EEE - 321: Signals and Systems

Lab Assignment 6

Yiğit Narter

22102718

Section 02

Part 1

$$\underline{n=0}: \quad y[0] = \underbrace{\sum_{\ell=1}^{N} a[\ell]\, y[-\ell]}_{0 \text{ since } y[n]=0 \text{ for } n<0} + \underbrace{\sum_{k=0}^{M} b[k]\, x[-k]}_{(\text{since } x[n]=0 \text{ for } n<0)} = \boxed{b[0].x[0]}$$

found prev.

$$\underline{n=1}: \quad y[1] = \underbrace{\sum_{\ell=1}^{N} a[\ell]\, y[1-\ell]}_{\substack{0 \text{ for } 1-\ell<0 \\ \Rightarrow 1<\ell}} + \sum_{k=0}^{M} \underbrace{b[k]\, x[1-k]}_{\substack{0 \text{ for } 1-k<0 \\ \Rightarrow 1<k}} = a[1]\, y[0] + b[0]\, x[1] + b[1]\, x[0]$$

$$= \boxed{a[1]\, b[0]\, x[0] + b[0].x[1] + b[1].x[0]}$$

$$\underline{\text{Rearrange eqn. (1)}}: \quad y[n] - \sum_{\ell=1}^{N} a[\ell]\, y[n-\ell] = \sum_{k=0}^{M} b[k]\, x[n-k]$$

Z transform

(using shift property)

$$Y(z) - \sum_{\ell=1}^{N} a[\ell].Y(z).z^{-\ell} = \sum_{k=0}^{M} b[k].X(z).z^{-k}$$

$$\Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b[k]\, z^{-k}}{1 - \sum_{\ell=1}^{N} a[\ell]\, z^{-\ell}} = \frac{\sum_{p=0}^{P} c_n[p]\, z^{-p}}{\sum_{q=0}^{Q} c_d[q]\, z^{-q}}$$

given in manual

Comparing, we obtain:

$$= \frac{\sum_{p=0}^{P} c_n[p]\, z^{-p}}{\sum_{q=0}^{Q} c_d[q]\, z^{-q}}$$

$$\boxed{P = M}, \quad \boxed{Q = N}$$

$$\boxed{c_n[p] = b[k]}, \quad \boxed{c_d[q] = -a[\ell]},\ \ell \geq 1$$

$$\boxed{\text{and } c_d[0] = 1}$$

but we need

$c_n[0] - a[0]$ 1

The function DTLTI(a,b,x,Ny):

```
function y = DTLTI(a, b, x, Ny)

    y = zeros(1, Ny);

    N = length(a); %corresponds to N in eqn 1
    M = length(b) - 1; %corresponds to M in eqn 2
    Nx = length(x);

    for n = 1:Ny
        term_y = 0;
        for l = 1:N
            if n - l > 0 %since y[n] is zero for n<0
                term_y = term_y + a(l) * y(n - l);
            end
        end

        term_x = 0;
        for m = 0:M
            if n - m <= Nx && n - m > 0 %since x[n] is zero for n<0
                term_x = term_x + b(m + 1) * x(n - m);
            end
        end

        y(n) = term_x + term_y;
    end
end
```

Part 2

a) The code used:

```
D4 = rem(22102718, 4);
M = 5+D4;

arr = 0:M-1;
b = exp(-1*arr);
a= zeros(1, length(b)),
Ny = 11;

impulse = zeros(1, Ny);
impulse(1) = 1; %Kronecker delta (impulse) function is 1 only at n=0, x[n] =
δ[n]

h = DTLTI(a, b, impulse, Ny);

figure;
stem(0:Ny-1, h, 'LineWidth', 1.5, 'Marker', 'o');
title('Impulse Response h[n]');
xlabel('n');
ylabel('h[n]');
grid on;
```

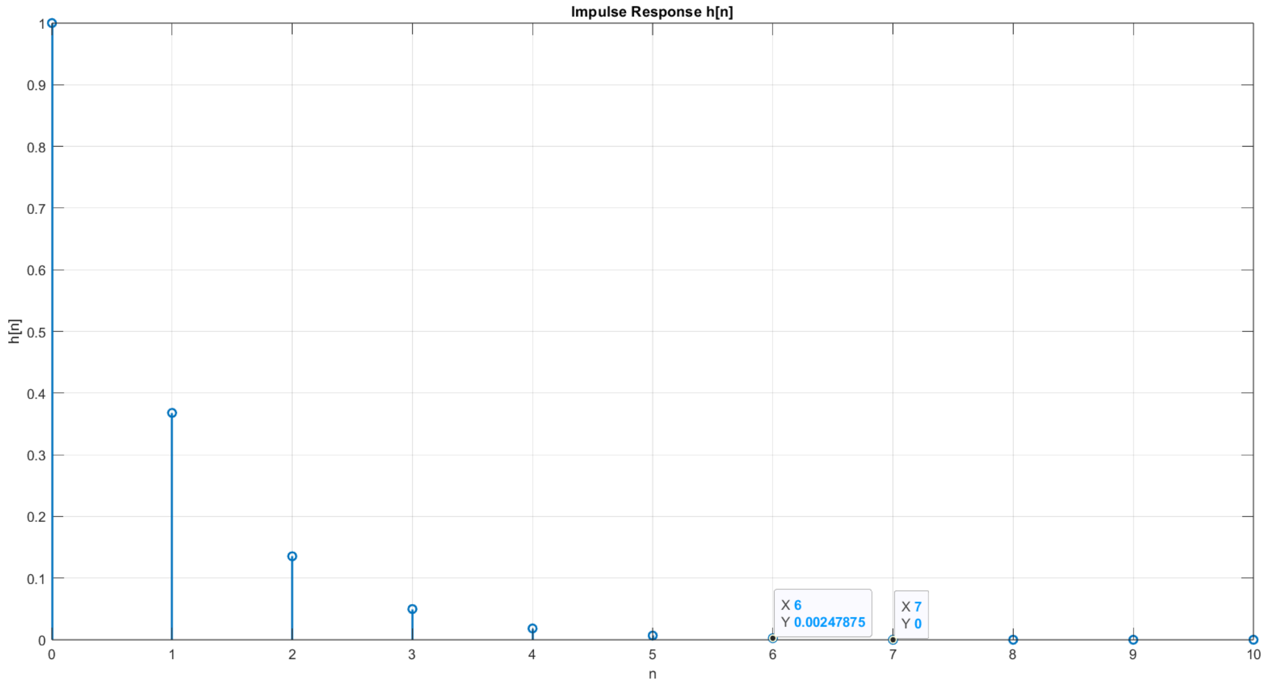


Figure 1: Impulse response h[n] of the filter, after n=7, it is 0

b) Computing h[n] and b[k] in MATLAB (by removing ";" from end of the line), the result is:

```
>> part_2

b =

    1.0000    0.3679    0.1353    0.0498    0.0183    0.0067    0.0025


h =

  Columns 1 through 10

    1.0000    0.3679    0.1353    0.0498    0.0183    0.0067    0.0025         0         0         0

  Column 11

         0
```

By observing the coefficients, it can be seen that the non-zero coefficients are equal to coefficients $b[k]=e^{-k}$, for $k \leq M-1$, which is 6 since M=7. After and including k=M=7, the coefficients are 0. The coefficients are decreasing (exponentially decaying), since $b[k]=e^{-k}$ is a decreasing function.

c) The impulse response has finite length since it is non-zero only for $k \leq M-1$. Therefore, the system is FIR (finite impulse response). The length of the impulse response is equal to M (which is 7 in my case), since b[k] is non-zero for k=0, 1, …, M-1.

d) Note that M=mod(22102718,4)+5=7

$$y[n] = \sum_{k=0}^{M-1} \underbrace{b[k]}_{e^{-k}} x[n-k] = 1.x[n] + e^{-1}x[n-1] + \dots + e^{M-1}x[n-(\underset{M=7}{M}-1)]$$

for $\boxed{k \leq M-1}$ and $k \geq 0$

$$\updownarrow \; z$$

$$Y(z) = X(z)\left[\sum_{k=0}^{M-1 \to 6} e^{-k}.z^{-k}\right] \rightarrow \text{using } \underline{\text{shifting property}}$$

$$\Rightarrow \boxed{H(z) = \frac{Y(z)}{X(z)} = \sum_{k=0}^{6} e^{-k}.z^{-k}} \xrightarrow{z^{-1}} \boxed{h[n] = \sum_{k=0}^{6} e^{-k}\delta[n-k]}$$

For DTFT, $H(e^{j\Omega}) = H(z)\Big|_{z=e^{j\Omega}} = \sum_{k=0}^{6} e^{-k}.(e^{j\Omega})^{-k}$

$$\Rightarrow \boxed{H(e^{j\Omega}) = \sum_{k=0}^{6} e^{-(j\Omega+1)k}}$$

e) The code:

```
omega = -pi:0.001:pi;
impulse_response = zeros(1, length(omega));

for m = 1:length(omega)
    sum = 0;
    for k= 0:6
        sum = sum + exp(-k) * exp(-1i * omega(m) * -k);
    end

    impulse_response(m) = sum;

end

figure;
plot(omega, abs(impulse_response));
title('Magnitude Response of the Impulse Response h[n]');
xlabel('Omega (Ω)');
ylabel('|H(e^j^Ω)|');
xlim([-pi, pi]);
grid on;
```

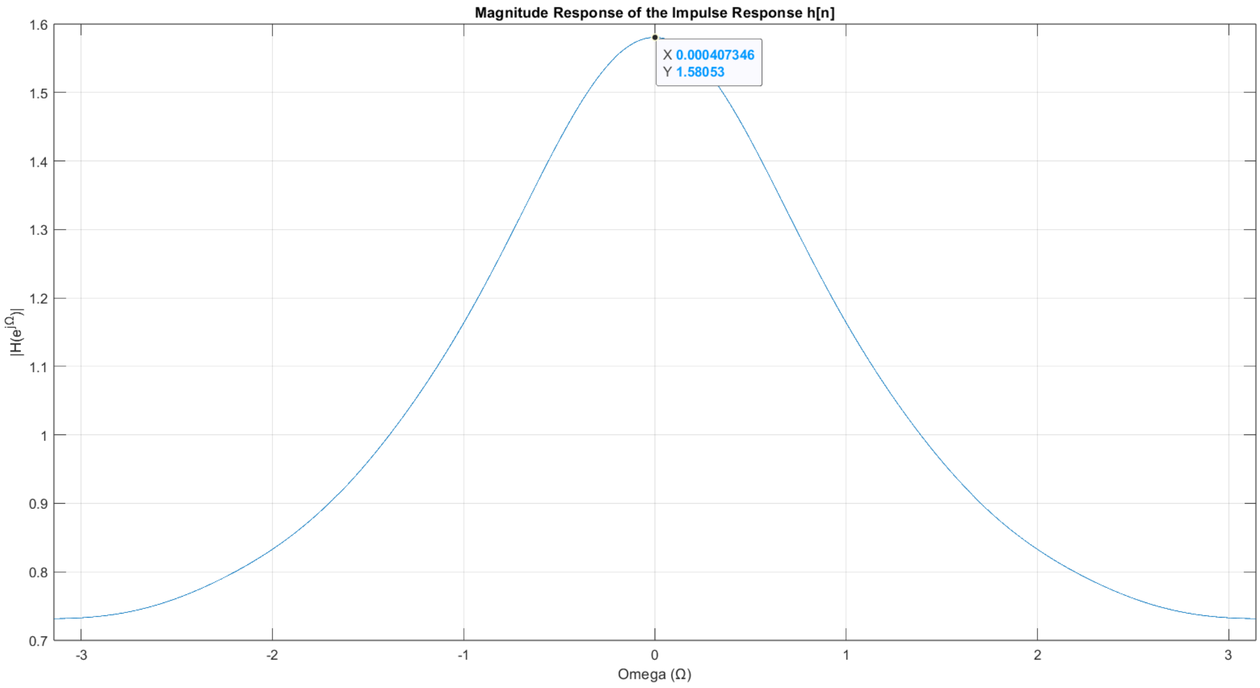


Figure 2: Magnitude response of the impulse response

The system is a low-pass filter since the magnitude increases at frequencies close to zero. 3 dB bandwidth is essentially a measure of the range of frequencies over which the filter responds to signals with less than 3 decibels (dB) below the maximum response, which is at 0 for LPFs. In the frequency response plot of this filter, the 3 dB bandwidth corresponds to the range of frequencies where the difference between the response magnitude and the maximum response magnitude is less than 3 dB.

The cut-off frequency is the point at which the response of a system or filter begins to deviate from its maximum response with a difference of more than 3 dB. Therefore, it is actually the frequency at which the response is 3 dB lower than the maximum response. In other words, they are also the limits of the 3 dB bandwidth. The dB magnitude is calculated with the formula $20\log|H(e^{j\Omega})|$, therefore in the $|H(e^{j\Omega})|$ plot I produced, at the cut-off frequencies are at the point where $20\log|H(e^{j\Omega})| - 3 = 20\log|H_{max}(e^{j\Omega})| - 20\log(\sqrt{2}) = 20\log|H_{max}(e^{j\Omega})/\sqrt{2}|$. In our case, since the filter is LPF, $H_{max}(e^{j\Omega})$ is at $\Omega=0$, and it is equal to 1.58 (from Fig. 2). So, at the cut-off frequencies the response magnitude is $1.58/\sqrt{2}$, which is approximately 1.117.

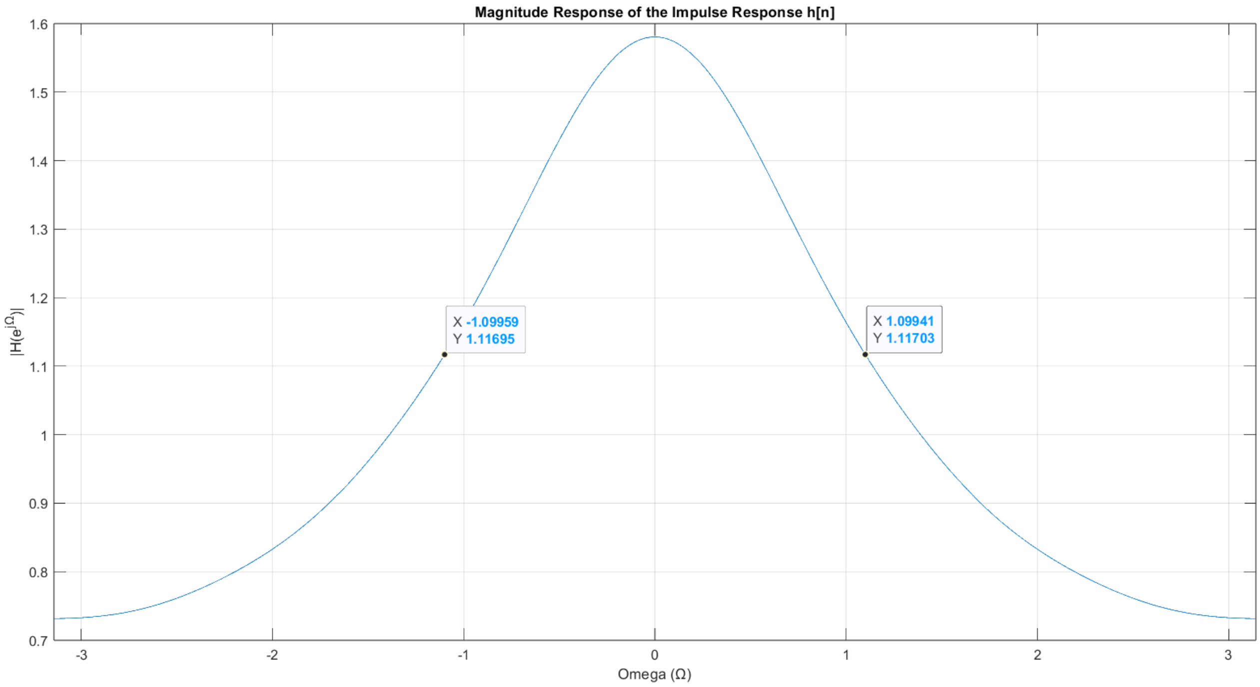


Figure 3: The cut-off frequencies are at Ω=1.1, -1.1 approximately

Therefore, the cut-off frequencies are Ω_{C1}=-1.1 rad/s, Ω_{C2}=1.1 rad/s, $f_{C1}=-1.1/2\pi = -0.175$ Hz, $f_{C2}=1.1/2\pi = 0.175$ Hz. Since the bandwidth is the interval between the two cut-off frequencies, its length is $\Omega_{C2} - \Omega_{C1}$=2.2 rad/s (or 0.35 Hz) and it is the interval [-1.1, 1.1] (in rad/s) or [-0.175, 0.175] (in Hz).

f) The code:

```
fs = 1400;
t1 = 0:(1/fs):1;
t2 = 0:(1/fs):10;
t3 = 0:(1/fs):1000;

f0 = 0;
f_final = 700;

D4 = rem(22102718, 4);
M = 5+D4;
a = zeros(1, 10);
arr = 0:M-1;
b = exp(-1*arr);

k1 = (f_final - f0) / 1; %at k=1/10/1000, f = f_ins, at k=0, f = f_0
k2 = (f_final - f0) / 10;
k3 = (f_final - f0) / 1000;

x1 = cos(2 * pi * ((k1/2) * (t1.^2)) + f0 * t1); %integrate to obtain
phi(t)
x2 = cos(2 * pi * ((k2/2) * (t2.^2)) + f0 * t2);
x3 = cos(2 * pi * ((k3/2) * (t3.^2)) + f0 * t3);


y1 = DTLTI(a, b, x1, length(x1));
y2 = DTLTI(a, b, x2, length(x2));
y3 = DTLTI(a, b, x3, length(x3));


N1 = length(y1);
N2 = length(y2);
N3 = length(y3);
f_axis1 = linspace(0, pi, N1);
f_axis2 = linspace(0, pi, N2);
f_axis3 = linspace(0, pi, N3);

figure;
subplot(1,3,1);
plot(f_axis1, abs(y1));
title('Linear chirp response for 0 ≤ t ≤ 1');
xlabel('Frequency (rad/sample)');
ylabel('Magnitude');
xlim([0, pi]);
grid on;

subplot(1,3,2);
plot(f_axis2, abs(y2));
title('Linear chirp response for 0 ≤ t ≤ 10');
xlabel('Frequency (rad/sample)');
ylabel('Magnitude');
```

```
xlim([0, pi]);
grid on;

subplot(1,3,3);
plot(f_axis3, abs(y3));
title('Linear chirp response for 0 ≤ t ≤ 100');
xlabel('Frequency (rad/sample)');
ylabel('Magnitude');
xlim([0, pi]);
grid on;
```

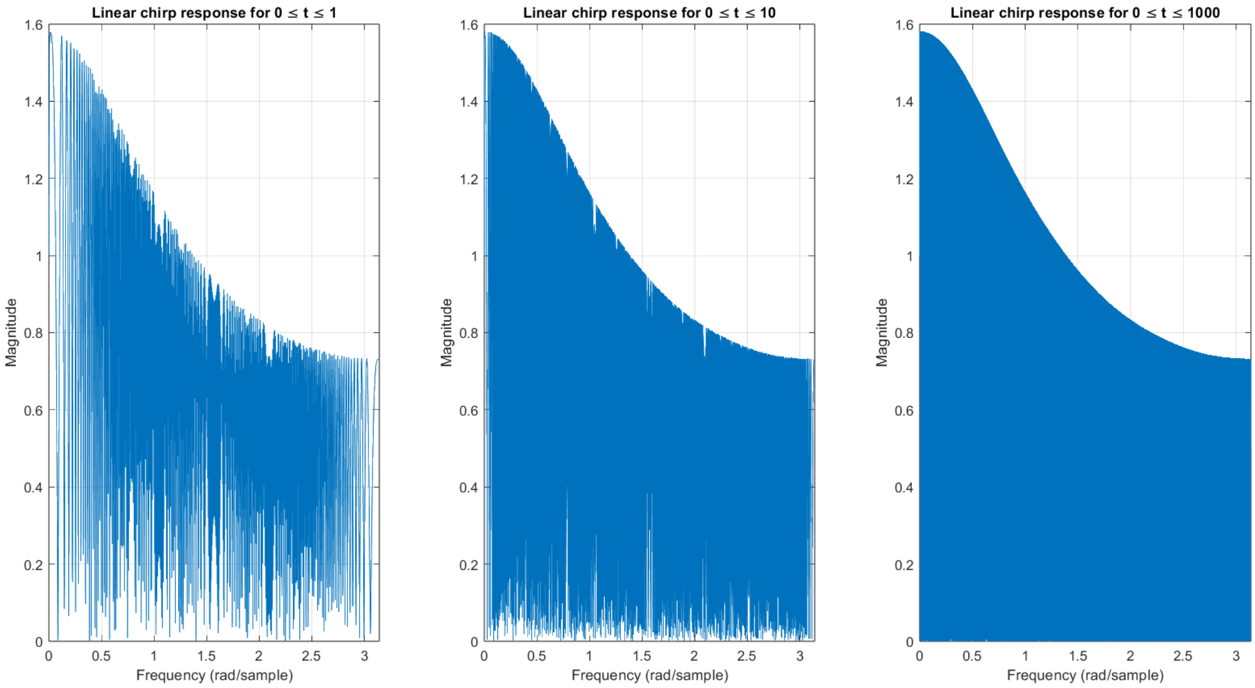


Figure 4: The output with linear chirp, see Appendix A for a closer look to the plots separately

K's are calculated by dividing the sweep frequency interval length (700-0=700) by the time interval length (t-0=t where t=1,10,1000), since the $f_{ins}(t)$ equation is linear, k is the slope. Overall, the plots resemble the previous magnitude plots, since in both cases the magnitude decreases from 0 to π, as in an LPF. The general trend is therefore in a decreasing manner from 0 to π. Also, the magnitudes are equal, which is approximately 1.6 at Ω=0.

For the first plot where $0 \leq t \leq 1$, there are sudden jumps which is because we are sampling this linear chirp and the frequency is changing continuously as time passes. Also, the sampling time interval is the shortest (1 second). As the interval is expanded for t=10 and t=1000, the graphs get a lot denser since the jumps occur much more frequently. By using time-domain chirp, the dynamic behavior of the system is revealed. It shows how the system reacts to changing frequencies over time, in other words, a frequency sweep.

Part 3

$$z_1 = \frac{4+3j}{\sqrt{4^2+3^2}} = 0.8+0.6j, \quad p_1 = \frac{4+4j}{\sqrt{1+4^2+4^2}} = 0.6963+0.6963j$$

$$p_2 = \frac{10+9j}{\sqrt{1+10^2+9^2}} = 0.7412+0.6671j$$

$n_1 = 2+2 = 4$
$n_2 = 2+2 = 4$
$n_3 = 2+1 = 3$
$n_4 = 2+0 = 2$
$n_5 = 2+2 = 4$
$n_6 = 2+7 = 9$
$n_7 = 2+1 = 3$
$n_8 = 2+8 = 10$

1 zero, 2 poles

a) We have, $\boxed{H(z) = \dfrac{1 - z_1 . z^{-1}}{(1-p_1 z^{-1})(1-p_2 z^{-1})}} = \dfrac{Y(z)}{X(z)}$

$$= \frac{1 - z_1 . z^{-1}}{1-(p_1+p_2)z^{-1} + p_1 p_2 z^{-2}} = \frac{Y(z)}{X(z)}$$

$$\Rightarrow \boxed{X(z)\left(1 - z_1 . z^{-1}\right) = Y(z)\left(1-(p_1+p_2)z^{-1} + p_1 p_2 z^{-2}\right)}$$

z. transform of filter

$\updownarrow$ Z^{-1} tr.

b) $\boxed{y[n] - (p_1+p_2)y[n-1] + p_1 p_2 y[n-2] = x[n] - z_1 x[n-1]}$

or $\boxed{y[n] = \sum_{\ell=1}^{2} a[\ell]\, y[n-\ell] + \sum_{k=0}^{1} b[k]\, x[n-k]}$

$\boxed{\begin{array}{l} a[1] = p_1+p_2 \\ a[2] = -p_1 . p_2 \end{array}}$ $\leftarrow$ where $\rightarrow$ $\boxed{b[0] = 1, \ b[1] = -z_1}$

z_1, p_1, p_2 found above!

c) $$H(z) = \frac{1 - z_1 z^{-1}}{(1-p_1 z^{-1})(1-p_2 z^{-1})} = \frac{A}{1-p_1 z^{-1}} + \frac{B}{1-p_2 z^{-1}} = \frac{p_1 - z_1}{p_1 - p_2} . \frac{1}{1-p_1 z^{-1}} + \frac{p_2 - z_1}{p_2 - p_1} . \frac{1}{1-p_2 z^{-1}}$$

$$A = \left.\frac{1 - z_1 z^{-1}}{1-p_2 z^{-1}}\right|_{z=p_1} = \frac{1 - \frac{z_1}{p_1}}{1-\frac{p_2}{p_1}} = \frac{p_1 - z_1}{p_1 - p_2}$$

$$B = \left.\frac{1 - z_1 z^{-1}}{1-p_1 z^{-1}}\right|_{z=p_2} = \frac{1 - \frac{z_1}{p_2}}{1-\frac{p_1}{p_2}} = \frac{p_2 - z_1}{p_2 - p_1}$$

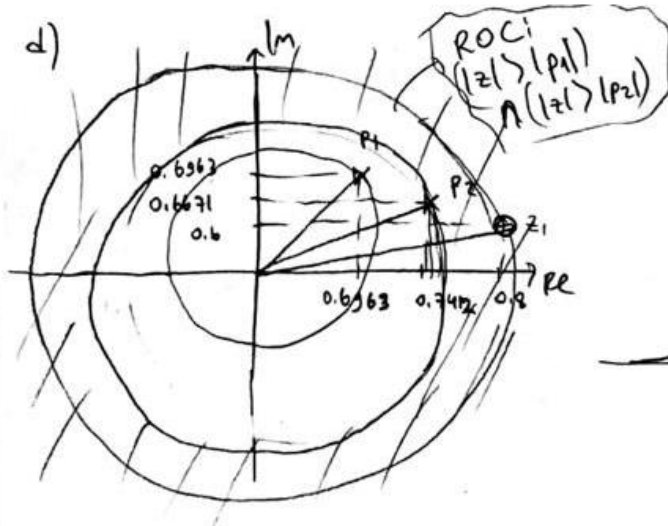
$\downarrow Z^{-1}$

$$\boxed{h[n] = \frac{p_1 - z_1}{p_1 - p_2}(p_1)^n u[n] + \frac{p_2 - z_1}{p_2 - p_1}(p_2)^n u[n]}$$

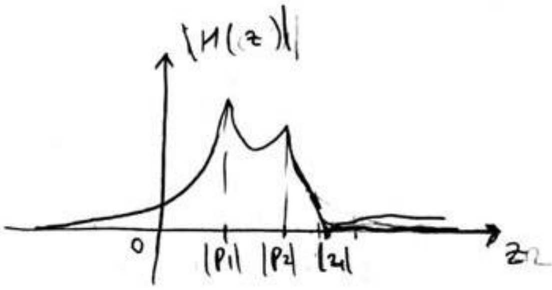
ROC: $|z| > |p_1|$ and $|z| > |p_2|$

$|p_1| = 0.9847$, $|p_2| = 0.9972$

d)



Mag. response $\cong \dfrac{|z-z_1|}{|z-p_1|\,|z-p_2|}$



e) The system is stable since all the poles lie within the unit circle. (i.e, $|p_1|<1$, $|p_2|<1$) we need the poles to lie inside the unit circle for stability.

f) The system is IIR (infinite impulse response). The reason is that the Impulse response has two poles (p_1, p_2), meaning that it will not decay completely i.e will have a lasting effect. This will make the response introduce feedback to the system. Therefore, a system is IIR if the impulse response involves both zeros and poles. In our case, therefore, the system is IIR.

g) To find DTFT, we have $H(e^{j\Omega}) = H(z)\big|_{z=e^{j\Omega}}$, and $H(z)$ found in item a.

$$\boxed{H(e^{j\Omega}) = H(z)\Big|_{z=e^{j\Omega}} = \frac{1-z_1\cdot e^{-j\Omega}}{(1-p_1\cdot e^{-j\Omega})(1-p_2 e^{-j\Omega})} = \frac{p_1-z_1}{p_1-p_2}\cdot\frac{1}{1-p_1 e^{-j\Omega}} + \frac{p_2-z_1}{p_2-p_1}\cdot\frac{1}{1-p_2 e^{-j\Omega}}}$$

p_1, p_2, z_1 found previously

To plot the magnitude in MATLAB:

```
n1 = 2+2;
n2 = 2+2;
n3 = 2+1;
n4 = 2+0;
n5 = 2+2;
n6 = 2+7;
n7 = 2+1;
n8 = 2+8;

z1 = (n2 + 1i*n3)/sqrt(n2^2 + n3^2);
p1 = (n1 + 1i*n5)/sqrt(1+ n1^2 + n5^2);
p2 = (n8 + 1i*n6)/sqrt(1+ n8^2 + n6^2);

omega = -pi:0.001:pi;
freq_response = zeros(1, length(omega));


for m = 1:length(omega)
    a1= 1-z1*exp(-1i*omega(m));
    a2 = (1-p1*exp(-1i*omega(m))) * (1-p2*exp(-1i*omega(m)));
    freq_response(m) = a1/a2 ;
end

figure;
plot(omega, abs(freq_response));
title('Magnitude response of the filter');
xlabel('Omega (Ω)');
ylabel('|H(e^j^Ω)|');
xlim([-pi, pi]);
grid on;
```

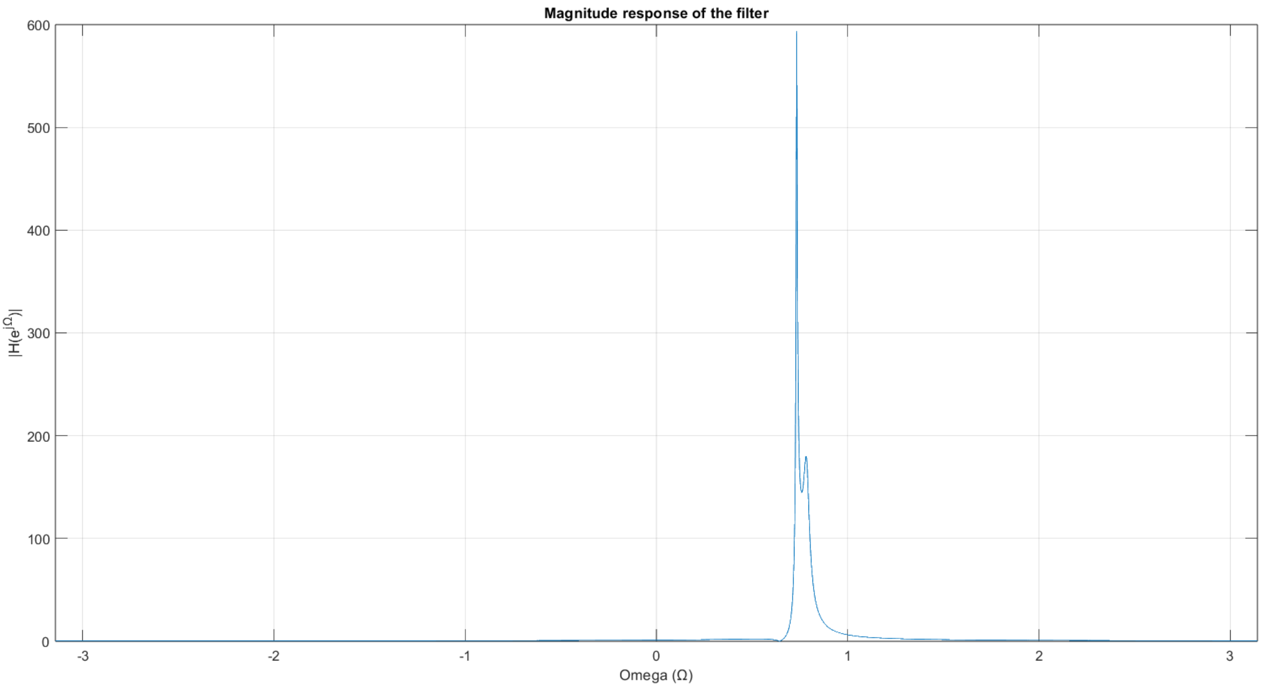


Figure 5: Magnitude of the filter’s DTFT

The system resembles a very narrow bandpass filter since it only passes the frequencies between the two peaks (corresponding to two poles). The shape also looks like that of a bandpass, but it is much narrower. The peaks represent the center frequencies at which the filter's magnitude response it maximum.

h) The code used:

```
n1 = 2+2;
n2 = 2+2;
n3 = 2+1;
n4 = 2+0;
n5 = 2+2;
n6 = 2+7;
n7 = 2+1;
n8 = 2+8;

z1 = (n2 + 1i*n3)/sqrt(n2^2 + n3^2);
p1 = (n1 + 1i*n5)/sqrt(1+ n1^2 + n5^2);
p2 = (n8 + 1i*n6)/sqrt(1+ n8^2 + n6^2);

fs = 1400;
t1 = 0:(1/fs):1;
t2 = 0:(1/fs):10;
t3 = 0:(1/fs):1000;

f0 = -700;
f_final = 700;

k1 = (f_final - f0) / 1; %at k=1/10/1000, f = f_ins, at k=0, f = f_0
k2 = (f_final - f0) / 10;
k3 = (f_final - f0) / 1000;

x1 = exp(2j * pi * ((k1/2) * (t1.^2) + f0 * t1));
x2 = exp(2j * pi * ((k2/2) * (t2.^2) + f0 * t2));
x3 = exp(2j * pi * ((k3/2) * (t3.^2) + f0 * t3));

a = [(p1+p2), -p1*p2];
b = [0, -z1];

y1 = DTLTI(a, b, x1, length(x1));
y2 = DTLTI(a, b, x2, length(x2));
y3 = DTLTI(a, b, x3, length(x3));

N1 = length(y1);
N2 = length(y2);
N3 = length(y3);
f_axis1 = linspace(-pi, pi, N1);
f_axis2 = linspace(-pi, pi, N2);
f_axis3 = linspace(-pi, pi, N3);
```

```
figure;
subplot(2,3,1);
plot(f_axis1, abs(y1));
title('Linear chirp magnitude response for 0 ≤ t ≤ 1');
xlabel('Frequency (rad/sample)');
ylabel('Magnitude');
xlim([-pi, pi]);
grid on;

subplot(2,3,2);
plot(f_axis2, abs(y2));
title('Linear chirp magnitude response for 0 ≤ t ≤ 10');
xlabel('Frequency (rad/sample)');
ylabel('Magnitude');
xlim([-pi, pi]);
grid on;

subplot(2,3,3);
plot(f_axis3, abs(y3));
title('Linear chirp magnitude response for 0 ≤ t ≤ 1000');
xlabel('Frequency (rad/sample)');
ylabel('Magnitude');
xlim([-pi, pi]);
grid on;

subplot(2,3,4);
plot(f_axis1, angle(y1));
title('Linear chirp phase response for 0 ≤ t ≤ 1');
xlabel('Frequency (rad/sample)');
ylabel('Phase');
xlim([-pi, pi]);
grid on;

subplot(2,3,5);
plot(f_axis2, angle(y2));
title('Linear chirp phase response for 0 ≤ t ≤ 10');
xlabel('Frequency (rad/sample)');
ylabel('Phase');
xlim([-pi, pi]);
grid on;

subplot(2,3,6);
plot(f_axis3, angle(y3));
title('Linear chirp phase response for 0 ≤ t ≤ 1000');
xlabel('Frequency (rad/sample)');
ylabel('Phase');
xlim([-pi, pi]);
grid on;
```

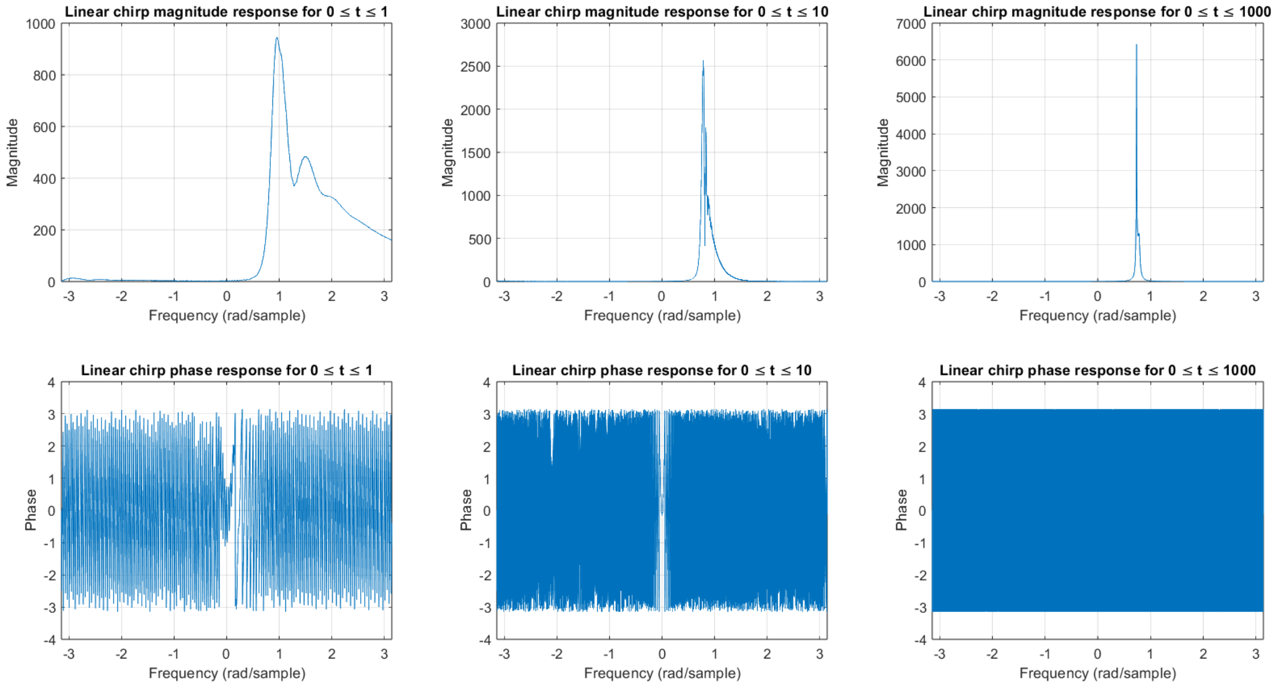


Figure 6: Magnitude and phase of the output, see Appendix B for a closer look (separate plots)

It can be seen that, again as the t interval is increased (from 1 to 1000), the plot resembles and in fact nearly becomes identical to the one found in the previous item. Still, even for the first plot where $0 \leq t \leq 1$, the peaks occur at nearly the same place. However, the width beneath the peaks is wider compared to the previous item, and the peaks are not as distinct. This is because the time interval of sampling is only 1 second. But as the time interval where we take samples becomes longer, the plot approaches the original one.

The magnitude plot is not symmetric with respect to the origin since the peaks are present only on the right side, and the left side is equal to almost zero. This happens due to the fact that both poles are in the first quadrant, and they are what causes the peaks.

If the chirp is sweeping from -600 to 800 Hz, to capture the entire frequency range of the chirp signal without aliasing, the Nyquist-Shannon sampling theorem should be satisfied. According to the theorem, the sampling frequency f_S should be at least twice the maximum frequency present in the signal. In our case, since the chirp signal is sweeping from -600 Hz to 800 Hz, the maximum frequency is 800 Hz. Therefore, the minimum sampling frequency required is $2 \times 800\text{Hz} = 1600\text{Hz}$ to understand the full behavior of the system.

Appendix A – Part 2 separate plots

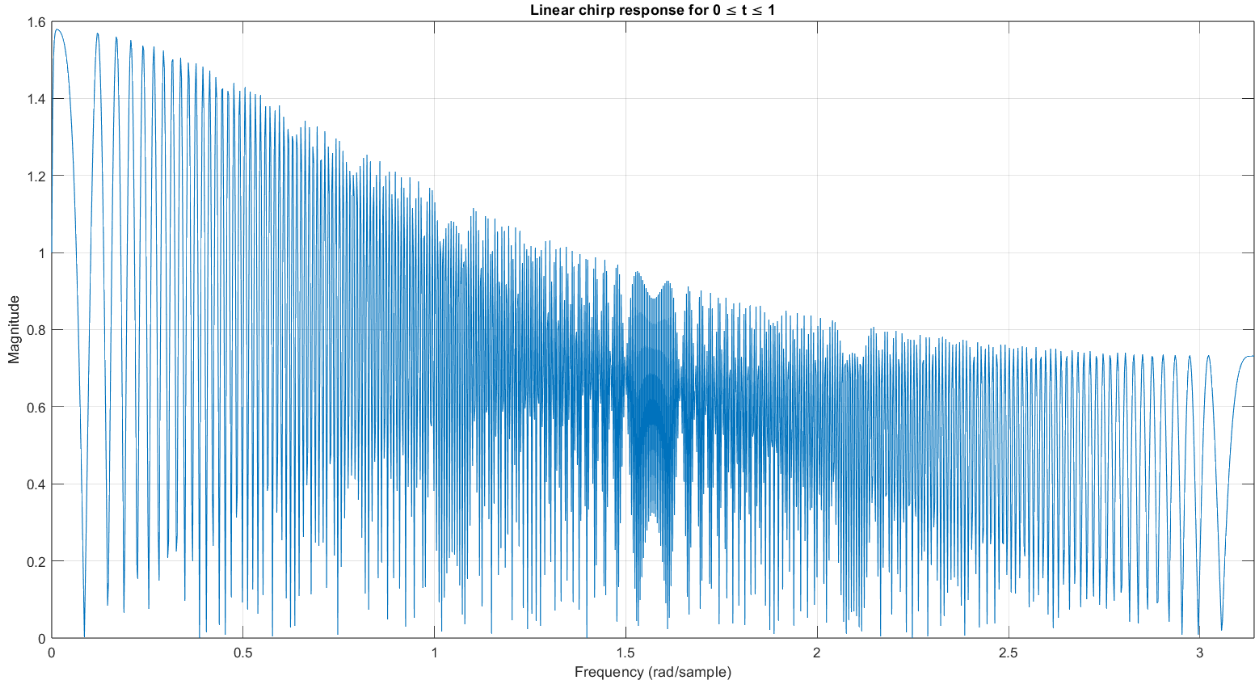


Figure 7: Linear chirp response for $0 \leq t \leq 1$

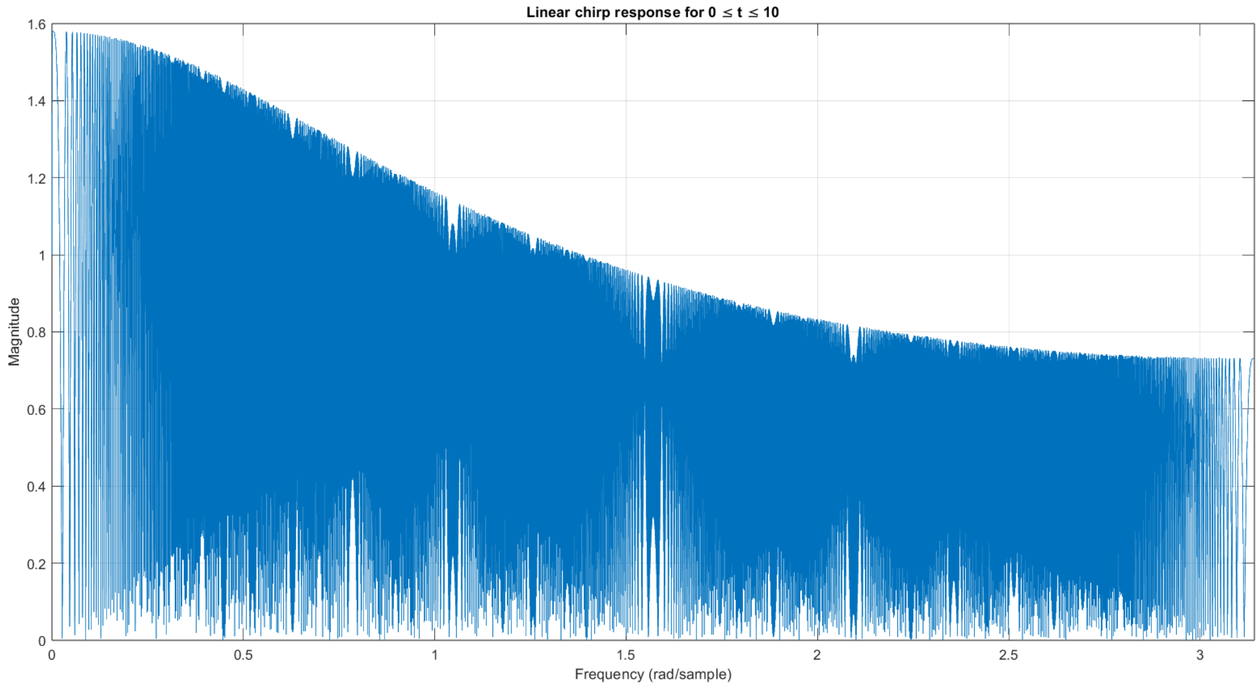


Figure 8: Linear chirp response for $0 \leq t \leq 10$

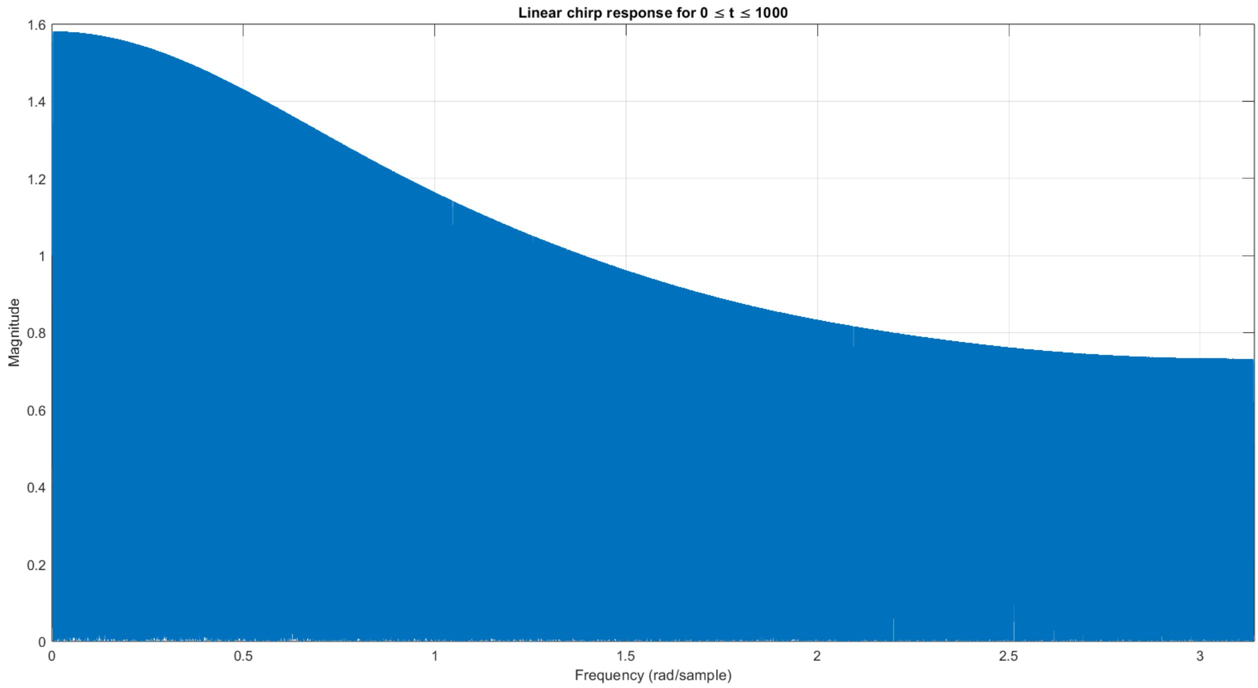


Figure 9: Linear chirp response for $0 \leq t \leq 1000$

Appendix B – Part 3 separate plots

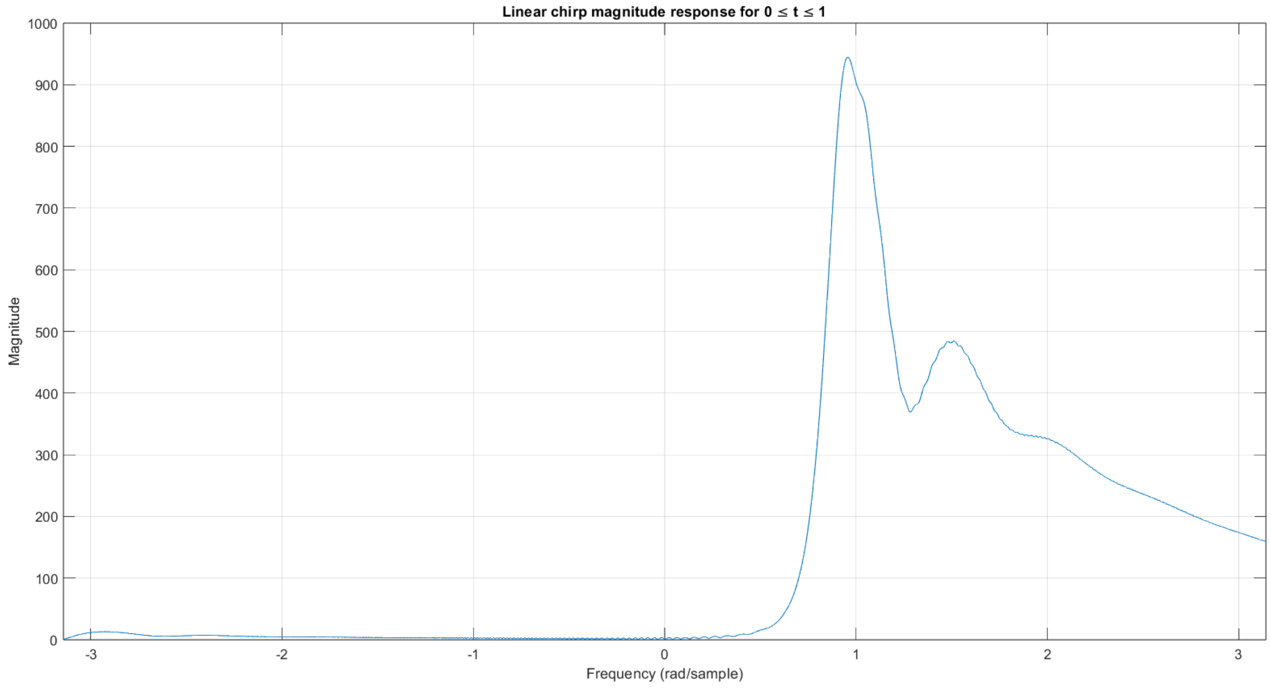


Figure 10: Magnitude response for $0 \leq t \leq 1$

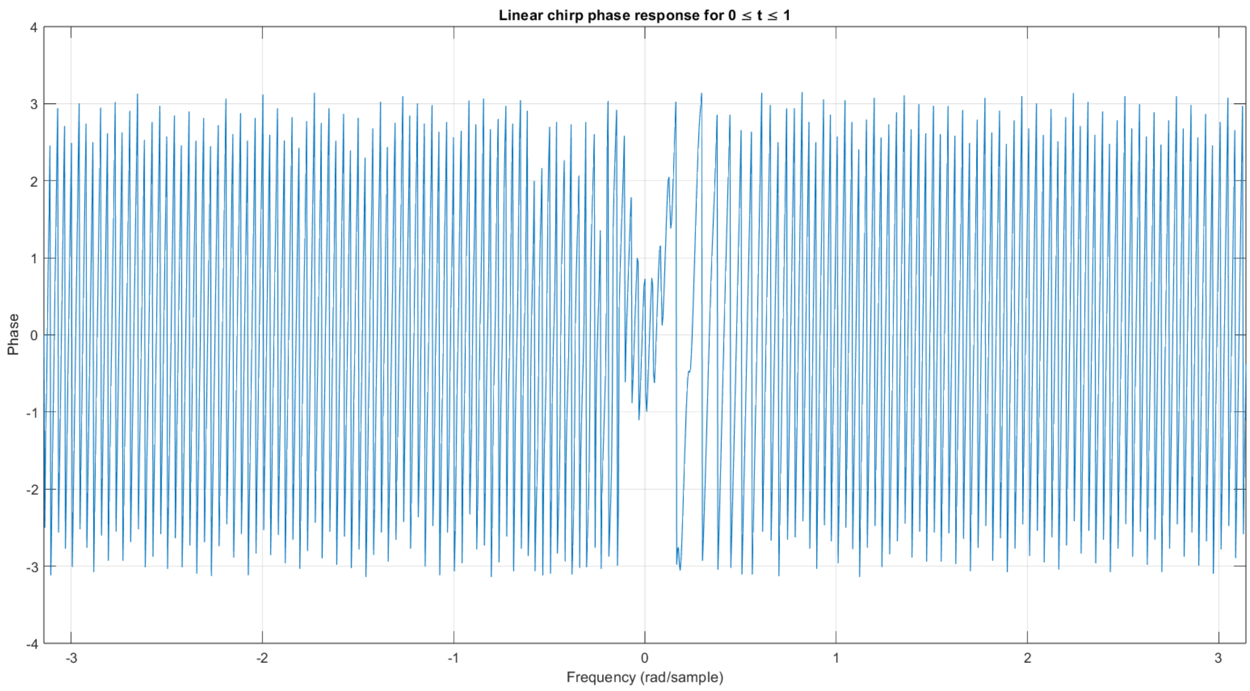


Figure 11: Phase response for $0 \leq t \leq 1$

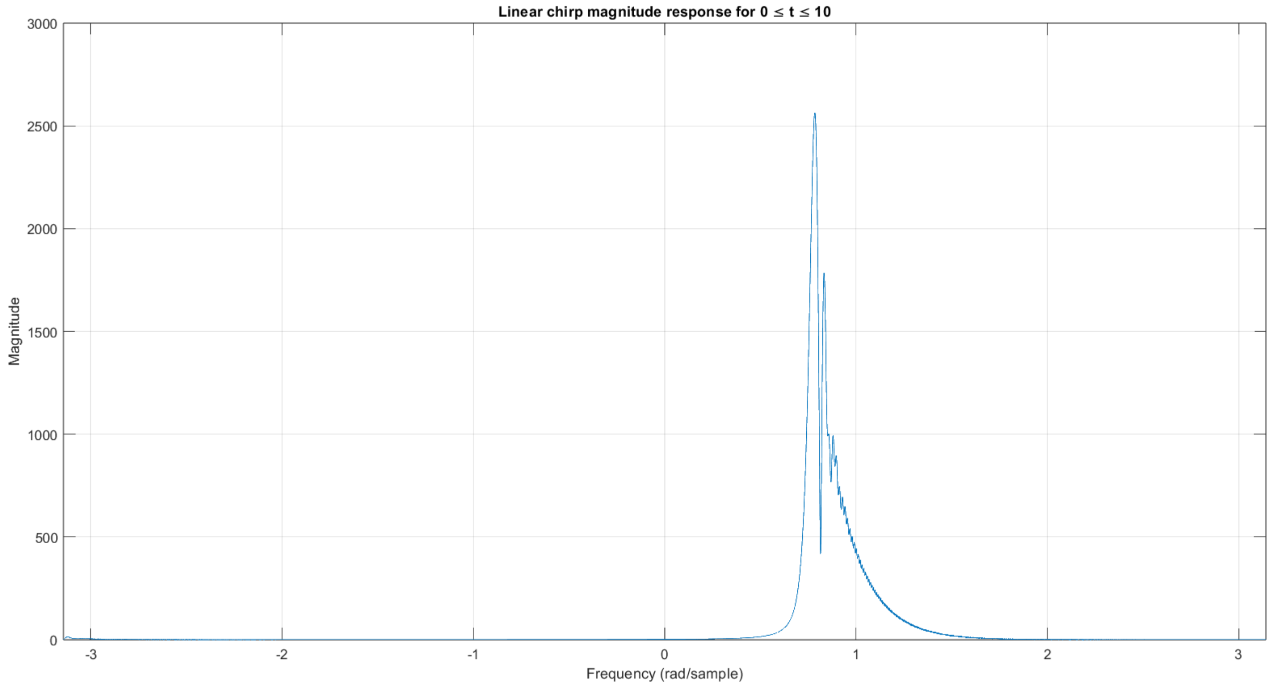


Figure 12: Magnitude response for $0 \leq t \leq 10$

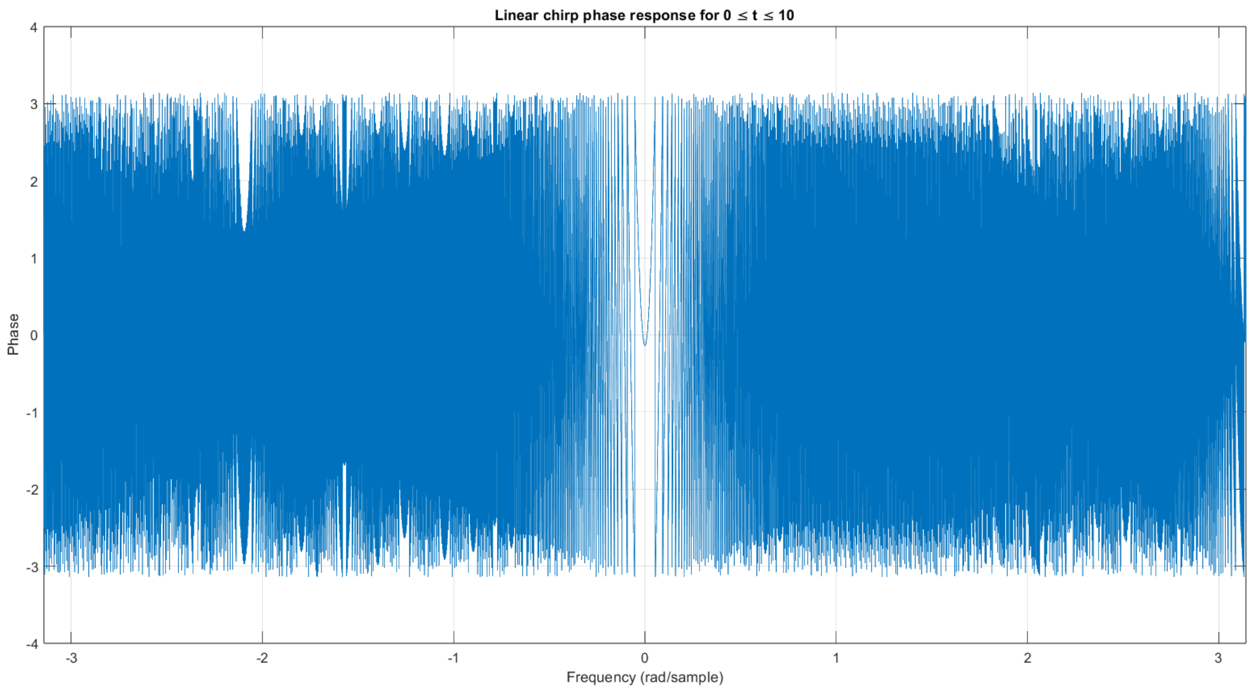


Figure 13: Phase response for $0 \leq t \leq 10$

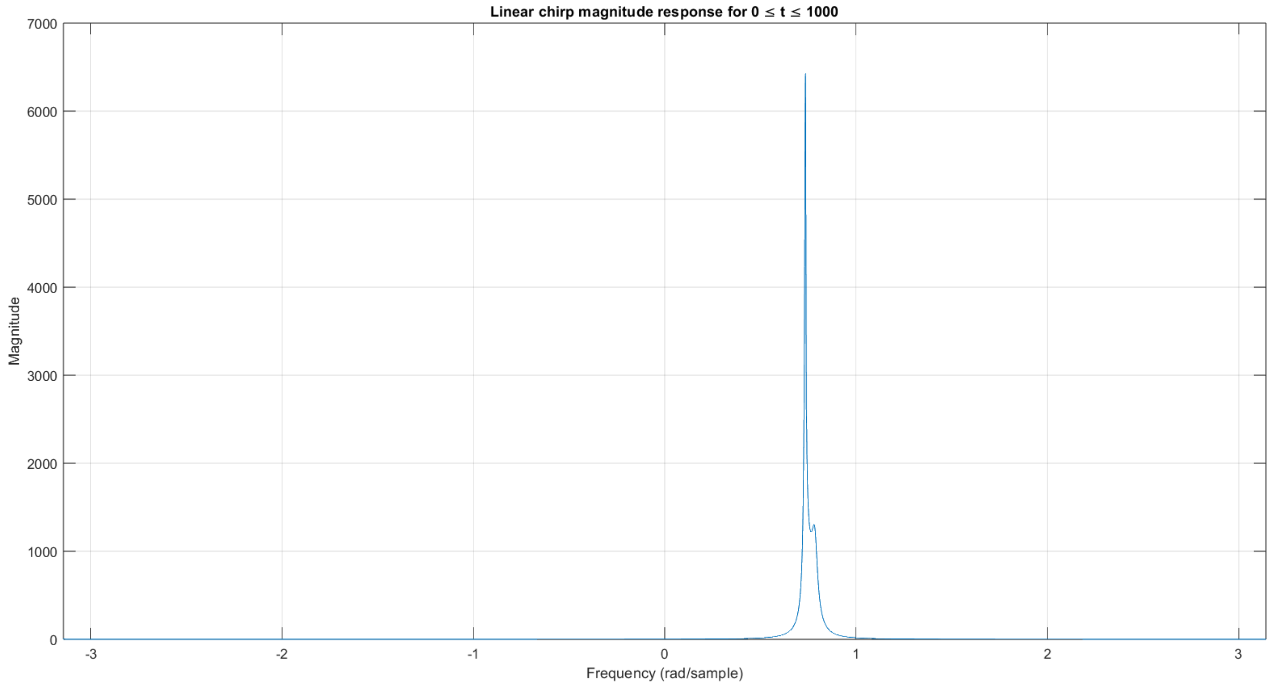


Figure 14: Magnitude response for $0 \leq t \leq 1000$

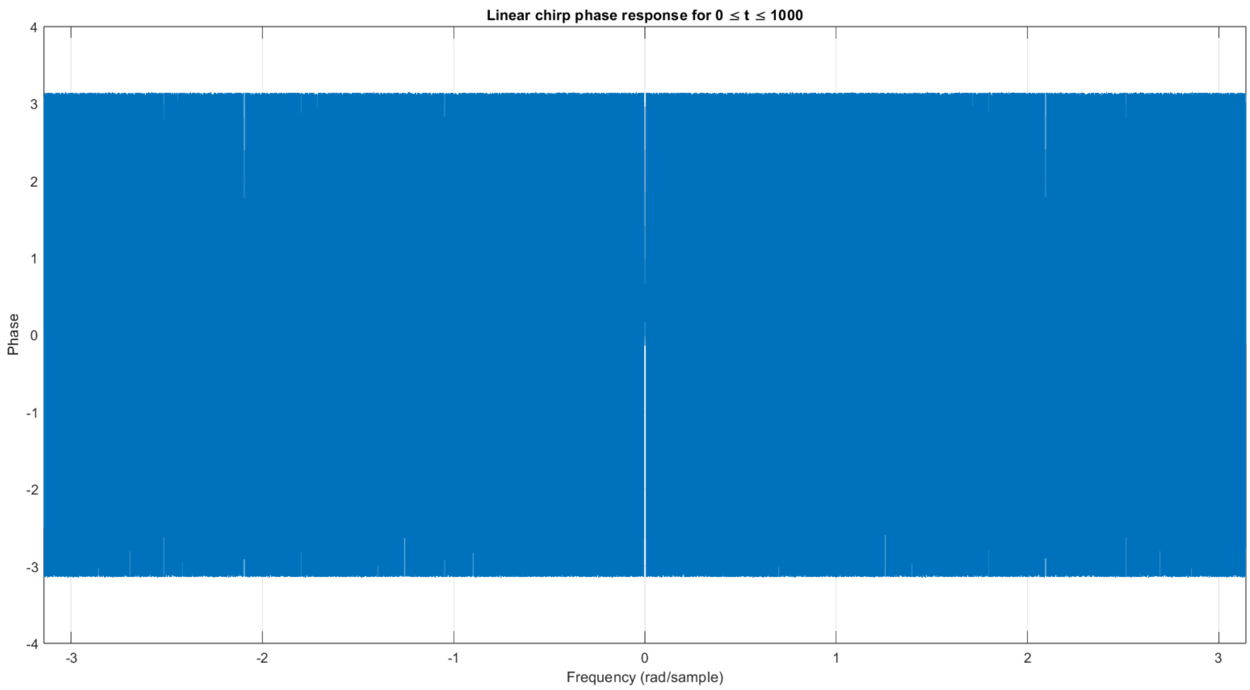


Figure 15: Phase response for $0 \leq t \leq 1000$