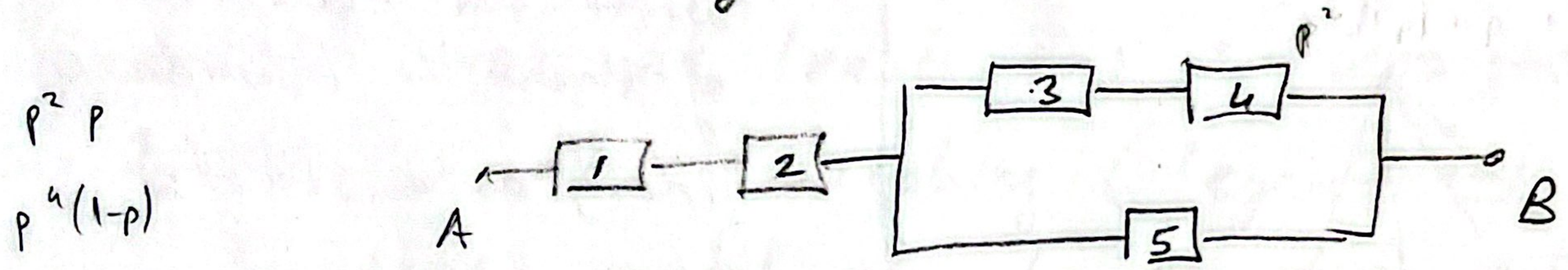
MATH 255 MIDTERM EXAM - 10 December 2023

150 min. Clearly justify all answers

1-) A network of identical switches is given below:




This network is inspected at discrete time instants $i = \ldots, 0, 1, 2, \ldots$. Let $X_{n,i}$ represent a random variable where $n$ is the label of the switch: $n = 1, 2, \ldots, 5$.

$$X_{n,i} = \begin{cases} 1 & \text{if switch } n \text{ is ON at instant } i. \\ 0 & \text{if the switch } n \text{ is OFF at instant } i. \end{cases}$$

$X_{n,i}$'s are Independent; $X_{n,i} = \begin{cases} 1 & \text{with } p \\ 0 & \text{" } 1-p \end{cases}$

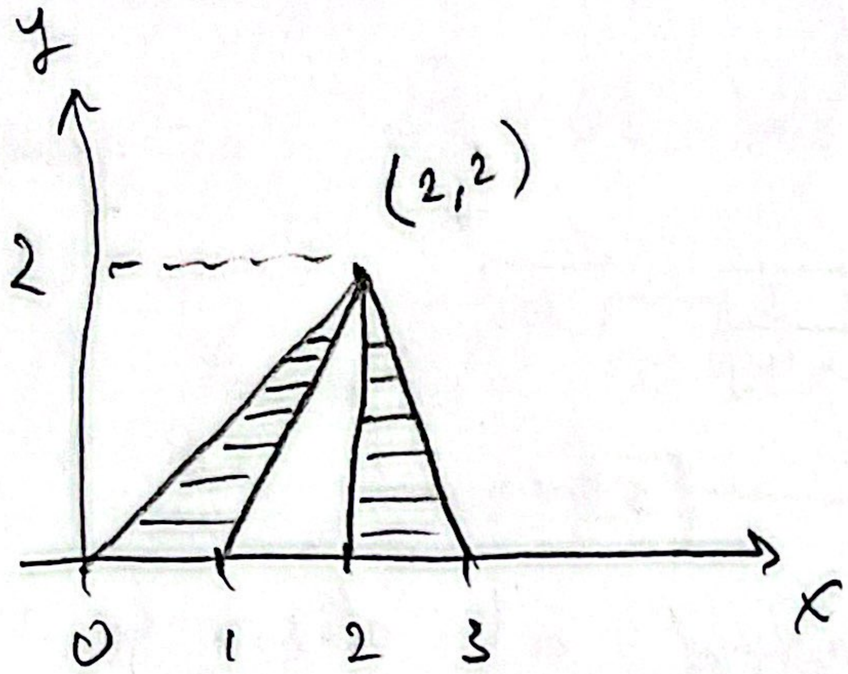
8pts a) Find the probability of a connection between A and B at a given instant i. in terms of p.

3pts b) Call the result of "(a) as q. Find the minimum value for q so that the connection between A and B is maintained over a given period of 10 instants with a probability greater than 0.9.

4pts c) $Y_i = \begin{cases} 1 & \text{if A and B is connected at instant } i \\ 0 & \text{" " " " is NOT " " " " " } \end{cases}$

Write $Y_i$ in terms of $X_{n,i}$'s.

2-) $X$ and $Y$ are jointly uniform distributed over the shaded area shown below:




(Hint for part (a): Write the equations of those 4 boundary lines, as a start.)

7pts a) Find $f_{X|Y=y}(x)$ for each $y \in [0,2]$. Plot $f_{X|Y=1}(x)$.

5pts b) Find minimum mean square error estimate $\hat{X}$ given $Y=y$, for each $y \in [0,2]$. Plot $\hat{X}$ as a function of $y$. (Hint: Symmetries may help)

5pts c) Find the pdf of $Z_1 = |X - \hat{X}|$ given $Y=y$ and $X > \hat{X}$.

5pts d) Find the pdf of $Z_2 = |X - \hat{X}|$ given $Y=y$ and $X < \hat{X}$.

8pts e) Find the pdf of $Z = |X - \hat{X}|$ given $Y=y_0$. (Hint: use your answers for parts (c) and (d).

3-) A Poisson process has a rate of $\lambda = 0.5$ per hour. We watch this process for 3 hours. At the end of 3 hours, we quit if there was at least one success; otherwise, we continue watching (beyond the first 3 hours) until at least one success.

5 pts a) What is the probability that our watching duration is more than 3 hours?

8 pts b) What is the probability that the watching time is between 3 and 5 hours?

7 pts c) What is the probability that there were exactly two successes are observed.

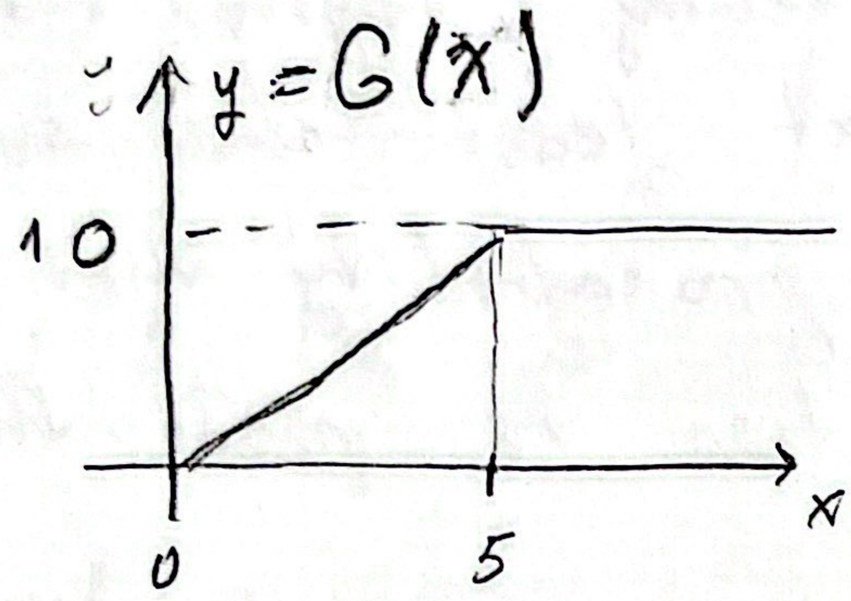
5 pts d) Find the expected number of successes.

5 pts e) Find the expected total watting time, given that we have already watted for 6 hours.

4-a) The pmf of a discrete random variable $\underline{X}$ is

$$P_{\underline{X}}(k) = (1-p)^{k-1} p \qquad k = 1, -- \infty$$

$Y = G(\underline{X})$ where $G(x)$ is given as:




Find the pdf of Y.
pmf

10 pts b) $\underline{X}$ is a continuous random variable whose pdf is:

$$f_{\underline{X}}(x) = \begin{cases} a\,\delta(x-1) + \frac{1}{2}e^{-x} & ; \text{if } x > 0 \\ 0 & \text{else} \end{cases}$$

3 pts i) Find a

12 pts ii) $G(x) = e^{-x}$, $Y = G(\underline{X})$

find the pdf of Y.