Bilkent University	Fall 2016-2017

Math255 Probability and Statistics
Midterm 1
3 Nov. 2016

- 100 minutes. Four problems. 30 points. Closed book. You may use one one-sided A4-size sheet of notes. No calculators.
- In each problem, you must show your work in the space provided for that problem and write your final answer in the designated box. You will receive no credit on correct answers if there is not sufficient justification or the answer does not appear in the box.
- Write legibly. If I cannot read your answer, do not expect to receive any credit.
- In problems where you are asked to "compute" a quantity, you are expected to simplify all expressions and provide a result such as 1.65, 0.0123, 1.23×10^{-2}, or 4/25. Do not leave a result as 12/75, simplify it to 4/25.

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Score (for instructor use)

	(a)	(b)	(c)	(d)	Sum
P1		—	—	—	
P2			—	—	
P3			—	—	
P4			—	—	
				Overall	

P1. (6 points) Let A, B, C be three events in a probability space. Suppose that A and C are conditionally independent given B.

Give a formula for $P(B|AC)$ in terms of $P(B)$, $P(A|B)$, $P(A|B^c)$, $P(C|B)$, and $P(C|B^c)$. Show your work and explain which laws of probability you are using. Write the final result in the box below. (Do not try to solve the problem inside the box!)

Formula:

$$P(B|AC) = \frac{P(A|B)\,P(C|B)\,P(B)}{P(A|B)\,P(C|B)\,P(B) + P(A|B^c)\,P(C|B^c)\,P(B^c)}$$

Suppose that $P(B) = 0.3$, $P(A|B) = 0.2$, $P(A|B^c) = 0.5$, $P(C|B) = 0.4$, and $P(C|B^c) = 0.3$. Evaluate your formula in the previous step to compute $P(B|AC)$.

Numerical result:

$$P(B|AC) = \frac{0.2 \cdot 0.4 \cdot 0.3}{0.2 \cdot 0.4 \cdot 0.3 + 0.5 \cdot 0.3 \cdot 0.7} = \frac{24}{24+105} = \frac{24}{129} = \frac{8}{43}.$$

$$P(B|AC) = P(ABC)/P(AC) = \frac{P(AC|B)\,P(B)}{P(AC)}$$

$$= \frac{P(A|B)\,P(C|B)\,P(B)}{P(ABC) + P(AB^cC)} \quad \begin{array}{l} \leftarrow \text{conditional independence} \\ \leftarrow \text{Law of total probability} \end{array}$$

$$= \frac{P(A|B)\,P(C|B)\,P(B)}{P(A|B)\,P(C|B)\,P(B) + P(A|B^c)\,P(C|B^c)\,P(B^c)}$$

P2. (8 points)

A system s consists of five subsystems a, b, c, d, e, each of which may be either in *working* or *failed* state at any point in time. Assume that s is configured so that it is in working state if and only if the following conditions are true simultaneously:

- Cond. 1: At least two of the subsystems $\{a, b, c\}$ are in working state,
- Cond. 2: At least two of the subsystems $\{a, d, e\}$ are in working state.

To analyze the reliability of the system, introduce a probabilistic model so that, at any given inspection point in time, each of the substems is found in working state with probability p, independently of each other. In this probability space, let W denote the event that the system s is in working state; likewise, let A, B, C, D, E, denote, respectively, the events that the subsystems a, b, c, d, e are in working state.

(a) (4 pts) Express the event W in terms of the events A, B, C, D, E.

$$W = \left[A(B \cup C)(D \cup E)\right] \cup \left[A^c (B \cap C)(D \cap E)\right]$$
$$= A(B \cup C)(D \cup E) \cup A^c BCDE$$

(b) (4 pts) Express $P(W)$ as a function of p. Simplify your result as much as possible. Show your work in detail and point out where the independence assumption is used.

$$P(W) = p(2p - p^2)^2 + (1-p)p^4 = p^3(4 - 3p)$$

Compute $P(W)$ for $p = 1/2$.

$$P(W)\Big|_{p=1/2} = \frac{5}{16}.$$

P3. (6 points)

Assume that (X, Y) are jointly distributed random variables with

$$p_X(x) = \begin{cases} 1/2, & x = 1, \\ 1/2, & x = 2, \end{cases} \qquad \text{and} \qquad p_{Y|X}(y|x) = \frac{x^y e^{-x}}{y!}, \quad y = 0, 1, 2, \ldots$$

(You may use without proof the facts that the expectation of a Poisson random variable with parameter λ is λ and its variance is also λ.)

(a) (3 pts) Compute $E[Y]$.

$E[Y] = 1.5$

(b) (3 pts) Compute $\mathrm{var}(Y)$.

$\mathrm{var}(Y) = \frac{7}{4} = 1.75$

$$E[Y] = P(X=1)\,E[Y|X=1] + P(X=2)\,E[Y|X=2]$$

$$= \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot 2 = \frac{3}{2} = 1.5$$

$$\mathrm{Var}(Y) = E[Y^2] - (E[Y])^2$$

$$E[Y^2] = P(X=1)\,E[Y^2|X=1] + P(X=2)\,E[Y^2|X=2]$$

$$= \frac{1}{2}\cdot[1+1^2] + \frac{1}{2}\cdot[2+2^2]$$

$$= 1 + 3 = 4.$$

$$\therefore\ E[Y^2] - (E[Y])^2 = 4 - \left(\frac{3}{2}\right)^2 = 4 - \frac{9}{4} = \underline{\underline{\frac{7}{4}}}$$

P4. (10 points)

Let Ω be the set of all 5-by-5 matrices such that $A \in \Omega$ if and only if A is a 0-1 matrix (i.e., each entry of A is either 0 or 1) with exactly 5 entries equal to 1. Thus, e.g., among the matrices

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

A and B belong to Ω but C doesn't.

(a) (5 pts) Let S be the event that a matrix chosen at random from Ω has exactly one 1 in each column and exactly one 1 in each row. (The matrix A above has this property, but B doesn't.) Compute $P(S)$. Show your reasoning clearly.

$$P(S) = \frac{5!}{\binom{25}{5}} = \frac{4}{1771} = 2.26 \times 10^{-3}$$

(b) (5 pts) For each $\omega \in \Omega$, define $X(\omega)$ as the number of rows of ω with exactly one 1. (For example, $X(A) = 5$ and $X(B) = 3$.) Compute $E[X]$.

$E[X] =$ 2.28

$X = X_1 + X_2 + \cdots + X_5$ where

$$X_i = \begin{cases} 1 & \text{if one 1 in row } i \\ 0 & \text{else} \end{cases}$$

$$E[X] = E[X_1] + E[X_2] + \cdots + E[X_5] = 5 \cdot E[X_1].$$

$$E[X_1] = P(X_1 = 1) = \frac{5 \cdot \binom{20}{4}}{\binom{25}{5}} = \frac{1615}{3542}$$

$$E[X] = \frac{8075}{3542} = 2.28$$