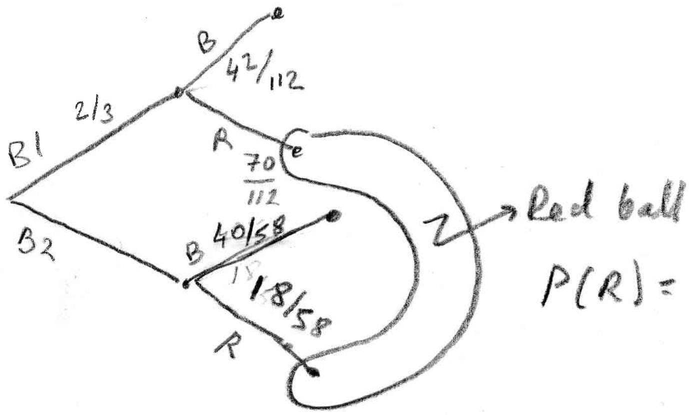
FINAL EXAM SOLUTIONS

1-) a)



$$P(R) = \frac{2}{3} \cdot \frac{70}{112} + \frac{1}{3}\,\frac{18}{58} = \boxed{\frac{181}{348}}$$

b) $$P(\text{Red ball comes from Box1}) = P(B_1|R) = \frac{P(B_1 \cap R)}{P(R)}$$

$$= \frac{\frac{2}{3} \cdot \frac{70}{112}}{\frac{2}{3} \cdot \frac{70}{112} + \frac{1}{3} \cdot \frac{18}{58}} = \boxed{\frac{145}{181}}$$

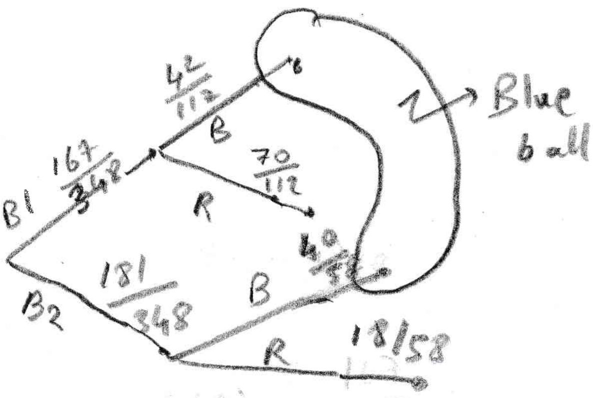
c) $$X = \begin{cases} 100 & 1-P(R) \\ -200 & P(R) \end{cases}$$

$$E\{X\} = 100\left(1-P(R)\right) - 200\,P(R) = 100 - 300\,P(R)$$

$$= 100 - 300 \cdot \frac{181}{348}$$

$$= \boxed{-\frac{6500}{116}\ TL}$$

d)



1st stage: $P(B) = 1 - P(R) = 1 - \frac{181}{348} = \frac{167}{348}$

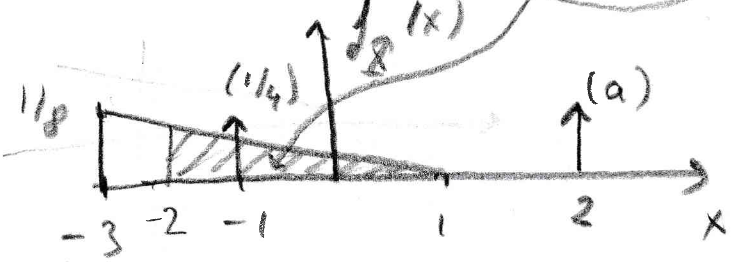
2nd stage: $$P(\text{Blue}) = \boxed{\frac{167}{348} \cdot \frac{42}{112} + \frac{181}{348} \cdot \frac{40}{58}}$$

2-) a) $\underbrace{\int_{-\infty}^{\infty} f_X(x)\,dx = 1}_{\text{Total area under } f_X(x).} \Rightarrow \underbrace{4\cdot\frac{1}{8}\cdot\frac{1}{2}}_{\text{Area of the triangle}} + \frac{1}{4} + a = 1$

$\frac{1}{4}$, a: area under the impulse

$\Rightarrow \boxed{a = \frac{1}{2}}$

b) $P\{-2 < X \le 1\} =$ This area.



Area of the shaded triangle: $\frac{3\cdot h}{2}$, $h = \frac{3}{4}\cdot\frac{1}{8}$

$= \frac{3\cdot \frac{3}{32}}{2} = \frac{9}{64}$

Total shaded area: $\frac{9}{64} + \frac{1}{4}$

$= \boxed{\frac{25}{64}}$ ($\frac{1}{4}$ from impulse)

c) $P\{-3 < X < 0.5 \mid X < 0\} = \frac{P\{-3 < X < 0.5 \text{ AND } X < 0\}}{P\{X < 0\}}$

Since $X < 0$, $(-3 < X < 0.5) \cap X < 0 = -3 < X < 0$

$= \frac{P\{-3 < X < 0\}}{P\{X < 0\}}$

$-3 < X < 0 \equiv X < 0$ since pdf is zero for $X < -3$

$= \frac{P\{X < 0\}}{P\{X < 0\}} = \boxed{1}$

3

3-) a)

i) $P_{X}(k) = \begin{cases} 0.6 & \text{if } k=1 \\ (0.4)\,0.7 = 0.28 & \text{if } k=2 \\ (0.4)(0.3)\,0.8 = 0.096 & \text{if } k=3 \\ (0.4)(0.3)(0.2)\,0.8^2 & \text{if } k=4 \\ (0.4)(0.3)(0.2)(1-0.8^2)\,0.8^3 & k=5 \\ \vdots & \\ 0.024 \cdot \prod_{i=5}^{k} (1-0.8^{i-3})\,0.8^{k-2} & k \geq 5 \end{cases}$

ii) $E\{X\} = \sum_{k=1}^{\infty} k\, P_X(k)$

$= \boxed{0.6 + 2(0.28) + 3(0.096) + 4(0.024)(0.64) + \sum_{k=5}^{\infty} k\,(0.024) \prod_{i=5}^{k} (1-0.8^{i-3})\, 0.8^{k-2}}$

4-) a) Throw the die n times independently. Count the occurances of each face : $k_1, k_2 \cdots k_6$ in those n trials.

$P(1) = \frac{k_1}{n}$, $P(2) = \frac{k_2}{n}$, ... $\boxed{P(i) = \frac{k_i}{n} \quad i=1,\cdots 6.}$

b) For each i, define a binary random variable $X_{ij} = \begin{cases} 1 & \text{if die}=i \\ 0 & \text{else} \end{cases}$ $j=1 \cdots n$ → trial number

Then $\underbrace{\frac{1}{n}\sum_{j=1}^{n} X_{ij}}_{\text{Sample mean}} = \frac{k_i}{n}$

Let $\underbrace{M_{n,i}}_{\text{sample mean}} = \frac{1}{n} \sum_{j=1}^{n} X_{ij}$

* $\underbrace{E\{M_{n,i}\} =}_{E\{\cdot\}\text{ is linear}} \frac{1}{n} \sum_{j=1}^{n} E\{X_{ij}\} = \frac{1}{\cancel{n}} \cancel{n} \cdot p_i = P(i)$

So: expected value of the estimate = true probability : Unbiased estimator "good"

* $Var\{M_{n,i}\} = n \cdot \frac{Var\{X_{ij}\}}{n^2} = \frac{p_i(1-p_i)}{n}$

$\Downarrow$

Variance of the estimate gets smaller as n increases.

$\Rightarrow$ Large n gives low variance : "good".

Note that as $n \to \infty$, variance $\to 0$
(Strong law of large numbers)

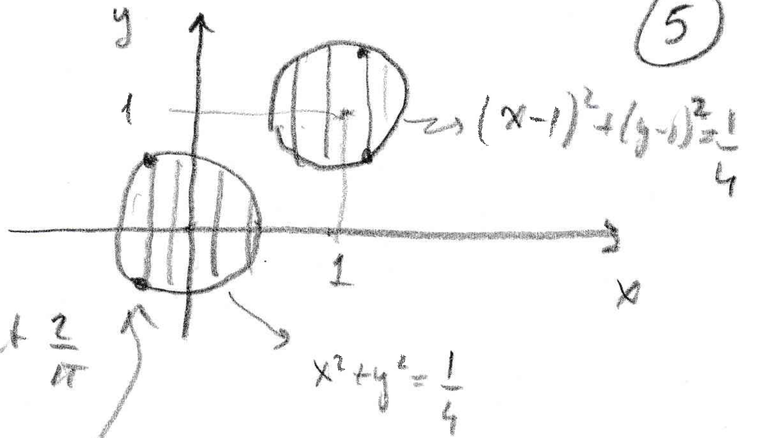
5-) $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$

Total area of $B = 2 \cdot \pi \left(\frac{1}{2}\right)^2 = \frac{\pi}{2}$

$\therefore f_{X,Y}(x,y) = \begin{cases} \frac{2}{\pi} & \text{if } (x,y) \in B \\ 0 & \text{else} \end{cases}$

$\longrightarrow$

$$f_{\underline{X}}(x) = \int_{-\infty}^{\infty} f_{\underline{X},Y}(x,y)\,dy =$$

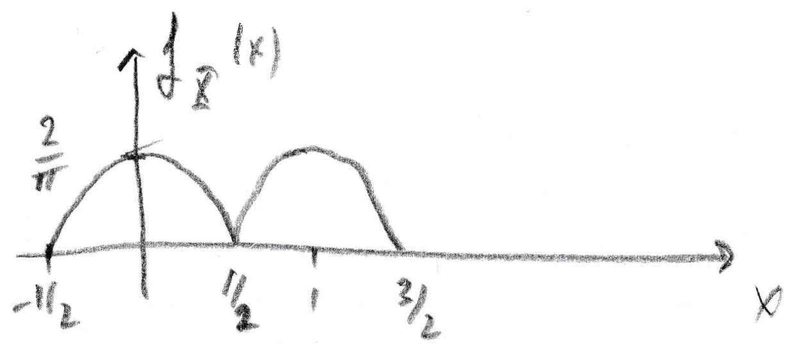


= integral of constant $\frac{2}{\pi}$ along those lines.

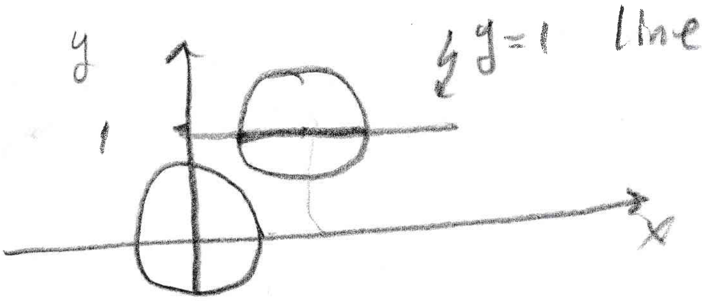
$$= \int_{-\sqrt{\frac{1}{4}-x^2}}^{\sqrt{1/4 - x^2}} \frac{2}{\pi}\,dy = 2\cdot\sqrt{1/4 - x^2}\cdot\frac{2}{\pi} \quad \text{for } x\in[-1/2, 1/2]$$

$$= \int_{1-\sqrt{\frac{1}{4}-(x-1)^2}}^{1+\sqrt{1/4-(x-1)^2}} \frac{2}{\pi}\,dy = 2\sqrt{1/4-(x-1)^2}\cdot\frac{2}{\pi} \quad \text{for } x\in[1/2, 3/2]$$

$$= 0 \qquad \text{for } x \notin [-1/2, 3/2]$$



b)

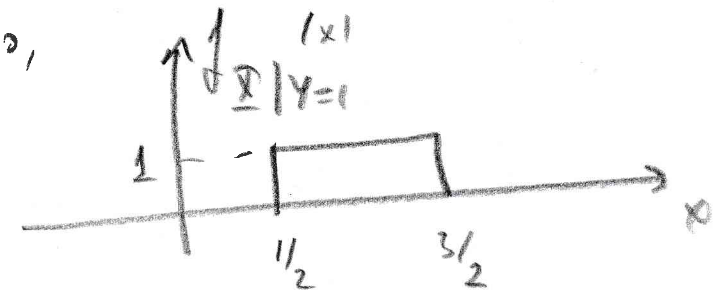


Since $Y=1$ is given, as we move along the $y=1$ line above, we see that

$$f_{X|Y=1}(x) = \begin{cases} \text{constant} & \text{if } x \in [1/2, 3/2] \\ 0 & \text{else} \end{cases}$$

since $\int_{-\infty}^{\infty} f_{X|Y=1}(x)\,dx = 1$, we find that constant $=1$

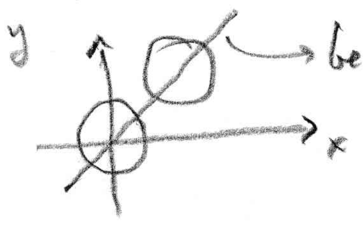
So,



$$f_{X|Y=1}(x) = \begin{cases} 1 & \text{if } x \in [1/2, 3/2] \\ 0 & \text{else} \end{cases}$$

c) From results of (a) and (b)

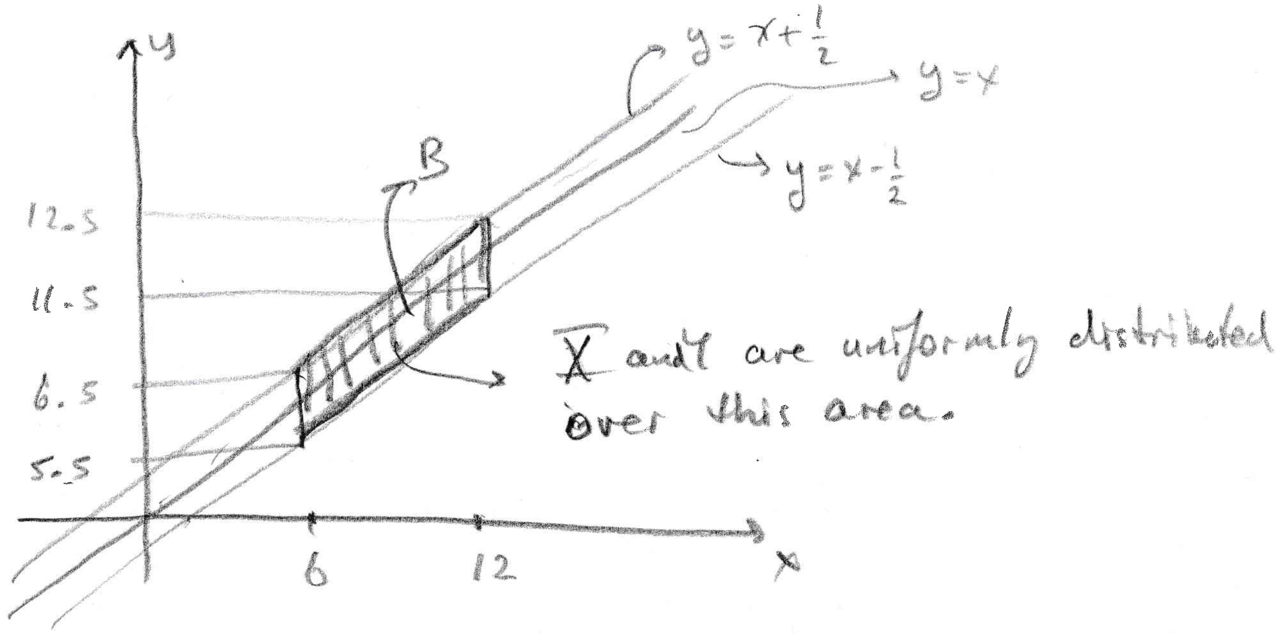
$f_X(x) \neq f_{X|Y=1}(x) \Rightarrow$ X & Y are <u>not</u> independent.



best fitting line.

Since there is a best fitting line whose slope $=1 \Rightarrow$ correlated.

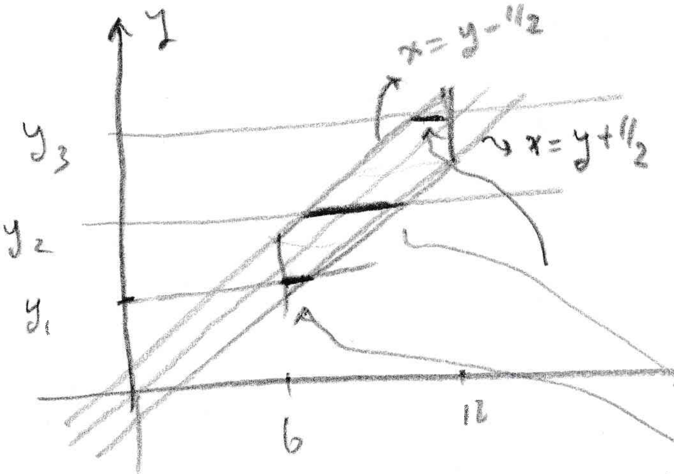
6-) a) $Y = X + N$, N uniform in $[-\frac{1}{2}, \frac{1}{2}]$, X uniform in $[6, 12]$ } So, Y is uniform in $[x-\frac{1}{2}, x+\frac{1}{2}]$ for each $X = x$.



Area of the parallelogram above: $6 \cdot 1 = 6$

So $$f_{X,Y}(x,y) = \begin{cases} \frac{1}{6} & \text{if } (x,y) \in B \\ 0 & \text{else} \end{cases}$$

b)



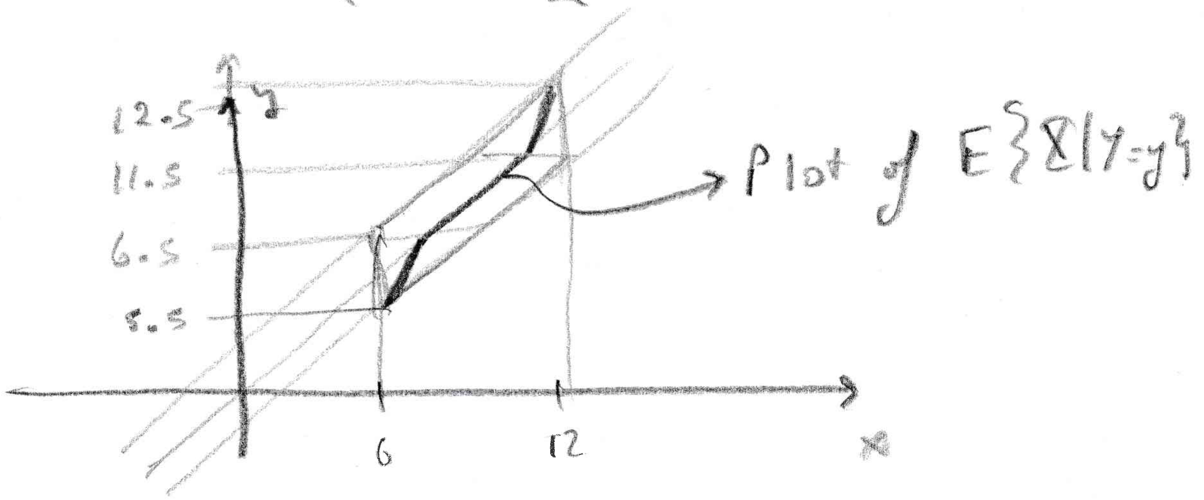
For each $Y = y_i$, X is uniformly distributed within the boundaries: examples

So: $$f_{X|Y=y_i}(x) = \begin{cases} \dfrac{1}{(y+\frac{1}{2})-6} & \text{if } x \in [6, y+1/2],\ y \in [5.5, 6.5] \\ 1 & \text{if } x \in [y-\frac{1}{2}, y+\frac{1}{2}],\ y \in [6.5, 11.5] \\ \dfrac{1}{12-(y-\frac{1}{2})} & \text{if } x \in [y-1/2, 12],\ y \in [11.5, 12.5] \end{cases}$$

length of non-zero (constant) range of pdf

Since for each $Y = y_i$, the conditional pdf $f_{X|Y=y_i}(x)$ is uniform within the given boundaries, $E\{X|Y=y_i\}$ is the mid-point of that uniform range:

$$E\{X|Y=y_i\} = \begin{cases} \dfrac{(y+\frac{1}{2})+6}{2} & \text{if } y \in [5.5, 6.5] \\ y & \text{if } y \in [6.5, 11.5] \\ \dfrac{(y-\frac{1}{2})+12}{2} & \text{if } y \in [11.5, 12.5] \end{cases}$$



c) $\underset{\hat{X}}{\arg\min}\, E\{(X-\hat{X})^2|Y=y\}$: $\dfrac{d}{d\hat{X}}$

$$-2E\{(X-\hat{X})|Y=y\} = 0$$

$$\hat{X}|Y=y = E\{X|Y=y\}$$

So, the answer as in (b); plot is the same as (b).