Bilkent University
Spring 2007-08
Math 250 Probability
Midterm II - 22/4/2008
Solutions

1. [10 pts] Let X be a r.v. with a p.d.f.

$$f_X(x) = \begin{cases} c & \text{if } |x| \le 1 \\ 2c & \text{if } 1 < |x| \le 2 \end{cases}$$

where c is a constant.

(a) [2] Determine the constant c so that the given function is a legitimate p.d.f.
Solution: We have

$$1 = \int_{-\infty}^{\infty} f_X(x)dx = \int_{-2}^{-1} 2cdx + \int_{-1}^{+1} cdx + \int_{1}^{2} 2cdx = 6c$$

So, $c = 1/6$.

(b) [2] Compute the expectation $E(X)$ and variance $\text{Var}(X)$.
Solution: Since $xf_X(x)$ is an odd function

$$E(X) = \int_{-\infty}^{\infty} xf_X(x)dx = 0.$$

For the variance, we have

$$\text{Var}(X) = E(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x)dx \quad = \int_{-2}^{-1} x^2/3dx + \int_{-1}^{+1} x^2/6dx + \int_{1}^{2} x^2/3dx = \frac{5}{3}.$$

(c) [2] Compute the expectation $E(X|X > 0)$.
Solution: By definition, we have

$$E(X|X > 0) = \frac{1}{P(X > 0)} \int_{X>0} xf_X(x)dx.$$

We have $P(X > 0) = 1/2$ by symmetry of the p.d.f. f_X around 0. We also have

$$\int_{X>0} xf_X(x)dx = \int_0^1 \frac{x}{6}dx + \int_1^2 \frac{2x}{6}dx = \frac{7}{12}.$$

So,

$$E(X|X > 0) = \frac{7}{6}.$$

(d) [2] Let $Y = X^2$. Determine the p.d.f. $f_Y(y)$ at the point $y = 2.25$.
Solution: We have $Y_g(X)$ with $g(x) = x^2$. Note that $g'(x) = 2x$.

$$f_Y(y) = \frac{f_X(\sqrt{y})}{|2\sqrt{y}|} + \frac{f_X(-\sqrt{y})}{|2\sqrt{y}|}$$

$$f_Y(2.25) = \frac{f_X(1.5)}{2 \cdot 1.5} + \frac{f_X(-1.5)}{2 \cdot 1.5} = \frac{2}{9}.$$

(e) [2] Give the joint p.d.f $f_{X,Y}(x, y)$. (You may use impulse functions if necessary.)
Solution:

$$f_{X,Y}(x, y) = f_X(x)\delta(y - x^2)$$

2. [10 pts] Let (X, Y) be jointly distributed r.v.'s with the p.d.f.

$$f_{X,Y}(x, y) = \begin{cases} 1/2 & \text{if } |x| + |y| \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) [2] Compute the probability of event $E = \{X \ge 1/2\}$ by integrating the joint p.d.f. over E. Sketch the event E in the $x - y$ plane. Show the limits of integration clearly.
Solution:

$$\begin{aligned} P(E) &= \int_{0.5}^{1} \int_{-(1-x)}^{1-x} \frac{1}{2} dy dx \\ &= \int_{0.5}^{1} (1 - x) dx \\ &= \frac{1}{8}. \end{aligned}$$

(b) [2] Determine the conditional p.d.f. $f_{X|Y}(x|y)$ for all x, y. Are X and Y independent. Justify your answer fully.
Solution: First compute $f_Y(y)$ as follows.

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x, y) dx \\ &= \begin{cases} 0 & \text{if } y > 1 \\ \int_{-(1-y)}^{1-y} \frac{1}{2} dx & \text{if } 0 \le y \le 1 \\ \int_{-(1+y)}^{1+y} \frac{1}{2} dx & \text{if } -1 \le y \le 0 \\ 0 & \text{if } y \le -1 \end{cases} \\ &= \begin{cases} 0 & \text{if } y > 1 \\ (1-y) & \text{if } 0 \le y \le 1 \\ (1+y) & \text{if } -1 \le y \le 0 \\ 0 & \text{if } y \le -1 \end{cases} \\ &= \begin{cases} (1 - |y|) & \text{if } |y| \le 1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Now,

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f_{X,Y}}{f_Y(y)} \\ &= \begin{cases} \frac{1}{2(1-|y|)} & \text{if } |x| + |y| \le 1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

No. $f_X(0.75) > 0$ but $f_{X|Y}(0.75|0.75) = 0$. Showing that we do not have $f_X(x) = f_{X|Y}(x|y)$ for all x, y, as it would be for independent random variables.

(c) [2] Compute $E[X|Y = y]$ for all $|y| \le 1$.
Solution: Since the conditional p.d.f. $f_{X|Y}(x|y)$ is symmetric around 0, the conditional expectation $E(X|Y = y) = 0$ for all y.

(d) [2] Compute $Var(X|Y = y)$ for all $|y| \le 1$.

Solution:

$$\begin{aligned}
\mathrm{Var}(X|Y=y) &= E(X^2|Y=y) \\
&= \int_{-(1-|y|}^{1-|y|} x^2 \frac{1}{2(1-|y|)} dx \\
&= \frac{(1-|y|)^2}{3}
\end{aligned}$$

(e) [2] Compute $E[Var(X|Y)]$.
Solution:

$$\begin{aligned}
E[\mathrm{Var}(X|Y)] &= E\left[\frac{(1-|Y|)^2}{3}\right] \\
&= \int_{-1} 1 \frac{(1-|y|)^3}{3} dy \\
&= 2\int_{0} 1 \frac{(1-y)^3}{3} dy \\
&= \frac{1}{6}
\end{aligned}$$

3. [10 pts] Let $\begin{bmatrix} X \\ Y \end{bmatrix} \sim N\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 & 0.5 \\ 0.5 & 2 \end{bmatrix}\right)$.

(a) [5] Let $\hat{X}_L(Y)$ denote the linear minimum mean-squared estimator (LMMSE) of X given Y. Determine $\hat{X}_L(y)$ as a function of y. Compute the resulting cost $E[(X - X_L(Y))^2]$. (A numerical value if required.)
Solution: We have

$$\begin{aligned}
\hat{X}_L(y) &= E[X] + \frac{\mathrm{Cov}(X,Y)}{\mathrm{Var}(Y)}(y - E(Y)) \\
&= 1 + \frac{0.5}{2}(y-2) \\
&= \frac{y+2}{4}
\end{aligned}$$

The resulting MSE is $(1-\rho^2)\mathrm{Var}(X) = \left[1 - \left(\frac{0.5}{\sqrt{2}}\right)^2\right] \cdot 1 = \frac{7}{8}$.

(b) [5] Let $Z = X - cY$ where $c \neq 0$ is an arbitrary constant. What is the joint pdf of (X, Z)? Specify the mean vector and covariance matrix of (X, Z). For which choice of c do we have $\mathrm{Cov}(X, Z) = 0$?
Solution:

$$E(Z) = E(X) - cE(Y) = 1 - 2c$$

$$\begin{aligned}
\mathrm{Var}(Z) &= \mathrm{Var}(X) + c^2\mathrm{Var}(Y) - 2c\,\mathrm{Cov}(X,Y) \\
&= 1 + 2c^2 - c
\end{aligned}$$

$$\begin{aligned}
\mathrm{Cov}(X,Z) &= \mathrm{Cov}(X, X - cY) \\
&= \mathrm{Var}(X) - c\,\mathrm{Cov}(X,Y) \\
&= 1 - c/2
\end{aligned}$$

$$\begin{bmatrix} X \\ Z \end{bmatrix} \sim N\left(\begin{bmatrix} 1 \\ 1-2c \end{bmatrix}, \begin{bmatrix} 1 & 1-c/2 \\ 1-c/2 & 1-c+2c^2 \end{bmatrix}\right)$$

For $c = 2$, we have $\mathrm{Cov}(X, Z) = 0$.

Midterm statistics: $N = 88$, $\mu = 8.89$, $\sigma = 5.90$.