Bilkent University	**Fall 2015-2016**

Math255 Probability and Statistics
Midterm 2 Solutions
5 Dec 2015, 1:30 - 3:30 pm

120 minutes. Three problems. 30 points. Closed book. You may use one two-sided A4-size sheet of notes. Good luck!

P1. (10 points) The two parts of this problem are independent.

(a) (5 pts) An urn has n white and m black balls that are removed one at a time in a randomly chosen order. Let X be the number of instances in which a white ball is immediately followed by a black one. For example, if $n = 2$, $m = 3$ and the balls are drawn in order $WBBWB$, then $X = 2$; if the order of drawing is $BBBWW$, then $X = 0$. Find the expectation of X as a function of m and n. Simplify your answer as much as possible. Explain your work in detail to receive full credit.

Solution. Let $X_i = 1$ if the ith ball is W and $(i + 1)$th ball is B; let $X_i = 0$ otherwise. We make this definition for $i = 1, 2, \ldots, n + m - 1$. Then, $X = X_1 + \ldots + X_{n+m-1}$ and $\mathbf{E}[X] = \mathbf{E}[X_1] + \mathbf{E}[X_2] + \cdots + \mathbf{E}[X_{n+m-1}]$. This is the main point: Since we are only interested in the expectation of X, we try to write X as the sum of sum simple random variables. To compute the expectations, let E_i denote the event that the ith ball is W; let E_i^c denote the complement of E_i. Now, we have

$$P(X_i = 1) = P(E_i \cap E_{i+1}^c) = P(E_i)P(E_{i+1}^c|E_i) = \frac{n}{m+n}\frac{m}{n+m-1}.$$

Thus,

$$\boxed{\mathbf{E}[X] = (n + m - 1)\mathbf{E}[X_1] = \frac{nm}{n+m}.}$$

(b) (2+3 pts) Let (X, Y, Z) be jointly distributed with

$$f_{X,Y,Z}(x, y, z) = \begin{cases} 6/(1 + x + y + z)^4, & x > 0, y > 0, z > 0; \\ 0, & \text{otherwise.} \end{cases}$$

Let $V = X+Y+Z$. Determine the PDFs $f_X(x)$ and $f_V(v)$. (Simplify as much as possible. Do not leave the results as integrals.)

Solution We have the general equation

$$f_X(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y,Z}(x, y, z)\,\mathrm{d}y\,\mathrm{d}z.$$

For the specific density here, we have, for $x > 0$,

$$\begin{aligned} f_X(x) &= \int_0^{\infty}\int_0^{\infty} \frac{6}{(1 + x + y + z)^4}\,\mathrm{d}y\,\mathrm{d}z = \int_0^{\infty} \frac{6/(-3)}{(1 + x + y + z)^3}\bigg|_0^{\infty}\mathrm{d}z \\ &= \int_0^{\infty} \frac{2}{(1 + x + z)^3}\,\mathrm{d}z = \frac{2/(-2)}{(1 + x + z)^2}\bigg|_0^{\infty} = \frac{1}{(1 + x)^2} \end{aligned}$$

$$\boxed{f_X(x) = \begin{cases} 1/(1+x)^2, & x > 0; \\ 0, & \text{o.w..} \end{cases}}$$

To compute $f_V(v)$, we note that

$$F_V(v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{v-z-y} f_{X,Y,Z}(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

Differentiating this, we obtain by elementary rules of calculus

$$f_V(v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y,Z}(v-y-z,y,z)\,\mathrm{d}y\,\mathrm{d}z.$$

Using the specific density here, we have

$$\begin{aligned} f_V(v) &= \int_0^v \int_0^{v-z} \frac{6}{(1+v)^4}\,\mathrm{d}y\,\mathrm{d}z = \frac{6}{(1+v)^4}\int_0^v \int_0^{v-z} \mathrm{d}y\,\mathrm{d}z \\ &= \frac{6}{(1+v)^4}\int_0^v (v-z)\,\mathrm{d}z = \frac{6}{(1+v)^4}\left.\frac{(v-z)^2}{-2}\right|_0^v = \frac{3v^2}{(1+v)^4} \end{aligned}$$

$$\boxed{f_V(v) = \begin{cases} 3v^2/(1+v)^4, & v > 0; \\ 0, & \text{o.w.} \end{cases}}$$

P2. (10 points) In a binary communication system, the transmitted signal is modeled as a random variable X that takes the values $\pm A$ with probability $1/2$ each, and the received signal is $Y = X + Z$ where Z is additive Gaussian noise, $Z \sim N(0, \sigma^2)$, independent of X. Assume that $A > 0$ in solving this problem.

(a) (3 pts) Compute the PDF $f_Y(y)$ of the received signal. Show your reasoning in detail to receive full credit. (Your answer must be an explicit function of y.)

We have $Y = X + Z$ with X and Z independent. Thus, $f_Y = f_X * f_Z$ where $*$ denotes convolution. The PDF of X can be written in terms of impulses as $f_X(x) = (1/2)\delta(x - A) + (1/2)\delta(x + A)$. So, we obtain

$$f_Y(y) = \frac{1}{2}f_Z(y-A) + \frac{1}{2}f_Z(y+A).$$

For the specific Gaussian Z here, we have

$$f_Y(y) = \frac{1}{2}\left[\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(z+A)^2/2\sigma^2} + \frac{1}{\sqrt{2\pi\sigma^2}}e^{-(z-A)^2/2\sigma^2}\right]$$

Alternatively, we may use the *method of events* and avoid using impulsive densities.

$$\begin{aligned}
F_Y(y) &= P(Y \le y) = P(X=-A)P(Y \le y|X=-A) + P(X=A)P(Y \le y|X=A) \\
&= \frac{1}{2}P(X+Z \le y|X=-A) + \frac{1}{2}P(X+Z \le y|X=A) \\
&= \frac{1}{2}P(Z \le y+A) + \frac{1}{2}P(Z \le y-A) \\
&= \frac{1}{2}F_Z(y+A) + \frac{1}{2}F_Z(y-A)
\end{aligned}$$

Differentiating this, we obtain

$$\begin{aligned}
f_Y(y) &= \frac{\mathrm{d}}{\mathrm{d}y}F_Y(y) \\
&= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}y}F_Z(y+A) + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}y}F_Z(y-A) \\
&= \frac{1}{2}f_Z(y+A) + \frac{1}{2}f_Z(y-A),
\end{aligned}$$

from which we obtain the same result.

$$\boxed{f_Y(y) = \frac{1}{2}\left[\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(y+A)^2/2\sigma^2} + \frac{1}{\sqrt{2\pi\sigma^2}}e^{-(y-A)^2/2\sigma^2}\right]}$$

(b) (3 pts) Compute the conditional probability $p_{X|Y}(-A|y)$ for all possible values of y. Show your work in detail and simplify your answer as much as possible for full credit.

Solution. We use the mixed form of the Bayes' rule to write

$$\begin{aligned}
p_{X|Y}(-A|y) &= \frac{f_{Y|X}(y|-A)p_X(-A)}{f_Y(y)} = \frac{f_{Y|X}(y|-A)p_X(-A)}{f_{Y|X}(y|-A)p_X(-A) + f_{Y|X}(y|A)p_X(A)} \\
&= \frac{\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(y+A)^2/2\sigma^2}\frac{1}{2}}{\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(y+A)^2/2\sigma^2}\frac{1}{2} + \frac{1}{\sqrt{2\pi\sigma^2}}e^{-(y-A)^2/2\sigma^2}\frac{1}{2}} = \frac{e^{-(y+A)^2/2\sigma^2}}{e^{-(y+A)^2/2\sigma^2} + e^{-(y-A)^2/2\sigma^2}} \\
&= \frac{1}{1+e^{2yA/\sigma^2}}.
\end{aligned}$$

$$\boxed{p_{X|Y}(-A|y) = \tfrac{1}{1+e^{2yA/\sigma^2}}}$$

(c) (4 pt) Compute the probability $P_e \triangleq P(Y > 0|X=-A)$ in terms of the CDF $\Phi(u)$ of the unit normal distribution $N(0,1)$. Show your work in detail and simplify the final result as much as possible for full credit.

Solution. We have

$$\begin{aligned}
P_e &= P(Y > 0|X=-A) = P(X+Z > 0|X=-A) \\
&= P(-A+Z > 0|X=-A) = P(Z > A|X=-A) = P(Z > 0) \\
&= P\left(\frac{Z}{\sigma} > \frac{A}{\sigma}\right) = 1 - \Phi\left(\frac{A}{\sigma}\right),
\end{aligned}$$

where we made use of the fact that X and Z are independent to drop the conditioning on $\{X = -A\}$, and also used the fact that $Z/\sigma \sim N(0,1)$.

$$\boxed{P_e = 1 - \Phi(A/\sigma)}$$

P3. (10 points) Let (X, Y) be jointly distributed random variables with

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{4}\lambda(y)e^{-\lambda(y)x}, & 0 \le y \le 4, x \ge 0; \\ 0, & \text{otherwise,} \end{cases}$$

where $\lambda(y) = 5 - y$. Thus, given $Y = y$, X is exponentially distributed with parameter $\lambda(y)$.

(a) (3 pts) Let $Z = \mathbf{E}[X|Y]$. Determine the PDF of Z.

Solution. We have

$$f_{X|Y}(x|y) = \begin{cases} \lambda(y)e^{-\lambda(y)x}, & x > 0; \\ 0, & \text{o.w.} \end{cases}$$

Recall that if X is an exponential RV with parameter λ, then $E[X] = 1/\lambda$ and $\text{var}(X) = 1/\lambda^2$. So, here we have

$$E[X|Y = y] = 1/\lambda(y) = 1/(5 - y),$$

which means that

$$Z \triangleq \mathbf{E}[X|Y] = \frac{1}{5 - Y}.$$

To determine the PDF of Z, we proceed in the usual manner.

$$\begin{aligned} F_Z(z)P(Z \le z) &= P(1/(5 - Y) \le z) \\ &= P(1 \le (5 - Y)z) = P(Y \le 5 - 1/z) = F_Y(5 - 1/z). \end{aligned}$$

This gives

$$f_Z(z) = \frac{\mathrm{d}}{\mathrm{d}z}F_Y(5 - 1/z) = f_Y(5 - 1/z)\,\frac{\mathrm{d}}{\mathrm{d}z}(5 - 1/z) = \frac{1}{z^2}f_Y(5 - 1/z).$$

Since Y takes values over $[0, 4]$, the range of $Z = 1/(5 - Y)$ is $[1/5, 1]$.

$$\boxed{f_Z(z) = \begin{cases} 1/(4z^2), & \frac{1}{5} \le z \le 1; \\ 0, & \text{o.w.} \end{cases}}$$

(b) (3 pts) Compute $\mathbf{E}[X]$ using the law of iterated expectation: $\mathbf{E}[X] = \mathbf{E}[\mathbf{E}[X|Y]]$. (A numerical result is required.)

Solution.

$$\mathbf{E}[X] = \mathbf{E}[\mathbf{E}[X|Y]] = \mathbf{E}[Z] = \int_{1/5}^{1} z\frac{1}{4z^2}\mathrm{d}z = \int_{1/5}^{1} \frac{1}{4z}\mathrm{d}z = \frac{1}{4}\ln(z)\Big|_{1/5}^{1} = \frac{\ln 5}{4}.$$

$\boxed{\mathbf{E}[X] = (\ln 5)/4}$

(c) (2+2 pts) Compute $\mathrm{var}(X)$ using the law of total variance: $\mathrm{var}(X) = \mathbf{E}[\mathrm{var}(X|Y)] + \mathrm{var}(\mathbf{E}[X|Y])$. (Numerical results required.)

Solution. By definition, $\mathrm{var}(X|Y = y)$ equals the variance of X conditional on $Y = y$. For the case here, $\mathrm{var}(X|Y = y) = 1/(\lambda(y))^2$. Thus, $\mathrm{var}(X|Y) = 1/(\lambda(Y))^2 = 1/(5-Y)^2$. Hence,

$$\mathbf{E}[\mathrm{var}(X|Y)] = \mathbf{E}[1/(5-Y)^2] = \int_0^4 \frac{1}{4}\frac{1}{(5-y)^2}\mathrm{d}y = \frac{1}{4}\frac{1}{(5-y)}\Big|_0^4 = \frac{1}{4}\left[\frac{1}{1} - \frac{1}{5}\right] = \frac{1}{5}.$$

$\boxed{\mathbf{E}[\mathrm{var}(X|Y)] = 1/5}$

As for $\mathrm{var}(\mathbf{E}[X|Y])$, we have

$$\mathrm{var}(\mathbf{E}[X|Y]) = \mathrm{var}(Z) = \mathbf{E}[Z^2] - (\mathbf{E}[Z])^2.$$

We have already computed that $\mathbf{E}[Z] = (\ln(5)/4)$. As for $\mathbf{E}[Z^2]$, we have

$$\mathbf{E}[Z^2] = \int_{1/5}^{1} z^2\frac{1}{4z^2}\mathrm{d}z = \frac{1}{4}(1 - \frac{1}{5}) = \frac{1}{5}.$$

$\boxed{\mathrm{var}(\mathbf{E}[X|Y]) = (1/5) - ((\ln 5)/4)^2}$