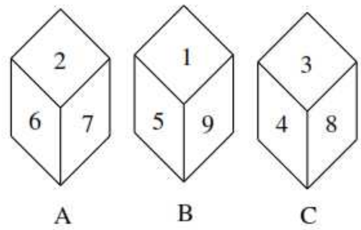
Bilkent University	Fall 2017-2018

Math255 Probability and Statistics
Midterm 1 Solutions
20 Oct. 2017

Problem 1. [6 pts]

There are three dice A, B, and C as shown. For each die, the number on the concealed face is the same as the number on the opposite exposed face. The dice are rolled. Let (X, Y, Z) denote the outcome, with X, Y, and Z being the number on the up face of dice A, B, and C, respectively.



Let $V = \{X > Y\}$ and $W = \{X > Y > Z\}$. Compute $P(W|V)$.

Solution. A sample space for this problem is $\Omega = \{2, 6, 7\} \times \{1, 5, 9\} \times \{3, 4, 8\}$ where $\times$ denotes Cartesian product. Elements of the sample space are ordered triples (x, y, z) where $x \in \{2, 6, 7\}$, $y \in \{1, 5, 9\}$, and $z \in \{3, 4, 8\}$. Each outcome is equally likely and has probability $1/27$ of occurring. The events of interest are identified as

$$\begin{aligned} V = \{X > Y\} = \{&(2,1,3),(2,1,4),(2,1,8),(6,1,3),(6,1,4),(6,1,8),(6,5,3),\\ &(6,5,4),(6,5,8),(7,1,3),(7,1,4),(7,1,8),(7,5,3),(7,5,4),(7,5,8)\} \end{aligned}$$

and

$$W = \{X > Y > Z\} = \{(6,5,3),(6,5,4),(7,5,3),(7,5,4)\}.$$

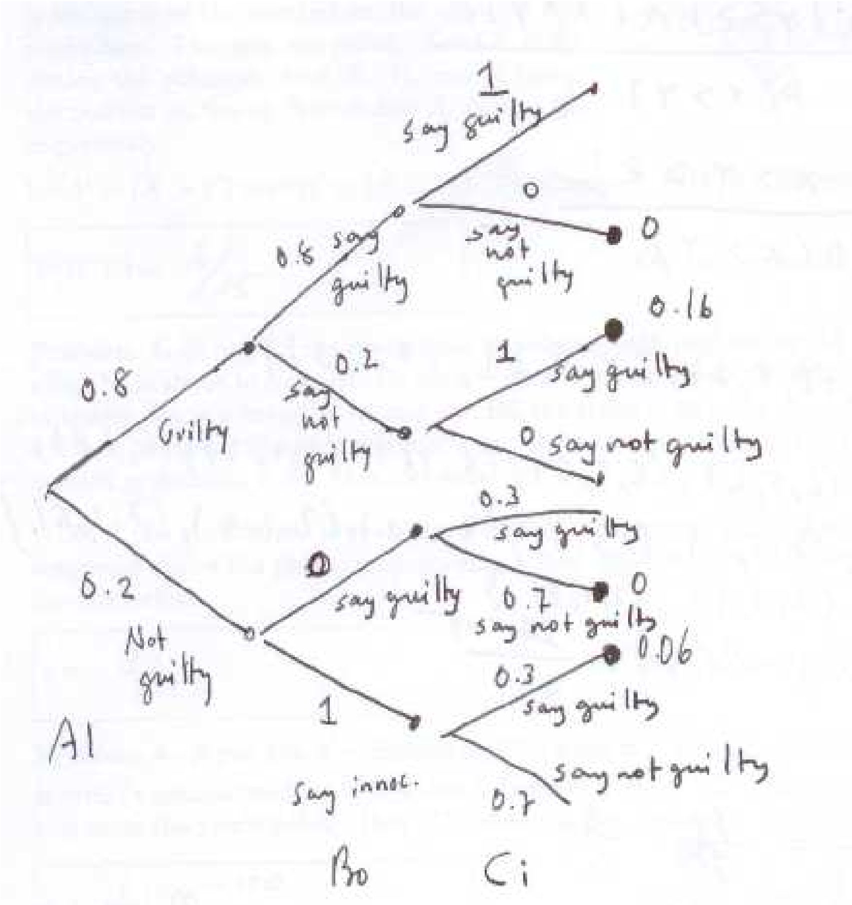
We note that $W \subset V$ so below we will have $W \cap V = W$.

We compute the probability of interest as

$$P(W|V) = \frac{P(W \cap V)}{P(V)} = \frac{P(W)}{P(V)} = \frac{4/27}{15/27} = 4/15.$$

Problem 2. [6 pts] To the best of our knowledge, with probability 0.8 Al is guilty of the crime for which he is about to be tried. Bo anc Ci, each of whom knows whether Al is guity, have been called to testify. Bo is a friend of Al and will tell the truth if Al is innocent but will lie with probability 0.2 if Al is guilty. Ci hates everybody but the judge and will tell the truth if Al is guilty and will lie with probability 0.3 if Al is innocent. What is the conditional probability q that Al is innocent, given that Bo and Ci gave conflicting testimony. Solve the problem by drawing a tree.

Solution. Let I be the event that Al is innoguilty and C be the event that Bo and Ci give conflicting testimonies. We are interested in the probability $q = P(I|C) = P(I \cap C)/P(C)$. To determine $P(I \cap C)$ and $P(C)$ we draw the tree as shown below. Leaf nodes marked with a solid dot correspond to the event C. We read off from the tree that $P(C) = 0.16 + 0.06 = 0.22$ and $P(I \cap C) = 0.06$. So, $q = \frac{0.06}{0.22} = \frac{3}{11}$.



Problem 3. [6 pts] Let $X \sim \text{Binom}(n, p)$ for some $n \geq 1$ and $0 < p < 1$. Let A be the event that X is even (a number such as 0, 2, 4, etc.). Let $\delta \triangleq P(A) - 1/2$. Compute δ for $n = 100$ and $p = 0.45$ and write the result below. *Hint: Consider the expansions of* $((1-p)+p)^n$ *and* $((1-p)-p)^n$.

Solution. Since $p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$ for $k = 0, 1, \ldots, n$, we have

$$P(A) = \sum_{k \text{ even}} \binom{n}{k} p^k (1-p)^{n-k}.$$

Observe that

$$((1-p)+p)^n = \sum_{k \text{ odd}} \binom{n}{k} p^k (1-p)^{n-k} + \sum_{k \text{ even}} \binom{n}{k} p^k (1-p)^{n-k}$$

$$((1-p)-p)^n = -\sum_{k \text{ odd}} \binom{n}{k} p^k (1-p)^{n-k} + \sum_{k \text{ even}} \binom{n}{k} p^k (1-p)^{n-k}$$

So,

$$\frac{((1-p)+p)^n + ((1-p)-p)^n}{2} = \sum_{k \text{ even}} \binom{n}{k} p^k (1-p)^{n-k}.$$

Thus,

$$P(A) = \frac{((1-p)+p)^n + ((1-p)-p)^n}{2} = \frac{1 + (1-2p)^n}{2},$$

and

$$\delta = P(A) - \frac{1}{2} = \frac{(1-2p)^n}{2}.$$

For $n = 100$, $p = 0.45$, we have

$$\delta = \frac{10^{-100}}{2},$$

which is almost 0.

Problem 4. [6 pts] Let Y be the waiting time until the nth success in a sequence of independent Bern(p) trials. Calculate $\mathbf{E}[Y]$ as a function of n and p. Show your reasoning and calculations in detail.

Solution. For a simple solution, we express Y as the sum of simpler random variables

$$Y = X_1 + X_2 + \cdots + X_n$$

where X_i is the waiting time from the $(i-1)$th success to the ith one. This implies

$$\mathbf{E}[Y] = \mathbf{E}[X_1] + \mathbf{E}[X_2] + \cdots + \mathbf{E}[X_n].$$

Next, we note that X_i is Geometric with parameter p, for each $i = 1, 2, \ldots, n$. (They are also independent but that we do not need to use here.) Recalling that the mean of a Geometric random variable is $1/p$, we obtain the answer as

$$\mathbf{E}[Y] = n/p.$$

Problem 5. [6 pts] Let (X, Y) be jointly distributed with

$$p_{X,Y}(x,y) = \begin{cases} c(2x+y), & \text{x=1,2; y=1,2} \\ 0, & \text{otherwise,} \end{cases}$$

where c is a constant (which you should determine as part of the problem). Compute the conditional expectation $\mathbf{E}[2XY + Y^2|Y = 2]$ (a numerical value is required).

Solution. First note that

$$\mathbf{E}[2XY + Y^2|Y = 2] = \mathbf{E}[4X + 4|Y = 2] = 4\mathbf{E}[X|Y = 2] + 4.$$

So, all that remains is to compute $\mathbf{E}[X|Y = 2]$.

$$\begin{aligned}
\mathbf{E}[X|Y = 2] &= \sum_{x=1}^{2} x p_{X|Y}(x|2) = p_{X|Y}(1|2) + 2p_{X|Y}(2|2) \\
&= \frac{p_{X,Y}(1,2) + 2p_{X,Y}(2,2)}{p_Y(2)} = \frac{p_{X,Y}(1,2) + 2p_{X,Y}(2,2)}{p_{X,Y}(1,2) + p_{X,Y}(2,2)} \\
&= \frac{c(2\cdot 1+2) + 2\cdot c(2\cdot 2+2)}{c(2\cdot 1+2) + c(2\cdot 2+2)} = \frac{4 + 2\cdot 6}{4+6} = \frac{16}{10} = 1.6
\end{aligned}$$

Thus, $\mathbf{E}[2XY + Y^2|Y = 2] = 4 \times 1.6 + 4 = 10.4$.