Deliverable 5 - More properties of quaternions (5 pts)

In the lecture notes, we have defined two linear maps $\Omega_1 : \mathbb{R}^4 \to \mathbb{R}^{4\times4}$, and $\Omega_2 : \mathbb{R}^4 \to \mathbb{R}^{4\times4}$, such that for any $q \in \mathbb{R}^4$, we have:

$$\Omega_1(q) = \begin{bmatrix} q_4 & -q_3 & q_2 & q_1 \\ q_3 & q_4 & -q_1 & q_2 \\ -q_2 & q_1 & q_4 & q_3 \\ -q_1 & -q_2 & -q_3 & q_4 \end{bmatrix}, \qquad \Omega_2(q) = \begin{bmatrix} q_4 & q_3 & -q_2 & q_1 \\ -q_3 & q_4 & q_1 & q_2 \\ q_2 & -q_1 & q_4 & q_3 \\ -q_1 & -q_2 & -q_3 & q_4 \end{bmatrix}.$$

The product between any two unit quaternions can then be explicitly computed as:

$$q_a \otimes q_b = \Omega_1(q_a) q_b = \Omega_2(q_b) q_a.$$

In fact, the two linear maps Ω_1 and Ω_2 have more interesting properties, and you are asked to prove the following equalities:

1. For any unit quaternion q, both $\Omega_1(q)$ and $\Omega_2(q)$ are *orthogonal* matrices, *i.e.*,

$$\Omega_1(q)^T \Omega_1(q) = \Omega_1(q)\Omega_1(q)^T = I_4,$$

$$\Omega_2(q)^T \Omega_2(q) = \Omega_2(q)\Omega_2(q)^T = I_4.$$

$$\|q\| = 1 \Rightarrow$$

$$\Rightarrow q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1$$

$$\Omega_1(q)^T \Omega_1(q) = \begin{bmatrix} q_4 & q_3 & -q_2 & -q_1 \\ -q_3 & q_4 & q_1 & -q_2 \\ q_2 & -q_1 & q_4 & -q_3 \\ q_1 & q_2 & q_3 & q_4 \end{bmatrix} \begin{bmatrix} q_4 & -q_3 & q_2 & q_1 \\ q_3 & q_4 & -q_1 & q_2 \\ -q_2 & q_1 & q_4 & q_3 \\ -q_1 & -q_2 & -q_3 & q_4 \end{bmatrix} =$$

$$= \begin{bmatrix} A & B & C & D \end{bmatrix},$$ where A, B, C, D are column vectors

$$A = \begin{bmatrix} q_4^2 + q_3^2 + q_2^2 + q_1^2 \\ -q_4 q_3 + q_3 q_4 - q_2 q_1 + q_2 q_1 \\ q_2 q_4 - q_1 q_3 - q_4 q_2 + q_1 q_3 \\ q_1 q_4 + q_2 q_3 - q_3 q_2 - q_4 q_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$B = \begin{bmatrix} -q_3 q_4 + q_3 q_4 - q_2 q_1 + q_1 q_2 \\ q_3^2 + q_4^2 + q_1^2 + q_2^2 \\ -q_2 q_3 - q_1 q_4 + q_4 q_1 + q_3 q_2 \\ -q_1 q_3 + q_2 q_4 + q_3 q_1 - q_4 q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} q_4 q_2 - q_3 q_1 - q_2 q_4 + q_1 q_3 \\ -q_3 q_2 - q_4 q_1 + q_1 q_4 + q_2 q_3 \\ q_2^2 + q_1^2 + q_4^2 + q_3^2 \\ q_1 q_2 - q_2 q_1 + q_3 q_4 - q_4 q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$D = \begin{bmatrix} q_4 q_1 + q_3 q_2 - q_2 q_3 - q_1 q_4 \\ -q_3 q_1 + q_4 q_2 + q_1 q_3 - q_2 q_4 \\ q_2 q_1 - q_1 q_2 + q_4 q_3 - q_3 q_4 \\ q_1^2 + q_2^2 + q_3^2 + q_4^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \Omega_1(q)^T \Omega_1(q) = \begin{bmatrix} A & B & C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I_4$$

$$\Omega_1(q)\Omega_1(q)^T = \begin{bmatrix} A^T \\ B^T \\ C^T \\ D^T \end{bmatrix} = I_4 \quad \Rightarrow \quad \Omega_1(q) \text{ is orthogonal}$$

similarly for $\Omega_2(q)$:

$$\Omega_2(q)^T \Omega_2(q) = \begin{bmatrix} A & B & C & D \end{bmatrix} =$$

$$= \begin{bmatrix} q_4 & -q_3 & q_2 & -q_1 \\ q_3 & q_4 & -q_1 & -q_2 \\ -q_2 & q_1 & q_4 & -q_3 \\ q_1 & q_2 & q_3 & q_4 \end{bmatrix} \begin{bmatrix} q_4 & q_3 & -q_2 & q_1 \\ -q_3 & q_4 & q_1 & q_2 \\ q_2 & -q_1 & q_4 & q_3 \\ -q_1 & -q_2 & -q_3 & q_4 \end{bmatrix} \Rightarrow$$

$$A = \begin{bmatrix} q_4^2 + q_3^2 + q_2^2 + q_1^2 \\ q_3 q_4 - q_4 q_3 - q_1 q_2 + q_2 q_1 \\ -q_2 q_4 - q_1 q_3 + q_4 q_2 + q_3 q_1 \\ q_1 q_4 - q_2 q_3 + q_3 q_2 - q_4 q_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$B = \begin{bmatrix} q_4 q_3 - q_3 q_4 - q_2 q_1 + q_2 q_1 \\ q_3^2 + q_4^2 + q_1^2 + q_2^2 \\ -q_2 q_3 + q_1 q_4 - q_4 q_1 + q_3 q_2 \\ q_1 q_3 + q_2 q_4 - q_3 q_1 - q_4 q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} -q_4 q_2 - q_3 q_1 + q_2 q_4 + q_3 q_1 \\ -q_3 q_2 + q_4 q_1 - q_1 q_4 + q_2 q_3 \\ q_2^2 + q_1^2 + q_4^2 + q_3^2 \\ -q_1 q_2 + q_2 q_1 + q_3 q_4 - q_4 q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$D = \begin{bmatrix} q_4 q_1 - q_3 q_2 + q_2 q_3 - q_1 q_4 \\ q_3 q_1 + q_4 q_2 - q_1 q_3 - q_2 q_4 \\ -q_2 q_1 + q_1 q_2 + q_4 q_3 - q_3 q_4 \\ q_1^2 + q_2^2 + q_3^2 + q_4^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \boxed{\begin{aligned} &\Omega_2(q)^T \Omega_2(q) = \begin{bmatrix} A & B & C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I_4 \\ &\Omega_2(q) \Omega_2(q)^T = \begin{bmatrix} A^T \\ B^T \\ C^T \\ D^T \end{bmatrix} = I_4 \quad \Rightarrow \quad \Omega_2(q) \text{ is orthogonal} \end{aligned}}$$

Intuitively, the result of $q_A \otimes q_B$ is also a unit quaternion. Therefore, the transformations must preserve the norm $\|q\| = 1$. This is only satisfied if transformations are orthogonal, meaning that it makes sense that $\Omega_1(q)$ and $\Omega_2(q)$ are orthogonal.

2 For any unit quaternion q, both $\Omega_1(q)$ and $\Omega_2(q)$ convert q to be the unit quaternion that corresponds to the 3D identity rotation, *i.e.*,

$$\Omega_1(q)^T q = \Omega_2(q)^T q = [0,0,0,1]^T.$$

$$\Omega_1(q)^T q = \begin{bmatrix} q_4 & q_3 & -q_2 & -q_1 \\ -q_3 & q_4 & q_1 & -q_2 \\ q_2 & -q_1 & q_4 & -q_3 \\ q_1 & q_2 & q_3 & q_4 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} \Rightarrow$$

$$\Omega_1(q)^T q = \begin{bmatrix} q_4 q_1 + q_3 q_2 - q_2 q_3 - q_1 q_4 \\ -q_3 q_1 + q_4 q_2 + q_1 q_3 - q_2 q_4 \\ q_2 q_1 - q_1 q_2 + q_4 q_3 - q_3 q_4 \\ q_1^2 + q_2^2 + q_3^2 + q_4^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

similarly:

$$\Omega_2(q)^T q = \begin{bmatrix} q_4 & -q_3 & q_2 & -q_1 \\ q_3 & q_4 & -q_1 & -q_2 \\ -q_2 & q_1 & q_4 & -q_3 \\ q_1 & q_2 & q_3 & q_4 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} q_4 q_1 - q_3 q_2 + q_2 q_3 - q_1 q_4 \\ q_3 q_1 + q_4 q_2 - q_1 q_3 - q_2 q_4 \\ -q_2 q_1 + q_1 q_2 + q_4 q_3 - q_3 q_4 \\ q_1^2 + q_2^2 + q_3^2 + q_4^2 \end{bmatrix}$$

$$\Rightarrow \Omega_2(q)^T q = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \Omega_1(q)^T q$$

3 For any two vectors $x, y \in \mathbb{R}^4$, show the two linear operators commute, *i.e.*,

$$\Omega_1(x)\Omega_2(y) = \Omega_2(y)\Omega_1(x),$$

$$\Omega_1(x)\Omega_2(y)^T = \Omega_2(y)^T\Omega_1(x).$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \quad y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

$$\Omega_1(x)\Omega_2(y) = \begin{bmatrix} x_4 & -x_3 & x_2 & x_1 \\ x_3 & x_4 & -x_1 & x_2 \\ -x_2 & x_1 & x_4 & x_3 \\ -x_1 & -x_2 & -x_3 & x_4 \end{bmatrix} \begin{bmatrix} y_4 & y_3 & -y_2 & y_1 \\ -y_3 & y_4 & y_1 & y_2 \\ y_2 & -y_1 & y_4 & y_3 \\ -y_1 & -y_2 & -y_3 & y_4 \end{bmatrix}$$

$$\Omega_2(y)\Omega_1(x) = \begin{bmatrix} y_4 & y_3 & -y_2 & y_1 \\ -y_3 & y_4 & y_1 & y_2 \\ y_2 & -y_1 & y_4 & y_3 \\ -y_1 & -y_2 & -y_3 & y_4 \end{bmatrix} \begin{bmatrix} x_4 & -x_3 & x_2 & x_1 \\ x_3 & x_4 & -x_1 & x_2 \\ -x_2 & x_1 & x_4 & x_3 \\ -x_1 & -x_2 & -x_3 & x_4 \end{bmatrix}$$

$$M_1 = \Omega_1(x)\Omega_2(y) = \Omega_2(y)\Omega_1(x) = [A_1 \; B_1 \; C_1 \; D_1]$$

A_1, B_1, C_1, D_1 are column vectors, see next page for definitions

M_1 components:

$$A_1 = \begin{bmatrix} x_4 y_4 - x_3 y_3 - x_2 y_2 - x_1 y_1 \\ x_3 y_4 + x_4 y_3 - x_1 y_2 - x_2 y_1 \\ -x_2 y_4 - x_1 y_3 + x_4 y_2 + x_3 y_1 \\ -x_1 y_4 + x_2 y_3 - x_3 y_2 + x_4 y_1 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} -x_4 y_3 - x_3 y_4 + x_2 y_1 + x_1 y_2 \\ x_4 y_4 - x_3 y_3 - x_2 y_2 - x_1 y_1 \\ x_2 y_3 - x_1 y_4 - x_4 y_1 - x_3 y_2 \\ -x_1 y_3 - x_2 y_4 + x_3 y_1 - x_4 y_2 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} x_4 y_2 - x_3 y_1 - x_2 y_4 - x_1 y_3 \\ -x_3 y_1 - x_4 y_2 - x_1 y_3 + x_2 y_4 \\ x_4 y_4 - x_3 y_3 - x_2 y_2 - x_1 y_1 \\ -x_1 y_2 + x_2 y_1 + x_3 y_4 + x_4 y_3 \end{bmatrix}$$

$$D_1 = \begin{bmatrix} -x_4 y_1 + x_3 y_2 - x_2 y_3 + x_1 y_4 \\ x_3 y_2 - x_4 y_1 + x_1 y_4 + x_2 y_3 \\ -x_2 y_1 - x_1 y_2 - x_4 y_3 + x_3 y_4 \\ x_4 y_4 - x_3 y_3 - x_2 y_2 - x_1 y_1 \end{bmatrix}$$

$$\Omega_1(x)\,\Omega_2^T(y) = \begin{bmatrix} x_4 & -x_3 & x_2 & x_1 \\ x_3 & x_4 & -x_1 & x_2 \\ -x_2 & x_1 & x_4 & x_3 \\ -x_1 & -x_2 & -x_3 & x_4 \end{bmatrix} \begin{bmatrix} y_4 & -y_3 & y_2 & -y_1 \\ y_3 & y_4 & -y_1 & -y_2 \\ -y_2 & y_1 & y_4 & -y_3 \\ y_1 & y_2 & y_3 & y_4 \end{bmatrix}$$

$$\Omega_2(y)^T\,\Omega_1(x) = \begin{bmatrix} y_4 & -y_3 & y_2 & -y_1 \\ y_3 & y_4 & -y_1 & -y_2 \\ -y_2 & y_1 & y_4 & -y_3 \\ y_1 & y_2 & y_3 & y_4 \end{bmatrix} \begin{bmatrix} x_4 & -x_3 & x_2 & x_1 \\ x_3 & x_4 & -x_1 & x_2 \\ -x_2 & x_1 & x_4 & x_3 \\ -x_1 & -x_2 & -x_3 & x_4 \end{bmatrix}$$

$$M_2 = \Omega_1(x)\,\Omega_2(y)^T = \Omega_2(y)^T\,\Omega_1(x) = \begin{bmatrix} A_2 & B_2 & C_2 & D_2 \end{bmatrix}$$

A_2, B_2, C_2, D_2 are column vectors

M_2 components:

$$A_2 = \begin{bmatrix} x_4 y_4 + x_3 y_3 + x_2 y_2 + x_1 y_1 \\ -x_3 y_4 + x_4 y_3 + x_1 y_2 - x_2 y_1 \\ x_2 y_4 - x_1 y_3 - x_4 y_2 + x_3 y_1 \\ -x_1 y_4 - x_2 y_3 - x_3 y_2 - x_4 y_1 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} x_4 y_3 - x_3 y_4 - x_2 y_1 + x_1 y_2 \\ x_4 y_4 + x_3 y_3 + x_2 y_2 + x_1 y_1 \\ x_2 y_3 + x_1 y_4 - x_4 y_1 + x_3 y_2 \\ -x_1 y_3 + x_2 y_4 + x_3 y_1 + x_4 y_2 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} -x_4 y_2 + x_3 y_1 - x_2 y_4 + x_1 y_3 \\ -x_3 y_1 - x_4 y_2 + x_1 y_3 - x_2 y_4 \\ x_4 y_4 + x_3 y_3 + x_2 y_2 + x_1 y_1 \\ x_1 y_2 - x_2 y_1 + x_3 y_4 - x_4 y_3 \end{bmatrix}$$

$$D_2 = \begin{bmatrix} x_4 y_1 + x_3 y_2 + x_2 y_3 + x_1 y_4 \\ -x_3 y_2 + x_4 y_1 - x_1 y_4 + x_2 y_3 \\ x_2 y_1 - x_1 y_2 + x_4 y_3 + x_3 y_4 \\ x_4 y_4 + x_3 y_3 + x_2 y_2 + x_1 y_1 \end{bmatrix}$$