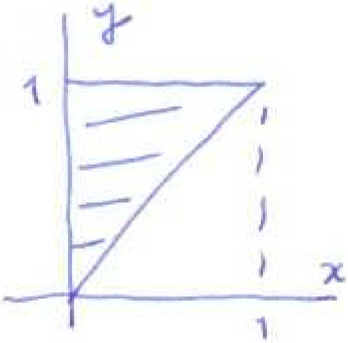
Math 285

Midterm 2 Solutions

1\.



$f_Z = c$ in the shaded region.

Area of shaded region $= \frac{1}{2}$

$c = 2$.

$$F_Z(z) = P(Z \le z) = P(2X \le z, Y \le z)$$
$$= P\left(X \le \frac{z}{2}, Y \le z\right).$$
$$= \int_{-\infty}^{z/2} dx \int_{-\infty}^{z} dy \cdot f_{X,Y}(x,y) = \int_0^{z/2} dx \int_0^{z} dy\, f_{X,Y}(x,y)$$

For $z < 0$, $f_{X,Y}(x,y) = 0$ in the range of integration,

so $F_Z(z) = 0$ for $z < 0$.

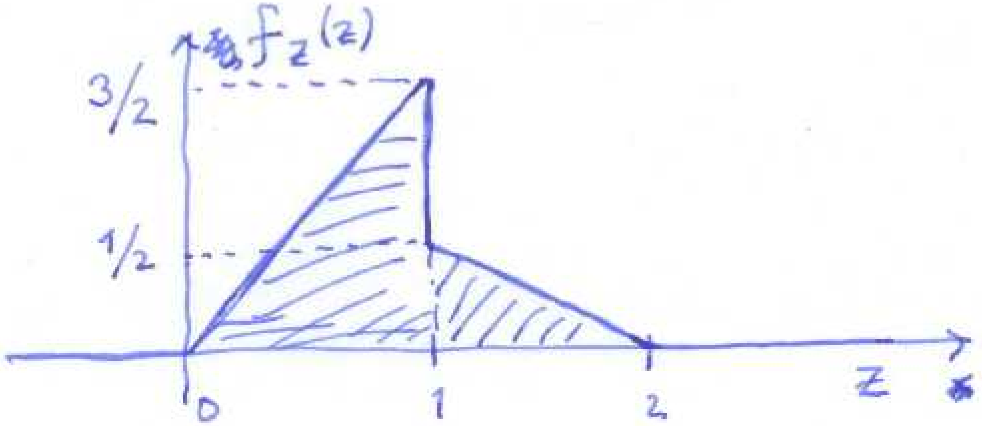
For $0 \le z \le 1$,
$$F_Z(z) = \int_0^{z/2} dx \int_x^{z} dy \cdot 2 = \int_0^{z/2} dx\, 2(z-x) = -(z-x)^2 \Big|_0^{z/2}$$
$$= -\frac{z^2}{4} + z^2 = \frac{3z^2}{4}$$

For $1 \le z \le 2$,
$$F_Z(z) = \int_0^{z/2} dx \int_x^{1} dy \cdot 2 = \int_0^{z/2} dx \cdot 2(1-x) = -(1-x)^2 \Big|_0^{z/2}$$
$$= -\left(1-\frac{z}{2}\right)^2 + 1 = z - \frac{z^2}{4} = z\left(1-\frac{z}{4}\right).$$

For $z > 2$, $F_Z(z) = \int_0^1 dx \int_x^1 dy\, 2 = 1$.

$$f_Z(z) = \frac{d}{dz} F_Z(z)$$

$$= \begin{cases} 0 & , \quad z < 0 \\ 3z/2 & , \quad 0 \le z \le 1 \\ 1 - z/2 & , \quad 1 \le z \le 2 \\ 0 & , \quad z > 2 \end{cases}$$



Check that the area under $f_Z(z)$ is 1

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2. $$f_{X,Y}(x,y) = \begin{cases} e^{-y} & , \quad 0 < x < y . \\ 0 & , \quad \text{o.w.} \end{cases}$$

$$f_{X|Y}(x|y) = \frac{e^{-y}}{f_Y(y)} \quad , \quad 0 < x < y .$$

$$f_Y(y) = \int_0^y dx \cdot e^{-y} = y e^{-y} , \quad 0 < y .$$

$$\therefore \; f_{X|Y}(x|y) = \frac{1}{y} \quad , \quad 0 < x < y .$$

Thus, X is uniform on $[0, y]$, conditional on $Y = y$. Hence, $\text{Var}(X \mid Y=y) = \frac{y^2}{12}$.

$$\text{Var}(X \mid Y) = \frac{Y^2}{12} \quad (\text{Uniform RV variance.})$$

$$E[\text{Var}(X \mid Y)] = E\left[\frac{Y^2}{12}\right] = \frac{1}{12} E[Y^2].$$

$$E[Y^2] = \int_{-\infty}^{\infty} y^2 f_Y(y)\, dy = \int_0^{\infty} y^2 \cdot y e^{-y}\, dy$$

$$= \int_0^{\infty} y^3 e^{-y}\, dy = \underbrace{\cancel{-y^3 e^{-y}\Big|_0^{\infty}}}_{0} + \int_0^{\infty} 3y^2 e^{-y}\, dy$$

integration by parts

$$= \underbrace{\cancel{-3y^2 e^{-y}\Big|_0^{\infty}}}_{0} + 6\int_0^{\infty} y e^{-y}\, dy$$

$$= \underbrace{\cancel{-6y e^{-y}\Big|_0^{\infty}}}_{0} + 6 \cdot \int_0^{\infty} e^{-y}\, dy = -6e^{-y}\Big|_0^{\infty} = \underline{\underline{6}}$$

$$\therefore \; E[\text{Var}(X \mid Y)] = \frac{6}{12} = \underline{\underline{\frac{1}{2}}}$$

3. $A = B \cup C$

$B = \{X_2 > X_3\}$, $C = \{X_1 > X_2\}$

$$
\begin{aligned}
P(A) &= \cancel{P}\, 1 - P(A^c) \\
&= 1 - P(B^c \cap C^c) \\
&= 1 - P\left(\{X_2 \le X_3\} \cap \{X_1 \le X_2\}\right) \\
&= 1 - P\left(\{X_1 \le X_2 \le X_3\}\right).
\end{aligned}
$$

Since X_1, X_2, X_3 are independent and identically-distributed, the event $\{X_1 \le X_2 \le X_3\}$ has the same probability as any of the ~~other~~ $3! = 6$ possible events

$\{X_1 \le X_2 \le X_3\}$, $\{X_1 \le X_3 \le X_2\}$, $\{X_2 \le X_1 \le X_3\}$,
$\{X_2 \le X_3 \le X_1\}$, $\{X_3 \le X_1 \le X_2\}$, $\{X_3 \le X_2 \le X_1\}$.

Since the random variables are continuous, the probability that any two of them are equal is zero; so, each of the above 6 events have probability $1/6$ (they form a partition of the space).

So, $P(A) = 1 - \frac{1}{6} = \frac{5}{6}$.