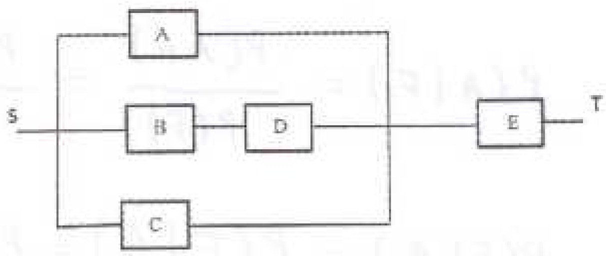
Math 255, Fall 2018
Midterm 1 - Solutions

Problem 1. [10 pts]

An electrical system (shown on the right) consists of 5 components, each of which is operational with probability $p = \frac{1}{2}$. Assume that the states of the components (whether operational or not) are jointly independent. The system is operational (call this event F) if there is a path from S to T such that all components on the path are operational. Let A be the event that component A is operational. Compute the conditional probability $P(A|F)$. Show your work. Give the result as a real number or a simple fraction.



$P(A|F) = 8/13$

Do not write in this space.

Problem 2. [10 pts] Consider tossing a fair six-sided die m times. Let X be the number of distinct faces that appear. (For example, if $m = 10$ and the outcomes are 2,3,3,5,6,3,2,6,2,4, then $X = 5$ since all outcomes except 1 appear at least once.) Give a **simple** formula for computing $E(X)$ as a function of m.

$E(X) = 6\left(1-\left(\frac{5}{6}\right)^m\right)$

Do not write in this space.

Problem 3. [10 pts] You toss a fair six-sided die twice. Let X_1, X_2 denote the outcomes, let $Y = X_1 + X_2$. Compute $\mathrm{var}(X_1|Y = 8)$, the conditional variance of the first outcome, given that the sum of the two outcomes is 8.

$\mathrm{var}(X_1|Y = 8) = 2$

Do not write in this space.

Solution of Problem 1 **only**.

$$P(A|F) = \frac{P(AF)}{P(F)} = \frac{P(F|A)\,P(A)}{P(F)}$$

(1) $\quad P(F|A) = P(E|A) = P(E) = p = \frac{1}{2}$

(2) $\quad P(A) = p = \frac{1}{2}$

$$P(F) = P\big((A \cup BD \cup C)E\big)$$
$$= P(A \cup BD \cup C)\,P(E)$$

$$P(A \cup BD \cup C) = P(A) + P(BD) + P(C)$$
$$- P(ABD) - P(AC) - P(BCD)$$
$$+ P(ABCD)$$
$$= \cancel{p} + \cancel{p^2} + \cancel{p} - p^3 - \cancel{p^2} - p^3 + p^4$$
$$= 2p - 2p^3 + p^4 = 2 \cdot \frac{1}{2} - 2 \cdot \frac{1}{8} + \frac{1}{16}$$
$$= 1 - 1/4 + 1/16 = \frac{13}{16}$$

(3) $\quad P(F) = \frac{13}{16} \cdot P(E) = 13/32$

$$P(A|F) = \frac{\frac{1}{2} \cdot \frac{1}{2}}{\frac{13}{32}} = \frac{8}{13}$$

Solution of Problem 2 **only**.

Let $X_i = \begin{cases} 1 & \text{if outcome } i \text{ appears in any of the } m \text{ trials} \\ 0 & \text{o.w.} \end{cases}$

$$X = X_1 + X_2 + \cdots + X_6$$

$$\begin{aligned} E(X) &= E(X_1 + X_2 + \cdots + X_6) \\ &= E(X_1) + E(X_2) + \cdots + E(X_6) \end{aligned}$$

$$\begin{aligned} E(X_i) &= 1 \cdot P(X_i = 1) + \cancelto{0}{0 \cdot P(X_i = 0)} \\ &= 1 - \left(\frac{5}{6}\right)^m \end{aligned}$$

$$\therefore \quad E(X) = 6 \cdot \left[1 - \left(\frac{5}{6}\right)^m\right].$$

Solution of Problem 3 **only**.

$$p_{X_1|Y}(x_1|8) = \begin{cases} 1/5 & , \quad x_1 = 2,3,4,5,6 \\ 0 & , \quad x_1 = 1 \end{cases}$$

$$E[X_1 | Y=8] = \frac{1}{5} \cdot (2+3+4+5+6) = 4$$

$$Var(X_1) = E(X_1^2) - (E(X_1))^2$$

$$E(X_1^2) = \frac{1}{5}(2^2+3^2+4^2+5^2+6^2) = 18$$

$$Var(X_1) = 18 - 4^2 = \underline{\underline{2}}$$