MATH 255 - HW 3

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$\underbrace{22102718}$ $\quad M = \dfrac{(0+2+7) \bmod 6}{3} + 3 = 6$

① a) Area of the $\underbrace{\text{shaded region}}_{\text{call it } A}$: $\dfrac{10M(5M+15M)}{2} = 100M^2$ (area of trapezoid)

$\Rightarrow$ Area$(A) = 3600$. Since joint pmf is uniformly dist., the pmf is constant over the area and we need, for a joint pmf:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1 \Rightarrow \int_A \underbrace{f_{X,Y}(x,y)}_{\text{constant (in A), call it C}}\,dx\,dy = C.(100M^2) = 1$$

$$\Rightarrow C = \frac{1}{100M^2} = \frac{1}{3600}.$$

$$\Rightarrow \boxed{f_{X,Y}(x,y) = \begin{cases} \frac{1}{3600}, & \text{if } (x,y) \in A \\ 0, & \text{else} \end{cases}}$$

(call the un. dist. area : A)

Note that the upper edge of A is part of line $\boxed{y = 90 - x}$ $(15M - x)$ can be seen from line eqn.

Now, using marginal pdf definitions:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \int_0^{90-x} \frac{1}{3600}\,dy = \frac{90-x}{3600} = -\frac{x}{3600} + \frac{1}{40}, \quad 0 \le x \le 60$$

④ in A, y changes from $y=0$ to the line $y = 15M - x = 90 - x$

$$\Rightarrow \boxed{f_X(x) = \begin{cases} \frac{-x}{3600} + \frac{1}{40}, & 0 \le x \le 60 \\ 0, & \text{else} \end{cases}}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \begin{cases} \int_0^{60} \frac{1}{3600}\,dx, & \text{when } 0 \le y \le \overset{5M}{30} \text{ since x changes from 0 to } 10M = 60 \\ \int_0^{90-y} \frac{1}{3600}\,dx, & \text{when } 30 \le y \le 90 \text{ since x changes from 0 to line } x = 90-y \\ 0, & \text{o.w} \end{cases}$$

$$\Rightarrow \boxed{f_Y(y) = \begin{cases} \frac{1}{60}, & 0 \le y \le 30 \\ \frac{-y}{3600} + \frac{1}{40}, & 30 < y \le 90 \\ 0, & \text{else} \end{cases}}$$

b) We know that $f_{X|Y=y}(x) = \dfrac{f_{X,Y}(x,y)}{f_Y(y)}$ and $f_{Y|X=x}(y) = \dfrac{f_{X,Y}(x,y)}{f_X(x)}$,

When $y=45$, $f_Y(45) = \left.\dfrac{90-y}{3600}\right|_{y=45} = \dfrac{1}{80}$, and $f_{X,Y}(x,y) = \dfrac{1}{3600}$ for $0 \le x \le 45$, since $(x,45) \in A$ when x is between zero and the line $x = 90-y\Big|_{y=45} = 45$.

$$\Rightarrow \boxed{f_{X|Y=45}(x) = \frac{f_{X,Y}(x,y)}{f_Y(45)} = \begin{cases} \dfrac{\frac{1}{3600}}{\frac{1}{80}} = \dfrac{1}{45}, & 0 \le x \le 45 \\ 0, & \text{else.} \end{cases}}$$

When $x=45$, $f_X(45) = \left.\dfrac{90-x}{3600}\right|_{y=45} = \dfrac{1}{80}$ and $f_{X,Y}(x,y) = \dfrac{1}{3600}$ for $0 \le y \le 45$, since $(45,y) \in A$ if y is between zero and the line $y = 90-x\Big|_{x=45} = 45$

$$\Rightarrow \boxed{f_{Y|X=45}(y) = \frac{f_{X,Y}(x,y)}{f_X(45)} = \begin{cases} \dfrac{\frac{1}{3600}}{\frac{1}{80}} = \dfrac{1}{45}, & 0 \le y \le 45 \\ 0, & \text{else} \end{cases}}$$

c) $$E\{X\} = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_0^{60} x\left(\frac{1}{40} - \frac{x}{3600}\right)dx = \frac{1}{40}\int_0^{60} x\,dx - \frac{1}{3600}\int_0^{60} x^2\,dx$$

Non-zero for $0 \le x \le 60$ as found in part a

$$= \frac{1}{40}\left[\frac{x^2}{2}\right]_0^{60} - \frac{1}{3600}\left[\frac{x^3}{3}\right]_0^{60}$$

$$= 45 - 20 = \underline{\underline{25}}$$

$$E\{Y\} = \int_{-\infty}^{\infty} y f_Y(y)\,dy = \int_0^{30} y \cdot \frac{1}{60}\,dy + \int_{30}^{90} y\left(\frac{1}{40} - \frac{y}{3600}\right)dy = \frac{1}{60}\left[\frac{y^2}{2}\right]_0^{30} + \frac{1}{40}\left[\frac{y^2}{2}\right]_{30}^{90} - \frac{1}{3600}\left[\frac{y^3}{3}\right]_{30}^{90}$$

as found in pt. a

$$= 7.5 + 90 - 65 = \underline{\underline{32.5}}$$

$$E\{X \mid Y=45\} = \int_{-\infty}^{\infty} x\, f_{X|Y=45}(x)\,dx = \int_0^{45} x \cdot \frac{1}{45}\,dx = \frac{1}{45}\left[\frac{x^2}{2}\right]_0^{45} = \underline{\underline{22.5}}$$

found in pt. b

d) i) $$P\{X<40\} = F_X(40) = \int_{-\infty}^{40} f_X(x)\,dx = \int_0^{40}\left(\frac{-x}{3600}+\frac{1}{40}\right)dx = 1 - \frac{1}{3600}\left[\frac{x^2}{2}\right]_0^{40}$$

pt. a

$$= 1 - \frac{2}{9} = \boxed{\frac{7}{9}}$$

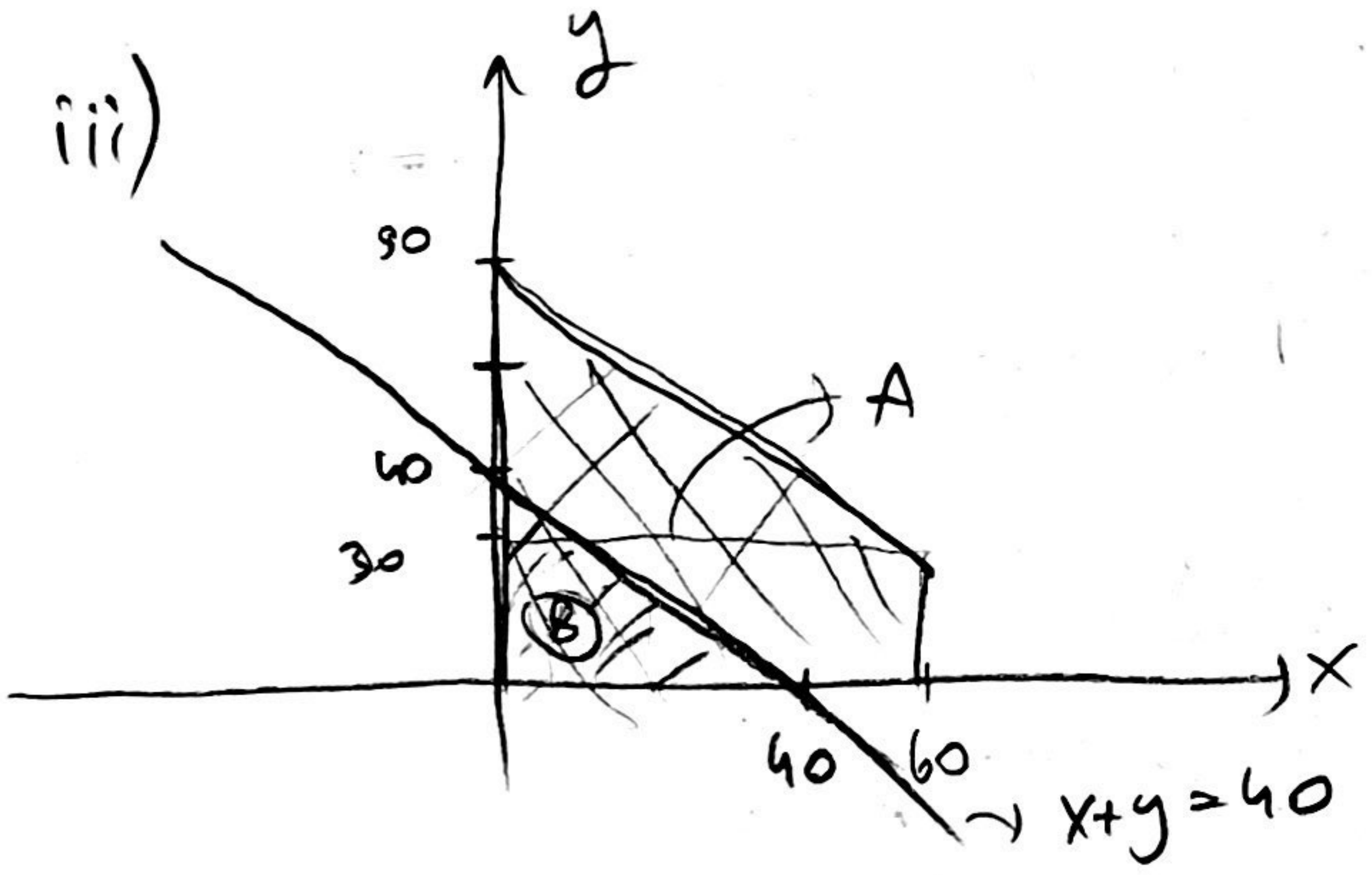
ii) $$P\{Y \geq 40\} = \int_{40}^{\infty} f_Y(y)\,dy = \int_{40}^{90}\left(\frac{-y}{3600}+\frac{1}{40}\right)dy = \frac{5}{4} - \underbrace{\frac{1}{3600}\left[\frac{y^2}{2}\right]_{40}^{90}}_{} = \frac{5}{4} - \frac{65}{72}$$

pt. a

(18)

$$= \boxed{\frac{25}{72}}$$

iii)



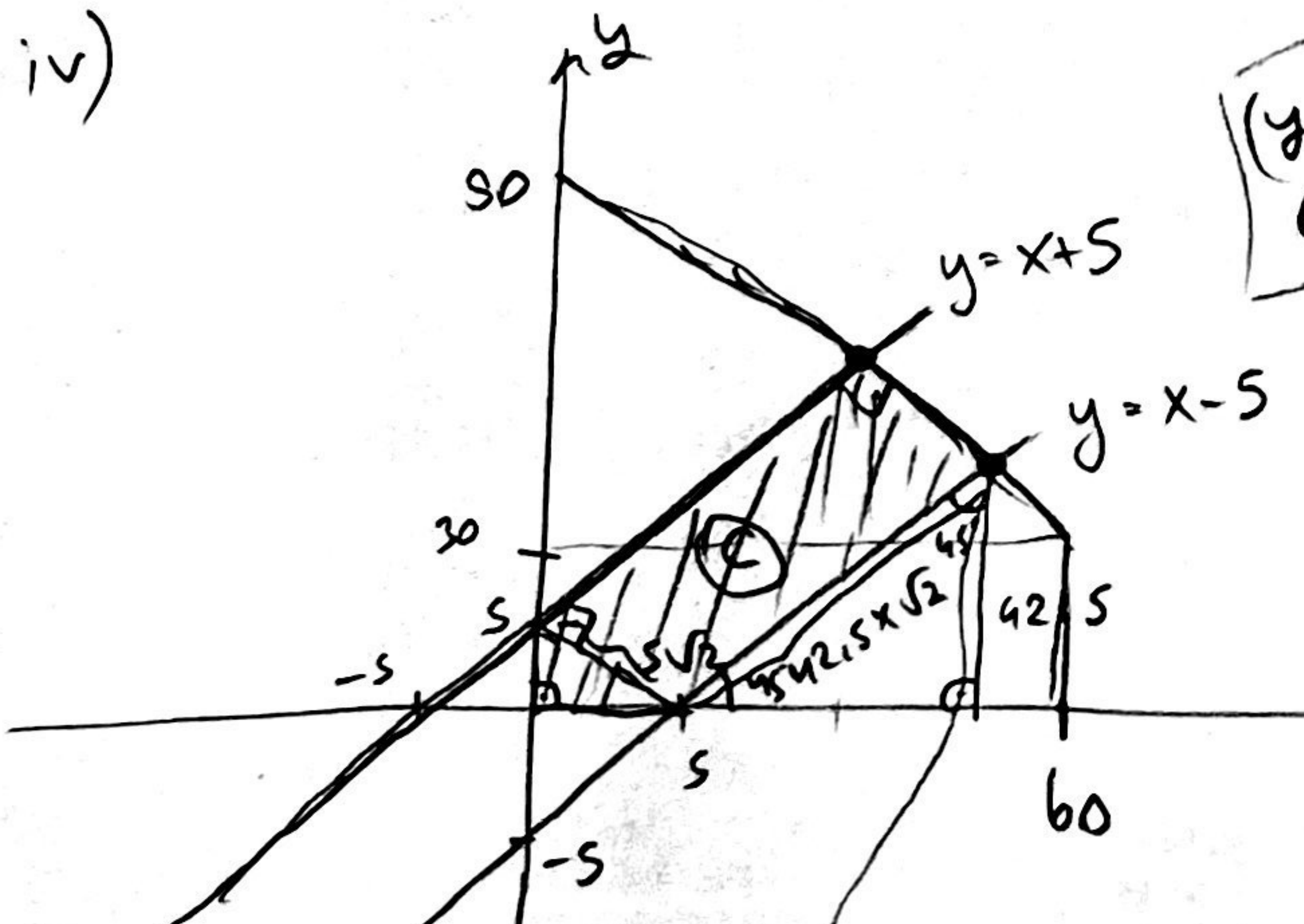
For $x+y<40$, we need the points lying under the line $x+y=40$. However, if $(x,y) \notin A$ (unif. distr. area), $f_{X,Y}(x,y)=0$ and if $(x,y)\in A$, $f_{X,Y}(x,y)=\frac{1}{3600}$. So, we can directly find prob. that $(x,y)\in B$, where $B = A \cap x+y<40$.

$$\Rightarrow P\{X+Y<40\} = \iint_B f_{X,Y}(x,y)\,dx\,dy = \int_0^{40}\int_0^{40-y} \frac{1}{3600}\,dx\,dy = \int_0^{40} \frac{40-y}{3600}\,dy = \underline{\underline{\frac{2}{9}}}$$

$f_{X,Y}(x,y)=0$ everywhere else.

x is between 0 and $x = 40-y$, y is bet. 0 and 40

iv)



Intersection point: $y=90-x \wedge y=x-5$

$x=47.5,\ y=42.5$

We need the points lying between $x-y=5$ and $x-y=-5$

$y=x-5$ $y=x+5$

$\boxed{(y\geq x-5) \wedge (y\leq x+5)}$

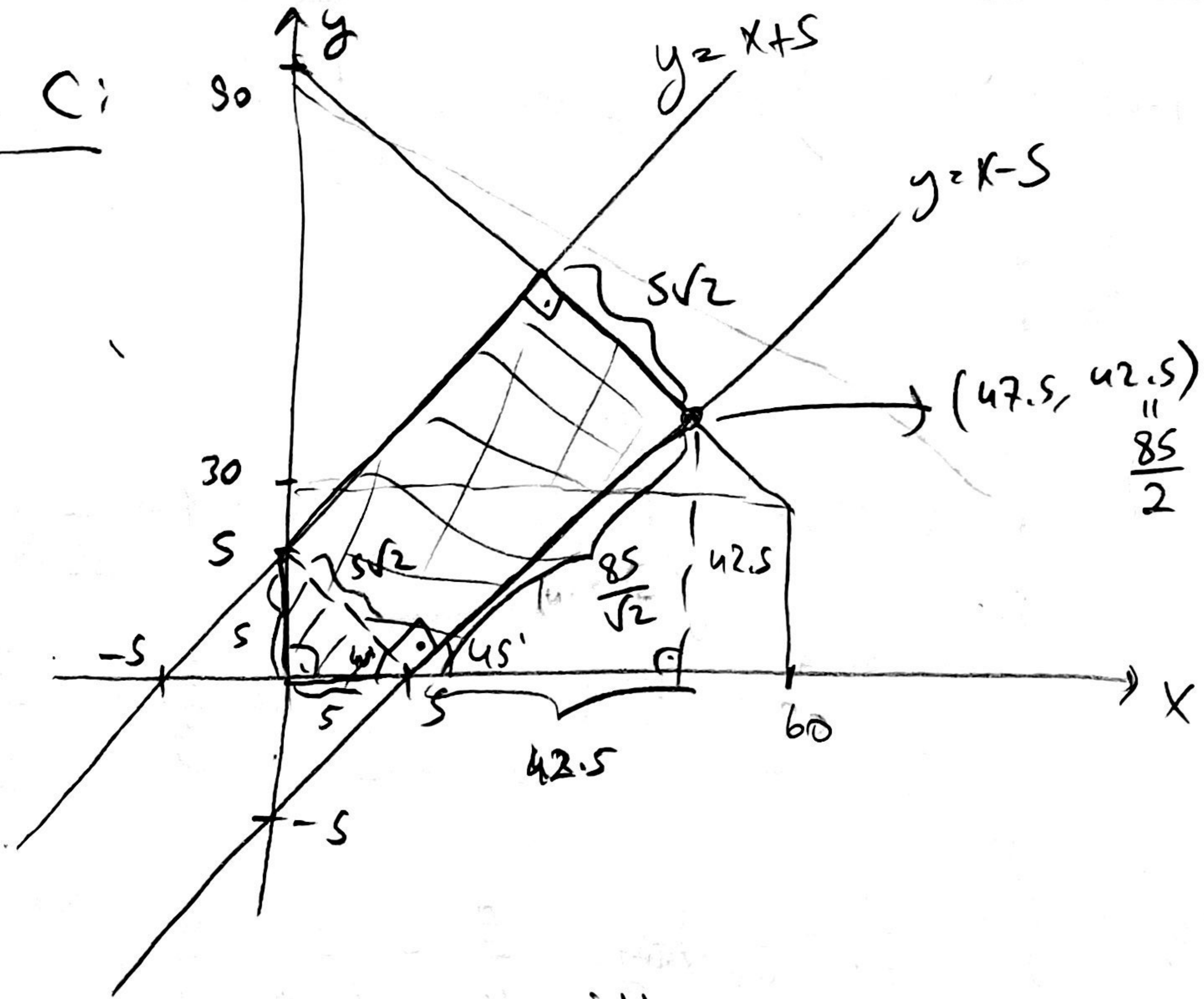
since $|x-y|\leq 5 \equiv \boxed{-5\leq x-y\leq 5}$

Due to having constant j.p.df only in A, (0 else), we can only integrate over C where $C = A \cap |x-y|\leq 5$.

$$\Rightarrow P\{|X+Y|\leq 5\} = \iint_C f_{X,Y}(x,y)\,dx\,dy = \iint_C \frac{1}{3600}\,dx\,dy$$

$$= \frac{1}{3600}\iint_C dx\,dy \rightarrow \underline{\text{Area of } C.}$$

Find area of C:



C is a combination of a right triangle and rectangle.

Area of triangle $= \frac{5.5}{2} = \frac{25}{2}$, Area of rectangle $= (5\sqrt{2})\left(\frac{85}{\sqrt{2}}\right) = 425$

$\Rightarrow$ Area of C $= 425 + 12.5 = 437.5$

$\Rightarrow P\{|X+Y| \leq 5\} = \frac{1}{3600} \underbrace{\iint_C dx\,dy}_{\text{Area of } C} = \frac{437.5}{3600} = 0.1215$

e) For independent r.v, we need $f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$ for any x, y. However, as $f_X(x)$, $f_Y(y)$ were found previously in part a, their multiplication cannot be equal to $f_{X,Y}(x,y)$, which is constant if $(x,y) \in A$, and zero else. Check:

$$f_X(x) \cdot f_Y(y) = \begin{cases} \frac{1}{60}\left(\frac{90-x}{3600}\right), & 0 \leq x \leq 60,\ 0 \leq y \leq 30 \\ \left(\frac{90-x}{3600}\right)\left(\frac{90-y}{3600}\right), & 0 \leq x \leq 60,\ 30 < y \leq 90 \\ 0 & \text{else} \end{cases}$$

but

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{3600}, & (x,y) \in A \\ 0, & \text{else} \end{cases}$$

$\Rightarrow f_{X,Y}(x,y) \neq f_X(x) \cdot f_Y(y)$

$\Rightarrow$ X and Y are not independent.

OR: for $x=0$, $y=0$ (pick a random (x,y))

$f_X(0) = \frac{1}{40}$

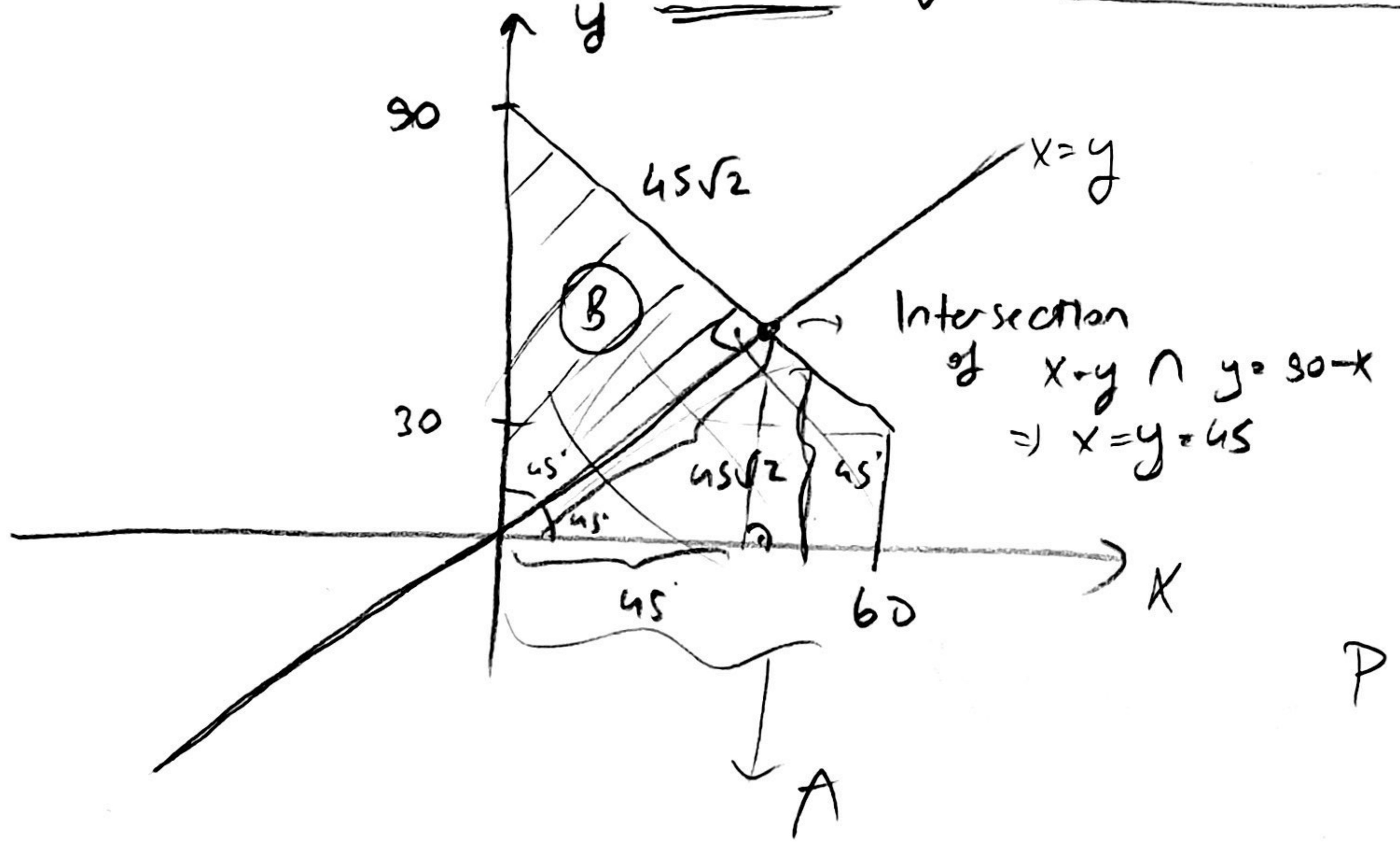
$f_Y(0) = \frac{1}{60}$

$\Rightarrow$ from part A, $f_{X,Y}(0,0) = \frac{1}{3600}$

$\Rightarrow f_X(0) \cdot f_Y(0) = \frac{1}{2400} \neq f_{X,Y}(0,0) = \frac{1}{3600}$

f) X: arrival time of A, Y: arrival time of B.

i) We need $Y > X$ if A arrives earlier than B, (arr. time X should be less than Y)



We need the upper points of line $x=y$, where $y>x$. Since $f_{X,Y}(x,y) \neq 0$ and constant for $(x,y) \in A$, integrate over B where

$B = A \cap y > x$.

$$P\{X<Y\} = \iint_B \underbrace{f_{X,Y}(x,y)}_{\frac{1}{3600} \text{ in } B \text{ (since } B \subset A)} \, dx\,dy$$

$$= \frac{1}{3600} \underbrace{\iint_B dx\,dy}_{\text{Area of } B} = \frac{2025}{3600} = \boxed{\frac{9}{16}}$$

$\underbrace{\text{Area of } B}_{\text{right triangle}} = \frac{45\sqrt{2} \cdot 45\sqrt{2}}{2} = 2025$

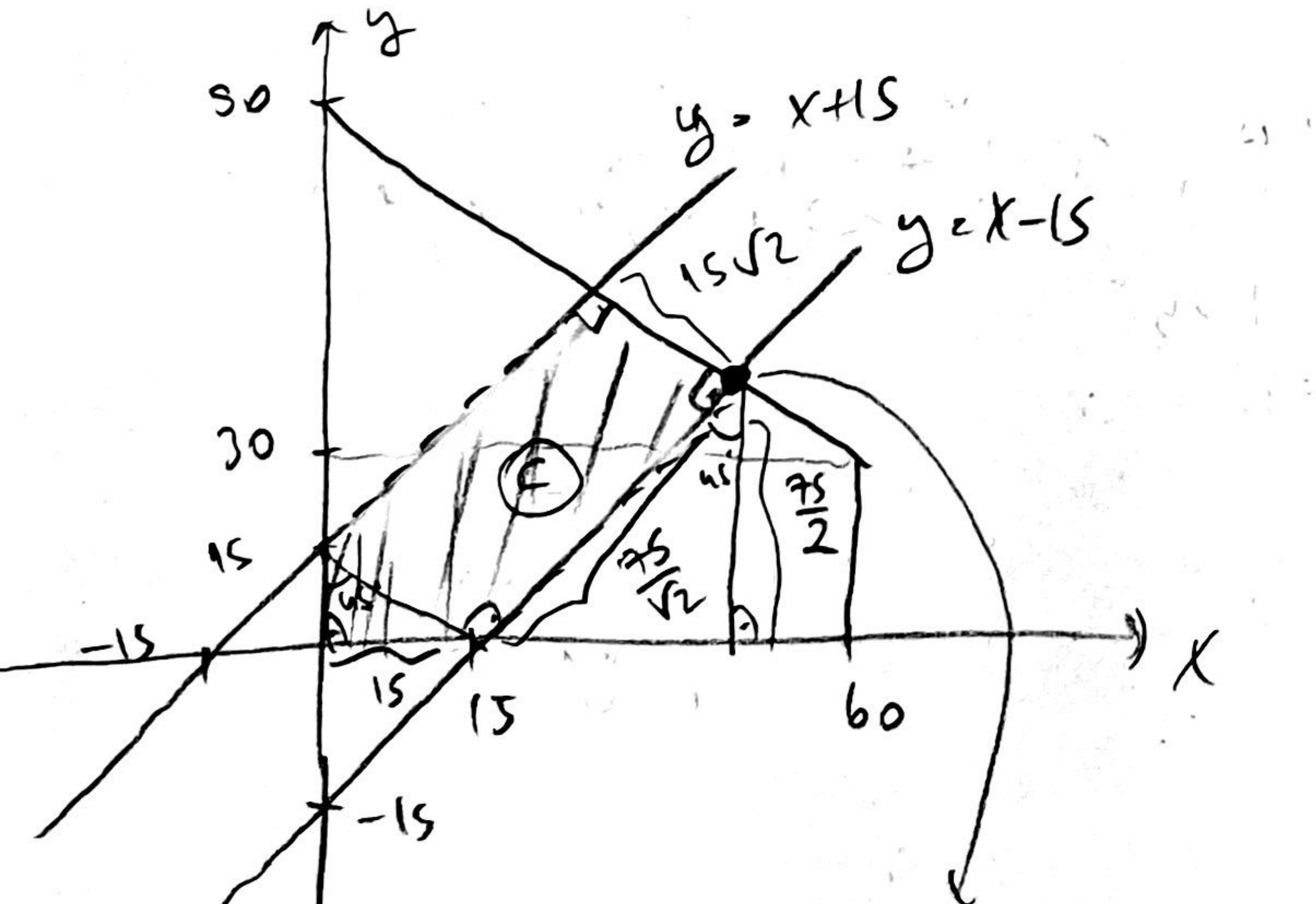
$\Rightarrow 2025 = \text{Area of } B$

ii) We are asked the difference without indication of whether $X \geq Y$ or $Y \geq X$. $\Rightarrow$ Find $P\{|X-Y| < 15\}$. $\rightarrow$ area between lines $\boxed{y = x+15}$ and $\boxed{y = x-15}$

since $|x-y| < 15$

$\Rightarrow -15 < x-y < 15$

$\Rightarrow y > x-15 \cap y < x+15$



intersection point

$y = x-15 \cap y = 90-x$

$\Rightarrow x = \frac{105}{2}$, $\boxed{y = \frac{75}{2}}$

Again, since $f_{X,Y}(x,y)$ is non-zero only in A, we can integrate only on C where $C = A \cap |x-y| < 15$.

$$\Rightarrow P\{|X-Y|<15\} = \iint_C \underbrace{f_{X,Y}(x,y)}_{\text{constant in } C} dx\,dy = \frac{1}{3600} \underbrace{\iint_C dx\,dy}_{\text{Area of } C.}$$

$$\boxed{\Rightarrow P\{|X-Y|<15\} = \frac{1237.5}{3600} = 0.34375}$$

Area of $C = \underbrace{\frac{15 \cdot 15}{2}}_{\text{area of triangle}} + \underbrace{(15\sqrt{2})\left(\frac{75}{\sqrt{2}}\right)}_{\text{area of rectangle}} = 112.5 + 1125$

$= \underline{1237.5}$