SOLUTIONS TO MIDTERM QUESTIONS

1-) a) Define event A as "wheter is bad"

P{conduct the meeting | A} =

P{3 members come | A} + P{4 members come | A} +

P(5 members come | A)

(Since each event above are disjoint, probabilities are added.)

$$P(n \text{ members come} \mid A) = \underbrace{\binom{5}{n}}_{\text{There are } \binom{5}{n} \text{ such sets}} \underbrace{(0.5)^n (0.5)^{5-n}}_{\text{probability of a given set of } n \text{ among 5 comes}}$$

So: $\left\{\binom{5}{3} + \binom{5}{4} + \binom{5}{5}\right\}(0.5)^5$

$$= \left(\frac{5!}{3!2!} + \frac{5!}{4!1!} + \frac{5!}{5!0!}\right)(0.5)^5 = 16(0.5)^5 = \underbrace{\frac{1}{2}}_{\text{call it } P_1}$$

b) Similarly P{conduct a meeting | A^c} =

$$\binom{5}{3}(0.9)^3(0.1)^2 + \binom{5}{4}(0.9)^4 0.1 + \binom{5}{5}(0.9)^5$$

$$= \boxed{0.1(0.9)^3 + 0.5(0.9)^4 + (0.9)^5} \rightarrow \text{call it } P_2$$

" call it P

So; using total probability

c) $\boxed{P\{\text{meeting}\} = 0.2 \cdot \frac{1}{2} + 0.8\left[0.1(0.9)^3 + 0.5(0.9)^4 + (0.9)^5\right]}$

d) $E\{\text{cost} \mid \text{bad}\} = 400 \cdot \underbrace{(1-p_1)}_{\substack{\text{probability} \\ \text{of not conducting} \\ \text{the meeting if} \\ \text{the weather is bad.}}} + 1000 \underbrace{p_1}_{\substack{\text{Probability of} \\ \text{conducting the} \\ \text{meeting if the} \\ \text{weather is bad}}}$

$$= 400 \cdot \frac{1}{2} + 1000 \cdot \frac{1}{2} = \boxed{700 \text{ TL}}$$

e) $E\{\text{cost} \mid \text{good}\} = 400(1-p_2) + 1000 p_2$

$$= \boxed{400 + 600 p_2} \text{ TL}$$

f) $E\{\text{cost}\} = 0.8\left(400 + 600 p_2\right) + (0.2)\, 700 \text{ TL}$

$$= \boxed{460 + 480 p_2} \text{ TL}$$

You may also get it from the results of (c):

$$E\{\text{cost}\} = 400(1-p) + 1000 p = \boxed{400 + 600 p \text{ TL}}$$

2-) a) Call "all four coins are H" as "success". (3)

$$P(\text{sucess in any toss}) = \left(\frac{1}{2}\right)^4 = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}$$

$$= \frac{1}{16} = \text{call it } p$$

(multiplied since coins are H or T independent of other coins)

So: N is number of trials until the first success:

$$f_N(k) = \underbrace{\overbrace{(1-p)^{k-1}}^{k-1 \text{ no sucesses}} \cdot \overbrace{p}^{\text{success at the end}}}_{\text{multiplied since trials are independent.}} \quad \text{where } p = \frac{1}{16}$$

$k = 1, \dots \infty$

b)

$$P\{N=5 \mid N>2\} = \frac{P\{N=5 \text{ and } N>2\}}{P\{N>2\}}$$

N=5 satisfies N>2

$$= \frac{P\{N=5\}}{P\{N>2\}}$$

$P\{N>2\} = P\{1^{st} \text{ and } 2^{nd} \text{ trials are not sucesses}\}$

$$= (1-p)^2$$

So: $$P\{N=5 \mid N>2\} = \frac{(1-p)^4 p}{(1-p)^2} = \boxed{(1-p)^2 p} = \boxed{\frac{15^2}{16^3}}$$

c) $\boxed{P(\text{last toss of } 1^{st} \text{ coin}) = 1}$ since the last toss has all 4 coins H.

3-) a) The total area under the pdf must be 1. Since the area under the impulse is 0.3, the rectangle must have an area = 0.7 (4)

$\Rightarrow \quad a = \frac{0.7}{2} = \boxed{0.35}$

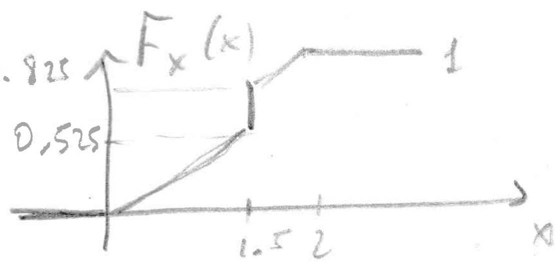
b) $P\{X = 1.5\}$ = area under the impulse = $\boxed{0.3}$

c) $P\{X = 0.5\} = 0$ since the area under the pdf at a single point $x = 0.5$ is zero.

d) $F_X(x) = \int_{-\infty}^{x} f(z)\,dz = \begin{cases} 0 & \text{if } x < 0 \\ ax & \text{if } 0 \le x < 1.5 \\ 0.3 + ax & \text{if } 1.5 \le x < 2 \\ 1 & \text{if } x > 2 \end{cases}$

if $0 \le x < 1.5$ $\quad F_X(x) = \int_0^x a\,dz$

$= az \Big|_0^x = ax$



if $1.5 \le x < 2$ $\quad F_X(x) = 0.3 + ax$

if $2 \le x$ $\quad F_X(x) = 1$

5

e) $$E\{X\} = \int_{-\infty}^{\infty} x \underbrace{f_X(x)}_{\downarrow} dx = \int_{-\infty}^{\infty} x \left[f_1(x) + f_2(x)\right] dx$$

$$f_X(x) = \underbrace{0.3\,\delta(x-\tfrac{3}{2})}_{f_1(x)} + \underbrace{\begin{cases} 0.35 & \text{if } x \in [0,2] \\ 0 & \text{else} \end{cases}}_{f_2(x)}$$

Call: $f_1(x)$, $f_2(x)$

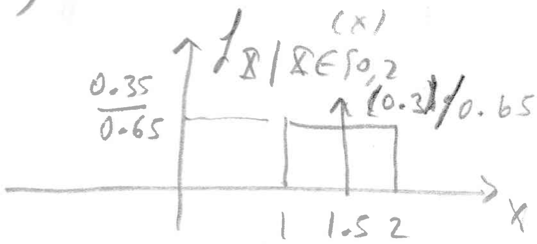
$$= \int_{-\infty}^{\infty} x\, 0.3\,\delta(x-\tfrac{3}{2})\,dx + \int_{-\infty}^{\infty} x f_2(x)\,dx$$

$$= \underbrace{0.3\cdot\frac{3}{2}}_{0.45} + \underbrace{\int_0^2 0.35\,x\,dx}_{=0.35\frac{x^2}{2}\Big|_0^2 = 0.7} = \boxed{1.15}$$

f) $$f_{X|X\in[1,2]}(x) = \begin{cases} \dfrac{f(x)}{P\{X\in[1,2]\}} & \text{if } x \in [1,2] \\ 0 & \text{else} \end{cases}$$

$$P\{X\in[1,2]\} = (0.35)\cdot(2-1) + 0.3 = 0.65$$

$\Rightarrow f_{X|X\in[1,2]}(x)$:



$$\| \qquad \frac{0.3}{0.65}\,\delta(x-\tfrac{3}{2}) + \begin{cases} \dfrac{0.35}{0.65} & \text{if } x \in [1,2] \\ 0 & \text{else} \end{cases}$$

You may also show it using the definition of $E\{\cdot\}$.

Due to symmetry of $f_{X|X\in[1,2]}(x)$ around 1.5 its mean value is $\boxed{1.5}$

$$\boxed{E\{X \mid X\in[1,2]\} = 1.5}$$

4- a) Total volume under the pdf must be equal to 1. Since "uniform distribution" is given pdf is constant over the area B. So the total volume is (6)

$$\underbrace{\iint_{-\infty}^{\infty} f_{X,Y}(x,y)}_{c}\,dx\,dy = 1 \Rightarrow c\cdot B = 1$$

$$c = \frac{1}{B} \quad \text{if } (x,y)\in B$$

$$B = 3 \Rightarrow \boxed{f_{X,Y}(x,y) = \begin{cases} \frac{1}{3} & \text{if } (x,y)\in B \\ 0 & \text{else} \end{cases}}$$

b) $$f_X(x) = \int_{-\infty}^{\infty} \underbrace{f_{X,Y}(x,y)}_{=\frac{1}{3} \text{ in } B,\ 0 \text{ else}}\,dy \Rightarrow$$

if $x \in [0,1]$, then:

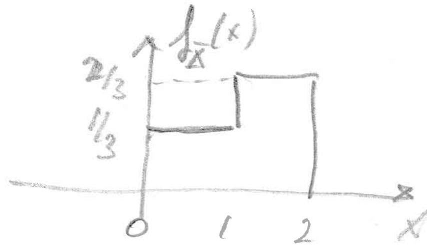
$$\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \int_0^1 \frac{1}{3}\,dy = \frac{1}{3}$$

if $x \in [0,2]$, then

$$\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \int_0^2 \frac{1}{3}\,dy = \frac{2}{3}$$

So:

$$f_X(x) = \begin{cases} \frac{1}{3} & \text{if } x\in[0,1] \\ \frac{2}{3} & \text{if } x\in[1,2] \\ 0 & \text{else} \end{cases}$$



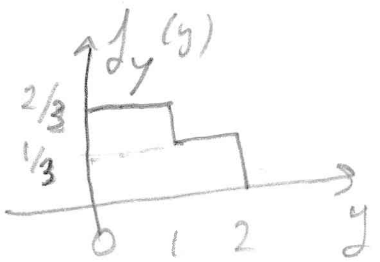
Similarly for $f_Y(y)$:

$$f_Y(y) = \int_{-\infty}^{\infty} \underbrace{f_{X,Y}(x,y)}_{\substack{1/3 \text{ in } B \\ 0 \text{ else}}} dx$$

if $y \in [0,1]$, then $f_Y(y) = \int_0^2 \frac{1}{3} dx = \frac{2}{3}$

if $y \in [1,2]$, then $f_Y(y) = \int_1^2 \frac{1}{3} dx = \frac{1}{3}$

So: $f_Y(y) = \begin{cases} \frac{2}{3} & \text{if } y \in [0,1) \\ \frac{1}{3} & \text{if } y \in [1,2] \\ 0 & \text{else} \end{cases}$

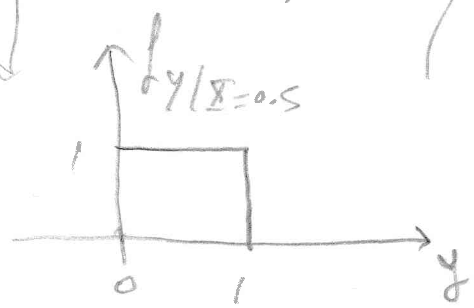


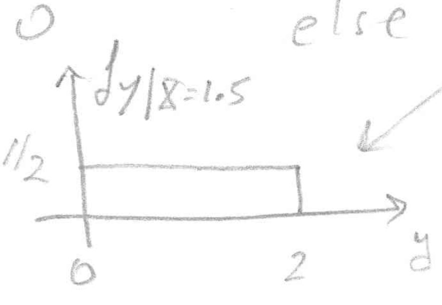
c) $f_{Y|X=x}(y) = \dfrac{f_{X,Y}(x,y)}{f_X(x)} \rightarrow \text{if } \neq 0$

so: if $x \in [0,1]$, then

$$f_{Y|X=x}(y) = \begin{cases} \dfrac{1/3}{1/3} = 1 & \text{if } y \in [0,1] \\ 0 & \text{else} \end{cases} \qquad x \in [0,1]$$

$$f_{Y|X=x} = \begin{cases} \dfrac{1/3}{2/3} = 1/2 & \text{if } y \in [0,2] \\ 0 & \text{else} \end{cases} \qquad x \in [1,2]$$





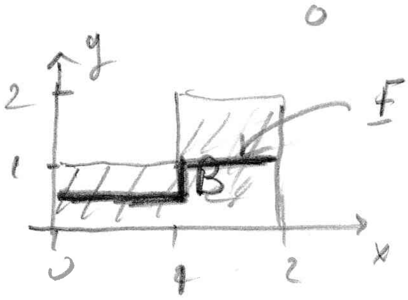
d) $f_{Y|X=x}(y)$ is alredy solved in part (c).

$$E\{Y|X=x\} = \int_{-\infty}^{\infty} y\, f_{Y|X=x}(y)\,dy$$

if $x \in [0,1]$:

$$\int_0^1 y \cdot 1\, dy = \frac{1}{2}$$

if $x \in [1,2]$: $\int_0^2 y \cdot \frac{1}{2}\, dy = 1$



$E\{Y|X=x\}$

e) $E\{Y\} = \int_{-\infty}^{\infty} y \underbrace{f_Y(y)}_{\text{known from part (b)}} dy = \int_0^1 y \cdot \frac{2}{3} dy + \int_1^2 y \cdot \frac{1}{3} dy$

$= \frac{1}{3} + \frac{1}{2} = \boxed{\frac{5}{6}}$

f) Check $f_{X,Y}(x,y) \overset{?}{=} f_X(x) f_Y(y) \;\forall x,y$

if $(x,y) = (0.5, 1.5)$

$\underbrace{f_{X,Y}(0.5, 1.5)}_{=0} \overset{?}{=} \underbrace{f_X(0.5)}_{1/3} \underbrace{f_Y(1.5)}_{1/3}$

No, they are not equal.

Note: one counter example is good enough to violate independence. You may choose any other (x,y) value to show one violation.

5-) First clearly define your experiment and outcomes. Note that the solution will be easy if the outcomes are equally likely:

* Assume each ball has an identifying label (like labels B1, ... B20)
* Each ball goes to a slot randomly. So, an example for the outcome is

	B1	B2	B3	B4	------	B20
	↓	↓	↓	↓		↓
Slot numbers →	2	1	1	4		2

* Count the number of such outcomes:

4 choices for 1 ball $\Rightarrow$ 4^{20} choices for 20 balls.

↳ Total number of outcomes

* All of those outcomes are equally likely since balls go to a slot randomly and independently

* Now count the number of <u>such</u> outcomes that satisfy the given event. $\longrightarrow$

4 balls to slot 3 among 20 balls

Choose 4 out of 20 (ordering is not important) due to definition of outcomes)

$$\downarrow$$

$$\binom{20}{4} = \frac{20!}{16!\,4!}$$

* The rest of the balls (16 of them) may go any one of the 3 remaining slots : 3^{16} possible cases:

$\therefore$ Total # of outcomes satisfying the event is $\frac{20!}{16!\,4!} \cdot 3^{16}$

* Since outcomes are equally likely:

$$P(4 \text{ balls in slot } 3) = \boxed{\frac{\frac{20!}{16!\,4!}\, 3^{16}}{4^{20}}}$$

$\longrightarrow$

Another solution for 5.a):

* A ball may go into slot 3 (probability of this is $\frac{1}{4}$) or to another slot (probability of this is $\frac{3}{4}$).

* Therefore each ball may end up with a "success" with $p=\frac{1}{4}$ or
into slot 3
"no success" with $p=\frac{3}{4}$
(Binary outcome for each ball)

* A sequence of balls from B1 -- B20 should have exactly 4 "successes" to satisfy the given event:

$$\left(\frac{1}{4}\right)^4 \left(\frac{3}{4}\right)^{16} = \text{probability of a sequence.}$$

* There are $\binom{20}{4}$ such sequences that all satisfy the given event (Those 4 successes may occur for any of those B1 -- B20 balls)

* Answer $= \binom{20}{4} \left(\frac{1}{4}\right)^4 \left(\frac{3}{4}\right)^{16} =$

$$\boxed{\frac{20!}{16!\,4!} \cdot \frac{3^{16}}{4^{20}}}$$

Note: Each ball and each slot is distinct and has an identifying label as described by the experiment and outcomes

b) Number of outcomes that satisfy the given event is

$$\binom{20}{5}\cdot\binom{15}{5}\cdot\binom{10}{5}\binom{5}{5}$$

Choose 5 among 20 to slot 1

Choose 5 among remaining 15 to Slot 2

Slot 3

Slot 4

$$= \frac{20!}{5!\,5!\,5!\,5!} = \frac{20!}{(5!)^4}$$

* Since all outcomes are equally likely

$$P(\text{5 balls in each slot}) = \frac{\frac{20!}{(5!)^4}}{4^{20}} =$$

# of total outcomes

$$\boxed{\frac{20!}{(5!)^4\,4^{20}}}$$