Math255: Probability and Statistics	**Bilkent University**
	Spring 2014-2015

Midterm 1
12 March 2015, 19:30 - 21:10

100 minutes. Three problems. 30 points. Closed book. You may use one one-sided A4-size sheet of notes. In each problem, you **must** show your work in the space provided for that problem and write your final answer in the designated box. **You may receive no credit on correct answers if you do not show your work or do not write your answer in the box.** Good luck!

Name and Lastname:

Bilkent ID No:

Math 255 Section No:

Score (for instructor use)

	(a)	(b)	(c)	(d)	Sum
P1				—	
P2					
P3				—	
				Overall	

P1. (10 points)

Consider the following random experiment. There are 5 boxes: 2 boxes of type a with 2 white balls and 1 red ball each; 1 box of type b with 10 red balls each; 2 boxes of type c with 3 white balls and 1 red ball each. A box is selected at random and one ball is drawn from it at random.

(a) (3 pt) Use a sequential model for the experiment in which outcomes are represented as pairs (x, y) where $x \in \{a, b, c\}$ is the type of box selected and $y \in \{r, w\}$ is the color of ball drawn (r for *red*, w for *white*). Determine the probability law P for this experiment and fill in the following table with all outcomes and the corresponding probabilities.

First consider the probability $P(\{a, w\})$ of choosing a box of type a and a ball of type w. The probability of choosing a box of type a is 2/5. The conditional probability of choosing a type w ball given that a box of type a is chosen is 2/3. So, $P(\{(a, w)\}) = (2/5)(2/3) = 4/15$. Using similar reasoning, we obtain the following entries for the table of probabilities.

Outcome (x, y)	$P(\{(x, y)\})$
(a, w)	4/15
(a, r)	2/15
(b, w)	0
(b, r)	1/5
(c, w)	6/20
(c, r)	2/20

As a check, we sum the probabilities and find that the sum is 1.

(b) (4 pt) Let W denote the event that the chosen ball is white. Compute the probability of event W **by using the law of total probability**. (Define other events as necessary.) Show your work in detail.

A direct solution is obtained by writing W in terms of its elements as $W = \{(a, w), (b, w), (c, w)\}$ and then using the table of part (a) to obtain $P(W) = (4/15) + 0 + (6/20) = 17/30$. This solution effectively uses the law of total probability since the entries in the table were obtained by conditioning. For a more explicit solution in terms of the law of total probability,define three events: A as the event that the chosen box is of type a; B as the box being of type b; and, C as the box being of type c. Then,

$$\begin{aligned} P(W) &= P(A)P(W|A) + P(B)P(W|B) + P(C)P(W|C) \\ &= (2/5)(2/3) + (1/5)(0/10) + (2/5)(3/4) = 17/30. \end{aligned}$$

$\boxed{P(W) = 17/30}$

(c) (3 pt) Compute the conditional probability that the box chosen is of type a (call this event A) given that the ball drawn is white (event W). Show your work in detail.

We use the Bayes' rule and the result of part (b):

$$\begin{aligned} P(A|W) &= \frac{P(W|A)P(A)}{P(W)} \\ &= \frac{(2/3)(2/5)}{(17/30)} = 8/17. \end{aligned}$$

$$\boxed{P(A|W) = 8/17}$$

P2. (10 points) Let (X, Y) be a pair of random variables with the joint PMF $p_{X,Y}(x, y)$ shown in the table below. Possible values of X are shown on the x-axis, those of Y on the y-axis.

y \ x	1	2	3	4
4	0.1	0	0.05	0.1
3	0.05	0	0.1	0.1
2	0	0.1	0.1	0.05
1	0.1	0.1	0.05	0

(a) (3 pt) Compute the probability of the event $A = \{X > Y\}$.

The table above already defines the probability model (Ω, P). The sample space Ω consists of pairs of integers (i, j) with $1 \leq i, j \leq 4$. The probability law P is as given in the table for each outcome. So, to find $P(A)$ all we need is to identify $A = \{X > Y\}$ as a subset of Ω. Clearly, $A = \{(2, 1), (3, 1), (4, 1), (3, 2), (4, 2), (4, 3)\}$. Hence,

$$P(A) = \sum_{(i,j)\in A} P(\{(i, j)\}) = 0.1 + 0.05 + 0 + 0.1 + 0.05 + 0.1 = 0.4.$$

$$\boxed{P(A) = 0.4}$$

(b) (2 pt) Compute the conditional PMF $p_{X|Y}(x|1)$ for each x.

$$p_{X|Y}(x|1) = \frac{p_{X,Y}(x, 1)}{p_Y(1)}.$$

We calculate $p_Y(1)$ as

$$p_Y(1) = \sum_{x'} p_{X,Y}(x', 1) = 0.1 + 0.1 + 0.05 + 0 = 0.4,$$

and obtain from the preceding formula the desired probabilities:

$$\boxed{p_{X|Y}(x|1) = \begin{cases} 0.4, & x = 1; \\ 0.4, & x = 2; \\ 0.2, & x = 3; \\ 0, & x = 4. \end{cases}}$$

(c) (3 pt) Compute the expectation of $Z = 2X + 3Y$.

There are three ways of solving this problem. We may use the formula

$$\mathbf{E}[Z] = \sum_z z p_Z(z).$$

This is not the right approach here.

The second method is to use the formula

$$\mathbf{E}[Z] = \sum_{x,y} (2x + 3y) p_{X,Y}(x, y).$$

This is not the right approach, either.

The best method is to use the linearity and write

$$\mathbf{E}[Z] = 2\mathbf{E}[X] + 3\mathbf{E}[Y].$$

Then, calculate the marginals of X and Y as

$$p_X(x) = \begin{cases} 0.25, & x = 1; \\ 0.2, & x = 2; \\ 0.3, & x = 3; \\ 0.25, & x = 4, \end{cases}$$

and

$$p_Y(y) = \begin{cases} 0.25, & y = 1; \\ 0.25, & y = 2; \\ 0.25, & y = 3; \\ 0.25, & y = 4. \end{cases}$$

From these, we obtain $\mathbb{E}[X] = (0.25) + (2)(0.2) + (3)(0.3) + (0.25)(4) = 2.55$ and $\mathbf{E}[Y] = (1 + 2 + 3 + 4)(0.25) = 2.5$. This yields

$$\mathbf{E}[Z] = (2)(2.55) + (3)(2.5) = 12.6.$$

$\boxed{\mathbf{E}[Z] = 12.6}$

(d) (2 pt) Compute the expectation of $U = XY$.

Here, we use the regular method and compute

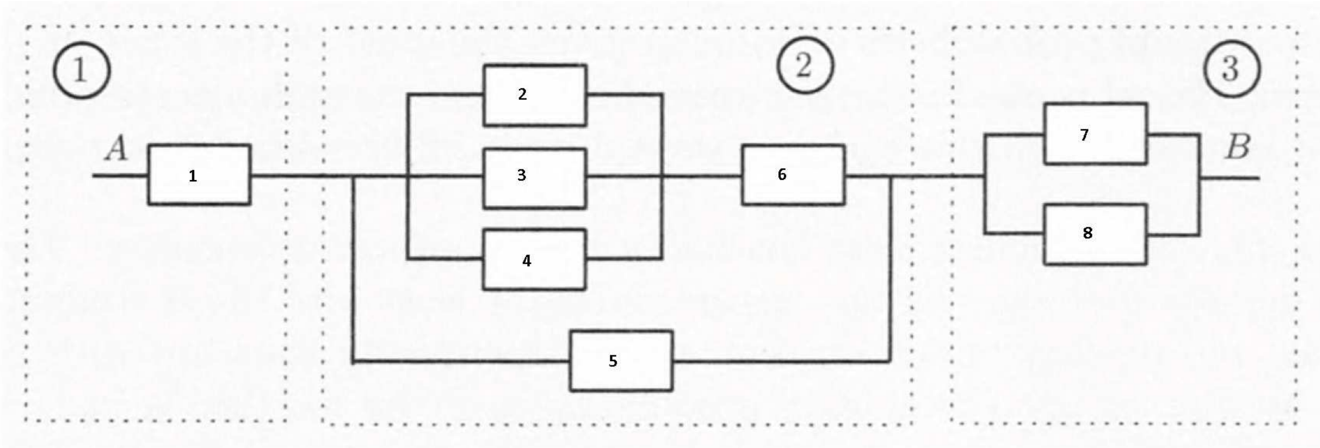
$$\mathbf{E}[U] = \sum_{x,y} xy p_{X,Y}(x, y).$$

Evaluating the sum, we obtain the result.

$\boxed{\mathbf{E}[U] = 6.7}$

P3. (10 points)

Consider the electrical system in the figure below, consisting of 8 identical components each of which is operational with probability p, independently of others. The system is operational if there is a path from A to B consisting of operational components. (In solving certain parts of the problem, it may be useful to observe that the system consists of three subsystems 1, 2, and 3. There is a path from A to B if and only if each subsystem is "operational".)



(a) (3 pt) Enumerate all possible paths (with no cycles, of course!) from A to B and write down the probability of that path being operational in the table below assuming that $p = 1/2$. Label the paths as (1,2,6,7), etc. Extend the table if necessary.

The probability that a path is operational is just $p^k = 1/2^k$ where k is the number of components along the path.

Path no	Path	Probability
1	(1,2,6,7)	1/16
2	(1,3,6,7)	1/16
3	(1,4,6,7)	1/16
4	(1,5,7)	1/8
5	(1,2,6,8)	1/16
6	(1,3,6,8)	1/16
7	(1,4,6,8)	1/16
8	(1,5,8)	1/8

(b) (4 pt) Let X be the number of operational paths between A and B. Compute $\mathbf{E}[X]$ for $p = 1/2$.

We may write $X = X_1 + X_2 + \cdots + X_8$ where X_i is the *indicator* function of path i being operational, *i.e.*,

$$X_i = \begin{cases} 1, & \text{if path } i \text{ is operational;} \\ 0, & \text{otherwise.} \end{cases}$$

By the linearity of expectation,

$$\mathbf{E}[X] = \mathbf{E}[X_1] + \mathbf{E}[X_2] + \cdots + \mathbf{E}[X_8].$$

We have $\mathbf{E}[X_i] = P(X_i = 1) = p^k$ where k is the length of path i. Thus, by adding the probabilities in the table in part (a), we obtain the result.

$$\boxed{\mathbf{E}[X] = 10/16.}$$

(c) (3 pt) Let $Y = 1$ if component 7 is operational; $Y = 0$ otherwise. Let E be the event that there exists an operational path from A to B. Compute the conditional PMF $p_{Y|E}(y)$ for $y = 0, 1$, assuming $p = 1/2$.

We may write

$$p_{Y|E}(y) = \frac{P(\{Y = 1\} \cap E)}{P(E)}.$$

The brute-force approach is to compute $P(E)$, which is not so simple. The point of the problem is that we can avoid computing $P(E)$ by making use of the independence properties of the components. Let us define E_1 as the event that there is an operational path in Section 1 (from A to the input of Section 2), E_2 as the event that there is an operational path across Section 2, and E_3 as the event that there is an operational path in Section 3, connecting the input of Section 3 to node B. We note that the events E_1, E_2, and E_3 are jointly independent since the occurrence of each each depends on a disjoint set of components and components fail independently. Since $E = E_1 \cap E_2 \cap E_3$, we have

$$P(E) = P(E_1)P(E_2)P(E_3).$$

Using independence, we can also write

$$P(\{Y = y\} \cap E) = P(\{Y = y\} \cap E_1 \cap E_2 \cap E_3) = P(\{Y = y\} \cap E_1)P(E_2)P(E_3).$$

This is justified by noting that the event $\{Y = y\} \cap E_3$ depends only on the components in Section 3, while E_1 and E_2 depend on the components in Sections 1 and 2. Substituting these back into the first equation above, we obtain

$$\begin{aligned} p_{Y|E}(y) &= \frac{P(\{Y = y\} \cap E_1)P(E_2)P(E_3)}{P(E_1)P(E_2)P(E_3)} \\ &= \frac{P(\{Y = y\} \cap E_3)}{P(E_3)} \\ &= \frac{P(\{Y = y\})P(E_3|\{Y = y\})}{P(E_3)}. \end{aligned}$$

The rest is a simple calculation. We observe that E_3 occurs if and only if component 7 or 8 is operational; so, $P(E_3) = 1 - (1 - p)^2 = 2p - p^2 = 3/4$. We also observe that

$$P(E_3|\{Y = y\}) = \begin{cases} 1, & \text{if } y = 1; \\ p = 1/2, & \text{if } y = 0. \end{cases}$$

Combining these, we obtain the answer.

$$\boxed{p_{Y|E}(y) = \begin{cases} \frac{2}{3}, & y = 1; \\ \frac{1}{3}, & y = 0. \end{cases}}$$