Bilkent University **Spring 2017-2018**

Math255 Probability and Statistics

Midterm 1 Solutions

5 March 2018

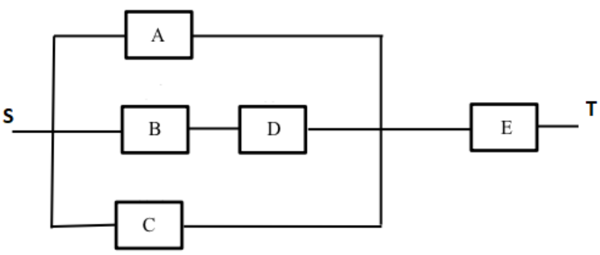
Problem 1. Consider the set all distinct 10-letter words that can be created by rearranging the letters of the word KAUNAKAKAI. Suppose a word is chosen at random from this set, what is the probability (call it p) that no K in the word is adjacent to another K?

Solution. To turn this probability problem into a counting problem we have to set up a sample space in which each outcome is equally likely. To that end, we need to make the identical letters in the given word distinguishable. This can be done by writing the given word as KAUNA'K'A"K"A"'I where we now have 10 distinguishable letters K,K',K",K"',A,A',A",N,U,I. Let Ω be the set of all permutations of K,K',K",K"',A,A',A",N,U,I. We will regard Ω as a sample space with equiprobable elements. Let A be the subset of Ω consisting of those permutations in which no two Ks (that is none of the three lettes K,K',K") are adjacent. A permutation of K,K',K",K"',A,A',A",N,U,I belongs to A if and only if the first two K's in that permutation have a non-K letter as their right-hand side neighbors. So, the size of A is the product of (the number of ways of choosing the two right-hand side neighbors) and (the number of ways the remaining 8 letters can be arranged).

$$p = \frac{|A|}{|\Omega|} = \frac{(7 \times 6) \times (8!)}{10!} = \frac{7 \cdot 6}{10 \cdot 9} = \frac{7}{15}.$$

Problem 2.

An electrical system (shown on the right) consists of 5 components, each of which is operational with probability $p = \frac{1}{2}$. Assume that the states of the components (whether operational or not) are jointly independent. The system is operational (call this event F) if there is a path from S to T such that all components on the path are operational. Find the probability $P(F)$ that the system is operational.



Solution. We may express the event F as

$$F = (A \cup (B \cap D) \cup C) \cap E$$

where A is the event that component A is operational, B is the event that component B is operational, etc. We have

$$\begin{aligned} P(F) &= P((A \cup (BD) \cup C) \cap E) \\ &= P(A \cup (BD \cup C)P(E) = P(A \cup (BD \cup C) \times p \end{aligned}$$

by independence. Using the inclusion-exclusion formula followed by the independence assumption,

$$\begin{aligned}P(A \cup (BD) \cup C) &= P(A) + P(BD) + P(C) - P(ABD) - P(AC) - P(BDC) + P(ABDC)\\ &= P(A) + P(B)P(D) + P(C) - P(A)P(B)P(D) - P(A)P(C) - \\ &\quad P(B)P(C)P(D) + P(A)P(B)P(C)P(D)\\ &= p + p^2 + p - p^3 - p^2 - p^3 + p^4\\ &= 2p - 2p^3 + p^4\end{aligned}$$

So, we get

$$\begin{aligned}P(F) &= (2p - 2p^3 + p^4)p = 2p^2 - 2p^4 + p^5\\ &= 2(1/2)^2 - 2(1/2)^4 + (1/2)^5 = 1/2 - 1/8 + 1/32 = 13/32.\end{aligned}$$

Problem 3. Suppose there are three coins that look identical, except the first coin has heads on both faces, the second has tails on both faces, and the third is a regular coin with heads on one face and tails on the other. Assume that each coin is fair in the sense that when flipped each face is equally likely to come up. The coins are put in a bag, a coin is picked at random without revealing the identity of the coin, the coin is flipped and allowed to land on the ground. Given that the up-face of the coin on the ground is heads (call this event B), what is the conditional probability that the coin on the ground is the third coin (call this event A)?

Solution. We may use the sample space $\Omega = \{(1,H),(2,T),(3,H),(3,T)\}$, where the first component of an outcome denotes the type of the selected coin, the second component denotes the up-face of the coin on the ground. The probality assignment is

$$P(\{(1,H)\}) = 1/3, \quad P(\{(2,T)\}) = 1/3, \quad , P(\{(3,H)\}) = 1/6, \quad P(\{(1,T)\}) = 1/6.$$

The events A and B as subsets of Ω are $A = \{(3,H),(3,T)\}$, $B = \{(1,H),(3,H)\}$. The probability of interest $P(A|B)$ is obtained by a straightforward calculation.

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(\{(3,H)\})}{P(\{(1,H),(3,H)\}} = \frac{1/6}{1/3 + 1/6} = 1/3.$$

Problem 4. A (six-faced) fair die is rolled twice. Let X and Y be the minimum and the maximum of the two rolls, respectively. Compute $\mathbf{E}[X + Y|Y = 3]$ and $\text{var}(X + Y)$.

Solution. Let U and V be the results of the first and second die roll, respectively. The pair (U, V) are independent with

$$p_{U,V}(u,v) = \frac{1}{36}, \quad 1 \le u, v \le 6.$$

In terms of these,

$$X = \min(U,V), \qquad Y = \max(U,V),$$

and the joint PMF of (X,Y) is computed as

$$p_{X,Y}(x,y) = \begin{cases} 0, & 1 \le y < x \le 6;\\ 1/36, & 1 \le x = y \le 6;\\ 2/36, & 1 \le x < y \le 6.\end{cases}$$

We are now ready to solve the problem. Note that $\mathbf{E}[X+Y|Y=3] = \mathbf{E}[X+3|Y=3] = 3+\mathbf{E}[X|Y=3]$. To calculate $E[X|Y=3]$, we need the conditional PMF $p_{X|Y}(x|3)$, which is derived from the joint PMF $p_{X,Y}(x,y)$ by a straightforward calculation.

$$p_{X|Y}(x|3) = \frac{p_{X,Y}(x,3)}{p_Y(3)} = \begin{cases} 2/5, & x=1; \\ 2/5, & x=2; \\ 1/5, & x=3; \\ 0, & x>3. \end{cases}$$

Now, we have $\mathbf{E}[X|Y=3] = 1\cdot(2/5)+2\cdot(2/5)+3\cdot(1/5) = 9/5$. So, $\mathbf{E}[X+Y|Y=3] = 24/5 = 4.8$.

Turning to $\mathrm{var}(X+Y)$, the calculation becomes simpler if you notice that $X+Y = U+V$, because then $\mathrm{var}(X+Y) = \mathrm{var}(U+V) = \mathrm{var}(U)+\mathrm{var}(V) = 2\,\mathrm{var}(U)$, where the last two equalities follow from the fact that U and V are independent and identically distributed. Recalling that the variance of a uniform discrete random variable over a set of n consecutive integers is $(n^2-1)/12$, we have $\mathrm{var}(U) = 35/12$. So, the final answer is $\mathrm{var}(X+Y) = 35/6$.

Problem 5. Consider an experiment that consists of tossing a fair four-faced die until all four faces appear at least once. Let X denote the duration of the experiment and compute $\mathbf{E}[X]$. Show your reasoning. Compute your answer as a real number or a simple fraction of integers. Do not leave any uncomputed sums.

Solution. We may write $X = Y_1+Y_2+Y_3+Y_4$ where Y_i is the interarrival time of a ith new type of face to appear. For example, in a sequence of outcomes such as 2,2,2,4,2,4,4,1,2,4,1,2,2,3 we have $X=14$, $Y_1=1$, $Y_2=3$, $Y_3=4$, and $Y_4=6$. Clearly, we have $Y_1=1$ with probability 1, since the first roll always gives a new type of face. We may think of Y_1 as a geometric random variable with probability of success $p_1=1$. The waiting time Y_2 between the first new face and the second new face is geometric with probability of success $p_2=3/4$. Likewise, Y_3 and Y_4 are geometric random variables with probabilities of success $p_3=2/4$ and $p_4=1/4$, respectively. So, we have

$$\begin{aligned}\mathbf{E}[X] &= \mathbf{E}[Y_1+Y_2+Y_3+Y_4] = \mathbf{E}[Y_1]+\mathbf{E}[Y_2]+\mathbf{E}[Y_3]+\mathbf{E}[Y_4] \\ &= 1/p_1+1/p_2+1/p_3+1/p_4 = 4/4+4/3+4/2+4/1 = 8\,\frac{1}{3}.\end{aligned}$$