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| **Math255: Probability and Statistics** | **Bilkent University** |
| | **Spring 2014-2015** |

**Midterm 2 Answers**
**16 April, 2015**

**P1.** (5 points) Ali arrives at the Bilkent bus stop to take a bus to Tunus. The chances that he finds a bus waiting upon arrival is 50%. If there is no bus, his waiting time is uniformly distributed between 0 and 10 mins. Let $X$ denote Ali's waiting time (in minutes) from the time of arrival at the bus stop until a bus arrival (if there is a bus waiting $X = 0$).

**(a)** (3 pt) Determine and sketch the CDF of $X$.

$$F_X(x) = \begin{cases} 0, & x < 0, \\ \frac{1}{2}, & x = 0, \\ \frac{1}{2} + \frac{1}{20}x, & 0 < x < 10, \\ 1, & x \geq 10 \end{cases}$$

**(b)** (2 pt) Determine the expectation of $X$.

$E[X] = 2.5$ min.

**P2.** (8 points) Let $X$ be a random variable with PDF

$$f_X(x) = \begin{cases} x^{-1}, & 1 \leq x \leq e, \\ 0, & \text{otherwise} \end{cases}$$

where $e \approx 2.718$ is the base of the natural logarithm.

**(a)** (2 pt) Let $A = \{X \geq \sqrt{e}\}$. Compute $P(A)$.

$$P(A) = \int_{\sqrt{e}}^{e} x^{-1}\, \mathrm{d}x = \ln(x)\Big|_{\sqrt{e}}^{e} = \ln(e) - \ln(\sqrt{e}) = 1 - \tfrac{1}{2} = \tfrac{1}{2}.$$

$P(A) = \frac{1}{2}$

**(b)** (2 pt) Compute $f_{X|A}(x)$ as a function of $x$.

$$f_{X|A}(x) = \begin{cases} 2x^{-1}, & \sqrt{e} \leq x \leq e, \\ 0, & \text{otherwise} \end{cases}$$

**(c)** (2 pt) Compute $E[X|A]$. (Leave the result as a function of $e$.)

$$E[X|A] = \int_{-\infty}^{\infty} x f_{X|A}(x);\mathrm{d}x = \int_{\sqrt{e}}^{e} x\,(2x^{-1})\,\mathrm{d}x = 2(e - \sqrt{e}).$$

$$E[X|A] = 2(e - \sqrt{e})$$

**(d)** (2 pt) Let $Y = X^2$. Calculate $E[Y]$. (Leave the result as a function of $e$.)

$$E[Y] = \int_{-\infty}^{\infty} x^2 f_X(x); \mathrm{d}x = \int_1^e x \ \mathrm{d}x = \frac{1}{2}(e^2 - 1).$$

$$E[Y] = \tfrac{1}{2}(e^2 - 1)$$

**P3.** (10 points)

Let $(X, Y)$ be jointly distributed random variables with a joint PDF

$$f_{X,Y}(x,y) = \begin{cases} xy, & 0 < x < 1, 0 < y < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

**(a)** (4 pt) Find the marginal PDF $f_X(x)$ and $f_Y(y)$.

$$f_X(x) = \begin{cases} 2x, & 0 < x < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{1}{2}y, & 0 < y < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

**(b)** (4 pt) Let $Z = X + Y$. Compute the CDF and PDF of $Z$.

$$F_Z(z) = \begin{cases} 0, & x \leq 0, \\ \frac{z^4}{24}, & 0 < z \leq 1, \\ \frac{z^2}{4} - \frac{z}{3} + \frac{1}{8}, & 1 < z \leq 2 \\ -\frac{z^4}{24} + \frac{5z^2}{4} - 3z + \frac{17}{8}, & 2 < z \leq 3 \\ 1, & z > 3. \end{cases}$$

$$f_Z(z) = \begin{cases} \frac{z^3}{6}, & 0 < z \leq 1, \\ \frac{z}{2} - \frac{1}{3}, & 1 < z \leq 2 \\ -\frac{z^3}{6} + \frac{5z}{2} - 3, & 2 < z \leq 3 \\ 0, & \text{otherwise.} \end{cases}$$

**(c)** (2 pt) Compute $E[2X + 3Y + XY]$. The result must be a numerical value. Do not leave it as an integral.

$$E(2X + 3Y + XY) = \frac{56}{9}$$

**P4.** (8 points)

Let $X, Y, Z$ be independent identically-distributed (i.i.d.) random variables drawn from the uniform distribution on $[0, 1]$. Compute $P(A)$ where $A$ is defined as the event

$$A = \{|X - Y| > a\} \cap \{|X - Z| > a\} \cap \{|Z - Y| > a\}.$$

Express $P(A)$ as a function of $a$. Note that $A$ is the event that the samples of $X$, $Y$, $Z$ are spaced at least $a$ apart from each other.

Let $B = \{Z \le Y \le X\}$.

$$P(A|B) = \int_{2a}^{1} dx \int_{a}^{x-a} dy \int_{0}^{y-a} dz = \frac{(1-2a)^6}{6}.$$

There are 6 permutations of $X, Y, Z$ and each has this probability. So, $P(A) = 6P(B|A) = (1-2a)^3$.

Note that the three $\{|X-Y| \ge a\}$, $\{|X-Z| \ge a\}$, $\{|Z-Y| \ge a\}$ are not independent. If you solved the problem by assuming that they were independent, you get the right answer by coincidence but the solution is not valid.

$\boxed{P(A) = (1-2a)^3}$