W - world frame

1 In the problem formulation, we mentioned that AV2's trajectory is an arc of parabola in the x-z plane of the world frame. Can you prove this statement?

- Hint: $cos(2t)$ can be written as...

$$x_2^w(t) = \sin(t)$$

$$z_2^w(t) = \cos(2t) = 1 - 2\sin^2(t) = 1 - 2{x_2^w}^2 \quad \Big| \Rightarrow$$

$$\Rightarrow \boxed{z_2^w = 1 - 2{x_2^w}^2 \Rightarrow \text{parabola in } x^w\text{-}z^w \text{ plane} \\ (z_2^w \text{ is a quadratic function of } x_2^w)}$$

2 Compute $o_2^1(t)$, i.e., the position of AV2 relative to AV1's body frame as a function of t.

- Hint: write down the homogeneous transformations and compose them accordingly...

$$o_1^w = \begin{bmatrix} \cos(t) \\ \sin(t) \\ 0 \end{bmatrix} ; \quad o_2^w = \begin{bmatrix} \sin(t) \\ 0 \\ \cos(2t) \end{bmatrix}$$

$$R_1^w = \begin{bmatrix} \cos(t) & -\sin(t) & 0 \\ \sin(t) & \cos(t) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_2^w = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T_w^1 = \begin{bmatrix} (R_1^w)^T & -(R_1^w)^T O_1^w \\ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}^T & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} \cos(t) & \sin(t) & 0 & -(\cos^2(t) + \sin^2(t)) \\ -\sin(t) & \cos(t) & 0 & -\sin(t)\cos(t) + \sin(t)\cos(t) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(t) & \sin(t) & 0 & -1 \\ -\sin(t) & \cos(t) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_2^w = \begin{bmatrix} R_2^w & O_1^w \\ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}^T & 1 \end{bmatrix}$$

$$T_2^w = \begin{bmatrix} 1 & 0 & 0 & \sin(t) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \cos(2t) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_2^1 = T_w^1 \; T_2^w = ?$$

$$T_2^1 = \begin{bmatrix} \cos(t) & \sin(t) & 0 & \cos(t)\sin(t)-1 \\ -\sin(t) & \cos(t) & 0 & -\sin^2(t) \\ 0 & 0 & 1 & \cos(2t) \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow$$

$$\Rightarrow \boxed{o_2^1(t) = \begin{bmatrix} \cos(t)\sin(t)-1 \\ -\sin^2(t) \\ \cos(2t) \end{bmatrix}}$$

3 Show that $o_2^1(t)$ describes a planar curve and find the equation of its plane Π.

- Hint: find a linear relation between z_2^1 and y_2^1

$$\left.\begin{aligned} x_2^1(t) &= \cos(t)\sin(t)-1 \\ y_2^1(t) &= -\sin^2(t) \\ z_2^1(t) &= \cos(2t) \end{aligned}\right| \Rightarrow \left.\begin{aligned} x_2^1(t) &= \frac{\sin(2t)}{2} - 1 \\ y_2^1(t) &= \frac{\cos(2t)-1}{2} \\ z_2^1(t) &= \cos(2t) \end{aligned}\right| \Rightarrow$$

used properties:

$$2\sin(t)\cos(t) = \sin(2t) \Rightarrow$$

$$\Rightarrow \sin(t)\cos(t) = \sin(2t)/2$$

$$\cos(2t) = 1 - 2\sin^2(t) \Rightarrow$$

$$\Rightarrow -\sin^2(t) = \frac{\cos(2t)-1}{2}$$

$$\Rightarrow y_2^1(t) = \frac{z_2^1(t)-1}{2} \Rightarrow$$

$$\Rightarrow \boxed{2y_2^1(t) - z_2^1(t) = -1} \Rightarrow$$

$\Rightarrow$ since $2y_2'(t) - z_2'(t) = -1$ is a linear relationship, then $o_2'(t)$ lies in the plane defined by that equation

$$\boxed{\tilde{\pi} = 2y' - z' + 1}$$

4 Rewrite the above trajectory explicitly using a 2D frame of reference (x_p, y_p) on the plane found before. Try to ensure that the curve is centered at the origin of this 2D frame and that x_p, y_p are axes of symmetry for the curve.

- Hints:

a. i) center the new 2D frame in $p^1 = (-1, -1/2, 0)$, these coordinates are in AV1's frame

b. ii) start with a 3D reference frame centered in p with axes (x_p, y_p, z_p), compute $o_2^p(t)$

c. iii) make sure that the z component vanishes after the change of coordinates

$$o_p^1 = \begin{bmatrix} -1 \\ -1/2 \\ 0 \end{bmatrix} \qquad z_p^1 = \begin{bmatrix} 0 \\ 2/\sqrt{2^2+1^2} \\ -1/\sqrt{2^2+1^2} \end{bmatrix} = \begin{bmatrix} 0 \\ 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix} \text{ (from } \tilde{\pi} \text{ equation)}$$

need to rotate with respect to x^1 to align $z^p \Rightarrow$

$$R_p^1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/\sqrt{5} & 2/\sqrt{5} \\ 0 & -2/\sqrt{5} & -1/\sqrt{5} \end{bmatrix}$$

$z_p^1 = z_p^*$ and $y_p^* \perp z_p^*$
rotation about x^1

$$\Rightarrow \boxed{T^1_p = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & -1/\sqrt{5} & 2/\sqrt{5} & -1/2 \\ 0 & -2/\sqrt{5} & -1/\sqrt{5} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \Rightarrow$$

$$\Rightarrow \boxed{T^p_1 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & -1/\sqrt{5} & -2/\sqrt{5} & -1/2\sqrt{5} \\ 0 & 2/\sqrt{5} & -1/\sqrt{5} & 1/\sqrt{5} \\ 0 & 0 & 0 & 1 \end{bmatrix}} \Rightarrow$$

$$\Rightarrow \tilde{O}^p_2 = T^p_1 \, \tilde{O}^1_2 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & -1/\sqrt{5} & -2/\sqrt{5} & -1/2\sqrt{5} \\ 0 & 2/\sqrt{5} & -1/\sqrt{5} & 1/\sqrt{5} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sin(2t)}{2} - 1 \\ (\cos(2t)-1)/2 \\ \cos(2t) \\ 1 \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{1}{2}\sin(2t) \\ -\frac{\sqrt{5}}{2}\cos(2t) \\ 0 \\ 1 \end{bmatrix} \Rightarrow \boxed{O^p_2 = \begin{bmatrix} \frac{1}{2}\sin(2t) \\ -\frac{\sqrt{5}}{2}\cos(2t) \\ 0 \end{bmatrix}}$$

Note that $O^P_{2_z}(t) = 0 \Rightarrow$ planar curve in x^P-y^P

<u>symmetry</u>: 1) y^P axis: take $t^* = -t$:

$$O^P_{2_x}(t^*) = \frac{1}{2}\sin(-2t) = -\frac{1}{2}\sin(2t) = -O^P_{2_x}(t)$$

$$O^P_{2_y}(t^*) = -\frac{5}{\sqrt{2}}\cos(-2t) = -\frac{5}{\sqrt{2}}\cos(2t) = O^P_{2_y}(t)$$

$$\boxed{\Rightarrow \left| \begin{array}{l} O^P_{2_x}(t^*) = -O^P_2(t) \\ O_{2_y}(t^*) = O^P_2(t) \\ \qquad \text{for any } t \text{ and } t^* = -t \end{array} \right| \Rightarrow y^P \text{ symmetry}}$$

2) x^P axis: take $t^* = \frac{\pi}{2} - t$

$$O^P_{2_x}(t^*) = \frac{1}{2}\sin\left(2\left(\frac{\pi}{2} - t\right)\right) = \frac{1}{2}\sin(2t) = O^P_{2_x}(t)$$

$$O^P_{2_y}(t^*) = -\frac{5}{\sqrt{2}}\cos\left(2\left(\frac{\pi}{2} - t\right)\right) = \frac{5}{\sqrt{2}}\cos(2t) = O^P_{2_y}(t)$$

$$\boxed{\Rightarrow \left| \begin{array}{l} O^P_{2_x}(t^*) = O^P_{2_x}(t) \\ O^P_{2_y}(t^*) = -O^P_{2_y}(t) \\ \qquad \text{for any } t \text{ and } t^* = \frac{\pi}{2} - t \end{array} \right| \Rightarrow x^P \text{ symmetry}}$$

5 Using the expression of $o_2^p(t)$, prove that the trajectory of AV2 relative to AV1 is an ellipse and compute the lengths of its semi-axes.

- Hint: what is the general form of the equation of an axis-aligned ellipse centered in the origin?

Equation for ellipse centered at origin: $\frac{x^2(t)}{a^2} + \frac{y^2(t)}{b^2} = 1$

set $a = \frac{1}{2}$ and $b = \frac{\sqrt{5}}{2}$ and substitute in $o^p_{2_x}(t)$ and $o^p_{2_y}(t)$

$$\Rightarrow \boxed{\frac{\frac{1}{4}\sin^2(2t)}{1/4} + \frac{\frac{5}{4}\cos^2(2t)}{5/4} = \sin^2(2t) + \cos^2(2t) = 1 \Rightarrow}$$

$\Rightarrow$ equation for ellipse satisfied! $\Rightarrow$

$\Rightarrow$ semi-minor axis: $a = \frac{1}{2}$

semi-major axis: $b = \frac{\sqrt{5}}{2}$