Bilkent University	Spring 2015-2016

Math255 Probability and Statistics
Midterm 1
29 February 2016, 18:30 - 20:30

120 minutes. Three problems. 30 points. Closed book. You may use one one-sided A4-size sheet of notes. In each problem, you **must** show your work in the space provided for that problem and write your final answer in the designated box. **You may receive no credit on correct answers if you do not show your work or do not write your answer in the box.** Good luck!

Name and Lastname: Uygar Uncuoglu

Bilkent ID No: 21201145

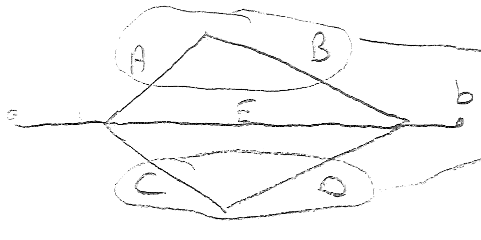
Score (for instructor use)

	(a)	(b)	(c)	(d)	Sum
P1	5	5	—	—	10
P2	5	5	—	—	10
P3	5	5	—	—	10
				Overall	30

Use this space to show your work for Problem 1 only.

$\boxed{p=\frac{1}{2}}$. $P(G) = P(F)P(G\setminus F) + P(F^c)P(G\setminus F^c)$

a) Lets start with the case that F is not operational (F^c). Calculate $P(G\setminus F^c)$



- A and B is serial, C and D is serial

$P(AB) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$

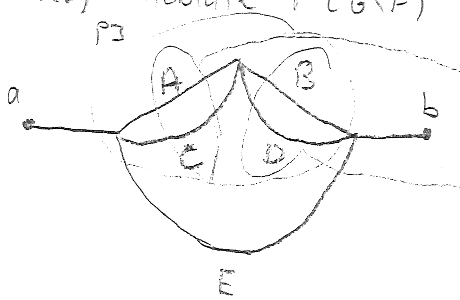
$P(CD) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$, $P(E) = \frac{1}{2}$

- AB, E, CD are parallel so if at least one of them is operational, the system is operational.

$P(G\setminus F^c) = 1-(1-AB)(1-E)(1-CD)$

$P(G\setminus F^c) = 1 - \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{3}{4} = 1-\frac{9}{32} = \boxed{\frac{23}{32}}$

Now, calculate $P(G\setminus F)$ by assuming F is operational.



- A and C is parallel and B and D is parallel.

$P_1 = 1-(1-A)(1-C) = 1-\frac{1}{2}\cdot\frac{1}{2} = \frac{3}{4}$

$P_2 = 1-(1-B)(1-D) = 1-\frac{1}{2}\cdot\frac{1}{2} = \frac{3}{4}$ //

- P_1 and P_2 are serial so

$P_3 = P_1\cdot P_2 = \frac{3}{4}\cdot\frac{3}{4} = \frac{9}{16}$ //

- P_3 and E are parallel

$P(G\setminus F) = 1-(1-P_3)(1-E) = 1-\frac{7}{16}\cdot\frac{1}{2} = 1-\frac{7}{32} = \frac{25}{32}$ //

So; $P(G) = P(F)P(G\setminus F) + P(F^c)P(G\setminus F^c)$

$= \frac{1}{2}\cdot\frac{25}{32} + \frac{1}{2}\cdot\frac{23}{32} = \frac{25}{64} + \frac{23}{64} = \boxed{\frac{48}{64}} = \boxed{\frac{3}{4}}$

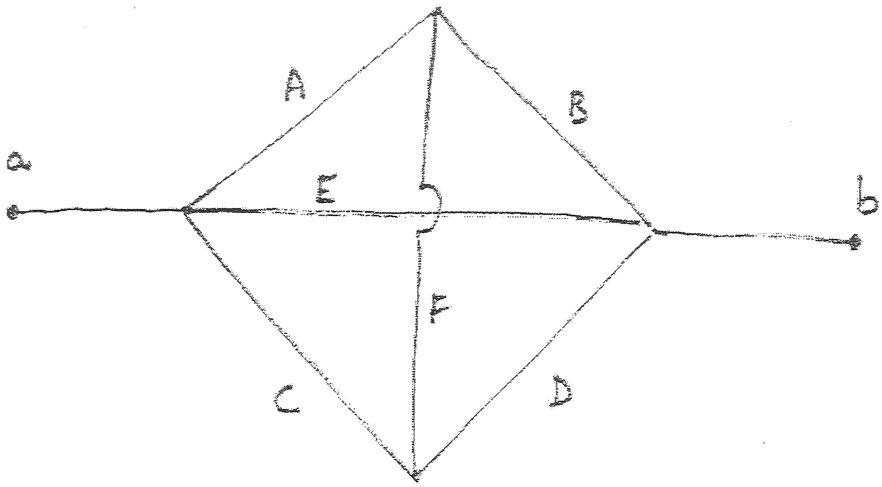
b) $P(F^c\setminus G) = \frac{P(F^c)P(G\setminus F^c)}{P(G)}$ using Baye's Rule

$P(F^c) = \frac{1}{2}$, $P(G) = \frac{3}{4}$ calculated at part a. $P(G\setminus F^c) = \frac{23}{32}$ calculated at part a

$P(F^c\setminus G) = \frac{\frac{1}{2}\cdot\frac{23}{32}}{\frac{3}{4}} = \frac{23}{\cancel{64}_{16}}\cdot\frac{\cancel{4}}{3} = \boxed{\frac{23}{48}}$

P1. (10 points)

Consider the electrical network shown in the figure below. The network consists of 6 links labeled $A, B, \ldots, F$. Assume that the state of each link is "conducting" with probability 1/2, or "non-conducting" with probability 1/2, independently of the states of the other links. We are interested in computing the probability that current flows from point a to point b.



To be more formal, let A denote the event that link A is conducting, let B be the event that link B is conducting, and define the events C, D, E, and F similarly. Let G denote the event that there is a conducting path from a to b. In other words, let $G = AB \cup E \cup CD \cup AFD \cup CFB$.

(a) (5 pt) Compute the probability of G. Show your work in detail. (Hint: Compute $P(G|F)$ and $P(G|F^c)$.)

$P(G) = \frac{48}{64} = \frac{3}{4}$ //

Leave this box empty

(b) (5 pt) Compute the conditional probability $P(F^c|G)$. Show your work in detail.

$P(F^c|G) = \frac{23}{48}$ //

Leave this box empty

Use this space to show your work for Problem 2 only.

a) 5 people chosen out of 9. 3 → born in January, 2 → born in June

p = among 5, 2 born in January, 1 born in June.

among 5 people 2 of them born in January → $\binom{3}{2}$ → 2 people chosen out of 3

1 of them born in June → $\binom{2}{1}$ → 1 people chosen

rest of 2 people will be chosen from remaining 4 people → $\binom{4}{2}$

- Different number of possibilities to choose people for given condition is

$\binom{3}{2}\binom{2}{1}\binom{4}{2} = 3, 2, 6 = 36$ //

- Number of all different possibilities is $= \binom{9}{5} = \frac{9!}{4!.5!} = 126$ //

So $p = \frac{36}{126} = \frac{4}{14} = \boxed{\frac{2}{7}}$

b) - Number of all arrangement is $= \boxed{52!}$

- For the desired case if we put ace of hearts and ace of diamonds adjacent to each other and count them 1. There will be 51! arrangement, However among these arrangements two aces may switch place between each other so the total arrangement possible is $\boxed{2 \times 51!}$

$q = \frac{2 . \cancel{51!}}{\cancel{52!}_{52}} = \boxed{\frac{2}{52}}$

P2. (10 points) The three parts of this problem are independent. Show your work on the facing page, write only the final answers in the boxes.

(a) (5 pt) Suppose 5 people are chosen at random from a group of 9 people of which 3 are born in January and 2 are born in June. Let p be the probability that, among the 5 people chosen, 2 are born in January and 1 is born in June. Compute p. Simplify your answer as much as possible.

$p = \boxed{\frac{2}{7}}$

[redacted]

(b) (5 pt) Find the probability q that in a random (linear) arrangement of an ordinary deck of 52 playing cards, the ace of hearts and the ace of diamonds are adjacent to each other.

$q = \boxed{\frac{2}{52}}$

[redacted]

Use this space to show your work for Problem 3 only.

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a) - Probability for each misprints to be on that specific page is $= \frac{1}{1000}$, So in order to have 3 of these misprints;

$$\binom{1000}{3}\left(\frac{1}{1000}\right)^3\left(\frac{999}{1000}\right)^{997}$$

$\binom{1000}{3}$: Choose 3 random misprint

$\left(\frac{1}{1000}\right)^3$: they will be on specific page

$\left(\frac{999}{1000}\right)^{997}$: remaining misprints should be on on other page.

- For Poisson approximation; $n = 1000$ $p = \frac{1}{1000}$, $k = 3$

$\lambda = n.p = \boxed{1}$

$$p = e^{-\lambda}\frac{\lambda^k}{k!} = e^{-1}\frac{1^3}{3!} = \frac{e^{-1}}{3!}$$

b) Town → N families → 100 TL per child child support up to 500 TL.

X → denote the number of children in the chosen family,

$$\sum_{k=0}^{\infty} p_X(k) = \sum_{k=0}^{\infty} a\left(\frac{3}{4}\right)^k = 1$$

$$= a\sum_{k=0}^{\infty}\left(\frac{3}{4}\right)^k = a\underbrace{\left(1+\frac{3}{4}+\left(\frac{3}{4}\right)^2+\cdots\right)}_{\text{since it is a geometric}} = 1$$

Serie $= \frac{1}{1-\frac{3}{4}} = 4 \Rightarrow 4a = 1$ $\boxed{a = \frac{1}{4}}$

$$p_Z(z) = \begin{cases} \frac{1}{4}\left(\frac{3}{4}\right)^{\frac{z}{100}} & \text{if } z = 0, 100, 200, 300, 400 \\ \sum_{k=5}^{\infty}\frac{1}{4}\left(\frac{3}{4}\right)^k = \left(\frac{3}{4}\right)^5 & \text{if } z = 500 \end{cases}$$

$$\frac{1}{4}\sum_{k=5}^{\infty}\left(\frac{3}{4}\right)^k = \frac{1}{4}\left(4-\left(1+\frac{3}{4}+\left(\frac{3}{4}\right)^2+\left(\frac{3}{4}\right)^3+\left(\frac{3}{4}\right)^4\right)\right)$$

$$= \frac{1}{4}\left(4 - \frac{4^5-3^5}{4^4}\right)$$

$$= 1 - \frac{4^5-3^5}{4^5}$$

$$= 1 - 1 + \left(\frac{3}{4}\right)^5 = \boxed{\left(\frac{3}{4}\right)^5}$$

$$E[Z] = 0.\frac{1}{4} + 100.\frac{1}{4}.\frac{3}{4} + 200.\frac{1}{4}\left(\frac{3}{4}\right)^2 + 300.\frac{1}{4}.\left(\frac{3}{4}\right)^3 + 400.\frac{1}{4}\left(\frac{3}{4}\right)^4 + 500.\left(\frac{3}{4}\right)^5$$

$$E[Z] \; 0 + \frac{300}{16} + \frac{1800}{64} + \frac{8100}{256} + \frac{32400}{1024} + \frac{121500}{1024}$$

$$E[Z] = 228,86$$

P3. (10 points)

The two parts of this problem are independent.

(a) (5 pt) A 1000-page book contains 1000 misprints (distributed independently at random throughout the book). Let p be the probability that a given page (say, page 10) contains exactly 3 misprints? Give an exact expression for p but do not try to simplify it.

$p = \binom{1000}{3}\left(\frac{1}{1000}\right)^3\left(\frac{999}{1000}\right)^{997}$ //

Leave this box empty

Use the Poisson approximation to compute p approximately.

$p \approx \boxed{\frac{e^{-1}}{3!}} = \frac{1}{e.3!}$ //

Leave this box empty

(b) (5 pt) Consider a town with N families. Suppose each family in the town receives child support of 100 TL per child up to a maximum of 500 TL (thus, a family with 3 children receives 300 TL, one with 7 children receives 500 TL). Suppose a family is chosen at random. Let X denote the number of children in the chosen family. Suppose X is modeled as a geometric random variable with $p_X(k) = \alpha(3/4)^k$, $k = 0, 1, 2, \ldots$ where α is some constant. Let Z denote the child support received by the family.

Determine the constant α.

$\alpha = \frac{1}{4}$ //

Leave this box empty

Compute the expectation of Z.

$E[Z] = 228,86$ //

Leave this box empty

$Z = 100 \cdot X$ if $X \leq 5$

$Z = 500$ if $X \geq 6$

$E[X] = \frac{1}{4} \cdot \frac{4}{3} = \boxed{\frac{1}{3}}$