Bilkent University	**Spring 2016-2017**

Math255 Probability and Statistics
Midterm 2 Solutions
20 April 2017

Problem 1. [6 pts] Let (X, Y) be jointly distributed random variables with

$$f_{X,Y}(x, y) = \frac{1}{a^2}, \quad 0 < x, y \le a,$$

where $a > 0$ is a constant. Compute the PDF $f_Z(z)$ of $Z = Y/X$.

Solution. First compute $F_Z(z)$. (Draw a picture to see what you are doing.)

$$\begin{aligned} F_Z(z) &= P(Z \le z) = P(Y \le zX) \\ &= \begin{cases} 0, & z < 0; \\ \frac{z}{2}, & 0 \le z \le 1; \\ 1 - \frac{1}{2z}, & z > 1. \end{cases} \end{aligned}$$

Next, differentiate $F_Z(z)$ to obtain

$$f_Z(z) = \begin{cases} 0, & z < 0; \\ \frac{1}{2}, & 0 \le z \le 1; \\ \frac{1}{2z^2}, & z > 1. \end{cases}$$

Problem 2. [6 pts] Compute $E[(X + Y)^2]$ for (X, Y) jointly distributed with

$$f_{X,Y}(x, y) = \begin{cases} 2, & 0 < y \le x < 1; \\ 0, & \text{otherwise.} \end{cases}$$

Solution. This is a simple exercise testing if you know how to compute the expectation of a function of two given random variables.

$$\begin{aligned} \mathbf{E}[(X + Y)^2] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x + y)^2 f_{X,Y}(x, y)\, dx\, dy \\ &= \int_0^1 \int_y^1 (x + y)^2\, 2\, dx\, dy \\ &= \int_0^1 \frac{2}{3}(x + y)^3 \Big|_y^1 dy \\ &= \int_0^1 \frac{2}{3}\left[(1 + y)^3 - 8y^3\right] dy \\ &= \frac{2}{3}\frac{1}{4}\left[(1 + y)^4 - 8y^4\right]\Big|_0^1 \\ &= \frac{7}{6} \end{aligned}$$

Problem 3. [6 pts] Compute $\mathbf{E}[\mathrm{var}(Y|X)]$ when (X, Y) are jointly distributed as in Problem 2.

Solution. First compute the random variable $\mathrm{var}(Y|X)$. For this note that, conditional on $X = x$ for $0 \le x \le 1$, Y is uniform on $[0, x]$, i.e.,

$$f_{Y|X}(y|x) = \frac{1}{x}, \quad 0 \le y \le x \le 1.$$

Thus, $\mathrm{var}(Y|X = x) = \frac{x^2}{12}$ by the formula for the variance of a uniform random variable. So, we have $\mathrm{var}(Y|X) = \frac{X^2}{12}$. The expectation is now computed in a manner similar to the computation in the previous problem.

$$\begin{aligned}
\mathbf{E}[\mathrm{var}(Y|X)] &= \frac{1}{12}\,\mathbf{E}[X^2] \\
&= \frac{1}{12}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2 f_{X,Y}(x,y)\,dx\,dy \\
&= \frac{1}{12}\int_{0}^{1}\int_{y}^{1} x^2\,2\,dx\,dy \\
&= \frac{1}{6}\int_{0}^{1} \frac{1}{3}x^3\Big|_{y}^{1}\,dy \\
&= \frac{1}{18}\int_{0}^{1}(1-y^3)\,dy \\
&= \frac{1}{18}\left(y - \frac{1}{4}y^4\right)\Big|_{0}^{1} \\
&= \frac{1}{18}\frac{3}{4} = \frac{1}{24}.
\end{aligned}$$

Problem 4. [6 pts] Let (X, Y) be jointly distributed with

$$f_{X,Y}(x,y) = \begin{cases} abe^{-(ax+by)}, & x > 0, y > 0; \\ 0, & \text{otherwise,} \end{cases}$$

where a and b are positive constants. Determine $P(X > Y)$ in terms of a and b.

Solution. This is a straightforward exercise in integration.

$$\begin{aligned}
P(X > Y) &= \int_{-\infty}^{\infty}\int_{y}^{\infty} f_{X,Y}(x,y)\,dx\,dy \\
&= \int_{0}^{\infty}\int_{y}^{\infty} ab\,e^{-(ax+by)}\,dx\,dy \\
&= \int_{0}^{\infty} be^{-by}\left[\int_{y}^{\infty} ae^{-ax}\,dx\right]dy \\
&= \int_{0}^{\infty} be^{-by}\left[-\,e^{-ax}\Big|_{y}^{\infty}\right]dy \\
&= \int_{0}^{\infty} b\,e^{-by}\,e^{-ay}\,dy \\
&= \int_{0}^{\infty} b\,e^{-(a+b)y}\,dy \\
&= -\frac{b}{a+b}\,e^{-(a+b)y}\Big|_{0}^{\infty} \\
&= \frac{b}{a+b}.
\end{aligned}$$

Problem 5. [6 pts] An insurance company has issued life insurance policies to $N = 10,000$ persons of the same age and the same social group. The probability of death during the year for each person is $q = 0.0064$. On January 1st each insured person deposits $D = 12$ TL on his/her policy and if he/she dies his/her beneficiaries receive $R = 1,000$ TL from the company. Let X be the number of policy holders who die during the year. Let $G = ND - RX$ denote the gain of the insurance company. Estimate the probability $P(G > 50,000)$ (to a precision of ± 0.01) using the table of $\Phi(x)$ on the next page.

Solution. This problem tests your knowledge of the central limit theorem (CLT).

$$\begin{aligned} P(G > 50,000) &= P\big(ND - RX > 50,000\big) \\ &= P\left(X < \frac{ND - 50,000}{R}\right) \\ &= P\left(X < \frac{10,000 \cdot 12 - 50,000}{1,000}\right) \\ &= P\big(X < 70\big). \end{aligned}$$

To estimate this probability we use the CLT. We have

$$\mu = \mathbf{E}[X] = N\,q = 10,000 \cdot 0.0064 = 64,$$

and

$$\sigma^2 = \mathrm{var}(X) = N\,q\,(1-q) = 10,000 \cdot 0.0064 \cdot (1 - 0.0064) \approx 64.$$

By the CLT, the normalized random variable $Z = (X - \mu)/\sigma$ has a CDF approximately equal to $\Phi(z)$. So,

$$\begin{aligned} P\big(X < 70\big) &= P\left(\frac{X-\mu}{\sigma} < \frac{70-\mu}{\sigma}\right) \\ &\approx P\left(Z < \frac{70-64}{\sqrt{64}}\right) \\ &= P(Z < 0.75) \approx \Phi(0.75) = 0.7734. \end{aligned}$$

So, we estimate the probability in question as

$$P(G > 50,000) \approx 0.77.$$