Bilkent University	**Spring 2017-2018**

Math255 Probability and Statistics
Midterm 2 Solutions
9 April 2018

Problem 1. [6 pts] Let X be exponentially distributed: $P(X > x) = e^{-\lambda x}$, $x \geq 0$, where $\lambda > 0$ is a fixed parameter. Let $Y = e^{-2X}$. Determine the PDF $f_Y(y)$ for all possible values of y as a function of λ.

Solution. We can solve this using the CDF approach. Note that Y takes values in the interval $(0, 1]$. So, $F_Y(y) = 0$ and $f_Y(y) = 0$ for $y < 0$; and, $F_Y(y) = 1$ for $y \geq 1$. For $0 < y \leq 1$, we have

$$F_Y(y) = P\big(Y \leq y\big) = P\big(e^{-2X} \leq y\big) = P\big(-2X \leq \ln y\big) = P\big(X \geq -\frac{2}{\ln} y\big) = e^{\frac{\lambda}{2}\ln y} = y^{\frac{\lambda}{2}}.$$

So, for $0 < y \leq 1$,

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} y^{\frac{\lambda}{2}} = \frac{\lambda}{2}\, y^{\frac{\lambda}{2}-1}.$$

To summarize,

$$f_Y(y) = \begin{cases} \frac{\lambda}{2}\, y^{\frac{\lambda}{2}-1}, & 0 < y \leq 1 \\ 0, & \text{elsewhere.} \end{cases}$$

Problem 2. [6 pts] Let X_1 and X_2 be independent identically distributed random variables. Let

$$f_{X_i}(x) = c\, x\, e^{-x}, \qquad x > 0,\ i = 1, 2,$$

where c is a constant. Compute the PDF $f_Z(z)$ of the sum $Z = X_1 + X_2$, and evaluate it at $z = 1$ to two significant digits. You should determine the constant c as part of solving the problem. (You may use $e^{-1} = 0.3679$ if needed.)

Solution. This is a straightforward convolution problem. The constant c is given by

$$c = \left(\int_0^\infty x e^{-x} dx\right)^{-1}.$$

The integral can be evaluated using integration by parts.

$$\int_0^\infty x e^{-2x} dx = -x e^{-x}\big|_{x=0}^\infty + \int_0^\infty e^{-x} dx = -e^{-x}\big|_{x=0}^\infty = 1.$$

Thus, $c = 1$.

Since Z is the sum of two non-negative random variables, we only need to compute $f_Z(z)$ for $z \geq 0$. For $z > 0$, the convolution integral is given by

$$\begin{aligned} f_Z(z) &= \int_0^z f_{X_1}(x) f_{X_2}(z-x) dx = \int_0^z (z-x) e^{-(z-x)} x e^{-x} dx \\ &= \int_0^z (z-x) x e^{-z} dx = e^{-z}\left(z\frac{x^2}{2} - \frac{x^3}{3}\right)\Bigg|_0^z = \frac{z^3 e^{-z}}{6}. \end{aligned}$$

Problem 3. [6 pts] Let $X_1, X_2, \ldots$ be independent normal random variables with mean 1 and variance 4. Let N a Poisson random variable with $p_N(n) = \lambda^n e^{-\lambda}/n!$, where $\lambda > 0$ is fixed and $n = 0, 1, 2, \ldots$. You may use the fact that $E[N] = \lambda$ and $\mathrm{var}(N) = \lambda$. Assume that N and $X_1, X_2, \ldots$ are jointly independent. Let $S = \sum_{i=1}^N X_i$. Compute $\mathbf{E}[S^2]$ as a function of λ.

Solution. This is an exercise in conditional expectations.

$$\mathbf{E}[S^2] = \mathbf{E}\Big[E\big[S^2 | N\big]\Big]$$

$$
\begin{aligned}
\mathbf{E}\big[S^2\big|N=n\big] &= \mathbf{E}\Big[\sum_{i=1}^{n}\sum_{j=1}^{n}X_iX_j\big|N=n\Big] = \mathbf{E}\Big[\sum_{i=1}^{n}\sum_{j=1}^{n}X_iX_j\Big] \\
&= \mathbf{E}\big[\sum_{i=1}^{n}X_i^2 + 2\sum_{i=1}^{n}\sum_{j=1}^{i-1}X_iX_j\Big] \\
&= n\mathbf{E}[X_1^2] + n(n-1)\mathbf{E}[X_1X_2] = n\big(4+1^2\big) + n(n-1)1^2 = n^2+4n.
\end{aligned}
$$

Thus, $\mathbf{E}[S^2|N] = N^2+4N$. It follows that

$$\mathbf{E}[S^2] = \mathbf{E}[N^2+4N] = \lambda+\lambda^2+4\lambda = \lambda^2+5\lambda = \lambda(\lambda+5).$$

Problem 4. [6 pts] A transport plane has a cargo capacity of 1000 kg. Suppose 50 boxes of cargo arrive for transport, with box number i containing X_i kgs of cargo, $i=1,2,\ldots,50$. Suppose $X_1,X_2,\ldots,X_{50}$ are independent normal random variables with mean $\mu=20.1$ kg and variance $\sigma^2=2$ kg^2. Compute to four significant digits the probability p that the sum of the weights in all 50 boxes does not exceed the cargo capacity of transport plane? (Use the table Φ given on the next page if needed.)

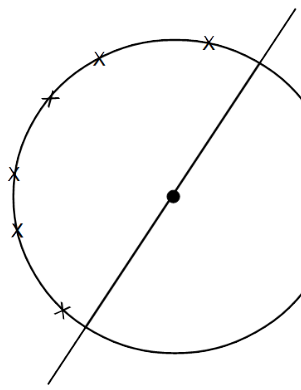
Solution. Let $S=X_1+\cdots+X_{50}$. We are interested in the probability that $P(S<1000)$. If the X_i were not normal, the solution would call for using the Central Limit Theorem (CLT) approximation. But here the X_i's are normal, so their sum S is normal, and we can compute the probability exactly. Since $S\sim N(50\times 20.1, 50\times 2) = N(1005,100)$, we have

$$P(S<1000) = P\bigg(\frac{S-1005}{\sqrt{100}} < \frac{1000-1005}{\sqrt{100}}\bigg) = \Phi\bigg(\frac{1000-1005}{\sqrt{100}}\bigg) = \Phi(-0.5) = 1-\Phi(0.5).$$

From table, $\Phi(0.5)=0.6915$, so the desired probability is $1-0.6915=0.3085$.

Problem 5. [6 pts]

Suppose that n points are chosen at random on the perimeter of a circle. What is the probability p that they all lie in some semicircle, such as in the picture on the right? Express your answer as a function of n. (Hint: the law of total probability.) (Correct answers without sufficient explanation will receive no credit.)



Solution. Let E be the desired event that all points lie on a semicircle. Number the points on the perimeter of the circle in the order they are chosen with integers 1 through n. Draw a straight line that passes through the ith point on the perimeter and the center of the circle. Let E_i be the event that all other $n-1$ points on the perimeter lie on the "clockwise" semicircle, *i.e.*, the semicircle that lies in the clockwise direction from the ith point. Convince yourself that the events $E_1,E_2,\ldots,E_n$ form a partition of E. Then,

$$P(E) = P(E_1)+P(E_2)+\cdots+P(E_n).$$

Next note that, by symmetry, $P(E_i)$ does not depend on i, hence $P(E)=nP(E_1)$. Finally, note that $P(E_1)=2^{-(n-1)}$ since each point, numbered 2 through n, has probability 1/2 of lying in the clockwise semicirle, independent of other points. So, the correct answer is $n2^{-(n-1)}$.