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MATH 255 - HW 5

(1) a) $Cov(y[n], x[n]) = E\{y[n].x[n]\} - E\{y[n]\}.E\{x[n]\}$

Find $E\{y[n].x[n]\}$:

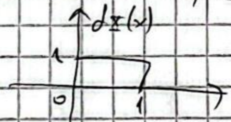
$y[n].x[n] = (0.5x[n] + x[n-1] + 0.5x[n-2])x[n]$

$= 0.5x^2[n] + x[n].x[n-1] + 0.5x[n-2].x[n] \ldots (1)$

$E\{x^2[n]\} = \int_0^1 x^2.1\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$ since $x[n]$ is uniformly dist. in $(0,1]$.

$\Downarrow$

Its pdf is $f_X(x) = \begin{cases} 1, & 0<x\leq 1 \\ 0, & o.w. \end{cases}$



$E\{x[n].x[n-1]\} \underset{\substack{x[n] \text{ is independent}}}{=} E\{x[n]\}.E\{x[n-1]\}$

$\underset{\substack{\text{shifting doesn't affect } x[n] \\ \text{since } x(n) \text{ is i.i.d}}}{=} E\{x[n]\}.E\{x[n]\}$

$= \left(\int_0^1 x(1)dx\right)^2$

$= \frac{1}{4}$

Similarly, $E\{x[n].x[n-2]\} = \frac{1}{4}$

Insert into (1):

$E\{y[n].x[n]\} \underset{E\{.\} \text{ is linear}}{=} 0.5\,E\{x^2(n)\} + E\{x(n).x[n-1]\} + 0.5\{x(n).x(n-2)\}$

$= 0.5\left(\frac{1}{3}\right) + \frac{1}{4} + (0.5)\left(\frac{1}{4}\right) = \frac{13}{24}$

and $E\{x[n]\} = \int_0^1 x.1.dx = \frac{1}{2}$

$E\{y[n]\} \underset{\text{linear}}{=} 0.5\,E\{x[n]\} + E\{x[n-1]\} + 0.5\,E\{x[n-2]\}$

$= 0.5\left(\frac{1}{2}\right) + \frac{1}{2} + (0.5)\frac{1}{2} = 1$

finally, we have:

$$Cov(y(n), x(n)) = \frac{13}{24} - 1 \cdot \frac{1}{2} = \boxed{\frac{1}{24}}$$

$$Cov(y[n], y(n-1]) = E\{y[n] \cdot y(n-1]\} - E\{y[n]\} \cdot E\{y[n-1]\}$$

$$y[n] \cdot y[n-1] = (0.5x(n] + x(n-1] + 0.5x(n-2])(0.5x[n-1] + x(n-2] + 0.5x(n-3])$$

$E\{.\}$ is linear

$$\Rightarrow E\{y[n] \cdot y(n-1]\} = 0.25 E\{x(n]x[n-1]\} + 0.5 E\{x(n]x(n-2]\} + 0.25 E\{x(n] \cdot x(n-3]\}$$
$$+ 0.5 E\{x^2(n-1]\} + E\{x(n-1]x(n-2]\} + 0.5 E\{x(n-1]x(n-3]\}$$
$$+ 0.25 E\{x(n-2]x(n-1]\} + 0.5 E\{x^2[n-2]\}$$
$$+ 0.25 E\{x(n-2]x(n-3]\}$$

Using the fact that $x(n)$ is iid, and using the calculations made previously:

$$E\{y(n] \cdot y(n-1]\} = 0.25 \cdot \frac{1}{4} + 0.5 \cdot \frac{1}{4} + 0.25 \cdot \frac{1}{4} + 0.5 \frac{1}{3} + \frac{1}{4} + 0.5 \cdot \frac{1}{4}$$
$$+ 0.25 \cdot \frac{1}{4} + 0.5 \frac{1}{3} + 0.25 \cdot \frac{1}{4}$$
$$= \boxed{\frac{13}{12}}$$

$E\{y(n]\} = 1$, computed previously

$E\{y[n-1]\} = 1$, since all $x(n)$'s will have the same mean

i.e. $E\{x(n]\} = E\{x[n-1]\}$

$$\Rightarrow Cov(y(n], y(n-1]) = \frac{13}{12} - 1 \cdot 1 = \boxed{\frac{1}{12}}$$

b) $\mathrm{Var}\{x[n]\} = \underbrace{E\{x^2[n]\}} - (\underbrace{E\{x[n]\}})^2$

$$= \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{3} - \frac{1}{4} = \boxed{\frac{1}{12}}$$

Computed in pt. a

$\mathrm{Var}\{y[n]\} = E\{y^2[n]\} - (E\{y(n)\})^2$

$$E\{y^2[n]\} \underset{\text{linear}}{=} 0.25\, E\{x^2[n]\} + 0.5\, E\{x[n]\cdot x(n-1)\} + 0.25\, E\{x[n]\cdot x[n-2]\}$$
$$+ 0.5\, E\{x[n]x[n-1]\} + E\{x^2[n-1]\} + 0.5\, E\{x[n-1]x[n-2]\}$$
$$+ 0.25\, E\{x[n]\,x[n-2]\} + 0.5\, E\{x[n-1]x[n-2]\} + 0.25\, E\{x^2[n-2]\}$$

from pct a

$$= 0.25\left(\frac{1}{3}\right) + 0.5\left(\frac{1}{4}\right) + 0.25\left(\frac{1}{4}\right) + 0.5\left(\frac{1}{4}\right) + \frac{1}{3} + 0.5\left(\frac{1}{4}\right) +$$
$$0.25\left(\frac{1}{4}\right) + 0.5\left(\frac{1}{4}\right) + 0.25\left(\frac{1}{3}\right) = \boxed{1.125}$$

$E\{y[n]\} = 1$

$\Rightarrow \mathrm{Var}\{y[n]\} = 1.125 - (1)^2 = \boxed{0.125 = \frac{1}{8}}$

c) Using the formula of LLMSE:

$$\hat{y}[n+1] = \underbrace{E\{y[n+1]\}}_{1} + \frac{\mathrm{Cov}(y(n), y(n+1))}{\underbrace{\mathrm{Var}(y(n))}_{\frac{1}{8}}}\,(y(n) - \underbrace{E\{y(n)\}}_{1})$$

from pt. a, b

from pct b

$$\mathrm{Cov}(y(n), y(n+1)) \underset{\substack{\text{c.o.v}\\ n=k}}{=} \mathrm{Cov}(y(k), y(k+1)) \underset{\substack{\text{put}\\ n=k+1}}{=} \mathrm{Cov}(y(n-1), y(n))$$

$$= \boxed{\frac{1}{12}}$$ found in pt. a

$$\Rightarrow \hat{y}[n+1] = 1 + \frac{\frac{1}{12}}{\frac{1}{8}}(y[n] - 1) = 1 + \frac{2}{3}(y(n) - 1) = \boxed{\frac{2}{3}y[n] + \frac{1}{3}}$$

$$\Rightarrow \boxed{a = \frac{2}{3},\ b = \frac{1}{3}}$$

3) a) $Cov(y[n], x[n]) = E\{y[n].x[n]\} - E\{y[n]\}.E\{x[n]\}$

$$E\{y[n].x[n]\} = E\{(-0.5x[n] + x[n-1] - 0.5x[n-2])x[n]\}$$

$$\xrightarrow{\text{Linearity}} = -0.5\underbrace{E\{x^2[n]\}}_{\frac{1}{3}} + \underbrace{E\{x[n]x[n-1]\}}_{\frac{1}{4}} - 0.5\underbrace{E\{x[n]x[n-1]\}}_{\frac{1}{4}}$$

shown in part 1a; since i.i.d

$$= -\frac{1}{2}\left(\frac{1}{3}+\frac{1}{4}\right) + \frac{1}{4} = -\frac{1}{6} - \frac{1}{8} + \frac{1}{4} = -\frac{1}{24}$$

$$E\{x[n]\} = \frac{1}{2}, \quad E\{y[n]\} \overset{\text{linearity}}{=} -0.5E\{x[n]\} + E\{x[n-1]\} - 0.5E\{x[n-2]\}$$

$$= -0.5\left(\frac{1}{2}\right) + \frac{1}{2} - 0.5\left(\frac{1}{2}\right)$$

$$= 0$$

$$\Rightarrow Cov(y[n], x[n]) = -\frac{1}{24} - \left(0 \cdot \frac{1}{2}\right) = \boxed{-\frac{1}{24}}$$

$Cov(y[n], y[n-1]) = E\{y[n].y[n-1]\} - E\{y[n]\}.E\{y[n-1]\}$

$$E\{y[n].y[n-1]\} \overset{\text{linearity}}{=} 0.25E\{x[n].x[n-1]\} - 0.5E\{x[n]x[n-2]\} + 0.25E\{x[n].x[n-3]\}$$
$$-0.5E\{x^2[n-1]\} + E\{x[n-1]x[n-2]\} - 0.5E\{x[n-1]x[n-3]\}$$
$$+0.25E\{x[n-2]x[n-1]\} - 0.5E\{x^2[n-2]\} + 0.25E\{x[n-2]x[n-3]\}$$

Since $x[n]$ is i.i.d $\rightarrow$

$$= (0.25)\frac{1}{4} - (0.5)\frac{1}{4} + (0.25)\frac{1}{4} - (0.5)\frac{1}{3} + \frac{1}{4} - (0.5)\frac{1}{4}$$
$$+ (0.25)\frac{1}{4} - (0.5)\frac{1}{3} + (0.25)\frac{1}{4} = \boxed{-\frac{1}{12}}$$

$E\{y[n]\} = E\{y[n-1]\} = 0$ (from prev. part) since x_i's have the same expected value, i.e. $E\{x[n]\} = E\{x[n-1]\}$

$$\Rightarrow Cov(y[n], y[n-1]) = -\frac{1}{12} - 0 = \boxed{-\frac{1}{12}}$$

b) $Var\{x[n]\} = \boxed{\frac{1}{12}}$ from Q1.b

$Var\{y[n]\} = Var\{-0.5x[n] + x[n-1] - 0.5x[n-2]\}$

since $x[n]$'s are **independent** and $Var\{aX\} = a^2 Var\{X\}$

$\rightarrow = (-0.5)^2 \underbrace{Var\{x[n]\}} + Var\underbrace{\{x[n-1]\}} + (-0.5)^2 Var\underbrace{\{x[n-2]\}}$

$= \frac{1}{4} \cdot \frac{1}{12} + \frac{1}{12} + \frac{1}{4} \cdot \frac{1}{12} = \boxed{\frac{1}{8}}$

$\frac{1}{12} \quad \frac{1}{24} \quad \frac{1}{2}$

c) Using LLMSE:

$\hat{y}[n+1] = \underbrace{E\{y[n+1]\}}_{0} + \frac{Cov(y[n], y[n+1])}{Var\{y[n]\}} (y[n] - \underbrace{E\{y[n]\}}_{0})$

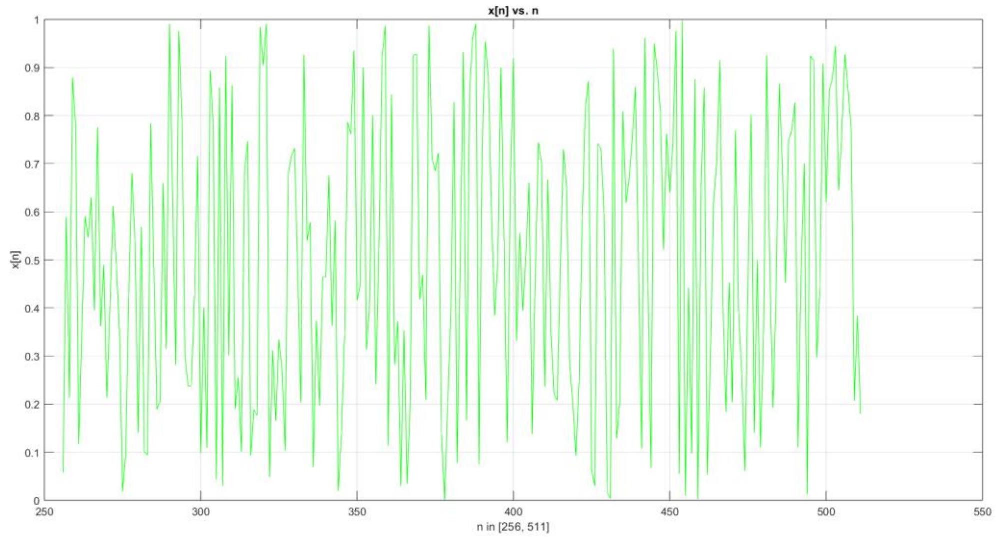
found previously $\rightarrow 0$

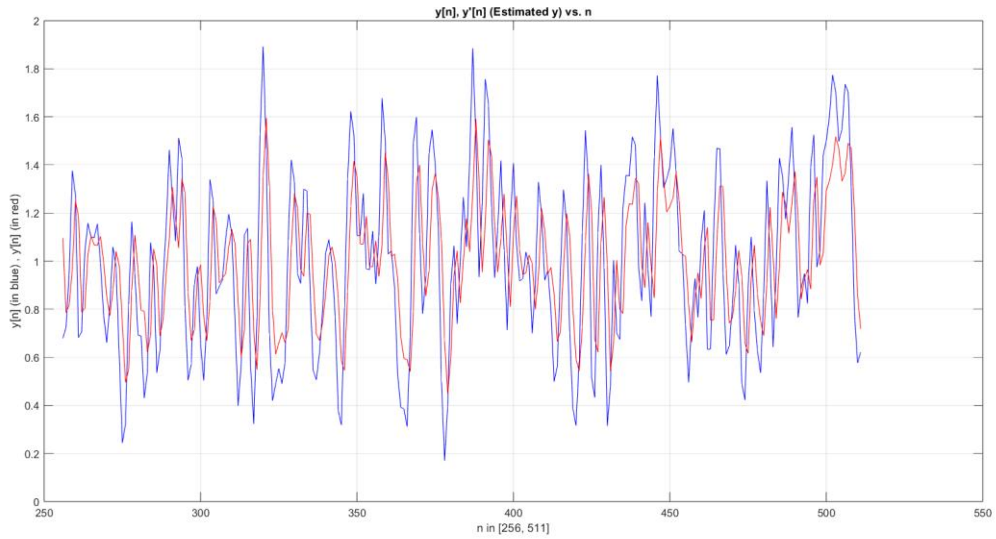
$Cov(y[n], y[n+1]) = Cov(y[n-1], y[n]) = \frac{-1}{12}$, since does not depend on n.

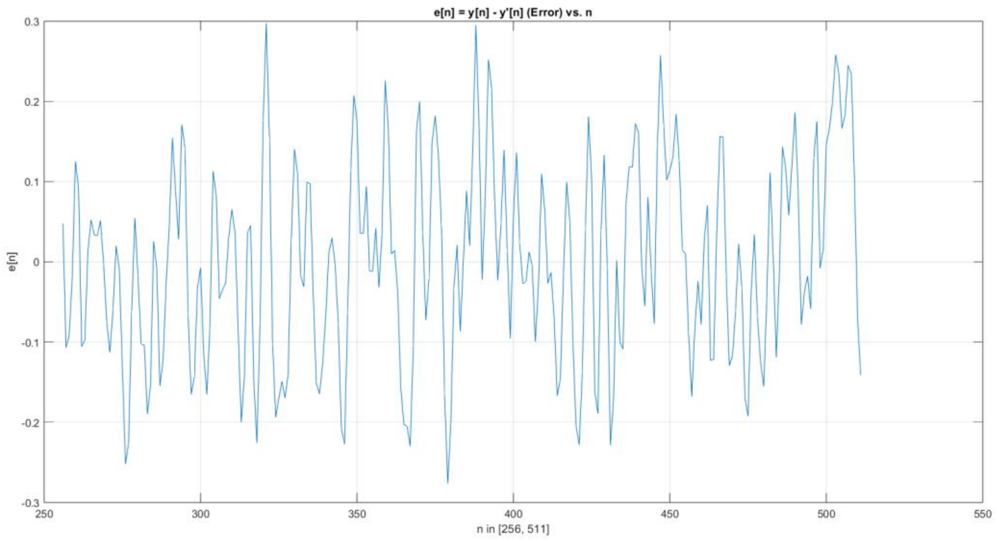
$Var\{y[n]\} = \frac{1}{8}$, as found previously

$\Rightarrow \boxed{\hat{y}[n+1] = \frac{\frac{-1}{12}}{\frac{1}{8}} y[n] = -\frac{2}{3} y}$, $\boxed{a = -\frac{2}{3}, \ b = 0}$

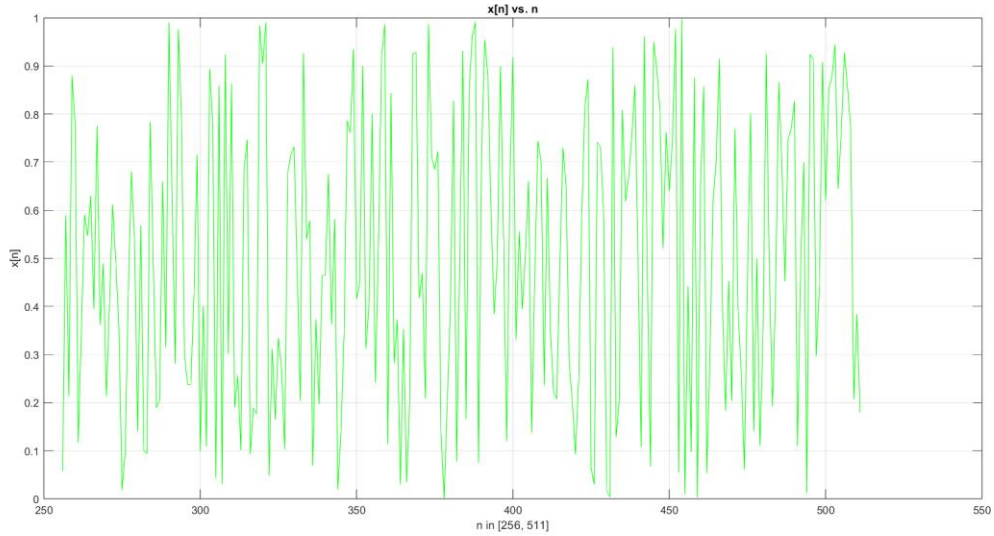
2) Plots (for the first FIR filter):

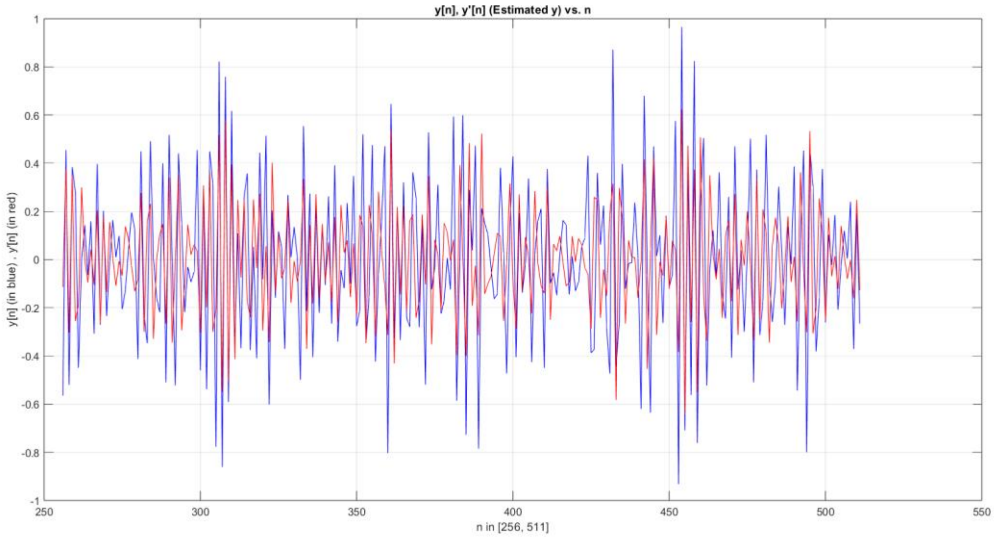


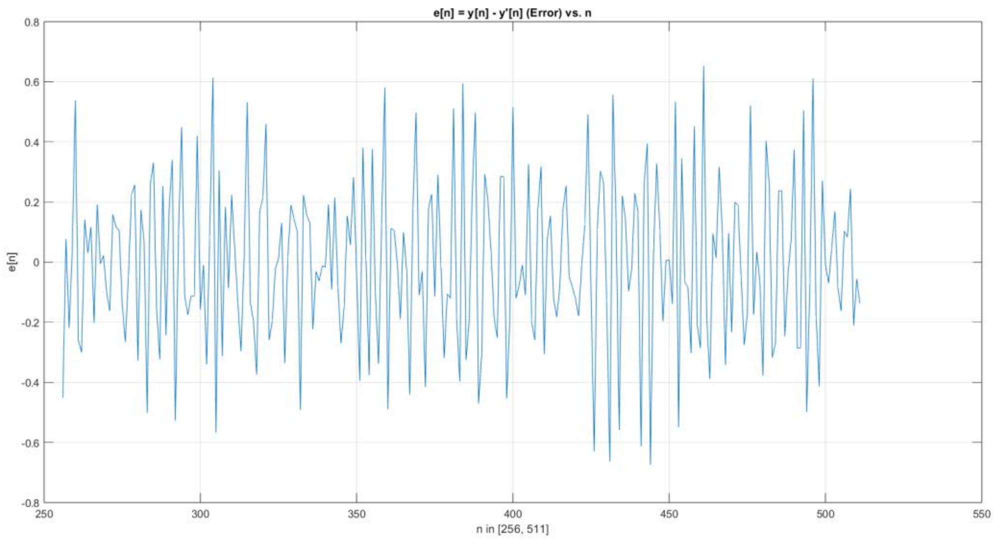




3) Plots (for the second FIR filter):







4) i) $x[n + 1]$ is not predictable from $x[n]$ since all $x[n]$'s are independent, as given in the question. Therefore, it is not possible to predict the future value n+1 from the present n.

ii) $y[n+1]$ is predictable from $y[n]$, because $y[n+1] = 0.5x[n+1] + x[n] + 0.5x[n - 1]$ has common elements (random variables) with $y[n] = 0.5x[n] + x[n - 1] + 0.5x[n - 2]$, which are $x[n]$ and $x[n-1]$. If they didn't have common random variables, then they would also be independent (since all $x[n]$'s are independent) but they are not. This is also shown by the fact that their covariance is not equal to zero, however for any two independent random variables their covariance must be zero, therefore they are not independent.

iii) Two consecutive elements of the output have covariance different than zero (calculated above) for both filters. Therefore, they are correlated. This can be observed from the fact that two consecutive elements, for example in the first filter, $y[1]= 0.5x[1] + x[0] + 0.5x[-1]$ and $y[2] = 0.5x[2] + x[1] + 0.5x[0]$ have two common elements, $x[0]$ and $x[1]$. Since correlation can be thought of as a measurement of similarity, the two common elements contribute to the similarity of these two consecutive outputs. For the second filter, this is also the case. Due to the common terms (random variables – $x[n]$'s), they are correlated (their covariance is not zero, meaning that they are not uncorrelated).

Appendix - Codes

```
%The numbers are stored in an array named "x"
y = zeros(1,2000);
y_1 = zeros(1,2000);
y_1_2 = zeros(1,2000);

for n=256:511
    y(n) = 0.5*x(n)+x(n-1)+0.5*x(n-2); %for the first
    y2(n) = -0.5*x(n)+x(n-1)-0.5*x(n-2); %for the second
end

for n=256:511
    y_1(n) = 0.5*x(n+1)+x(n)+0.5*x(n-1); %for the first
    y_1_2(n) = -0.5*x(n+1)+x(n)-0.5*x(n-1); %for the second
end

y_est = (2/3)*y + 1/3; %for the first
y_est_2 = (-2/3)*y; %for the second
err = y_1(256:511) - y_est(256:511);
err2 = y_1_2(256:511) - y_est_2(256:511);
n = 256:1:511;

figure;
plot(n, x(256:511), "green");
xlabel("n in [256, 511]");
ylabel("x[n]");
title("x[n] vs. n");
grid("on");

figure;
plot(n, y_1(256:511), "blue");
hold;
plot(n, y_est(256:511), "red");
xlabel("n in [256, 511]");
ylabel("y[n] (in blue) , y'[n] (in red)");
title("y[n], y'[n] (Estimated y) vs. n");
grid("on");

figure;
plot(n, err);
xlabel("n in [256, 511]");
ylabel("e[n]");
title("e[n] = y[n] - y'[n] (Error) vs. n");
grid("on");

figure;
plot(n, y_1_2(256:511), "blue");
hold;
plot(n, y_est_2(256:511), "red");
```

```
xlabel("n in [256, 511]");
ylabel("y[n] (in blue) , y'[n] (in red)");
title("y[n], y'[n] (Estimated y) vs. n");
grid("on");

figure;
plot(n, err_2);
xlabel("n in [256, 511]");
ylabel("e[n]");
title("e[n] = y[n] - y'[n] (Error) vs. n");
grid("on");
```