# MAXIMAL INEQUALITIES FOR NONCOMMUTATIVE MARTINGALES AND APPLICATIONS

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Abstract. In the present paper, we study asymmetric maximal inequalities for noncommutative martingales, and show that for each noncommutative martingale $x = (x_n)_{n\geq 1}$ associated with a regular filtration there exists a universal constant $c > 0$ such that

$$\|x\|_{H_1(\mathcal{M})} \leq c\|x\|_{L_1(\mathcal{M},\ell_\infty)},$$

which provides a positive answer of an open question raised by Junge and Xu 20 years ago. Our results apply to triangular truncation, noncommutative Vilenkin-Fourier series, Mei's operator-valued Hardy spaces and Garnett-Jones' dyadic Carleson decomposition of noncommutative BMO spaces.

## 1. Introduction

Since the remarkable noncommutative Burkholder-Gundy inequality was established by Pisier and Xu [59] in 1997, noncommutative martingale theory has obtained a rapid development, and many classical inequalities have been reformulated to include noncommutative martingales. For instance, Junge obtained in [38] a noncommutative Doob maximal inequality; Randrianantoanina [62] proved that noncommutative martingale transform is of weak $(1,1)$; strong and weak type noncommutative Burkholder/Rosenthal inequalities for conditioned square functions can be found in [36, 42, 45, 64, 65, 67]; noncommutative John-Nirenberg theorem was proved in [40]. We also refer the reader to [31, 32, 34], [9], [66] for recent results concern noncommutative differential subordinate martingales, atomic decomposition and interpolation.

In the present paper, we study several noncommutative asymmetric maximal inequalities for noncommutative martingales. Our first motivation comes from the recent study of noncommutative (asymmetric) Doob maximal inequalities. Let $(\mathcal{M},\tau)$ be a noncommutative probability associated with a weak-$*$ dense increasing filtration $(\mathcal{M}_n)_{n\geq 1}$, and let $\mathcal{E}_n$ denote the conditional expectation from $\mathcal{M}$ onto $\mathcal{M}_n$, $n \geq 1$. Denote by $\mathcal{P}(\mathcal{M})$ the collection of all projections in $\mathcal{M}$. For any sequence $(a_n)_{n\geq 1} \subset L_{p,\infty}(\mathcal{M})$ with $0 < p < \infty$, define

$$\|(a_n)_{n\geq 1}\|_{\Lambda_{p,\infty}(\mathcal{M},\ell_\infty)} = \inf_{e\in\mathcal{P}(\mathbb{R}_+,\mathcal{M})} \{\sup_{t>0} t\tau(\mathbf{1} - e^{(t)})^{\frac{1}{p}} : \sup_n \|e^{(t)} a_n e^{(t)}\|_{L_\infty(\mathcal{M})} \leq t\}, \tag{1.1}$$

where

$$\mathcal{P}(\mathbb{R}_+,\mathcal{M}) := \{e^{(\cdot)} : e^{(t)} \in \mathcal{P}(\mathcal{M}), \quad \forall t \in \mathbb{R}_+\}. \tag{1.2}$$

This is one of the noncommutative maximal functions used in the paper, for the others like $\Lambda_{p,\infty}(\mathcal{M},\ell_\infty^c)$, we refer the reader to Section 2 below. Recall that the first form of Doob's maximal inequality for noncommutative martingales goes back to Cuculescu who established in [12] that for each $x \in L_1(\mathcal{M})$,

$$\|(\mathcal{E}_n(x))_{n\geq 1}\|_{\Lambda_{1,\infty}(\mathcal{M},\ell_\infty)} \leq c\|x\|_{L_1(\mathcal{M})}. \tag{1.3}$$

In 2002, Junge [38] established the following strong type Doob inequality

$$\|(\mathcal{E}_n(x))_{n\geq 1}\|_{L_p(\mathcal{M},\ell_\infty)} \leq c_p\|x\|_{L_p(\mathcal{M})}, \quad 1 < p \leq \infty, \tag{1.4}$$

where the noncommutative maximal function $\|\cdot\|_{L_p(\mathcal{M},\ell_\infty)}$ is defined in (2.6) below. It is different with the classical martingale setting that now the best order is $c_p = O(\frac{1}{(p-1)^2})$ as $p \to 1$; this is proved by Junge and Xu in [43, Theorem 8]. As for the case $p = 1$, the famous Davis inequality (see [13]) for commutative martingales defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ states that there exists a universal constant $c > 0$ such that

$$\frac{1}{c}\|x\|_{H_1(\Omega)} \le \|\sup_n |\mathcal{E}_n(x)|\|_{L_1(\Omega)} \le c\|x\|_{H_1(\Omega)}, \tag{1.5}$$

where the martingale Hardy space $H_1(\Omega)$ is referred to Section 2.3. However, for noncommutative martingales, it was shown by Junge and Xu [43, Lemma 13] that there does NOT exist a universal constant $c > 0$ such that

$$\|(\mathcal{E}_n(x))_{n\ge1}\|_{L_1(\mathcal{M},\ell_\infty)} \le c\|x\|_{H_1(\mathcal{M})}. \tag{1.6}$$

Motivated by this and the open problem raised in [43] (see Problem 1.1 below), people considered noncommutative Davis inequality with noncommutative conditioned square function in place of the maximal function, and obtained several noncommutative Davis decomposition results. For this line of investigation, we refer the reader to [56] and [39] via duality arguments, and to [41] and [67] via constructive approach. A break through was made by Hong et al. who proved an asymmetric maximal inequality in [27, Theorem B], which can be viewed a noncommutative version of the right-hand side inequality of (1.5). However, it remains open whether the converse of (1.6) holds or one can find a noncommutative analogy of left-hand side inequality of (1.5). This is actually an open problem raised by Junge and Xu 20 years ago ([43]). We rewrite it as follows.

**Problem 1.1** ([43, Problem 16])**.** *It is entirely open that whether there exists a universal constant $c > 0$ such that*

$$\|x\|_{H_1(\mathcal{M})} \le c\|(\mathcal{E}_n(x))_{n\ge1}\|_{L_1(\mathcal{M},\ell_\infty)}.$$

Our first main result of the paper provides positive answer to this problem under the assumption that the related filtration is regular; see Theorem 1.5 below.

Now we turn to describe the second object of the paper. To this end, let us first recall the definition of noncommutative almost uniform convergence. The following definition, which goes back at least Lance [48], is taken from [11] and [14, p. 109].

**Definition 1.2.** *Consider $(x_k)_{k\ge1} \subseteq L_0(\mathcal{M})$ and $x \in L_0(\mathcal{M})$.*

(i) *We say $(x_k)_{k\ge1}$ converges to $x$ column almost uniformly (c.a.u. in short) if for any $\varepsilon > 0$, there is a projection $e \in \mathcal{P}(\mathcal{M})$ such that*

$$\tau(\mathbf{1} - e) < \varepsilon, \quad \lim_{k\to\infty} \|(x_k - x)e\|_{\mathcal{M}} = 0.$$

(ii) *We say $(x_k)_{k\ge1}$ converges to $x$ row almost uniformly (r.a.u. in short) if $(x_k^*)_{k\ge1}$ converges to $x^*$ column almost uniformly.*

(iii) *We say $(x_k)_{k\ge1}$ converges to $x$ column + row almost uniformly (c+r a.u.) provided the sequence $(x_k - x)_{k\ge1}$ decomposes into a sum $(a_k)_{k\ge1} + (b_k)_{k\ge1}$ of two sequences such that $(a_k)_{k\ge1}$ converges to $0$ column almost uniformly and $(b_k)_{k\ge1}$ converges to $0$ row almost uniformly.*

(iv) *We say $(x_k)_{k\ge1}$ converges to $x$ bilaterally almost uniformly (b.a.u. in short) if for any $\varepsilon > 0$, there is a projection $e \in \mathcal{P}(\mathcal{M})$ such that*

$$\tau(\mathbf{1} - e) < \varepsilon, \quad \lim_{k\to\infty} \|e(x_k - x)e\|_{\mathcal{M}} = 0.$$

The column almost uniform convergence is the usual almost uniform convergence used in [44] and [50]. It was shown in [14, Chapter 3.1.8] that the below implications hold, however the converse implications do not hold:

$$\text{a.u. or r.a.u.} \Rightarrow c + r \text{ a.u.} \Rightarrow \text{b.a.u.}$$

The above definition generalizes the notion of almost everywhere convergence in the case of finite measure spaces. Actually, for finite abelian von Neumann algebra $\mathcal{M}$ (i.e., $\tau(1) < \infty$), the almost uniform convergences in the definition above are all equivalent to the usual almost everywhere convergence by virtue of Egorov's theorem.

Nowadays, it is known that for each $x \in L_p(\mathcal{M})$ with $2 \le p < \infty$, $(\mathcal{E}_n(x))_n$ converges to $x$ column or row almost uniformly. In fact, this can be deduced from the asymmetric maximal inequalities [38, Corollary 4.6] and [27, Theorem A]. As for $1 < p < 2$, recall that (see [27, Theorem A])

$$\|(\mathcal{E}_n(x))_n\|_{\Lambda_{p,\infty}(\mathcal{M},\ell_\infty^c)} \le c\|x\|_{H_p^c(\mathcal{M})}, \quad \|(\mathcal{E}_n(x))_n\|_{\Lambda_{p,\infty}(\mathcal{M},\ell_\infty^r)} \le c\|x\|_{H_p^r(\mathcal{M})},$$

which together with [34, Theorem 4.10] gives us that there exist two martingales $x^c$ and $x^r$ such that $x = x^c + x^r$

$$\|(\mathcal{E}_n(x^c))_n\|_{\Lambda_{p,\infty}(\mathcal{M},\ell_\infty^c)} + \|(\mathcal{E}_n(x^r))_n\|_{\Lambda_{p,\infty}(\mathcal{M},\ell_\infty^r)} \le c_p\|x\|_{L_p(\mathcal{M})}, \quad 1 < p < 2. \tag{1.7}$$

From this asymmetric maximal inequality, we can deduce that for each $x \in L_p(\mathcal{M})$ with $1 < p < 2$, $(\mathcal{E}_n(x))_n$ converges to $x$ column + row almost uniformly. This kind of almost uniform convergence may be best possible because very recently, Hong and Ricard [30] provided a counterexample to show that the almost uniform convergence can not happen for truly noncommutative $L_p$ martingales with $1 \le p < 2$. For the case $p = 1$, what we know is that Cuculescu [12] deduced from the weak type maximal inequality (1.3) that for each $x \in L_1(\mathcal{M})$, $(\mathcal{E}_n(x))_n$ converge bilaterally almost uniformly. Thus it is natural to ask the following question:

**Problem 1.3.** *For each $x \in L_1(\mathcal{M})$, whether $(\mathcal{E}_n(x))_n$ converge column + row almost uniformly?*

Our second main result of the present paper provides positive answers to the above Problem 1.1 under the assumption that the filtration $(\mathcal{M}_n)_{n\ge1}$ is regular. We say a filtration $(\mathcal{M}_n)_{n\ge1}$ is regular with constant $c_{\mathrm{reg}}$ if for any positive $x \in \mathcal{M}$,

$$\mathcal{E}_n(x) \le c_{\mathrm{reg}}\mathcal{E}_{n-1}(x), \quad \forall n \ge 1. \tag{1.8}$$

Our main results are listed as below.

**Theorem 1.4.** *Let $(\mathcal{M},\tau)$ be a noncommutative probability space associated with the regular filtration $(\mathcal{M}_n)_{n\ge1}$. For each martingale $x = (x_n)_{n\ge1}$, there exist two martingales $x^c$ and $x^r$ such that $x = x^c + x^r$ and the following holds:*

(i) *We have*

$$\|x^c\|_{\Lambda_{1,\infty}(\mathcal{M},\ell_\infty^c)} + \|x^r\|_{\Lambda_{1,\infty}(\mathcal{M},\ell_\infty^r)} \le 8(6c_{\mathrm{reg}} + 18)\|x\|_{L_1(\mathcal{M})}.$$

(ii) *There exists a constant $c_p > 0$ such that*

$$\|x^c\|_{\Lambda_p(\mathcal{M},\ell_\infty^c)} + \|x^r\|_{\Lambda_p(\mathcal{M},\ell_\infty^r)} \le c_p\|x\|_{L_p(\mathcal{M})}, \quad 1 < p < 2.$$

*The order of the constant, $O(\frac{1}{p-1})$ as $p \to 1$, is already best possible for commutative martingales.*

(iii) *Consequently, for each martingale $x = (x_n)_{n\ge1} \in L_p(\mathcal{M})$ with $1 \le p < 2$, $(x_n)_{n\ge1}$ converges column + row almost uniformly.*

**Theorem 1.5.** *Let $(\mathcal{M},\tau)$ be a noncommutative probability space associated with the regular filtration $(\mathcal{M}_n)_{n\ge1}$. For each martingale $x = (x_n)_{n\ge1} \in L_p(\mathcal{M},\ell_\infty)$ with $0 < p \le 1$, there exists positive constant $c_{p,\mathrm{reg}}$ depends on $p$ and the regularity constant $c_{\mathrm{reg}}$ such that*

$$\|x\|_{H_p(\mathcal{M})} \le c_{p,\mathrm{reg}}\|x\|_{L_p(\mathcal{M},\ell_\infty)}.$$

**Remark 1.6.** *We put a few comments below.*

(i) *Note that $\|z\|_{\Lambda_{p,\infty}(\mathcal{M},\ell_\infty^c)} \le \|z\|_{\Lambda_p(\mathcal{M},\ell_\infty^c)}$ for each $z$ and $1 \le p < \infty$. Thus item (ii) in Theorem 1.4 is stronger than (1.7). At the time of this writing, we do not know whether items (i) and (ii) in Theorem 1.4 hold without the assumption.*

(ii) *Consider the Orlicz space $L\log L(\mathcal{M})$ which satisfies $L_p(\mathcal{M}) \subset L\log L(\mathcal{M}) \subset L_1(\mathcal{M})$, $1 < p < \infty$. Without the assumption of regularity, we can show that for each martingale $x = (x_n)_{n\ge1} \in L\log L(\mathcal{M})$, $(x_n)_{n\ge1}$ converges column + row almost uniformly; see Section 5.2 for details.*

(iii) *It is not surprise that Theorem 1.5 holds for $0 < p \le 1$. Actually, for commutative martingales associated with regular filtration, different kinds of martingale Hardy spaces are equivalent for $0 < p \le 1$; see e.g.* [75, Corollary 2.23].

(iv) *With the same proof, the above results hold for general semi-finite von Neumann algebras associated with a regular filtration.*

For further applications, we will prove Theorem 1.4 and Theorem 1.5 in a more general framework. This is motivated by the recent advance of noncommutative differential subordination and weak domination martingales [31, 32, 34, 53]. Let us briefly recall the basic definitions in the classical martingale setting. For a given martingale $x = (x_n)_{n\geq 1} \in L_1(\mathcal{M})$, write its martingale difference by $d_1 x = x_1$ and

$$d_n x = x_n - x_{n-1}, \quad n \geq 2.$$

Considering martingales defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, following Burkholder [5], we say martingale $y = (y_n)_{n\geq 1} \in L_1(\Omega)$ is differentially subordinate to $x = (x_n)_{n\geq 1} \in L_1(\Omega)$ if

$$|d_n y| \leq |d_n x|, \quad \forall n \geq 1.$$

Burkholder proved that for any martingale $x$, if the martingale $y$ is differentially subordinate to $x$, then the following sharp inequalities hold:

$$\|y\|_{L_{1,\infty}(\Omega)} \leq 2\|x\|_{L_1(\Omega)} \tag{1.9}$$

and

$$\|y\|_{L_p(\Omega)} \leq (p^* - 1)\|x\|_{L_p(\Omega)}, \quad 1 < p < \infty, \tag{1.10}$$

where $p^* = \max\{p, \frac{p}{p-1}\}$. These inequalities have deep influence in both probability and harmonic setting; we refer the reader to the survey [3] and the book [52] for more information about this topic.

Burkholder's results have many extensions; for example the type of domination considered [47, 51] is less restrictive: a martingale $y$ is weakly dominated by $x$, if for any $n \geq 1$ and any nondecreasing convex function $\Phi : [0, \infty) \to \mathbb{R}$ with a linear growth at infinity we have

$$\mathcal{E}_{n-1}\Phi(|d_n y|) \leq \mathcal{E}_{n-1}\Phi(|d_n x|). \tag{1.11}$$

Recently, the notion of subordination and domination were also introduced in noncommutative martingale setting [31, 33]. The following definition is taken from [31, Definition 3.1] and [33, Definition 2.4], respectively.

**Definition 1.7.** *Let $x = (x_n)_{n\geq 1}$ and $y = (y_n)_{n\geq 1}$ be two self-adjoint $L_2(\mathcal{M})$-bounded martingales.*

(i) *We say $y$ is differentially subordinate to $x$ if for any $Q \in \mathcal{M}_{n-1}$,*

$$Qdy_n Qdy_n Q \leq Qd_n x Qd_n x Q.$$

(ii) *We a martingale $y$ is weakly dominated by $x$, if for any $n \geq 1$ and any nondecreasing convex function $\Phi : [0, \infty) \to \mathbb{R}$ with a linear growth at infinity we have*

$$\tau(\Phi(Qd_n yQ)) \leq \tau(\Phi(Qd_n xQ)).$$

Under these appropriate dominations, it was shown in [31, 33] that the aforementioned two inequalities (1.9) and (1.10) remain valid (but with different constants) in the noncommutative martingale setting.

In this paper, we prove Theorem 1.4 and Theorem 1.5 for the following noncommutative differential subordination martingales; see Theorem 1.10 and Theorem 1.11 below. The following new definition is weaker than Definition 1.7.

**Definition 1.8.** *Let $x = (x_n)_{n\geq 1}$ and $y = (y_n)_{n\geq 1}$ be two $L_2(\mathcal{M})$-bounded martingales. We say a martingale $y = (y_n)_{n\geq 1}$ (not necessary self-adjoint) is $L_2$-dominated by $x$ if for any projection $Q \in \mathcal{M}_{n-1}$, we have*

$$\|Qdy_n Q\|_{L_2(\mathcal{M})} \leq \|Qdx_n Q\|_{L_2(\mathcal{M})}.$$

As we will see, this definition is good enough for our purpose of the paper.

**Remark 1.9.** *It is clear that our definition is weaker than Osekowski's version. Also is weaker than [31]. The advantage of our definition is that: $P_{\mathrm{j}}$ in [18] satisfies our definition but does not satisfies theirs.*

*In the classical setting, Condition (ii) is weaker than $|dy_n| \leq |dx_n|$. However, if both $y$ and $x$ are conditionally symmetric martingales, then Condition (ii) is equivalent to $|dy_n| \leq |dx_n|$.*

**Theorem 1.10.** *Let $(\mathcal{M}, \tau)$ be a noncommutative probability space associated with the regular filtration $(\mathcal{M}_n)_{n\geq 1}$. Suppose that martingale $x \in L_2(\mathcal{M})$ is positive, and martingale $y$ is $L_2$-dominated by $x$. There exist two martingales $y^c$ and $y^r$ such that $y = y^c + y^r$, and moreover*

(i) *We have*
$$\|y^c\|_{\Lambda_{1,\infty}(\mathcal{M},\ell_\infty^c)} + \|y^r\|_{\Lambda_{1,\infty}(\mathcal{M},\ell_\infty^r)} \leq (6c_{\mathrm{reg}} + 18)\|x\|_{L_1(\mathcal{M})}.$$

(ii) *There exists a constant $c_p > 0$ such that*
$$\|y^c\|_{\Lambda_p(\mathcal{M},\ell_\infty^c)} + \|y^r\|_{\Lambda_p(\mathcal{M},\ell_\infty^r)} \leq c_p\|x\|_{L_p(\mathcal{M})}, \quad 1 < p < 2.$$
*The order of the constant, $O(\frac{1}{p-1})$ as $p \to 1$, is already best possible for commutative martingales.*

(iii) *There exists a universal constant $c > 0$ such that*
$$\|y^c\|_{\Lambda_1(\mathcal{M},\ell_\infty^c)} + \|y^r\|_{\Lambda_1(\mathcal{M},\ell_\infty^r)} \leq 1 + (10c_{\mathrm{reg}} + 394)\|x\|_{L\log L(\mathcal{M})}.$$

**Theorem 1.11.** *Let $(\mathcal{M}, \tau)$ be a noncommutative probability space associated with the regular filtration $(\mathcal{M}_n)_{n\geq 1}$. Suppose that martingale $x \in L_2(\mathcal{M})$ is positive, and martingale $y$ is $L_2$-dominated by $x$. There exist two martingales $y^c$ and $y^r$ such that $y = y^c + y^r$, and moreover*
$$\|y^c\|_{H_p^c} + \|y^r\|_{H_p^r} \leq c_{p,\mathrm{reg}}\|x\|_{L_p(\mathcal{M},\ell_\infty)}, \quad 0 < p \leq 1.$$
*Here, the constant $c_{p,\mathrm{reg}}$ only depends on $p$ and the regularity constant $c_{\mathrm{reg}}$*

Our proofs depend on very careful estimates of Cuculescu projections and a new type Gundy decomposition. Besides the results mentioned above, our estimates of Cuculescu projections also allow us to get the following

(i) Let $x = (x_n)_{n\geq 1}$ and $y = (y_n)_{n\geq 1}$ be two self-adjoint $L_2(\mathcal{M})$-bounded martingales. If $y$ is weakly differentially subordinate to $x$ in the sense of Definition 1.7, then
$$\|y\|_{L_1(\mathcal{M})} \leq c\|x\|_{L\log L(\mathcal{M})}.$$
This is a supplement of [31] and is a generalization of [62, Theorem 6.2].

(ii) Let $x = (x_n)_{n\geq 1}$ and $y = (y_n)_{n\geq 1}$ be two self-adjoint $L_2(\mathcal{M})$-bounded martingales. If $y$ is weakly differentially subordinate to $x$ in the sense of Definition 1.7, then there exist a decomposition of $y = y^c + y^r$ such that
$$\|S_c(y^c)\|_{L_1(\mathcal{M})} + \|S_r(y^r)\|_{L_1(\mathcal{M})} \leq c\|x\|_{L\log L(\mathcal{M})}.$$
This is a supplement of [34] and is stronger than [64, Theorem 5.7].

The above two results are discussed details in Section 4.2 and Section 5.2. Our results have many applications, here we only mention several of them

(i) Triangular truncation is a typical example which satisfies Definition 1.8.
(ii) A modified partial sum operator of Vilenkin Fourier series $P_{\mathrm{j}}$ satisfies Definition 1.8.
(iii) Theorem 1.5 applies to Mei's operator-valued Hardy space.
(iv) A Carleson's decomposition of noncommutative BMO space can be deduced from Theorem 1.5.

Hence, Theorem 1.10 and Theorem 1.11 and their corollaries apply to triangular truncation operator $\mathcal{T}$ and $P_{\mathrm{j}}$. The details are given in Section 6.

The organization of the paper is as follows. Definitions, notions and basic lemmas are presented in Section 2. We collect useful estimates for Cuculescu projections in Section 3. These estimates will be used in the whole paper. Section 4 devotes to proving Theorems 1.10 and Theorem 1.4. Several interesting remarks and questions are also given in this section. Section 5 contains the proofs of Theorem 1.5 and Theorem 1.11. Finally, in Section 6, we provide applications of our main results.

## 2. Preliminary

Throughout this paper, $c_p$ always denotes a positive constant only depends on symbol $p$.

2.1. **Noncommutative symmetric spaces.** Let $\mathcal{M}$ be a finite von Neumann algebra equipped with a normal faithful normalized trace $\tau$. We always assume $\mathcal{M} \subset B(H)$ for some Hilbert space $H$. Denote by $L_0(\mathcal{M})$ the topological $*$-algebra of all $\tau$-measurable operators. Suppose that $a$ is a self-adjoint $\tau$-measurable operator and let $a = \int_{-\infty}^{\infty} \lambda de_\lambda$ stand for its spectral decomposition. For any Borel subset $B$ of $\mathbb{R}$, the spectral projection of $a$ corresponding to the set $B$ is defined by $\chi_B(a) = \int_{-\infty}^{\infty} \chi_B(\lambda) de_\lambda$. For $x \in L_0(\mathcal{M})$, we denote by $r(x)$ and $l(x)$ the right and left support projections of $x$, respectively. In fact, the projection $r(x)$ (resp. $l(x)$) is the least projection $e$ satisfying $xe = x$ (resp. $ex = x$). For $x \in L_0(\mathcal{M})$, the spectral distribution function $\mathsf{d}(\cdot, x)$ is defined by

$$\mathsf{d}(s,x) = \tau\big(\chi_{(s,\infty)}(|x|)\big),$$

and the generalized singular value function $\mu(x)$ is defined by

$$\mu(t,x) = \inf\big\{s > 0 : \mathsf{d}(s,x) \le t\big\}, \quad t > 0.$$

The function $t \mapsto \mu(t,x)$ is decreasing and right-continuous. In the case that $\mathcal{M}$ is the abelian von Neumann algebra $L_\infty(0,1)$ with the trace given by integration with respect to the Lebesgue measure, $L_0(\mathcal{M})$ is the space of all measurable functions, with non-trivial distribution, and $\mu(f)$ is the decreasing rearrangement of a measurable function $f$. For more information of the singular value function we refer the reader to [20, 16].

We collect several basic facts which are often used in the present paper. The proof of item (iv) below is referred to [16, Proposition 3.2.10].

**Fact 2.1.** *Let $x \in L_0(\mathcal{M})$.*

(i) *For $s_1, s_2 > 0$, if $s_1 \le s_2$, then $\chi_{(s_2,\infty)}(|x|) \le \chi_{(s_1,\infty)}(|x|)$.*
(ii) *It follows from item (i) above that $\mathsf{d}(s,x)$ is decreasing on $s$.*
(iii) *It follows from the definition that $t \le \mathsf{d}(s,x)$ if and only if $s \le \mu(t,x)$.*
(iv) *If $e \in L_0(\mathcal{M})$ is a projection, then*

$$\mu(t,e) = \chi_{[0,\tau(e)]}(t), \quad t > 0.$$

A Banach (or quasi-Banach) function space $(E, \|\cdot\|_E)$ on the interval $(0,1)$ is called symmetric if for every $g \in E$ and for every measurable function $f \in L_0(0,1)$ with $\mu(f) \le \mu(g)$, we have $f \in E$ and $\|f\|_E \le \|g\|_E$. Following [46], for a given symmetric Banach (or quasi-Banach) function space $(E, \|\cdot\|_E)$ , we define the corresponding noncommutative space on $(\mathcal{M}, \tau)$ by setting

$$E(\mathcal{M}) := \big\{x \in L_0(\mathcal{M}) : \mu(x) \in E\big\}.$$

The associated norm (or quasi-norm) is

$$\|x\|_{E(\mathcal{M})} = \|\mu(x)\|_E. \tag{2.1}$$

It is shown in [46] (resp. [69]) that if $E(0,1)$ is a symmetric Banach (resp. quasi-Banach) function space, then $E(\mathcal{M})$ is a Banach space (resp. a quasi-Banach).

In the present paper, we concern the following concrete examples of symmetric (quasi-) Banach function spaces.

Consider the usual Lebesgue space $L_p(0,1)$, $0 < p < \infty$. The noncommutative Lebesgue space $L_p(\mathcal{M})$ can be defined according to (2.1), that is,

$$\|x\|_{L_p(\mathcal{M})}^p = \int_0^1 \mu(t,x)^p dt, \quad 0 < p < \infty.$$

Moreover, for each $x \in L_p(\mathcal{M})$, we have the following relationship ([16, Proposition 3.3.9] or [20, Corollary 2.8])

$$\tau(|x|^p) = \int_0^1 \mu(t,x)^p dt = p\int_0^\infty \lambda^{p-1}\tau\big(\chi_{(\lambda,\infty)}(|x|)\big)d\lambda. \tag{2.2}$$

As usual, $L_\infty(\mathcal{M})$ is $\mathcal{M}$ with the usual operator norm. We refer the reader to [60] for more information about the noncommutative Lebesgue spaces.

Consider the weak Lebesgue space $L_{p,\infty}(0,1)$, $0<p<\infty$, equipped with the quasi-norm

$$\|f\|_{L_{p,\infty}(0,1)} = \sup_{t>0} t^{1/p}\mu(t,f).$$

The corresponding noncommutative weak Lebesgue space $L_{p,\infty}(\mathcal{M})$ can be defined according to (2.1). Moreover, for each $x \in L_{p,\infty}(\mathcal{M})$, we have (see also [16] or [20]),

$$\|x\|_{L_{p,\infty}(\mathcal{M})} = \sup_{t>0} t^{1/p}\mu(t,f) = \sup_{\lambda>0} \lambda[\tau\big(\chi_{(\lambda,\infty)}(|x|)\big)]^{1/p}.$$

The commutative $L\log L(0,1)$ space is defined by

$$L\log L(0,1) = \{f \in L_0(0,1) : \int_0^1 |f(t)|\log^+|f(t)|dt < \infty\},$$

equipped with the norm

$$\|f\|_{L\log L(0,1)} = \int_0^1 |f(t)|\log^+|f(t)|dt.$$

Here $\log^+(t) = \max\{0, \log(t)\}$. This space $L\log L(0,1)$ is symmetric (see e.g. [4]). The corresponding noncommutative weak Lebesgue space $L\log L(\mathcal{M})$ can be defined according to (2.1), i.e., for each $x \in L\log L(\mathcal{M})$, we have

$$\|x\|_{L\log L(\mathcal{M})} = \int_0^1 |\mu(t,x)|\log^+|\mu(t,x)|dt.$$

In the sequel of the paper, if there is no confusion appears, for each $x \in L_0(\mathcal{M})$, we may write $\|x\|_p$, $\|x\|_{L\log L}$ instead of $\|x\|_{L_p(\mathcal{M})}$ and $\|x\|_{L\log L(\mathcal{M})}$, respectively.

We conclude this subsection with the following result which is taken from [16, Lemma 3.4.27] (see also [20, Theorem 4.2]).

**Lemma 2.2.** *If $0 \le a, b \in \mathcal{M}$, then*

$$\tau(ab) \le \int_0^\infty \mu(t,a)\mu(t,b)dt. \tag{2.3}$$

2.2. **Noncommutative maximal functions.** In this subsection, following [38, 58] (see also [27, 11]), we recall several different noncommutative maximal functions, i.e., $L_p(\mathcal{M},\ell_\infty)$ and $\Lambda_{p,\infty}(\mathcal{M},\ell_\infty^c)$ and $\Lambda_p(\mathcal{M},\ell_\infty^c)$.

We begin with the weak type noncommutative maximal functions. Recall that $\mathcal{P}(\mathbb{R}_+,\mathcal{M})$ is given in (1.2). For any sequence $a=(a_n)_{n\ge1} \subset L_{p,\infty}(\mathcal{M})$ with $0<p<\infty$, define

$$\|a\|_{\Lambda_{p,\infty}(\mathcal{M},\ell_\infty^c)} = \inf_{e\in\mathcal{P}(\mathbb{R}_+,\mathcal{M})} \{\sup_{t>0} t\tau(\mathbf{1}-e^{(t)})^{\frac{1}{p}} : \sup_n \|a_n e^{(t)}\|_\infty \le t\}, \tag{2.4}$$

and define

$$\|a\|_{\Lambda_{p,\infty}(\mathcal{M},\ell_\infty^r)} = \|a^*\|_{\Lambda_{p,\infty}(\mathcal{M},\ell_\infty^c)}.$$

The following definition is taken from [11, Remark 5.3]. For a sequence of operators $a=(a_n)_{n\ge1} \subset L_p(\mathcal{M})$ with $0<p<\infty$, define

$$\|a\|_{\Lambda_p(\mathcal{M},\ell_\infty)} = \inf_{e\in\mathcal{P}(\mathbb{R}_+,\mathcal{M})} \Big\{\Big[p\int_0^\infty t^{p-1}\tau(1-e^{c,t})dt\Big]^{1/p} : \sup_n \|e^t a_n e^t\|_\infty \le t\Big\},$$

Similar to (2.4), we define for a sequence of operators $a=(a_n)_{n\ge1} \subset L_p(\mathcal{M})$ with $0<p<\infty$,

$$\|a\|_{\Lambda_p(\mathcal{M},\ell_\infty^c)} = \inf_{e\in\mathcal{P}(\mathbb{R}_+,\mathcal{M})} \Big\{\Big[p\int_0^\infty t^{p-1}\tau(1-e^{c,t})dt\Big]^{1/p} : \sup_n \|a_n e^{c,t}\|_\infty \le t\Big\}, \tag{2.5}$$

and define

$$\|a\|_{\Lambda_p(\mathcal{M},\ell_\infty^r)} = \|a^*\|_{\Lambda_p(\mathcal{M},\ell_\infty^c)}.$$

For $0 < p \leq \infty$, define $L_p(\mathcal{M}, \ell_\infty)$ as the space of all sequences $a = (a_n)_{n\geq 1}$ in $L_p(\mathcal{M})$ for which there exist $y \in L_{2p}(\mathcal{M})$, $z \in L_{2p}(\mathcal{M})$ and a bounded sequence $v = (v_n)_{n\geq 1} \subset \mathcal{M}$ such that

$$a_n = y v_n z, \qquad \forall n \in \mathbb{N}.$$

For $a = (a_n)_{n\geq 1} \in L_p(\mathcal{M}, \ell_\infty)$, define

$$\|a\|_{L_p(\mathcal{M},\ell_\infty)} = \inf\{\|y\|_{2p} \sup_{n\geq 1} \|v_n\|_\infty \|z\|_{2p}\}, \tag{2.6}$$

where the infimum is taken over all possible factorizations of $a$ as above. According to the definitions, it is obvious that

$$\|a\|_{\Lambda_{p,\infty}(\mathcal{M},\ell_\infty)} \leq \|a\|_{L_p(\mathcal{M},\ell_\infty)}, \quad \forall 0 < p < \infty.$$

Besides, if $a = (a_n)_{n\geq 1} \in L_p(\mathcal{M}, \ell_\infty)$ and $a_n = a_n^*$ for each $n$, then (see [15, p. 518] or [44])

$$\|a\|_{L_p(\mathcal{M},\ell_\infty)} = \inf\{\|A\|_{L_p(\mathcal{M})} : A \in L_p(\mathcal{M}), -A \leq a_n \leq A,\ \forall n \geq 1\}. \tag{2.7}$$

In the commutative setting $\mathcal{M} = L_\infty(\Omega)$, one can see that

$$\|a\|_{L_p(\mathcal{M},\ell_\infty)} = \|a\|_{\Lambda_p(\mathcal{M},\ell_\infty)} = \|a\|_{\Lambda_p(\mathcal{M},\ell_\infty^c)} = \|a\|_{\Lambda_p(\mathcal{M},\ell_\infty^r)} = \|\sup_n |a_n|\|_{L_p(\Omega)}.$$

It is clear that for each $a = (a_n) \subset L_{p,\infty}(\mathcal{M})$

$$\sup_n \|a_n\|_{L_{p,\infty}(\mathcal{M})} \leq \|a\|_{\Lambda_{p,\infty}(\mathcal{M},\ell_\infty)} \leq \min\{\|a\|_{\Lambda_{p,\infty}(\mathcal{M},\ell_\infty^c)}, \|a\|_{\Lambda_{p,\infty}(\mathcal{M},\ell_\infty^r)}\}, \tag{2.8}$$

where the left-hand side inequality was implicit in [35, Remark 3.4 (i)]. Moreover, we further have the following

**Lemma 2.3.** *Let $0 < p < \infty$. For each sequence $a = (a_n)_{n\geq 1} \subset L_{p,\infty}(\mathcal{M})$, we have*

$$\|a\|_{\Lambda_{p,\infty}(\mathcal{M},\ell_\infty)} \leq 2^{1/p} \inf_{a=a^c+a^r} \{\|a^c\|_{\Lambda_{p,\infty}(\mathcal{M},\ell_\infty^c)} + \|a^r\|_{\Lambda_{p,\infty}(\mathcal{M},\ell_\infty^r)}\}.$$

*Proof.* Let $a = a^c + a^r$. According to the definition of $\|\cdot\|_{\Lambda_{p,\infty}(\mathcal{M},\ell_\infty^c)}$, we can take $e_c^{(\cdot)} \in \mathcal{P}(\mathbb{R}_+, \mathcal{M})$ such that $\sup_n \|a_n^c e_c^t\|_\infty \leq t$ and

$$\sup_t t[\tau(1 - e_c^t)]^{1/p} \leq \|a^c\|_{\Lambda_{p,\infty}(\mathcal{M},\ell_\infty^c)}.$$

Similarly, we can take $e_r^{(\cdot)} \in \mathcal{P}(\mathbb{R}_+, \mathcal{M})$ such that $\sup_n \|e_r^t a_n^c\|_\infty \leq t$ and

$$\sup_t t[\tau(1 - e_r^t)]^{1/p} \leq \|a^c\|_{\Lambda_{p,\infty}(\mathcal{M},\ell_\infty^c)}.$$

For each $t > 0$, set $e^t = e_c^t \wedge e_r^t$. Then

$$\sup_n \|e^t a_n e^t\|_\infty \leq \sup_n \|e^t a_n^c e^t\|_\infty + \sup_n \|e^t a_n^r e^t\|_\infty \leq 2t, \quad \forall t > 0.$$

In addition, for each $t > 0$, we have

$$\begin{aligned} t[\tau(1 - e^t)]^{1/p} &\leq t[\tau(1 - e_c^t) + \tau(1 - e_r^t)]^{1/p} \\ &\leq 2^{1/p-1}\Big(t[\tau(1 - e_c^t)]^{1/p} + t[\tau(1 - e_r^t)]^{1/p}\Big) \\ &\leq 2^{1/p-1}(\|a^c\|_{\Lambda_{p,\infty}(\mathcal{M},\ell_\infty^c)} + \|a^r\|_{\Lambda_{p,\infty}(\mathcal{M},\ell_\infty^r)}), \end{aligned}$$

which together with the definition of $\|\cdot\|_{\Lambda_{p,\infty}(\mathcal{M},\ell_\infty)}$ implies

$$\|z\|_{\Lambda_{p,\infty}(\mathcal{M},\ell_\infty)} \leq 2^{1/p} \inf_{z=z^c+z^r} \{\|z^c\|_{\Lambda_{p,\infty}(\mathcal{M},\ell_\infty^c)} + \|z^r\|_{\Lambda_{p,\infty}(\mathcal{M},\ell_\infty^r)}\}.$$

□

Similarly, we can prove

**Lemma 2.4.** *Let $0 < p < \infty$. For each sequence $a = (a_n)_{n\geq 1} \subset L_p(\mathcal{M})$, we have*

$$\sup_n \|a_n\|_{L_p(\mathcal{M})} \leq \|a\|_{\Lambda_p(\mathcal{M},\ell_\infty)} \leq 2^{1/p} \inf_{a=a^c+a^r} \{\|a^c\|_{\Lambda_p(\mathcal{M},\ell_\infty^c)} + \|a^r\|_{\Lambda_p(\mathcal{M},\ell_\infty^r)}\}.$$

The below result with $p=1$ is essentially due to Junge and Xu [44, Proposition 2.1(i)]. The same argument used in [44, Proposition 2.1(i)] also works for $0<p<1$. We still include the details for the reader's convenience.

**Lemma 2.5.** *Let $0<p\leq 1$. Assume that $a=(a_n)_{n\geq 1}\in L_p(\mathcal{M},\ell_\infty)$ with $\|a\|_{L_p(\mathcal{M},\ell_\infty)}\leq 1$. Then there exist positive $\{b^k\}_{k=1}^{16}\subset L_p(\mathcal{M},\ell_\infty)$ such that $a$ is a linear combination of $\{b^k\}_{k=1}^{16}$ and*

$$\|b^k\|_{L_p(\mathcal{M},\ell_\infty)}\leq c_p,\quad \forall 1\leq k\leq 16.$$

*Proof.* Note that $L_p(\mathcal{M},\ell_\infty)$ is closed with respect to involution. Hence, it suffices to consider self-adjoint elements of $L_p(\mathcal{M},\ell_\infty)$. Assume that $a=(a_n)_{n\geq 1}\in L_p(\mathcal{M},\ell_\infty)$ with $a=a^*$, $\|a\|_{L_p(\mathcal{M},\ell_\infty)}\leq 1$. Write a factorization of $a=(a_n)$:

$$a_n=y^*v_nz,\quad \|y\|_{2p}\leq 1,\quad \|z\|_{2p}\leq 1,\quad \sup_n\|v_n\|_\infty\leq 1.$$

Following the proof of [44, Proposition 2.1], by a standard polorization argument, we have

$$\begin{aligned}
a_n&=\frac{1}{4}\sum_{k=0}^{3}i^{-k}(y+i^kz)^*v_n(y+i^kz)\\
&=\frac{1}{4}\sum_{k=0}^{3}(y+i^kz)^*u_n(y+i^kz)=\sum_{k=0}^{3}(\frac{y+i^kz}{2})^*u_n(\frac{y+i^kz}{2})\\
&=\sum_{k=0}^{3}(\frac{y+i^kz}{2})^*u_n^+(\frac{y+i^kz}{2})-\sum_{k=0}^{3}(\frac{y+i^kz}{2})^*u_n^-(\frac{y+i^kz}{2}),
\end{aligned}$$

where $u_n^+$ and $u_n^-$ denote the positive part and negative part of $u_n$, and

$$u_n=\frac{i^{-k}v_n+(i^{-k}v_n)^*}{2}.$$

Note that for $0<r<1$, $\|\cdot\|_r$ is a quasi-norm with constant $2^{1/r-1}$. Hence, for each $k$,

$$\|\frac{y+i^kz}{2}\|_{2p}\leq\max\{2^{\frac{1}{2p}-1},1\}.$$

Taking $c_p=\max\{2^{\frac{1}{p}-2},1\}$, the desired assertion follows. □

2.3. **Noncommutative, conditional expectations, martingales and Hardy spaces.** Throughout the paper, let $(\mathcal{M}_n)_{n\geq 1}$ be a sequence of increasing von Neumann sub-algebras of $\mathcal{M}$ satisfying $\cup_n\mathcal{M}_n$ is weak$^*$ dense in $\mathcal{M}$. Assume that for each $n\geq 1$ there exists conditional expectation $\mathcal{E}_n:\mathcal{M}\to\mathcal{M}_n$. The conditional expectations $(\mathcal{E}_n)_{n\geq 1}$ satisfy
(a) $\mathcal{E}_n(xy)=\mathcal{E}_n(x)y$, $\mathcal{E}_n(yx)=y\mathcal{E}_n(x)$, $n\geq 1$ and $y\in\mathcal{M}_n$;
(b) $\mathcal{E}_n\mathcal{E}_m=\mathcal{E}_n$ for $m\geq n$;
(c) $\tau(\mathcal{E}_n(x))=\tau(x)$, $n\geq 1$.
Since each $\mathcal{E}_n$ preserves the trace, it extends to a contractive projection from $L_p(\mathcal{M},\tau)$ onto $L_p(\mathcal{M}_n,\tau)$ for all $1\leq p\leq\infty$.

A sequence $x=(x_n)_{n\geq 1}\subset L_1(\mathcal{M})$ is called a martingale if

$$\mathcal{E}_n(x_{n+1})=x_n,\quad n\geq 1.$$

If in addition, $(x_n)_{n\geq 1}\subset L_p(\mathcal{M})$ with $1\leq p<\infty$, then $x$ is called an $L_p(\mathcal{M})$-martingale. In this case, set

$$\|x\|_p=\sup_{n\geq 1}\|x_n\|_p.$$

If $\|x\|_p<\infty$, then $x=(x_n)_{n\geq 1}$ is called a bounded $L_p(\mathcal{M})$-martingale. A martingale is called finite if there exists some $N$ such that $x_m=x_N$ for $m\geq N$. In what follows, the sequence of martingale differences corresponding to the martingale $x=(x_n)_{n\geq 1}$ is denoted by $(d_nx)_{n\geq 1}$ with $d_1x=x_1$ and

$$d_nx=x_n-x_{n-1},\quad \forall n\geq 2. \tag{2.9}$$

For an element $y \in L_1(\mathcal{M})$, note that the sequence $(\mathcal{E}_n(y))_{n\geq 1}$ is a martingale.

**Remark 2.6.** *If $x \in L_p(\mathcal{M})$ with $1 \leq p < \infty$, then $(\mathcal{E}_n(x))_{n\geq 1}$ is a bounded $L_p(\mathcal{M})$-martingale and $\mathcal{E}_n(x)$ converges to $x$ in $L_p(\mathcal{M})$. Conversely, if $(x_n)_{n\geq 1}$ is a bounded $L_p(\mathcal{M})$-martingale with $1 < p < \infty$, then there exists $x_\infty \in L_p(\mathcal{M})$ such that $x_n = \mathcal{E}_n(x_\infty)$ for each $n \geq 1$. In this paper, we often identify a martingale with its final value $x_\infty$ whenever the latter exists.*

For $0 < p < \infty$ and any finite sequence $a = (a_n)_{n\geq 1} \subset L_p(\mathcal{M})$, define

$$\|a\|_{L_p(\mathcal{M},\ell_2^c)} = \Big\|\Big(\sum_{n\geq 1}|a_n|^2\Big)^{1/2}\Big\|_p, \quad \|a\|_{L_p(\mathcal{M},\ell_2^r)} = \Big\|\Big(\sum_{n\geq 1}|a_n^*|^2\Big)^{1/2}\Big\|_p.$$

This gives two quasi-norms on the family of all finite sequences in $L_p(\mathcal{M})$. The corresponding completion is a Banach space, denoted by $L_p(\mathcal{M},\ell_2^c)$ or $L_p(\mathcal{M},\ell_2^r)$, respectively. It is easy to check that a sequence $a = (a_n)_{n\geq 1} \subset L_p(\mathcal{M})$ belong to $L_p(\mathcal{M},\ell_2^c)$ if and only if $\sup_N \|(\sum_{n=1}^N |a_n|^2)^{1/2}\|_p < \infty$, if this is the case, then $(\sum_{n=1}^N |a_n|^2)^{1/2}$ converges to $(\sum_{n\geq 1}|a_n|^2)^{1/2}$ in $L_p(\mathcal{M})$.

For $0 < p \leq 2$, define $H_p^r(\mathcal{M})$ (resp. $H_p^c(\mathcal{M})$) as the completion of all finite $L_2(\mathcal{M})$-martingales under the quasi-norm

$$\|x\|_{H_p^r} := \|(d_nx)_{n\geq 1}\|_{L_p(\mathcal{M},\ell_2^r)} \quad (\text{resp. } \|x\|_{H_p^c} := \|(d_nx)_{n\geq 1}\|_{L_p(\mathcal{M},\ell_2^c)}).$$

Define

$$H_p(\mathcal{M}) = H_p^c(\mathcal{M}) + H_p^r(\mathcal{M})$$

equipped with the quasi-norm

$$\|x\|_{H_p(\mathcal{M})} = \inf_{x=y+z}\{\|y\|_{H_p^c(\mathcal{M})} + \|z\|_{H_p^r(\mathcal{M})}\},$$

where the infimum is taken over all the decomposition of $x = y+z$, where $y$ and $z$ are also martingales. If $x \in H_p^c(\mathcal{M})$ or $x \in H_p^r(\mathcal{M})$, then $S_c(x) \in L_p(\mathcal{M})$ or $S_r(x) \in L_p(\mathcal{M})$, where the column and row square functions $S_c(x)$ and $S_r(x)$ are defined by

$$S_c(x) = \Big(\sum_{n\geq 1}|d_nx|^2\Big)^{1/2}, \quad S_r(x) = \Big(\sum_{n\geq 1}|d_nx^*|^2\Big)^{1/2}. \tag{2.10}$$

We conclude this section with the following lemma which is taken from [27, Theorem A] (see [27, p. 1004] there for details).

**Lemma 2.7.** *For each martingale $x = (x_n)_{n\geq 1} \in L_2(\mathcal{M})$, we have*

$$\|x\|_{\Lambda_{2,\infty}(\mathcal{M},\ell_\infty^c)} \leq \|x\|_2.$$

## 3. Useful estimates for Cuculescu projections

In this section, we introduce the Cuculescu projections and provide useful estimates for them. Here we only work with noncommutative probability space $(\mathcal{M},\tau)$ (actually, this assumption is only used in Subsection 3.3). The regularity for the filtration $(\mathcal{M}_n)_{n\geq 1}$ is not necessary for this section. For a bounded $L_1(\mathcal{M})$-martingale $x = (x_n)_{n\geq 1} \in L_1(\mathcal{M})$, we always assume that it is generated by an element $x \in L_1(\mathcal{M})$, that is, for each $n \geq 1$, $x_n = \mathcal{E}_n(x)$. This assumption is not necessary, but we adhere to it in order to simply the discussion.

3.1. **Cuculescu projections and basic estimates.** Let $x = (x_n)_{n\geq 1}$ be a positive bounded $L_1(\mathcal{M})$-martingale. The Cuculescu projections (see [12]) are defined by induction as follows. For any fixed $\lambda > 0$, define $q_0 = 1$,

$$q_1^{(\lambda)} = q_0^{(\lambda)}\chi_{[0,\lambda]}(q_0^{(\lambda)}x_1q_0^{(\lambda)}), \quad q_n^{(\lambda)} = q_{n-1}^{(\lambda)}\chi_{[0,\lambda]}(q_{n-1}^{(\lambda)}x_nq_{n-1}^{(\lambda)}), \quad n \geq 2. \tag{3.1}$$

It is clear that $(q_n^{(\lambda)})_{n\geq 1}$ is decreasing. The Cuculescu projections have the following properties; see [12], [55, Proposition 1.4] or [62, Proposition 2.3].

**Lemma 3.1.** *Let $x = (x_n)_{n\geq 1}$ be a positive bounded $L_1(\mathcal{M})$-martingale. For any given $\lambda > 0$, the related Cuculescu projections $(q_n^{(\lambda)})_{n\geq 1}$ defined in (3.1) satisfy the following properties:*

(i) *for every $n \geq 1$, $q_n^{(\lambda)} \in \mathcal{M}_n$;*
(ii) *for every $n \geq 1$, $q_n^{(\lambda)}$ commutes with $q_{n-1}^{(\lambda)} x_n q_{n-1}^{(\lambda)}$;*
(iii) *for every $n \geq 1$, $q_n^{(\lambda)} x_n q_n^{(\lambda)} \leq \lambda q_n^{(\lambda)}$;*
(iv) *if we set $q^\lambda = \wedge_{n=1}^\infty q_n^{(\lambda)}$, then $\lambda\tau(1 - q^\lambda) \leq \tau((1 - q^\lambda)x) \leq \|x\|_{L_1(\mathcal{M})}$.*

In what follows, for fixed $\lambda > 0$, we always let $(q_n^{(\lambda)})_{n\geq 1}$ be the Cuculescu projections associated to a positive bounded $L_1(\mathcal{M})$-martingale $x = (x_n)_{n\geq 1}$. Set

$$p_n^{(\lambda)} = q_{n-1}^{(\lambda)} - q_n^{(\lambda)}, \quad n \geq 1. \tag{3.2}$$

It is clear that $(p_n^{(\lambda)})_{n\geq 1}$ are disjoint,

$$\sum_{n=1}^N p_n^{(\lambda)} = 1 - q_N^{(\lambda)}, \quad N \geq 1 \tag{3.3}$$

and (in the sense of strong operator topology)

$$\sum_{n\geq 1} p_n^{(\lambda)} = 1 - q^{(\lambda)}. \tag{3.4}$$

In addition, for each $k \geq 1$ (see also [62, p. 186]),

$$p_k^{(\lambda)} x_k p_k^{(\lambda)} = (q_{k-1}^{(\lambda)} - q_k^{(\lambda)}) q_{k-1}^{(\lambda)} x_k q_{k-1}^{(\lambda)} (q_{k-1}^{(\lambda)} - q_k^{(\lambda)}) > \lambda p_k^{(\lambda)}. \tag{3.5}$$

From this, we obtain the following key lemma to prove the noncommutative Davis inequality.

**Lemma 3.2.** *For each $\lambda > 0$ and positive $a \in L_0(\mathcal{M})$ such that $x_n \leq a$ for each $n \geq 1$, we have*

$$\tau(1 - q^{(\lambda)}) \leq \tau(\chi_{(\lambda,\infty)}(a)).$$

*Proof.* It suffices to show $1 - q^\lambda$ is equivalent to a sub-projection of $\chi_{(\lambda,\infty)}(a)$, and by Proposition 1.6 in [70, p. 292], it is enough to check (denote the projection $\chi_{[0,\lambda]}(a)$ by $f$)

$$[1 - q^{(\lambda)}] \wedge f = 0.$$

Assume that $[1 - q^{(\lambda)}] \wedge f \neq 0$. Then $[1 - q^{(\lambda)}](H) \cap f(H)$ contains nonzero elements of $H$ (recall that we always assume $\mathcal{M} \subset B(H)$ for some Hilbert space $H$). Note that the sets $\{[1 - q_N^{(\lambda)}](H)\}_{N\geq 1}$ are increasing with the limit $[1 - q^{(\lambda)}](H)$. Hence, there exists big enough $N_0 \in \mathbb{N}$ such that $[1 - q_{N_0}^{(\lambda)}](H) \cap f(H)$ contains nonzero elements of $H$.

Since $[1 - q_{N_0}^{(\lambda)}](H) = \cup_{n=1}^{N_0} p_n^{(\lambda)}(H)$ by (3.3), it follows that there at least exists $k \in \mathbb{N}$ such that $1 \leq k \leq N_0$ and $p_k^{(\lambda)}(H) \cap f(H)$ contains nonzero elements of $H$. Take nonzero element $\xi \in p_k^{(\lambda)}(H) \cap f(H)$. By (3.5), we have

$$\langle x_k \xi, \xi \rangle = \langle p_k^{(\lambda)} x_k p_k^{(\lambda)} \xi, \xi \rangle > \lambda \|\xi\|_H^2.$$

However, this is a contradiction to the following

$$\langle x_k \xi, \xi \rangle = \langle f x_k f \xi, \xi \rangle \leq \langle f a f \xi, \xi \rangle \leq \lambda \|\xi\|_H^2.$$

Thus, $[1 - q^{(\lambda)}] \wedge f = 0$. $\square$

The following interesting and important result is taken from [11, Lemma 5.2].

**Lemma 3.3.** *For each $\lambda > 0$, we have*

$$\tau(1 - q^\lambda) \leq \frac{2}{\lambda} \tau\Big(x \chi_{(\lambda/2,\infty)}(x)\Big).$$

**Lemma 3.4.** *For each $\lambda > 0$, we have*

$$\tau\Big((1-q^{(\lambda)})x\Big) \leq 2\tau\Big(x\chi_{(\lambda/2,\infty)}(x)\Big).$$

*Proof.* We have

$$\begin{aligned}\tau\Big((1-q^{(\lambda)})x\Big) &= \tau\Big((1-q^{(\lambda)})x\chi_{[0,\frac{\lambda}{2}]}(x)\Big) + \tau\Big((1-q^{(\lambda)})x\chi_{(\frac{\lambda}{2},\infty)}(x)\Big)\\ &\leq \frac{\lambda}{2}\tau(1-q^{(\lambda)}) + \tau\Big(x\chi_{(\lambda/2,\infty)}(x)\Big)\\ &\leq 2\tau\Big(x\chi_{(\lambda/2,\infty)}(x)\Big),\end{aligned}$$

where the last inequality is due to Lemma 3.3. □

We also consider collection of projections derived from the Cuculescu projections. For each $\ell \in \mathbb{Z}$ and each $n \geq 1$, set

$$e_{\ell,n} = \bigwedge_{k\geq \ell} q_n^{(2^k)}, \quad e_\ell = \bigwedge_{k\geq\ell} q^{(2^k)}. \tag{3.6}$$

The family of projections $(e_{\ell,n})_{\ell,n}$ is decreasing on $n$ and increasing on $\ell$. Of course, $(e_\ell)_\ell$ is increasing on $\ell$. For each $\ell \in \mathbb{Z}$ and $n \geq 1$, we have

$$\tau(e_{\ell,n}x_ne_{\ell,n}x_ne_{\ell,n}) \leq 2\sum_{j=-\infty}^{\ell} 2^{2j}\tau(e_{j,n}-e_{j-1,n}).$$

We refer to [34, eq. (2.5)] for a detailed proof of the last inequality, and by the same argument as for [34, eq. (2.5)], we can show

$$\tau(e_\ell xe_\ell xe_\ell) \leq 2\sum_{j=-\infty}^{\ell} 2^{2j}\tau(e_j - e_{j-1}) \leq 2\sum_{j=-\infty}^{\ell} 2^{2j}\tau(1-e_{j-1}). \tag{3.7}$$

The following lemma is implicit in [31]. For completeness, we still include the proof.

**Lemma 3.5.** *For each $\ell \in \mathbb{Z}$, we have*

$$\tau\Big(q^{(2^\ell)}xq^{(2^\ell)}xq^{(2^\ell)}\Big) \leq \tau(e_\ell xe_\ell xe_\ell) + 3\cdot 2^{2\ell}\tau(1-e_\ell).$$

*Proof.* By the tracial property of the trace $\tau$, we have

$$\begin{aligned}\tau\Big(q^{(2^\ell)}xq^{(2^\ell)}xq^{(2^\ell)}\Big) &= \tau\Big(q^{(2^\ell)}xq^{(2^\ell)}x\Big)\\ &= \tau\Big([q^{(2^\ell)}-e_\ell+e_\ell]x[q^{(2^\ell)}-e_\ell+e_\ell]x\Big)\\ &= \tau(e_\ell xe_\ell xe_\ell) + 2\tau\Big([q^{(2^\ell)}-e_\ell]xe_\ell x\Big) + \tau\Big([q^{(2^\ell)}-e_\ell]x[q^{(2^\ell)}-e_\ell]x\Big).\end{aligned}$$

Note that, by Lemma 3.1 (iii), $q^{(2^\ell)}xq^{(2^\ell)} \leq 2^\ell$. Thus

$$\begin{aligned}\tau\Big([q^{(2^\ell)}-e_\ell]x[q^{(2^\ell)}-e_\ell]x\Big) &= \tau\Big([q^{(2^\ell)}-e_\ell]x[q^{(2^\ell)}-e_\ell]x[q^{(2^\ell)}-e_\ell]\Big)\\ &= \tau\Big([q^{(2^\ell)}-e_\ell]q^{(2^\ell)}xq^{(2^\ell)}[q^{(2^\ell)}-e_\ell]q^{(2^\ell)}xq^{(2^\ell)}[q^{(2^\ell)}-e_\ell]\Big)\\ &\leq 2^{2\ell}\tau(q^{(2^\ell)}-e_\ell) \leq 2^{2\ell}\tau(1-e_\ell).\end{aligned}$$

Similarly, we have

$$\tau\Big([q^{(2^\ell)}-e_\ell]xe_\ell x\Big) = \tau\Big([q^{(2^\ell)}-e_\ell]q^{(2^\ell)}xq^{(2^\ell)}e_\ell q^{(2^\ell)}xq^{(2^\ell)}[q^{(2^\ell)}-e_\ell]\Big) \leq 2^{2\ell}\tau(1-e_\ell).$$

The desired inequality follows from the above argument. □

**Lemma 3.6.** *For each $\ell \in \mathbb{Z}$, we have*

$$\tau\Big(q^{(2^\ell)}xq^{(2^\ell)}xq^{(2^\ell)}\Big) \leq 2\sum_{j=-\infty}^{\ell} 2^{2j} \sum_{k\geq j-1} \tau(1-q^{(2^k)}) + 3\cdot 2^{2\ell}\sum_{k\geq \ell}\tau(1-q^{(2^k)}).$$

*Proof.* The assertion follows from (3.7), Lemma 3.6 and the basic inequality below: for each $j \in \mathbb{Z}$,

$$\tau(1-e_j) = \tau(1-\bigwedge_{k\geq j} q^{(2^k)}) \leq \sum_{k\geq j}\tau(1-q^{(2^k)}).$$

□

By Lemma 3.3, we know that $\tau(1-q^{(2^k)}) \leq 2^{1-k}\tau(x\chi_{(2^{k-1},\infty)}(x))$ for each $k$. Applying this to Lemma 3.6, we further obtain

**Lemma 3.7.** *For each $\ell \in \mathbb{Z}$, we have*

$$\tau\Big(q^{(2^\ell)}xq^{(2^\ell)}xq^{(2^\ell)}\Big) \leq 2\sum_{j=-\infty}^{\ell} 2^{2j} \sum_{k\geq j-1} 2^{1-k}\tau\Big(x\chi_{(2^{k-1},\infty)}(x)\Big) + 3\cdot 2^{2\ell}\sum_{k\geq j} 2^{1-k}\tau\Big(x\chi_{(2^{k-1})}(x)\Big).$$

3.2. **$L_p$-estimates of Cuculescu projections with $1<p<2$.** The estimates in this subsection will be used to show strong type $(p,p)$ inequality with $1<p<2$ for noncommutative martingales. We begin with the following lemma.

**Lemma 3.8.** *For each positive $x \in L_p(\mathcal{M})$ with $1<p<\infty$, we have*

$$\int_0^\infty \lambda^{p-2}\tau\Big(x\chi_{(\lambda,\infty)}(x)\Big)d\lambda \leq \frac{1}{p-1}\|x\|_p^p. \tag{3.8}$$

*Proof.* Denote the left-hand side formula by LHS. Note that by the definition of the singular value function, for $t>0$, $t \leq \tau(\chi_{(\lambda,\infty)}(x))$ if and only if $\lambda \leq \mu(t,x)$. Thus, by (2.3) and Fact 2.1(iv), we have

$$\begin{aligned}
\text{LHS} &\leq \int_0^\infty \lambda^{p-2}\int_0^\infty \mu(x,t)\chi_{(0,\tau(\chi_{(\lambda,\infty)}(x))]}(t)dtd\lambda \\
&= \int_0^\infty \mu(x,t)\int_0^{\mu(x,t)} \lambda^{p-2}d\lambda dt \\
&= \frac{1}{p-1}\int_0^\infty \mu(x,t)^p dt = \frac{1}{p-1}\|x\|_p^p,
\end{aligned}$$

where the last equality follows from (2.2). □

**Lemma 3.9.** *For each positive $x \in L_p(\mathcal{M})$ with $1<p<\infty$, we have*

$$\frac{1}{4}\sum_{\ell\in\mathbb{Z}} 2^{\ell(p-1)}\tau\Big(x\chi_{(2^{\ell-1},\infty)}(x)\Big) \leq \int_0^\infty \lambda^{p-2}\tau\Big(x\chi_{(\lambda,\infty)}(x)\Big)d\lambda \leq 2\sum_{\ell\in\mathbb{Z}} 2^{\ell(p-1)}\tau\Big(x\chi_{(2^{\ell-1},\infty)}(x)\Big). \tag{3.9}$$

*Proof.* We have

$$\begin{aligned}
\int_0^\infty \tau\Big(x\chi_{(\lambda,\infty)}(x)\Big) &= \sum_{\ell\in\mathbb{Z}}\int_{2^{\ell-1}}^{2^\ell} \lambda^{p-2}\tau\Big(x\chi_{(\lambda,\infty)}(x)\Big)d\lambda \\
&\leq \sum_{\ell\in\mathbb{Z}}\int_{2^{\ell-1}}^{2^\ell} \lambda^{p-2}\tau\Big(x\chi_{(2^{\ell-1},\infty)}(x)\Big)d\lambda \\
&\leq \max\{1,2^{2-p}\}\sum_{\ell\in\mathbb{Z}}\int_{2^{\ell-1}}^{2^\ell} 2^{\ell(p-2)}\tau\Big(x\chi_{(2^{\ell-1},\infty)}(x)\Big)d\lambda \\
&\leq 2\sum_{\ell\in\mathbb{Z}} 2^{\ell(p-1)}\tau\Big(x\chi_{(2^{\ell-1},\infty)}(x)\Big).
\end{aligned}$$

This is the right-hand side of (3.9), and the left-hand side of (3.9) can be established similarly. □

**Lemma 3.10.** *For each positive $x \in L_p(\mathcal{M})$ with $1 < p < \infty$, we have*

$$\sum_{\ell\in\mathbb{Z}} 2^{\ell(p-1)}\tau\Big(x\chi_{(2^{\ell-1},\infty)}(x)\Big) \le \frac{4}{p-1}\|x\|_p^p,$$

*and*

$$\sum_{\ell\in\mathbb{Z}} 2^{\ell p}\sum_{k\ge\ell} 2^{1-k}\tau\Big(x\chi_{(2^{k-1},\infty)}(x)\Big) \le \frac{2^{p+3}}{(2^p-1)(p-1)}\|x\|_p^p.$$

*Moreover, for $1 < p < 2$, we have*

$$\sum_{\ell\in\mathbb{Z}} 2^{\ell(p-2)}\sum_{j=-\infty}^{\ell} 2^{2j}\sum_{k\ge j-1} 2^{1-k}\tau\Big(x\chi_{(2^{k-1},\infty)}(x)\Big) \le \frac{2^p}{1-2^{p-2}}\frac{2^{p+3}}{(2^p-1)(p-1)}\|x\|_p^p.$$

*Proof.* Combining the left-hand side inequality of (3.9) and Lemma 3.8, the first inequality follows. The second inequality can be deduced from the first one and the observation below:

$$\begin{aligned}\sum_{\ell\in\mathbb{Z}} 2^{\ell p}\sum_{s\ge\ell} 2^{1-k}\tau(x\chi_{(2^{k-1},\infty)}(x)) &= \sum_{k\in\mathbb{Z}}\sum_{\ell\le k} 2^{\ell p}2^{1-k}\tau(x\chi_{(2^{k-1},\infty)}(x))\\ &= \frac{2}{1-2^{-p}}\sum_{s\in\mathbb{Z}} 2^{k(p-1)}\tau(x\chi_{(2^{k-1},\infty)}(x))\\ &= \frac{2^{p+1}}{2^p-1}\sum_{s\in\mathbb{Z}} 2^{k(p-1)}\tau(x\chi_{(2^{k-1},\infty)}(x)).\end{aligned}$$

Now we deal with the third inequality. We have

$$\begin{aligned}&\sum_{\ell\in\mathbb{Z}} 2^{\ell(p-2)}\sum_{j=-\infty}^{\ell} 2^{2j}\sum_{k\ge j-1} 2^{1-k}\tau(x\chi_{(2^{k-1},\infty)}(x))\\ &\quad= \sum_{j\in\mathbb{Z}}\sum_{\ell\ge j} 2^{(p-2)\ell}2^{2j}\sum_{k\ge j-1} 2^{1-k}\tau(x\chi_{(2^{k-1},\infty)}(x))\\ &\quad= \frac{1}{1-2^{p-2}}\sum_{j\in\mathbb{Z}} 2^{jp}\sum_{k\ge j-1} 2^{1-k}\tau(x\chi_{(2^{k-1},\infty)}(x)) \le \frac{2^p}{1-2^{p-2}}\frac{2^{p+3}}{(2^p-1)(p-1)}\|x\|_p^p,\end{aligned}$$

where the last inequality follows from the second inequality of the lemma (note that we need $p < 2$ in the second equality). □

Now we are ready to show the $L_p$-estimates of Cuculescu projections. The results below are implicit in [31] and [34]. However, the argument here is more simpler.

**Proposition 3.11.** *For $\lambda > 0$, consider the Cuculescu projections $q^{(\lambda)}$ associated with the positive martingale $x = (\mathcal{E}_n(x))_{n\ge1} \in L_p(\mathcal{M})$, $1 < p < \infty$. We have*

$$\sum_{\ell\in\mathbb{Z}} 2^{\ell(p-1)}\tau\Big((1-q^{(2^\ell)})x\Big) \le \frac{8}{p-1}\|x\|_p^p,$$

*and*

$$\sum_{\ell\in\mathbb{Z}} 2^{\ell p}\sum_{k\ge\ell}\tau(1-q^{(2^k)}) \le \frac{2^{p+3}}{(2^p-1)(p-1)}\|x\|_p^p.$$

*Moreover, for $1 < p < 2$, we have*

$$\sum_{\ell\in\mathbb{Z}} 2^{\ell(p-2)}\tau\Big(q^{(2^\ell)}xq^{(2^\ell)}xq^{(2^\ell)}\Big) \le \Big(3 + \frac{2^{p+1}}{1-2^{p-2}}\Big)\frac{2^{p+3}}{(2^p-1)(p-1)}\|x\|_p^p.$$

*Proof.* The first inequality follows by combining Lemma 3.4 and Lemma 3.10, and the second one follows by combining Lemma 3.3 and Lemma 3.10. As for the third assertion, by Lemma 3.7, we have

$$\sum_{\ell\in\mathbb{Z}} 2^{\ell(p-2)}\tau\Big(q^{(2^\ell)}xq^{(2^\ell)}xq^{(2^\ell)}\Big)$$
$$\leq 2\sum_{\ell\in\mathbb{Z}} 2^{\ell(p-2)}\sum_{j=-\infty}^{\ell} 2^{2j}\sum_{k\geq j-1} 2^{1-k}\tau\Big(x\chi_{(2^{k-1},\infty)}(x)\Big) + 3\sum_{\ell\in\mathbb{Z}} 2^{\ell p}\sum_{k\geq \ell} 2^{1-k}\tau\Big(x\chi_{(2^{k-1},\infty)}(x)\Big).$$

The third assertion of the proposition now follows from Lemma 3.10. □

3.3. $L\log L$**-estimates of Cuculescu projections.** The results in this subsection will be used to prove Theorem 1.10 (iii). The assumption that $(\mathcal{M},\tau)$ is a noncommutative probability space (i.e., $\tau(1)=1$) is used here; see the proof of Lemma 3.13.

**Lemma 3.12.** *For each positive $x\in L\log L(\mathcal{M})$, we have*

$$\int_1^\infty \frac{1}{\lambda}\tau\Big(x\chi_{(\lambda,\infty)}(x)\Big)d\lambda \leq \|x\|_{L\log L}.$$

*Proof.* By (2.3) and Fact 2.1(iv), we have

$$\begin{aligned}\int_1^\infty \frac{1}{\lambda}\tau\Big(x\chi_{(\lambda,\infty)}(x)\Big)d\lambda &\leq \int_1^\infty \frac{1}{\lambda}\int_0^1 \mu(x,t)\chi_{[0,\tau(\chi_{(\lambda,\infty)}(x))]}(t)dt\\ &= \int_0^1 \mu(x,t)\int_1^{\mu(x,t)}\frac{1}{\lambda}d\lambda dt\\ &= \int_0^1 \mu(x,t)\log^+(\mu(x,t))dt = \|x\|_{L\log L},\end{aligned}$$

where in the first equality we used the fact that $t\leq\tau(\chi_{(\lambda,\infty)}(x))$ if and only if $\lambda\leq\mu(x,t)$. □

**Lemma 3.13.** *For each positive $x\in L\log L(\mathcal{M})$, we have*

$$\sum_{\ell\geq 0}\tau\Big(x\chi_{(2^{\ell-1},\infty)}(x)\Big)\leq 5\|x\|_{L\log L}.$$

*Proof.* We have

$$\begin{aligned}\sum_{\ell\geq 0}\tau\Big(x\chi_{(2^{\ell-1},\infty)}(x)\Big) &= \sum_{\ell\geq 0}2^{2-\ell}\int_{2^{\ell-2}}^{2^{\ell-1}}\tau\Big(x\chi_{(2^{\ell-1},\infty)}(x)\Big)d\lambda\\ &\leq 2\sum_{\ell\geq 0}\int_{2^{\ell-2}}^{2^{\ell-1}}\frac{1}{\lambda}\tau\Big(x\chi_{(\lambda,\infty)}(x)\Big)d\lambda\\ &= 2\int_{\frac14}^\infty\frac{1}{\lambda}\tau(x\chi_{(\lambda,\infty)}(x))d\lambda\\ &= 2\int_{\frac14}^1\frac{1}{\lambda}\tau\Big(x\chi_{(\lambda,\infty)}(x)\Big)d\lambda + \int_1^\infty\frac{1}{\lambda}\tau\Big(x\chi_{(\lambda,\infty)}(x)\Big)d\lambda\\ &\leq 2\|x\|_1\int_{\frac14}^1\frac{1}{\lambda}d\lambda + \|x\|_{L\log L} = 2\ln 4\|x\|_1 + \|x\|_{L\log L},\end{aligned}$$

where we applied Lemma 3.12 in last inequality. Note that $\|x\|_1\leq\|x\|_{L\log L}$ since $(\mathcal{M},\tau)$ is a noncommutative probability space. The desired assertion follows. □

**Lemma 3.14.** *For $\lambda>0$, consider the Cuculescu projections $q^{(\lambda)}$ associated with the positive martingale $x=(\mathcal{E}_n(x))_{n\geq 1}\in L\log L(\mathcal{M})$. We have*

$$\sum_{\ell\geq 0}\tau\Big((1-q^{(2^\ell)})x\Big)\leq 10\|x\|_{L\log L}. \tag{3.10}$$

*Consequently,*

$$\sum_{\ell\geq 0} 2^\ell \tau(1-q^{(2^\ell)}) \leq 10\|x\|_{L\log L},$$

*and*

$$\sum_{\ell\geq 0} 2^\ell \sum_{k\geq \ell} \tau(1-q^{(2^k)}) \leq 20\|x\|_{L\log L}.$$

*Proof.* By Lemma 3.4, we have

$$\sum_{\ell\geq 0} \tau\Big((1-q^{(2^\ell)})x\Big) \leq 2\sum_{\ell\geq 0} \tau(x\chi_{(2^{\ell-1},\infty)}(x)).$$

Hence, the inequality (3.10) follows from Lemma 3.13. The second inequality follows from Lemma 3.1 (iv) and (3.10). The third inequality follows from the second one. □

**Proposition 3.15.** *For $\lambda > 0$, consider the Cuculescu projections $q^{(\lambda)}$ associated with the positive martingale $x = (\mathcal{E}_n(x))_{n\geq 1} \in L\log L(\mathcal{M})$. We have*

$$\sum_{\ell\geq 0} 2^{-\ell}\tau\Big(q^{(2^\ell)}xq^{(2^\ell)}xq^{(2^\ell)}\Big) \leq 384\|x\|_{L\log L}.$$

*Proof.* By Lemma 3.6, we have

$$\begin{aligned}&\sum_{\ell\geq 0}2^{-\ell}\tau\Big(q^{(2^\ell)}xq^{(2^\ell)}xq^{(2^\ell)}\Big)\\ &\quad\leq 2\sum_{\ell\geq 0}2^{-\ell}\sum_{j=-\infty}^{\ell}2^{2j}\sum_{k\geq j-1}\tau(1-q^{(2^k)}) + 3\sum_{\ell\geq 0}2^\ell\sum_{k\geq \ell}\tau(1-q^{(2^k)}).\end{aligned}$$

We claim that for each $\ell \geq 0$, we have

$$\sum_{j=-\infty}^{\ell} 2^{2j}\sum_{k\geq j-1}\tau(1-q^{(2^k)}) \leq \|x\|_1 + 4\sum_{j=0}^{\ell}2^{2j}\sum_{k\geq j}\tau(1-q^{(2^k)}).$$

Once this claim is proved, we conclude from the above argument that

$$\begin{aligned}&\sum_{\ell\geq 0}2^{-\ell}\tau\Big(q^{(2^\ell)}xq^{(2^\ell)}xq^{(2^\ell)}\Big)\\ &\quad\overset{L.\,3.14}{\leq} 2\sum_{\ell\geq 0}2^{-\ell}\Big(\|x\|_1 + 4\sum_{j=0}^{\ell}2^{2j}\sum_{k\geq j}\tau(1-q^{(2^k)})\Big) + 60\|x\|_{L\log L}\\ &\quad= 4\|x\|_1 + 16\sum_{j\geq 0}2^j\sum_{k\geq j}\tau(1-q^{(2^k)}) + 60\|x\|_{L\log L}\\ &\quad\leq 4\|x\|_{L\log L} + 320\|x\|_{L\log L} + 60\|x\|_{L\log L} = 384\|x\|_{L\log L},\end{aligned}$$

where we used the fact $\|x\|_1 \leq \|x\|_{L\log L}$ and Lemma 3.14 in the last inequality.

It remains to show the claim. Observe that

$$\begin{aligned}\sum_{j=-\infty}^{\ell} 2^{2j}\sum_{k\geq j-1}\tau(1-q^{(2^k)}) &\leq 4\sum_{j=-\infty}^{\ell}2^{2j}\sum_{k\geq j}\tau(1-q^{(2^k)})\\ &= 4\sum_{j=-\infty}^{-1}2^{2j}\sum_{k\geq j}\tau(1-q^{(2^k)}) + 4\sum_{j=0}^{\ell}2^{2j}\sum_{k\geq j}\tau(1-q^{(2^k)})\end{aligned}$$

and by Lemma 3.1 (iv)

$$\sum_{j=-\infty}^{-1}2^{2j}\sum_{k\geq j}\tau(1-q^{(2^k)}) \leq \sum_{j=-\infty}^{-1}2^{2j}\sum_{k\geq j}2^{-k}\|x\|_1 \leq \|x\|_1.$$

Hence, the claim follows. □

3.4. **$L_p$-estimates of Cuculescu projections with $0<p\le 1$.** This subsection contains necessary results which will be used to prove Theorem 1.11.

**Lemma 3.16.** *For $\lambda>0$, consider the Cuculescu projections $q^{(\lambda)}$ associated with the positive martingale $x=(\mathcal{E}_n(x))_{n\ge1}\in L_1(\mathcal{M})$. Assume that positive $a\in L_0(\mathcal{M})$ satisfies $x_n=\mathcal{E}_n(x)\le a$ for each $n\ge1$. We have*

$$\sum_{k\in\mathbb{Z}}2^{kp}\tau(1-q^{(2^k)})\le\frac{2}{p}\|a\|_p^p,\quad 0<p\le1.$$

*Proof.* By Lemma 3.2, we first have $\tau(1-q^{(2^k)})\le\tau(\chi_{(2^k,\infty)}(a))$ for each $k\in\mathbb{Z}$. Hence,

$$\begin{aligned}\sum_{k\in\mathbb{Z}}2^{kp}\tau(1-q^{(2^k)})&\le\sum_{k\in\mathbb{Z}}2^{kp}\tau(\chi_{(2^k,\infty)}(a))\\&=2\sum_{k\in\mathbb{Z}}\int_{2^{k-1}}^{2^k}2^{k(p-1)}\tau(\chi_{(2^k,\infty)}(a))d\lambda\\&\le2\sum_{k\in\mathbb{Z}}\int_{2^{k-1}}^{2^k}\lambda^{p-1}\tau(\chi_{(\lambda,\infty)}(a))d\lambda\\&=2\int_0^\infty\lambda^{p-1}\tau(\chi_{(\lambda,\infty)}(a))d\lambda=\frac{2}{p}\|a\|_p^p,\end{aligned}$$

where the last equality is due to (2.2). □

**Proposition 3.17.** *For $\lambda>0$, consider the Cuculescu projections $q^{(\lambda)}$ associated with the positive martingale $x=(\mathcal{E}_n(x))_{n\ge1}\in L_1(\mathcal{M})$. Assume that positive $a\in L_0(\mathcal{M})$ satisfies $x_n=\mathcal{E}_n(x)\le a$ for each $n\ge1$. We have*

$$\sum_{\ell\in\mathbb{Z}}2^{\ell p}\sum_{k\ge\ell}\tau(1-q^{(2^k)})\le\frac{2^{p+1}}{p(2^p-1)}\|a\|_p^p,\quad 0<p\le1$$

*and*

$$\sum_{\ell\in\mathbb{Z}}2^{\ell(p-2)}\tau\Big(q^{(2^\ell)}xq^{(2^\ell)}xq^{(2^\ell)}\Big)\le\frac{2^{p+4}}{p(2^p-1)}\|a\|_p^p,\quad 0<p\le1.$$

*Proof.* The first inequality follows from Lemma 3.16 and the below basic observation:

$$\sum_{\ell\in\mathbb{Z}}2^{p\ell}\sum_{k\ge\ell}\tau(1-q^{(2^k)})=\sum_{k\in\mathbb{Z}}\sum_{\ell\le k}2^{p\ell}\tau(1-q^{(2^k)})=\frac{2^p}{2^p-1}\sum_{k\in\mathbb{Z}}2^{kp}\tau(1-q^{(2^k)}).$$

As for the second assertion, by Lemma 3.6, we have

$$\begin{aligned}&\sum_{\ell\in\mathbb{Z}}2^{\ell(p-2)}\tau\Big(q^{(2^\ell)}xq^{(2^\ell)}xq^{(2^\ell)}\Big)\\&\le2\sum_{\ell\in\mathbb{Z}}2^{\ell(p-2)}\sum_{j=-\infty}^{\ell}2^{2j}\sum_{k\ge j-1}\tau(1-q^{(2^k)})+3\sum_{\ell\in\mathbb{Z}}2^{\ell p}\sum_{k\ge\ell}\tau(1-q^{(2^k)})\\&=2\sum_{j\in\mathbb{Z}}\sum_{\ell\ge j}2^{\ell(p-2)}2^{2j}\sum_{k\ge j-1}\tau(1-q^{(2^k)})+3\sum_{\ell\in\mathbb{Z}}2^{\ell p}\sum_{k\ge\ell}\tau(1-q^{(2^k)})\\&=\frac{2^{p+1}}{1-2^{p-2}}\sum_{j\in\mathbb{Z}}2^{jp}\sum_{k\ge j}\tau(1-q^{(2^k)})+3\sum_{\ell\in\mathbb{Z}}2^{\ell p}\sum_{k\ge\ell}\tau(1-q^{(2^k)})\end{aligned}$$

Note that $\frac{2^{p+1}}{1-2^{p-2}}\le4$ provided $0<p\le1$. Hence, the second assertion of the proposition follows from the first one. □

## 4. Asymmetric Doob maximal inequalities

In this section, we study asymmetric maximal inequalities for noncommutative martingales. Particularly, we provide proofs of Theorem 1.10 and Theorem 1.4. The definitions for $\Lambda_{p,\infty}(\mathcal{M},\ell_\infty^c)$, $\Lambda_p(\mathcal{M},\ell_\infty^c)$ are referred to Section 2.

4.1. **Proofs of Theorems 1.10 and 1.4.** In this subsection, we always assume that the filtration $(\mathcal{M}_n)_{n\geq 1}$ is regular with constant $c_{\mathrm{reg}}$, that is, for any positive $x\in\mathcal{M}$,

$$\mathcal{E}_n(x)\leq c_{\mathrm{reg}}\mathcal{E}_{n-1}(x).$$

For each positive $x\in L_2(\mathcal{M})$ and $\lambda>0$, the Cuculescu projections related to the martingale $x=(\mathcal{E}_n(x))_{n\geq 1}$ are still denoted by $(q_n^{(\lambda)})_n$ and $q^{(\lambda)}=\wedge_n q_n^{(\lambda)}$; see Lemma 3.1 for more details.

We first prove Theorem 1.10, and then Theorem 1.4 can be deduced from it. We begin with the decomposition of the martingale $y$. To this end, let us first recall the basic definition of triangular truncation. Let $(R_k)_{k\geq 1}$ be a sequence of mutually orthogonal projections in $\mathcal{M}$. For each $x\in L_2(\mathcal{M})$, the triangular truncation are defined by

$$\mathcal{T}^c(x)=\sum_{j\geq 1}\sum_{i=1}^{j}R_ixR_j,\quad \mathcal{T}^r(x)=x-\mathcal{T}^c(x)=\sum_{i>j}R_ixR_j. \tag{4.1}$$

Observe that $\mathcal{T}^c(x)$ is well defined. Indeed, it is easy to check that $\sum_{j=1}^N\sum_{i=1}^j R_jxR_k$ converges to $\mathcal{T}^c(x)$ in $L_2(\mathcal{M})$, moreover,

$$\|\mathcal{T}^c(x)\|_{L_2(\mathcal{M})}\leq\|x\|_{L_2(\mathcal{M})}. \tag{4.2}$$

For $j\geq 1$ and fixed $k\geq 1$, define

$$\pi_{0,k}=e_{0,k}=\bigwedge_{i\geq 0}q_k^{(2^i)}\qquad\text{and}\qquad \pi_{j,k}=e_{j,k}-e_{j-1,k}=\bigwedge_{i\geq j}q_k^{(2^i)}-\bigwedge_{i\geq j-1}q_k^{(2^i)}.$$

The family of projections $(e_{j,k})_{j,k}$ are defined in (3.6). Note that $\sum_{j\geq 0}\pi_{j,k}=1$ in the sense of strong operator topology for every fixed $k\geq 1$ (see [63, Proposition 1.4]). In what follows, for each $k\geq 1$, we denote $\mathcal{T}_{k-1}^c$ and $\mathcal{T}_{k-1}^r$ the triangular truncation defined according to (4.1) with respect to the mutually orthogonal projections $(\pi_{j,k-1})_j$.

**Lemma 4.1.** *Suppose that martingale $x\in L_2(\mathcal{M})$ is positive, and martingale $y$ is $L_2$-dominated by $x$. For each $k\geq 1$, $d_ky=\mathcal{T}_{k-1}^c(d_ky)+\mathcal{T}_{k-1}^r(d_ky)$, $\mathcal{T}_{k-1}^c(d_ky)$ and $\mathcal{T}_{k-1}^r(d_ky)$ are martingale differences. Consequently,*

$$y=y^c+y^r,$$

*where $y^c=(y_n^c)_{n\geq 1}$ and $y^r=(y_n^r)_{n\geq 1}$ are two martingales. Here, for each $n\geq 1$,*

$$y_n^c=\sum_{k=1}^n\mathcal{T}_{k-1}^c(d_ky),\quad y_n^r=\sum_{k=1}^n\mathcal{T}_{k-1}^r(d_ky). \tag{4.3}$$

*Proof.* It follows from Lemma 3.1 (i) that $q_{k-1}^{(2^i)}\in\mathcal{M}_{k-1}$ for every $i$, and consequently for each $j$ and $k$, $\pi_{j,k-1}\in\mathcal{M}_{k-1}$. Thus, for each $k\geq 1$, $\mathcal{T}_{k-1}^c(d_ky)=\sum_{j\geq 1}\sum_{i\leq j}\pi_{i,k-1}d_ky\pi_{j,k-1}$ is a martingale difference. Similarly, each $\mathcal{T}_{k-1}^r(d_ky)$ is a martingale difference. □

We here point out that for $y=x$ the above decomposition was used in [64] and [55]. The present decomposition is taken from [34, eq. (4.10)].

We also need the following noncommutative Gundy decomposition which is one of our key tool to prove Theorem 1.10. Comparing our Gundy decomposition Lemma 4.2 with previous versions like [55, Theorem 2.1] and [34, Theorem 3.1], there is no $L_1$-estimates in Lemma 4.2 with the help of regularity.

**Lemma 4.2.** *Let $(\mathcal{M},\tau)$ be a noncommutative probability space associated with the regular filtration $(\mathcal{M}_n)_{n\geq 1}$ (the related regularity constant is denoted by $c_{\mathrm{reg}}$). Suppose that martingale $x\in L_2(\mathcal{M})$ is positive, and martingale $y$ is $L_2$-dominated by $x$. For any $\lambda>0$, there exist three martingales $\alpha$, $\beta$ and $\gamma$ such that*

(i) *$y = \alpha + \beta + \gamma$;*
(ii) *the martingale $\alpha$ satisfies $\|\alpha\|_2^2 \leq c_{\mathrm{reg}}\lambda\tau((1-q^{(\lambda)})x) + \tau(q^{(\lambda)}xq^{(\lambda)}xq^{(\lambda)})$.*
(iii) *the martingales $\beta$ and $\gamma$ satisfy*

$$\max\Big\{\tau\Big(\bigvee_n \mathrm{supp}(|d\beta_n|)\Big), \tau\Big(\bigvee_n \mathrm{supp}(|d\gamma_n^*|)\Big)\Big\} \leq \tau(1-q^{(\lambda)}).$$

*Here, $q^{(\lambda)}$ is given in Lemma 3.1.*

*Proof.* Let $\lambda > 0$. Consider the Cuculescu projections $(q_n^{(\lambda)})_{n\geq 1}$ relative to the positive martingale $x$. For convenience, we just write $(q_n)_{n\geq 1}$ below. For each $n \geq 1$, we decompose $dy_n$ as follows:

$$(\mathbf{G}_\lambda) \qquad \begin{cases} d\alpha_n & := q_{n-1}dy_nq_{n-1}, \\ d\beta_n & := dy_n(\mathbf{1}-q_{n-1}), \\ d\gamma_n & := (\mathbf{1}-q_{n-1})dy_nq_{n-1}. \end{cases}$$

It is clear that $d\alpha_n$, $d\beta_n$ and $d\gamma_n$ are martingale differences. Denote by $\alpha$, $\beta$ and $\gamma$ the martingales generated by $(d\alpha_n)_n$, $(d\beta_n)_n$ and $(d\gamma_n)_n$, respectively. Then, clearly $y = \alpha+\beta+\gamma$. Items (i) and (iii) in the lemma are clearly satisfied.

It remains to verify item (ii). According to Definition 1.8 , we have

$$\|\alpha\|_2^2 = \sum_{n\geq 1}\|d\alpha_n\|_2^2 = \sum_{n\geq 1}\|q_{n-1}dy_nq_{n-1}\|_2^2 \leq \sum_{n\geq 1}\|q_{n-1}dx_nq_{n-1}\|_2^2. \tag{4.4}$$

We shall estimate $\sum_{n=1}^N \|q_{n-1}dx_nq_{n-1}\|_2^2$ for each $N \geq 1$. Note that

$$\tau(q_{n-1}x_{n-1}q_{n-1}dx_nq_{n-1}) = \tau(q_{n-1}x_{n-1}q_{n-1}\mathcal{E}_{n-1}(dx_n)q_{n-1}) = 0.$$

Then, for each $1 \leq n \leq N$, we write

$$\begin{aligned} \|q_{n-1}dx_nq_{n-1}\|_2^2 &= \tau(q_{n-1}dx_nq_{n-1}dx_nq_{n-1}) \\ &= \tau(q_{n-1}x_nq_{n-1}dx_nq_{n-1}) \\ &= \tau(q_{n-1}x_nq_{n-1}x_nq_{n-1}) - \tau(q_{n-1}x_nq_{n-1}x_{n-1}q_{n-1}) \\ &= \tau(q_{n-1}x_nq_{n-1}x_nq_{n-1}) - \tau(q_{n-1}x_{n-1}q_{n-1}x_{n-1}q_{n-1}). \end{aligned} \tag{4.5}$$

We claim that

$$\tau(q_{n-1}x_nq_{n-1}x_nq_{n-1}) \leq c_{\mathrm{reg}}\lambda\tau((q_{n-1}-q_n)x_n) + \tau(q_nx_nq_nx_nq_n). \tag{4.6}$$

Once this claim have been proved, we combine (4.5) to obtain

$$\begin{aligned} &\sum_{n=1}^N \|q_{n-1}dx_nq_{n-1}\|_2^2 \\ &\leq c_{\mathrm{reg}}\lambda\sum_{n=1}^N \tau((q_{n-1}-q_n)x_n) + \tau(q_nx_nq_nx_nq_n) - \tau(q_{n-1}x_{n-1}q_{n-1}x_{n-1}q_{n-1}). \\ &= c_{\mathrm{reg}}\lambda\sum_{n=1}^N \tau((q_{n-1}-q_n)x_n) + \tau(q_Nx_Nq_Nx_Nq_N) \\ &= c_{\mathrm{reg}}\lambda\sum_{n=1}^N \tau((q_{n-1}-q_n)x) + \tau(q_Nx_Nq_Nx_Nq_N) \\ &= c_{\mathrm{reg}}\tau((1-q_N)x) + \tau(q_Nx_Nq_Nx_Nq_N), \end{aligned}$$

where we used basic property of conditional expectation in the second equality. Recall that $(q_n)_n$ is decreasing and $\lim_{n\to\infty} q_n = q$. Hence,

$$\lim_{N\to\infty}\tau((1-q_N)x) = \tau((1-q)x).$$

Since $x_n$ converges to $x$ in $L_2(\mathcal{M})$, it follows that $\lim_{N\to\infty}\tau(q_Nx_Nq_Nx_Nq_N)=\tau(qxqxq)$. Therefore, we conclude from (4.4) and the above argument that

$$\begin{aligned}\|\alpha\|_2^2 &= \sum_{n=1}^{N}\|q_{n-1}dx_nq_{n-1}\|_2^2\\ &\le \lim_{N\to\infty} c_{\mathrm{reg}}\lambda\tau((1-q_N)x)+\lim_{N\to\infty}\tau(q_Nx_Nq_Nx_Nq_N)\\ &= c_{\mathrm{reg}}\lambda\tau((1-q)x)+\tau(qxqxq).\end{aligned}$$

Item (ii) of the lemma is justified.

It remains to prove the claim (4.6). Note that by Lemma 3.1 (ii), we have

$$q_nx_n(q_{n-1}-q_n)=q_nq_{n-1}x_nq_{n-1}(q_{n-1}-q_n)=q_{n-1}x_nq_{n-1}q_n(q_{n-1}-q_n)=0.$$

Also $(q_{n-1}-q_n)x_nq_n=0$ for each $n\ge 1$. Thus, for each $n$, we have

$$\begin{aligned}&\tau(q_{n-1}x_nq_{n-1}x_nq_{n-1})\\ &=\tau([q_{n-1}-q_n+q_n]x_nq_{n-1}x_nq_{n-1})\\ &=\tau([q_{n-1}-q_n]x_nq_{n-1}x_nq_{n-1})+\tau(q_nx_n[q_{n-1}-q_n+q_n]x_nq_{n-1})\\ &=\tau([q_{n-1}-q_n]x_n[q_{n-1}-q_n]q_{n-1}x_nq_{n-1})+\tau(q_nx_nq_nx_nq_n)\\ &\le \|q_{n-1}x_nq_{n-1}\|_\infty\tau([q_{n-1}-q_n]x_n[q_{n-1}-q_n])+\tau(q_nx_nq_nx_nq_n)\\ &\le c_{\mathrm{reg}}\lambda\tau((q_{n-1}-q_n)x_n)+\tau(q_nx_nq_nx_nq_n),\end{aligned}$$

where the last inequality is due to the fact that $(x_n)_{n\ge1}$ are positive and the related filtration is regular:

$$\|q_{n-1}x_nq_{n-1}\|_\infty\le c_{\mathrm{reg}}\|q_{n-1}x_{n-1}q_{n-1}\|_\infty\le c_{\mathrm{reg}}\lambda.$$

The proof is complete. □

With the help of Lemma 4.2, we establish a distribution estimate for asymmetric noncommutative maximal function.

**Lemma 4.3.** *Let $(\mathcal{M},\tau)$ be a noncommutative probability space associated with the regular filtration $(\mathcal{M}_n)_{n\ge1}$ (the related regularity constant is denoted by $c_{\mathrm{reg}}$). Suppose that martingale $x\in L_2(\mathcal{M})$ is positive, and martingale $y$ is $L_2$-dominated by $x$. For $\lambda>0$, denote by $(q_k^{(\lambda)})_{k\ge1}$ the Cuculescu projections with respect to the martingale $x=(\mathcal{E}_n(x))_{n\ge1}$. Consider the decomposition $y=y^c+y^r$ as in Lemma 4.1. For each $\ell\in\mathbb{Z}$, there exist projections $e^{c,\ell}$ and $e^{r,\ell}$ in $\mathcal{M}$ satisfying $\sup_n\|y_n^ce^{c,\ell}\|_\infty\le 3\cdot 2^\ell$,*

$$\tau(1-e^{c,\ell})\le c_{\mathrm{reg}}\frac{1}{2^\ell}\tau\Big((1-q^{(2^\ell)})x\Big)+\frac{1}{2^{2\ell}}\tau\Big(q^{(2^\ell)}xq^{(2^\ell)}xq^{(2^\ell)}\Big)+\sum_{k\ge\ell}\tau(1-q^{(2^k)}),$$

*and $\sup_n\|e^{r,\ell}y_n^r\|_\infty\le 3\cdot2^\ell$,*

$$\tau(1-e^{r,\ell})\le c_{\mathrm{reg}}\frac{1}{2^\ell}\tau((1-q^{(2^\ell)})x)+\frac{1}{2^{2\ell}}\tau(q^{(2^\ell)}xq^{(2^\ell)}xq^{(2^\ell)})+\sum_{k\ge\ell}\tau(1-q^{(2^k)}).$$

In the commutative martingale setting (consider $\mathcal{M}=L_\infty(\Omega)$), we can view $e^{c,\ell}=\chi_{\{\sup_n|y_n|\le3\cdot2^\ell\}}$, and hence,

$$1-e^{c,\ell}=\chi_{\{\sup_n|y_n|>3\cdot2^\ell\}}.$$

In this way, we say the above lemma is actually a distribution estimate for noncommutative maximal function. The word "asymmetric" is due to the projection appears in the right-hand or left-hand side of the original data.

*Proof of Lemma 5.2.* We only deal with $y^c$ and the estimate for $y^r$ can be proved by a similar way. Fix $\ell\in\mathbb{Z}$. Applying the Gundy decomposition Lemma 4.2 to $y$ and $\lambda=2^\ell$, we have

$$y=\alpha+\beta+\gamma.$$

Hence, for each $n \geq 1$,

$$y_n^c = \sum_{k=1}^n \mathcal{T}_{k-1}^c(d_k\alpha) + \sum_{k=1}^n \mathcal{T}_{k-1}^c(d_k\beta) + \sum_{k=1}^n \mathcal{T}_{k-1}^c(d_k\gamma) =: \alpha_n^c + \beta_n^c + \gamma_n^c.$$

To finish the proof, we shall show that there exists projection $e^{c,\ell,\alpha} \in \mathcal{M}$ such that

$$\tau(1-e^{c,\ell,\alpha}) \leq c_{\mathrm{reg}}\frac{1}{2^\ell}\tau((1-q^{(2^\ell)})x) + \frac{1}{2^{2\ell}}\tau(q^{(2^\ell)}xq^{(2^\ell)}xq^{(2^\ell)}), \quad \sup_n \|\alpha_n^c e^{\ell,z}\|_\infty \leq 2^\ell \tag{4.7}$$

and for each $z \in \{\beta,\gamma\}$, there exists a projection $e^{c,\ell,z} \in \mathcal{M}$ such that

$$\tau(1-e^{c,\ell,z}) \leq \sum_{k\geq\ell}\tau(1-q^{(2^k)}), \quad \sup_n \|z_n^c e^{c,\ell,z}\|_\infty \leq 2^\ell. \tag{4.8}$$

Once (4.7) and (4.8) are proved, then, letting $e^{c,\ell} = e^{c,\ell,\alpha} \wedge e^{c,\ell,\beta} \wedge e^{c,\ell,\gamma}$, we have

$$\sup_n \|y_n^c e^{c,\ell}\|_\infty \leq \sup_n \sum_{z\in\{\alpha,\beta,\gamma\}} \|z_n^c e^{c,\ell}\|_\infty \leq 3\cdot 2^\ell$$

and

$$\begin{aligned}\tau(1-e^{c,\ell}) &\leq \sum_{z\in\{\alpha,\beta,\gamma\}} \tau(1-e^{c,\ell,z}) \\ &\leq c_{\mathrm{reg}}\frac{1}{2^\ell}\tau((1-q^{(2^\ell)})x) + \frac{1}{2^{2\ell}}\tau(q^{(2^\ell)}xq^{(2^\ell)}xq^{(2^\ell)}) + 2\sum_{k\geq\ell}\tau(1-q_k^{(2^k)}).\end{aligned}$$

This is the desired estimate of the lemma.

It remains to show (4.7) and (4.8). According to the definition of $\|\cdot\|_{\Lambda_{2,\infty}(\mathcal{M},\ell_\infty^c)}$ and applying Lemma 2.7 to the martingale $\alpha = (\alpha_n^c)_n$, there exists a projection $e^{c,\ell,\alpha} \in \mathcal{M}$ such that

$$2^\ell[\tau(1-e^{c,\ell,\alpha})]^{1/2} \leq \|\alpha^c\|_{L_2(\mathcal{M})}, \quad \sup_n \|\alpha_n^c e^{c,\ell,\alpha}\|_\infty \leq 2^\ell. \tag{4.9}$$

On the other hand side, we have

$$\begin{aligned}\|\alpha^c\|_{L_2(\mathcal{M})}^2 &= \sup_n \|\alpha_n^c\|_{L_2(\mathcal{M})}^2 \\ &= \sup_n \Big\|\sum_{k=1}^n \mathcal{T}_{k-1}^c(d_k\alpha)\Big\|_{L_2(\mathcal{M})}^2 \\ &\overset{\mathrm{a}}{=} \sup_n \sum_{k=1}^n \|\mathcal{T}_{k-1}^c(d_k\alpha)\|_{L_2(\mathcal{M})}^2 \\ &\overset{\mathrm{b}}{\leq} \sup_n \sum_{k=1}^n \|d_k\alpha\|_{L_2(\mathcal{M})}^2 \\ &= \|\alpha\|_{L_2(\mathcal{M})}^2,\end{aligned} \tag{4.10}$$

where "a" is due to the orthogonality of the martingale differences $(\mathcal{T}_{k-1}^c(d_k\alpha))_k$, "b" is due to the fact that triangular truncation is bounded on $L_2(\mathcal{M})$ with norm less than 1 (see (4.2) above). Therefore, we conclude from (4.9), (4.10) and Lemma 4.2 (ii) that

$$\begin{aligned}\tau(1-e^{c,\ell,\alpha}) &\leq \frac{1}{2^{2\ell}}\|\alpha^c\|_{L_2(\mathcal{M})}^2 \\ &\leq c_{\mathrm{reg}}\frac{1}{2^\ell}\tau((1-q^{(2^\ell)})x) + \frac{1}{2^{2\ell}}\tau(q^{(2^\ell)}xq^{(2^\ell)}xq^{(2^\ell)}).\end{aligned}$$

This shows (4.7).

Now we turn to the proofs of (4.8) for $z=\beta$ or $z=\gamma$. Set

$$e^{c,\ell,\beta}=e^{c,\ell,\gamma}=e_\ell=\bigwedge_{k\geq\ell}q^{(2^k)}.$$

We immediately have

$$\tau(1-e_\ell)=\tau(1-\bigwedge_{k\geq\ell}q^{(2^k)})\leq\sum_{k\geq\ell}\tau(1-q^{(2^k)}).$$

We shall prove below that for each $n\geq1$,

$$\beta_n^c e_\ell=0 \tag{4.11}$$

and

$$\gamma_n^c e_\ell=0. \tag{4.12}$$

If this is proved, then (4.8) for $z\in\{\beta,\gamma\}$ follows. Note that

$$\beta_n^c e_\ell=\sum_{k=1}^n \mathcal{T}_{k-1}^c(d_k\beta)e_\ell=\sum_{k=1}^n \mathcal{T}_{k-1}^c(d_k\beta)e_{\ell,k-1}e_\ell,\quad n\geq1$$

and

$$\gamma_n^c e_\ell=\sum_{k=1}^n \mathcal{T}_{k-1}^c(d_k\gamma)e_\ell=\sum_{k=1}^n \mathcal{T}_{k-1}^c(d_k\gamma)e_{\ell,k-1}e_\ell,\quad n\geq1,$$

where $e_{\ell,k-1}$ is defined as in (3.9) such that $e_{\ell,k-1}\geq e_\ell$. Now to see (4.11) and (4.12), it suffices to show

$$\mathcal{T}_{k-1}^c(d_k\beta)e_{\ell,k-1}=0\quad\text{and}\quad \mathcal{T}_{k-1}^c(d_k\gamma)e_{\ell,k-1}=0, 1\leq k\leq n. \tag{4.13}$$

Note that $e_{i,k-1}\geq e_{\ell,k-1}$ for $i>\ell$. Recall that $\pi_{i,k-1}=e_{i,k-1}-e_{i-1,k-1}$. Thus,

$$\pi_{i,k-1}e_{\ell,k-1}=e_{\ell,k-1}\pi_{i,k-1}=0,\qquad\text{for } i>\ell,\ k\geq1,$$

which further implies

$$\mathcal{T}_{k-1}^c(d_k\beta)e_{\ell,k-1}=\sum_{i\leq j}\pi_{i,k-1}d_k\beta\pi_{j,k-1}e_{\ell,k-1}=\sum_{i\leq j\leq\ell}\pi_{i,k-1}d_k\beta\pi_{j,k-1}$$

and similarly

$$\mathcal{T}_{k-1}^c(d_k\gamma)e_{\ell,k-1}=\sum_{i\leq j\leq\ell}\pi_{i,k-1}d_k\gamma\pi_{j,k-1}.$$

As given in ($\mathbf{G}_\lambda$), $d_k\beta=d_ky(1-q_{k-1}^{(2^\ell)})$. It follows from the fact $\pi_{i,k-1}\leq e_{i,k-1}\leq q_{k-1}^{(2^\ell)}$ for $i\leq\ell$ that

$$\mathcal{T}_{k-1}^c(d_k\beta)e_{\ell,k-1}=\sum_{i\leq j\leq\ell}\pi_{i,k-1}d_ky(1-q_{k-1}^{(2^\ell)})\pi_{j,k-1}=\sum_{i\leq j\leq\ell}\pi_{i,k-1}d_ky(1-q_{k-1}^{(2^\ell)})q_{k-1}^{(2^\ell)}\pi_{j,k-1}=0.$$

Similarly,

$$\begin{aligned}\mathcal{T}_{k-1}^c(d_k\gamma)e_{\ell,k-1}&=\sum_{i\leq j\leq\ell}\pi_{i,k-1}(1-q_{k-1}^{(2^\ell)})d_kyq_{k-1}^{(2^\ell)}\pi_{j,k-1}\\&=\sum_{i\leq j\leq\ell}\pi_{i,k-1}q_{k-1}^{(2^\ell)}(1-q_{k-1}^{(2^\ell)})d_kyq_{k-1}^{(2^\ell)}\pi_{j,k-1}=0.\end{aligned}$$

Hence (4.13) is proved. The proof is complete. □

Now we are in the position to prove Theorem 1.10.

*Proof of Theorem 1.10.* Let $x \in L_2(\mathcal{M})$ be positive. For the whole proof, we let $y = y^c + y^r$ be the decomposition as in Lemma 4.1. We shall only deal with $y^c$ since the estimates for $y^r$ can be similarly established.

According to Lemma 4.3, for each $\ell \in \mathbb{Z}$, there is a projection $e^{c,\ell} \in \mathcal{M}$ satisfying

$$\sup_n \|y_n^c e^{c,\ell}\|_\infty \leq 3 \cdot 2^\ell$$

and

$$\tau(1 - e^{c,\ell}) \leq c_{\mathrm{reg}} \frac{1}{2^\ell} \tau((1 - q^{(2^\ell)})x) + \frac{1}{2^{2\ell}} \tau(q^{(2^\ell)} x q^{(2^\ell)} x q^{(2^\ell)}) + \sum_{k \geq \ell} \tau(1 - q^{(2^k)}), \tag{4.14}$$

where $(q_k^{(\lambda)})_{k\geq 1}$ denote the Cuculescu projections with respect to the martingale $x = (\mathcal{E}_n(x))_{n\geq 1}$ and $\lambda > 0$. Note that for each fixed $t > 0$, there exists unique $\ell \in \mathbb{Z}$ such that $2^{\ell-1} < t \leq 2^\ell$. Define

$$e^{c,t} = e^{c,\ell}, \quad 2^{\ell-1} < t \leq 2^\ell. \tag{4.15}$$

Hence, for each $t > 0$

$$\sup_n \|y_n^c e^{c,t}\|_\infty \leq 6t.$$

We first prove item (i) of the theorem, i.e., the weak type asymmetric maximal inequality. According to the definition of $\|\cdot\|_{\Lambda_{1,\infty}(\mathcal{M},\ell_\infty^c)}$ (see (2.4)), we have

$$\|y^c\|_{\Lambda_{1,\infty}(\mathcal{M},\ell_\infty^c)} \leq 6 \sup_t t\tau(1 - e^{c,t}). \tag{4.16}$$

Fix $t > 0$. Without loss of generality, we assume $2^{\ell-1} < t \leq 2^\ell$ for some $\ell \in \mathbb{Z}$. Then

$$t\tau(1 - e^{c,t}) \leq 2^\ell \tau(1 - e^{c,\ell}). \tag{4.17}$$

On the other hand, by Lemma 3.1 (iii), we know that $q^{(2^\ell)} x q^{(2^\ell)} \leq 2^\ell$, and hence

$$\tau(q^{(2^\ell)} x q^{(2^\ell)} x q^{(2^\ell)}) \leq 2^\ell \tau(q^{(2^\ell)} x q^{(2^\ell)}) \leq 2^\ell \|x\|_1.$$

Again, by Lemma 3.1 (iv), we have

$$\sum_{k\geq \ell} \tau(1 - q_k^{(2^k)}) \leq \sum_{k \geq \ell} \frac{1}{2^k} \|x\|_1 = \frac{1}{2^{\ell-1}} \|x\|_1.$$

Consequently, combining the above argument with (4.14), we have

$$\tau(1 - e^{c,\ell}) \leq c_{\mathrm{reg}} \frac{1}{2^\ell} \|x\|_1 + \frac{1}{2^{2\ell}} 2^\ell \|x\|_1 + \frac{1}{2^{\ell-1}} \|x\|_1,$$

which implies

$$t 2^\ell \tau(1 - e^{c,\ell}) \leq (c_{\mathrm{reg}} + 3)\|x\|_1.$$

Since $t$ is arbitrary, the last inequality together with (4.16) and (4.17) gives us

$$\|y^c\|_{\Lambda_{1,\infty}(\mathcal{M},\ell_\infty^c)} \leq (6c_{\mathrm{reg}} + 18)\|x\|_{L_1(\mathcal{M})}.$$

Similarly,

$$\|y^r\|_{\Lambda_{1,\infty}(\mathcal{M},\ell_\infty^r)} \leq (6c_{\mathrm{reg}} + 18)\|x\|_{L_1(\mathcal{M})}.$$

Now we turn to prove item (ii) of the theorem, i.e., the strong type asymmetric maximal inequality. Let $1 < p < 2$. We still use the projection $e^{c,t}$ defined in (4.15). According to the definition of $\|\cdot\|_{\Lambda_p(\mathcal{M},\ell_\infty^c)}$ given in (2.5), we have

$$\|y^c\|^p_{\Lambda_p(\mathcal{M},\ell_\infty^c)} \leq 6p \int_0^\infty t^{p-1} \tau(1 - e^{c,t}) dt \tag{4.18}$$

By (4.14), we have

$$
\begin{aligned}
&\int_0^\infty t^{p-1}\tau(1-e^{c,t})dt \\
&= \sum_{\ell\in\mathbb{Z}}\int_{2^{\ell-1}}^{2^\ell} t^{p-1}\tau(1-e^{c,t})dt \le \sum_{\ell\in\mathbb{Z}} 2^{\ell p}\tau(1-e^{c,\ell}) \\
&\le c_{\mathrm{reg}}\sum_{\ell\in\mathbb{Z}} 2^{\ell(p-1)}\tau((1-q^{(2^\ell)})x) + \sum_{\ell\in\mathbb{Z}} 2^{\ell(p-2)}\tau(q^{(2^\ell)}xq^{(2^\ell)}xq^{(2^\ell)}) + \sum_{\ell\in\mathbb{Z}} 2^{\ell p}\sum_{k\ge\ell}\tau(1-q^{(2^k)})
\end{aligned}
\tag{4.19}
$$

Combining (4.18), (4.19) and Proposition 3.11, item (ii) of the theorem follows.

Finally, let us prove the asymmetric maximal inequality associated with $L\log L$ space, i.e., item (iii) of the theorem. We still use the projection $e^{c,t}$ defined in (4.15). According to the definition of $\|\cdot\|_{\Lambda_1(\mathcal{M},\ell_\infty^c)}$, we have

$$
\|y^c\|_{\Lambda_1(\mathcal{M},\ell_\infty^c)} \le 6\int_0^\infty \tau(1-e^{c,t})dt \tag{4.20}
$$

By (4.14), we have

$$
\begin{aligned}
&\int_0^\infty \tau(1-e^{c,t})dt \le \int_0^1 \tau(1-e^{c,t})dt + \int_1^\infty \tau(1-e^{c,t})dt \\
&\le 1 + \sum_{\ell\ge 0}\int_{2^{\ell-1}}^{2^\ell} \tau(1-e^{c,t})dt \le 1 + \sum_{\ell\ge 0} 2^\ell\tau(1-e^{c,\ell}) \\
&\le 1 + c_{\mathrm{reg}}\sum_{\ell\ge 0}\tau((1-q^{(2^\ell)})x) + \sum_{\ell\ge 0} 2^{-\ell}\tau(q^{(2^\ell)}xq^{(2^\ell)}xq^{(2^\ell)}) + \sum_{\ell\ge 0} 2^\ell\sum_{k\ge\ell}\tau(1-q^{(2^k)}).
\end{aligned}
\tag{4.21}
$$

Now item (iii) follows from Proposition 3.15 and Lemma 3.14. The proof is complete. □

**Remark 4.4.** *In the above proofs, we use triangular truncation to decompose martingale $y$ (see Lemma 4.1). Triangular truncation is just weak $(1,1)$ (see e.g. [17]) which prevent us removing the assumption of regularity .*

Combining (2.8), Lemma 2.3 and Lemma 2.4, we immediately get the following consequence of Theorem 1.10

**Corollary 4.5.** *Let $(\mathcal{M},\tau)$ be a noncommutative probability space associated with the regular filtration $(\mathcal{M}_n)_{n\ge1}$. Suppose that martingale $x\in L_2(\mathcal{M})$ is positive, and martingale $y$ is $L_2$-dominated by $x$. The following hold:*

(i) *There exists a constant $c$ such that*

$$\|y\|_{L_{1,\infty}(\mathcal{M})} \le \|y\|_{\Lambda_{1,\infty}(\mathcal{M},\ell_\infty)} \le c\|x\|_{L_1(\mathcal{M})}.$$

(ii) *There exists a constant $c_p>0$ such that*

$$\|y\|_{L_p(\mathcal{M})} \le \|y\|_{\Lambda_p(\mathcal{M},\ell_\infty)} \le c_p\|x\|_{L_p(\mathcal{M})}, \quad 1<p<2.$$

*The order of the constant, $O(\frac{1}{p-1})$ as $p\to1$, is already best possible for commutative martingales.*

(iii) *There exist universal constants $c_1,c_2>0$ such that*

$$\|y\|_{L_1(\mathcal{M})} \le \|y\|_{\Lambda_1(\mathcal{M},\ell_\infty)} \le c_1 + c_2\|x\|_{L\log L(\mathcal{M})}.$$

*Here all constants depend on the regularity constant $c_{\mathrm{reg}}$.*

We can easily prove Theorem 1.4 via Theorem 1.10.

*Proof of Theorem 1.4.* We only prove item (i) of the theorem and item (ii) can be proved similarly. Once we have the maximal inequalities (i) and (ii), item (iii) can be proved by standard argument (see e.g. [44, Section 6]).

Applying Theorem 1.10 (i) to the case $y = x$, we then know that for each positive $L_1(\mathcal{M})$-bounded martingale $x = (x_n)_{n\geq 1}$, there exist two martingales $x^c$ and $x^r$ such that $x = x^c + x^r$ and

$$\|x^c\|_{\Lambda_{1,\infty}(\mathcal{M},\ell_\infty^c)} + \|x^r\|_{\Lambda_{1,\infty}(\mathcal{M},\ell_\infty^r)} \leq (6c_{\mathrm{reg}} + 18)\|x\|_{L_1(\mathcal{M})}.$$

Recall that the noncommutative analogue of Krickeberg's decomposition (see e.g. [12, p. 20]) says that each $L_1(\mathcal{M})$-bounded martingale can be written into 4 positive $L_1(\mathcal{M})$-bounded martingales. Also note that $\|\cdot\|_{\Lambda_{1,\infty}(\mathcal{M},\ell_\infty^c)}$ is a quasi-norm with constant 2. The desired assertion follows. □

4.2. **Related remarks and questions.** In this subsection, we do not assume that the filtration $(\mathcal{M}_n)_{n\geq 1}$ is regular. Let $x = (x_n)_{n\geq 1}$ and $y = (y_n)_{n\geq 1}$ be two self-adjoint $L_2(\mathcal{M})$-bounded martingales. The Cuculescu projections (still denoted by $(q_n^{(\lambda)})_{n\geq 1}$) related to the self-adjoint martingale $x = (x_n)_{n\geq 1}$ are also well-defined; see e.g. [55, Proposition 1.4]. If $y$ is differentially subordinate to $x$ in the sense of Definition 1.7, then it is proved in [31, Lemma 5.4] that

$$\tau\Big(\chi_{(2^k,\infty)}(|y|)\Big) \leq 9 \cdot \frac{1}{2^\ell}\tau((1-q^{(2^\ell)})|x|) + 2\cdot \frac{1}{2^{2\ell}}\tau(q^{(2^\ell)}xq^{(2^\ell)}xq^{(2^\ell)}).$$

Similar to proof of Theorem 1.10 (iii), we can apply Proposition 3.15 and Lemma 3.14 to get

$$\|y\|_{L_1(\mathcal{M})} \leq c_1 + c_2\|x\|_{L\log L(\mathcal{M})}. \tag{4.22}$$

This is a supplement to [31] and is a generalization of [62, Theorem 6.2]. Actually, when $y$ is a martingale transform as in [62], i.e., $y_n = \sum_{k=1}^n \varepsilon_k d_k x$, $\varepsilon_k = 1$ or $-1$, Randrianantoanina proved the last inequality by extrapolation as usual in the classical setting. We provide a direct proof via distribution estimate given in Lemma 4.3.

Furthermore, motivated by Lemma 4.3, instead of Lemma 4.2 and Lemma 2.7, using the noncommutative Gundy decomposition established in [34, Theorem 3.1] and the Doob's maximal inequality [38, Theorem 0.2], we can show that (the details are left to the reader), for each $\ell \in \mathbb{Z}$, there exists a projection $e^\ell$ such that $\sup_n \|e^\ell y_n e^\ell\|_\infty \leq 2^\ell$ and

$$\tau(1-e^\ell) \leq 9\cdot \frac{1}{2^\ell}\tau((1-q^{(2^\ell)})|x|) + 2\cdot\frac{1}{2^{2\ell}}\tau(q^{(2^\ell)}xq^{(2^\ell)}xq^{(2^\ell)}). \tag{4.23}$$

Following the argument used in the proof of Theorem 1.10, combining the last inequality and the useful estimates proved in Section 3, we have

**Theorem 4.6.** *Let $(\mathcal{M},\tau)$ be a noncommutative probability space associated with the filtration $(\mathcal{M}_n)_{n\geq 1}$ (which is not necessary regular). Suppose that martingales $x$ and $y$ are self-adjoint, and martingale $y$ is differentially subordinate to $x$ in the sense of Definition 1.7 (i). We have the following inequalities*

(i) *There exists a universal constant $c > 0$ such that*

$$\|y\|_{\Lambda_{1,\infty}(\mathcal{M},\ell_\infty)} \leq c\|x\|_{L_1(\mathcal{M})}.$$

(ii) *There exists a constant $c_p > 0$ such that*

$$\|y\|_{\Lambda_p(\mathcal{M},\ell_\infty)} \leq c_p\|x\|_{L_p(\mathcal{M})}, \quad 1 < p < 2.$$

*The order of the constant, $c_p = O(\frac{1}{p-1})$ as $p \to 1$, is sharp.*

(iii) *There exists a universal constant $c_1, c_2 > 0$ such that*

$$\|y\|_{\Lambda_1(\mathcal{M},\ell_\infty)} \leq c_1 + c_2\|x\|_{L\log L(\mathcal{M})}.$$

**Remark 4.7.** *If $y$ is weakly dominated by $x$ in the sense of Definition 1.7 (ii), then Theorem 4.6 also holds. This is also left to the interested reader.*

The above theorem and Theorem 1.10 are comparable. However, we can not say one is stronger than the other one. Indeed, there is no regularity assumption in Theorem 4.6, but we can not get asymmetric maximal inequalities; even through we obtain asymmetric maximal inequalities in Theorem 1.10, we assume that the related filtration is regular. Moreover, the following question is unknown

**Question 4.8.** *Does Theorem 1.4 hold for general filtration $(\mathcal{M}_n)_{n\geq 1}$?*

To answer the above question for $1 < p < 2$, with the help of [34, Theorem 4.10], it suffices to provide a positive answer for the following question

**Question 4.9.** *For general noncommutative martingale $x$, do we have*

$$\|x\|_{\Lambda_p(\mathcal{M},\ell_\infty^c)} \leq c\|x\|_{H_p^c}, \quad 1 \leq p \leq 2?$$

Note that the positive answer of the last question will be an improvement of [27, Theorem A].

Let us also pay attention to the noncommutative maximal inequalities with best order constants. Consider $y$ is a martingale transform as in [62], i.e., $y_n = \sum_{k=1}^n \varepsilon_k d_k x$, $\varepsilon_k = 1$ or $-1$. It is clear that $y$ is differentially subordinate to $x$ in the sense of Definition 1.7 (i). Theorem 4.6 (ii) tells us that

$$\|(\sum_{k=1}^n \varepsilon_k d_k x)_{n\geq 1}\|_{\Lambda_p(\mathcal{M},\ell_\infty)} \leq c_p\|x\|_{L_p(\mathcal{M})}, \quad 1 < p < 2,$$

where $c_p = O(\frac{1}{p-1})$ as $p \to 1$ is of best order. Actually, in the commutative setting $\mathcal{M} = L_\infty(\Omega)$ for some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the below inequality is best

$$\|\sup_n |\sum_{k=1}^n \varepsilon_k d_k x|\|_{L_p(\Omega)} \leq \frac{cp^2}{p-1}\|x\|_{L_p(\Omega)}, \quad 1 < p < \infty.$$

Note that $\|a\|_{\Lambda_p(\mathcal{M},\ell_\infty)} \leq \|a\|_{L_p(\mathcal{M},\ell_\infty)}$ for any sequence $a = (a_n) \subset L_p(\mathcal{M})$. Motivated by (1.4), it is natural to study the best order constant $\beta_p$ (when $p \to 1$) of the following inequality

$$\|(\sum_{k=1}^n \varepsilon_k d_k x)_{n\geq 1}\|_{L_p(\mathcal{M},\ell_\infty)} \leq \beta_p\|x\|_{L_p(\mathcal{M})}, \quad 1 < p < 2.$$

Let us state this question for differential subordination martingales.

**Question 4.10.** *Assume that martingale $y$ is differentially subordinate to martingale $x$ in the sense of Definition 1.7 (i). For $1 < p < \infty$, we have the following inequality*

$$\|y\|_{L_p(\mathcal{M},\ell_\infty)} \leq \beta_p\|x\|_{L_p(\mathcal{M})}.$$

*As $p \to 1$, is the best order of $\beta_p$ the same to the usual Doob maximal inequality studied by Junge and Xu* [43, Theorem 8(iii)]*?*

## 5. Noncommutative Davis inequalities

In this section, prove Theorem 1.11 and Theorem 1.5. Under the regularity assumption, Theorem 1.5 provides a positive answer to Junge and Xu's open problem raised in [43, Problem 16] (see also Problem 1.1 above).

### 5.1. Proof of Theorems 1.11 and 1.5.

To prove Theorem 1.11, we begin with the following estimate for Cuculescu projections.

**Lemma 5.1.** *Consider the Cuculescu projections $(q_n^{(\lambda)})_n$ (introduced in Lemma 3.1) with respect to the positive martingale $x = (\mathcal{E}_n(x))_{n\geq 1}$. Assume that $(\mathcal{M}_n)$ is regular with regularity constant $c_{\text{reg}}$. For each $\lambda > 0$, we have*

$$\tau\Big((1 - q^{(\lambda)})x\Big) \leq c_{\text{reg}}\lambda\tau(1 - q^\lambda).$$

*Proof.* It follows from the regularity assumption that (see (1.8))

$$\mathcal{E}_n(x) \leq c_{\text{reg}}\mathcal{E}_{n-1}(x).$$

Recall that $p_n^{(\lambda)} = q_{n-1}^{(\lambda)} - q_n^{(\lambda)} \in \mathcal{M}_n$ is defined in (3.2). Thus, for each $n \geq 1$, we have

$$p_n^{(\lambda)}\mathcal{E}_n(x)p_n^{(\lambda)} \leq c_{\text{reg}}p_n^{(\lambda)}\mathcal{E}_{n-1}(x)p_n^{(\lambda)} = c_{\text{reg}}p_n^{(\lambda)}q_{n-1}^{(\lambda)}\mathcal{E}_{n-1}(x)q_{n-1}^{(\lambda)}p_n^{(\lambda)} \leq c_{\text{reg}}\lambda p_n^{(\lambda)},$$

where we used Lemma 3.1 (iii) in the last inequality. Consequently, by the tracial property of $\tau$ and basic property of conditional expectation, we obtain

$$\begin{aligned}\tau((1-q^{(\lambda)})x) &= \sum_{n\geq 1} \tau(p_n^{(\lambda)}x) \\ &= \sum_{n\geq 1} \tau\Big(p_n^{(\lambda)}\mathcal{E}_n(x)p_n^{(\lambda)}\Big) \\ &\leq c_{\mathrm{reg}}\lambda \sum_{n\geq 1}\tau(p_n^{(\lambda)}) = c_{\mathrm{reg}}\lambda\tau(1-q^{(\lambda)}).\end{aligned}$$

The proof is complete. □

We also need the following distribution estimates for noncommutative square functions.

**Lemma 5.2.** *Let $(\mathcal{M},\tau)$ be a noncommutative probability space associated with the regular filtration $(\mathcal{M}_n)_{n\geq 1}$. Suppose that martingale $x \in L_2(\mathcal{M})$ is positive, and martingale $y$ is $L_2$-dominated by $x$. Consider the decomposition $y = y^c + y^r$ as in Lemma 4.1. For each $\ell \in \mathbb{Z}$, we have*

$$\tau\Big(\chi_{(2^\ell,\infty)}(S_c(y^c))\Big) \leq \frac{2}{2^{2\ell}}\tau\Big(q^{(2^\ell)}xq^{(2^\ell)}xq^{(2^\ell)}\Big) + (2c_{\mathrm{reg}}^2+2)\sum_{k\geq \ell}\tau(1-q^{(2^k)})$$

*and*

$$\tau\Big(\chi_{(2^\ell,\infty)}(S_r(y^r))\Big) \leq \frac{2}{2^{2\ell}}\tau\Big(q^{(2^\ell)}xq^{(2^\ell)}xq^{(2^\ell)}\Big) + (2c_{\mathrm{reg}}^2+2)\sum_{k\geq \ell}\tau(1-q^{(2^k)}).$$

*Proof.* We only deal with $y^c$ and the estimate for $y^r$ can be similarly done. Applying the Gundy decomposition Lemma 4.2 to $y$ and $\lambda = 2^\ell$ for some $\ell \in \mathbb{Z}$, we have

$$y = \alpha + \beta + \gamma.$$

Hence, for each $n \geq 1$,

$$\begin{aligned} y_n^c &= \sum_{k=1}^n \mathcal{T}_{k-1}^c(d_k\alpha) + \sum_{k=1}^n \mathcal{T}_{k-1}^c(d_k\beta) + \sum_{k=1}^n \mathcal{T}_{k-1}^c(d_k\gamma) \\ &=: \alpha_n^c + \beta_n^c + \gamma_n^c, \end{aligned}$$

and by the basic inequality $|a+b|^2 \leq 2|a|^2 + 2|b|^2$, we have

$$\begin{aligned} S_c(y^c)^2 &= \sum_{k\geq 1} |\mathcal{T}_{k-1}^c(d_k\alpha) + \mathcal{T}_{k-1}^c(d_k\beta) + \mathcal{T}_{k-1}^c(d_k\gamma)|^2 \\ &\leq 2\sum_{k\geq 1}|\mathcal{T}_{k-1}^c(d_k\alpha)|^2 + 4\sum_{k\geq 1}|\mathcal{T}_{k-1}^c(d_k\beta)|^2 + 4\sum_{k\geq 1}|\mathcal{T}_{k-1}^c(d_k\gamma)|^2. \end{aligned}$$

Consequently,

$$\begin{aligned} &\tau\Big(\chi_{(2^\ell,\infty)}(S_c(y^c))\Big) \\ &= \tau\Big(\chi_{(2^{2\ell},\infty)}(S_c(y^c)^2)\Big) \\ &\leq \tau\Big(\chi_{(2^{2\ell},\infty)}(2\sum_{k\geq 1}|\mathcal{T}_{k-1}^c(d_k\alpha)|^2)\Big) + \\ &\quad + \tau\Big(\chi_{(2^{2\ell},\infty)}(4\sum_{k\geq 1}|\mathcal{T}_{k-1}^c(d_k\beta)|^2)\Big) + \tau\Big(\chi_{(2^{2\ell},\infty)}(4\sum_{k\geq 1}|\mathcal{T}_{k-1}^c(d_k\gamma)|^2)\Big) \\ &=: \mathcal{X}_{\alpha,\ell} + \mathcal{X}_{\beta,\ell} + \mathcal{X}_{\gamma,\ell}. \end{aligned}$$

According to (4.13), for each $k$, the right support of $\mathcal{T}^c_{k-1}(d_k\beta)$ is less than $1-e_{\ell,k-1}\leq 1-e_\ell$. Thus the support of $|\mathcal{T}^c_{k-1}(d_k\beta)|^2$ is less than $1-e_\ell$, which implies

$$\mathcal{X}_{\beta,\ell}\leq \tau(1-e_\ell)=\tau(1-\bigwedge_{k\geq \ell} q_{(2^k)})\leq \sum_{k\geq\ell}\tau(1-q^{(2^k)}).$$

Similarly, by (4.13),

$$\mathcal{X}_{\gamma,\ell}\leq \sum_{k\geq\ell}\tau(1-q^{(2^k)}).$$

As for $\mathcal{X}_{\alpha,\ell}$, using the Chebyshev inequality and similar argument used in (4.10), we get

$$\begin{aligned}\mathcal{X}_{\alpha,\ell}&\leq 2\cdot 2^{-2\ell}\|\sum_{k\geq 1}|\mathcal{T}^c_{k-1}(d_k\alpha)|^2\|_1\\ &\leq 2\cdot 2^{-2\ell}\sum_{k\geq 1}\|\mathcal{T}^c_{k-1}(d_k\alpha)\|_2^2\\ &\leq 2\cdot 2^{-2\ell}\|\alpha\|_2^2\\ &\leq 2c_{\mathrm{reg}}2^{-\ell}\tau((1-q^{(2^\ell)})x)+2\cdot 2^{-2\ell}\tau(q^{(-2\ell)}xq^{(-2\ell)}xq^{(-2\ell)})\\ &\leq 2c_{\mathrm{reg}}^2\tau(1-q^{(2^\ell)})+2\cdot 2^{-2\ell}\tau(q^{(-2\ell)}xq^{(-2\ell)}xq^{(-2\ell)}),\end{aligned}$$

where we used Lemma 4.2 (ii) in the fourth inequality and Lemma 5.1 in the last inequality. Combining the estimates for $\mathcal{X}_{\alpha,\ell}, \mathcal{X}_{\beta,\ell}$, and $\mathcal{X}_{\gamma,\ell}$, we get the desired result. □

Now we are in a position to prove Theorem 1.11.

*Proof of Theorem 1.11.* Let $0<p\leq 1$. Take positive $x\in L_2(\mathcal{M})$ and assume $\|(\mathcal{E}_n(x))_{n\geq1}\|_{L_p(\mathcal{M},\ell_\infty)}$ is finite. According to the definition of $\|\cdot\|_{L_p(\mathcal{M},\ell_\infty)}$, we can take positive $a\in L_p(\mathcal{M})$ such that $x_n=\mathcal{E}_n(x)\leq a$ for all $n\geq 1$ and

$$\|a\|_p\leq \|x\|_{L_p(\mathcal{M},\ell_\infty)}.$$

By Lemma 5.2, we have

$$\begin{aligned}\|y^c\|^p_{H_p^c}=\|S_c(y^c)\|_p^p&=p\int_0^\infty \lambda^{p-1}\tau(\chi_{(\lambda,\infty)}(S_c(y^c)))d\lambda\\ &\leq p\sum_{\ell\in\mathbb{Z}}2^{\ell p}\tau\Big(\chi_{(2^\ell,\infty)}(S_c(y^c))\Big)\\ &\leq 2p\sum_{\ell\in\mathbb{Z}}2^{\ell(p-2)}\tau(q^{(2^\ell)}xq^{(2^\ell)}xq^{(2^\ell)})+p(2c_{\mathrm{reg}}^2+2)\sum_{\ell\in\mathbb{Z}}2^{\ell p}\sum_{k\geq\ell}\tau(1-q^{(2^k)}).\end{aligned}$$

It follows from Proposition 3.17 that

$$\|y^c\|^p_{H_p^c}\leq \frac{2^{p+5}}{(2^p-1)}\|a\|_p^p+(2c_{\mathrm{reg}}^2+2)\frac{2^{p+1}}{(2^p-1)}\|a\|_p^p\leq c_{p,\mathrm{reg}}\|x\|^p_{L_p(\mathcal{M},\ell_\infty)}.$$

The same estimate holds for $y^r$, and the proof of the theorem is complete. □

With the help of Theorem 1.11, we prove Theorem 1.5.

*Proof of Theorem 1.5.* Let $0<p\leq 1$. Applying Theorem 1.11 with $y=x$, we deduce that for each positive martingale $x\in L_p(\mathcal{M},\ell_\infty)$,

$$\|x\|_{H_p(\mathcal{M})}\leq c_{p,\mathrm{reg}}\|x\|_{L_p(\mathcal{M},\ell_\infty)},$$

which together with Lemma 2.5 proves the desired assertion for general martingales in $L_p(\mathcal{M},\ell_\infty)$. □

**Remark 5.3.** *We also realize that Lemma 5.1 (together with Lemma 3.3) is actually a noncommutative version of Gundy's two-side inequality* [26, Theorem 2]. *It was proved in* [26] *that for a regular martingale, if* $\sup_n|x_n|\in L_1(\Omega)$, *then the martingale* $x=(x_n)_{n\geq1}$ *belongs to* $L\log L(\Omega)$. *We do not know whether this holds for noncommutative martingale.*

If one can provide a affirmative answer to the above question, then the below is also true. we may delete the question below.

**Question 5.4.** *For a regular martingale $x = (x_n)_{n\geq 1}$, do we have*

$$\|x\|_{L\log L(\mathcal{M})} \leq c\|x\|_{L_1(\mathcal{M},\ell_\infty)}?$$

By the distribution estimate obtained in Lemma 5.2, using identical argument to Theorem 1.10, we immediately have

**Theorem 5.5.** *Let $(\mathcal{M},\tau)$ be a noncommutative probability space associated with the regular filtration $(\mathcal{M}_n)_{n\geq 1}$. Suppose that martingale $x \in L_2(\mathcal{M})$ is positive, and martingale $y$ is $L_2$-dominated by $x$. There exists a decomposition $y = y^c + y^r$ such that the following holds:*

(i) *We have*
$$\|S_c(y^c)\|_{L_{1,\infty}(\mathcal{M})} + \|S_r(y^r)\|_{L_{1,\infty}(\mathcal{M})} \leq (6c_{\mathrm{reg}} + 18)\|x\|_{L_1(\mathcal{M})}.$$
(ii) *There exists a constant $c_p > 0$ such that*
$$\|S_c(y^c)\|_{L_p(\mathcal{M})} + \|S_r(y^r)\|_{L_p(\mathcal{M})} \leq c_p\|x\|_{L_p(\mathcal{M})}, \quad 1 < p < 2.$$
*The order of the constant, $O(\frac{1}{p-1})$ as $p \to 1$, is already best possible for commutative martingales.*
(iii) *There exists a universal constant $c > 0$ (depends on $c_{\mathrm{reg}}$) such that*
$$\|S_c(y^c)\|_{L_1(\mathcal{M})} + \|S_r(y^r)\|_{L_1(\mathcal{M})} \leq 2 + c\|x\|_{L\log L(\mathcal{M})}.$$

5.2. **Discussion for non-regular martingales.** Similar to Section 4.2, we here collect some comments for differential subordination martingales in the sense of Definition 1.7 (i). Again, we do not need to assume that the filtration $(\mathcal{M}_n)_{n\geq 1}$ is regular in the subsection. Let $x = (x_n)_{n\geq 1}$ and $y = (y_n)_{n\geq 1}$ be two self-adjoint $L_2(\mathcal{M})$-bounded martingales. Still denote by $(q_n^{(\lambda)})_{n\geq 1}$ the Cuculescu projections related to the self-adjoint martingale $x = (x_n)_{n\geq 1}$. Assume that $y$ is differentially subordinate to $x$ in the sense of Definition 1.7 (i). Consider the decomposition of $y = y^c + y^r$ given in Lemma 4.1. Then the following distribution estimate holds: for each $\ell \in \mathbb{Z}$, there exist universal constants $c_1, c_2 > 0$ such that

$$\tau\Big(\chi_{(2^\ell,\infty)}(S_c(y^c))\Big) \leq c_1 \cdot \frac{1}{2^\ell}\tau((1-q^{(2^\ell)})|x|) + c_2\frac{1}{2^{2\ell}}\tau(q^{(2^\ell)}xq^{(2^\ell)}xq^{(2^\ell)}) + c_3\sum_{k\geq \ell}\tau(1-q^{(2^k)}).$$

This is implicit in [34], and can be also proved similarly to Lemma 5.2. The same estimate applies to the distribution function of $S_r(y^r)$. Therefore, with the same argument used in the proof of Theorem 1.10 (iii), one can apply Proposition 3.15 and Lemma 3.14 to show

$$\|S_c(y^c)\|_{L_1(\mathcal{M})} + \|S_r(y^r)\|_{L_1(\mathcal{M})} \leq c_1 + c_2\|x\|_{L\log L(\mathcal{M})}, \tag{5.1}$$

where $c_1, c_2$ are universal constants. This is a supplement to [34]. Recall that there exists a universal constant $\beta > 0$ such that

$$\|y\|_{L_1(\mathcal{M})} \leq \beta\|y\|_{H_1(\mathcal{M})} \leq \beta\|S_c(y^c)\|_{L_1(\mathcal{M})} + \beta\|S_r(y^r)\|_{L_1(\mathcal{M})},$$

where the first inequality can be found in [59, eq. (A3)] (or [42, Corollary 4.3]) and the second inequality is due to the definition of $H_1(\mathcal{M})$. Hence, (5.1) implies (4.22). Also recall that

$$\|x\|_{H_1(\mathcal{M})} \leq c_1 + c_2\|x\|_{L\log L(\mathcal{M})} \tag{5.2}$$

was firstly proved by Randrianantoanina in [64, Theorem 5.7] via extrapolation. Thus, the above inequality (5.1) (even for $y = x$) is stronger than (5.2).

Finally, combining (5.1) with $y = x$ and [27, Theorem A], we obtain that for each martingale $x = (x_n)_{n\geq 1} \in L\log L(\mathcal{M})$, there exist two martingales $x^c$ and $x^r$ such that $x = x^c + x^r$ and

$$\|x^c\|_{\Lambda_{1,\infty}(\mathcal{M},\ell_\infty^c)} + \|x^r\|_{\Lambda_{1,\infty}(\mathcal{M},\ell_\infty^r)} \leq c\|x\|_{L\log L(\mathcal{M})}.$$

Moreover, we can deduce from this asymmetric inequality that $(x_n)_{n\geq 1}$ converges column + row almost uniformly provided $x = (x_n)_{n\geq 1} \in L\log L(\mathcal{M})$.

**Remark 5.6.** *Assume that $y$ is differentially subordinate to $x$ in the sense of Definition 1.7 (i). It is different from the commutative martingale case that one can not deduce from (5.2) that*

$$\|y\|_{H_1(\mathcal{M})} \leq c\|x\|_{L\log L(\mathcal{M})}.$$

*Indeed, the below question raised in* [34, Problem 5.7] *is still open: is there a universal constant $c > 0$ such that*

$$\|y\|_{H_1(\mathcal{M})} \leq c\|x\|_{H_1(\mathcal{M})}?$$

## 6. Applications

In this section, we collect four typical applications of our main results. The first two applications given in Sections 6.1 and 6.2 are are concrete examples of noncommutative $L_2$-domination martingales in the sense of Definition 1.8. The other two applications are from Theorem 1.5.

We here point out that the example presented in Section 6.2 satisfies Definition 1.8 but does not fulfill Definition 1.7. The triangular truncation studied in Section 6.1 is an example of Definition 1.8, however, if we do not care about the self-adjointness, it is also an example of Definition 1.7.

6.1. **Triangular truncation.** We again recall the definition of triangular truncation which is already introduced in (4.1). To simply the discussion, let $(R_k)_{k=1}^N$ be a sequence of mutually orthogonal projections in $\mathcal{M}$ such that $\sum_{k=1}^N R_k = 1$. For each $x \in L_2(\mathcal{M})$, the triangular truncation are defined by

$$\mathcal{T}^c(x) = \sum_{j=1}^N \sum_{i=1}^j R_i x R_j, \quad \mathcal{T}^r(x) = x - \mathcal{T}^c(x) = \sum_{i\geq j} R_i x R_j. \tag{6.1}$$

Here we still assume that $(\mathcal{M},\tau)$ is a noncommutative probability space.

The study of triangular truncation goes back at least to [25], and nowadays plays an important role in noncommutative analysis. For example, it can be used to study Lipschitz continuity of the absolute value mapping in noncommutative space. We refer the reader to [8, 17] for more details. The below result is well-known (see e.g. [17]).

**Theorem 6.1.** *The following inequalities hold:*

(i) *There exists a universal constant $c > 0$ such that*

$$\|\mathcal{T}^c(x)\|_{L_{1,\infty}(\mathcal{M})} \leq c\|x\|_{L_1(\mathcal{M})}.$$

(ii) *For $1 < p < \infty$, there exists a constant $c > 0$ such that*

$$\|\mathcal{T}^c(x)\|_{L_p(\mathcal{M})} \leq \frac{cp^2}{p-1}\|x\|_{L_p(\mathcal{M})}.$$

(iii) *There exist universal constants $c_1, c_2 > 0$ such that*

$$\|\mathcal{T}^c(x)\|_{L_1(\mathcal{M})} \leq c_1 + c_2\|x\|_{L\log L(\mathcal{M})}.$$

We shall explain that the above result is just a special case of Corollary 4.5. Indeed, each triangular truncation is a martingale transform with respect to the shell filtration. Denote $\widetilde{R}_n = \sum_{k=1}^n R_k$, $n \geq 1$. For each $k \geq 1$, let

$$M_k = \{a : a \in \widetilde{R}_k\mathcal{M}\widetilde{R}_k \quad \text{or} \quad a \in \bigcup_{n>k} R_n\mathcal{M}R_n\}, \quad \mathcal{M}_k = \mathrm{VNA}(M_k), \tag{6.2}$$

where $\mathrm{VNA}(M_k)$ denotes the von Neumann algebra generated by $M_k$. Then $(\mathcal{M}_k)_{k\geq 1}$ is a sequence of increasing von Neumann subalgebras of $\mathcal{M}$, which is referred to the shell filtration in [61]. Moreover, the map $\mathcal{E}_k : \mathcal{M} \to \mathcal{M}_k$ defined by

$$\mathcal{E}_k(x) = \widetilde{R}_k x \widetilde{R}_k + \sum_{n>k} R_n x R_n \tag{6.3}$$

is the corresponding conditional expectation.

It was shown by the authors and Potapov in [61] that the shell filtration is regular in the sense of (1.8). We still include its proof here for convenience.

**Lemma 6.2.** *The shell filtration $(\mathcal{M}_k)_{k\geq 1}$ given in (6.2) is regular with constant 2.*

*Proof.* Take positive $x \in \mathcal{M}$. We shall show

$$\mathcal{E}_k(x) \leq 2\mathcal{E}_{k-1}(x), \quad k \geq 2.$$

We start with

$$(\widetilde{R}_{k-1} - R_k)x(\widetilde{R}_{k-1} - R_k) \geq 0.$$

Opening the brackets, we arrive at

$$R_k x \widetilde{R}_{k-1} + \widetilde{R}_{k-1} x R_k \leq \widetilde{R}_{k-1} x \widetilde{R}_{k-1} + R_k x R_k.$$

Hence, for each $k$,

$$\begin{aligned}\widetilde{R}_k x \widetilde{R}_k &= (R_k + \widetilde{R}_{k-1})x(R_k + \widetilde{R}_{k-1}) \\ &= R_k x \widetilde{R}_{k-1} + \widetilde{R}_{k-1} x R_k + \widetilde{R}_{k-1} x \widetilde{R}_{k-1} + R_k x R_k \\ &\leq 2\Big(\widetilde{R}_{k-1} x \widetilde{R}_{k-1} + R_k x R_k\Big).\end{aligned}$$

Furthermore, according to (6.3),

$$\begin{aligned}\mathcal{E}_k(x) &= \widetilde{R}_k x \widetilde{R}_k + \sum_{n\geq k+1} R_n x R_n \\ &\leq 2\Big(\widetilde{R}_{k-1} x \widetilde{R}_{k-1} + R_k x R_k + \sum_{n\geq k+1} R_n x R_n\Big) = 2\mathcal{E}_{k-1}(x).\end{aligned}$$

This verifies the assertion. □

Recall that the martingale difference is defined by $d_n := \mathcal{E}_n - \mathcal{E}_{n-1}$, $n \geq 2$. The below result is also included in [61].

**Lemma 6.3.** *Let $x \in L_2(\mathcal{M})$ and consider the martingale $x = (\mathcal{E}_n(x))_{n=1}^N$ with respect to the shell filtration $(\mathcal{M}_k)_{k=1}^N$. We have $\mathcal{E}_1(\mathcal{T}^c(x)) = \mathcal{E}_1(x)$,*

$$d_n(\mathcal{T}^c(x)) = \widetilde{R}_{n-1} d_n(x), \quad n \geq 2, \tag{6.4}$$

*and consequently,*

$$\mathcal{T}^c(x) = \mathcal{E}_1(x) + \sum_{k=2}^N \widetilde{R}_{k-1} d_k(x).$$

Note that $\widetilde{R}_{k-1} \in \mathcal{M}_{k-1}$ for each $k$. Let $\mathcal{T}_n^c(x) = \mathcal{E}_1(x) + \sum_{k=2}^n \widetilde{R}_{k-1} d_k(x)$, $1 \leq n \leq N$. Then $\mathcal{T}^c(x) = (\mathcal{T}_n^c(x))_{n=1}^N$ is also a martingale, and thus $(\mathcal{T}_n^c(x))_{n=1}^N$ is a martingale transform of $x = (\mathcal{E}_n(x))_{n=1}^N$ (see for instance [62], [55] for related definition). Moreover, since $\widetilde{R}_{k-1} y = y \widetilde{R}_{k-1}$ for any $y \in \mathcal{M}_{k-1}$, it follows from (6.4) that for each projection $Q \in \mathcal{M}_{k-1}$,

$$\|Q d_n(\mathcal{T}_n^c(x)) Q\|_{L_2(\mathcal{M})} = \|Q \widetilde{R}_{n-1} d_n(x) Q\|_{L_2(\mathcal{M})} = \|\widetilde{R}_{n-1} Q d_n(x) Q\|_{L_2(\mathcal{M})} \leq \|Q d_n(x) Q\|_{L_2(\mathcal{M})}.$$

Thus, the martingale $\mathcal{T}^c(x) = (\mathcal{T}_n^c(x))_{n=1}^N$ is $L_2$-dominated by $x = (\mathcal{E}_n(x))_{n=1}^N$ in the sense of Definition 1.8, and hence, Theorem 6.1 can be deduced from Corollary 4.5.

6.2. **Partial sums of noncommutative Vilenkin-Fourier series.** The study of noncommutative Vilenkin-Fourier series goes back to [2, 18, 19]. These are noncommutative analogue of the classical results due to R. Paley, W.S. Young, F. Schipp, P. Simon and J. Gosselin (one can find related references in last three mentioned papers). Recently, weak type estimates for partial sums of noncommutative bounded Vilenkin-Fourier series were investigated in [37, 68, 72]. We below shall show that these weak type estimates can be deduced from our Theorem 1.10 or its consequence Corollary 4.5.

To simply the discussion, let us put the argument in the operator-valued setting. We firstly recall the Vilenkin system defined on the interval $[0,1)$ (equipped with the usual Lebesgue measure $m$). Let $\mathbf{m}=(m_k)_{k\geq 0}\subset\mathbb{N}$ be a sequence of natural numbers with entries at least 2, and let $M_0=1$ and

$$M_{n+1}:=\prod_{k=0}^{n} m_k, \qquad n\in\mathbb{N}. \tag{6.5}$$

In this subsection, we assume that $\sup_k m_k<\infty$. The generalized Rademacher functions on $[0,1)$ are defined by

$$r_n(t):=\exp\frac{2\pi i t_n}{m_n}, \qquad \forall t\in[0,1), \qquad n\in\mathbb{N}.$$

The product system generated by the Rademacher functions is the Vilenkin system (see e.g. [57]):

$$\psi_n:=\prod_{k=0}^{\infty} {r_k}^{n_k}, \qquad n\in\mathbb{N},$$

where $n=\sum_{k=0}^{\infty} n_k M_k$, $0\leq n_k<m_k$, $n_k\in\mathbb{N}$. For the natural numbers $n=\sum_{k=0}^{\infty} n_k M_k$ and $l=\sum_{k=0}^{\infty} l_k M_k$, we define $n\dot{+}l:=\sum_{k=0}^{\infty}(n_k\oplus l_k)M_k$. Moreover, we have $\psi_n\psi_l=\psi_{n\dot{+}l}$.

Let $(\mathcal{N},\varphi)$ be a finite von Neumann algebra (for convenience, assume $\varphi(1_{\mathcal{N}})=1$). In what follows, consider $\mathcal{M}=L_\infty(0,1)\bar{\otimes}\mathcal{N}$ equipped with the trace $\tau=\int\otimes\varphi$. For $f\in L_1(\mathcal{M})$, the $n$-th Vilenkin-Fourier coefficient of $f$ is defined by

$$\widehat{f}(n):=\int_0^1 f(t)\overline{\psi}_n(t)dt, \qquad n\in\mathbb{N}.$$

Denote by $S_n(f)$ the $n$-th partial sum of the Vilenkin-Fourier series of $f\in L_1(\mathcal{N})$, namely,

$$S_n(f):=\sum_{k=0}^{n-1}\widehat{f}(k)\psi_k. \tag{6.6}$$

To apply the result in the present paper, we introduce the following regular filtration. Consider $([0,1),\mathcal{F},m)$, where $m$ is the Lebesgue measure on $[0,1)$ and $\mathcal{F}$ is the $\sigma$-algebra generated by all open sets in $[0,1)$. We simply write $L_\infty([0,1),\mathcal{F},m)$ by $L_\infty(0,1)$. Define

$$F_n=\sigma\{[\frac{k}{M_n},\frac{k+1}{M_n}):0\leq k<M_n\}, \quad \mathcal{F}_n=\sigma(F_n), \quad n\geq 0, \tag{6.7}$$

where $M_k$ is defined in the begin of the subsection. Then $(\mathcal{F}_n)_{n\geq 1}$ is a regular filtraion of $\mathcal{F}$. The conditional expectations with respect to $(\mathcal{F}_n)_{n\geq 1}$ are denoted by $(\mathbb{E}_n)_{n\geq 1}$. In fact, for $f\in L_1(0,1)$,

$$\mathbb{E}_n(f)=\sum_{Q\in F_n}\frac{1}{m(Q)}\int_Q f(x)dm(x)\chi_Q, \quad n\geq 1. \tag{6.8}$$

For each $n$, we simply write $L_\infty([0,1),\mathcal{F}_n,m)$ by $L_\infty(\mathcal{F}_n)$, and let

$$\mathcal{M}_n=L_\infty(\mathcal{F}_n)\bar{\otimes}\mathcal{N}. \tag{6.9}$$

Then $(\mathcal{N}_n)_{n\geq 1}$, an increasing sequence of von Neumann subalgebras of $\mathcal{M}$, is regular with regularity constant $c_{\text{reg}}\leq\sup_k m_k<\infty$; see e.g. [72]. The conditional expectation of $\mathcal{M}$ onto $\mathcal{M}_n$ is $\mathbb{E}_n\otimes\mathbb{I}_{\mathcal{N}}$ (simply $\mathbb{E}_n$), where $\mathbb{I}_{\mathcal{N}}$ is the identity map from $\mathcal{N}$ to $\mathcal{N}$.

For each $f \in L_1(\mathcal{M})$, it is known that (see e.g. [77, p. 312] or [57, Theorem 2.3.1])

$$\mathbb{E}_n(f) = S_{M_n}(f) = \sum_{k=0}^{M_n-1} \widehat{f}(k)\psi_k, \quad \forall n \geq 1.$$

Thus, for each $k \geq 1$,

$$d_k(f) = \sum_{n=M_{k-1}}^{M_k-1} \widehat{f}(n)\psi_n.$$

For each $k \in \mathbb{N}$, define

$$\Delta_{k,l}(f) = \sum_{n=lM_{k-1}}^{(l+1)M_{k-1}-1} \widehat{f}_n\psi_n, \quad 1 \leq l \leq m_{k-1} - 1.$$

It is obvious that for each pair $(k, l)$,

$$\Delta_{k,l}(f) = \Delta_{k,l}(d_k f). \tag{6.10}$$

For $\mathbf{j} = (j_0, j_1 \cdots, j_{N-1})$ with $0 \leq j_k \leq m_k - 1$ for each $0 \leq k \leq N-1$, define

$$P_{\mathbf{j}}(f) = \sum_{k=0}^{N-1} P_k^{j_k}(f) = \sum_{k=0}^{N-1}\sum_{l=1}^{j_k} \Delta_{k,l}(f).$$

By the Parseval's identity $\|g\|_{L_2(\mathcal{M})} = \sum_{k\geq 1} \tau(|\widehat{g}(k)|^2)$ for any $g \in L_2(\mathcal{M})$, it is obvious that

$$\|\Delta_{k,l}(f)\|_{L_2(\mathcal{M})} \leq \|f\|_{L_2(\mathcal{M})}$$

and

$$\|P_k^{j_k}(f)\|_{L_2(\mathcal{M})} \leq \|f\|_{L_2(\mathcal{M})}.$$

**Remark 6.4.** *It was proved in* [73, Lemma 2] *that if $f\chi_E = f$ with $E \in \mathcal{F}_{k-1}$, then*

$$\Delta_{k,l}(f) = \Delta_{k,l}(f)\chi_E.$$

**Lemma 6.5.** *Fix $k \geq 1$ and $1 \leq l \leq m_{k-1} - 1$. For each $f \in L_2(\mathcal{M})$ and $I \in F_{k-1} \subset \mathcal{F}_{k-1}$, we have*

$$\Delta_{k,l}(f)\chi_I = \Delta_{k,l}\Big(d_k(f)\chi_I\Big)\chi_I,$$

*and for each projection $p \in \mathcal{N}$, we further have*

$$p\Delta_{k,l}(f)p\chi_I = p\Delta_{k,l}\Big(pd_k(f)p\chi_I\Big)p\chi_I.$$

*Proof.* By the definition, it is clear that $\Delta_{k,l}(f) = \Delta_{k,l}(d_k(f))$. According to Remark 6.4, we have

$$\begin{aligned}\Delta_{k,l}(f) &= \Delta_{k,l}(d_k(f)) = \Delta_{k,l}(d_k(f)\chi_I) + \Delta_{k,l}(d_k(f)\chi_{I^c})\\ &= \Delta_{k,l}(d_k(f)\chi_I)\chi_I + \Delta_{k,l}(d_k(f)\chi_{I^c})\chi_{I^c}.\end{aligned}$$

Thus

$$\Delta_{k,l}(f)\chi_I = \Delta_{k,l}\Big(d_k(f)\chi_I\Big)\chi_I.$$

The second assertion follows from the first one. □

**Lemma 6.6.** *Let $f \in L_2(\mathcal{M})$. For $\mathbf{j} = (j_0, j_1 \cdots, j_{N-1})$ with $0 \leq j_k \leq m_k - 1$ for each $0 \leq k \leq N-1$, consider the martingale generated by $P_{\mathbf{j}}f$, i.e., $P_{\mathbf{j}}f = (\mathcal{E}_n(P_{\mathbf{j}}f))_{n\geq 1}$. For each $k \geq 1$ and each projection $Q \in \mathcal{M}_{k-1}$, we have*

$$\|Qd_k(P_{\mathbf{j}}f)Q\|_{L_2(\mathcal{M})} \leq \|Qd_k(f)Q\|_{L_2(\mathcal{M})}.$$

*Proof.* Fix $k \geq 1$, and take projection $Q \in \mathcal{M}_{k-1}$. We first write

$$d_k(P_{\mathbf{j}}f) = \sum_{l=1}^{j_k} \Delta_{k,l}(f) = \sum_{l=1}^{j_k} \Delta_{k,l}(d_k f).$$

Since the projection $Q \in \mathcal{M}_{k-1} = L_\infty(\mathcal{F}_n)\bar{\otimes}\mathcal{N}$, we may write

$$Q = \sum_{I \in F_{k-1}} q_I \chi_I,$$

where each $q_I$ is a projection in $\mathcal{N}$. Therefore, by Lemma 6.5, we get

$$\begin{aligned} Q d_k(P_{\mathbf{j}}f)Q &= \sum_{I \in F_{k-1}} \sum_{l=1}^{j_k} q_I \Delta_{k,l}(f) q_I \chi_I \\ &= \sum_{I \in F_{k-1}} \sum_{l=1}^{j_k} q_I \Delta_{k,l}\Big(q_I d_k(f) q_I \chi_I\Big) q_I \chi_I \\ &= \sum_{I \in F_{k-1}} \sum_{l=1}^{j_k} q_I \Delta_{k,l}\Big(Q d_k(f) Q\Big) q_I \chi_I \\ &= \sum_{I \in F_{k-1}} q_I P_k^{j_k}\Big(Q d_k(f) Q\Big) q_I \chi_I = Q P_k^{j_k}\Big(Q d_k(f) Q\Big) Q. \end{aligned}$$

Thus, we have

$$\|Q d_k(P_{\mathbf{j}}f)Q\|_{L_2(\mathcal{M})} = \|Q P_k^{j_k}\Big(Q d_k(f) Q\Big) Q\|_{L_2(\mathcal{M})} \leq \|P_k^{j_k}\Big(Q d_k(f) Q\Big)\|_{L_2(\mathcal{M})} \leq \|Q d_k(f) Q\|_{L_2(\mathcal{M})}.$$

The proof is complete. □

Therefore, by the last lemma, the martingale $(\mathcal{E}_n(P_{\mathbf{j}}f))_{n\geq 1}$ is $L_2$-dominated by $f = (\mathcal{E}_n(f))_{n\geq 1}$ in the sense of Definition 1.8. Also note that the related filtration is regular. Hence, we can deduce from Corollary 4.5 that

**Theorem 6.7.** *Consider $\mathcal{M} = L_\infty(0,1)\bar{\otimes}\mathcal{N}$. Let $\mathbf{j} = (j_0, j_1 \cdots, j_{N-1})$ be arbitrary. The following inequalities hold:*

(i) *There exists a universal constant $c > 0$ such that*

$$\|P_{\mathbf{j}}(f)\|_{L_{1,\infty}(\mathcal{M})} \leq c\|f\|_{L_1(\mathcal{M})}.$$

(ii) *For $1 < p < \infty$, there exists a constant $c$ such that*

$$\|P_{\mathbf{j}}(f)\|_{L_p(\mathcal{M})} \leq \frac{cp^2}{p-1}\|f\|_{L_p(\mathcal{M})}.$$

(iii) *There exist universal constants $c_1, c_2 > 0$ such that*

$$\|P_{\mathbf{j}}(f)\|_{L_1(\mathcal{M})} \leq c_1 + c_2\|f\|_{L\log L(\mathcal{M})}.$$

Consider $n = \sum_{k=0}^\infty n_k M_k$ with $0 \leq n_k \leq m_k - 1$. Let $\mathbf{j} = (j_0, j_1 \cdots, j_{N-1})$ be defined by setting $j_k = m_k - n_k - 1$, $0 \leq k \leq N-1$. The following equality can be found in [18, p. 422]:

$$S_n(f) = \psi_n(\mathbb{I} - P_{\mathbf{j}} - \mathbb{E})(\bar{\psi}_n f),$$

where $\mathbb{I}$ and $\mathbb{E}$ denote the identity map and the integral over interval $[0,1)$, respectively. Note that $|\psi_n| = 1$ for each $n$. The following result, which goes back [72, Theorem 1.1] and [68, Theorem 3.8], can be deduced from the above equality and last theorem.

**Theorem 6.8.** *Consider $\mathcal{M} = L_\infty(0,1)\bar{\otimes}\mathcal{N}$. Let $n \in \mathbb{N}$ be arbitrary. The following inequalities hold:*

(i) *There exists a universal constant $c > 0$ such that*

$$\|S_n(f)\|_{L_{1,\infty}(\mathcal{M})} \leq c\|f\|_{L_1(\mathcal{M})}.$$

(ii) *For $1 < p < \infty$, there exists a constant $c$ such that*

$$\|S_n(f)\|_{L_p(\mathcal{M})} \leq \frac{cp^2}{p-1}\|f\|_{L_p(\mathcal{M})}.$$

(iii) *There exist universal constants $c_1, c_2 > 0$ such that*

$$\|S_n(f)\|_{L_1(\mathcal{M})} \leq c_1 + c_2\|f\|_{L\log L(\mathcal{M})}.$$

6.3. **One-side maximal function characterization of Mei's operator-valued Hardy spaces.** In last decades, operator-valued harmonic analysis has gained rapid developments. We refer the reader to [50, 76] for operator-valued Hardy spaces; to [6, 54, 28] for noncommutative CZ theory and so on. It is also known that operator-valued harmonic analysis often provides deep insights in harmonic analysis in the general noncommutative setting, and sometimes plays essential role based on the so-called transference principles (see e.g. [10, 29, 76] and the references therein).

We shall apply Theorem 1.5 to get a one-side maximal function characterization of operator-valued Hardy spaces introduced by Mei [50]. We first recall related definitions. In this subsection, we always consider the Euclidean space $\mathbb{R}$ equipped the usual Lebesgue measure $m$. Let $(\mathcal{N}, \varphi)$ be a semifinite von Neumann algebra equipped with a normal semifinite faithful trace $\varphi$ and let $S_{\mathcal{N}}$ be the set of all positive $x$ in $\mathcal{N}$ such that $\varphi(\mathrm{supp}(x)) < \infty$. By a $S_{\mathcal{N}}$-valued simple function $f$ on $(\mathbb{R}, m)$ we mean $f$ has the following form:

$$f = \sum_{i=1}^n a_i \chi_{A_i},$$

where $a_i \in S_{\mathcal{N}}$ and $A_i$'s are measurable disjoint subsets of $\mathbb{R}$ with $m(A_i) < \infty$. For a $S_{\mathcal{N}}$-valued simple function $f$, its Poisson integral on the upper half plane $\mathbb{R}^2_+ = \{(s,\varepsilon) | \varepsilon > 0\}$ is defined by:

$$P_\varepsilon(f)(s) = \int_{\mathbb{R}} P_\varepsilon(s-t) f(t) dt, \quad (s,\varepsilon) \in \mathbb{R}^2_+,$$

where $P_\varepsilon(s) = \frac{1}{\pi}\frac{\varepsilon}{s^2+\varepsilon^2}$ is the Poisson kernel. Then $P_\varepsilon(f)$ is a harmonic function with values in $S_{\mathcal{N}}$, and so in $\mathcal{N}$. Following [50, p. 7-8], for each $S_{\mathcal{N}}$-valued simple function $f$ on $\mathbb{R}$, define the noncommutative analogues of the classical Littlewood-Paley $g$-function

$$G_c(f)(s) = \Big(\int_0^\infty \varepsilon |\frac{\partial}{\partial \varepsilon} P_\varepsilon(f)(s)|^2 d\varepsilon\Big)^{1/2}, \quad s \in \mathbb{R}.$$

and

$$\|f\|_{\mathcal{H}_1^c(\mathbb{R},\mathcal{N})} = \|G_c(f)\|_{L_1(\mathcal{M})}, \quad \mathcal{M} := L_\infty(\mathbb{R})\bar{\otimes}\mathcal{N}.$$

Set the column Hardy space $\mathcal{H}_1^c(\mathbb{R},\mathcal{N})$ to be the completion of the space of all $S_{\mathcal{N}}$-valued simple function with finite $\mathcal{H}_1^c(\mathbb{R},\mathcal{N})$ norm. The row Hardy space $\mathcal{H}_1^r(\mathbb{R},\mathcal{N})$ is the space of all $f$ such that $f^* \in \mathcal{H}_1^c(\mathbb{R},\mathcal{N})$ equipped with the norm

$$\|f\|_{\mathcal{H}_1^r(\mathbb{R},\mathcal{N})} = \|f^*\|_{\mathcal{H}_1^c(\mathbb{R},\mathcal{N})}.$$

Define the Hardy space $\mathcal{H}_1(\mathbb{R},\mathcal{N})$ as follows:

$$\mathcal{H}_1(\mathbb{R},\mathcal{N}) = \mathcal{H}_1^c(\mathbb{R},\mathcal{N}) + \mathcal{H}_1^r(\mathbb{R},\mathcal{N})$$

equipped with the norm

$$\|f\|_{\mathcal{H}_1(\mathbb{R},\mathcal{N})} = \inf_{f=g+h}\{\|g\|_{\mathcal{H}_1^c(\mathbb{R},\mathcal{N})} + \|h\|_{\mathcal{H}_1^r(\mathbb{R},\mathcal{N})}.$$

For each $S_{\mathcal{N}}$-valued simple function $f$ on $\mathbb{R}$ and $h = (h_1, h_2) \in \mathbb{R}_+ \times \mathbb{R}_+$, define

$$A_h(f)(t) = \frac{1}{h_1+h_2}\int_{t-h_1}^{t+h_2} f(s) ds.$$

In the commutative case, i.e., $\mathcal{N} = \mathbb{C}$, it is well-known that the Hardy space has two-side maximal function characterization, that is, there exists universal constant $c > 0$ such that

$$\frac{1}{c}\|\sup_h |A_h(f)|\|_{L_1(\mathbb{R})} \leq \|f\|_{\mathcal{H}_1(\mathbb{R})} \leq c\|\sup_h |A_h(f)|\|_{L_1(\mathbb{R})}.$$

According to [43, Lemma 13], the operator-valued version of the left hand side inequality no longer holds. However, we can apply Theorem 1.5 to get the operator-valued analogue of the above right hand side inequality.

To apply Theorem 1.5, we also need to introduce noncommutative dyadic martingale Hardy space. For each $n \in \mathbb{Z}$, let $\mathcal{D}_n$ be the $\sigma$-algebra generated by $D_n = \left\{D_n^k : k \in \mathbb{Z}\right\}$, where the dyadic intervals $D_n^k$ is defined as

$$D_n^k = (k2^{-n}, (k+1)2^{-n}],$$

and let

$$\mathcal{M}_n = L_\infty(\mathcal{D}_n)\bar{\otimes}\mathcal{N}, \quad n \in \mathbb{Z}.$$

Denote by $\mathbb{E}_n$ the conditional expectation $L_\infty(\mathbb{R}) \to L_\infty(\mathcal{D}_n)$:

$$\mathbb{E}_n(f) = \sum_{I \in D_n} \frac{1}{m(I)} \int_I f(t)dt\chi_I.$$

Then each $\mathbb{E}_n \otimes \mathrm{id}$ is the conditional expectation from $\mathcal{M} = L_\infty(\mathbb{R})\bar{\otimes}\mathcal{N}$ onto $\mathcal{M}_n = L_\infty(\mathcal{D}_n)\bar{\otimes}\mathcal{N}$, where id is the identity map on $\mathcal{N}$. The noncommutative dyadic martingale Hardy space with respect to the filtration $(\mathcal{M}_n)_{n\in\mathbb{Z}}$ is denoted by $H_1(\mathcal{M}, \mathcal{D})$. The following fact is well-known; see e.g. [54, p. 561] for its proof.

**Lemma 6.9.** *The filtration $(\mathcal{M}_n)_{n\in\mathbb{Z}}$ is regular with constant 2, where $\mathcal{M}_n = L_\infty(\mathcal{D}_n)\bar{\otimes}\mathcal{N}$, $n \in \mathbb{Z}$.*

The lemma below follows from [50, Corollary 5.3].

**Lemma 6.10.** *For each $f \in H_1(\mathcal{M}, \mathcal{D})$, there exists a universal constant $c > 0$ such that*

$$\|f\|_{\mathcal{H}_1(\mathbb{R},\mathcal{N})} \le c\|f\|_{H_1(\mathcal{M},\mathcal{D})}.$$

Now we are ready to provide our one-side maximal function characterization of the operator-valued Hardy space $\mathcal{H}_1(\mathbb{R}, \mathcal{N})$.

**Theorem 6.11.** *For each $f \in L_1(\mathcal{M})$, there exists a universal constant $c > 0$ such that*

$$\|f\|_{\mathcal{H}_1(\mathbb{R},\mathcal{N})} \le c\|(A_h(f))_{h\in\mathbb{R}_+\times\mathbb{R}_+}\|_{L_1(\mathcal{M},\ell_\infty)}.$$

*Proof.* Firstly, it is clear that

$$\|(\mathbb{E}_n(f))_n\|_{L_1(\mathcal{M},\ell_\infty)} \le c\|(A_h(f))_{h\in\mathbb{R}_+\times\mathbb{R}_+}\|_{L_1(\mathcal{M},\ell_\infty)}.$$

Therefore, we have

$$\|f\|_{\mathcal{H}_1(\mathbb{R},\mathcal{N})} \le c\|f\|_{H_1(\mathcal{M},\mathcal{D})} \le c\|(\mathbb{E}_n(f))_n\|_{L_1(\mathcal{M},\ell_\infty)} \le c\|(A_h(f))_{h\in\mathbb{R}_+\times\mathbb{R}_+}\|_{L_1(\mathcal{M},\ell_\infty)},$$

where the first inequality is due to Lemma 6.10, and the second inequality is due to Theorem 1.5 (see also Remark 1.6(iv)). We can apply Theorem 1.5 is because of the related filtration is regular as explained above. □

**Remark 6.12.** *The same idea applies to operator-valued Hardy space defined on a homogeneous metric measure space; see for instance [21].*

Our results can be also applied other models in operator-valued harmonic analysis. See below for an example.

**Remark 6.13.** *For each $I \in D_n$ for some $n \in \mathbb{Z}$, there are $I_L, I_R \in D_{n+1}$ such that $I_L \cup I_R = I$, where $I_L, I_R$ are referred to the left and right parts of $I$, respectively. The Haar function associated with the interval is defined by setting*

$$h_I = m(I)^{-\frac{1}{2}}\chi_{I_L} - m(I)^{\frac{1}{2}}\chi_{I_R}.$$

*Then, for each $f \in L_2(\mathcal{M})$, it is not hard to see that $(T_N(f))_{N\in\mathbb{Z}}$ forms a martingale which is $L_2$-dominated by $f = (\mathcal{E}_N(f))_{N\in\mathbb{Z}}$ in the sense of Definition 1.8 (also Definition 1.7 (i)), where the dyadic Haar multiplier $T_N(f)$ in operator-valued setting is defined by setting*

$$T_N(f) = \sum_{n\le N} \sum_{I\in D_n} \epsilon_I \langle f, h_I\rangle h_I.$$

*Hence, our Theorem 1.10 and Theorem 1.11 apply to the Haar multiplier. Our results also apply to Haar shift operators studied in* [74] *since these kind of operators fulfill Definition 1.7 (ii) (to check this, one needs to borrow some argument from* [1, Section 3]*).*

6.4. **Carleson's decomposition of noncommutative martingale BMO spaces.** We here study the Carleson decomposition of noncommutative martingale BMO spaces via Theorem 1.5. In the classical setting, Carleson [7] firstly provided a decomposition for functions in $BMO(\mathbb{R})$, and its dyadic version was given by Garnett and Jones [22, Theorem 2.1]. To better state the their result, let us first recall related notations. The results (Theorems 6.15 and 6.16) below actually work for semifinite von Neumann algebras, and to simply the discussion, let us only consider the finite case. Following the symbols introduced in Section 2.3, let $(\mathcal{M},\tau)$ be a noncommutative probability space and let $(\mathcal{M}_n)_{n\geq 1}$ be a sequence of increasing von Neumann sub-algebras of $\mathcal{M}$ satisfying $\cup_n\mathcal{M}_n$ is weak$^*$ dense in $\mathcal{M}$. Assume that for each $n\geq 1$ there exists conditional expectation $\mathcal{E}_n:\mathcal{M}\to\mathcal{M}_n$. The column noncommutative martingale BMO space is defined by (let $\mathcal{E}_0=0$ for convenience):

$$\mathcal{BMO}^c(\mathcal{M})=\{x\in L_2(\mathcal{M}):\sup_n\|(\mathcal{E}_n|x-\mathcal{E}_{n-1}(x)|^2)_{n\geq 1}\|_\infty<\infty\}$$

equipped with the norm

$$\|x\|_{\mathcal{BMO}^c(\mathcal{M})}=\sup_{n\geq 1}\|(\mathcal{E}_n|x-\mathcal{E}_{n-1}(x)|^2)_{n\geq 1}\|^{\frac12}_{L_{\frac{q}{2}}(\mathcal{M},\ell_\infty)},$$

and similarly, we define the row BMO space as

$$\mathcal{BMO}^r(\mathcal{M})=\{x:x^*\in\mathcal{BMO}^c(\mathcal{M})\}$$

equipped with the norm $\|x\|_{\mathcal{BMO}^r(\mathcal{M})}=\|x^*\|_{\mathcal{BMO}^c(\mathcal{M})}$. Finally, define

$$\mathcal{BMO}(\mathcal{M})=\mathcal{BMO}^c(\mathcal{M})\cap\mathcal{BMO}^r(\mathcal{M})$$

equipped with the norm

$$\|x\|_{\mathcal{BMO}(\mathcal{M})}=\max\{\|x\|_{\mathcal{BMO}^c(\mathcal{M})},\|x\|_{\mathcal{BMO}^r(\mathcal{M})}\}.$$

For the case $\mathcal{M}=L_\infty(0,1)$, and $\mathcal{M}_n=L_\infty(\mathcal{F}_n)$ ($\mathcal{F}_n$ is defined as in (6.7) with $M_n=2^n$ there), we write $\mathcal{BMO}(0,1)$ instead of $\mathcal{BMO}(L_\infty(0,1))$. Garnett and Jones' famous dyadic Carleson's decomposition is read as follows

**Theorem 6.14** ([22, Theorem 2.1])**.** *For each $f\in\mathcal{BMO}(0,1)$, there exist $g\in L_\infty(0,1)$, a sequence $(I_k)_k$ of dyadic intervals and a sequence of real numbers $(a_k)_k$ such that*

$$f-\mathbb{E}(f)=g+\sum_k a_k\chi_{I_k}$$

*with $\|g\|_\infty\leq\|f\|_{\mathcal{BMO}(0,1)}$ and for each dyadic $I$, there exists a universal constant $c>0$*

$$\frac{1}{m(I)}\sum_{I_k\subset I}|a_k|m(I_k)\leq c\|f\|_{\mathcal{BMO}(0,1)}.$$

In the sequel, we shall show a noncommutative version of this result. Under the assumption that $\|x\|_{H_1(\mathcal{M})}\leq\|x\|_{L_1(\mathcal{M},\ell_\infty)}$ for each martingale $x=(x_n)_{n\geq1}$, it was recently established by Talebi et al. [71, Theorem 2.3] that for each positive $f\in\mathcal{BMO}(\mathcal{M})$, there exist positive $v\in L_\infty(\mathcal{M})$ and $(u_k)_{k\geq1}\subset L_\infty(\mathcal{M})$ satisfying (in the sense of the strong operator topology)

$$f=v+\sum_{k\geq1}\mathcal{E}_k(u_k)$$

and

$$\|u\|_\infty+\|\sum_{k\geq1}u_k\|_\infty\leq c\|f\|_{\mathcal{BMO}(\mathcal{M})}.$$

Combining the above decomposition result and our Theorem 1.5, we deduce the following decomposition result for positive elements in $\mathcal{BMO}(\mathcal{M})$.

**Theorem 6.15.** *Consider noncommutative probability space associated with a regular filtration $(\mathcal{M},\tau)$. For each positive $f\in\mathcal{BMO}(\mathcal{M})$, there exist positive $g\in L_\infty(\mathcal{M})$ and positive $(u_k)_{k\geq1}\subset L_\infty(\mathcal{M})$ satisfying (in the sense of the strong operator topology)*

$$f=g+\sum_{k\geq1}\mathcal{E}_k(u_k)$$

*and*

$$\|g\|_\infty+\|\sum_{k\geq1}u_k\|_\infty\leq c\|f\|_{\mathcal{BMO}(\mathcal{M})}.$$

For last theorem, we refer the reader to [24, Theorem II.4.1] for its classical martingale version. This kind of decomposition result has its own interest and applications. For example, it can be used to study the distance between martingale $BMO$ space and $L_\infty$ space, which goes back to [23]. See [49, Chapter 4.6] for more information.

We here apply Theorem 6.15 to deduce a noncommutative version of Theorem 6.14. The dyadic sigma-algebra $\mathcal{F}_n$ is defined as in (6.7) with $M_n=2^n$ there.

**Theorem 6.16.** *Let $\mathcal{M}=L_\infty(0,1)\bar{\otimes}\mathcal{N}$ where $(\mathcal{N},\varphi)$ is a noncommutative probability space. Consider the regular filtration $\mathcal{M}_n=L_\infty(\mathcal{F}_n)\bar{\otimes}\mathcal{N}$ as in (6.9). For each positive $f\in\mathcal{BMO}(\mathcal{M})$, there exist $g\in L_\infty(\mathcal{M})$, a sequence $(I_k)_k$ of dyadic intervals and a sequence of positive elements $(a_k)_k\subset\mathcal{N}$ such that*

$$f=g+\sum_i a_i\chi_{I_i}$$

*with $\|g\|_\infty\leq\|f\|_{\mathcal{BMO}(\mathcal{M})}$, and for each dyadic $I_0$, there exists a universal constant $c>0$*

$$\Big\|\frac{1}{m(I_0)}\sum_{i:I_i\subset I_0}a_km(I_i)\Big\|_{\mathcal{N}}\leq c\|f\|_{\mathcal{BMO}(\mathcal{M})}.$$

*Proof.* By Theorem 6.15, there exist positive $g\in L_\infty(\mathcal{M})$ and positive $(u_k)_{k\geq1}\subset L_\infty(\mathcal{M})$ satisfying (in the sense of the strong operator topology)

$$f=g+\sum_{k\geq1}\mathcal{E}_k(u_k)$$

and

$$\|g\|_\infty+\|\sum_k u_k\|_\infty\leq c\|f\|_{\mathcal{BMO}(\mathcal{M})}.$$

Note that the conditional expectation now is given by $\mathcal{E}_n=\mathbb{E}_n\otimes 1_{\mathcal{N}}$, where $\mathbb{E}_n$ is as in (6.8). Hence,

$$\sum_{k\geq1}\mathcal{E}_k(u_k)=\sum_{k\geq1}\sum_{I\in F_k}\frac{1}{m(I)}\int_I u_k(t)dt\chi_I.$$

For each $k$ and $I\in F_k$, set $b_{k,I}=\frac{1}{m(I)}\int_I u_k(t)dt\in\mathcal{M}$. Therefore, we can rewrite

$$\sum_{k\geq1}\mathcal{E}_k(u_k)=\sum_k\sum_{I\in F_k}b_{k,I}\chi_I=\sum_i a_i\chi_{I_i}.$$

On the other hand, concerning dyadic $I_0\in F_n$ for some $n\in\mathbb{N}$, we have

$$\begin{aligned}\sum_{i:I_i\subset I_0}a_im(I_i)&=\sum_{k\geq n+1}\sum_{I\in F_k,I\subset I_0}a_im(I_i)\\&=\sum_{k\geq n+1}\sum_{I\in F_k,I\subset I_0}\int_I u_k(t)dt=\sum_{k\geq n+1}\int_{I_0}u_k(t)dt\\&\leq\int_{I_0}\sum_{k\geq1}u_k(t)dt.\end{aligned}$$

Thus, we have

$$\Big\|\frac{1}{m(I_0)}\sum_{i:I_i\subset I_0} a_k m(I_i)\Big\|_{\mathcal{N}} \le \frac{1}{m(I_0)}\int_{I_0}\|\sum_{k\ge1}u_k\|_\infty dt \le c\|f\|_{\mathcal{BMO}(\mathcal{M})}.$$

The proof is complete. ☐

**Remark 6.17.** *A few comments are listed as below.*

(i) *Recall that noncommutative version of $BMO(\mathbb{R})$ was also introduced by Mei* [50, Chapter 1.3]*: for a function $f:\mathbb{R}\to\mathcal{N}$, define*

$$\|f\|_{BMO^c(\mathbb{R},\mathcal{N})} = \sup_I \Big\|\Big(\frac{1}{m(I)}\int_I |f-f_I|^2 dm\Big)^{1/2}\Big\|_{\mathcal{N}},$$

*where $I$ denotes the interval of $\mathbb{R}$, and $f_I = \frac{1}{m(I)}\int_I f dm$. Similarly, define $\|f\|_{BMO^r(\mathbb{R},\mathcal{N})} = \|f^*\|_{BMO^c(\mathbb{R},\mathcal{N})}$, and*

$$\|f\|_{BMO(\mathbb{R},\mathcal{N})} = \max\{\|f\|_{BMO^c(\mathbb{R},\mathcal{N})}, \|f\|_{BMO^r(\mathbb{R},\mathcal{N})}\}.$$

*It is possible to get a decomposition*

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