Bilkent University		Spring 2015-2016
	Math255 Probability and Statistics Final Exam 9 May 2016, 15:30 - 18:00	

150 minutes. Three problems. 25 points. Closed book. You may use two two-sided A4-size sheet of notes. Good luck!

In each problem, you must show your work in the space provided for that problem and write your final answer in the designated box. You will receive no credit on correct answers if there is not sufficient justification. Write legibly. If I cannot read your answer, do not expect to receive any credit.

Name and Lastname: SOLUTIONS

Bilkent ID No:

Score (for instructor use)

	(a)	(b)	(c)	(d)	Sum
P1		—	—	—	
P2			—	—	
P3			—	—	
				Overall	

P1. (5 points) Ali is away from home, watching TV in his hotel room, when he sees a news flash that an earthquake took place near his home moments ago. Two minutes later, he receives a phone call from his neighbor Hasan who assures him that the earthquake did not cause any damage to the neighborhood but that the burglar alarm in Ali's home is ringing. Would Ali be justified in thinking that the alarm is triggered by the earthquake and not by a burglar breaking into his house?

Let us formalize the problem as follows. Let A be the event that Ali's burglar alarm is ringing, B the event that a burglar broke into Ali's house, E the event that an earthquake took place near Ali's house, N the event that an earthquake in the town where Ali lives is reported on national TV. Suppose B and E are independent events, and that N is conditionally independent of A and B given E. Let the following probabilities be provided.

- $P(B) = 0.01$
- $P(E) = 0.01$
- $P(A|B^cE^c) = 0$
- $P(A|B^cE) = 0.3$
- $P(A|BE^c) = 0.8$
- $P(A|BE) = 0.99$
- $P(N|E) = 0.5$
- $P(N|E^c) = 0$

Compute $P(B^c|AN)$. Give a numerical result after simplifying it to a simple fraction. Show your work in full detail for partial credit. Just a correct numerical answer will receive zero credit.

$P(B^c|AN) =$ $30/31$

$$P(B^c|AN) = \frac{P(AN|B^c)\,P(B^c)}{P(AN)} = \frac{P(AN|B^c)\,P(B^c)}{P(AN|B^c)\,P(B^c) + P(AN|B)\,P(B)}$$

$$\begin{aligned} P(AN|B^c) &= P(ANE|B^c) + P(ANE^c|B^c) \\ &= P(AN|EB^c)\,P(E|B^c) + P(AN|E^cB^c)\,P(E^c|B^c) \\ &= P(A|EB^c)\,P(N|AEB^c)\,P(E|B^c) + P(A|E^cB^c)\,P(N|AE^cB^c)\,P(E^c|B^c) \end{aligned}$$

using independence properties $\rightarrow$ $= P(A|EB^c)\,P(N|E)\,P(E) + P(A|E^cB^c)\,\underbrace{P(N|E^c)}_{0}\,P(E^c)$

$$= P(A|EB^c)\,P(N|E)\,P(E).$$

Likewise, $P(AN|B) = P(A|EB)\,P(N|E)\,P(E)$.

Thus,
$$P(B^c|AN) = \frac{P(A|EB^c)\,P(N|E)\,P(E)\,P(B^c)}{P(A|EB^c)\,P(N|E)\,P(E)\,P(B^c) + P(A|EB)\,P(N|E)\,P(E)\,P(B)}$$

$$= \frac{P(A|EB^c)\,P(B^c)}{P(A|EB^c)\,P(B^c) + P(A|EB)\,P(B)}$$

$$= \frac{1}{1 + \frac{P(A|EB)}{P(A|EB^c)} \cdot \frac{P(B)}{P(B^c)}} = \frac{30}{31}$$

P2. (10 points) (The two parts of this problem can be solved independently.)

Assume that X and Y are two independent random variables, each having an exponential PDF with parameter λ:

$$f_{X,Y}(x,y) = f_X(x) f_Y(y) = \lambda^2 e^{-\lambda(x+y)}, \quad x \geq 0,\ y \geq 0.$$

Let $Z \triangleq X + Y$.

(Useful fact: For the exponential PDF, the mean and variance are, respectively, $1/\lambda$ and $1/\lambda^2$.)

2.(a) (5 pts) Assume that (X, Y, Z) are as defined above. Compute the PDF $f_Z(z)$, $z \geq 0$. Show your work.

$$\boxed{f_Z(z) = \lambda^2 z e^{-\lambda z}, \quad z \geqslant 0.}$$

Since X, Y are independent, we have

$$f_Z = f_X * f_Y \quad \text{(convolution)}.$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\, dx$$

For $z \geqslant 0$, we have

$$f_Z(z) = \int_0^z \underbrace{\lambda e^{-\lambda x}}_{f_X(x)} \cdot \underbrace{\lambda e^{-\lambda(z-x)}}_{f_Y(z-x)} dx$$

$$= \int_0^z \lambda^2 e^{-\lambda z} dx = \lambda^2 z e^{-\lambda z}.$$

$$\therefore\ f_Z(z) = \begin{cases} \lambda^2 z e^{-\lambda z}, & z \geq 0 \\ 0, & \text{o.w.} \end{cases}$$

2.(b) (5 pts) In this part assume $\lambda = 2$. Consider a random variable of the form $\hat{X} = aZ + b$ where a and b are real numbers. Determine a and b such that the mean squared error (MSE) $E[(\hat{X} - X)^2]$ is minimized. In other words, find the best linear least mean squares (LLMS) estimator $\hat{X}$ of X given Z. **(No derivation of the LLMS formula is required but you must show your computations of a and b in full detail.)**

$a = \frac{1}{2}$	$b = 0$

$$\hat{X} = E[X] + \frac{Cov(X,Z)}{Var(Z)}(Z - E(Z))$$

$$= \frac{1}{\lambda} + \frac{Cov(X, X+Y)}{Var(X+Y)}(Z - E(X+Y))$$

$$= \frac{1}{\lambda} + \frac{Var(X)}{Var(X) + Var(Y)}\left(Z - \frac{2}{\lambda}\right)$$

$$= \frac{1}{\lambda} + \frac{1}{2}\left(Z - \frac{2}{\lambda}\right) = \frac{Z}{2}$$

P3. (5 points) The two parts of this problem are independent.

3.(a) (5 pts) Consider the following classical estimation problem. Let a PDF be specified in terms of an unknown parameter $\theta > 0$ as follows

$$f_{X,Y}(x, y; \theta) = \begin{cases} \frac{1}{\pi\theta^2}, & \text{if } x^2 + y^2 \leq \theta^2; \\ 0, & \text{otherwise.} \end{cases}$$

[3 pts] Given a sample (x, y) from the preceding PDF, compute the realization value of the maximum likelihood (ML) estimator $\hat{\Theta}(x, y)$ of θ as a function of (x, y). Write in the box below the ML estimator $\hat{\Theta}(X, Y)$ as a function of (X, Y).

[2 pts] Compute the bias $b_\theta = E_\theta[\hat{\Theta}(X, Y)] - \theta$.

$\hat{\Theta}(X, Y) = \sqrt{X^2+Y^2}$.	$b_\theta = -\theta/3$

$$\hat{\theta}(x,y) = \underset{\theta \geq \sqrt{x^2+y^2}}{\operatorname{argmax}} \frac{1}{\pi\theta^2} = \sqrt{x^2+y^2}.$$

$$b_\theta = E_\theta\left[\sqrt{X^2+Y^2}\right] - \theta$$

$$E_\theta\left[\sqrt{X^2+Y^2}\right] = \iint_{x^2+y^2 \leq \theta^2} \frac{\sqrt{x^2+y^2}}{\pi\theta^2}\, dx\, dy$$

$$= \int_0^{2\pi}\int_0^{\theta} \frac{r}{\pi\theta^2} \cdot \underset{\substack{\uparrow \\ \text{Jacobian}}}{r} \cdot dr \cdot d\phi$$

where $r = \sqrt{x^2+y^2}$

$\phi = \tan^{-1}\left(\frac{y}{x}\right)$

are polar coordinates

$$= \int_0^{2\pi} \frac{r^3}{3\pi\theta^2}\Bigg|_0^{\theta} d\phi = \int_0^{2\pi} \frac{\theta}{3\pi}\, d\phi = \frac{2\theta}{3}.$$

$$b_\theta = \frac{2\theta}{3} - \theta = -\frac{\theta}{3}$$

3.(b) (5 pts) The received signal in a radar system is $X = 2 + W$ if a targer airplane is present, otherwise (if no target is present), it is $X = W$. Here, W models noise and is assumed Gaussian with mean 0 and variance 4, i.e., $W \sim N(0,4)$. Accordingly, we treat this problem as a binary hypothesis testing problem with

$$f_X(x; H_0) = \frac{1}{\sqrt{8\pi}} e^{-x^2/8}, \quad f_X(x; H_1) = \frac{1}{\sqrt{8\pi}} e^{-(x-\cancel{10}\,2)^2/8}.$$

Consider the likelihood ratio tests (LRT)

$$L(x) = \frac{f_X(x; H_1)}{f_X(x; H_0)} \geq \gamma$$

where $\gamma \geq 0$ is a real number. Compute the probability of false alarm (type I error) α and the probability of miss (type II error) β as functions of γ. Express the results explicitly (in as simple a form as possible) in terms of the CDF $\Phi(x)$ of the unit normal PDF $N(0,1)$. Let γ_0 denote the value of γ for which $\alpha = \beta$. Determine γ_0.

$\alpha = 1 - \Phi\left(\ln \gamma + \frac{1}{2}\right)$	$\beta = \Phi\left(\ln \gamma - \frac{1}{2}\right)$	$\gamma_0 = 1$

$$L(x) = \frac{e^{-(x-2)^2/8}}{e^{-x^2/8}} = e^{\frac{4x-4}{8}} = e^{\frac{x-1}{2}}.$$

$$L(x) \underset{H_0}{\overset{H_1}{\gtrless}} \gamma \iff \ln L(x) = \frac{x-1}{2} \underset{H_0}{\overset{H_1}{\gtrless}} \ln \gamma.$$

$$\alpha = P\left(\frac{X-1}{2} \geq \ln \gamma ; H_0\right)$$

$$= P\left(\frac{W-1}{2} \geq \ln \gamma\right) = P(W \geq 2\ln\gamma + 1)$$

$$= P\Big(\underbrace{\frac{W-0}{\sqrt{4}}}_{N(0,1)} \geq \frac{2\ln\gamma + 1}{\sqrt{4}}\Big) = 1 - \Phi\left(\ln\gamma + \frac{1}{2}\right)$$

$$\beta = P\left(\frac{X-1}{2} < \ln\gamma ; H_1\right) = P\left(\frac{2+W-1}{2} < \ln\gamma\right)$$

$$= P\Big(\underbrace{\frac{W}{2}}_{N(0,1)} < \ln\gamma - \frac{1}{2}\Big) = \Phi\left(\ln\gamma - \frac{1}{2}\right)$$

We have $\alpha = \beta$ iff $\gamma = 1$.