# MIDTERM 2 SOLUTIONS

1-) a) Let event $C$ be the connection between A and B:

$C = S_1 \cap S_2 \cap \left( (S_3 \cap S_4) \cup S_5 \right)$ where $S_k$ is the event "switch $s_k$ conducts (ON)".

$$P(C) = P\left(S_1 \cap S_2 \cap \left((S_3 \cap S_4) \cup S_5\right)\right)$$

due to independence of switches:

$$P(C) = \underbrace{P(S_1)\cdot P(S_2)}_{= p^2} \cdot \underbrace{P\left((S_3 \cap S_4) \cup S_5\right)}_{= P(S_3 \cap S_4) + P(S_5) - P(S_3 \cap S_4 \cap S_5)}$$

$$P(S_3 \cap S_4) = P(S_3)P(S_4) = p^2, \quad P(S_5) = p, \quad P(S_3 \cap S_4 \cap S_5) = P(S_3)P(S_4)P(S_5) = p^3$$

So, $P(C) = p^2\left[p + p^2 - p^3\right] = \boxed{p^3 + p^4 - p^5}$

b) Since each instant operation is independent:

P(connection over a period of 10 instants) $= q^{10}$

So: $q^{10} > 0.9$

$$\boxed{q > (0.9)^{\frac{1}{10}}}$$

c) $Y_i = \begin{cases} 1 & \text{if there is a connection} \\ 0 & \text{else} \end{cases}$

$$\boxed{Y_i = X_{1,i} \cdot X_{2,i} \left( X_{3,i} \cdot X_{4,i} + X_{5,i} - X_{3,i} X_{4,i} X_{5,i} \right)}$$

Check: $Y_i = 1$ if $X_{1,i}$, $X_{2,i}$ and $(X_{3,i}X_{4,i} + X_{5,i} - X_{3,i}X_{4,i}X_{5,i})$ are all 1

$(X_{3,i}X_{4,i} + X_{5,i} - X_{3,i}X_{4,i}X_{5,i}) = 1$ if $X_3 X_4$ is on OR $X_{5,i}$ is ON ✓

2-a) The boundary line equations are: 



* $y = x$
* $y = 2x - 2$
* $x = 2$
* $y = -2x + 6$

$\Downarrow$

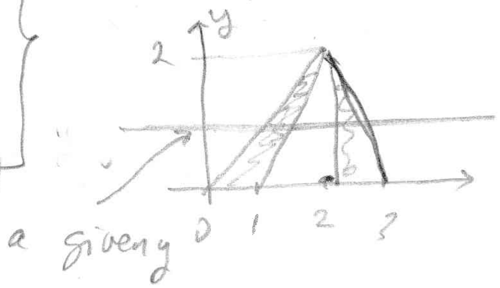
$x = y$

$x = \frac{y+2}{2}$

$x = 2$

$x = \frac{6-y}{2}$

Easy approach:



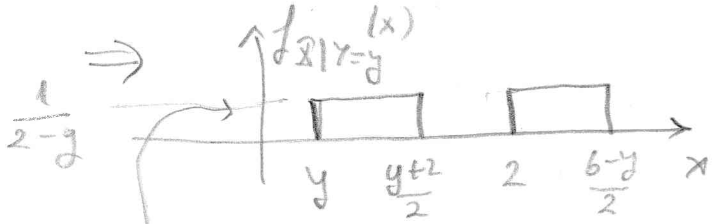

If a $y$ is given, $x$ takes values between the boundaries of shaded region at that $y$, and all those values are equally likely since the joint pdf is equally likely in the shaded region.

The bounderies of those shaded region for the given $y$ are:

$$x \in \left[y, \frac{y+2}{2}\right] \cup \left[2, \frac{6-y}{2}\right] \Rightarrow \text{total length} = \left(\frac{y+2}{2} - y\right) + \left(\frac{6-y}{2} - 2\right)$$

$$= \boxed{2 - y}$$

$\Rightarrow$




Normalize to have total area under the curve to be 1.

So: $$\boxed{f_{X|Y=y}(x) = \begin{cases} \frac{1}{2-y} & \text{if } x \in \left[y, \frac{y+2}{2}\right] \cup \left[2, \frac{6-y}{2}\right] \\ 0 & \text{else} \end{cases}}$$

for $y \in [0, 2)$

Note that division by $2-y$ is not possible if $y = 2$, therefore, above result is valid if $y \neq 2$.

If $y = 2$, from the given joint pdf, $x$ can take only one value: $x = 2$. $\Rightarrow$

$$\boxed{f_{X|Y=2}(x) = \delta(x-2)} \text{ if } y = 2.$$

$\longrightarrow$

Alternative formal solution:

$$f_{X|Y=y}(x) = \frac{f_{X,Y}(x,y)}{f_Y(y)} \quad , \quad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$

First find $f_{X,Y}(x,y)$: Since joint pdf is uniform,

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{\text{area}} & \text{if } (x,y) \in B \\ 0 & \text{else} \end{cases}$$

$\to$ shaded region

area of shaded region $B$.

$$\text{Area of } B = \frac{1 \cdot 2}{2} + \frac{1 \cdot 2}{2} = 2$$

So,

$$f_{X,Y}(x,y) = \begin{cases} 1/2 & \text{if } (x,y) \in B \\ 0 & \text{else} \end{cases}$$

Then find

$$f_Y(y) = \int_{y}^{\frac{y+2}{2}} \frac{1}{2}\,dx + \int_{2}^{\frac{6-y}{2}} \frac{1}{2}\,dx$$

$$= \left(\frac{y+2}{2} - y\right) + \left(\frac{6-y}{2} - 2\right) = (2-y)\cdot\frac{1}{2}$$

for $y \in [0,2)$

$$f_Y(y) = \begin{cases} \frac{2-y}{2} & y \in [0,2) \\ 0 & \text{else} \end{cases}$$

division by 0 is not possible

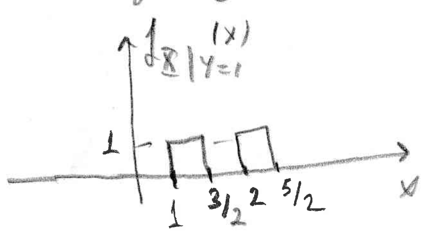
So,

$$f_{X|Y=y}(x) = \begin{cases} \dfrac{1/2}{\frac{2-y}{2}} & \text{if } y \in [0,2) \text{ and } x \in \left[y, \frac{y+2}{2}\right] \cup \left[2, \frac{6-y}{2}\right] \\ 0 & \text{else} \end{cases}$$

$$= \begin{cases} \dfrac{1}{2-y} & \text{if} \\ 0 & \text{else} \end{cases}$$

$$f_{X|Y=y}(x) = \delta(x-2) \quad \text{if} \quad y = 2$$

Plot for $y=1$



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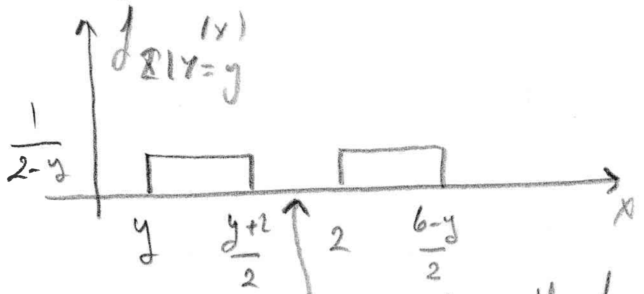
b) $\hat{X}_{MMSE} \mid Y=y = E\{X \mid Y=y\}$

$f_{X|Y=y}(x)$ is already found in (a):




for $y \in [0,2)$

Note that $f_{X|Y=y}(x)$ is symmetric.

Easy solution:

Center of symmetry is : $\dfrac{\frac{6-y}{2} + y}{2} = \dfrac{6+y}{4}$

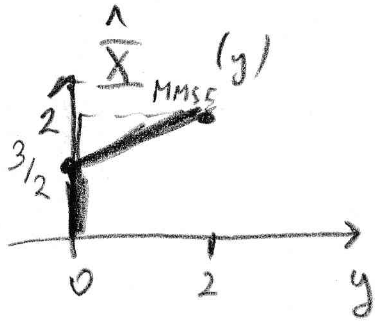
For a symmetric pdf, the center of symmetry is the expected value :

$$\boxed{\hat{X}_{MMSE} \mid Y=y = \frac{6+y}{4}} \quad , \; y \in [0,2]$$

Alternative formal solution:

$$E\{X \mid Y=y\} = \int_{-\infty}^{\infty} x \underbrace{f_{X|Y=y}(x)}_{\text{found in part (a) (above plot)}} dx = \int_{y}^{\frac{y+2}{2}} \frac{x}{2-y}\, dx + \int_{2}^{\frac{6-y}{2}} \frac{x}{2-y}\, dx$$

$$= \frac{1}{2-y}\left[ \frac{x^2}{2}\Bigg|_{y}^{\frac{y+2}{2}} + \frac{x^2}{2}\Bigg|_{2}^{\frac{6-y}{2}} \right]$$




$$= \boxed{\frac{6+y}{4}} = \hat{X}_{MMSE}$$

c) if $Y=y$ and $\bar{X} > \hat{\bar{X}} \Rightarrow Z_1 = \bar{X} - \hat{\bar{X}}$

found in part (b) (a constant)

Therefore $Z_1$ is the shifted version of $\bar{X}$.

$\bar{X}$, when $\bar{X} > \hat{\bar{X}}$ condition is given, uniformly distributed in the interval $\left[2, \frac{6-y}{2}\right]$.

Shifting that region to the left by $\hat{\bar{X}} = \frac{6+y}{4}$ gives:

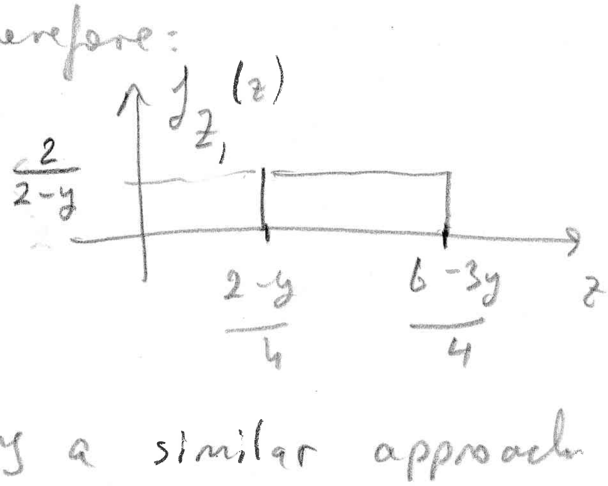
$Z_1$ is uniform in $\left[2 - \frac{6+y}{4}, \frac{6-y}{2} - \frac{6+y}{4}\right]$

$$= \left[\frac{2-y}{4}, \frac{6-3y}{4}\right]$$

length of the interval is:

$$\frac{6-3y}{4} - \frac{2-y}{4} = \frac{4-2y}{4} = \frac{2-y}{2}$$

Therefore:




$$\boxed{f_{Z_1}(z) = \begin{cases} \frac{2}{2-y} & \text{if } z \in \left[\frac{2-y}{4}, \frac{6-3y}{4}\right] \\ 0 & \text{else} \end{cases} \quad \text{for } y \in [0,2]}$$

d) Using a similar approach as in (c):

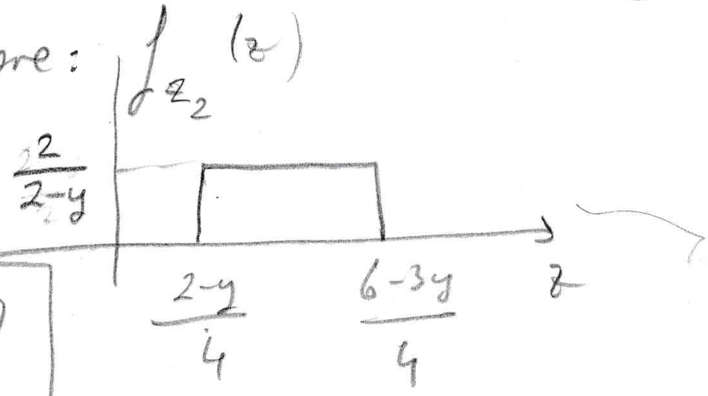
If $Y=y$ and $\bar{X} < \hat{\bar{X}} \Rightarrow Z_2 = \hat{\bar{X}} - \bar{X}$

constant $= \frac{6+y}{4}$

$\Rightarrow Z_2$ is uniform in: $\left[-\left(\frac{y+2}{2}\right) + \frac{6+y}{4}, -y + \frac{6+y}{4}\right] \longrightarrow$

$$= \left[ \frac{2-y}{4}, \frac{6-3y}{4} \right]$$

Therefore:




$$\boxed{f_{Z_2}(z) = \begin{cases} \frac{2}{2-y} & \text{if } z \in \left[\frac{2-y}{4}, \frac{6-3y}{4}\right] \\ 0 & \text{else} \end{cases} \quad \text{for } y \in [0,2]}$$

e) Using total probability:

$$f_Z(z) = f_{Z_1}(z) \cdot \underbrace{P(X > \hat{X})}_{\substack{\text{area of the right shaded} \\ \text{triangle of the joint pdf.} \\ = 1/2}} + f_{Z_2}(z) \underbrace{P(X < \hat{X})}_{\substack{\text{area of the left shaded} \\ \text{triangle of the joint pdf.} \\ = 1/2}}$$

So:

$$\boxed{f_Z(z) = f_{Z_1}(z) = f_{Z_2}(z) = \begin{cases} \frac{2}{2-y} & \text{if } z \in \left[\frac{2-y}{4}, \frac{6-3y}{4}\right] \\ 0 & \text{else} \end{cases} \quad \text{for } y \in [0,2]}$$

3-) a) Means no arrival in the first three hours:

pmf of Poisson = $P_X(k) = e^{-\lambda\tau} \frac{(\lambda\tau)^k}{k!}$

(for duration of 3 hours)

$\tau = 3h$ $\lambda = 0.5$ per hour.

$\lambda\tau = 1.5$

No arrival : $k = 0$

So: P(waiting more than 3 hours) = $e^{-1.5} \cdot \frac{1}{1}$

$= \boxed{e^{-1.5}}$

b) P(watching time $\in [3,5]$ h) =

P(No arrival in the first 3 hours AND at least one arrival in the next 2 hours)

Since Poisson process: non overlapping time interval arrivals are <u>independent</u>.

$\Longrightarrow$

$= \underbrace{P(k=0 \text{ in the first } 3h)}_{= e^{-(0.5)3} \cdot \frac{1}{1}} \underbrace{P(k \neq 0 \text{ in the next } 2h)}_{= 1 - \underbrace{e^{-(0.5)2} \cdot \frac{1}{1}}_{\text{No arrival probability } (k=0)}}$

$= e^{-1.5} \cdot (1 - e^{-1}) = \boxed{e^{-1.5} - e^{-2.5}}$

Alternative solutions for 3-a and 3-b: (8)

Since the process is Poisson, interarrival times are exponentially distributed

$$P_T(t) = \lambda e^{-\lambda \tau}, \quad \lambda = 0.5$$

a) No arrival in 3 hours $\equiv$ interarrival time is $> 3h$.

$$P(t>3) = \int_3^\infty \lambda e^{-\lambda \tau} = 0.5 \left. \frac{e^{-0.5t}}{-0.5} \right|_3^\infty$$

$$= \boxed{e^{-1.5}}$$

b) $$P_T(3 \le t \le 5) = \int_3^5 0.5 e^{-0.5t} dt$$

$$= 0.5 \left. \frac{e^{-0.5t}}{-0.5} \right|_3^5$$

$$= \boxed{e^{-1.5} - e^{-2.5}}$$

c) If waiting time is $> 3$, only 1 arrival will be observed. Therefore waiting time is $\le 3$.

$\equiv$ 2 arrivals in the first 3 hours:

$$P_X(2) = e^{-1.5} \frac{(1.5)^2}{2!} = e^{-1.5} \frac{2.25}{2} = \boxed{\frac{9}{8} e^{-1.5}}$$

$\lambda \tau = (0.5) \cdot 3 = 1.5$

d) $E\{\text{successes}\} = E\{\text{successes} \mid \text{waiting time} < 3\} \cdot P(\text{waiting time} < 3)$

$+ E\{\text{successes} \mid \text{waiting time} \geq 3\} \cdot P(\text{w. time} \geq 3)$

(Total expectation)

$$E\{\text{success} \mid \text{waiting time} < 3\} = E\{\underbrace{X}_{\text{Poisson with } \lambda\tau = 1.5} \mid T < 3\} = 1.5$$

$$P\{\text{waiting time} < 3\} = \int_0^3 0.5 e^{-0.5t} dt = 0.5 \left. \frac{e^{-0.5t}}{-0.5} \right|_0^3$$

$$= 1 - e^{-1.5}$$

$$E\{\text{success} \mid \text{waiting time} \geq 3\} = 1$$

There is always only one observation if $t \geq 3h$, so $\underbrace{E\{\text{constant}\}}_{1} = 1$

$$P\{\text{waiting time} \geq 3\} = 1 - (1 - e^{-1.5}) = e^{-1.5}$$

So: $E\{\text{success}\} = 1.5(1 - e^{-1.5}) + 1 \cdot e^{-1.5}$

$$= \boxed{1.5 - 0.5 e^{-1.5}}$$

e) Poisson process $\equiv$ waiting times are exponentially distributed; memoriless: past will not affect the future $\equiv$ fresh start

→ "Already waited for 6 hours" will not affect the statistics of what will happen from now on. Therefore, expected value of waiting "after the first 6 hours" is the same as the unconditional waiting time:

$$E\{T\} = \int_{-\infty}^{\infty} t f_T(t)\,dt = \int_0^{\infty} t \lambda e^{-\lambda t} dt$$

$$= \frac{1}{\lambda} = \frac{1}{0.5} = 2 \text{ hours}$$

$\Rightarrow E\{\text{total waiting time}\} = 6 + 2 = \boxed{8 \text{ hours}}$

4-) a) discrete random variable

$X$ takes integer values $k = 1, 2 \ldots \infty$.

Therefor $Y$ takes values $2, 4, 6, 8$ when $k = 1, 2, 3, 4$, since $Y = 2X$.

$P\{Y = 2k\} = P\{X = k\} = (1-p)^{k-1} p$ for $k = 1, 2, 3, 4$.

for $k = 5, 6 \ldots \infty$, $Y$ is always 10

So $P\{Y = 10\} = P\{X = 5 \text{ or } 6 \text{ or } \ldots \infty\}$

$= 1 - P\{X \in [1, 2, 3, 4]\}$

$= 1 - \left[p + p(1-p) + p(1-p)^2 + p(1-p)^3\right] = (1-p)^4$

→

$$\longrightarrow P_Y(k) = \begin{cases} (1-p)^{\frac{k}{2}-1} p & k = 2,4,6,8 \\ (1-p)^4 & k = 10 \\ 0 & \text{else (other integer } k\text{'s)} \end{cases}$$

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* Note: $P_X(k) = (1-p)^{k-1} p$ is a geometric r.v.

$\equiv$ number of trials until first success.

So: $P(k \geq k_0)$ means no successes in the first $k_0 - 1$ attempts. $= (1-p)^{k_0 - 1}$

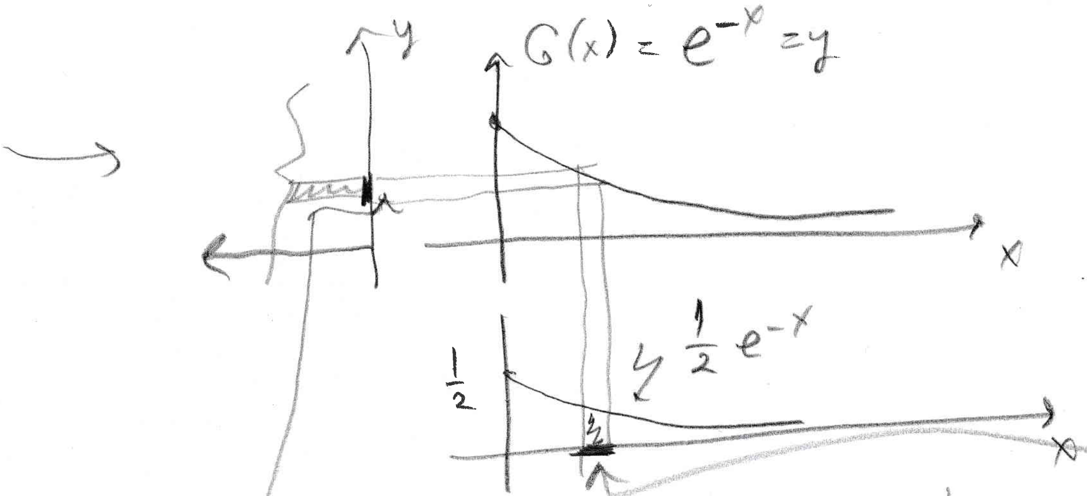
b) $$\int_{-\infty}^{\infty} f_X(x)\,dx = 1 = \int_{-\infty}^{\infty} a\,\delta(x-1)\,dx + \int_{-\infty}^{\infty} \frac{1}{2} e^{-x}\,dx$$

$$= a + \frac{1}{2} \Rightarrow \boxed{a = \frac{1}{2}}$$

c) There is an impulse located at $x=1$ with an amplitude $\frac{1}{2}$ in the pdf of $X$. $\Longrightarrow$ There is also an impulse in the pdf of $Y$.

$x = 1 \Rightarrow G(x) = e^{-x} \Rightarrow G(1) = e^{-1}$ = location of impulse in the pdf of $Y$; its amplitude does not change $\Rightarrow f_Y(y)$ will have an impulsive component $\frac{1}{2}\delta(y - e^{-1})$.

Now, let's work on the other component $\frac{1}{2}e^{-x}$, of $f_X(x)$: $\longrightarrow$




$$\underbrace{f_Y(y)\,|dy|}_{\text{probability of } Y \in \text{this interval}} = \underbrace{f_X(x)\,|dx|}_{\text{probability of } X \in \text{this interval}}$$

intervals do not include the impulse for this case

So: $$f_Y(y) = \left|\frac{dx}{dy}\right| f_X(x)$$

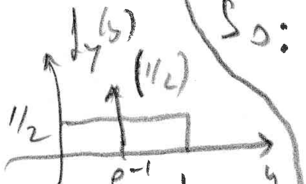
$$y = e^{-x} \longrightarrow \frac{dy}{dx} = -e^{-x} = -y \longrightarrow \left|\frac{dx}{dy}\right| = \frac{1}{y}$$

$$\Downarrow$$

$$x = -\ln y \longrightarrow f_X(x) = f_X(-\ln y)$$

So:

$$f_Y(y) = \frac{1}{y} f_X(-\ln y) = \frac{1}{y}\cdot\frac{1}{2}e^{\ln y} = \frac{1}{y}\frac{1}{2}\cdot y = \frac{1}{2} \quad \text{for } y \in [0,1]$$




So: Combining the impulsive and non impulsive components found above

$$\boxed{f_Y(y) = \frac{1}{2}\delta(y - e^{-1}) + \begin{cases} \frac{1}{2} & \text{for } y \in [0,1] \\ 0 & \text{else} \end{cases}}$$