Math255: Probability and Statistics **Bilkent University**
Fall 2015-2016

Midterm 1
24 October 2015, 10:00 - 12:00

120 minutes. Three problems. 30 points. Closed book. One one-sided A4-size sheet of notes allowed.

Solutions

P1. (10 points) The two parts of this problem are independent.

(a) (5 pt) Let A be the event that the birthdays of 6 people fall in two calendar months leaving exactly 10 months free. (Assume independence of birthdays and equal probabilities for all months.) Compute the probability $P(A)$. Simplify the result.

Solution: There are $\binom{12}{2} = 66$ different ways 2 months can be chosen out of 12. Having chosen two months, say, May and July, 6 people can be assigned birthdays to May and July in $2^6 = 64$ different ways. In two of these 64 assignments all 6 people are born in May or in July, which violates the requirement that exactly 10 months will be free of birthdays. So, the number of valid assignments of birthdays to May and July is $64 - 2 = 62$. On the other hand, the total number of assignments of 6 birthdays to 12 months is $12^6 = 2,985,984$. So, the answer is found as:

$$\boxed{P(A) = \frac{\binom{12}{2}(2^6 - 2)}{12^6} \approx 0.00137}$$

(b) (5 pt) Suppose the integers $1, 2, \ldots, n$ are rearranged in random order (each permutation equally likely). Let B be the event that the numbers 1 and 2 appear as neighbors after the rearrangement (either as 12 or as 21). Compute $P(B)$.

Solution: If you "stick" 1 and 2 together in the order 12, there are $(n-1)!$ permutations in which 1 and 2 are neighbors as 12. Similarly, there are $(n-1)!$ permutations in which 1 and 2 are neighbors in the order 21. There are $n!$ permutations in all. So, we obtain the answer as:

$$\boxed{P(B) = \frac{2(n-1)!}{n!} = \frac{2}{n}.}$$

P2. (10 points)

The two parts of this problem are independent.

(a) (5 pt) Consider a population consisting of N men and $2N$ women. Suppose the frequency of color-blindness in the population is 5% among men and 1% among women. A person is chosen at random from the population. What is the conditional probability that the chosen person is male (event A) given that the person is color blind (event B)?

Solution: By Bayes' rule,

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)},$$

where

$$P(A) = N/(N + 2N) = 1/3.$$

and, by the law of total probability,

$$P(B) = P(B|A)P(A) + P(B|A^c)P(A^c) = 0.05 \cdot 1/3 + 0.01 \cdot 2/3 = 0.07/3.$$

Thus,

$$\begin{aligned} P(A|B) &= \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)} \\ &= \frac{0.05/3}{0.07/3} \\ &= 5/7. \end{aligned}$$

$$\boxed{P(A|B) = 5/7}$$

(b) (5 pt) In a bolt factory machines A, B, C manufacture, respectively, 25, 35, and 40 percent of the total. Of the outputs of machines A, B, and C, the percentage of defective bolts are 5, 4, and 2 percent, respectively. A bolt is drawn at random from the produce and is found defective. What is the probability p that it was manufactured by machine A?

Solution: Let D be the event that the randomly chosen bolt is defective. As in the previous part, we have

$$\begin{aligned} p = P(A|D) &= \frac{P(D|A)P(A)}{P(D|A)P(A) + P(D|B)P(B) + P(D|C)P(C)} \\ &= \frac{0.05 \cdot 0.25}{0.05 \cdot 0.25 + 0.04 \cdot 0.35 + 0.02 \cdot 0.40} = 25/69. \end{aligned}$$

$$\boxed{p = 25/69}$$

P3. (10 points)

The two parts of this problem are independent.

(a) (5 pt) Let X be a random variable that takes values from 0 to 19 with equal probability 1/20. Let $Z = X \bmod(6)$. Find the PMF $p_Z(k)$ for $k = 0, 1, 2, 3, 4, 5$, and the expectation $E[Z]$.

Solution: $Z = 0$ if and only if (iff) $X = 0, 6, 12, 18$; $Z = 1$ iff $X = 1, 7, 13, 19$; $Z = 2$ iff $X = 2, 8, 14$; $Z = 3$ iff $X = 3, 9, 15$; $Z = 4$ iff $X = 4, 10, 16$; and $Z = 5$ iff $X = 5, 11, 17$. Since X is uniform on integers 0 to 19, from the above frequencies for values of Z, we obtain

$$\boxed{p_Z(k) = \begin{cases} 4/20, & \text{if } k = 0, 1, \\ 3/20, & \text{if } k = 2, 3, 4, 5 \end{cases}}$$

The expectation is computed as

$$E[Z] = \sum_{k=0}^{5} k p_Z(k) = 0 \cdot \frac{4}{20} + 1 \cdot \frac{4}{20} + 2 \cdot \frac{3}{20} + 3 \cdot \frac{3}{20} + 4 \cdot \frac{3}{20} + 5 \cdot \frac{3}{20} = \frac{46}{20}$$

$$\boxed{E[Z] = 2.3}$$

(b) (5 pt) Assume that the number of eggs laid by an insect is a Poisson random variable X with parameter λ, $p_X(k) = \lambda^k e^{-\lambda}/k!$, $k = 0, 1, \ldots$. Suppose that each egg develops into an insect with probability p, independently of others. Let Y be the number of eggs that develop. Find $p_Y(k)$ for $k = 0$, $k = 1$, and for general k.

Solution: By the law of total probability

$$p_Y(\ell) = \sum_{k=0}^{\infty} p_X(k) P(Y = \ell | X = k) = \sum_{k=\ell}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} \binom{k}{\ell} p^\ell (1-p)^{k-\ell},$$

where we observed that, conditional on k eggs being laid, the number of eggs that develop is a $\text{Binom}(k, p)$ random variable, for $0 \le \ell \le k$. We also started the sum at $k = \ell$ since $P(Y = \ell | X = k) = 0$ for $k < \ell$. With a little algebra (see Problem 2.37 of textbook), the sum can be simplied and one obtains that Y is Poisson with parameter λp! This is an important fact to remember: If you put a sieve at the output of a Poisson source, you get a Poisson source.

$$\boxed{p_Y(\ell) = \tfrac{(\lambda p)^\ell}{\ell!} e^{-\lambda p}, \quad \ell = 0, 1, \ldots}$$

From this general form, we obtain the special cases.

$$\boxed{p_Y(0) = e^{-\lambda p}}$$

$$\boxed{p_Y(1) = (\lambda p) e^{-\lambda p}}$$