Math 255
Spring 2019-20
Midterm 2 Solutions

1. X, Y, Z iid $U\{1,2,3,4,5,6\}$.

$A = \{X < Y < Z\}$, $B = \{X < Z\}$.

$$P(A) = \frac{\binom{6}{3}}{6^3} = \frac{5}{54}.$$

We can choose three distinct numbers from $\Omega' = \{1,2,3,4,5,6\}$ in $\binom{6}{3}$ ways and assign them to X, Y, Z in one way to satisfy event A.

There are 6^3 different ways of choosing three numbers from Ω', all equally likely.

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(A)}{P(B)} \quad \text{since } \cancel{B \subset A} \ A \subset B.$$

$$P(B) = \frac{\binom{6}{2}}{6^2} = \frac{15}{36} = \frac{5}{12}.$$

$$\therefore P(A|B) = \frac{5/54}{5/12} = \frac{12}{54} = \frac{2}{9}$$

The reasoning here is similar to finding $P(A)$. There are $\binom{6}{2} = 15$ ways of choosing two distinct numbers to satisfy $B = \{X < Z\}$.

P2 X, Y iid $U[0,1]$, $Z = 2X+Y$.

We can use the convolution formula to write

$$f_Z(1) = \int_{-\infty}^{\infty} f_X(x)\, f_Y(1-2x)\, dx$$

$$= \int_0^1 1 \cdot \underbrace{f_Y(1-2x)}_{\substack{\text{non-zero} \\ \text{only over } [0,\frac{1}{2}]}} dx = \int_0^{1/2} 1 \cdot dx = \underline{\underline{\frac{1}{2}}}$$

It is also possible to write $Z = V + Y$ with $V \triangleq 2X$ and $f_V(v) = \frac{1}{2} f_X\left(\frac{v}{2}\right)$.

Then $f_Z(1) = \int_{-\infty}^{\infty} f_V(v)\, f_Y(1-v)\, dv$ which

gives the same result.

A third possibility is to compute f_Z by

$$f_Z(1) = \int_{-\infty}^{\infty} f_X\left(\frac{1-y}{2}\right) f_Y(y)\, dy.$$

Finally, one may first compute $F_Z(z)$ and obtain $f_Z(1)$ by evaluating $\frac{d}{dz} F_Z(z)$ at $z = 1$.

$$Var(Z) = Var(2X+Y) \overset{\text{independence}}{=} Var(2X) + Var(Y)$$

$$= 4 \cdot Var(X) + Var(Y) = 4 \cdot \frac{1}{12} + \frac{1}{12}$$

$$= \underline{\underline{5/12}}. \quad \left(\text{Variance of } U[0,1] = \frac{1}{12}.\right)$$

P3 $E[X+Y] = E[X] + E[Y] = 2E[X]$.

$$E[X] = E\left[E[X|Q]\right] = E[Q]$$

$$= \int_0^1 q \cdot 3q^2 dq = \frac{3q^4}{4}\Big|_0^1 = \frac{3}{4}.$$

Hence, $E[X+Y] = 2 \cdot E[Q] = \frac{3}{2}$

$$Var(Z) = Var(E[Z|Q]) + E[Var(Z|Q)]$$

$$Var(E[Z|Q]) = Var(E[X+Y|Q])$$

$$= Var(2 \cdot Q) = 4 \cdot Var(Q).$$

$$Var(Q) = E[Q^2] - (E[Q])^2$$

$$= \int_0^1 q^2 \cdot 3q^2 dq - \left(\frac{3}{4}\right)^2 = \frac{3 \cdot q^5}{5}\Big|_0^1 - \left(\frac{3}{4}\right)^2$$

$$= \frac{3}{5} - \frac{9}{16} = \frac{3}{80}.$$

$$Var(Z|Q) = 1.$$

$$\therefore Var(Z) = 4 \cdot Var(Q) + 2 = 4 \cdot \frac{3}{80} + 2 = \frac{3}{20} + 2$$

$$= \frac{43}{20}$$

Scanned with CamScanner