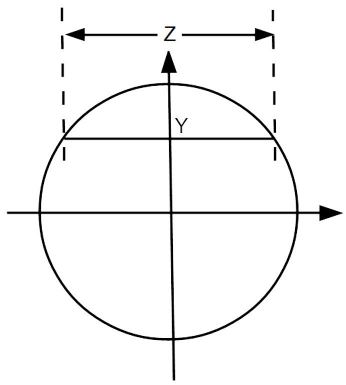
Bilkent University	Fall 2017-2018

Math255 Probability and Statistics
Midterm 2 Solutions
23 Nov. 2017

Problem 1. [6 pts]

Consider the unit circle $x^2+y^2=1$ on the x-y plane. Select a point Y at random on the y-axis so that $-1 \le Y \le 1$. Draw a chord through the point Y perpendicular to the y-axis. Let Z denote the length of the chord. Find the CDF $F_Z(z)$ for all $-\infty < z < \infty$.



$$F_Z(z) = \begin{cases} 0 & z \le 0 \\ 1-\sqrt{1-z^2/4} & 0 \le z \le 2 \\ 1 & z \ge 2 \end{cases}$$

Solution. From the geometry of the two random variables, we have $Z = 2\sqrt{(1-Y^2)}$. Clearly, $F_Z(z) = 0$ for $z < 0$ and $F_Z(z) = 1$ for $z \ge 2$. So, let us suppose $0 \le z \le 2$. Then,

$$\begin{aligned} F_Z(z) &= P(Z \le z) \\ &= P(2\sqrt{(1-Y^2)} \le z) \\ &= P(1-Y^2 \le z^2/4) \\ &= P(Y^2 \ge 1-z^2/4) \\ &= P(|Y| \ge \sqrt{1-z^2/4}) \\ &= 1-\sqrt{1-z^2/4}. \end{aligned}$$

Problem 2. [6 pts] Let (X, Y) be uniformly distributed in the triangle in the $x-y$ plane with corners $(0,0)$, $(0,1)$ and $(1,0)$. Let $Z = X+Y$. Determine the variance of Z. (A numerical answer is required.)

$$\text{var}(Z) = \tfrac{1}{18}$$

Solution. You can solve this problem either by deriving the PDF of Z or by using the properties of expectation. In the first approach, we have

$$\begin{aligned} F_Z(z) &= P(Z \le z) \\ &= P(X+Y \le z) \\ &= \begin{cases} 0 & z \le 0 \\ z^2 & 0 \le z \le 1 \\ 1 & z \ge 1 \end{cases} \end{aligned}$$

Therefore,

$$f_Z(z) = \begin{cases} 2z & 0 \le z \le 1; \\ 0 & \text{otherwise.} \end{cases}$$

Now, we have

$$\mathbf{E}[Z] = \int_0^1 2z^2 dz = \frac{2}{3}.$$

$$\mathbf{E}[Z^2] = \int_0^1 2z^3 dz = \frac{1}{2}.$$

Hence,

$$\operatorname{var}(Z) = \mathbf{E}[Z^2] - (\mathbf{E}[Z])^2 = \frac{1}{18}.$$

In the second approach, we use the relation

$$\operatorname{var}(X+Y) = \operatorname{var}(X) + 2\operatorname{cov}(X,Y) + \operatorname{var}(Y).$$

Now, we need to compute $\mathbf{E}[X^2]$, $\mathbf{E}[X]$, $\mathbf{E}[XY]$, $\mathbf{E}[Y]$ and $\mathbf{E}[Y^2]$. For this we note that

$$f_X(x) = \int_0^{1-x} 2dy = 2(1-x), \quad 0 \le x \le 1.$$

This gives

$$\mathbf{E}[X] = \int_0^1 2x(1-x)dx = (x^2 - 2x^3/3)\big|_0^1 = 1/3$$
$$\mathbf{E}[X^2] = \int_0^1 2x^2(1-x)dx = (2x^3/3 - 2x^4/4)\big|_0^1 = 1/6$$
$$\operatorname{var}(X) = 1/6 - (1/3)^2 = 1/18.$$

Likewise, we have $\operatorname{var}(Y) = 1/18$. Finally,

$$\mathbf{E}[XY] = \int_0^1 \int_0^{1-x} 2xy\,dy\,dx = \int_0^1 x(1-x)^2\,dx = (x^2/2 - 2x^3/3 + x^4/4)\big|_0^1 = 1/12.$$

So,

$$\operatorname{cov}(X,Y) = \mathbf{E}[XY] - \mathbf{E}[X]\mathbf{E}[Y] = 1/12 - (1/3)^2 = -1/36.$$

Thus,

$$\operatorname{var}(Z) = (1/18) - 2 * (1/36) + (1/18) = 1/18.$$

Problem 3. [6 pts] Compute $\mathbf{E}[\operatorname{var}(Y|X)]$ when (X,Y) are jointly distributed with

$$f_{X,Y}(x,y) = \frac{2}{x\sqrt{2\pi}}e^{-x^2/2}, \quad 0 \le y \le x < \infty.$$

(A numerical answer is required.)

$$\boxed{\mathbf{E}[\operatorname{var}(Y|X)] = \tfrac{1}{12}}$$

Solution. We first note that

$$f_X(x) = \int_0^x \frac{2}{x\sqrt{2\pi}}e^{-x^2/2}dy = \frac{2}{\sqrt{2\pi}}e^{-x^2/2}.$$

Thus,

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{1}{x}, \quad 0 \le y \le x.$$

Thus, the conditional distribution of Y given $X = x$ is uniform on $[0,x]$. This means

$$\operatorname{var}(Y|X=x) = \frac{x^2}{12},$$

or

$$\operatorname{var}(Y|X) = \frac{X^2}{12},$$

and

$$\mathbf{E}[\operatorname{var}(Y|X)] = \mathbf{E}[\frac{X^2}{12}].$$

Now note that

$$\begin{aligned}\mathbf{E}[X^2] &= \int_0^\infty x^2 f_X(x)\,dx \\ &= \int_0^\infty x^2 \frac{2}{\sqrt{2\pi}} e^{-x^2/2}\,dx \\ &= \int_{-\infty}^\infty x^2 \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx \\ &= 1.\end{aligned}$$

In the last step, we noted that the integral equals the variance of $N(0,1)$. Thus,

$$\mathbf{E}[\mathrm{var}(Y|X)] = \frac{1}{12}.$$

Problem 4. [6 pts] In an optical communication system, a light pulse consisting of a number of photons is sent. Assume that individual photon energies are i.i.d. random variables $X_1, X_2, \ldots$ with an exponential density $\mu e^{-\mu x}$, $x \geq 0$. Suppose the number of photons reaching the receiver is a Poisson random variable N with $p_N(n) = (\lambda^n/n!)e^{-\lambda}$, $n = 0, 1, \ldots$. Assume that N and $X_1, X_2, \ldots$ are jointly independent. Let $Z = X_1 + X_2 + \cdots + X_N$ denote the total photon energy reaching the receiver. Compute $\mathbf{E}[Z]$ as a function of the parameters λ and μ (both strictly positive constants).

$\mathbf{E}[Z] = \lambda/\mu$

Solution. Use the law of iterated expectations to write

$$\mathbf{E}[Z] = \mathbf{E}\left[\mathbf{E}[Z|N]\right] = \mathbf{E}[\frac{1}{\mu}N] = \lambda/\mu.$$

(Recall that the mean of an exponential random variable is μ^{-1}.

Problem 5. [6 pts] Let $X_1, X_2, \ldots$ be i.i.d. uniform on $[-1, 1]$. Let $Y_n = X_1^2 + X_2^2 + \cdots + X_n^2$. Approximate the probability $P(Y_{180} > 65)$ using the table of the CDF Φ of $N(0,1)$ on the next page. (The answer must be a numerical value.)

$P(Y_{180} > 65) \approx 0.1056$

Solution. First compute the mean and variance of Y_n. For the mean, we have

$$\mathbf{E}[Y_n] = n\mathbf{E}[X_1^2] = n\,\mathrm{var}(X_1) = n(2^2/12) = n/3$$

where we used the fact that the variance of a uniform random variable equals the square of the length of the interval divided by twelve. For the variance, we have

$$\mathrm{var}[Y_n] = n\,\mathrm{var}(X_1^2) = \mathbf{E}[X_1^4] - (\mathbf{E}[X_1^2])^2,$$

where

$$\mathbf{E}[X_1^4] = \int_{-1}^1 x^4(1/2)dx = x^5/10\big|_{-1}^1 = 1/5.$$

So,

$$\mathrm{var}(X_1^2) = (1/5) - (1/3)^2 = 4/45,$$

and

$$\mathrm{var}(Y_n) = 4n/45.$$

For $n = 180$, we have

$$\mathbf{E}[Y_{180}] = 60, \quad \mathrm{var}(Y_{180}) = 16.$$

We can now approximate the probability in question as follows.

$$P(Y_{180} \geq 65) = P\left(\frac{Y_{180} - 60}{\sqrt{16}} \geq \frac{65 - 60}{\sqrt{16}} = 1.25\right) \approx 1 - \Phi(1.25) = 1 - 0.8944 = 0.1056$$