spectrum. Such models violate one or more of the assuptiions behind the standard treatment outlined in the last section, and some of the claims in the literature made by, for instance, [43] and [44] are incorrect because the standard formula was used inappropriately.

The more elaborate treatment discussed here has been developed by [36]. The main features are:

1. Allows for two or more interacting scalar fields;
2. Includes dissipation terms to convert scalar field energy to thermal energy;
3. Includes metric perturbations from the beginning as part of the quantum fluctuations;
4. Does not assume the slow rolldown approximation, $-\dot{H}$ can be comparable to H^2;
5. Allows the effective masses of the scalar fields to be comparable to H.

The quantum fluctuations are still treated as first-order fluctuations about a classical mean field background, and in the discussion here the scalar fields are assumed minimally coupled to gravity.

The classical background equations involve N scalar fields Φ_i satisfying equations of motion

$$\ddot{\Phi}_i + (3H + \Gamma_i)\dot{\Phi}_i + \frac{\partial V}{\partial \Phi_i} = 0, \tag{3.21}$$

and the thermal energy density E_r, which evolves according to

$$\dot{E}_r = \sum_i r_i \dot{\Phi}_i^2 - 4HE_r. \tag{3.22}$$

The potential V is a function of all the Φ_i, and the *ad hoc* damping coefficients Γ_i are, for simplicity, assumed to be constants, independent of the Φ_i and $\dot{\Phi}_i$. The Γ_i should be small compared with H while the Fourier modes of interest are expanding beyond the horizon, so that dissipation is unimportant during the quantum regime.

The background energy density is

$$E_0 = \sum_i \frac{1}{2}\dot{\Phi}_i^2 + V + E_r, \tag{3.23}$$

and the background pressure is

$$P_0 = \sum_i \frac{1}{2}\dot{\Phi}_i^2 - V + \frac{E_r}{3}, \tag{3.24}$$

The evolution of the scale factor a is determined from E_0 by Eq. (2.24).

The fluctuations in Φ are quantum Heisenberg operators $\phi_i(\vec{x}, t)$. Since a spatially flat background is assumed, the ϕ_i can be expanded in Fourier modes,

$$\phi_i(\vec{x}, t) = \int \frac{d^3k}{(2\pi)^3}\left[\phi_i(\vec{k}, t)e^{i\vec{k}\cdot\vec{x}} + \phi_i^\dagger(\vec{k}, t)e^{-i\vec{k}\cdot\vec{x}}\right], \tag{3.25}$$

and similarly for the metric perturbations and the perturbations in the energy density and momentum density of the radiation, ε_r and ψ_r, all regarded as quantum operators. In a synchronous gauge (and with $N_0 = 1$), the operator equations for the $\phi_i(\vec{k}, t)$ are

$$\frac{d^2\phi_i}{dt^2} + (3H + \Gamma_i)\frac{d\phi_i}{dt} + \frac{k^2}{a^2}\phi_i + \frac{\partial^2 V}{\partial\Phi_i \partial\Phi_j}\phi_j - \dot{\Phi}_i\kappa = 0, \tag{3.26}$$

and the operator κ is related to the operator ϕ and the total energy density perturbation operator ε by

$$\kappa = \frac{1}{H}\left[\left(\frac{k}{a}\right)^2\varphi - 4\pi G\varepsilon\right]. \tag{3.27}$$

The operator ϕ is found from the total momentum density scalar operator ψ by Eq. (2.50),

$$\dot{\varphi} = 4\pi G\psi. \tag{3.28}$$

Sound waves in the radiation are in principle an independent dynamical degree of freedom, described by quantum operators ψ_r and ε_r independent of the scalar field operators. However, the radiation is not really important until reheating, far into the classical regime, and I consider sound waves only as they are generated in response to the scalar field perturbations. Previous treatments of quantized sound waves in the literature (*e.g.*, [45]) assume the sound waves are the only degree of freedom in the perturbations, and are not applicable here.

Using ε_r and ψ_r as variables means keeping track of instantaneous energy and momentum transfer as the scalar field fluctuations oscillate in time. Using the total energy and momentum scalars ε and ψ as variables in place of ε_r and ψ_r can have significant numerical advantages, since they evolve in time more smoothly. It is better still to use $\varepsilon_C \equiv \varepsilon - 3H\psi$ in place of

ε, since ε_C is gauge invariant and is unaffected by the gauge mode at the beginning of the Friedmann epoch.

One advantage of the synchronous gauge is the absence of any metric perturbations in relating ε to the ϕ_i and ε_r,

$$\varepsilon = \sum_i \left[\dot{\Phi}_i\dot{\phi}_i + \left(\frac{\partial V}{\partial \Phi_i}\right)\phi_i\right] + \varepsilon_r. \tag{3.29}$$

Since the total isotropic stress perturbation is

$$\pi = \frac{\varepsilon}{3} + \frac{2}{3}\sum_i \left[\dot{\Phi}_i\dot{\phi}_i - 2\left(\frac{\partial V}{\partial \Phi_i}\right)\phi_i\right], \tag{3.30}$$

Eq. (2.47) becomes

$$\dot{\psi} = -4H\psi - \frac{\varepsilon_C}{3} - \frac{2}{3}\sum_i \left[\dot{\Phi}_i\dot{\phi}_i - 2\left(\frac{\partial V}{\partial \Phi_i}\right)\phi_i\right]. \tag{3.31}$$

Eq. (2.54) with $\sigma = 0$ is

$$\dot{\varepsilon_c} = -3H\left[1 + \frac{1}{2}\left(1 + \frac{P_0}{E_0}\right)\right]\varepsilon_C + \left(\frac{k}{a}\right)^2\left[(E_0 + P_0)\frac{\varphi}{H} + \psi\right]. \tag{3.32}$$

The quantum operators $\phi_i(\vec{k}, t)$ in Eq. (3.25) are expressed as linear combinations of N annihilation operators $a_j(\vec{k})$,

$$\phi_i(\vec{k}, t) = \sum_j \phi_{ij}(\vec{k}, t)a_j(\vec{k}), \quad \phi_i^\dagger = \sum_j \phi_{ij}^* a_j^\dagger. \tag{3.33}$$

The a_j and $a_j^\dagger$ satisfy the commutation relations

$$\left[a_j(\vec{k}), a_l^\dagger(\vec{k}')\right] = (2\pi)^3\delta_{jl}\delta^3(\vec{k} - \vec{k}'). \tag{3.34}$$

The remaining perturbation variables, as quantum operators, are written as linear combinations of the same set of a_j's:

$$\kappa(\vec{k}, t) = \sum_j \kappa_j(\vec{k}, t)a_j(\vec{k}), \quad \varphi(\vec{k}, t) = \sum_j \varphi(\vec{k}, t)a_j(\vec{k}), \tag{3.35}$$

are similarly for ε_C and ψ, since no other degrees of freedom are being considered. The coefficients of each a_j are complex solutions of the classical perturbation equations satisfying appropriate initial conditions.

Consider an initial time when the physical wavenumber k/a is large compared with H and $|H^{-1}(dH/dt)|$, and

$$\left(\frac{k}{a}\right)^2 \gg m_{ij}^2 \equiv \frac{\partial^2 V}{\partial\Phi_i\partial\Phi_j}. \tag{3.36}$$

Assume that the Γ_i are all much less than H. The classical mode functions ϕ_{ij} are then accurately approximated by WKB solutions to Eq. (3.26). Through first order in WKB

$$\phi_{ij}(\vec{k},t) \cong \delta_{ij}\frac{A(k)}{a(t)}\exp\left[-ik\int_{t_0}^{t}\frac{dt'}{a(t')}\right], \tag{3.37}$$

if the initial hypersurface in the synchronous gauge is chosen to make the metric scalar φ purely oscillatory to zeroth order,

$$\varphi_j(\vec{k},t) \cong -4\pi iG\left(\frac{a}{k}\right)\dot{\Phi}_j\phi_{jj}(\vec{k},t). \tag{3.38}$$

To zeroth order in WKB Eq. (3.38) solves Eq. (3.28), since, neglecting ψ_r,

$$\psi \cong -\sum \dot{\Phi}_i\phi_i. \tag{3.39}$$

The leading contributions of φ and ε to κ in Eq. (3.27) *cancel*, with

$$\varepsilon \cong \sum \dot{\Phi}_i\dot{\phi}_i \cong -i\left(\frac{k}{a}\right)\sum \dot{\Phi}_i\phi_i. \tag{3.40}$$

The avoidance of complications from metric perturbations in the WKB solutions for ϕ_{jj} is a big plus for the synchronous gauge.

The mode functions $\phi_j j$ are normalized by imposing the equal time commutation relation

$$\left[\phi_i(\vec{x},t), \frac{\partial\phi_j(\vec{x}',t)}{\partial t}\right] = \frac{i}{a^3}\delta_{ij}\delta^3(\vec{x}-\vec{x}'), \tag{3.41}$$

with the result

$$A(k) = \frac{1}{\sqrt{2k}}, \tag{3.42}$$

within an irrelevant phase factor.

The numerical solution for the ϕ_{ij} is now straightforward. Initial conditions are imposed at a time t_0 when $k/aH \sim 50$, so the WKB approximation is quite accurate. There must be $2N$ integrations of the classical perturbation equations for each k, the jth pair starting from the real and imaginary parts of ϕ_{jj}, $\dot{\phi}_{jj}$, and φ_j, as determined from Eqs. (3.37) and (3.38), with all the other scalar fields unexcited and $\varepsilon_r = \psi_r = 0$ initially. The values of the ith pair of scalar field perturbations in the jth pair of integrations are, within a phase, the real and imaginary parts of ϕ_{ij} at later times, and similarly for the other perturbation variables.

Denote the initial quantum state of the fluctuations by $|\psi\rangle$. The standard assumption is that this is a state of *zero excitation*, with

$$a_j|\psi\rangle = 0 \tag{3.43}$$

for all j. Ultimately this relates to the initial quantum state of the universe, but is difficult to justify in any rigorous way. One major problem is that the amplitudes of the quantum fluctuations, which must be small as the modes leave the horizon, become very large at early times, calling into question the validity of treating the fluctuations as linear perturbations. Still, it is difficult to see why the ultimate classical perturbations should be any *smaller* than what one finds using Eq. (3.43), at least if inflation takes place well after the Planck time, with $H \ll M_p$. Eq. (3.43) gives the minimum required by the quantum uncertainty principle.

One measure of the state of the scalar field fluctuation is the *cross-correlation power spectrum*, defined in terms of the equal time two-point function by

$$P_{\phi|ij}(k) \equiv \frac{k^3}{2\pi^2} \int d^3x e^{-i\vec{k}\cdot\vec{x}} \langle \Psi | \phi_i(\vec{x}) \phi_j(\vec{0}) | \Psi \rangle = \frac{k^3}{2\pi^3} \sum \phi_{il} \phi^*_{jl}. \tag{3.44}$$

The second equality assumes Eq. (3.43). While the WKB solution is still valid,

$$P_{\phi|ij}(k) = \left(\frac{k}{2\pi a} \right)^2 \delta_{ij}, \tag{3.45}$$

which is the same as the high-k limit in flat space. However it must be remembered that Eq. (3.44) is a *gauge-dependent* expression in cosmology, and that Eq. (3.45) is specific to our particular choice of gauge.

What we are interested in as a measure of the primordial amplitude of adiabatic perturbations for extrapolation to the growth of structure in the recent universe is the gauge-invariant scalar ζ defined by Eq. (2.56). As a quantum operator, ζ for each wavenumber $\vec{k}$ is a linear combination of the a_j, with coefficients ζ_j, similar to Eq. (3.35). The expectation value of the equal-time two-point function for ζ is related to the power spectrum per unit logarithmic interval in k by

$$\langle \Psi | \zeta(\vec{x}, t) \zeta(\vec{0}, t) | \Psi \rangle = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \sum |\zeta_j|^2 \equiv \int \frac{d^3k}{4\pi k^3} e^{i\vec{k}\cdot\vec{x}} P_\zeta(k), \tag{3.46}$$

so the power spectrum is

$$P_\zeta(k) = \frac{k^3}{2\pi^2} \sum |\zeta_j(k)|^2. \tag{3.47}$$

The ζ_j are calculated from the φ_j and the $\varepsilon_j = (\varepsilon_C)_j + 3H\psi_j$ in the same way as ζ from φ and ε in Eq. (2.56).

Once the wavelengths are large compared to the horizon the quantum two-point function for ζ goes over into a classical ensemble average of a random field. It is normally assumed the statistics of the random field are Gaussian, on the grounds that the fluctuations are linear to a good approximation as they expand beyond the horizon if the eventual amplitude of ζ is consistent with isotropy of the microwave background radiation (CMB). Whether the large amplitudes of the mode functions at earlier times can cause deviations from Gaussian statistics is an important open question.

I will not go into the details of actual calculations for particular non-standard inflation models, but will discuss some general features. The simplest such models are "double inflation" models with two non-interacting scalar fields Φ_1 and Φ_2. One of the fields (Φ_1) dominates the potential energy at early times and drives the first stage of inflation.

In chaotic inflation, for example, one might have $V = (1/4)\lambda_1\Phi_1^4 + (1/4)\lambda_2\Phi_2^4$, with $\lambda_1 \gg \lambda_2$ and the initial Φ_1 and Φ_2 both greater than $M_{\rm p}$. Then Φ_1 rolls down while Φ_2 is frozen in place. As Φ_1 becomes less than $M_{\rm p}$, the Φ_1 field starts to oscillate and the first stage of inflation comes to an end. The expansion rate H decreases rather rapidly until the potential energy density of Φ_2 dominates the energy density. The approximately exponential expansion resumes and Φ_2 begins a slow rolldown.

The number of e-foldings of expansion in the second stage of inflation depends on the initial value of Φ_2. If this has just the right value, a few times $M_{\rm p}$, then the scales relevant to present large scale structure leave the horizon around the time of transition. The sharp decrease in H at the transition means that Fourier modes leaving earlier (larger comoving wavelengths) can have substantially larger fluctuation amplitudes than Fourier modes leaving later. The result can be a fairly sharp break in the power spectrum for ζ (see [43]).

Ref. [44] considered different forms of potentials which they claimed would give the opposite result, larger amplitudes on smaller scales. However, they overlooked the fluctuations generated in Φ_2 during the first stage of inflation. These at first have a negligible effect on the total energy density perturbation and on ζ. As the potential energy density of Φ_2 becomes dominant, the long wavelength fluctuations in Φ_2 are converted into large amplitude “adiabatic" fluctuations. The final result is more power on large scales than small scales.

I myself don’t find these simple double inflation models very appealing. To get a break in the ζ power spectrum at an interesting wavelength requires that the initial Φ_2 be in a very narrow range. Also, it is difficult to avoid violating large angular scale CMB anisotropy constraints, since the increase in amplitude with wavelength does not level out very sharply.

More complicated effects are possible if the two fields interact, in particular if the interactio can cause the effective mass of one or both of the fields to vary rapidly. However, here it is particularly important to do a careful job of calculating the evolution of the fluctuations. The standard formula can fail very badly. Some claims in the literature of being able to generate sharp peaks in the power spectrum of ζ are not valid.

One model I have played with uses a temporary instability of one field to produce through a non-linear interaction (in the classical regime) adiabatic perturbations with a peak in the power spectrum and strongly non-Gaussian statistics. Considerable fine tuning of potential parameters, but not initial values, is required.

All of these models are very contrived, and the approximately scale-invariant spectrum produced by standard inflation models should still be considered a rather strong, but not inevitable, prediction of inflation.

4 Transfer Functions

4.1 Definition of the Transfer Function

The primordial power spectrum for adiabatic perturbations is characterized by the power spectrum for ζ, $P_\zeta(k)$, as it is established during the inflation epoch and is preserved unchanged over the many e-foldings of expansion until the scales of interest for large scale structure come back within the horizon in the relatively recent universe. As a given Fourier mode comes back within the horizon it begins to evolve dynamically again, as the matter and radiation present respond to gravitational and non-gravitational forces.

The formation of large scale structure is determined by the growth of density perturbations in the non-relativistic matter. Once the total energy density becomes dominated by this non-relativistic matter the fractional perturbation in the mass density, δ, grows as the scale factor a for all scales will within the horizon, $k/aH \gg 1$, and larger than the Jeans mass associated with the velocity dispersion of the matter particles. The final stage of growth preserves the shape of the power spectrum until δ becomes larger than one, leading to gravitational collapse.

In hierarchical clustering collapse occurs first on small scales, and linear perturbation theory still applies to the large scales after the small scales have gone nonlinear. Within the context of linear perturbation theory it makes sense to talk about the last stage of evolution in terms of the linear density perturbation δ extrapolated to the present, $\delta_{\rm p}$. A value of the $\delta_{\rm p}$ greater than one means that δ actually became nonlinear at a redshift $z_{\rm nl}$ such that $1 + z_{\rm nl} \approx \delta_{\rm p}$.

Define the "present" density perturbation power spectrum, P_δ, by

$$\langle \delta_{\rm p}(x)\delta_{\rm p}(0)\rangle = \int \frac{d^3k}{4\pi k^3} e^{ik\cdot x} P_\delta(k). \tag{4.1}$$

Since the evolution in linear by definition, one can write

$$\sqrt{P_\delta(k)} = T(k)\sqrt{P_\zeta(k)}, \tag{4.2}$$

where $T(k)$ is the *transfer function*, defined as the ratio of a Fourier component of $\delta_{\rm p}$ to the corresponding Fourier component of the primordial ζ. The transfer function depends only on the physics of the recent universe, whether the dark matter is cold or hot, *etc.*, while the primordial power spectrum is entirely determined by what went on before the adiabatic perturbations got established, presumably during inflation.

4.2 Transfer Function for a Matter-Dominated Universe

The transfer function is easily calculated analytically for long wavelength perturbations which come back within the horizon after the universe is matter dominated. The calculation is a useful exercise in applying the synchronous gauge formalism. Assume the matter has negligible pressure and the background has zero spatial curvature.

The momentum conservation equation in the synchronous gauge is

$$\dot\psi = -3H\psi. \tag{4.3}$$

If the freedom of choice of initial hypersurface is used to make the initial ψ zero, then Eq. (4.3) says that ψ will remain zero. This particular synchronous gauge is also a comoving gauge.

With $\psi = 0$, Eq. (2.50) says that $\dot\varphi$ is zero, so φ is constant in time. Eq. (2.53) reduces to

$$\dot\delta = \frac{k^2}{a^2H}\varphi - \frac{3}{2}H\delta, \tag{4.4}$$

after a change of variables from $\varepsilon_c = \varepsilon$ to

$$\delta = \frac{\varepsilon}{E_0}. \tag{4.5}$$

Since $a \sim t^{2/3}$, $H = 2/(3t)$, the general solution of Eq. (4.4) is

$$\delta = \frac{2}{5}\left(\frac{k}{aH}\right)^2 \varphi + \frac{B}{a^{3/2}}. \tag{4.6}$$

The second term in Eq. (4.6) is a *physical* decaying mode, not a gauge mode as claimed by [46]. A different choice of initial hypersurface can cancel the decaying behavior in δ, but the *hypersurface-invariant* quantity ζ still has the decaying behavior in it,

$$\zeta = \varphi + \frac{\delta}{3} = \left[1 + \frac{2}{15}\left(\frac{k}{aH}\right)^2\right]\varphi + \frac{1}{3}\frac{B}{a^{3/2}}, \tag{4.7}$$

regardless of the choice of gauge. Of course, if the perturbations are generated during inflation, the decaying mode is long gone by the time the perturbations are back within the horizon.

From Eq. (4.7), the primordial amplitude of the perturbation, as measured by ζ when $k/aH \ll 1$ and after transients have died out, is just φ in our chosen gauge. The transfer function $T(k)$ defined by Eq. (4.2) is

$$T(k) = \frac{2}{5}\left(\frac{k}{aH}\right)^2. \tag{4.8}$$

The longer wavelengths have smaller final amplitudes because they have less time to grow after coming inside the horizon.

4.3 Newtonian Description of Perturbations

The physics of perturbations within the horizon in a matter-dominated universe is essentially Newtonian. In a Newtonian description of the perturbations the extrinsic curvature of the constant-t hypersurfaces is unperturbed, so a Newtonian gravitational potential can be defined as α in zero-shear gauge. In gauge-invariant form,

$$\Phi_N = \alpha - \frac{\dot{\chi}}{N_0} = -(\varphi - H\chi), \tag{4.9}$$

using Eq. (2.45) with $\sigma = 0$. Eqs. (2.52) and (4.6) relate the present χ to the primordial ζ (ζ_0),

$$\chi = \frac{\varphi}{H} - \frac{3}{2}\left(\frac{k}{aH}\right)^{-2}\frac{\delta}{H} = \frac{2}{5}\frac{\varphi}{H} = \frac{2}{5}\frac{\zeta_0}{H}, \tag{4.10}$$

and the Newtonian potential is

$$\Phi_N = -\frac{3}{5}\zeta_0. \tag{4.11}$$

Similarly, the Newtonian peculiar velocity 3-vector is

$$v_i = D_i \left(\frac{\psi}{E_0 + P_0} + \chi \right) = \frac{2}{5} \frac{D_i \zeta_0}{H}. \tag{4.12}$$

To compare with observations in the present universe, it is convenient to use "astronomical" units. Distances are measured by Hubble velocities (redshifts) on cosmological scales. The ratio between velocity and distance, the Hubble constant H_0, is uncertain by a factor two, an uncertainty absorbed in Hubble parameter h defined by

$$H_0 = 100h \ \mathrm{km\,s^{-1}\,Mpc^{-1}}. \tag{4.13}$$

As a unit of distance take 1 $\mathrm{Mpc}\,h^{-1}$, corresponding to 100 $\mathrm{km\,s^{-1}}$ of recession velocity. Take as a unit of velocity 100 $\mathrm{km\,s^{-1}}$. Then the unit of time is one Hubble time, $3.09 \times 10^{17} h^{-1}$ seconds or about $10^{10} h^{-1}$ yr. With the scale factor $a = 1$ at the present, the comoving wavenumber k is in units of $h\,\mathrm{Mpc}^{-1}$.

In these units for k transfer function is

$$T(k) = 3.6 \times 10^6 k^2, \tag{4.14}$$

so

$$\delta(\vec{k}) = 3.6 \times 10^6 k^2 \zeta_0(\vec{k}), \tag{4.15}$$

and the Newtonian potential is

$$\Phi_N = -5.4 \times 10^6 \zeta_0 \ (100 \ \mathrm{km\,s^{-1}})^2. \tag{4.16}$$

The scalar ζ is dimensionless. The Fourier components of the peculiar velocity at the present epoch are given by

$$v(\vec{k}) = 3.6 \times 10^6 \ i\vec{k} \ \zeta_0(\vec{k}). \tag{4.17}$$

The expressions are only valid on very large scales, with $k \lesssim 0.01 \ h\,\mathrm{Mpc}^{-1}$.

On smaller scales, larger k, the transfer function must be calculated numerically. The physics which goes into these calculations, as it depends on the nature of the dark matter, is discussed by Fang (this volume). Briefly, perturbations, even if inside the horizon, are not able to grow very much until the universe becomes matter dominated, at $t \simeq t_{\mathrm{eq}}$. In hot dark matter the free streaming of the neutrinos damps out perturbations with comoving wavelengths less than a damping length roughly corresponding to the horizon at t_{eq}.

4.4 Scaling of CDM Models

The scaling behavior of CDM models is worth looking at in more detail. The matter, including baryons and CDM, has negligible pressure after recombination, and the dominant CDM is always pressureless, so the matter mass density scales as a^{-3}. The present mass density is a fraction Ω_{nr} of the cosmological critical density, giving

$$a^3 \rho = 1.88 \times 10^{-29} \, \Omega_{\mathrm{nr}} h^2 \ \mathrm{g\,cm^{-3}}. \tag{4.18}$$

The radiation consists of photons and neutrinos. Adopting 2.74 K as the current best value of the CMB temperature [47], the photons have a present energy density of $4.26 \times 10^{-13}\,\mathrm{erg\,cm^{-3}}$. For three species of massless neutrinos the total energy density in relativistic particles is 1.68 times this (see [33]). Absorb the uncertainties in a parameter θ, so that the total radiation energy density is

$$a^4 E_r = 7.15 \times 10^{-13}\,\theta\ \mathrm{erg\,cm^{-3}}. \tag{4.19}$$

Define $a_{\rm eq}$ to be the scale factor when $\rho c^2 = E_r + P_r = 4E_r/3$. From Eqs. (4.18) and (4.19),

$$a_{\rm eq} = 5.64 \times 10^{-5}\,\theta\,\Omega_{\rm nr}^{-1}\,h^{-2}. \tag{4.20}$$

Let $k_{\rm eq}$ be the comoving wavenumber such that $k_{\rm eq}\int dt/a = 2\pi$ when $a = a_{\rm eq}$. The corresponding length scale in the present universe is

$$k_{\rm eq}^{-1} = 3.7\,\theta^{1/2}\Omega_{\rm nr}^{-1}\,h^{-1}\ \mathrm{Mpc}\,h^{-1}. \tag{4.21}$$

4.5 Analytic Fits to Transfer Functions

Analytic expressions as fits to numerical results for certain transfer functions are given by [26]. With $q \equiv k/k_{\rm eq}$, the transfer function as defined here for adiabatic CDM perturbations is

$$T_A = 3.6 \times 10^6\,k^2 \frac{\ln(1+0.70q)}{0.70q}\left[1 + 1.17q + (4.83q)^2 + (1.64q)^3 + (2.01q)^4\right]^{-1/4}. \tag{4.22}$$

As k increases, the initial growth of the transfer function as k^2 flattens out into a slow logarithmic growth for $k > k_{\rm eq}$ [48]. The latter comes from the gravitational attraction of the radiation on the CDM generating a peculiar velocity as the mode comes inside the horizon. The CDM density perturbation grows logarithmically as the peculiar velocity decays.

For HDM with one massive neutrino species $\Omega_\nu h^2$ is proportional to the mass of the neutrino, since the number density is fixed from initial thermal equilibrium. Free streaming of the neutrinos determines a characteristic damping length

$$R_f = 2.6(\Omega_\nu h)^{-1}\ \mathrm{Mpc}\,h^{-1}. \tag{4.23}$$

With $q \equiv kR_f$, the analytic fit to the transfer function is

$$T_H = 3.6 \times 10^6 k^2 e^{-0.16q-q^2}\left[1 + 0.615q + (1.54q)^{1.5} + (0.354q)^2\right]^{-1}. \tag{4.24}$$

Since the comoving scale corresponding to a massive galaxy is about 1 $mpch$, galaxies must form by fragmentation of collapse on larger scales. The HDM models have considerable difficulty in forming galaxies early enough without greatly overdoing clustering of galaxies [49], but can't be ruled out completely [50]. The HDM model becomes more attractive if galaxy formation is seeded by cosmic strings (Turok, this volume).

In some models of CDM it is possible to generate perturbations in the ratio of the CDM mass density to the thermal entropy density (for axions, see [51]). The absence of any adiabatic perturbation initially is enforced by setting the initial ζ asymptotically to zero. Since in standard gauge ζ is approximately equal to the curvature perturbation φ at early times, this type of perturbation is usually called an *isocurvature mode*. The initial amplitude of an isocurvature mode is given by

$$\delta_I = \frac{\varepsilon_m}{E_m} - \frac{3}{4}\frac{\varepsilon_r}{E_r}, \tag{4.25}$$

where the subscript m refers to the CDM. δ_I is hypersurface invariant, since the background entropy per unit CDM mass is time independent.

Note that the isocurvature mode is *not* necessarily isothermal ($\varepsilon_r = 0$) at early times. There is no natural gauge-independent definition of isothermal initial conditions. An attempt by [52] to impose isothermal initial conditions actually generated an adiabatic mode with a much larger amplitude than the isocurvature mode [53].

The isocurvature CDM transfer function is defined as the ratio of Fourier components of the final density perturbation $\delta(k)$ and the initial $\delta_I(k)$, and an analytic fit to numerical calculations [26] is

$$T_I(k) = 1.2 \times 10^6\, k^2 \left[1 + \frac{(12q)^2}{1 + 65q + \frac{(4.8q)^2}{1+0.15q}} + (1.7q)^{8/5}\right]^{-5/4} \tag{4.26}$$

, with k and q defined the same way as for adiabatic CDM. The isocurvature transfer function is considerably flatter than the adiabatic one for $k > k_{\rm eq}$. With a scale-invariant power spectrum for δ_I, galaxy formation is uncomfortably late [26], and the CMB anisotropy on large angular scales is above observational upper limits [54].

Cosmological models with comparable mass densities in baryons and CDM are interesting because the transfer function for adiabatic perturbations has considerably higher amplitude on scales of tens of Mpc compared to pure CDM, while preserving the shape of the CDM transfer function on scales of a few Mpc, where the CDM model is most sucessful in explaining clustering properties of galaxies (see [42, 55]). The baryons are unable to follow the growth of the CDM density perturbations until after recombination. FOr scales which come inside the horizon and begin oscillating before recombination, the growth of the CDM perturbation is impeded because the baryons dilute the self-gravity of the CDM. For larger wavelength modes the baryons are able to clump along with the CDM as soon as the perturbation comes inside the horizon. When the amplitude is normalized by fitting galaxy clustering on scale of a few $mpch$, the result is more power around $k_{\rm rec}$, such that $k_{\rm rec} \int dt/a = 2\pi$ at recombination.

Recombination occurs roughly when $a = 9 \times 10^{-4}$, or when $a/a_{\rm eq} = 16\Omega_{\rm nr}h^2/\theta$. The value of k, $k_{\rm rec}$, such that $\int dt(k/a) = 2\pi$ at recombination gives a characteristic length scale

$$k_{\rm rec}^{-1} \approx 29\, \Omega_{\rm nr}^{-1/2}(1 - 0.2\, \theta^{1/2}\Omega_{\rm nr}^{-1/2}\, h^{-1})\, {\rm Mpc}\, h^{-1}, \tag{4.27}$$

for $\Omega_{\rm nr}h^2/\theta \gg 0.05$.

The problem here is that primordial nucleosynthesis does not predict correct abundance for the light elements unless $\Omega_b \leq 0.1$ for $h > 0.5$ [56]. To have $\Omega_b \approx \Omega_{\rm CDM}$ and a total $\Omega = 1$

as expected from inflation requires making up the difference with vacuum energy density (*i.e.*, a cosmological constant), which is very unpleasant theoretically.

4.6 Entropy and wavenumber scaling

If the primordial power spectrum is not scale invariant, it is necessary to determine the relationship between wavenumbers in the primordial perturbation calculation of §3 and wavenumbers in astronomical units. Unfortunately, the relationship depends on the efficiency of reheating following inflation, so it is sensitive to some of the more obscure details of the particle physics of inflation. The match is made by comparing the entropy in the primordial calculation after reheating is completed with the present entropy.

If g is the effective number of degrees of freedom in relativistic particles, the entropy in a volume (a/k^3) is

$$S = \frac{4}{3}\left(\frac{g\pi^2}{30}\right)^{1/4}\left(\frac{a}{k}\right)^3\left(\frac{E_r}{\hbar c}\right)^{3/4}. \tag{4.28}$$

Evaluating this at present, with $a = 1$ and k in units of $hmpc$, must be done separately for the photons and neutrinos, since they are at different temperatures. For photons, $g = 2$, and the temperature is 2.74 K. For three species of neutrinos, $g = 21/4$, $E_\nu = 0.68E_\gamma$, and $S_\nu = 0.953S_\gamma$. The total present entropy in a volume k^{-3} is

$$S_{\rm now} = 8.67\times 10^{76}\, h^{-3}(k\,{\rm Mpc}\,h^{-1})^{-3}. \tag{4.29}$$

The primordial entropy is found at some time after a^4E_r is constant by evaluating Eq. (4.28) in terms of the comoving wavenumber k, with k/a and E_r in Planck units. On the assumption of negligible entropy generation between then and now, equating the two entropies determines the relation between the primordial k and the astronomical k.

5 Statistical predictions of large scale structure

5.1 Overview

In conventional inflation models the primordial perturbations should approximate Gaussian random fields on all astrophysically relevant scales. The limits on the anisotropy of the CMB require that the primordial amplitude as measured by ζ be less than about 10^{-5} on comoving scales larger than the horizon at the time of last scattering. This small amplitude should mean individual Fourier modes evolve independently. Statistically independent Fourier modes imply Gaussian statistics for the random field by the Central Limit Theorem, regardless of the statistics of the individual modes, through this should be Gaussian if they arise from zero-point quantum fluctuations. The only real question is whether phase correlations can be established when the quantum mode wavelengths are small compared with the horizon and quantum fluctuations do have large amplitudes by the uncertainty principle. This seems doubtful, but needs to be checked by more sophisticated quantum field theoretic calculations. A recent claim by [57] that non-Gaussian statistics are generic in power-law inflation is incorrect.

In some non-standard inflation models large amplitude perturbations in small components of the energy-momentum tensor make the primordial perturbations highly non-Gaussian, while still consistent with CMB isotropy [58, 59]. I will only consider Gaussian statistics here, because that assumption greatly simplifies calculations and deserves to be pursued as long as it is not decisively rejected by observations.

Of course, as the perturbations grow dynamically and begin to become nonlinear in the recent universe, the statistics of, say, the density perturbation field becomes non-Gaussian in any case. In principle, numerical simulations can be started with Gaussian initial conditions when the perturbation amplitudes are small on all scales. In practice this may be difficult or impossible, because of limited dynamic range. Normally it is assumed that nonlinearities on small scales, less than the resolution of the calculation, do not significantly affect the dynamical evolution on larger scales.

What astronomers actually observe are galaxies. There is no reason to believe that galaxies are good tracers of the mass distribution, given that the mass in the optically visible portions of the galaxies amounts to less than one percent of the closure density and that perhaps only about ten percent of the closure density is in dark halos associated with the galaxies. On very large scales one might hope to use the galaxies as tracers of the *velocity* field and infer the density perturbation field from linear perturbation theory, but this may also have problems [60].

The "biased galaxy formation" hypothesis of [61] and [26] associates sites of galaxy formation with high peaks, above a threshold, in the linear density perturbation field δ as smoothed on an appropriate length scale. The statistical analysis of these peaks [26], plus some accounting for dynamical evolution, results in predictions for such things as correlation functions and mass-to-light ratios in clusters of galaxies. The hypothesis is plausible for CDM scenarios, in which galaxy formation is hierarchical, and makes no sense at all for galaxies in HDM scenarios, which form out of a highly nonlinear fragmentation process. On sufficiently large scales in the present universe the density perturbation amplitudes must still be smaller than one, and linear perturbation theory can be applied more directly.

The height of the peak is correlated with the time of collapse. Modeled as a spherical, uniform-density "top hat" with a density excess $\delta_{\rm TH}$ extrapolated to the present by linear perturbation theory, the peak collapses to infinite density at a redshift z_c such that

$$1 + z_c = \frac{\delta_{\rm TH}}{1.69}. \tag{5.1}$$

Infinite density is reached somewhat sooner if the collapse is two dimensional (filament) or one dimensional (pancake). A threshold in δ corresponds to a threshold in time, in that only peaks which collapse before a certain time (above a certain redshift) form galaxy or cluster.

For galaxy clusters there is a natural time threshold. Only a cluster with $z_c > 0$ will have reached the large overdensity required for it to be identified as an Abell cluster [62, 63], and in simple CDM models this implies $\nu = \delta/\sigma_\delta \geq 2.5 - 3$ when the density perturbation field is smoothed on a comoving scale corresponding to the mass in the core of a typical Abell cluster.

The existence of a sharp threshold for galaxy formation is more problematic. While various ideas have been explored (see [64] for a review), it is difficult to see what physical process, such as heating the IGM, would shut off galaxy formation *uniformly* at a certain cosmological epoch. The most plausible way of producing a threshold is what has been called *natural* biasing [65, 66]. For a given mass, the height of the peak is correlated with the depth of the gravitational potential well and the velocity dispersion or circular velocity of the final collapsed "halo".If the escape velocity from the halo is too small, supernovae may blow most of the gas out of the halo, cutting off star formation after only a small fraction of the gas has turned into stars.

The mathematical theory of Gaussian random fields is given in books by [67, 68]. What I will do here is review a few of the basic concepts in [26], trying to avoid the mathematical contortions which make [26] hard to read, and consider a couple of newer applications to large scale streaming velocities.

5.2 Gaussian Random Fields

Any linear operation on a Gaussian random field gives another Gaussian random field, so, for example, δ and Φ_N extrapolated to the present by *linear* perturbation theory are Gaussian random fields if the primordial amplitude ζ is a Gaussian random field. Furthermore, they should be statistically homogeneous and isotropic random fields. The 1-point probability distribution of a field f with power spectrum $P(k)$ is

$$P(f)df = \frac{1}{\sqrt{2\pi}\sigma_0} e^{-\frac{f^2}{2\sigma_0^2}}, \tag{5.2}$$

where σ_0 is the dispersion in the field f,

$$\sigma_0^2 = \left\langle f^2 \right\rangle = \int d(\ln k) P(k). \tag{5.3}$$

The 2-point correlation function is

$$\xi(\vec{x}_1, \vec{x}_2) = \langle f(\vec{x}_1) f(\vec{x}_2) \rangle = \int d(\ln k) P(k) \frac{\sin(kr)}{kr} \tag{5.4}$$

where r is the distance between the two points. All statistical properties of f can be expressed in terms of the correlation function $\xi(r)$ and its derivatives, or alternatively in terms of integrals over the power spectrum $P(k)$.

Behind many statistical calculations is the *joint probability distribution* in N variables $y_1, \ldots, y_N$. The probability that y_1 is in the range dy_1, y_2 is in the range of dy_2, *etc.*, is $p(y_1, \ldots, y_N) dy_1 \ldots dy_N$. The variables y_i might be the field f and various of its derivatives at one or more points in space. To calculate $p(y_1, \ldots, y_N)$ construct the *correlation matrix*

$$M_{ij} = \langle y_i y_j \rangle \tag{5.5}$$

and its inverse $(M^{-1})_{ij}$. Then

$$p(y_1, \ldots, y_N) = (2\pi)^{-N/2} (\det M)^{-1/2} e^{-\frac{1}{2} y_i (M^{-1})_{ij} y_j}. \tag{5.6}$$

A trivial example is the joint probability distribution that f have a value f_1 at $\vec{x}_1$ and a value f_2 at $\vec{x}_2 = \vec{x}_1 + \vec{r}$. To make things as simple as possible use normalized variables $\nu_1 = f_1/\sigma_0$, $\nu_2 = f_2/\sigma_0$, and define a normalized correlation function

$$\psi(r) = \frac{\langle f_1 f_2 \rangle}{\sigma_0^2} = \frac{\xi(r)}{\sigma_0^2}. \tag{5.7}$$

The correlation matrix and its inverse are

$$M = \begin{pmatrix} 1 & \psi \\ \psi & 1 \end{pmatrix}, \quad M^{-1} = \frac{1}{1-\psi^2} \begin{pmatrix} 1 & -\psi \\ -\psi & 1 \end{pmatrix}, \tag{5.8}$$

so

$$p(\nu_1, \nu_2) = \frac{1}{2\pi} \frac{1}{\sqrt{1-\psi^2}} e^{-\frac{\nu_1^2 - 2\psi\nu_1\nu_2 + \nu_2^2}{2(1-\psi^2)}}. \tag{5.9}$$

5.3 Peaks in Gaussian Random Fields

The raw density perturbation power spectrum for CDM has enough power on small scales that its dispersion σ_0 is infinite, and smoothing is necessary to talk about peaks at all. For a general field f, smoothing is done by convolution integral over a normalized weight function $W(r)$,

$$f_s(\vec{x}) = \int d^3x'\, W(|\vec{x} - \vec{x}'|) f(\vec{x}'), \quad \int dx^3\, W(|\vec{x}|) = 1. \tag{5.10}$$

Note that f_s is a Gaussian random field in its own right, with a power spectrum

$$P_s(k) \equiv |W(k)|^2 P(k), \tag{5.11}$$

where $W(k)$ is the Fourier transform of $W(r)$. The scale of the smoothing is characterized by a smoothing length R_s. The two conventional types of smoothing are Gaussian smoothing, with

$$W(r) = \frac{1}{(2\pi)^{3/2}} \frac{1}{R_s^3} e^{-\frac{r^2}{2R_s^2}}, \tag{5.12}$$

and top-hat smoothing, with

$$W(r) = \begin{cases} \frac{3}{4\pi R_s^3} & r < R_s \\ 0 & r > R_s \end{cases} \tag{5.13}$$

The Fourier transform of $W(r)$ is

$$W(k) = e^{-\frac{R_s^2 k^2}{2}} \tag{5.14}$$

for Gaussian smoothing. The moderately sharp cutoff in wavenumber as well as in radius is a nice feature of Gaussian smoothing as opposed to top hat smoothing.

The arbitrariness of the smoothing procedure means one must be cautious about making quantitative predictions, particularly when these are sensitive to R_s. The mass in the smoothing window, with R_s in units of $\mathrm{Mpc}\,h^{-1}$, is

$$M_s = (2\pi)^{3/2}\rho R_f^3 = 4.37 \times 10^{12}\, R_f^3 h^{-1} M_\odot \tag{5.15}$$

for Gaussian smoothing, but just how this relates to the mass of the bound system formed by collapse of the peak is not clear. For a high peak there is likely to be a density excess surrounding it out to a distance a few times R_s, and M_s could be a considerable underestimate of the mass associated with the peak.

A maximum of f_s is characterized by:

1. the relative height $\nu = f_s/\sigma_0(R_s)$, $\sigma_0 \equiv \sqrt{\langle f_s^2 \rangle}$;
2. $\eta_i \equiv \partial_i f_s = 0$;
3. a second derivative matrix $\zeta_{ij} = \partial_i \partial_j f_s$ which is negative definite.

The statistical properties of peaks are based on the joint probability distribution in ν, the three η_i, and the six independent elements of the symmetric matrix ζ_{ij}, such that the probability is $p(\nu, \eta_i, \zeta_{ij}) d\nu d^3\eta d^6\zeta$. Near an extremum at $\vec{x} = 0$, say, $\eta_i \simeq \zeta_{ij} x^j$, so $\vec{\eta}$ in the range $d^3\eta$ about $\vec{\eta} = 0$ means the extremum is in the volume element d^3x, with

$$d^3\eta = |\det \zeta| d^3x. \tag{5.16}$$

Therefore, the probability per unit volume of a peak with height in range $d\nu$ about ν is

$$n_{\mathrm{pk}}(\nu) d\nu = d\nu \int d^6\zeta\, |\det \zeta| p(\nu, \vec{0}, \zeta_{ij}). \tag{5.17}$$

The domain of integration is over the region where all the eigenvalues of the ζ_{ij} matrix are negative.

It helps to take full advantage of the isotropy of the random field in constructing the correlation matrix. Change variables from ζ_{11}, ζ_{22}, ζ_{33} to

$$x \equiv -\frac{\zeta_{11} + \zeta_{22} + \zeta_{33}}{\sigma_2}, \tag{5.18}$$

$$y \equiv -\frac{\zeta_{11} - \zeta_{33}}{2\sigma_2}, \tag{5.19}$$

$$z \equiv -\frac{\zeta_{11} - 2\zeta_{22} + \zeta_{33}}{2\sigma_2}, \tag{5.20}$$

with

$$\sigma_2^2 = \left\langle (\Delta f_s)^2 \right\rangle = \int d(\ln k)\, k^4 P_s(k). \tag{5.21}$$

The statistical isotropy of the random field f_s gives

$$3\left\langle \eta_i^2 \right\rangle = \langle \eta_i \eta_i \rangle = \sigma_1^2 \equiv \int d(\ln k)\, k^2 P_s(k) \tag{5.22}$$

and

$$\langle x^2 \rangle = 1, \quad \langle y^2 \rangle = \frac{1}{15}, \quad \langle z^2 \rangle = \frac{1}{5}, \quad \langle \zeta_{12}^2 \rangle = \langle \zeta_{13}^2 \rangle = \langle \zeta_{23}^2 \rangle = \frac{\sigma_2^2}{15}. \tag{5.23}$$

The only nonzero cross-correlation is

$$\langle x\nu \rangle = \gamma \equiv \frac{\sigma_1^2}{\sigma_0 \sigma_2} = \frac{\langle k^2 \rangle}{\sqrt{\langle k^4 \rangle}}, \tag{5.24}$$

and it is only through γ that the peak height distribution depends on the shape of the power spectrum.

If the smoothed power spectrum $P_s(k)$ is sharply peaked about a particular wavenumber, γ is close to one. If $P_s(k)$ is large over many e-folds in k, then γ is close to zero. For $P(k)$ a power law, $P(k) \sim k^n$, and Gaussian smoothing,

$$\gamma^2 = \frac{n}{n+2}. \tag{5.25}$$

The constraint that ζ_{ij} be negative definite is simplified by a choice of coordinate axes along the principal axes of ζ_{ij}, with

$$-\zeta_{11} = \lambda_1 \geq -\zeta_{22} = \lambda_2 \geq -\zeta_{33} = \lambda_3 > 0. \tag{5.26}$$

The integral in Eq. (5.17) then becomes an integral over x, y, and z and the Euler angles specifying the orientation of the principal axes. The range of x, y, z are determined by the conditions Eq. (5.26). The integrals over y and z can be done analytically, but the last integral over x must be done numerically (see [26]). The result has the form

$$n_{\rm pk}(\nu) = \frac{1}{(2\pi)^2 R_*^3} e^{-\frac{\nu^2}{2}} G(\gamma, \gamma\nu), \tag{5.27}$$

where

$$R_* = 3^{1/2} \frac{\sigma_1}{\sigma_2} = \sqrt{\frac{6}{n+2}} R_s \tag{5.28}$$

for a smoothed power law power spectrum. Ref. [26] give an accurate numerical interpolation formula for $G(\gamma, w)$. In the limit $w \gg 1$, $G \approx w(w^2 - 3\gamma^2)$.

For the adiabatic CDM density perturbation power spectrum, the effective local power law index ranges from $n \sim 1$ on scales around 0.5 Mpc h^{-1} to $n \sim 4$ on very large scales. Note that $n = 2$ here corresponds to $n = -1$ in [26].

5.4 Global Peak Densities

The threshold hypothesis of [26, 61, 62] is implemented by combining $n_{\rm pk}(\nu)$ for the density perturbation field with a threshold function $t(\nu/\nu_t)$, the probability that a peak of height ν will end up as a luminous galaxy or a rich cluster, as appropriate to the smoothing length chosen. While a simple idealization for $t(\nu/\nu_t)$ is a step function, zero for $\nu < \nu_t$ and one for $\nu > \nu_t$, it is more realistic to make t vary continuously from zero to one over a range of ν about

ν_t. The choice of a form for t is rather arbitrary, given our ignorance of the astrophysics, but t should vary sharply enough that the distribution of “objects" in ν is peaked around ν_t despite the rapid decrease of $n_{\rm pk}(\nu)$ with ν.

The theoretical number density of “objects",

$$n_{\rm obj} = \int_0^\infty d\nu\, t\left(\frac{\nu}{\nu_t}\right) n_{\rm pk}(\nu) \tag{5.29}$$

must be equal to the observed number density. This determines ν_t given the threshold function and the smoothing length R_s. For galaxies, Ref. [26] take the number density of luminous galaxies counted in the first CfA redshift survey, $n_{\rm gal} \approx 0.01\,(h\,{\rm Mpc}^{-1})^3$ [69]. The number density of Abell clusters of richness class $R \geq 1$ is about $10^{-5}(h\,{\rm Mpc}^{-1})^3$ [70]. In both cases reasonable choices for the smoothing length given $\nu_t \approx 2.5 \sim 3.5$.

5.5 Local Peak Densities and Biasing

If the random field has a significant amount of power on scales considerably larger than the smoothing scale R_s which defines the peaks, it makes sense to ask how the *local* distribution of peaks with height ν depends on the local value of a “background” field f_b, defined as the original random field smoothed on a scale R_b considerably larger than R_s. This is the *constrained* peak density $n_{\rm pk}(\nu|f_b)$. The overall peak density distribution can be reconstructed from the constrained density distribution through

$$n_{\rm pk}(\nu) = \int df_b\, n_{\rm pk}(\nu|f_b) p(f_b), \tag{5.30}$$

where $p(f_b)$ is the (Gaussian) probability distribution for the background field, whose dispersion is σ_{0b}. The calculation of $\nu_{\rm pk}(\nu|f_b)$ is described in Appendix E of [26]. If the power spectrum for f is significantly flatter than white noise, $P(k) \sim k^n$ with $0 < n \leq 2$, then in the limit $R_b \gg R_s$, $n_{\rm pk}(\nu|f_b)$ is well approximated by Eq. (5.27) for $n_{\rm pk}(\nu)$ when ν and γ are replaced by

$$\tilde\nu = \frac{\nu - \epsilon\nu_b}{\sqrt{1-\epsilon^2}}, \quad \tilde\gamma = \frac{\gamma}{\sqrt{1-\epsilon^2}}, \tag{5.31}$$

with

$$\epsilon \equiv \int d(\ln k)\, \frac{W_s(k)W_b(k)P(k)}{\sigma_{0s}\sigma_{0b}}, \tag{5.32}$$

and $R_\star$ is left unchanged.

The interpretation is relatively simple when $W_b(k)$ cuts off sharply at $k = k_b = R_b^{-1}$. Then the overall field smoothed only by W_s can be split into two statistically independent parts, the background field with $k < k_b$ and a “peak field” with $k > k_b$ whose dispersion is $\sigma_{0p} \approx \sqrt{1-\epsilon^2}\sigma_{0s}$, since $\epsilon \approx \sigma_{0b}/\sigma_{0s}$. The peaks in the overall field f_s are peaks in the peak field f_p whose height is *biased by* the local height of the background field f_b,

$$f_s = \left(\sqrt{1-\epsilon^2}\tilde\nu + \epsilon\nu_b\right)\sigma_{0s} \approx f_p + f_b. \tag{5.33}$$

The ratio $\tilde{\gamma}/\gamma$ is just σ_{0s}/σ_{0p}.

With Gaussian or top hat smoothing the peak and background fields are correlated, in that peaks in the peak field tend to be higher in regions where the background field is larger. Gaussian smoothing for both f_s and f_b and a k^n unsmoothed power spectrum gives $\epsilon \approx 2^{n/2}(\sigma_{0b}/\sigma_{0s})$.

Substitute Eq. (5.30) into Eq. (5.29) and perform the integral over ν first to find the local density of "objects" as a function of δ_b,

$$n_{\rm obj}(\delta_b) = \int_0^\infty d\nu\, t\left(\frac{\nu}{\nu_t}\right) n_{\rm pk}\left(\nu|\delta_b\right). \tag{5.34}$$

This density is a number per unit *comoving* volume, and thus is proportional to the number of objects (galaxies or clusters) per unit *mass* even after the background fluctuations (clusters or superclusters) have evolved to large amplitude.

Define the *enhancement factor* $E(\delta_b)$ to be the ratio of $n_{\rm obj}(\delta_b)$ to $n_0 \equiv n_{\rm obj}(0)$. In the limit of a *high, sharp* threshold [62, 71],

$$E(\delta_b) \approx \exp\left(\frac{\nu_t \delta_b}{\sigma_{0s}}\right). \tag{5.35}$$

Numerical calculations (see [26]) show that the enhancement factor appropriate for galaxies is reasonably well approximated by

$$E(\delta_b) \approx e^{\alpha\left(\frac{\delta_b}{\sigma_{0s}}\right)-\frac{\beta}{2}\left(\frac{\delta_b}{\sigma_{0s}}\right)^2}, \tag{5.36}$$

with α a bit less than ν_t, $\alpha/\sigma_{0s} \sim 1$, and β roughly one.

The implication of Eq. (5.36) is that any region where the clustering of mass is substantially nonlinear today, so $\delta_b \gtrsim 1$, will have a substantial enhancement in the number of galaxies per unit mass, and therefore in the luminosity/mass ratio. There is at least qualitative success in explaining why local measurements of mass-to-light ratios extrapolating to a global density parameter $\Omega < 1$ can be consistent with $\Omega = 1$, and this success has been confirmed by numerical simulations [66].

When the smoothing is on the scale of clusters of galaxies, σ_{0s} is substantially smaller for a similar ν_t, and substantial statistical enhancements in the density of rich clusters can be the result of relatively small mass density fluctuations on supercluster scales.

On very large scales, where $\delta_b \ll 1$, the perturbation in the space density of galaxies in the present universe is

$$\frac{\delta n_{\rm gal}}{n_{\rm gal}} \approx (1+\delta_b)E - 1 \approx (1+\frac{\alpha}{\sigma_{0s}})\delta_b. \tag{5.37}$$

The ratio of the galaxy density perturbation to the mass density perturbation in the linear regime is called the *linear biasing factor* b. Clearly b depends on the normalization of the density perturbation field. In the quasi-analytic modeling of [26], matching the observed amplitude of galaxy-galaxy correlations gave b about $1.6 - 1.7$, while the most successful numerical simulations of [61] had $b \approx 2.5$.

5.6 Correlation Functions

Eq. (5.34) directly relates the number density fluctuations of a class of objects and mass density fluctuations on large scales in the background. The 2-point correlation amplitude of the objects in comoving coordinates,

$$\xi_{\rm com} = \frac{\langle n_{\rm obj}(\delta_b(x)) n_{\rm obj}(\delta_b(0))\rangle}{\langle n_{\rm obj}\rangle^2} - 1, \tag{5.38}$$

can be calculated as an integral over the joint probability distribution in $\nu_b(x) = \delta_b(x)/\sigma_{0b}$ and $\nu_b(0)$ as given by Eq. (5.9). The integral is easily done analytically when $n_{\rm obj}$ is expressed as n_0 times the enhancement factor in the form of Eq. (5.36). For consistency, $\langle n_{\rm obj}\rangle$ should also be found by an integral over Eq. (5.36). The result is [26]

$$1 + \xi_{\rm com}(r) = \frac{1}{\sqrt{1 - \left(\frac{\beta\psi}{1+\beta}\right)^2}} \exp\left[\frac{\left(\frac{\alpha}{1+\beta}\right)^2 \psi}{1 + \frac{\beta\psi}{1+\beta}}\right], \tag{5.39}$$

where $\psi(r)$ is the normalized mass correlation function for the background field δ_b as calculated from Eqs. (5.4) and (5.7).

Eq. (5.39) is at beast accurate only for r larger than about four times R_b, so that background mass correlations reflect the true mass correlations, and in turn R_b must be at least about three times R_s for the approximations behind the peak-background split to be valid. For galaxies this is not so bad, because the dynamical evolution is nonlinear on the smaller scales anyway. Eq. (5.39) does not work at all for correlations of rich clusters in the context of conventional CDM models, because the density perturbation spectrum steepens too rapidly going out to scales substantially larger than cluster R_s. Ref. [26] describe more accurate approximations to the peak correlations which are too complicated to describe here.

To observed correlation function is in real space, not comoving coordinate space, and reflects the dynamical evolution of the density perturbations. When both the statistical correlation amplitude $\xi_{\rm com}$ and the mass correlation amplitude ξ_ρ are small, the physical correlation amplitude is

$$\xi(r) \approx \left(\sqrt{\xi_{\rm com}(r)} + \sqrt{\xi_\rho(r)}\right)^2. \tag{5.40}$$

With a conventional CDM perturbation spectrum, scaled for a Hubble parameter $h \approx 0.5$, the result for $\xi(r)$ agrees in slope fairly well with the observed $r^{-1.8}$ power law [72, 73] on scales of a few ${\rm Mpc}\,h^{-1}$. Matching $\xi(r)$ from Eq. (5.40) to the observed correlation amplitude on scales large enough that $\xi \leq 1$, or alternatively to the observed J_3 integral, normalizes the amplitude of the density perturbations.

For the choice of parameters $h = 0.5$, $R_s = 0.356\,{\rm Mpc}\,h^{-1}$, Ref. [26] find $\sigma_{0s} \approx 2.4$. A typical galaxy forms from a peak with height $\nu \approx 2.8$, implying a collapse redshift $z_c \approx 3$. The bias factor $b = 1 + \alpha/\sigma_0 \approx 1.7$.

While the approximations we have discussed are not really appropriate for calculating cluster-cluster correlations, the qualitative picture is clear. The CDM mass correlation function becomes *negative* beyond $r \approx 20\,(\Omega h^2)^{-1}$Mpc. The high-peak correlation function reflects the sign of the mass correlation function. The Abell cluster correlations, as analyzed by [70], apparently maintain an $r^{-1.8}$ power law out to *at least* $r \sim 50\,\mathrm{Mpc}\,h^{-1}$. The Abell clusters were picked out by eye, and may well not form a good statistical sample. On the other hand, (see [74] for a review) no one has been able to identify a bias in the data that seems strong enough to remedy the conflict with CDM models. The issue remains unresolved, pending analysis of new computer-generated cluster catalogues.

5.7 Large Scale Streaming Motions

The development of efficient techniques for measuring large numbers of redshifts, the calibration of distance indicators based on the Fisher-Tully and Faber-Jackson relations between circular velocity or velocity dispersion and luminosity, and the IRAS catalogue of galaxies with its nearly complete sky coverage, all have begun to give intriguing glimpses of the large-scale distribution of galaxies, of their streaming motions, and of the source of the peculiar acceleration which has generated the 600 km/s peculiar velocity of the Local Group with respect to the CMB. Again, there are indications, not yet conclusive, of conflict with the predictions of simple CDM models.

The peculiar velocity is related to the smoothed gravitational potential perturbation by

$$v_i = -\frac{2}{3}\frac{1}{H} D_i \Phi_N, \tag{5.41}$$

and in terms of the density perturbation power spectrum,

$$\left\langle v^2 \right\rangle = 10^4 \int d(\ln k)\, \frac{P_\delta(k)}{k^2}\, (\mathrm{km\,s^{-1}})^2. \tag{5.42}$$

If the density perturbation spectrum is normalized so $\sqrt{\langle (\delta M/M)^2 \rangle} = 1/b$ in a sphere of radius $8\,\mathrm{Mpc}\,h^{-1}$, corresponding to $\left\langle (\delta n_{\rm gal}/n_{\rm gal})^2 \right\rangle = 1$ [18], then a Gaussian smoothing length R_s of $1\,\mathrm{Mpc}\,h^{-1}$, as perhaps appropriate for the Local Group, gives

$$\sqrt{\langle v^2 \rangle} \approx \frac{900}{b}\,\mathrm{km\,s^{-1}}. \tag{5.43}$$

This is perfectly consistent with the CMB dipole if $b \approx 1.5$. However, $\sqrt{\langle v^2 \rangle}$ decreases fairly rapidly as the smoothing length is increased. With $R_s = 5\,\mathrm{Mpc}\,h^{-1}$, perhaps appropriate for a rich cluster, $\sqrt{\langle v^2 \rangle} \approx 600/b\,\mathrm{km\,s^{-1}}$, which makes the $1000\,\mathrm{km\,s^{-1}}$ velocities seen for some clusters by [75] seem rather improbable with any substantial amount of biasing.

On the other hand, a direct maximum likelihood test for the value of b applied to the Ref. [75] data by [76] is consistent with $b = 1.4$, a bit smaller than what seems to be needed for galaxy clustering, but perhaps not totally unreasonable. The distant clusters with the large peculiar velocities are given low weight in [76].

The "Great Attractor" model of [77] fit to the same data has coherent infall velocities of $\sim 1000\,\mathrm{km\,s^{-1}}$ near the Great Attractor in the Hydra-Centaurus region and an infall velocity of over $500\,\mathrm{km\,s^{-1}}$ at the $40\,\mathrm{Mpc}\,h^{-1}$ distance of the local group. Ref. [**?**] found only one out of 10^6 random samples drawn from a CDM power spectrum with coherent infall velocities exceeding the Great Attractor model at all r.

A Great Attractor can be viewed as a deep gravitational potential well. We will use the peak profile techniques of App. D of [26] to find the mean profile of the radial velocity around the extremum in Φ_N. Consider the field $f = -\Phi_N$, as smoothed on a scale R_s. As discussed earlier, a peak is characterized by a height $\nu = f/\sigma_0$ and the variables x, y, z which describe the eigenvalues of the second derivative matrix $\xi_{ij} = -D_i D_j f$ evaluated at the peak according to Eq. (5.20). The radial velocity is, by Eq. (5.41), $v_r = (2/3)H^{-1}\partial_r f$. The general formula for the mean profile of a quantity $q(r)$ around the peak is

$$\langle q\rangle = \frac{(\nu-\gamma x)\langle q\nu\rangle + (x-\gamma\nu)\langle qx\rangle}{(1-\gamma^2)} + 15y\langle qy\rangle + 5z\langle qz\rangle\,. \tag{5.44}$$

The gravitational potential field is very different from the density perturbations discussed in [26] because the integral over the power spectrum for σ_0^2 diverges at large wavelengths, where Φ_N is just proportional to ζ. The integrals for σ_1^2 and σ_2^2 contain additional powers of k and are finite, as is the integral for $\langle v_r(r) f\rangle$. Thus $\gamma = -$, $\langle v_r(r)\nu\rangle = 0$. Averaged over all directions, the last two terms are zero. From $-\Delta f = 4\pi G\varepsilon$, $x = \delta/\sqrt{\langle\delta^2\rangle} \equiv \nu_\delta$, and

$$\langle v_r(r)\rangle = \nu_\delta(0)\,\langle v_r(r)\nu_\delta(0)\rangle\,. \tag{5.45}$$

As an integral over the density perturbation power spectrum,

$$\langle v_r(r)\delta(0)\rangle = Hr\int d(\ln k)\,P_\delta(k)\frac{\cos(kr) - \frac{\sin(kr)}{kr}}{(kr)^2}. \tag{5.46}$$

For $h = 0.5$, $R_s = 4\,\mathrm{Mpc}\,h^{-1}$, and [18] normalization of the CDM spectrum, the quantity $-b\langle v_r\rangle/\nu_\delta$ rises to a maximum of about $180\,\mathrm{km\,s^{-1}}$ at $r \approx 10\mathrm{Mpc}\,h^{-1}$ and then falls to only $35\,\mathrm{km\,s^{-1}}$ at $r \approx 40\mathrm{Mpc}\,h^{-1}$, the distance of the Local Group from the Great Attractor. Even if $b = 1$ and $\nu_\delta = 3$, the infall velocities are well below those in the Great Attractor model. Inconsistency with the model does not necessarily mean inconsistency with the data, however.

Preliminary analysis of the IRAS redshift survey data by [78] suggests that the large scale distribution of galaxies, assuming that it traces the mass, comes reasonably close to predicting the correct direction of the Local Group's velocity with respect to the CMB, within about 20° to 30°, and gives roughly the right magnitude for an assumed $\Omega \approx 1$. The analysis calculates the peculiar velocity field from the inferred peculiar acceleration field using linear perturbation theory. Only the almost complete sky coverage of the IRAS survey makes this possible. But there are still substantial uncertainties, particularly from the shot noise associated with the sparse sampling at the larger distances. Half of the Local Group's peculiar velocity seems to be generated within about $20\,\mathrm{Mpc}\,h^{-1}$ and perhaps 80% within $40\,\mathrm{Mpc}\,h^{-1}$, in apparent conflict with the Great Attractor model.

Ref. [79] calculates from the CDM theoretical model the average contribution to the peculiar velocity at the origin from mass fluctuations out to radius R, as a function of R, and also the root mean square statistical deviations from the mean profile. On average, for $\Omega h = 1/2$, half of the peculiar velocity should come from within $10\,\mathrm{Mpc}\,h^{-1}$, but the results are consistent with the IRAS data at the 1σ level.

6 Conclusions

For the most part these lectures have focussed on techniques for calculating the origin and evolution of density perturbations in the early universe, and for making statistical predictions of present large scale structure, rather than the results of such calculations. Ultimately, the best theoretical predictions for large scale structure will come from numerical simulations, since they can follow the nonlinear dynamical evolution, but quasi-analytic methods play a crucial role in providing initial conditions, physical insight, statistical predictions for rare events, *etc.*.

The one theoretical model that has received the most attention is the "biased" CDM model with adiabatic perturbations, a scale-invariant primordial spectrum, and galaxies identified with high peaks of the density perturbation field. The model works very well in explaining clustering properties of galaxies on scales less than about $10\,\mathrm{Mpc}\,h^{-1}$ if the Hubble parameter is small, $h \sim 0.5$. The properties of the bound objects themselves scale in about the right way with mass [80]. However, CDM is in serious trouble if h is close to one, since k_{eq}^{-1} becomes smaller and galaxy and cluster mass scales are moved to steeper parts of the density perturbation spectrum.

Even with h as small as it possibly can be ($h \sim 0.4$), there are strong indications that the simple CDM model conflicts with observations of structure on scales larger than $10 - 15\,\mathrm{Mpc}\,h^{-1}$. While CDM does predict a substantially higher amplitude for cluster-cluster correlations than for galaxy-galaxy correlations, it does not account for the full amplitude of the cluster-cluster correlations determined from the Abell catalogue at a separation of $r \sim 20\,\mathrm{Mpc}\,h^{-1}$, and the discrepancy grows rapidly as r increases. Any primordial density perturbation model with k_{eq} as the only scale (such as HDM) has similar problems with at least the slope of the cluster-cluster correlation function. The very large scale distribution of Abell clusters, compared with numerical simulations by [81] also seems incompatible with both CDM and HDM.

The peculiar velocity field seen within $30\,\mathrm{Mpc}\,h^{-1}$ in spiral galaxy data [82] and inferred from the IRAS redshift survey [78] may be at least roughly consistent with CDM if the biasing factor $b < 2$. However, the coherent velocities of $600\,\mathrm{km\,s^{-1}}$ on scales $\sim 50\,\mathrm{Mpc}\,h^{-1}$ that seem to be present in the elliptical galaxy data of [75] are not at all compatible with conventional CDM, particularly if interpreted as a single "Great Attractor" [77].

Taken at face value, the large scale streaming velocities imply density pertrubations large enough to produce CMB anisotropies on angular scales of a few degrees very close to prsent upper limits from observation. The measurement of the pattern of CMB anisotropy has tremendous potential for revealing the spectrum and statistics of primordial density perturbations [83, 84].

Further refinements in the observations of large scale structure may well eventually force us to abandon simple cosmological scenarios based on scale-invariant primordial perturbations with Gaussian statistics. Potential alternatives include generating large scale structure from cosmic strings or primordial explosions [85, 86]. Some nonstandard inflation models are also capable of producing more power on large scales and highly non-Gaussian statistics. Sorting out the possibilities will require both more observational data and careful theoretical analysis.

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A Notation and convention

J. Bardeen, V. Mukhanov and S. Dodelson use different notations and conventions in their books and notes. For the metric, Bardeen use $(-+++)$ signature and

$$ds^2 = -N_0^2(1+2\alpha)dt^2 - 2a^2(D_i\beta)dx^i dt + a^2\left(\gamma_{ij}(1+2\phi) + 2D_iD_j\gamma\right)dx^i dx^j. \tag{A.1}$$

Mukhanov use $(+---)$ instead and

$$ds^2 = a^2\left[(1+2\phi)d\eta^2 + 2(D_iB)dx^i d\eta - \left(\gamma_{ij}(1-2\psi) - 2D_iD_jE\right)dx^i dx^j\right]. \tag{A.2}$$

Dodelson use $(-+++)$ and

$$ds^2 = -(1+2A)dt^2 - 2a^2(D_iB)dx^i dt + a^2\left(\gamma_{ij}(1+2\psi) - 2D_iD_jE\right)dx^i dx^j. \tag{A.3}$$

The differences in the metric perturbation convention are summarized in the following list. Note that Bardeen's potentials are defined by $\Phi_H = \phi - H\chi$ and $\Phi_A = \alpha - \dot{\chi}/N_0$.

Bardeen	Mukhanov	Dodelson
α	ϕ	A
β	B	B
ϕ	$-\psi$	ψ
γ	$-E$	$-E$
Φ_H	$-\Psi$	$-\Phi$
Φ_A	Φ	Ψ

Table 1. Dictionary for notations used by Badeen, Mukhanov and Dodelson.

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