MATH 255 Homework 4

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The MATLAB code used for the homework:

```
%the numbers are stored in a 1x2000 array called "rand"

writematrix(rand, 'hw4_0'); %to store in a file
random_numbers_50 = rand(1:50) %to print first 50 of them,
omit semicolon
figure;
histogram(rand, 20);
title('3, uniformly distributed in [0,1]');


q1 = -0.5*log(1-(rand));
writematrix(q1, 'hw4_1'); %to store in a file
exp_dist_50 = q1(1:50) %to print first 50 of them
figure;
histogram(q1, 20);
title('4.1, exponentially distributed with λ=2');

q2 = zeros(1, 2000);
for k=1:2000
    if rand(k)<0.25
        q2(k) = 1+2*sqrt(rand(k));
    elseif rand(k)<0.5
        q2(k) = 3-sqrt(2-4*rand(k));
    elseif rand(k)<0.75
        q2(k) = 4+sqrt(4*rand(k)-2);
    else
        q2(k) = 6-2*sqrt(1-rand(k));
    end
end
writematrix(q2, 'hw4_2'); %to store in a file
tr_dist_50 = q2(1:50) %to print first 50 of them
figure;
histogram(q2,20);
title('4.2, triangular PDF');

q3 = zeros(1, 2000);
```

```
for k=1:2000
    if rand(k)<0.5
        q3(k) = 0.5*log(2*rand(k));
    else
        q3(k) = -0.5*log(2-2*rand(k));
    end
end
writematrix(q3, 'hw4_3'); %to store in a file
biexp_dist_50 = q3(1:50) %to print first 50 of them
figure;
histogram(q3,20);
title('4.3, two-sided exponentially distributed with
λ=2');

M0 = mean(rand)
M1 = mean(q1)
M2 = mean(q2)
M3 = mean(q3)

V0 = var(rand)
V1 = var(q1)
V2 = var(q2)
V3 = var(q3)
```

3) First 50 numbers are printed:

```
random_numbers_50 =

  Columns 1 through 10

    0.5686    0.2406    0.4931    0.9937    0.4688    0.8669    0.7895    0.3278    0.4793    0.1431

  Columns 11 through 20

    0.5686    0.7516    0.3238    0.5816    0.8928    0.1970    0.7772    0.4939    0.4879    0.5445

  Columns 21 through 30

    0.9520    0.3180    0.7131    0.5494    0.8997    0.2574    0.2518    0.0421    0.1602    0.5298

  Columns 31 through 40

    0.0099    0.4723    0.1397    0.9364    0.5352    0.0938    0.9903    0.9650    0.1025    0.2663

  Columns 41 through 50

    0.6940    0.3353    0.8088    0.9408    0.6394    0.1593    0.7846    0.6639    0.1817    0.5095
```

Figure 1: First 50 numbers of the uniform numbers

4) The conversion procedure and calculations are attached at the end of this document, please see after page 6. For each distribution, first 50 numbers are printed.

4.1)

```
exp_dist_50 =

  Columns 1 through 10

    0.4203    0.1376    0.3398    2.5317    0.3163    1.0084    0.7791    0.1986    0.3263    0.0772

  Columns 11 through 20

    0.4204    0.6964    0.1956    0.4356    1.1164    0.1097    0.7507    0.3405    0.3346    0.3932

  Columns 21 through 30

    1.5180    0.1914    0.6243    0.3986    1.1497    0.1488    0.1450    0.0215    0.0873    0.3773

  Columns 31 through 40

    0.0050    0.3196    0.0752    1.3774    0.3831    0.0492    2.3179    1.6757    0.0541    0.1548

  Columns 41 through 50

    0.5921    0.2042    0.8272    1.4138    0.5100    0.0868    0.7677    0.5451    0.1003    0.3561
```

Figure 2: First 50 numbers of the exponential distribution

4.2)

```
tr_dist_50 =

  Columns 1 through 10

    4.5236    1.9810    2.8344    5.8410    2.6467    5.2704    5.0823    2.1701    2.7123    1.7566

  Columns 11 through 20

    4.5238    5.0033    2.1605    4.5713    5.3451    1.8876    5.0560    2.8441    2.7801    4.4220

  Columns 21 through 30

    5.5617    2.1468    4.9233    4.4446    5.3665    2.0149    2.0035    1.4105    1.8005    4.3450

  Columns 31 through 40

    1.1994    2.6673    1.7475    5.4956    4.3754    1.6125    5.8030    5.6257    1.6403    2.0332

  Columns 41 through 50

    4.8810    2.1883    5.1254    5.5135    4.7467    1.7982    5.0719    4.8096    1.8525    4.1946
```

Figure 3: First 50 numbers of the triangular distribution

4.3)

```
biexp_dist_50 =

  Columns 1 through 10

    0.0737   -0.3658   -0.0069    2.1851   -0.0322    0.6618    0.4325   -0.2111   -0.0211   -0.6255

  Columns 11 through 20

    0.0738    0.3498   -0.2172    0.0891    0.7699   -0.4658    0.4042   -0.0061   -0.0122    0.0466

  Columns 21 through 30

    1.1714   -0.2262    0.2778    0.0520    0.8031   -0.3320   -0.3431   -1.2368   -0.5691    0.0307

  Columns 31 through 40

   -1.9590   -0.0285   -0.6376    1.0309    0.0365   -0.8368    1.9713    1.3292   -0.7925   -0.3149

  Columns 41 through 50

    0.2456   -0.1998    0.4806    1.0672    0.1634   -0.5719    0.4211    0.1985   -0.5062    0.0096
```

Figure 4: First 50 numbers of the bi-exponential distribution

5) For 3, the histogram looks like a uniform distribution. For 4.1, the distribution looks exponential and for 4.3 it looks exponential from both sides. For 4.2, distribution looks triangular. Even though all histograms have certain defects (since we need infinitely many numbers for a true uniform distribution), the results conform to the desired distributions in 3, 4.1, 4.2, 4.3.

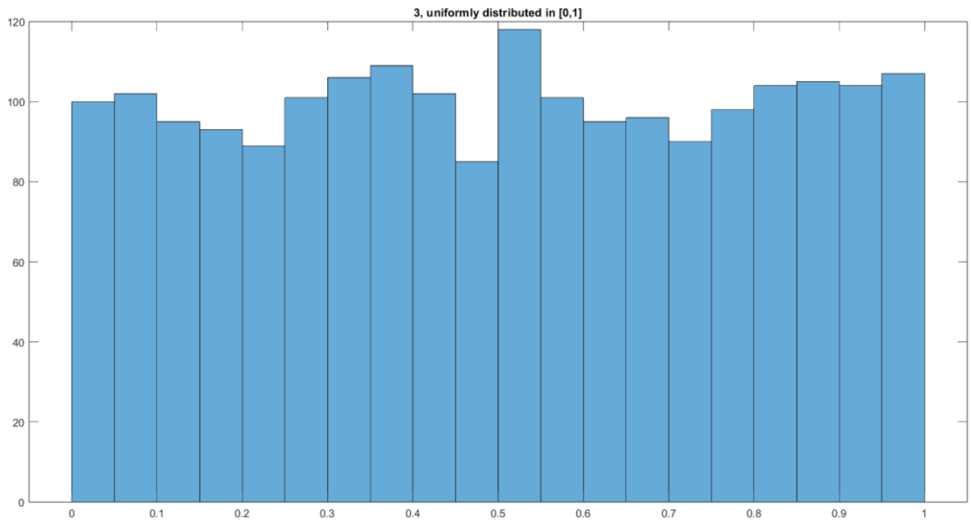


Figure 5: Histogram of 2000 random numbers in 3, uniform distribution

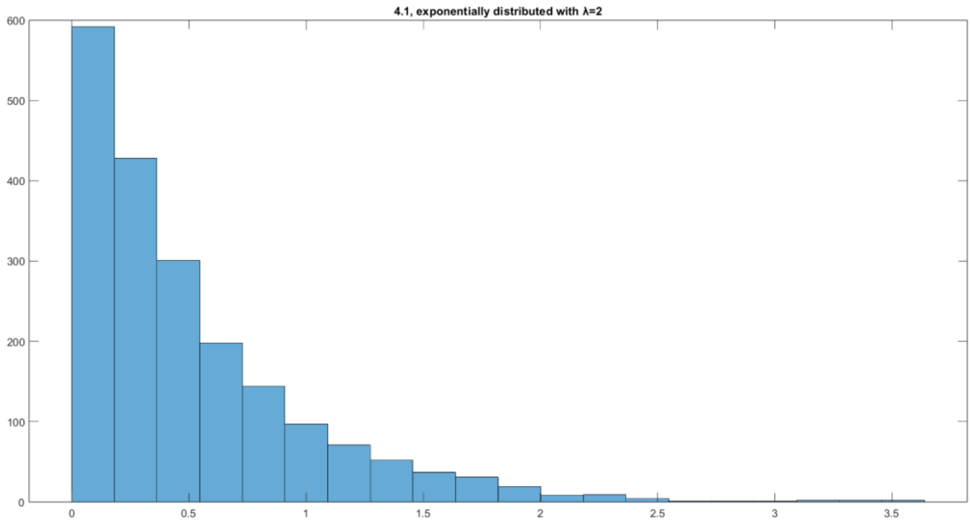


Figure 6: Histogram of 2000 random numbers in 4.1, exponential distribution

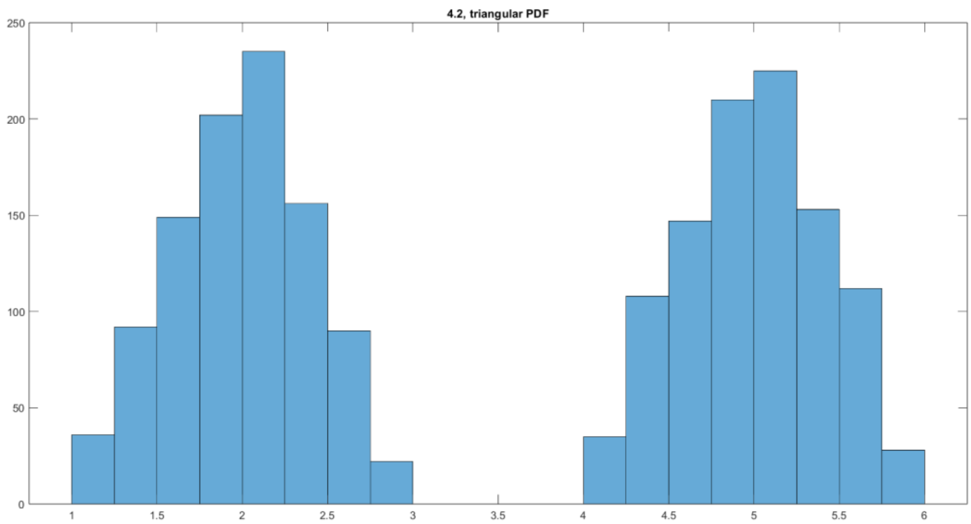


Figure 7: Histogram of 2000 random numbers in 4.2, triangular distribution as given in the manual

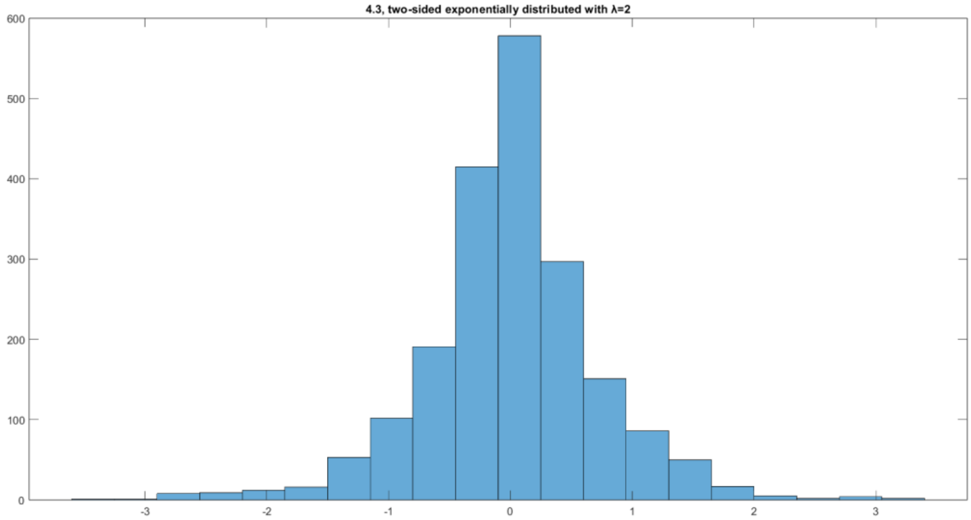


Figure 8: Histogram of 2000 random numbers in 4.3, bi-exponential distribution

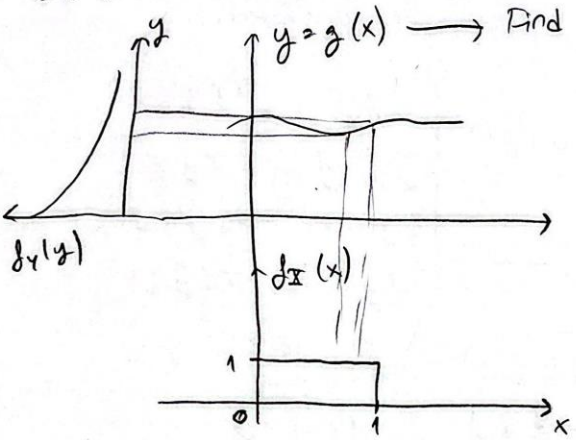
6) The mean and the variance of the 2000-sized arrays are computed using the "mean" and "var" functions of MATLAB and they are printed, using the code given at the beginning. The calculations of these pdfs' mean and variances, as well as a comparison between the MATLAB estimation and the calculated values are given at the end of this document.

```
M0 =

    0.5041


M1 =

    0.5068


M2 =

    3.5234


M3 =

    0.0062
```

```
V0 =

    0.0840


V1 =

    0.2498


V2 =

    2.4266


V3 =

    0.5099
```

Figure 9: Estimated means and variances, M0 (mean) and V0 (variance) are for part 3, M1 and V1 are for part 4.1, M2 and V2 are for part 4.2, M3 and V3 are for part 4.3.

4.1) $f_Y(y) = \lambda e^{-\lambda y} \cdot u(y)$



$\longrightarrow$ Find $g(x)$ such that we can convert the random numbers to distribution Y.

$Y = G(X) \Rightarrow G^{-1}(Y) = X$, $\oplus$
assuming that $g(x)$ is one-to-one

$$F_Y(y) = P\{Y \le y\} = P\{G(X) \le y\} = P\{X \le G^{-1}(y)\} = F_X(G^{-1}(y))$$

and
$$F_X(x) = \begin{cases} \int_0^1 1\,dx = x, & \text{for } 0 \le x \le 1 \\ 1, & \text{for } x > 1 \\ 0, & \text{for } x < 0 \end{cases}$$

$$F_Y(y) = \int_{-\infty}^{y} f_Y(z)\,dz = \int_0^y \lambda e^{-\lambda z}\,dz = \left[e^{-\lambda z}\right]_y^0 = 1 - e^{-\lambda y} = \begin{cases} 1 - e^{-2y}, & y > 0 \\ 0, & \text{else} \end{cases}$$

non-zero for $y \geq 0$.

Now, we have $F_Y(y) = F_X(G^{-1}(y))$

$$\Rightarrow 1 - e^{-2y} = G^{-1}(y) \quad \text{if} \quad 0 \le G^{-1}(y) \le 1 \quad \triangle$$

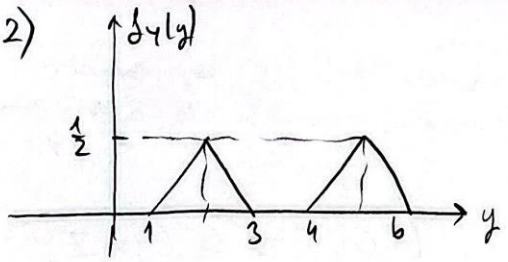
$$\Rightarrow 1 - G^{-1}(y) = e^{-2y} \Rightarrow y = -\frac{1}{2}\ln\big(1 - \underbrace{G^{-1}(y)}_{\text{equal to } x \text{ from } \oplus}\big)$$

$$\Rightarrow y = \frac{-\ln(1-x)}{2}$$

and we are given $0 \le x = G^{-1}(y) \le 1$ so $\triangle$ is satisfied.

$$\Rightarrow \boxed{Y = G(X) = \frac{-\ln(1-X)}{2}}$$

4.2)

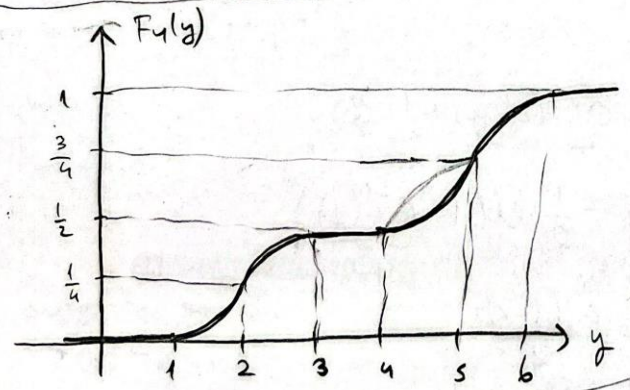


To find $F_Y(y)$, area under $f_Y(y)$ will be considered:

$$f_Y(y) = \begin{cases} \frac{1}{2}y - \frac{1}{2}, & 1 \le y < 2 \\ -\frac{1}{2}y + \frac{3}{2}, & 2 \le y \le 3 \\ \frac{1}{2}y - 2, & 4 \le y < 5 \\ -\frac{1}{2}y + 3, & 5 \le y \le 6 \\ 0, & \text{else} \end{cases}$$

line eqns. $\longleftarrow$

$$\Rightarrow F_Y(y) = \int_{-\infty}^{y} f_Y(z)\,dz = \begin{cases} 0, & y < 1 \\ \int_{-\infty}^{y}\left(\frac{1}{2}z - \frac{1}{2}\right)dz = \frac{y^2}{4} - \frac{y}{2} + \frac{1}{4}, & 1 \le y < 2 \\ \int_{-\infty}^{y}\left(-\frac{1}{2}z + \frac{3}{2}\right)dz = -\frac{y^2}{4} + \frac{3y}{2} - \frac{7}{4}, & 2 \le y \le 3 \\ \frac{1}{2}, & 3 < y < 4 \\ \int_{-\infty}^{y}\left(\frac{1}{2}z - 2\right)dz = \frac{y^2}{4} - 2y + \frac{9}{2}, & 4 \le y < 5 \\ \int_{-\infty}^{y}\left(-\frac{1}{2}z + 3\right)dz = -\frac{y^2}{4} + 3y - 8, & 5 \le y \le 6 \\ 1, & y > 6 \end{cases}$$



and $F_X(x) = \begin{cases} 0, & x \le 0 \\ x, & 0 \le x \le 1 \\ 1, & x \ge 1 \end{cases}$ from prev. part.

$F_Y(y) = F_X(G^{-1}(y)) = \underbrace{G^{-1}(y)}_{=x}$ as shown previously, and since $G^{-1}(y) \in [0,1]$, $= x$

- For $1 \le y < 2$:

$\frac{y^2}{4} - \frac{y}{2} + \frac{1}{4} = \frac{(y-1)^2}{4} = G^{-1}(y) \Rightarrow |y-1| = 2\sqrt{G^{-1}(y)}$

since $y \ge 1$ $\rightarrow$ $\boxed{y = 1 + 2\sqrt{G^{-1}(y)}}$

$\Rightarrow \boxed{y = 1 + 2\sqrt{x}}$ defined for $x \ge 0$

- for $2 \le y \le 3$:

$-\frac{y^2}{4} + \frac{3y}{2} - \frac{7}{4} = \frac{-(y-3)^2 + 2}{4} = G^{-1}(y)$

$\Rightarrow 2 - 4G^{-1}(y) = (y-3)^2$ since $y \le 3$

$\Rightarrow \boxed{y = 3 - \sqrt{2 - 4G^{-1}(y)} = 3 - \sqrt{2 - 4x}}$ defined for $x \le \frac{1}{2}$

- For $4 \leq y < 5$: $\frac{y^2}{4} - \underset{(4)}{2y} + \underset{(2)}{\frac{9}{2}} = \frac{(y-4)^2+2}{4} = G^{-1}(x)$

$$\Rightarrow |y-4| = \sqrt{4G^{-1}(x)-2}$$

$$\underset{y \geq 4}{\Rightarrow} \boxed{y = 4 + \sqrt{4G^{-1}(x)-2} = 4 + \sqrt{4x-2}}$$ defined for $x \geq \frac{1}{2}$

- For $5 < y \leq 6$: $\frac{-y^2}{4} + 3y - 8 = \frac{-(y-6)^2+4}{4} = G^{-1}(x)$

$$\Rightarrow \sqrt{4-4G^{-1}(x)} = |y-6| \underset{y \leq 6}{\Rightarrow} \boxed{y = 6 - 2\sqrt{1-G^{-1}(x)} = 6 - 2\sqrt{1-x}}$$

defined for $x \leq 1$

Check:

For $0 \leq x < \frac{1}{4}$, y takes values between $1 \leq y < 2$,

For $\frac{1}{4} \leq x < \frac{1}{2}$, y " " $2 \leq y < 3$

For $\frac{1}{2} < x < \frac{3}{4}$, y " " $4 < y < 5$

For $\frac{3}{4} \leq x \leq 1$, y " " $5 \leq y \leq 6$

in accordace with the limits of y in defn. Also y's are defined in these intervals (inside of square roots are gr. than or equal to 0)

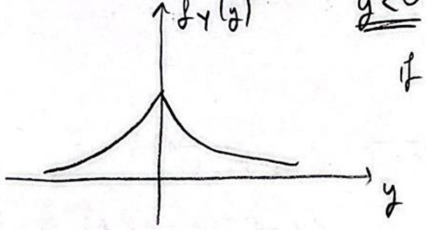
$$\Rightarrow \boxed{Y = G(X) = \begin{cases} 1 + 2\sqrt{X}, & 0 \leq x < \frac{1}{4} \\ 3 - \sqrt{2-4X}, & \frac{1}{4} \leq x < \frac{1}{2} \\ 4 + \sqrt{4X-2}, & \frac{1}{2} < x < \frac{3}{4} \\ 6 - 2\sqrt{1-X}, & \frac{3}{4} \leq x \leq 1 \end{cases}}$$

Also we have for $3 \leq y \leq 4$, we have $x = \frac{1}{2}$ $\Rightarrow$ Corresponds to an impulse with amplitude $\frac{1}{2}$

or a discontinuity at $x = \frac{1}{2}$

4.3) $f_Y(y) = \frac{\lambda}{2} e^{-\lambda |y|}$

$$F_Y(y) = \int_{-\infty}^{y} f_Y(z)\,dz \underset{\substack{\text{if}\\ y<0}}{=} \int_{-\infty}^{y} \frac{\lambda}{2} e^{\lambda z}\,dz = \frac{1}{2}\left[e^{\lambda z}\right]_{-\infty}^{y} = \frac{e^{\lambda y}}{2}, \quad \underline{y<0}$$



$$\text{if } y \geq 0 \Rightarrow \int_{-\infty}^{y} f_Y(z)\,dz = \int_{-\infty}^{0} \frac{\lambda}{2} e^{\lambda z}\,dz + \int_{0}^{y} \frac{\lambda}{2} e^{-\lambda z}\,dz$$

$$= \frac{1}{2}\underbrace{\left[e^{\lambda z}\right]_{-\infty}^{0}}_{1} + \frac{1}{2}\underbrace{\left[e^{-\lambda z}\right]_{y}^{0}}_{1-e^{-\lambda y}} = 1 - \frac{1}{2}e^{-\lambda y}, \quad y \geq 0$$

$$\Rightarrow \boxed{F_Y(y) = \begin{cases} \frac{e^{\lambda y}}{2}, & y<0 \\ 1 - \frac{e^{-\lambda y}}{2}, & y \geq 0 \end{cases}} \quad \text{and} \quad F_X(x) = \begin{cases} 0, & x<0 \\ x, & 0 \leq x \leq 1 \\ 1, & x>1 \end{cases}$$

We have $F_Y(y) = P\{Y \leq y\} = P\{G(X) \leq y\} = P\{X \leq G^{-1}(y)\} = F_X(G^{-1}(y))$

$\Rightarrow F_Y(y) = F_X(G^{-1}(y)) = G^{-1}(y)$, since $G^{-1}(y) = x$ is between $[0,1]$

$\Rightarrow$ For $\underline{y<0}$: $\frac{e^{\lambda y}}{2} = G^{-1}(y) \Rightarrow y = \frac{\ln(2\overset{x}{\overline{G^{-1}(y)}})}{\underset{\lambda=2}{\lambda}} = \frac{\ln(2x)}{2}$ (1)

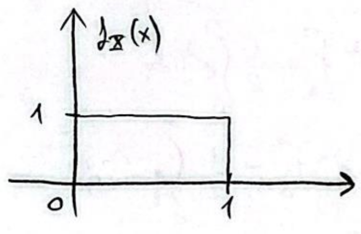
For $\underline{y \geq 0}$: $1 - \frac{e^{-\lambda y}}{2} = G^{-1}(y) \Rightarrow y = -\frac{\ln(2-2G^{-1}(y))}{\lambda} = -\frac{\ln(2-2x)}{2}$ (2)

For $0 \leq x < \frac{1}{2}$, $\underline{y = g(x) \text{ takes values smaller than } 0}$ for both (1), (2)

For $\frac{1}{2} < x \leq 1$, $\underline{y = g(x) \text{ takes values bigger than } 0}$ " "

$$\Rightarrow \boxed{Y = G(X) = \begin{cases} \frac{\ln(2X)}{2}, & 0 \leq x < \frac{1}{2} \\ -\frac{\ln(2-2X)}{2}, & \frac{1}{2} \leq x \leq 1 \end{cases}} \quad \text{since } 0 \leq x \leq 1$$

6) - For un. dist. in $[0,1]$



$$E\{X\} = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_0^1 x\,dx = \left[\frac{x^2}{2}\right]_0^1 = \boxed{0.5}$$

$$E\{X^2\} = \int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

$$Var\{X\} = E\{X^2\} - (E\{X\})^2 = \frac{1}{3} - \frac{1}{4} = \boxed{\frac{1}{12} = 0.083}$$

- For exp. dist, $f_X(x) = \lambda e^{-\lambda x} \cdot u(x)$

$$E\{X\} = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_0^{\infty} \underbrace{x}_{u} \underbrace{\lambda e^{-\lambda x}}_{dv} dx = \underbrace{\left[-x e^{-\lambda x}\right]_0^{\infty}}_{0} + \int_0^{\infty} e^{-\lambda x} dx = -\frac{1}{\lambda}\left[e^{-\lambda x}\right]_0^{\infty}$$

by L'Hôpital:

$$\lim_{x \to \infty} -\frac{e^{-\lambda x}}{\frac{1}{x}} = 0$$

$$= \frac{1}{\lambda} = \boxed{0.5} \; (\lambda = 2)$$

$$E\{X^2\} = \int_0^{\infty} \underbrace{x^2}_{u} \underbrace{\lambda e^{-\lambda x}}_{dv} dx = \underbrace{\left[-x^2 e^{-\lambda x}\right]_0^{\infty}}_{0} + \int_0^{\infty} \underbrace{2x}_{u} \underbrace{e^{-\lambda x}}_{dv} dx = \underbrace{\left[-\frac{2}{\lambda} x e^{-\lambda x}\right]_0^{\infty}}_{0} + \frac{2}{\lambda}\int_0^{\infty} e^{-\lambda x} dx$$

$$= \frac{2}{\lambda^2}$$

$$\Rightarrow Var\{X\} = E\{X^2\} - (E\{X\})^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2} = \frac{1}{4} = \boxed{0.25}$$

- For tr. dist, $f_X(x)$ was computed previously:

$$E\{X\} = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_1^2 x\left(\frac{1}{2}x - \frac{1}{2}\right)dx + \int_2^3 x\left(-\frac{1}{2}x + \frac{3}{2}\right)dx + \int_4^5 x\left(\frac{1}{2}x - 2\right)dx + \int_5^6 x\left(-\frac{1}{2}x + 3\right)dx$$

$$= \underbrace{\frac{5}{12} + \frac{7}{12}}_{1} + \underbrace{\frac{7}{6} + \frac{4}{3}}_{\frac{15}{6} = 2.5} = 1 + 2.5 = \boxed{3.5}$$

$$E\{X^2\} = \int_1^2 x^2\left(\frac{1}{2}x - \frac{1}{2}\right)dx + \int_2^3 x^2\left(-\frac{1}{2}x + \frac{3}{2}\right)dx + \int_4^5 x^2\left(\frac{1}{2}x - 2\right)dx + \int_5^6 x^2\left(-\frac{1}{2}x + 3\right)dx$$

$$= \frac{17}{24} + \frac{-111}{8} + \frac{131}{24} + \frac{57}{8} = \frac{148}{24} + \frac{68}{8} = \frac{352}{24} \cong 14.67$$

$$Var\{X\} = E\{X^2\} - (E\{X\})^2 = 14.67 - (3.5)^2 = \boxed{2.42}$$

- For bi exp. dist: $f_X(x) = \frac{\lambda}{2} e^{-\lambda|x|}$

$$E\{X\} = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_{-\infty}^{0} \underbrace{x}_{u} \underbrace{\frac{\lambda}{2} e^{\lambda x} dx}_{dv} + \int_{0}^{\infty} \underbrace{x}_{u} \underbrace{\frac{\lambda}{2} e^{-\lambda x} dx}_{dv} = \overbrace{\left[x \cdot \frac{1}{2} e^{\lambda x}\right]_{-\infty}^{0}}^{0} + \frac{1}{2} \int_{-\infty}^{0} e^{\lambda x} dx + \underbrace{\left[x \cdot -\frac{1}{2} e^{-\lambda x}\right]_{0}^{\infty}}_{0} - \frac{1}{2} \int_{0}^{\infty} e^{-\lambda x} dx$$

IBP

$$= \frac{1}{2} \cdot \frac{1}{\lambda} \left(\underbrace{\left[e^{\lambda x}\right]_{-\infty}^{0}}_{1} + \underbrace{\left[e^{-\lambda x}\right]_{0}^{\infty}}_{-1} \right) = \underline{\underline{0}}$$

$$E\{X^2\} = \int_{-\infty}^{0} \underbrace{x^2}_{u} \underbrace{\frac{\lambda}{2} e^{\lambda x} dx}_{dv} + \int_{0}^{\infty} x^2 \frac{\lambda}{2} e^{-\lambda x} dx = \overbrace{\left[x^2 \frac{e^{2x}}{2}\right]_{-\infty}^{0}}^{0} + \int_{-\infty}^{0} \underbrace{2x}_{u} \underbrace{e^{2x} dx}_{dv} + \overbrace{\left[x^2 \cdot -\frac{e^{-2x}}{2}\right]_{0}^{\infty}}^{0} - \frac{1}{2} \int_{0}^{\infty} \underbrace{2x}_{u} \underbrace{e^{-2x} dx}_{dv}$$

put $\lambda = 2$ IBP

$$= \left[x e^{2x}\right]_{-\infty}^{0} + \underbrace{2 \cdot \frac{1}{2} \int_{-\infty}^{0} e^{2x} dx}_{1} + \left[x e^{-2x}\right]_{0}^{\infty} - \underbrace{\int_{0}^{\infty} \frac{e^{-2x}}{-2} dx}_{-\frac{1}{2}} = 1 - \frac{1}{2} = \frac{1}{2}$$

$$Var\{X\} = E\{X^2\} - (E\{X\})^2 = \frac{1}{2} = \underline{\underline{0.5}}$$

	Calculated	Estimated (by MATLAB functions)
M0 (uniform)	0.5	0.5041
M1 (exp. dist.)	0.5	0.5068
M2 (tri. dist.)	3.5	3.5234
M3 (bi exp.)	0	0.0062
V0	0.083	0.0840
V1	0.25	0.2498
V2	2.42	2.4266
V3	0.5	0.5099

mean(.), var(.)

$\Rightarrow$ All values are nearly equal.