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A thermodynamic analysis of an enhanced theory of heat conduction model: Extended influence of finite strain and heat flux

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ABSTRACT

An Eulerian form of thermodynamically consistent generalized thermoelastic model capable of accounting for thermal signals is proposed. The heat flux is assumed to consist of both energetic and dissipative components. The constitutive relations for the stress, entropy and the heat fluxes are derived in the spatial coordinate system. Later on, a domain of dependence inequality for the proposed hyperbolic type heat conduction model is proved. The linearised form of this improved thermoelasticity theory of finite deformation is employed to study the thermoelastic interactions due a continuous source of heat in isotropic elastic solids. This newly developed model is also applied in a thermoelastic saturated porous medium. In order to obtain the exact analytical expressions of the field functions, appropriate integral transformations are employed in a convenient way.

1. Introduction

In mechanics and physics, invariance plays a significant role. There are some conservation laws in the all continuum theory; i.e., quantities that are constant in time, namely, mass and energy or balance laws such as balance of linear and angular momentum. There exist two ways to develop the continuum theory. In one hand building of continuum theory is to postulate those conservation or balance laws. On the other way, conservation laws and the balance laws can be obtained as a result of postulating invariance of a quantity such as energy or Lagrangian density, under some group of transformations.

In the classical theory of elasticity a deformation is termed as infinitesimal when the space derivatives of the components of the displacement vector of an arbitrary particle of the medium are so small that their squares and products may be neglected. Many attempts have been made to extend the classical theory of infinitesimal strain to the case of finite strains i.e., strains in which the fundamental hypothesis which serves to define an infinite strain is not legitimate. In the theory of finite strain, there is two essentially different perspectives which coalesce in the case of infinitesimal strain. We may use as the independent variables in terms of which the strains are described either in reference or spatial coordinate system. Most of the authors on the subject of finite strain have, probably for the reason of mathematical convenience, adopted the Lagrangian system. According to Seth (1935),"*Like the body-stress equation these (the components of strain) should be referred to the actual position of the material point in the strained condition, and not to the position of a point considered before strain. Apparently* Coker and Filon (1931) *were first to notice it and to stress its importance".*

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From the principle of energy conservation, in the classical theory, the relation between stress and strain can be expressed as: *The stress tensor equals to the gradient of the elastic-energy-density with respect to the strain tensor*. In infinitesimal theory, for a virtual displacement of the strained elastic medium, the virtual work done by all the forces, both surface as well as body, acting upon the medium can be obtained by integrating the scalar product of the stress tensor and the variation of the strain tensor over the medium. This principle is merely an approximation, it is not holds good in the finite-strain theory. The exact principle is that the virtual work is obtained by integrating the scalar product of the stress tensor with the space-derivative of the virtual displacement vector over the medium (for details see Ref. Murnaghan, 1937). So in this view point domain of dependence of the proposed model is very significant.

In the small deformation, physical observations and results of the conventional coupled dynamic thermoelasticity theories, which were based on the mixed parabolic-hyperbolic governing equations of Biot (1956b), involved infinite speed of thermal signals. This paradoxical circumstances led many researchers to advance various generalizations on the coupled dynamic thermoelasticity theory. They are based on different modifications of the classical Fourier law of heat conduction. The refinements aim at derivations of hyperbolic partial differential equations of coupled thermoelasticity. Those are to simultaneously satisfy the following criteria: (i) Finiteness of heat signal propagation speed, (ii) Spatial propagation of thermoelastic waves without attenuation, and (iii) Existence of distortion less wave forms akin to the classical d'Alembert type waves. The first was due to Cattaneo (1958), in which a wave type heat equation was derived by postulating a new law of heat conduction to replace the classical Fourier law. Later on, Lord and Shulman (1967), Green and Lindsay (1972) and Green and Naghdi (1991, 1992, 1993) have proposed three different generalized heat conduction equations which are the most discussed hyperbolic type heat equations in the literature. All of these modifications, so called thermoelasticity with second sound, have been performed in the infinitesimal theory of elasticity.

In the context of non-linear continuum mechanics, after the pioneering work of Lee (1969), using the multiplicative decomposition of the deformation gradient into mechanical and thermal parts, several authors (Darijani & Naghdabadi, 2010; Imam & Johnson, 1998; Lubarda, 2004) have placed their articles for modeling the material behavior under various mechanical and geometrical boundary conditions in the context of elasto-plastic deformation.

In this article, based on the laws of conservation of thermodynamics, a spatial description of a fully non-linear coupled problem of non-classical thermoelasticity is proposed. There are two distinctions from the classical theory. The heat flux vector is additively decomposed into two components: dissipative and energetic. Secondly, the material derivative of the absolute temperature with respect to the time is assumed to be proportionate with the heat and the entropy of the system. Finally, the linearised form of the proposed model is applied successfully in elastic and poroelastic medium with various circumstances.

2. Kinematics

Kinematics is the study of the motion of bodies, without regard to the cause of the motion. It is purely a study of geometry and is an exact science within the hypothesis of continuum.

We consider $\mathbf{x} = (x_1, x_2, x_3)$ be a fixed coordinate system with respect to the fixed point $\mathbf{o}$ in R^3 and an orthogonal basis $e_i (i = 1, 2, 3)$ is defined there. When a physical body with its reference configuration Ω_0 at, say, $t = 0$, moves over a period of time and occupies a configuration Ω_t at time t, the material point $\mathbf{X} = X_i e_i$ in $\bar{\Omega}_0$ (the closer of Ω_0) is mapped into the position $\mathbf{x}$ in Ω_t by a smooth vector valued function:

$$\mathbf{x} = \xi(\mathbf{X}, t).$$

Thus, $\xi(\mathbf{X}, t)$ is the spatial position of the material point $\mathbf{X}$ at time t. The one parameter family $\{\xi(\mathbf{X}, t)\}$ of position is called the trajectory of $\mathbf{X}$. We assume that the function ξ be differentiable, injective, and orientation preserving. The function ξ is called the motion of the body.

Since $det(\frac{\partial \chi_i}{\partial X^j}) > 0$, so there exist another mapping $\Xi : \mathbf{X} \xrightarrow{t} \mathbf{x}$ i.e. $\mathbf{X} = \Xi(\mathbf{x}, t)$ for every material point in the spatial configuration $\mathbf{x}$ there is one point $\Xi(\mathbf{x}, t)$ in the reference configuration.

The displacement of the point $\mathbf{x}$ is

$$\mathbf{u} = \Xi(\mathbf{x}) - \mathbf{x}$$

and

$d\mathbf{X} = \nabla \Xi(\mathbf{x}) d\mathbf{x}$ i.e. $dX_i = \frac{\partial \Xi_i}{\partial x^j} dx_j$

The tensor $\mathbf{E(x)} = \nabla \Xi(\mathbf{x})$ is called the deformation gradient.

Thus, $\mathbf{E(x)} = I + \nabla \mathbf{u(x)}$ where I is the identity tensor $\nabla \mathbf{u}$ is the displacement gradient.

3. Governing equations in generalized thermodynamic framework

3.1. Balance laws:

Balance of linear momentum yields the following equation

$$t_{ji,j} + \rho(f_j - \ddot{u}_j) = 0 \tag{1}$$

Balance of moment of momentum yields

$$t_{ij} = t_{ji} \tag{2}$$

where *tij* is the Cauchy stress tensor. ρ denotes the density of the material. f_j and u_j are the jth components of the body force density and displacement vector **u**. The superposed dot denotes the differentiation with respect to time t and 'comma' in the subscript denotes the differentiation with respect to spatial coordinates.

According to the first law of thermodynamics, balance of energy is given by

$$\dot{U} = t_{ij}\dot{e}_{ij} - div\mathbf{q} + Q \tag{3}$$

Here U, q and Q denote the internal energy, heat flux vector, and source of heat respectively. e_{ij} represents the strain tensor.

Second law of thermodynamics gives the balance of entropy inequality as follows:

$$\dot{\eta} \geq -div\mathbf{H} + S \tag{4}$$

where η, **H** and S are respectively the entropy, entropy flux vector, and source of entropy.

In Green-Naghdi Type-II and Type-III (for details see Refs. Green & Naghdi, 1992; Green & Naghdi, 1993) models of generalized thermoelasticity an extra thermal state variable, named thermal displacement, is defined by

$$\alpha(x,t) = \alpha_0(x) + \int_0^t \theta(x,s)ds \tag{5}$$

Here $\theta(x, t)$ is an empirical temperature scale, not necessarily the absolute one.

Now, assuming the summability of θ on R^-, we can extend the definition of thermal displacement as

$$\alpha(x,t) = \int_{-\infty}^t \theta(x,s)ds$$

If we consider $\alpha_0(x) = \alpha(x,0)$, in Eq. (5), summarizes the temperature history upto the initial time $t = 0$ i.e.

$$\alpha_0(x) = \int_{-\infty}^0 \theta(x,s)ds$$

Therefore, the temperature above the reference temperature i.e. the absolute temperature is defined by

$$\theta = \dot{\alpha} \quad and \quad \dot{\alpha} > 0. \tag{6}$$

Consequently,

$$\mathbf{H} = \frac{\mathbf{q}}{\dot{\alpha}} \quad and \quad S = \frac{Q}{\dot{\alpha}} \tag{7}$$

Now, substituting Eq. (7) in Eq. (4) we obtain,

$$\dot{\eta} \geq -div\left[\frac{\mathbf{q}}{\dot{\alpha}}\right] + \frac{Q}{\dot{\alpha}}$$

Using Eq. (6);

$$\frac{d}{dt}(U - \theta\eta) \leq -\dot{\theta}\eta - \frac{1}{\theta}\mathbf{q}.(\nabla\theta) + t_{ij}\dot{e}_{ij}$$

i.e.,

$$\dot{\psi} + \dot{\theta}\eta + \frac{1}{\theta}\mathbf{q}.(\nabla\theta) - t_{ij}\dot{e}_{ij} = -\theta\chi \leq 0 \tag{8}$$

where χ denotes the rate of entropy production and $\theta\chi$ represents the rate of energy dissipation. In which ψ denotes the Helmoltz's free energy function and Eq. (8) gives the balance of free energy.

3.2. Constitutive equations:

The modern research on thermodynamics to strengthening the hardening mechanisms for micro/nano structured materials indicate that the adaptation of only one type energetic or dissipative description may be insufficient to accurately describe the size effects exhibited in metallic components. therefore, in order to have a better understanding of the thermoelastic characteristics for micro/nano structured materials, it is important to incorporate more than one description of the thermodynamic processes into the modelling.

Without any loss of generality, as mentioned in the above motivation, we assume that the heat flux **q** is split additively as,

$$\mathbf{q} = \mathbf{q}_E + \mathbf{q}_D \tag{9}$$

where $\mathbf{q}_E$ and $\mathbf{q}_D$ are the energetic and dissipative components of the heat flux $\mathbf{q}$.

Consequently, the free energy balance Eq. (8) becomes,

$$\dot{\psi} - t_{ij}\dot{e}_{ij} + \dot{\theta}\eta + \frac{1}{\theta}\mathbf{q}_E.(\nabla\theta) + \frac{1}{\theta}\mathbf{q}_D.(\nabla\theta) = -\theta\chi \leq 0 \tag{10}$$

Here the function $\dot{\psi}$ and all other functions under consideration can be expressed in terms of the set of state variables $\{e_{ij}, \theta, \nabla\alpha, \nabla\theta\}$. Now by the chain rule of differentiation we get,

$$\dot{\psi} = \frac{\partial\psi}{\partial e_{ij}}\dot{e}_{ij} + \frac{\partial\psi}{\partial\theta}\dot{\theta} + \frac{\partial\psi}{\partial\Lambda}.\dot{\Lambda} + \frac{\partial\psi}{\partial\mathbf{G}}.\dot{\mathbf{G}} \tag{11}$$

where $\Lambda = \nabla\alpha, \mathbf{G} = \nabla\theta$.

Substitution of the time derivative of the free energy (11) into the Eq. (10) yields

$$\left(\frac{\partial\psi}{\partial e_{ij}} - t_{ij}\right)\dot{e}_{ij} + \left(\frac{\partial\psi}{\partial\theta} + \eta\right)\dot{\theta} + \left(\frac{\partial\psi}{\partial\Lambda} + \frac{1}{\theta}\mathbf{q}_E\right).\dot{\Lambda} + \frac{\partial\psi}{\partial\mathbf{G}}.\dot{\mathbf{G}} + \frac{1}{\theta}\mathbf{q}_D.(\nabla\theta) \leq 0 \tag{12}$$

which must be satisfied for all states.

As $\dot{e}_{ij}, \dot{\theta}, \dot{\Lambda}$ and $\dot{\mathbf{G}}$ can be chosen arbitrarily, it is sufficient to choose the constitutive equations as;

$$t_{ij} = \frac{\partial\psi}{\partial e_{ij}}, \quad \eta = -\frac{\partial\psi}{\partial\theta}, \quad \frac{1}{\theta}\mathbf{q}_E = -\frac{\partial\psi}{\partial\Lambda}, \quad \frac{\partial\psi}{\partial\mathbf{G}} = \mathbf{0} \quad \textit{and} \quad \mathbf{q}_D.(\nabla\theta) \leq 0 \tag{13}$$

in order to maintain the inequality (12). As a result, the rate of entropy production in any thermodynamically admissible process becomes

$$\chi = -\frac{1}{\theta^2}\mathbf{q}_D.(\nabla\theta) \tag{14}$$

Eq. (13) reveals that the free energy does not depend on $\nabla\theta$. Such a class of free energy function in the non-classical theory may be taken in the following form:

$$\psi = \psi_c + \frac{1}{2}K_1\nabla\alpha.\nabla\alpha$$

in which ψ_c represents the free energy function for classical thermoelasticity and K_1 is the non-classical symmetric, positive definite, second order conductivity tensor.

3.3. Field eqs. for heat conduction:

From the expressions obtained in the constitutive eqs. (13), the time derivative of the free energy function becomes

$$\dot{\psi} = t_{ij}\dot{e}_{ij} - \eta\dot{\theta} - \frac{1}{\theta}\dot{\mathbf{q}}_E.(\nabla\dot{\alpha})$$

or,

$$\dot{U} = t_{ij}\dot{e}_{ij} + \theta\dot{\eta} - \frac{1}{\theta}\dot{\mathbf{q}}_E.(\nabla\dot{\alpha})$$

Using Eq. (3);

$$-div\mathbf{q} + Q = \theta\dot{\eta} - \frac{1}{\theta}\dot{\mathbf{q}}_E.(\nabla\dot{\alpha}) \tag{15}$$

Now applying the constitutive relations (13) we obtain,

$$\rho C_e\dot{\theta} = -div\mathbf{q} + Q + \frac{2}{\theta}\dot{\mathbf{q}}_E.(\nabla\theta) + \theta\dot{e}_{ij}\frac{\partial t_{ij}}{\partial\theta} \tag{16}$$

where C_e denotes the specific heat and is defined by

$$\rho C_e = -\theta\frac{\partial^2\psi}{\partial\theta^2} \tag{17}$$

3.3.1. The linearised form:

In this section we summarize the initial boundary value problem (IBVP) for generalized thermoelasticity in linearised form.

The balance of linear momentum Eq. (1), the thermal displacement-temperature relation together with the balance of energy in the conservation form Eq. (15) lead to the system of PDEs governs the thermoelastic behaviour in the non-classical regime, given by

$$\begin{aligned} t_{ij,j} + \rho(f_i - \ddot{u}_i) &= 0, \\ \dot{\alpha} &= \theta, \\ T_0\dot{\eta} &= -div\mathbf{q} + Q, \end{aligned} \tag{18}$$

in which $\mathbf{u} = \dot{\mathbf{u}} = \mathbf{0}$, $\alpha = 0$, $\theta = T_0$ are assumed to be the natural state of the body.

The constitute equations can be derived from the classical Helmholtz free energy function, such that

$$t_{ij} = \frac{\partial\psi}{\partial e_{ij}}, \quad \eta = -\frac{\partial\psi}{\partial\theta}, \quad \mathbf{q} = \mathbf{q}_E + \mathbf{q}_D, \quad \mathbf{q}_E = -T_0\frac{\partial\psi}{\partial\Lambda}, \quad \mathbf{q}_D = -K_2\nabla\theta \tag{19}$$

and

$$\psi = \frac{1}{2}C_{ijkl} - T_0\beta_{ij}e_{ij} - \frac{\rho C_e}{2T_0}\theta^2 + \frac{1}{2}K_1(\nabla\alpha).(\nabla\alpha)$$

where C_{ijkl} is the forth order elasticity tensor, β_{ij} is the thermodynamic coupling tensor, K_2 is the classical heat conduction tensor and K_1 is the material constant characteristic of the body i.e., the non-classical conductivity tensor.

The linearised form of this theory has a wonderful similarity with Green-Naghdi Type-III model. In Green-Naghdi model if we substitute the non-classical conductivity tensor by T_0K_1 then this new model can be revealed.

3.4. Domain of dependence inequality:

Here we confine our attention on the statement and proof of the domain of dependence inequality for the solutions of the dynamic thermoelastic problem.

By a solution of the mixed initial boundary value problem in $\Omega = \Omega_t \times (0, \tau)$ we mean the pair $(\mathbf{u}, \theta)$ satisfy the Eqs. (18) in Ω together with initial conditions

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \dot{\mathbf{u}}(\mathbf{x}, 0) = \dot{\mathbf{u}}_0(\mathbf{x}), \theta(\mathbf{x}, 0) = T_0(\mathbf{x}), \tag{20}$$

and boundary conditions

$$\mathbf{u}(\mathbf{x}, t)|_{\partial\Omega_t^{(1)}} = \mathbf{0}, t_{ij}(\mathbf{x}, t)|_{\partial\Omega_t^{(2)}} = 0, \quad \theta(\mathbf{x}, t)|_{\partial\Omega_t^{(3)}} = 0, \quad [\mathbf{q}(\mathbf{x}, t).\mathbf{n}(\mathbf{x})]|_{\partial\Omega_t^{(4)}} = 0 \tag{21}$$

in which $\mathbf{n}$ is the unit normal to $\partial\Omega_t$ and $\partial\Omega_t^{(i)}$ fixed subset of $\partial\Omega_t$ such that

$$\partial\bar{\Omega}_t^{(1)} \cup \partial\Omega_t^{(2)} = \partial\bar{\Omega}_t^{(3)} \cup \partial\Omega_t^{(4)} = \partial\Omega_t \ ;$$

$$\partial\bar{\Omega}_t^{(1)} \cap \partial\Omega_t^{(2)} = \partial\bar{\Omega}_t^{(3)} \cap \partial\Omega_t^{(4)} = \emptyset.$$

If $\partial\Omega_t^{(1)} = \emptyset$, then there exists a family of rigid motions which are the solutions of Eq. (18). To avoid the occurrence of the trivial solution in the case of such boundary conditions it is necessary to impose the following normalization restrictions on the initial data and boundary forces:

$$\begin{aligned} &\int_{\Omega_t}\mathbf{u}_0(\mathbf{x})d\mathbf{x} = \int_{\Omega_t}\dot{\mathbf{u}}_0(\mathbf{x})d\mathbf{x} = \int_{\Omega_t}\mathbf{u}_0(\mathbf{x}) \times \mathbf{x}d\mathbf{x} = \int_{\Omega_t}\dot{\mathbf{u}}_0(\mathbf{x}) \times \mathbf{x}d\mathbf{x} = \mathbf{0}, \\ &\int_{\Omega_t}\mathbf{f}(\mathbf{x}, t)d\mathbf{x} = \int_{\Omega_t}\mathbf{f}(\mathbf{x}, t) \times \mathbf{x}d\mathbf{x} = \mathbf{0}; \quad 0 < t < \tau. \end{aligned} \tag{22}$$

With the similar analogy, if $\partial\Omega_t^{(3)} = \emptyset$, we assume

$$\begin{aligned} &\int_{\Omega_t}\left[T_0\beta_{ij}e_{ij}(\mathbf{x}, 0) + \rho C_e T_0(\mathbf{x})\right]d\mathbf{x} = 0, \\ &\int_{\Omega_t}\rho Q(\mathbf{x}, t)d\mathbf{x} = 0, \quad 0 < t < \tau. \end{aligned} \tag{23}$$

To analyse the obtained solutions of the mixed initial boundary value problem including all possible boundary conditions defined in Eq. (21), here we define following spaces:

$$\tilde{\mathbf{H}}^1(\Omega_t) = \left\{\mathbf{u}(\mathbf{x}, t) \in \mathbf{H}^1(\Omega_t) : \mathbf{u}|_{\partial\Omega_t^{(1)}} = \mathbf{0} \quad or \quad if, \quad \partial\Omega_t^{(1)} = \emptyset \quad then \quad (22)_1 \quad holds\right\},$$

$$\tilde{H}^1(\Omega_t) = \left\{\theta(\mathbf{x}, t) \in H^1(\Omega_t) : \theta|_{\partial\Omega_t^{(3)}} = 0 \quad or \quad if, \quad \partial\Omega_t^{(3)} = \emptyset \quad then \quad (23)_1 \quad holds\right\},$$

$$\tilde{\mathbf{H}}^2(\Omega_t) = \mathbf{H}^2(\Omega_t) \cap \tilde{\mathbf{H}}^1(\Omega_t); \quad \tilde{H}^2(\Omega_t) = H^2(\Omega_t) \cap \tilde{H}^1(\Omega_t).$$

$$\tilde{\mathbf{L}}^1(\Omega_t) = \left\{\mathbf{f}(\mathbf{x}, t) \in \mathbf{L}^2(\Omega_t); \quad or \quad if, \quad \partial\Omega_t^{(1)} = \emptyset \quad then \quad (22)_2 \quad holds\right\},$$

$$\tilde{\mathbf{L}}^2(\Omega_t) = \left\{\mathbf{G}(\mathbf{x}, t) \in \mathbf{L}^2(\Omega_t); \quad or \quad if, \quad \partial\Omega_t^{(3)} = \emptyset \quad then \quad (23)_2 \quad holds\right\},$$

$$\mathscr{L}(\Omega_t) = \tilde{\mathbf{L}}^1(\Omega_t) \times \tilde{\mathbf{L}}^2(\Omega_t), \quad \mathscr{L}(0, \tau : \Omega_t) = \tilde{\mathbf{L}}^2(0, \tau; \mathscr{L}(\Omega_t)).$$

$$\mathscr{W}(0, \tau; \Omega_t) = \Big[H^2\big(0, \tau; \mathbf{L}^2(\Omega_t)\big) \cap H^1\big(0, \tau; \mathbf{H}^1(\Omega_t)\big) \cap L^2\big(0, \tau; \tilde{\mathbf{H}}^1(\Omega_t)\big)\Big]\Big[H^1\big(0, \tau; \mathbf{L}^2(\Omega_t)\big) \cap L^2\big(0, \tau; \tilde{\mathbf{H}}^2(\Omega_t)\big)\Big].$$

In which L^1, L^2, H^2, etc. represent the usual space of scalar functions, while $\mathbf{L}^1, \mathbf{L}^2, \mathbf{H}^1$ etc. the set of vectorial or tensorial functions.

Definition 1: A pair $(\mathbf{u}, \theta)$ is said to be a strong solution of the IBVP (19)-(21) in Ω with the initial conditions $(\mathbf{u}_0, \theta_0, \dot{\mathbf{u}}_0) \in \tilde{\mathbf{H}}^2(\Omega_t) \times \tilde{H}^2(\Omega_t) \times \tilde{\mathbf{H}}^1(\Omega_t)$ and source $(\mathbf{f}, \mathbf{G}) \in \mathscr{L}(0, \tau; \Omega_t)$, if $(\mathbf{u}, \theta) \in \mathscr{W}(0, \tau; \Omega_t)$ and satisfy Eq. (19) almost everywhere in Ω and Eq. (21) almost everywhere in Ω_t.

Definition 2: If $(\mathbf{u}, \theta)$ be a solution of (19)-(21), then for every domain $\mathscr{A} \subset \Omega_t$, we define the total energy

$$\mathscr{E}(\mathscr{A}, t) = \int_{\mathscr{A}} \rho\Big[\frac{1}{2}|\dot{\mathbf{u}}(\mathbf{x}, t)|^2 + \psi(\mathbf{x}, t)\Big]d\mathbf{x} \tag{24}$$

We now employ the function $\mathscr{E}$ to state and proof the domain of dependence inequality for the solution of (19)-(21).

Lemma: Prove that

$$|\eta\dot{\theta}(t) + \frac{1}{\theta}\dot{\mathbf{q}}_E(t).(\nabla\theta(t))| \le \zeta\Big[\frac{1}{2}\dot{\mathbf{u}}^2 + \psi(t)\Big] \tag{25}$$

where ζ is a constant depends upon the material moduli.

Proof: The non-negativity of the entropy production defined in Eq. (14) is satisfied because of the inequality $(13)_4$ and it gives the insurance to satisfy the Eqs. (8) and (12). Hence, it follows that the temperature has to be strictly positive. Therefore,

$$\begin{aligned}|\eta\dot{\theta}(t) + \frac{1}{\theta}\dot{\mathbf{q}}_E(t).(\nabla\theta(t))| &\le |\dot{\psi} - t_{ij}\dot{e_{ij}}| \\ &\le |\frac{\partial\psi}{\partial\theta}\dot{\theta} + \frac{\partial\psi}{\partial\Lambda}.\dot{\Lambda} + \frac{\partial\psi}{\partial\mathbf{G}}.\dot{\mathbf{G}}| \\ &\le \frac{\partial}{\partial t}\Big(Total \quad energy \quad of \quad the \quad system\Big)\end{aligned}$$

Therefore, $|\eta\dot{\theta}(t) + \frac{1}{\theta}\dot{\mathbf{q}}_E(t).(\nabla\theta(t))| \le \zeta[\frac{1}{2}\dot{\mathbf{u}}^2 + \psi(t)]$. □

Theorem: Let $(\mathbf{u}, \theta)$ be a solution of (19)-(21), then for every $(\mathbf{x}_0, t_0) \in \Omega$ we have

$$\mathscr{E}(\mathscr{B}(\mathbf{x}_0, a), t_0) \le \mathscr{E}(\mathscr{B}(\mathbf{x}_0, a + \zeta t_0), 0) + \int_0^{t_0}\int_{\Omega_t \cap \mathscr{B}(\mathbf{x}_0, a+\zeta t_0)} \rho\Big[\dot{\mathbf{u}}.\ddot{\mathbf{u}} + t_{ij}\dot{e_{ij}}\Big]d\mathbf{x}dt \tag{26}$$

with $\mathscr{B}(\mathbf{x}_0, a) = \{\mathbf{x} \in \Omega_t; |\mathbf{x} - \mathbf{x}_0| < a\}$.

Proof: Without any loss of generality, let us consider a function ϕ defined in $C_0^\infty(\mathbb{R}^3, \mathbb{R})$ and $(\mathbf{u}, \theta)$ be a solution of (19)-(21) such that

$$\mathscr{E}_\phi(\Omega_t, t) = \int_{\Omega_t} \rho\Big[\frac{1}{2}|\dot{\mathbf{u}}(\mathbf{x}, t)|^2 + \psi(\mathbf{x}, t)\Big]\phi(\mathbf{x}, t)d\mathbf{x}$$

Therefore,

$$\dot{\mathscr{E}}_\phi(\Omega_t, t) = \int_{\Omega_t} \rho\Big[\dot{\mathbf{u}}.\ddot{\mathbf{u}} + \dot{\psi}(\mathbf{x}, t)\Big]\phi(\mathbf{x}, t)d\mathbf{x} + \int_{\Omega_t} \rho\Big[\frac{1}{2}|\dot{\mathbf{u}}(\mathbf{x}, t)|^2 + \psi(\mathbf{x}, t)\Big]\dot{\phi}(\mathbf{x}, t)d\mathbf{x}$$

Let $\mathbf{x}_0$ be a fixed point in Ω_t for $\mathbf{x} \in \Omega_t, \quad a > 0, \quad t < T$. We define

$$\phi(\mathbf{x}, t) = \phi_\delta\Big(|\mathbf{x} - \mathbf{x}_0| - a - \zeta(T - t)\Big)$$

with $\phi_\delta \in C^\infty(\mathbb{R}), \phi_\delta' < 0$ and

$$\begin{aligned}\phi_\delta(z) &= 1 \quad if \quad z \le -\delta \\ &= 0 \quad if \quad z \ge \delta\end{aligned}$$

So that

$$\begin{aligned}\dot{\phi}_\delta(\mathbf{x}, t) &= \zeta\phi_\delta'(\mathbf{x}, t) \\ \nabla\phi_\delta(\mathbf{x}, t) &= \nabla|\mathbf{x} - \mathbf{x}_0|\phi_\delta'(\mathbf{x}, t)\end{aligned}$$

Thus,

$$\dot{\mathscr{E}}_{\phi}(\Omega_t,t) = \int_{\Omega_t}\rho\Big[\dot{\mathbf{u}}.\ddot{\mathbf{u}}+t_{ij}\dot{e_{ij}}\Big]\phi_\delta(\mathbf{x},t)d\mathbf{x}-\int_{\Omega_t}\rho\Big\{\Big[\eta\dot{\theta}+\frac{1}{\theta}\mathbf{q}_E.(\nabla\theta)\Big]\phi_\delta(\mathbf{x},t)-\zeta\Big[\frac{1}{2}|\dot{\mathbf{u}}(\mathbf{x},t)|^2+\psi(\mathbf{x},t)\Big]\phi'_\delta(\mathbf{x},t)\Big\}d\mathbf{x}$$
$$\leq \int_{\Omega_t}\rho\Big[\dot{\mathbf{u}}.\ddot{\mathbf{u}}+t_{ij}\dot{e_{ij}}\Big]\phi_\delta(\mathbf{x},t)d\mathbf{x}$$

On integration over $(0, T)$ we obtain,

$$\mathscr{E}_{\phi_\delta}(\Omega_t,T)-\mathscr{E}_{\phi_\delta}(\Omega_t,0)\leq\int_0^T\int_{\Omega_t}\rho\Big[\dot{\mathbf{u}}.\ddot{\mathbf{u}}+t_{ij}\dot{e_{ij}}\Big]\phi_\delta(\mathbf{x},t)d\mathbf{x}dt$$

When δ tend to 0; $\phi_\delta(\mathbf{x},t)$ tends to the characteristic function of $\mathscr{B}(\mathbf{x}_0, a+\zeta(t-T))$, and the expression under the integral sign is given by Eq. (26). □

4. Applications

Thermo-acoustics in gases is a mechanism by which work done by the acoustic waves can be applied to transfer heat across the medium (Garrett, 2004; Swift, 1988; Wheatley, Hofler, Swift & Migliori, 1985). Though acoustic waves themselves do not transport heat on the average, it is possible to have such a transport if the phase angle between the particle velocity and temperature oscillations is altered by the presence of a thermal reservoir. The oscillation of fluid particles results in instantaneous heat transport. This transport can produce a heat transport if its time average at a given location is non-zero. Due to the intrinsic higher thermal conductivity, similar phenomenon is also exits in solids. In fact, given thermoelastic propagating waves in a homogeneous isotropic solid, the phase angle between the particle velocity and temperature oscillations enables motion-induced heat flux. This heat flux component is in addition to the Fourier conduction of heat flux.

4.1. Isotropic elastic solid

This section is devoted to analyse the thermoelastic interactions caused by a continuous heat source in a homogeneous and isotropic unbounded thermoelastic solid, by employing the proposed new thermoelasticity theory. The problem is solved by using suitable integral transformations. After some tedious mathematical manipulations, we have obtained the exact expressions of the temperature and stress fields in closed form.

4.1.1. Governing equations:

The governing equations (linear form) of the thermodynamically consistent generalized thermoelasticity at finite deformation are

$$\mu\nabla^2\mathbf{u}+(\lambda+\mu)\nabla(div\mathbf{u})-\beta\nabla\theta+\rho\mathbf{f}=\rho\ddot{\mathbf{u}} \tag{27}$$

$$T_0K_1\nabla^2\theta+K_2\nabla^2\dot{\theta}+\rho\dot{Q}=\rho C_e\ddot{\theta}+\beta T_0(\nabla.\ddot{\mathbf{u}}) \tag{28}$$

Here $\mathbf{u}$ is the displacement vector; θ is the temperature change with respect to the reference temperature T_0; $\mathbf{f}$ is the external force; and Q is the external rate of heat supply, both measured per unit mass; ρ is the mass density; C_e is the specific heat; λ and μ are the lame constants; $\beta=(3\lambda+2\mu)\alpha_t$; α_t being the coefficient of volume expansion.

The strain tensor $\mathbf{E}$ and the stress tensor $\mathbf{T}$ associated with $\mathbf{u}$ and θ are given by the following geometrical and constitutive relations, respectively, as

$$\mathbf{E}=\frac{1}{2}(\nabla\mathbf{u}+\nabla\mathbf{u}^T),$$
$$\mathbf{T}=\lambda(div\mathbf{u})\mathbf{I}+\mu(\nabla\mathbf{u}+\nabla\mathbf{u}^T)-\beta\theta\mathbf{I} \tag{29}$$

In all the above equations, the direct vector/tensor notation is employed. A superposed dot denotes the partial derivative with respect to the time variable t.

We suppose that the constants appearing in Eqs. (27) and (28) satisfy the inequalities:

$$\mu>0,\ \lambda+\mu>0,\ \rho>0,\ T_0>0,\ C_e>0,\ \beta>0,\ K_1>0,\ K_2>0$$

Then Eqs. (27) and (28) represent a fully hyperbolic system that permits finite speed for both elastic and thermal waves, which are coupled together in general.

In the problem of discussing a specific initial boundary value problem, it is convenient to rewrite the Eqs. (27–29) in non-dimensional form. For this we consider the transformations:

$$\mathbf{x}'=\frac{1}{l}\mathbf{x},\quad t'=\frac{v}{l}t,\quad \mathbf{u}'=\frac{\lambda+2\mu}{l\beta T_0}\mathbf{u},\quad \theta'=\frac{\theta}{T_0},\quad \mathbf{E}'=\frac{\lambda+2\mu}{\beta T_0}\mathbf{E},\quad \mathbf{T}'=\frac{1}{\beta T_0}\mathbf{T},$$
$$\mathbf{f}'=\frac{l(\lambda+2\mu)}{\beta T_0\rho v^2}\mathbf{f},\quad Q'=\frac{l}{\rho C_e v T_0}Q$$

Here l is a standard length and v is a standard speed.

Applying the above mentioned transformations and suppressing primes, Eqs. (27–29) can be recast in the following manner:

$$C_S^2\nabla^2\mathbf{u} + (C_P^2 - C_S^2)\nabla(div\mathbf{u}) - C_P^2\nabla\theta + \rho\mathbf{f} = \ddot{\mathbf{u}} \tag{30}$$

$$C_T^2\nabla^2\theta + C_D^2\nabla^2\dot{\theta} + \rho\dot{Q} = \ddot{\theta} + \epsilon(\nabla.\ddot{\mathbf{u}}) \tag{31}$$

$$\mathbf{E} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$$
$$\mathbf{T} = \left(1 - 2\frac{C_S^2}{C_P^2}\right)(div\mathbf{u})\mathbf{I} + \frac{C_S^2}{C_P^2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T) - \theta\mathbf{I} \tag{32}$$

where $C_P^2 = \frac{\lambda+2\mu}{\rho v^2}$, $C_S^2 = \frac{\mu}{\rho v^2}$, $C_T^2 = \frac{T_0K_1}{\rho C_e v^2}$, $C_D^2 = \frac{K_2}{l\rho C_e v}$ and $\epsilon = \frac{\beta^2 T_0}{\rho C_e(\lambda+2\mu)}$.

Eqs. (30) and (31) represent the governing equations in coupled form for displacement and temperature fields. We observe that C_P and C_S, respectively, denote the speeds of purely elastic dilatational and shear waves. C_T represents the speed of the thermal waves without energy dissipation at finite deformation and C_D denotes the speed of the purely thermal waves in the material body. ϵ is the usual thermoelastic coupling parameter.

4.1.2. Statement of the problem:

Here we consider a homogeneous, isotropic unbounded thermoelastic medium and the conventional cylindrical polar coordinate system is adopted to characterise the position of the particles. With the assumption of axisymmetric condition, the non-dimensional governing equations, in absence of body forces, become

$$C_P^2\left(\nabla^2\phi - \theta\right) = \frac{\partial^2\phi}{\partial t^2} \tag{33}$$

$$\left(C_T^2 + C_D^2\frac{\partial}{\partial t}\right)\nabla^2\theta = \frac{\partial^2\theta}{\partial t^2} + \epsilon\frac{\partial^2}{\partial t^2}\left(\nabla^2\phi\right) - \rho\frac{\partial Q}{\partial t} \tag{34}$$

$$\sigma_r = \frac{1}{C_P^2}\frac{\partial^2\phi}{\partial t^2} - \frac{2}{r}\frac{C_S^2}{C_P^2}\frac{\partial\phi}{\partial r} \tag{35}$$

$$\sigma_\Phi = \frac{1}{C_P^2}\frac{\partial^2\phi}{\partial t^2} - \frac{2C_S^2}{C_P^2}\frac{\partial^2\phi}{\partial r^2} \tag{36}$$

where r is the radial distance measured from the axis of symmetry $(z - axis)$, ∇^2 is the Laplacian operator, ϕ is the thermoelastic potential, defined by

$$u = \frac{\partial\phi}{\partial r}.$$

In which u is the radial displacement, σ_r is the radial stress, σ_Φ is the circumferential stress.

Again we suppose that the medium was at rest in a undeformed and unstressed state at a uniform reference temperature. There was no body force acted upon the medium. The deformation occurred due to the presence of internal heat sources. The strength of the internal source of heat is given by

$$Q = \frac{Q_0}{r}\delta(r)H(t) \tag{37}$$

where $\delta(r)$ is the Dirac-delta function, $H(t)$ is the Heaviside unit step function and Q_0 is a constant.

4.1.3. Solution in integral transform domain:

Elimination of θ from Eqs. (33) and (34) gives

$$\left[\left(C_T^2 + C_D^2\frac{\partial}{\partial t}\right)C_P^2\nabla^4 - \left\{\left(C_T^2 + C_D^2\frac{\partial}{\partial t}\right) + (1+\epsilon)C_P^2\right\}\nabla^2\frac{\partial^2}{\partial t^2} + \frac{\partial^4}{\partial t^4}\right]\phi + C_P^2\rho\frac{\partial Q}{\partial t} = 0 \tag{38}$$

As, initially the medium was at rest with undeformed and unstressed state, so all the field functions satisfy homogeneous initial conditions. Taking Laplace transformation of the Eq. (38) under the homogeneous initial conditions we obtain

$$\left[(C_T^2 + C_D^2\, s)C_P^2\nabla^4 - \left\{(C_T^2 + C_D^2\, s) + (1+\epsilon)C_P^2\right\}s^2\nabla^2 + s^4\right]\bar{\phi} + \rho C_P^2\frac{Q_0}{r}\delta(r) = 0 \tag{39}$$

Here an over bar denotes the Laplace transformation of the field functions and s is the transform parameter.

Eq. (39) may be written in the following form:

$$\left(\nabla^2-\lambda_1^2\right)\left(\nabla^2-\lambda_2^2\right)\bar{\phi}=-\frac{Q_0\rho}{\left(C_T^2+C_D^2\ s\right)r}\delta(r) \tag{40}$$

where λ_1 and λ_2 are the roots of the bi-quadratic equation:

$$\lambda^4-\frac{s^2}{\left(C_T^2+C_D^2\ s\right)C_P^2}\left\{\left(C_T^2+C_D^2\ s\right)+(1+\epsilon)C_P^2\right\}\lambda^2+\frac{s^4}{\left(C_T^2+C_D^2\ s\right)C_P^2}=0 \tag{41}$$

Now, applying Hankel transformation in both sides of Eq. (40) we get

$$\left(\xi^2+\lambda_1^2\right)\left(\xi^2-\lambda_2^2\right)\tilde{\phi}=-\frac{Q_0\rho}{\left(C_T^2+C_D^2\ s\right)} \tag{42}$$

where

$$\tilde{\phi}(\xi,s)=H_0\left[\bar{\phi}(r,s);r\to\xi\right]=\int_0^\infty rJ_0(\xi r)\bar{\phi}(r,s)dr$$

in which J_0 is the Bessel function of 1st kind and of order zero.

Taking inverse Hankel transformation defined by

$$\bar{\phi}(r,s)=\int_0^\infty \xi J_0(\xi r)\tilde{\phi}(\xi,s)d\xi=H_0^{-1}\left[\tilde{\phi}(\xi,s);\xi\to r\right]$$

from Eq. (42) we obtain

$$\bar{\phi}(r,s)=\frac{Q_0\rho}{\left(C_T^2+C_D^2\ s\right)\left(\lambda_1^2-\lambda_2^2\right)}\left[K_0\left(\lambda_1 r\right)-K_0\left(\lambda_2 r\right)\right] \tag{43}$$

where $K_0(z)$ is the modified Bessel function of second kind and of order zero.

With the help of Eq. (43) from Eq. (33) we get

$$\bar{\theta}(r,s)=\frac{Q_0\rho}{\left(C_T^2+C_D^2\ s\right)\left(\lambda_1^2-\lambda_2^2\right)}\left[\left(\lambda_1^2-\frac{s^2}{C_P^2}\right)K_0\left(\lambda_1 r\right)-\left(\lambda_2^2-\frac{s^2}{C_P^2}\right)K_0\left(\lambda_2 r\right)\right] \tag{44}$$

and from Eqs. (35) and (36) we obtain the expressions of stresses in the Laplace transform domain

$$\bar{\sigma}_r(r,s)=\frac{Q_0\rho}{\left(C_T^2+C_D^2\ s\right)\left(\lambda_1^2-\lambda_2^2\right)}\sum_{\alpha=1}^{2}(-1)^{\alpha-1}\left[\frac{s^2}{C_P^2}K_0\left(\lambda_\alpha r\right)-\frac{2C_S^2}{C_P^2 r}\lambda_\alpha I_1\left(\lambda_\alpha r\right)\right] \tag{45}$$

$$\bar{\sigma}_\Phi(r,s)=\frac{Q_0\rho}{\left(C_T^2+C_D^2\ s\right)\left(\lambda_1^2-\lambda_2^2\right)}\sum_{\alpha=1}^{2}(-1)^{\alpha-1}\left[\left(\frac{s^2}{C_P^2}-\frac{2C_S^2}{C_P^2 r}\lambda_\alpha^2\right)K_0\left(\lambda_\alpha r\right)-\frac{2C_S^2}{C_P^2 r}\lambda_\alpha I_1\left(\lambda_\alpha r\right)\right] \tag{46}$$

where $I_1(z)$ denotes the modified Bessel function of 1st kind and of order one.

4.1.4. Special case:

Eqns. (44)–(46) represents the expressions of the temperature and stresses in the Laplace transform domain. Now, when there is no energy dissipating from the system i.e., $K_2=0$ we obtain the following exact solutions, in closed form, for θ, σ_r and σ_Φ :

$$\theta(r,t)=\frac{\rho Q_0 V_1^2 V_2^2}{C_T^2\left(V_1^2-V_2^2\right)}\sum_{\alpha=1}^{2}\left[(-1)^\alpha\left(\frac{1}{V_\alpha^2}-\frac{1}{C_P^2}\right)\left(t^2-\frac{r^2}{V_\alpha^2}\right)^{-1/2}H\left(t-\frac{r}{V_\alpha}\right)\right] \tag{47}$$

$$\sigma_r(r,t)=\frac{\rho Q_0 V_1^2 V_2^2}{C_T^2\left(V_1^2-V_2^2\right)}\sum_{\alpha=1}^{2}(-1)^\alpha\left[\frac{1}{C_P^2}\left(t^2-\frac{r^2}{V_\alpha^2}\right)^{-1/2}+\frac{2C_S^2}{C_P^2 r^2}\left(t^2-\frac{r^2}{V_\alpha^2}\right)^{1/2}\right]H\left(t-\frac{r}{V_\alpha}\right) \tag{48}$$

$$\sigma_\Phi(r,t)=\frac{\rho Q_0 V_1^2 V_2^2}{C_T^2\left(V_1^2-V_2^2\right)}\sum_{\alpha=1}^{2}(-1)^\alpha\left[\left(\frac{1}{C_P^2}-\frac{2C_S^2}{C_P^2 V_\alpha^2}\right)\left(t^2-\frac{r^2}{V_\alpha^2}\right)^{-1/2}-\frac{2C_S^2}{C_P^2 r^2}\left(t^2-\frac{r^2}{V_\alpha^2}\right)^{1/2}\right]H\left(t-\frac{r}{V_\alpha}\right) \tag{49}$$

where $V_\alpha=\frac{1}{\sqrt{2}}\left[\{C_T^2+(1+\epsilon)C_P^2\}+(-1)^{\alpha+1}\Delta\right]^{1/2}$ and

$\Delta=\left[\{C_T^2-(1+\epsilon)C_P^2\}^2+4\epsilon C_P^2 C_T^2\right]^{1/2}$

From the above expressions of temperature distribution and stress field it is observed that each of the solutions consist of two components corresponding to two distinct coupled waves propagating with the respective speeds V_1 and V_2. It is also noticed that the faster wave with speed V_1 is a predominantly elastic wave $(e-wave)$ or a predominantly thermal wave $(\theta-wave)$ according as $C_P > C_T$ or $C_P < C_P$. A Similar analysis holds for the slower wave with speed V_2 that is, elastic wave for $C_P < C_T$ and thermal wave for $C_P > C_T$.

4.2. Poroelastic solid

The theories of thermoelasticity as well as poroelasticity were established by Biot (1956a,b). He was very much aware of the isomorphism between thermoelastic continua and mechanics of porous media. Biot (1964) also shows that the theory of porous media applies immediately to thermoelasticity. Through the experimental study, Gurevich, Kelder and Smeulders (1999) confirms that the poroelasticity theory of Biot is adequate to describe the behaviour of porous material. There is another analogy, to translate the results from thermoelasticity in solving the problems of poroelasticity, in which the fluid pressure replaces the temperature and the relative fluid displacement is applied instead of the thermal displacement. In this section, the theory of Biot is adopted in a thermally conducting, isotropic porous solid saturated with a non-viscous fluid.

4.2.1. Governing equations:

The governing equations for isotropic thermally conducting fluid saturated porous medium is given by (in absence of body forces)

$$\mu\nabla^2\mathbf{u} + (\lambda+\mu+\alpha^2 M)\nabla(div\mathbf{u}) + \alpha M\nabla(div\mathbf{w}) - \beta_s\nabla\theta = \rho\ddot{\mathbf{u}} + \rho_f\ddot{\mathbf{w}} \tag{50}$$

$$\alpha M\nabla(div\mathbf{u}) + M\nabla(div\mathbf{w}) - \beta_f\nabla\theta = \rho_f\ddot{\mathbf{u}} + q\ddot{\mathbf{w}} \tag{51}$$

$$T_0K_1\nabla^2\theta + K_2\nabla^2\dot{\theta} + \rho\dot{Q} = \rho C_e\ddot{\theta} + \beta T_0(\nabla.\ddot{\mathbf{u}} + \nabla.\ddot{\mathbf{w}}) \tag{52}$$

Here λ and μ are the Lame constants for porous medium; M is the elastic parameter for isotropic bulk coupling of fluid and solid particles; w_i denotes the component of the averaged fluid motion relative to solid frame and is defined as $w_i = f(U_i - u_i)$, in which f is the porosity of the solid, u_i and U_i are the displacement components in solid and fluid phases respectively. $\beta = \beta_s + \alpha\beta_f$; ρ and ρ_f are the densities of porous aggregate and pore-fluid respectively. q is the parameter represents inertial coupling between pore-fluid and solid matrix of porous aggregate.

As it was done in the previous section, the dimensionless form of the eqs. (50) - (52) are as follows:

$$C_S^2\nabla^2\mathbf{u} + (C_P^2 - C_S^2 + \alpha^2 C_F^2)\nabla(div\mathbf{u}) + \alpha C_F^2\nabla(div\mathbf{w}) - \frac{\beta_s}{\beta}C_P^2\nabla\theta = \ddot{\mathbf{u}} + \frac{\rho_f}{\rho}\ddot{\mathbf{w}} \tag{53}$$

$$\alpha C_F^2\nabla(div\mathbf{u}) + C_F^2\nabla(div\mathbf{w}) - \frac{\beta_f}{\beta}C_P^2\nabla\theta = \frac{\rho_f}{\rho}\ddot{\mathbf{u}} + \frac{q}{\rho}\ddot{\mathbf{w}} \tag{54}$$

$$C_T^2\nabla^2\theta + C_D^2\nabla^2\dot{\theta} + \rho\dot{Q} = \ddot{\theta} + \epsilon(\nabla.\ddot{\mathbf{u}} + \nabla.\ddot{\mathbf{w}}) \tag{55}$$

where $C_F^2 = \frac{M}{\rho v^2}$.

Eqs. (53)–(55) represent the governing equations in coupled form of displacements of solid and fluid phases and the temperature field. C_F denotes the speed of the fluid motion relative to solid frame.

Similar to the previous section, due to axial-symmetric the above set of equations becomes;

$$\Big(C_P^2 + (\alpha - f)\alpha C_F^2\Big)\nabla^2\phi - \frac{\beta_s}{\beta}C_P^2\theta + f\alpha C_F^2\nabla^2\Phi = \Big(1 - \frac{\rho_f}{\rho}f\Big)\ddot{\phi} + f\frac{\rho_f}{\rho}\ddot{\Phi} \tag{56}$$

$$\big(\alpha - f\big)C_F^2\nabla^2\phi - \frac{\beta_s}{\beta}C_P^2\theta + fC_F^2\nabla^2\Phi = \Big(\frac{\rho_f}{\rho} - \frac{q}{\rho}f\Big)\ddot{\phi} + f\frac{q}{\rho}\ddot{\Phi} \tag{57}$$

$$\Big(C_T^2 + C_D^2\frac{\partial}{\partial t}\Big)\nabla^2\theta = \frac{\partial^2\theta}{\partial t^2} + \epsilon\frac{\partial^2}{\partial t^2}\Big((1-f)\nabla^2\phi - f\nabla^2\Phi\Big) - \rho\frac{\partial Q}{\partial t} \tag{58}$$

where r is the radial distance measured from the axis of symmetry $(z-axis)$, ∇^2 is the Laplacian operator, ϕ is the thermoelastic potential, defined by

$$u = \frac{\partial\phi}{\partial r}.$$

In which u is the radial displacement, and Φ is the potential function for fluid motion over the solid such that

$$U = \frac{\partial\Phi}{\partial r}.$$

Now from the eqs. (56) - (58) we obtain,

$$\left(A\nabla^6 + B\nabla^4 + C\nabla^2 + D\right)\phi = \left[\left\{fC_F^2\nabla^2 - \frac{q}{\rho}f\frac{\partial^2}{\partial t^2}\right\}\left\{f\alpha C_F^2\nabla^2 - \frac{\rho_f}{\rho}f\frac{\partial^2}{\partial t^2}\right\} - \frac{\beta_s\beta_f}{\beta^2}C_P^4\right]\rho\frac{\partial Q}{\partial t} \tag{59}$$

where

$$\begin{aligned}
A &= f\alpha C_F^4(\alpha - f)\left(C_D^2\frac{\partial}{\partial t} - C_T^2\right)\\
B &= C_F^2(\alpha - f)\left\{\frac{\rho_f}{\rho}f\left(C_T^2 - C_D^2\frac{\partial}{\partial t}\right)\frac{\partial^2}{\partial t^2} + f\left(\alpha C_F^2 + \frac{\beta_s}{\beta}\epsilon C_P^2\right)\frac{\partial^2}{\partial t^2}\right\} + f\alpha C_F^2\left(\frac{\rho_f}{\rho} - \frac{q}{\rho}f\right)\left(C_T^2 - C_D^2\frac{\partial}{\partial t}\right)\frac{\partial^2}{\partial t^2}\\
&\quad - fC_P^2C_F^2\epsilon(1 - f)\left(\frac{\beta_s}{\beta} - \alpha\frac{\beta_f}{\beta}\right)\frac{\partial^2}{\partial t^2}\\
C &= \epsilon f(1 - f)\left(\frac{q\beta_s}{\rho\beta} - \frac{\rho_f\beta_f}{\rho\beta}\frac{\partial^4}{\partial t^4}\right)\left(\frac{\rho_f}{\rho} - f\frac{q}{\rho}\right)\left\{f\frac{\rho_f}{\rho}\left(C_T^2 - C_D^2\frac{\partial}{\partial t}\right)\frac{\partial^2}{\partial t^2} - f\alpha C_F^2\frac{\partial^2}{\partial t^2} - f\epsilon C_P^2\frac{\beta_s}{\beta}\frac{\partial^2}{\partial t^2}\right\}\frac{\partial^2}{\partial t^2}\\
&\quad - fC_F^2\frac{\rho_f}{\rho}(\alpha - f)\frac{\partial^2}{\partial t^2}\\
D &= f\frac{\rho_f}{\rho}\left(\frac{\rho_f}{\rho} - f\frac{q}{\rho}\right)\frac{\partial^4}{\partial t^4}
\end{aligned}$$

This gives the governing equation for ϕ. Once ϕ is determined by solving this equation (with appropriate boundary conditions), then u, U and θ are follows from the eqs. (56) - (58).

4.2.2. Solution of the problem:

We suppose that initially the porous medium was at rest in an unstressed and unstrained state at a uniform reference temperature T_0. Also, we consider the strength of the heat source is given by

$$Q = \frac{Q_0}{r}\delta(r)H(t) \tag{60}$$

Now employing Laplace transformation with respect to the time variable t and after some mathematical manipulations eqs. (56) - (59) yields

$$\begin{aligned}
\bar{\phi}(r,s) &= Q_0\Big[A_3K_0(\lambda_3 r) + A_4K_0(\lambda_4 r) + A_5K_0(\lambda_5 r)\Big]\\
\bar{\Phi}(r,s) &= Q_0V\Big[A_3K_0(\lambda_3 r) + A_4K_0(\lambda_4 r) + A_5K_0(\lambda_5 r)\Big]\\
\bar{\theta}(r,s) &= Q_0\frac{\beta}{\beta_f C_P^2}\sum_{i=3}^{5}A_i\left[\Big\{\alpha - (V-1)f\Big\}C_F^2\frac{1-\lambda_i}{r}K_1(\lambda_i r) + \left\{\Big(\alpha - (V-1)f\Big)\lambda_3 C_F^2 - \left(\frac{\rho_f}{\rho} - (V-1)\frac{q}{\rho}\right)s^2\right\}K_0(\lambda_i r)\right]
\end{aligned} \tag{61}$$

In which

$$\begin{aligned}
A_3 &= \frac{-1}{\left(\lambda_3^2 - \lambda_4^2\right)\left(\lambda_3^2 - \lambda_5^2\right)}\left[\Big(\rho_f\beta_f - q\beta_s\Big)s^2 - \rho\lambda_3^2C_F^2\Big(\alpha\beta_f - \beta_s\Big)\right]\frac{f}{\beta}C_P^2\\
A_4 &= \frac{-1}{\left(\lambda_3^2 - \lambda_4^2\right)\left(\lambda_3^2 - \lambda_5^2\right)}\left[\Big(\rho_f\beta_f - q\beta_s\Big)s^2 - \rho\lambda_4^2C_F^2\Big(\alpha\beta_f - \beta_s\Big)\right]\frac{f}{\beta}C_P^2\\
A_5 &= \frac{-1}{\left(\lambda_3^2 - \lambda_4^2\right)\left(\lambda_3^2 - \lambda_5^2\right)}\left[\Big(\rho_f\beta_f - q\beta_s\Big)s^2 - \rho\lambda_5^2C_F^2\Big(\alpha\beta_f - \beta_s\Big)\right]\frac{f}{\beta}C_P^2
\end{aligned}$$

$$V = \frac{C_P^2 + (\alpha - f)(\alpha - 1)C_F^2}{f(1-\alpha)C_F^2}$$

and λ_i^2 $(i = 3, 4, 5)$ are the roots of the equation

$$\bar{A}\lambda^6 + \bar{B}\lambda^4 + \bar{C}\lambda^2 + \bar{D} = 0 \tag{62}$$

The roots are given by

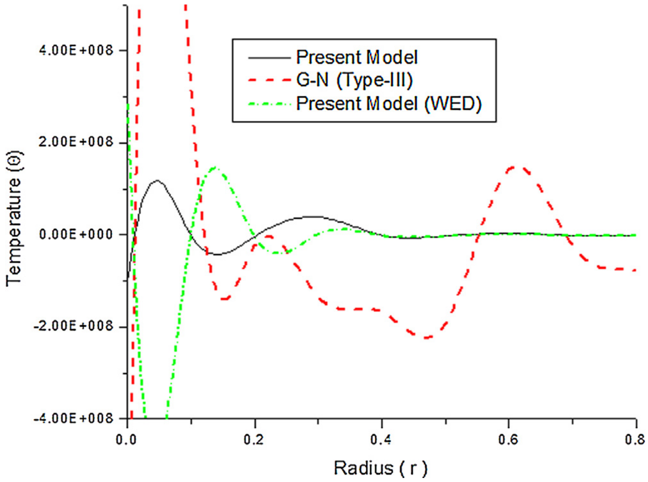

Fig. 1. Temperature distribution near the source of heat at $t = 0.25$.

$$\begin{aligned}
\lambda_3^2 &= \frac{1}{3}\left(2\bar{P}\sin(\bar{Q}) + \bar{A}\right)\\
\lambda_4^2 &= -\frac{\bar{P}}{3}\left(\sqrt{3}\cos(\bar{Q}) + \sin(\bar{P})\right) + \frac{\bar{A}}{3}\\
\lambda_5^2 &= \frac{\bar{P}}{3}\left(\sqrt{3}\cos(\bar{Q}) - \sin(\bar{P})\right) + \frac{\bar{A}}{3}
\end{aligned} \tag{63}$$

where
$\bar{P} = \frac{\sqrt{\bar{B}^2 - 3\bar{A}\bar{C}}}{\bar{A}}$, $\bar{Q} = \frac{1}{3}\sin^{-1}(\bar{R})$ and $\bar{R} = -\frac{2\bar{B}^3 - 9\bar{A}\bar{B}\bar{C} + 27\bar{A}^2\bar{D}}{2\sqrt{2\bar{A}^3\bar{P}^3}}$.

4.2.3. Numerical simulations:

With the view of illustrating the theoretical results obtained in the preceding sections and comparing those obtained in the context of Green-Naghdi (GN Type-III) (Green and Naghdi, 1993) model of generalized thermoelasticity theory, here in this section we now present some numerical simulating results on both elastic as well as poroelastic solids.

Isotropic elastic solid

To investigate the characteristics of the energetic and dissipative heat fluxes in solids, a quantitative example is set up. The medium is assumed to be made out of a material with the following properties: (i) Excellent heat conductivity, (ii) Excellent electric conductivity, (iii) Good corrosion resistance, (iv) Good machinability and (v) Retention of mechanical and electrical properties of cryogenic temperature. For numerical simulation, the copper type material is considered with the following physical data values:

$$\lambda = 77.6\ GPa, \quad \mu = 38.6\ GPa, \quad \alpha_t = 1.78 \times 10^{-5} m.K^{-1}, \quad \rho = 8945 Kg.m^{-3},$$
$$C_e = 381\ J.Kg^{-1}.K^{-1}, \quad K_1 = 300\ W.m^{-2}.K^{-2}, \quad K_2 = 400\ W.m^{-1}.K^{-1}, \quad T_0 = 298\ K.$$

Further we assume, $l = 1$ m, $v = 7.7 m.s^{-1}$ and $Q_0 = 1$.

In Fig. 1 we illustrate the nature of the temperature distribution in the vicinity of the source of heat applied. A comparison is made with the present analysis, proposed model including without energy dissipation (WED) and the Green-Naghdi Type-III model. Figs. 2–5 represent the variation of radial and hoop stresses near the applied heat source. In addition, comparisons are also made with the results obtained in absence of energy dissipation from the system. These plots were obtained maintaining all the parameters constant at the reference values above, expect the radial distance r that varies in predefined increments.

From Fig. 1, we note that, as expected, the temperature profiles of the proposed model, without energy dissipation theory and Green-Naghdi model are largely different. In Green-Naghdi theory as well as without energy dissipation cases the amplitudes of the temperature profile are abruptly very high near the origin. But, if both the heat fluxes are taken into account then we found a smooth wave form of the temperature. Thus, Green-Naghdi model and without energy dissipation theory of generalized thermoelasticity are not well appropriate in the vicinity of the applied heat source. Therefore, in order to obtain more accurate information about the temperature distribution into the solids in high temperature region we shall have to adopt both the aforementioned heat fluxes.

We also observe that the large temperature oscillation amplitude (as well as the corresponding heat fluxes) should be discarded because they violate the linearisation approximations. It is also of interest to note the physical principle of the energetic heat flux through the asymptotic behaviour of the material properties. (i) when $K_2 \to 0$, i.e. dissipative heat flux

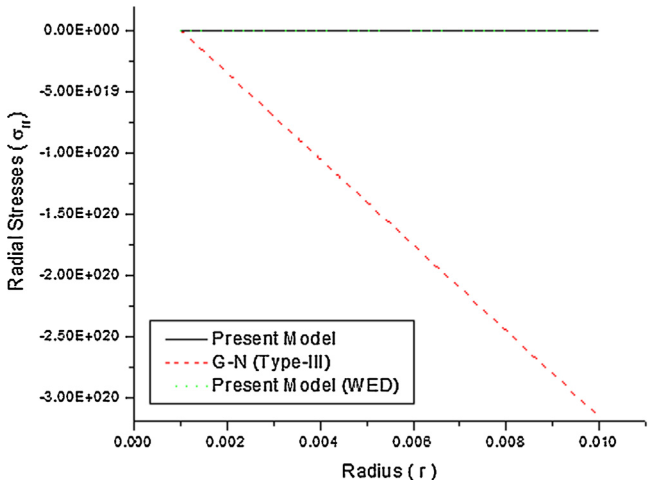

Fig. 2. Variation of radial stress near the source of heat at $t = 0.25$.

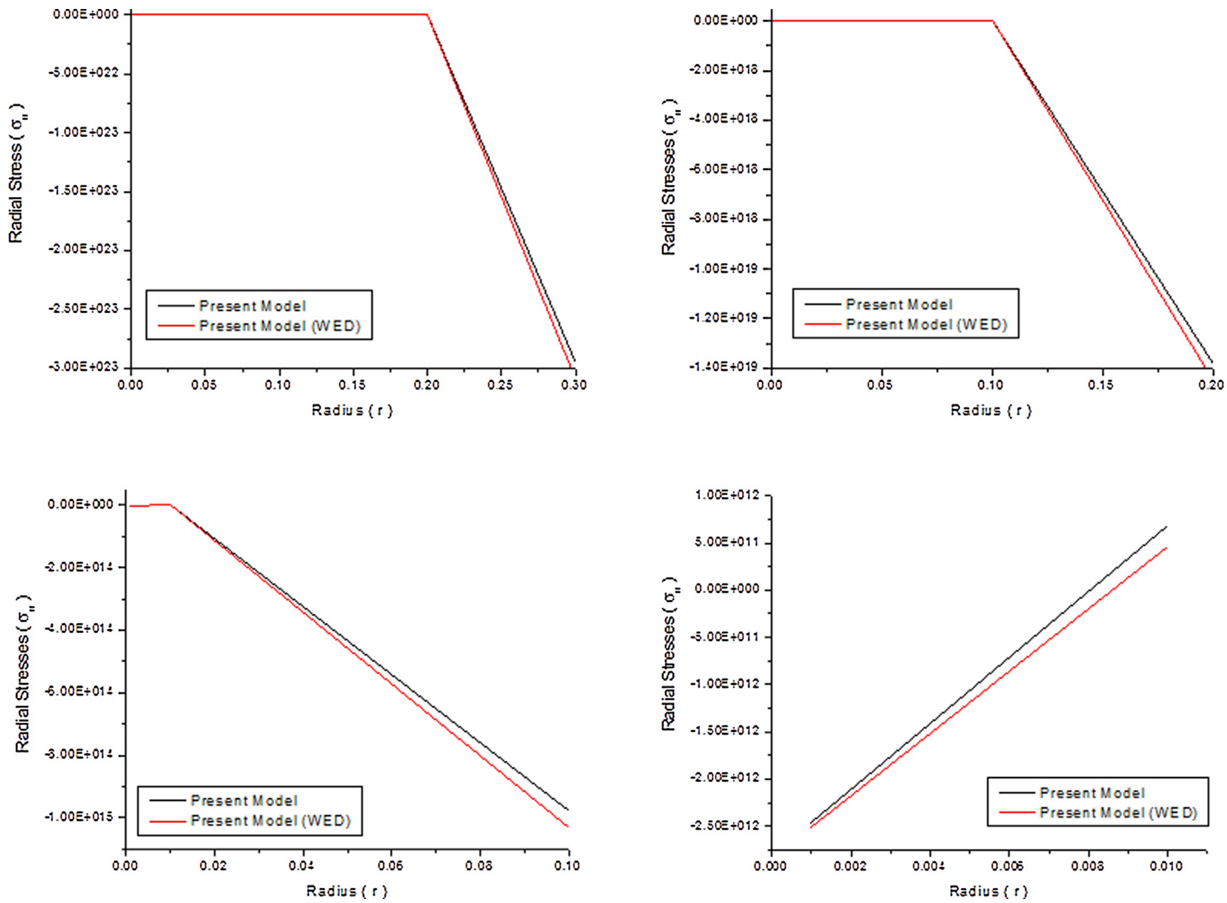

Fig. 3. Comparison of radial stresses for with (without) energy dissipation at $t = 0.25$.

is discarded, which gives the highest heat flux possible and consequently very large amplitude of the temperature profile. On the other hand (*ii*) when $K_2 \to \infty$, Eq. (28) shows that the amplitude of the temperature oscillation $\theta \to 0$ and $\mathbf{q}_E = \mathbf{0}$.

From the Eq. (19) it is clear that the energetic heat flux depends upon the material parameters β_{ij}, ρ, C_e and K_2. This relation suggest that it could be possible, in principle, to select the material properties and forcing conditions (input frequency) that produce large variation of the energetic heat fluxes. It is envisioned that the specific material properties could be properly tailored by engineering so to obtain desired heat flux performance.

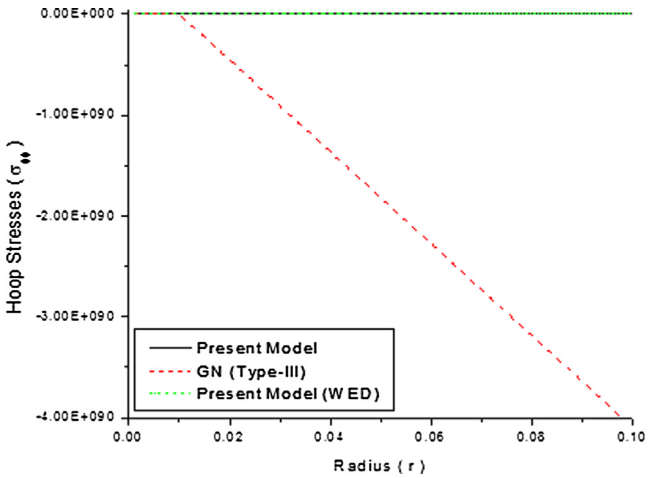

Fig. 4. Variation of hoop stress near the source of heat at $t = 0.25$.

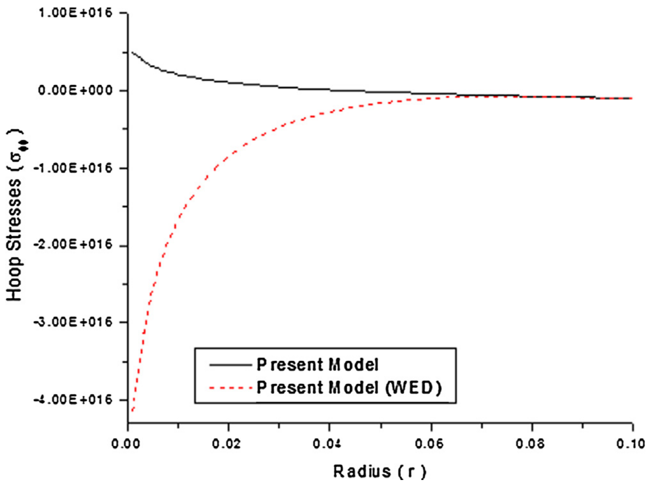

Fig. 5. Comparison of hoop stresses for with (without) energy dissipation at $t = 0.25$.

From Figs. 2 and 3, it is observed that the radial stress is decaying very rapidly with the distance from the origin. But there is a significant difference in stress profile with Green-Naghdi model. In Green-Naghdi model the generated stresses are started to decay from very first point, but the stresses due to the proposed model are increasing initially and then start to decay as distance goes. Moreover, in presence of the dissipative heat fluxes the produced stresses in the medium, in principle, should have been lesser. This phenomenon over stress distribution is confirmed by the Fig. 3.

Similar type of stress profiles are shown in hoop stresses (see Fig. 4) as it was observed in radial stress. Form Fig. 5, it is evident that impact of heat fluxes on the tangential stresses is energetic in the low-frequency region (i.e. far from the applied heat source) and predominantly dissipative at high frequency range.

Poroelastic Solid:

The effect of the heat fluxes upon the temperature distribution in a specific model of porous medium is considered here. A liquid-saturated reservoir rock (North-sea Sandstone) is chosen for the numerical computation purpose. The values of the elastic and dynamical constants for the porous rock are taken from the anisotropic constants in Rasolofosaon and Zinszner (2002). Those are given by,

$$\lambda = 3.7\ GPa, \quad \mu = 7.9\ GPa, \quad M = 6\ GPa, \quad \alpha = 0.4, \quad \rho = 2216 Kg.m^{-3}, \quad f = 0.16$$

The saturating fluid is assumed with the density $\rho_f = 950$ Kg.m^{-3} and $q = 1.05\frac{\rho_f}{f}$.

The numerical values

$$C_e = 381\ J.Kg^{-1}.K^{-1}, \quad K_1 = 150\ W.m^{-2}.K^{-2}, \quad K_2 = 170\ W.m^{-1}.K^{-1}, \quad T_0 = 298\ K,$$
$$\beta_f = 2.37 \times 10^{-3} GPa.K^{-1}, \quad \beta_s = 2\beta_f$$

define the thermoelastic characteristics of the porous aggregate.

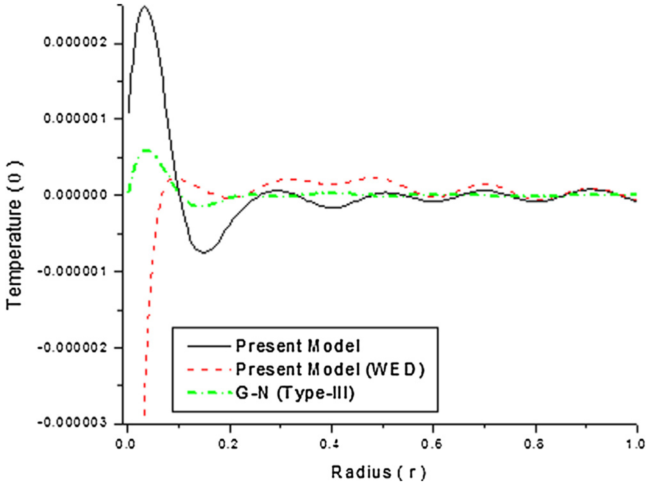

Fig. 6. Temperature distribution in poroelastic solid near the source of heat at $t = 0.25$.

In addition we consider, the characteristic length $l = 1$ m, to make the speed (non-dimensional) of the elastic dilatation wave $C_P = 1$ we assume $\nu = 7.7$ m.s^{-1} and the strength of the heat source $Q_0 = 1$.

In general, the thermal conductivity of a porous medium in a complex fashion on the geometry of the medium. In the case of a liquid-saturated porous medium temperature slip occurs in the fluid at the pore boundaries. In these circumstances one could expect that the fluid conductivity would tend to zero. Then in the case of external heating the heat would be be conducted almost entirely through the solid matrix. For internal heating in the fluid, the situation is reversed as the fluid phase becomes thermally isolated from the solid phase.

Thus the motion-induced heat fluxes are nominal and the average heat transfer occurred due to the dissipative heat fluxes. For this reason, the temperature profile for without energy dissipation is very low initially and the present model gives a high note near the heat sources (Fig. 6). Whereas, Green-naghdi model gives a moderate temperature distribution.

5. Conclusion

In this article, a spatial form of a coupled thermoelasticity theory at finite strain is proposed in thermodynamically consistent manner. Based on the Fourier's law of heat conduction, the overall heat flux is decomposed into two components: one is due to the motion of the particles (motion-induced heat fluxes) and other one for dissipation of heat energy. The domain of dependence inequality for the solution of the proposed dynamical thermoelastic problem is proved successfully. The linearised form of this newly developed theory at finite stain is applied in isotropic as well as poroelastic solids and the salient features are emphasized in throughout the application section. It has been observed that, in order to obtain more precise information about the temperature profile in solids (specially near the heat sources) both the heat fluxes are very much required. The energetic and dissipative heat fluxes have a great impact on stress distribution in solids. The stress profile is influenced by energetic heat transfer in the low-frequency region and predominantly dissipative at high frequency range.

In addition, a quantitative results have shown that the two characteristic thermoelastic modes can transfer a significant amount of heat. unlike in gases where de-phasing between the velocity and temperature oscillations is usually achieved by heat exchange with a thermal reservoir, in solids that effect is enabled due to thermal conduction. In fact, a large amount of heat flux may be obtained after certain combinations of material properties and input frequency.

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