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# Non-Commutative Martingale Inequalities

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**Abstract:** We prove the analogue of the classical Burkholder-Gundy inequalites for non-commutative martingales. As applications we give a characterization for an Ito-Clifford integral to be an $L^p$-martingale via its integrand, and then extend the Ito-Clifford integral theory in $L^2$, developed by Barnett, Streater and Wilde, to $L^p$ for all $1 < p < \infty$. We include an appendix on the non-commutative analogue of the classical Fefferman duality between $H^1$ and $BMO$.

## 0. Introduction

Recently, non-commutative (=quantum) probability theory has developed considerably. In particular, all sorts of non-commutative analogues of Brownian motion and martingales have been studied following the basic work of Parthasarathy and Schmidt. We refer the reader to P. A. Meyer's exposition ([M]) and to the proceedings of the successive conferences on quantum probability [AvW] for more details and references. There are also intimate connections with Harmonic Analysis (cf. e.g. [Mi]).

Motivated by quantum physics, and after the pioneer works of Gross (cf. [Gr1-2]), a Fermionic version of Brownian motion and stochastic integrals was developed (see [BSW1]), and the optimal hypercontractive inequalities have been finally proved ([CL]).

In this paper we will prove the non-commutative analogue of the classical Burkholder-Gundy inequalities from martingale theory. We should point out that what follows was originally inspired by some recent work of Carlen and Krée, who had considered Fermionic versions of the Burkholder-Gundy inequalities. They obtained the inequality in Theorem 4.1 below in some special cases, as well as some sufficient conditions for the convergence of stochastic integrals in the case $p \le 2$ (see Sect. 4 below for more on this).

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One interesting feature of our work, is that the square function is defined differently (and it must be changed!) according to $p < 2$ or $p > 2$. This surprising phenomenon was already discovered by F. Lust-Piquard in [LP] (see also [LPP]) while establishing non-commutative versions of Khintchine's inequalities.

Let us briefly describe our main inequality. Let $\mathcal{M}$ be a finite von Neumann algebra with a normalized normal faithful trace $\tau$, and $(\mathcal{M}_n)_{\geq 0}$ be an increasing filtration of von Neumann subalgebras of $\mathcal{M}$. Let $1 < p < \infty$ and $(x_n)$ be a martingale with respect to $(\mathcal{M}_n)_{\geq 0}$ in the usual $L^p$-space $L^p(M, \tau)$ associated to $(M, \tau)$. Set $d_0 = x_0,\ d_n = x_n - x_{n-1}$. Then our main result reads as follows. If $p \geq 2$, we have (with equivalence constants depending only on $p$)

$$\sup_n \|x_n\|_p \approx \max\{\|(\sum d_n^* d_n)^{1/2}\|_p,\ \|(\sum d_n d_n^*)^{1/2}\|_p\}. \tag{0.1}$$

This is no longer valid for $p < 2$; however for $p < 2$ the "right" inequalities are

$$\sup_n \|x_n\|_p \approx \inf\{\|(\sum a_n^* a_n)^{1/2}\|_p + \|(\sum b_n b_n^*)^{1/2}\|_p\}, \tag{0.2}$$

where the infimum runs over all decompositions $d_n = a_n + b_n$ of $d_n$ as a sum of martingale difference sequences adapted to the same filtration.

In particular, this applies to martingale transforms: given a martingale $(x_n)$ as above and an adapted bounded sequence $\xi = (\xi_n)$, *i.e.* such that $\xi_n \in \mathcal{M}_n$ for all $n \geq 0$, we can form the martingale

$$y_n = x_0 + \sum_1^n \xi_{k-1}(x_k - x_{k-1}).$$

Then, if $(x_n)$ is a martingale which converges in $L^p(M, \tau)$ $(1 < p < \infty)$, if the sequence $\xi = (\xi_n)$ is bounded in $\mathcal{M}$ and if $\xi_{n-1}$ commutes with $\mathcal{M}_n$ for all $n$, the transformed martingale $(y_n)$ also converges in $L^p(M, \tau)$. Indeed, by duality, it suffices to check this for $p \geq 2$, and then it is an easy consequence of (0.1). Note however that the preceding statement can fail if one does not assume that $\xi_{n-1}$ commutes with $\mathcal{M}_n$. In the case $p \geq 2$, it suffices to assume that $\xi_{n-1}$ commutes with $x_n - x_{n-1}$ for all $n$. The latter assumption is used to show that if, say $\|\xi_{n-1}\| \leq 1$, we have $(y_n - y_{n-1})(y_n - y_{n-1})^* \leq (x_n - x_{n-1})(x_n - x_{n-1})^*$. Of course, this assumption can be relaxed further, all that is needed is to be able to compare the "square functions" associated to $(y_n)$ and $(x_n)$ appearing on the right in (0.1).

In Sect. 2 the above inequalities (0.1) and (0.2) are proved. The key point of our proof is the following passage: assuming the above inequalities for some $1 < p < \infty$, then we deduce them for $2p$. The rest of the proof can be accomplished by iteration (starting from $p = 2$), interpolation and duality. We would like to emphasize that this proof is entirely self-contained.

The style of proof of (0.1) and (0.2) is rather old fashioned: it is reminiscent of Marcel Riesz's classical argument for the boundedness of the Hilbert transform on $L^p$ $(1 < p < \infty)$, and also of Paley's proof of (0.1) in the classical dyadic case ([Pa]), *i.e.* when $\mathcal{M}_n = L_\infty(\{-1, +1\}^n)$. It has been known for many years that Marcel Riesz's argument could be easily adapted to prove the boundedness of the Hilbert transform on the vector valued $L^p$-space $(p \geq 2)$ $L^p(X)$, when the Banach space $X$ is the Schatten $p$-class $S_p$, or a non-commutative $L^p$-space associated to a trace (the first author learned this from P. Muhly back in 1976). More recently, Bourgain ([B1]) used this to show the unconditionality of martingale differences with values in $S_p$. In other words, he showed that $S_p$ is a UMD space, in the terminology of [Bu2]. (See [BGM] for the case of more general non-commutative $L^p$-spaces.) Recall that a Banach space $X$ is called a UMD space if, for any $1 < q < \infty$, there is a constant $C$ such that, for any $q$-integrable $X$-valued finite martingale $(x_n)$ on a probability space $(\Omega, \mathcal{A}, P)$ and for any choice of sign $\epsilon_n = \pm 1$, we have (here we write briefly $L_q(X)$ instead of $L_q(\Omega, \mathcal{A}, P; X)$)

$$\| \sum \varepsilon_n (x_n - x_{n-1}) \|_{L_q(X)} \leq C \| \sum x_n - x_{n-1} \|_{L_q(X)} = C \sup_n \| x_n \|_{L_q(X)}. \quad (0.3)$$

We will denote by $C_q(X)$ the best constant $C$ satisfying this. By well known stopping time arguments (the so-called “good $\lambda$ inequalities”, see [Bu1]) it suffices to have this for *some* $1 < q < \infty$, for instance for $q = 2$ say, and there is a positive constant $K_q$ depending only on $q$ such that for all $1 < q < \infty$,

$$K_q^{-1} C_2(X) \leq C_q(X) \leq K_q C_2(X). \quad (0.4)$$

Of course, when $X$ is a non-commutative $L^p$-space, the choice of $q = p$ gives a nicer form to (0.3). The reader is referred to [Bu2] for more information on UMD spaces.

The fact that non-commutative $L^p$-spaces are UMD ([B1-2, BGM]), which is of course a corollary of our main result, can also be used to prove, by some kind of transference argument, several special cases of it. This is explained in Sect. 3. However, although it seems to give better behaved constants (when $p \to \infty$), we do not see how to use this transference idea in the situation of an arbitrary filtration, as treated in Sect. 2.

In Sect. 3 we give three examples. They are respectively the tensor products, Clifford algebras and algebras of free groups. For all of them the preceding inequalities admit a different proof, that we outline in the tensor product case. Its main idea is to transfer a non-commutative martingale to a *commutative* martingale with values in the corresponding non-commutative $L^p$-space $L^p(\mathcal{M}, \tau)$, and to use its unconditionality. This alternate method is, in fact, our first approach to non-commutative martingale inequalities, as announced in [PX].

Section 4 is devoted to the Ito-Clifford integral. There we apply our main inequalities to give a characterization for an Ito-Clifford integral to be a $L^p$-martingale via its integrand. This is the Fermionic analogue of the square function inequality for the classical Ito integrals. As a consequence, we extend the Ito-Clifford integral theory in $L^2$, developed by Barnett, Streater and Wilde, to $L^p$ for all $1 < p < \infty$.

We include an appendix on the non-commutative analogue of the classical Fefferman duality between $H^1$ and $BMO$.

## 1. Preliminaries

Let $\mathcal{M}$ be a finite von Neumann algebra with a normalized faithful trace $\tau$. For $1 \leq p \leq \infty$ let $L^p(\mathcal{M}, \tau)$ or simply $L^p(\mathcal{M})$ denote the associated non-commutative $L^p$-space. Note that if $p = \infty$, $L^p(\mathcal{M})$ is just $\mathcal{M}$ itself with the operator norm; also recall that the norm in $L^p(\mathcal{M})$ $(1 \leq p < \infty)$ is defined as

$$\|x\|_p = (\tau(|x|^p))^{1/p}, \qquad x \in L^p(\mathcal{M}),$$

where

$$|x| = (x^* x)^{1/2}$$

is the usual absolute value of $x$.

Let $a = (a_n)_{n\geq 0}$ be a finite sequence in $L^p(\mathcal{M})$. Define

$$\|a\|_{L^p(\mathcal{M};l^2_C)} = \Big\|\Big(\sum_{n\geq 0}|a_n|^2\Big)^{1/2}\Big\|_p, \quad \|a\|_{L^p(\mathcal{M};l^2_R)} = \Big\|\Big(\sum_{n\geq 0}|a_n^*|^2\Big)^{1/2}\Big\|_p. \tag{1.1}$$

This gives two norms on the family of all finite sequences in $L^p(\mathcal{M})$. To see that, denoting by $B(l^2)$ the algebra of all bounded operators on $l^2$ with its usual trace tr, let us consider the von Neumann algebra tensor product $\mathcal{M}\otimes B(l^2)$ with the product trace $\tau\otimes$tr. $\tau\otimes$tr is a semifinite faithful trace. The associated non-commutative $L^p$-space is denoted by $L^p(\mathcal{M}\otimes B(l^2))$. Now any finite sequence $a = (a_n)_{n\geq 0}$ in $L^p(\mathcal{M})$ can be regarded as an element in $L^p(\mathcal{M}\otimes B(l^2))$ via the following map:

$$a \mapsto T(a) = \begin{pmatrix} a_0 & 0 & \dots \\ a_1 & 0 & \dots \\ \vdots & \vdots & \end{pmatrix},$$

that is, the matrix of $T(a)$ has all vanishing entries except those in the first column which are the $a_n$'s. Such a matrix is called a column matrix, and the closure in $L^p(\mathcal{M}\otimes B(l^2))$ of all column matrices is called the column subspace of $L^p(\mathcal{M}\otimes B(l^2))$ (when $p = \infty$, we take the $w^*$-closure of all column matrices). Then

$$\|a\|_{L^p(\mathcal{M};l^2_C)} = \|\,|T(a)|\,\|_{L^p(\mathcal{M}\otimes B(l^2))} = \|T(a)\|_{L^p(\mathcal{M}\otimes B(l^2))}.$$

Therefore, $\|\cdot\|_{L^p(\mathcal{M};l^2_C)}$ defines a norm on the family of all finite sequences of $L^p(\mathcal{M})$. The corresponding completion (for $1\leq p<\infty$) is a Banach space, denoted by $L^p(\mathcal{M};l^2_C)$. Then $L^p(\mathcal{M};l^2_C)$ is isometric to the column subspace of $L^p(\mathcal{M}\otimes B(l^2))$. For $p=\infty$ we let $L^\infty(\mathcal{M};l^2_C)$ be the Banach space of sequences in $L^\infty(\mathcal{M})$ isometric by the above map $T$ to the column subspace of $L^\infty(\mathcal{M}\otimes B(l^2))$. It is easy to check that a sequence $a = (a_n)_{n\geq 0}$ in $L^p(\mathcal{M})$ belongs to $L^p(\mathcal{M};l^2_C)$ iff

$$\sup_{n\geq 0}\Big\|\Big(\sum_{k=0}^{n}|a_k|^2\Big)^{1/2}\Big\|_p < \infty;$$

if this is the case, $\big(\sum_{k=0}^{\infty}|a_k|^2\big)^{1/2}$ belongs to $L^p(\mathcal{M})$ and $\big(\sum_{k=0}^{n}|a_k|^2\big)^{1/2}$ converges to it in $L^p(\mathcal{M})$ (relative to the $w^*$-topology for $p=\infty$).

Similarly (or passing to adjoints), we may show that $\|\cdot\|_{L^p(\mathcal{M};l^2_R)}$ is a norm on the family of all finite sequences in $L^p(\mathcal{M})$. As above, it defines a Banach space $L^p(\mathcal{M};l^2_R)$, which now is isometric to the row subspace of $L^p(\mathcal{M}\otimes B(l^2))$ consisting of matrices whose non-zero entries lie only in the first row.

Observe that the column and row subspaces of $L^p(\mathcal{M}\otimes B(l^2))$ are 1-complemented subspaces. Therefore, from the classical duality between $L^p(\mathcal{M}\otimes B(l^2))$ and $L^q(\mathcal{M}\otimes B(l^2))$ $\big(\frac{1}{p}+\frac{1}{q}=1,\ 1\leq p<\infty\big)$ we deduce that

$$L^p(\mathcal{M};l^2_C)^* = L^q(\mathcal{M};l^2_C) \quad\text{and}\quad L^p(\mathcal{M};l^2_R)^* = L^q(\mathcal{M};l^2_R).$$

This complementation also shows that the families $\{L^p(\mathcal{M};l^2_C)\}$ and $\{L^p(\mathcal{M};l^2_R)\}$ are two interpolation scales, say, for instance, relative to the complex interpolation method.

Note that, for any finite sequence $(a_n)_{n\geq 0}$ in $L^p(\mathcal{M})$, we have, using tensor product notation and denoting again by $\|.\|_p$ the norm in $L^p(\mathcal{M}\otimes B(l^2))$,

$$\|(\sum a_n^* a_n)^{1/2}\|_p = \|\sum a_n \otimes e_{n0}\|_p \quad \text{and} \quad \|(\sum a_n a_n^*)^{1/2}\|_p = \|\sum a_n \otimes e_{0n}\|_p.$$

The following is an extension of a non-commutative version of Hölder's inequality from [LP], which can be established (perhaps at the cost of an extra factor 2) by arguing as in [LP]. For completeness, we include a direct elementary proof (without any extra factor) based on the three lines lemma.

**Lemma 1.1.** *Let $2\leq p\leq\infty$. For any finite sequence $a=(a_n)_{n\geq 0}$ in $L^{2p}(\mathcal{M})$ and any $A\in L^{2p}(\mathcal{M})$ we set $B(a,A)=(a_nA)_{n\geq 0}$. Then*

$$\|B(a,A)\|_{L^p(\mathcal{M};l^2_R)} \leq \max\left\{\|a\|_{L^{2p}(\mathcal{M};l^2_C)}\ ,\ \|a\|_{L^{2p}(\mathcal{M};l^2_R)}\right\}\ \|A\|_{2p}\,. \tag{1.2}$$

*Proof.* By definition, the left side of (1.2) is equal to $\|\sum a_nAA^*a_n^*\|_{p/2}^{1/2}$ and, on the other hand, by duality, we have

$$\|\sum a_nAA^*a_n^*\|_{p/2} = \sup|\psi(B)| \tag{1.3}$$

with

$$\psi(B)=\tau(\sum a_nAA^*a_n^*B)$$

and where the supremum in (1.3) runs over the set of all $B\geq 0$ in $\mathcal{M}$ such that $\tau(B^r)\leq 1$ with $r$ conjugate to $p/2$, or equivalently with $1/r=1-2/p$.

We will apply the three lines lemma to the analytic function $F$ defined for $0\leq\Re(z)\leq 1$ by

$$F(z)=\tau\left(\sum a_n\ (AA^*)^{zp/p'}\ a_n^*\ B^{(1-z)r/p'}\right).$$

Let $\theta=p'/p$ so that $1-\theta=p'/r$. Note that $0\leq\theta\leq 1$ and $F(\theta)=\psi(B)$. Hence, by the three lines lemma, we have

$$|\psi(B)|=|F(\theta)|\leq(\sup_{t\in\mathbb{R}}|F(it)|)^{1-\theta}(\sup_{t\in\mathbb{R}}|F(1+it)|)^{\theta}. \tag{1.4}$$

But, by an easy application of Hölder's inequality, we have

$$\sup_{t\in\mathbb{R}}|F(it)|\leq\sup\{\|\sum a_nUa_n^*\|_p\,|\,U\in\mathcal{M},\ \|U\|\leq 1\}, \tag{1.5}$$

and since $\tau$ is a trace, we also find

$$\sup_{t\in\mathbb{R}}|F(1+it)|\leq\|(AA^*)^{p/p'}\|_{p'}\sup\{\|\sum a_n^*Ua_n\|_p\ |\ U\in\mathcal{M},\ \|U\|\leq 1\}. \tag{1.6}$$

Note that, if $\|U\|\leq 1$, we have $\|\sum a_nUa_n^*\|_p\leq\|\sum a_na_n^*\|_p$, and similarly with $a_n^*$ instead of $a_n$. Indeed, $\|\sum a_nUa_n^*\|_p=\|(\sum a_nU\otimes e_{0n})(\sum a_n^*\otimes e_{n0})\|_p$, hence

$$\|\sum a_nUa_n^*\|_p\leq\|\sum a_nU\otimes e_{0n}\|_{2p}\|\sum a_n^*\otimes e_{n0}\|_{2p}=\|\sum a_na_n^*\|_p.$$

Therefore the inequalities (1.4), (1.5) and (1.6) combined with (1.3) immediately yield the announced result (1.2). $\square$

*Remark 1.2.* The following example shows that the right side of (1.2) cannot be simplified too much: let $\mathcal{M}$ be the algebra of all $N \times N$ complex matrices equipped with its usual trace, let $A = e_{11}$ and let $a_n = e_{n1}$ for $n = 1, ..., N$. Then $(\sum a_n AA^* a_n^*)^{1/2} = \sum_1^N e_{nn} = (\sum a_n a_n^*)^{1/2}$ and $\sum a_n^* a_n = Ne_{11}$ so that $\|(\sum a_n AA^* a_n^*)^{1/2}\|_p = N^{1/p}$, $\|A\|_{2p} = 1$ and $\|(\sum a_n^* a_n)^{1/2}\|_{2p} = N^{1/2}$. Thus, if $2 \leq p < \infty$, for no constant $C$ can the inequality $\|B(a, A)\|_{L^p(\mathcal{M};l_R^2)} \leq C\|a\|_{L^{2p}(\mathcal{M};l_R^2)} \|A\|_{2p}$ be true. This example also shows that (1.2) fails for $p < 2$. Similarly, the inequality $\|B(a, A)\|_{L^p(\mathcal{M};l_R^2)} \leq C\|a\|_{L^{2p}(\mathcal{M};l_C^2)} \|A\|_{2p}$ also fails if $2 < p \leq \infty$ (take $A = 1$ and $a_n = e_{1n}$).

We now turn to the description of non-commutative martingales and their square functions. Let $(\mathcal{M}_n)_{n\geq 0}$ be an increasing sequence of von Neumann subalgebras of $\mathcal{M}$ such that $\bigcup\limits_{n\geq 0} \mathcal{M}_n$ generates $\mathcal{M}$ (in the $w^*$-topology). $(\mathcal{M}_n)_{n\geq 0}$ is called a filtration of $\mathcal{M}$. The restriction of $\tau$ to $\mathcal{M}_n$ is still denoted by $\tau$. Let $\mathcal{E}_n = \mathcal{E}(\cdot|\mathcal{M}_n)$ be the conditional expectation of $\mathcal{M}$ with respect to $\mathcal{M}_n$. $\mathcal{E}_n$ is a norm 1 projection of $L^p(\mathcal{M})$ onto $L^p(\mathcal{M}_n)$ for all $1 \leq p \leq \infty$, and $\mathcal{E}_n(x) \geq 0$ whenever $x \geq 0$. A non-commutative $L^p$-martingale with respect to $(\mathcal{M}_n)_{n\geq 0}$ is a sequence $x = (x_n)_{n\geq 0}$ such that $x_n \in L^p(\mathcal{M}_n)$ and

$$\mathcal{E}_m(x_n) = x_m, \qquad \forall\, m = 0, 1, ..., n.$$

Let $\|x\|_p = \sup\limits_{n\geq 0} \|x_n\|_p$. If $\|x\|_p < \infty$, $x$ is said to be bounded.

*Remark 1.3.* Let $x_\infty \in L^p(\mathcal{M})$. Set $x_n = \mathcal{E}_n(x_\infty)$ for all $n \geq 0$. Then $x = (x_n)$ is a bounded $L^p$-martingale and $\|x\|_p = \|x_\infty\|_p$; moreover, $x_n$ converges to $x_\infty$ in $L^p(\mathcal{M})$ (relative to the $w^*$-topology in the case $p = \infty$). Conversely, if $1 < p < \infty$, every bounded $L^p$-martingale converges in $L^p(\mathcal{M})$, and so is given by some $x_\infty \in L^p(\mathcal{M})$ as previously. Thus one can identify the space of all bounded $L^p$-martingales with $L^p(\mathcal{M})$ itself in the case $1 < p < \infty$.

Let $x$ be a martingale. Its difference sequence, denoted by $dx = (dx_n)_{n\geq 0}$, is defined as (with $x_{-1} = 0$ by convention)

$$dx_n = x_n - x_{n-1}, \qquad n \geq 0.$$

Set

$$S_{C,n}(x) = \left(\sum_{k=0}^{n} |dx_k|^2\right)^{1/2} \quad \text{and} \quad S_{R,n}(x) = \left(\sum_{k=0}^{n} |dx_k^*|^2\right)^{1/2}.$$

By the preceding discussion $dx$ belongs to $L^p(\mathcal{M}; l_C^2)$ (resp. $L^p(\mathcal{M}; l_R^2)$) iff $(S_{C,n}(x))_{n\geq 0}$ (resp. $(S_{R,n}(x))_{n\geq 0}$) is a bounded sequence in $L^p(\mathcal{M})$; in this case,

$$S_C(x) = \left(\sum_{k=0}^{\infty} |dx_k|^2\right)^{1/2} \quad \text{and} \quad S_R(x) = \left(\sum_{k=0}^{\infty} |dx_k^*|^2\right)^{1/2}$$

are elements in $L^p(\mathcal{M})$. These are the non-commutative analogues of the usual square functions in the commutative martingale theory. It should be pointed out that one of $S_C(x)$ and $S_R(x)$ may exist as an element of $L^p(\mathcal{M})$ without the other making sense; in other words, the two sequences $S_{C,n}(x)$ and $S_{R,n}(x)$ may not be bounded in $L^p(\mathcal{M})$ at the same time.

Let $1 \leq p < \infty$. Define $\mathcal{H}_C^p(\mathcal{M})$ (resp. $\mathcal{H}_R^p(\mathcal{M})$) to be the space of all $L^p$-martingales $x$ with respect to $(\mathcal{M}_n)_{n\geq 0}$ such that $dx \in L^p(\mathcal{M}; l_C^2)$ (resp. $dx \in L^p(\mathcal{M}; l_R^2)$), and set

$$\|x\|_{\mathcal{H}_C^p(\mathcal{M})} = \|dx\|_{L^p(\mathcal{M};l_C^2)} \quad \text{and} \quad \|x\|_{\mathcal{H}_R^p(\mathcal{M})} = \|dx\|_{L^p(\mathcal{M};l_R^2)}.$$

Equipped respectively with the previous norms, $\mathcal{H}_C^p(\mathcal{M})$ and $\mathcal{H}_R^p(\mathcal{M})$ are Banach spaces. Note that if $x \in \mathcal{H}_C^p(\mathcal{M})$,

$$\|x\|_{\mathcal{H}_C^p(\mathcal{M})} = \sup_{n\geq 0} \|S_{C,n}(x)\|_p = \|S_C(x)\|_p \,,$$

and similar equalities hold for $\mathcal{H}_R^p(\mathcal{M})$. Then we define the Hardy spaces of non-commutative martingales as follows: if $1 \leq p < 2$,

$$\mathcal{H}^p(\mathcal{M}) = \mathcal{H}_C^p(\mathcal{M}) + \mathcal{H}_R^p(\mathcal{M})$$

equipped with the norm

$$\|x\|_{\mathcal{H}^p(\mathcal{M})} = \inf\{\|y\|_{\mathcal{H}_C^p(\mathcal{M})} + \|z\|_{\mathcal{H}_R^p(\mathcal{M})}\colon\ x = y + z, \quad y \in \mathcal{H}_C^p(\mathcal{M}), z \in \mathcal{H}_R^p(\mathcal{M})\}; \tag{1.7}$$

and if $2 \leq p < \infty$,

$$\mathcal{H}^p(\mathcal{M}) = \mathcal{H}_C^p(\mathcal{M}) \cap \mathcal{H}_R^p(\mathcal{M})$$

equipped with the norm

$$\|x\|_{\mathcal{H}^p(\mathcal{M})} = \max\{\|x\|_{\mathcal{H}_C^p(\mathcal{M})}, \quad \|x\|_{\mathcal{H}_R^p(\mathcal{M})}\}. \tag{1.8}$$

The reason that we have defined $\mathcal{H}^p(\mathcal{M})$ differently according to $1 \leq p < 2$ or $2 \leq p < \infty$ will become clear in the next section, where we will show that $\mathcal{H}^p(\mathcal{M}) = L^p(\mathcal{M})$ with equivalent norms for all $1 < p < \infty$.

## 2. The Main Result

In this section $(\mathcal{M}, \tau)$ always denotes a finite von Neumann algebra equipped with a normalized faithful trace, and $(\mathcal{M}_n)_{n\geq 0}$ an increasing filtration of subalgebras of $\mathcal{M}$ which generate $\mathcal{M}$. We keep all notations introduced in the last section.

In the sequel $\alpha_p$, $\beta_p$, *etc.*, denote positive constants depending only on $p$. The following is the main result of this paper.

**Theorem 2.1.** *Let $1 < p < \infty$. Let $x = (x_n)_{n\geq 0}$ be an $L^p$-martingale with respect to $(\mathcal{M}_n)_{n\geq 0}$. Then $x$ is bounded in $L^p(\mathcal{M})$ iff $x$ belongs to $\mathcal{H}^p(\mathcal{M})$; moreover, if this is the case,*

$$\alpha_p^{-1}\|x\|_{\mathcal{H}^p(\mathcal{M})} \leq \|x\|_p \leq \beta_p\|x\|_{\mathcal{H}^p(\mathcal{M})}. \tag{$BG_p$}$$

Identifying bounded $L^p$-martingales with their limits, we may reformulate Theorem 2.1 as follows.

**Corollary 2.2.** *Let $1 < p < \infty$. Then $\mathcal{H}^p(\mathcal{M}) = L^p(\mathcal{M})$ with equivalent norms.*

Corollary 2.2 explains why we have defined, in (1.7) and (1.8), the space $\mathcal{H}^p(\mathcal{M})$ and its norm differently for $p$ in [1,2) and $[2,\infty)$. One should note that such a different behavior in the non-commutative case already appears in the non-commutative Khintchine inequalities obtained by F. Lust-Piquard, which we will recall later on.

Before proceeding to the proof of Theorem 2.1, let us briefly explain our strategy. Firstly, we prove the implication "$(BG_p) \Longrightarrow (BG_{2p})$" (this is the key point of the proof). Then by iteration (noting that $(BG_2)$ is trivial) and interpolation we deduce $(BG_p)$ for all $2 \le p < \infty$. Finally, duality yields $(BG_p)$ for $1 < p < 2$. This is a well-known approach to the classical Burkholder-Gundy inequalities in the *commutative* martingale theory. However, in order to adapt it to the non-commutative setting, one encounters several substantial difficulties. Perhaps the main one is the lack of a reasonable maximal function in the non-commutative case. (Note that all the truncation arguments that appeal to stopping times appear unavailable or inefficient.)

In the course of the proof we will show (and also need) the following result, which is the non-commutative analogue of a classical inequality due to Stein [St]. (See also [B1, Lemma 8] for a similar result in the case of commutative martingales with values in a UMD space.)

**Theorem 2.3.** *Let $1 < p < \infty$. Define the map $Q$ on all finite sequences $a = (a_n)_{n\ge 0}$ in $L^p(\mathcal{M})$ by $Q(a) = (\mathcal{E}_n a_n)_{n\ge 0}$. Then*

$$\|Q(a)\|_{L^p(\mathcal{M};l^2_C)} \le \gamma_p \|a\|_{L^p(\mathcal{M};l^2_C)}\,, \quad \|Q(a)\|_{L^p(\mathcal{M};l^2_R)} \le \gamma_p \|a\|_{L^p(\mathcal{M};l^2_R)}\,. \qquad (S_p)$$

*Thus $Q$ extends to a bounded projection on $L^p(\mathcal{M}; l^2_C)$ and $L^p(\mathcal{M}; l^2_R)$; consequently, $\mathcal{H}^p(\mathcal{M})$ is complemented in $L^p(\mathcal{M}; l^2_C) + L^p(\mathcal{M}, l^2_R)$ or $L^p(\mathcal{M}; l^2_C) \cap L^p(\mathcal{M}; l^2_R)$ according to $1 < p \le 2$ or $2 \le p < \infty$.*

*Remark 2.4.* The inequalities $(BG_p)$ imply that all martingale difference sequences are unconditional in $L^p(\mathcal{M})$, *i.e.* there is a positive constant $\beta'_p$ such that for all finite martingales $x$ in $L^p(\mathcal{M})$ we have

$$\|\sum_n \varepsilon_n dx_n\|_p \le \beta'_p \|\sum_n dx_n\|_p\,, \quad \forall\, \varepsilon_n = \pm 1. \qquad (BG'_p)$$

Moreover $\beta'_p \le \alpha_p \beta_p$.

We begin the proof of Theorems 2.1 and 2.3 with some elementary lemmas.

The inequality below is well known: indeed, it is a consequence of the UMD property of $L^p(\mathcal{M})$. One can also use the Hilbert transform instead. For the sake of completeness, we will show that it follows from $(BG_p)$. The following proof is similar to an argument presented in [HP].

**Lemma 2.5.** *Let $\varepsilon = (\varepsilon_n)_{n\ge 0}$ be a sequence of independent random variables on some probability space $(\Omega, \mathcal{F}, P)$ such that $P(\varepsilon_n = 1) = P(\varepsilon_n = -1) = 1/2$ for all $n \ge 0$. Let $\varepsilon' = (\varepsilon'_n)_{n\ge 0}$ be an independent copy of $\varepsilon$. Let $1 < p < \infty$. Suppose $(BG_p)$. Then for all finite double sequences $(a_{ij})_{i,j\ge 0}$ in $L^p(\mathcal{M})$,*

$$\Big(\int_\Omega \|\sum_{0\le i\le j} \varepsilon_i \varepsilon'_j a_{ij}\|_p^p dP(\varepsilon) dP(\varepsilon')\Big)^{1/p}$$
$$\le \alpha_p \beta_p \Big(\int_\Omega \|\sum_{i,j\ge 0} \varepsilon_i \varepsilon'_j a_{ij}\|_p^p dP(\varepsilon) dP(\varepsilon')\Big)^{1/p}.$$

*Proof.* Given $n \geq 0$ let $\mathcal{F}_{2n}$ and $\mathcal{F}_{2n+1}$ be the sub-$\sigma$-fields of $\mathcal{F}$ generated respectively by $\{\varepsilon_0, \cdots, \varepsilon_n\} \cup \{\varepsilon'_0, \cdots, \varepsilon'_n\}$ and $\{\varepsilon_0, \cdots, \varepsilon_n, \varepsilon_{n+1}\} \cup \{\varepsilon'_0, \cdots, \varepsilon'_n\}$. Then $(\mathcal{F}_n)_{n\geq 0}$ is an increasing filtration of sub-$\sigma$-fields of $\mathcal{F}$. Let $\mathbb{E}$ denote the expectation viewed as a (tracial!) functional on $L^\infty(\Omega, \mathcal{F}, P)$. We consider the tensor product $(\mathcal{M}, \tau) \otimes (L^\infty(\Omega, \mathcal{F}, P), \mathbb{E})$ and its increasing filtration $\mathcal{M} \otimes L^\infty(\Omega, \mathcal{F}_n, P)$. Hence we have $(BG_p)$ for the corresponding martingales (noting that such martingales are in fact *commutative* martingales with values in $L^p(\mathcal{M})$). Now given a finite double sequence $(a_{ij})_{i,j\geq 0}$ in $L^p(\mathcal{M})$ we define a martingale $f = (f_n)_{n\geq 0}$ by

$$f_n = \mathrm{id}_{\mathcal{M}} \otimes \mathbb{E}_n \big( \sum_{i,j\geq 0} \varepsilon_i \varepsilon'_j a_{ij} \big),$$

where $\mathbb{E}_n$ stands for the conditional expectation of $\mathcal{F}$ with respect to $\mathcal{F}_n$. Then $(BG_p)$ yields

$$\| \sum_{n\geq 0} \varepsilon''_n df_n \|_p \leq \alpha_p \beta_p \|f\|_p, \quad \forall\, \varepsilon''_n = \pm 1,$$

where the norm $\| \cdot \|_p$ is understood as it should be, that is, it is the norm on $L^p\big(\mathcal{M} \otimes L^\infty(\Omega), \tau \otimes \mathbb{E}\big)$. Consequently,

$$\| \sum_{n\geq 0} df_{2n} \|_p \leq \alpha_p \beta_p \|f\|_p.$$

However,

$$\sum_{n\geq 0} df_{2n} = \sum_{0\leq i\leq j} \varepsilon_i \varepsilon'_j a_{ij},$$

whence the announced result. $\square$

**Lemma 2.6.** *Let $1 \leq p \leq \infty$. Then for all finite sequences $a = (a_n)_{n\geq 0} \subset L^p(\mathcal{M})$ we have*

$$\|(\sum_{n\geq 0} |a_n|^4)^{1/2}\|_p \leq \|(\sum_{n\geq 0} |a_n|^2)^{1/2}\|_{2p} (\sum_{n\geq 0} \|a_n\|_{2p}^{2p})^{1/(2p)}.$$

*Proof.* Let $e_{i,j}$ be the matrix in $B(l^2)$ whose entries all vanish but the one on the position $(i, j)$ which equals 1. Using the tensor product $\mathcal{M} \otimes B(l^2)$ (already considered in Sect. 1) we have

$$\begin{aligned}
\|(\sum_{n\geq 0} |a_n|^4)^{1/2}\|_p &= \| \sum_{n\geq 0} |a_n|^2 \otimes e_{n,0} \|_{L^p(\mathcal{M}\otimes B(l^2))} \\
&= \| \big( \sum_{n\geq 0} a_n^* \otimes e_{n,n} \big) \big( \sum_{n\geq 0} a_n \otimes e_{n,0} \big) \|_{L^p(\mathcal{M}\otimes B(l^2))} \\
&\leq \| \sum_{n\geq 0} a_n^* \otimes e_{n,n} \|_{L^{2p}(\mathcal{M}\otimes B(l^2))} \| \sum_{n\geq 0} a_n \otimes e_{n,0} \|_{L^{2p}(\mathcal{M}\otimes B(l^2))} \\
&= (\sum_{n\geq 0} \|a_n\|_{2p}^{2p})^{1/(2p)} \|(\sum_{n\geq 0} |a_n|^2)^{1/2}\|_{2p}.
\end{aligned}$$

$\square$

In particular, for martingale differences we get the following

**Lemma 2.7.** *Let* $1 \le p \le \infty$. *Then for all finite martingales* $x = (x_n)_{n\ge 0} \subset L^{2p}(\mathcal{M})$ *we have*

$$\|(\sum_{n\ge 0} |dx_n|^4)^{1/2}\|_p \le 2^{1-1/p}\|x\|_{2p}\|x\|_{\mathcal{H}_C^{2p}(\mathcal{M})}\,.$$

*Proof.* By Lemma 2.6, it suffices to show

$$\big(\sum_{n\ge 0} \|dx_n\|_{2p}^{2p}\big)^{1/(2p)} \le 2^{1-1/p}\|x\|_{2p}\,.$$

This is trivial for $p = 1$ and $p = \infty$. Then the general case follows by interpolation. $\square$

Now we are prepared to prove Theorems 2.1 and 2.3. The proof is divided into several steps.

*Proof of Theorems 2.1 and 2.3. Step 1.* $(BG_p)$ *implies* $(S_p)$. Let $1 < p < \infty$. Suppose $(BG_p)$ holds. We will show $(S_p)$ holds as well.

To this end, fix a finite sequence $a = (a_k)_{0\le k\le n} \subset L^p(\mathcal{M})$. We consider the tensor product $(\mathcal{M}, \tau) \otimes (\mathcal{N}, \sigma)$, where $\mathcal{N} = B(l^2_{n+1})$ and $\sigma = (n+1)^{-1}\mathrm{tr}$ is the normalized trace on $B(l^2_{n+1})$. Let $\tilde{\mathcal{E}}_k = \mathcal{E}_k \otimes \mathrm{id}_{\mathcal{N}}$ denote the conditional expectation of $\mathcal{M}\otimes\mathcal{N}$ with respect to $\mathcal{M}_k \otimes \mathcal{N}$. Then we have $(BG_p)$ for all martingales relative to the filtration $(\mathcal{M}_k \otimes \mathcal{N})_{k\ge 0}$. Now set

$$A_k = (n+1)^{1/p} a_k \otimes e_{k,0}\,, \quad 0 \le k \le n.$$

Let $\varepsilon = (\varepsilon_n)_{n\ge 0}$ and $\varepsilon' = (\varepsilon'_n)_{n\ge 0}$ be the sequences in Lemma 2.5. Then, with $\|.\|_p$ denoting here the norm in the space $L^p(\mathcal{M}\otimes\mathcal{N})$, we have

$$\begin{aligned}\|Q(a)\|_{L^p(\mathcal{M};l^2_C)} &= \|\sum_{k=0}^n \tilde{\mathcal{E}}_k(\varepsilon_k A_k)\|_p = \|\sum_{k=0}^n\sum_{j=0}^k (\tilde{\mathcal{E}}_j - \tilde{\mathcal{E}}_{j-1})(\varepsilon_k A_k)\|_p \\ &= \|\sum_{j=0}^n (\tilde{\mathcal{E}}_j - \tilde{\mathcal{E}}_{j-1})\big(\sum_{k=j}^n \varepsilon_k A_k\big)\|_p\,,\end{aligned}$$

hence by $(BG_p)$ (*cf*. Remark 2.4)

$$\le \alpha_p\beta_p\Big(\int_\Omega \|\sum_{j=0}^n \varepsilon'_j(\tilde{\mathcal{E}}_j - \tilde{\mathcal{E}}_{j-1})\big(\sum_{k=j}^n \varepsilon_k A_k\big)\|_p^p dP(\varepsilon)dP(\varepsilon')\Big)^{1/p},$$

so by Lemma 2.5,

$$\le (\alpha_p\beta_p)^2\Big(\int_\Omega \|\sum_{j=0}^n \varepsilon'_j(\tilde{\mathcal{E}}_j - \tilde{\mathcal{E}}_{j-1})\big(\sum_{k=0}^n \varepsilon_k A_k\big)\|_p^p dP(\varepsilon)dP(\varepsilon')\Big)^{1/p}.$$

On the other hand, applying $(BG_p)$ once again, this is

$$\begin{aligned}&\le (\alpha_p\beta_p)^3\Big(\int_\Omega \|\sum_{j=0}^n (\tilde{\mathcal{E}}_j - \tilde{\mathcal{E}}_{j-1})\big(\sum_{k=0}^n \varepsilon_k A_k\big)\|_p^p dP(\varepsilon)\Big)^{1/p} \\ &= (\alpha_p\beta_p)^3\Big(\int_\Omega \|\sum_{k=0}^n \varepsilon_k A_k\|_p^p d\varepsilon\Big)^{1/p} = (\alpha_p\beta_p)^3\|a\|_{L^p(\mathcal{M};l^2_C)}.\end{aligned}$$

Thus, we conclude

$$\|Q(a)\|_{L^p(\mathcal{M};l^2_C)} \le (\alpha_p\beta_p)^3\|a\|_{L^p(\mathcal{M};l^2_C)} .$$

Hence $Q$ is bounded on $L^p(\mathcal{M};l^2_C)$. Passing to adjoints yields the boundedness of $Q$ on $L^p(\mathcal{M};l^2_R)$.

*Step 2. $(BG_p)$ implies $(BG_{2p})$.* Let $1 < p < \infty$ and suppose $(BG_p)$. Let $x = (x_n)_{n\ge 0}$ be a martingale in $L^{2p}(\mathcal{M})$. We must show $x$ satisfies $(BG_{2p})$. Clearly, we can assume $x$ finite, that is, there exists $n \in \mathbb{N}$ such that $x_k = x_n$ for all $k \ge n$. For simplicity, set $d_k = dx_k$ (so $d_k = 0$ for all $k > n$). Then we write the classical "Doob identity":

$$|x_n|^2 = x_n^* x_n = S_C(x)^2 + \sum_{k\ge 0} d_k^* x_{k-1} + \sum_{k\ge 0} x_{k-1}^* d_k . \tag{2.1}$$

Hence

$$\begin{aligned} \|x\|_{2p}^2 &= \||x_n|^2\|_p \\ &\le \|S_C(x)^2\|_p + \|\sum_{k\ge 0} d_k^* x_{k-1}\|_p + \|\sum_{k\ge 0} x_{k-1}^* d_k\|_p \\ &= \|x\|^2_{\mathcal{H}^{2p}_C(\mathcal{M})} + 2\|\sum_{k\ge 0} d_k^* x_{k-1}\|_p . \end{aligned} \tag{2.2}$$

Observe that $(d_k^* x_{k-1})_{k\ge 0}$ is a martingale difference sequence. Letting $y = (y_k)$ be the corresponding martingale, then by $(BG_p)$, we get

$$\|y\|_p \le \beta_p\|y\|_{\mathcal{H}^p(\mathcal{M})} . \tag{2.3}$$

Now note that

$$dy_k = d_k^* x_k - d_k^* d_k = \mathcal{E}_k(d_k^* x_n) - |d_k|^2, \quad 0 \le k \le n. \tag{2.4}$$

Let us first consider the case $1 < p < 2$. Then $\|y\|_{\mathcal{H}^p(\mathcal{M})} \le \|y\|_{\mathcal{H}^p_C(\mathcal{M})}$, so by (2.3), (2.4), Lemma 2.7 and $(S_p)$ (which, by Step 1, holds under $(BG_p)$), we get

$$\begin{aligned} \|y\|_{\mathcal{H}^p(\mathcal{M})} &\le \|(\sum_{k=0}^n |d_k|^4)^{1/2}\|_p + \|\big(\sum_{k=0}^n |\mathcal{E}_k(d_k^* x_n)|^2\big)^{1/2}\|_p \\ &\le 2^{1-1/p}\|x\|_{2p}\|x\|_{\mathcal{H}^{2p}_C(\mathcal{M})} + \gamma_p\|\big(\sum_{k=0}^n x_n^* d_k d_k^* x_n\big)^{1/2}\|_p \\ &\le 2^{1-1/p}\|x\|_{2p}\|x\|_{\mathcal{H}^{2p}_C(\mathcal{M})} + \gamma_p\|x\|_{2p}\|x\|_{\mathcal{H}^{2p}_R(\mathcal{M})} \\ &\le (2^{1-1/p} + \gamma_p)\|x\|_{2p}\|x\|_{\mathcal{H}^{2p}(\mathcal{M})}. \end{aligned} \tag{2.5}$$

If $2 \le p < \infty$, again by (2.3) and (2.4),

$$\begin{aligned} \|y\|_{\mathcal{H}^p(\mathcal{M})} \le &\|(\sum_{k=0}^n |d_k|^4)^{1/2}\|_p \\ &+ \sup\Big\{\|\big(\sum_{k=0}^n |\mathcal{E}_k(d_k^* x_n)|^2\big)^{1/2}\|_p, \|\big(\sum_{k=0}^n |\mathcal{E}_k(d_k^* x_n)^*|^2\big)^{1/2}\|_p\Big\}. \end{aligned}$$

The first two terms on the right are dealt with as before; while by $(S_p)$ and Lemma 1.1, the third term is majorized by $\gamma_p\|x\|_{2p}\|x\|_{\mathcal{H}^{2p}(\mathcal{M})}$. Thus in the case $2 \le p < \infty$, we have

$$\|y\|_{\mathcal{H}^p(\mathcal{M})} \le (2^{1-1/p} + \gamma_p)\|x\|_{2p}\|x\|_{\mathcal{H}^{2p}(\mathcal{M})}. \tag{2.6}$$

Putting together (2.2), (2.3), (2.5) and (2.6), we obtain finally

$$\begin{aligned}\|x\|_{2p}^2 &\le \|x\|^2_{\mathcal{H}_C^{2p}(\mathcal{M})} + 2\beta_p(2^{1-1/p} + \gamma_p)\|x\|_{2p}\|x\|_{\mathcal{H}^{2p}(\mathcal{M})}\\ &\le \|x\|^2_{\mathcal{H}^{2p}(\mathcal{M})} + \delta_p\|x\|_{2p}\|x\|_{\mathcal{H}^{2p}(\mathcal{M})}\,,\end{aligned}$$

where $\delta_p = 2\beta_p(2^{1-1/p} + \gamma_p)$. Therefore, it follows that

$$\|x\|_{2p} \le \beta_{2p}\|x\|_{\mathcal{H}^{2p}(\mathcal{M})}$$

with $\beta_{2p} = \frac{1}{2}(\delta_p + \sqrt{4 + \delta_p^2})$. Thus we have proved the second inequality of $(BG_{2p})$. The first one can be obtained in a similar way. Indeed, again by (2.1) and the previous argument, we get

$$\|x\|_{2p}^2 \ge \|x\|^2_{\mathcal{H}_C^{2p}(\mathcal{M})} - \delta_p\|x\|_{2p}\|x\|_{\mathcal{H}^{2p}(\mathcal{M})}\,.$$

Replacing $x_n$ by $x_n^*$ in (2.1), we also have

$$\|x\|_{2p}^2 \ge \|x\|^2_{\mathcal{H}_R^{2p}(\mathcal{M})} - \delta_p\|x\|_{2p}\|x\|_{\mathcal{H}^{2p}(\mathcal{M})}\,.$$

Therefore,

$$\|x\|^2_{\mathcal{H}^{2p}(\mathcal{M})} \le \|x\|_{2p}^2 + \delta_p\|x\|_{2p}\|x\|_{\mathcal{H}^{2p}(\mathcal{M})}\,,$$

which gives the first inequality of $(BG_{2p})$.

*Step 3.* $(BG_p)$ *for* $2 \le p < \infty$ *and* $(S_p)$ *for* $1 < p < \infty$. Evidently, $(BG_2)$ holds with $\alpha_2 = \beta_2 = 1$. Then by Step 2 and iteration we get $(BG_{2^n})$ for all positive integers $n$, and so also $(S_{2^n})$ by virtue of Step 1.

Now we use interpolation to cover all values of $p$ in $[2,\infty)$. This is easy for $(S_p)$ and the first inequality of $(BG_p)$. Let us consider, for instance, the first inequality of $(BG_p)$. By what we already know about $(BG_{2^n})$, the linear map $x \mapsto dx$ is bounded from $L^{2^n}(\mathcal{M})$ into $L^{2^n}(\mathcal{M}; l_C^2)$ for every positive integer $n$. Then by complex interpolation, it is bounded from $L^p(\mathcal{M})$ into $L^p(\mathcal{M}; l_C^2)$ for $2^n < p < 2^{n+1}$, and so for all $p \in [2,\infty)$. Hence

$$\|dx\|_{L^p(\mathcal{M};l_C^2)} \le \alpha_p\|x\|_p\,.$$

Passing to adjoints, we get the same inequality with $L^p(\mathcal{M}; l_R^2)$ instead of $L^p(\mathcal{M}; l_C^2)$. Thus the first inequality of $(BG_p)$ holds for all $2 \le p < \infty$. A similar argument applies to $(S_p)$ for all $2 \le p < \infty$. However, the projection $Q$ in Theorem 2.3 is self-adjoint; hence, we get $(S_p)$ for all $1 < p < \infty$, which completes the proof of Theorem 2.3.

Concerning the second inequality of $(BG_p)$, we observe that by duality and the first inequality of $(BG_p)$ just proved in $[2,\infty)$, we deduce that for every $1 < p \le 2$ and any martingale $x$ in $L^p(\mathcal{M})$ we have

$$\|x\|_p \le \beta_p \inf\left\{\|x\|_{\mathcal{H}_C^p(\mathcal{M})}\,, \|x\|_{\mathcal{H}_R^p(\mathcal{M})}\right\}.$$

(Here if $1/p + 1/q = 1$ (so $2 \leq q < \infty$), $\beta_p = \alpha_q$ with $\alpha_q$ being the constant in the first inequality of $(BG_q)$; see the next step for more on this). Examining the proof in Step 2, we see that the implication "$(BG_p) \Longrightarrow (BG_{2p})$" still holds now with the help of $(S_p)$ and the above inequality for all $1 < p \leq 2$. It follows that the second inequality of $(BG_p)$ holds for all $2 \leq p \leq 4$. Then Step 2 and iteration yield the second inequality of $(BG_p)$ for all $2 \leq p < \infty$.

*Step 4.* $(BG_p)$ *for* $1 < p < 2$. Dualizing $(BG_p)$ in the case $2 < p < \infty$, we obtain that if $1 < p < 2$, then for all martingales $x$ in $L^p(\mathcal{M})$,

$$\|x\|_p \approx \|dx\|_{L^p(\mathcal{M};l^2_C)+L^p(\mathcal{M};l^2_R)} .$$

On the other hand, by Theorem 2.3 (already proved), $\mathcal{H}^p(\mathcal{M})$ is complemented in $L^p(\mathcal{M};l^2_C) + L^p(\mathcal{M};l^2_R)$, so the norm of $dx$ in the latter space is equivalent to the norm of $x$ in the former.

Therefore, the proof of Theorems 2.1 and 2.3 is now complete. $\square$

*Remarks.* (i) In Step 3 above, for the proof of the second inequality of $(BG_p)$ we have avoided interpolating the intersection spaces $L^p(\mathcal{M};l^2_C) \cap L^p(\mathcal{M};l^2_R)$ for $p \geq 2$, although it is shown in [P] that they form an interpolation scale for the complex method.
(ii) The constants $\alpha_p$ and $\beta_p$ given by the above proof are not good. In fact, they grow exponentially as $p \to \infty$ (see also Remark 3.2 below).

The inequalities $(BG_p)$ are intimately related to the non-commutative Khintchine inequalities, which played an important rôle in our first approach to $(BG_p)$ for the examples considered in the next section. Let us recall them here for the convenience of the reader. Let $\varepsilon = (\varepsilon_n)_{n\geq 0}$ be a sequence of independent random variables on some probability space $(\Omega, P)$ such that $P(\varepsilon_n = 1) = P(\varepsilon_n = -1) = 1/2$ for all $n \geq 0$.

**Theorem 2.8. (Non-commutative Khintchine inequalities, [LP, LPP]).** *Let* $1 \leq p < \infty$. *Let* $a = (a_n)_{n\geq 0}$ *be a finite sequence in* $L^p(\mathcal{M})$.

(i) *If* $2 \leq p < \infty$,

$$\begin{aligned}\|a\|_{L^p(\mathcal{M};l^2_C)\cap L^p(\mathcal{M};l^2_R)} &\leq \Big(\int\limits_\Omega \|\sum_{n\geq 0}\varepsilon_n a_n\|_p^2 dP(\varepsilon)\Big)^{1/2}\\ &\leq \delta_p\|a\|_{L^p(\mathcal{M};l^2_C)\cap L^p(\mathcal{M};l^2_R)} .\end{aligned}$$

(ii) *If* $1 \leq p < 2$,

$$\begin{aligned}\alpha\|a\|_{L^p(\mathcal{M};l^2_C)+L^p(\mathcal{M};l^2_R)} &\leq \Big(\int\limits_\Omega \|\sum_{n\geq 0}\varepsilon_n a_n\|_p^2 dP(\varepsilon)\Big)^{1/2}\\ &\leq \|a\|_{L^p(\mathcal{M};l^2_C)+L^p(\mathcal{M};l^2_R)} ,\end{aligned}$$

*where* $\alpha > 0$ *is an absolute constant.*

This result was first proved in [LP] for $1 < p < \infty$ for the Schatten classes. The general statement as above (including $p = 1$) is contained in [LPP]. Let us also mention that, as observed in [P] (independently observed by Marius Junge), a combination of the main result in [LPP] with the type 2 estimate from [TJ] yields that $\delta_p$ is of order $\sqrt{p}$ (the best possible) as $p \to \infty$. One should emphasize that for $1 < p < \infty$ the above non-commutative Khintchine inequalities all follow from $(BG_p)$ (with some worse constants, of course). In that special case however, our proof essentially reduces to the original one in [LP].

*Remark 2.9.* (i) Note that, by Theorem 2.8, the unconditionality of martingale differences expressed in $(BG'_p)$ actually implies (hence is equivalent to) $(BG_p)$. Evidently, $(BG_p)$ or $(BG'_p)$ is no longer valid for $p = 1$. However, in this case $p = 1$, the second inequality of $(BG_p)$ remains true (see the corollary in the appendix). Consequently, by the above non-commutative Khintchine inequalities ($p = 1$), we deduce the following substitute for $(BG'_1)$: for any finite martingale $x$ in $L^1(\mathcal{M})$

$$\sup_{\varepsilon_n=\pm 1} \|\sum_n \varepsilon_n dx_n\|_1 \approx \|dx\|_{L^1(\mathcal{M};l^2_C)+L^1(\mathcal{M};l^2_R)} \,.$$

(ii) Clearly, $(BG'_p)$ implies the well-known fact (*cf.* [B1, BGM]) that $L^p(\mathcal{M})$ is a UMD space for all $1 < p < \infty$ (take $q = p$ in (0.3)). In particular if $f = (f_n)_{n\geq 0}$ is a finite *commutative* martingale defined on some probability space with values in $L^p(\mathcal{M})$, then

$$\begin{aligned} &\Big(\int \|\sum_{n\geq 0} \varepsilon_n\big(f_n(\omega) - f_{n-1}(\omega)\big)\|_p^p d\omega\Big)^{1/p} \\ &\leq \beta'_p \sup_{n\geq 0} \Big(\int \|f_n(\omega)\|_p^p d\omega\Big)^{1/p}, \quad \forall\, \varepsilon_n = \pm 1. \end{aligned} \tag{2.7}$$

## 3. Examples

In this section, we give some examples for which the corresponding inequalities $(BG_p)$ can be proved by a different method from the one given in Sect. 2. The key idea of this alternate method is to transfer a non-commutative martingale in $L^p(\mathcal{M})$ to a *commutative* martingale with values in $L^p(\mathcal{M})$. This then enables us to use the unconditionality of *commutative* martingale differences with values in $L^p(\mathcal{M})$. (Recall that $L^p(\mathcal{M})$ is a UMD space; see Remark 2.9 in Sect. 2). Although it does not seem suitable in the general case, this transference approach might be of interest in other situations. This explains why we will give a sketch of this second method in the tensor product case below. Let us also point out that we have first obtained the non-commutative martingale inequalities for these examples, before proving the general Theorem 2.1 (see [PX]).

*I. Tensor products.* Let $(\mathcal{A}_n)$ be a sequence of hyperfinite von Neumann algebras, $\mathcal{A}_n$ being equipped with a normalized faithful trace $\sigma_n$. Let

$$(\mathcal{M}_n, \tau_n) = \bigotimes_{k=0}^{n} (\mathcal{A}_k, \sigma_k) \quad \text{and} \quad (\mathcal{M}, \tau) = \bigotimes_{k=0}^{\infty} (\mathcal{A}_k, \sigma_k)$$

be the tensor products in the sense of von Neumann algebras. Thus we have an increasing filtration $(\mathcal{M}_n)_{n\geq 0}$ of subalgebras of $\mathcal{M}$ which allows us to consider martingales. Let us reformulate Theorem 2.1 in this case as follows.

**Theorem 3.1.** *Let $1 < p < \infty$ and $(\mathcal{M}_n)_{n\geq 0}$ be as above. Then $L^p(\mathcal{M}) = \mathcal{H}^p(\mathcal{M})$ with equivalent norms.*

*Remark.* A special case of Theorem 3.1 is the one where all $\mathcal{A}_n$'s are equal to the algebra of all $2 \times 2$ matrices with its normalized trace. Then $\mathcal{M}$ is the hyperfinite $\mathrm{II}_1$ factor, and $(\mathcal{M}_n)_{n\geq 0}$ is its natural filtration.

*Sketch of the transference proof of Theorem 3.1.* It is not hard to reduce Theorem 3.1 to the case where all $\mathcal{A}_n$'s are finite dimensional and simple. Thus we will consider this special case only. Then let $\Omega_n$ be the unitary group of $\mathcal{A}_n$, equipped with its normalized Haar measure $\mu_n$ (noting that since $\dim\mathcal{A}_n < \infty$, $\Omega_n$ is compact). Set

$$(\Omega, \mu) = \prod_{n\geq 0} (\Omega_n, \mu_n)\,.$$

For $\omega = (\omega_0, \omega_1, \cdots) \in \Omega$, we denote by $\pi_{\omega_n}$ the automorphism of $\mathcal{A}_n$ induced by $\omega_n$, i.e.

$$\pi_{\omega_n}(a) = \omega_n^* a \omega_n\,, \quad \forall\, a \in \mathcal{A}_n,$$

and we let

$$\pi_\omega = \bigotimes_{n\geq 0} \pi_{\omega_n}\,.$$

Then $\pi_\omega$ is an automorphism of $\mathcal{M}$, and extends to an isometry on $L^p(\mathcal{M})$ for all $1 \leq p \leq \infty$.

Now for $a \in L^p(\mathcal{M})$ we define

$$f(a, \omega) = \pi_\omega(a), \qquad \forall\, \omega \in \Omega.$$

Then $f(a,\omega)$ is strongly measurable as a function from $\Omega$ to $L^p(\mathcal{M})$ for every $1 \leq p < \infty$. Let $\Sigma_n$ be the $\sigma$-field on $\Omega$ generated by $(\omega_k)_{k=0}^n$, and $\mathbb{E}_n = \mathbb{E}(\,\cdot\,|\Sigma_n)$ the corresponding conditional expectation. The key point here is the following observation:

$$\mathbb{E}_k f(a,\omega) = f(\mathcal{E}_k(a), \omega) \quad \text{a.e. on } \Omega, \qquad \forall\, k \geq 0,\ \forall\, a \in L^1(\mathcal{M}).$$

(Roughly speaking, the automorphism $\pi_\omega$ intertwines the two conditional expectations $\mathbb{E}_k$ and $\mathcal{E}_k$.) Then let $x$ be a finite $L^p$-martingale (so there is an $n$ such that $x_k = x_n$ for all $k \geq n$). Let $f(\omega) = f(x_n, \omega)$ be the function defined above. Then $(\mathbb{E}_k f)_{k\geq 0}$ is a *commutative* martingale on $\Omega$ with values in $L^p(\mathcal{M})$, and by the above observation,

$$\mathbb{E}_k f - \mathbb{E}_{k-1} f = \pi_\omega(dx_k), \quad a.e.$$

Therefore, since $L^p(\mathcal{M})$ is a UMD space (see [B1, B2, BGM]), with constant $C_p = C_p(L^p(\mathcal{M}))$ in (0.3), we have

$$\int_\Omega \|\sum_{k\geq 0} \varepsilon_k \pi_\omega(dx_k)\|_p^p d\omega \leq (C_p)^p \int_\Omega \|\pi_\omega(x_n)\|_p^p d\omega\,, \quad \forall\, \varepsilon_k = \pm 1.$$

But $\pi_\omega$ is an isometry on $L^p(\mathcal{M})$; hence

$$\|\sum_{k\geq 0} \varepsilon_k dx_k\|_p \leq C_p \|x\|_p\,, \quad \forall\, \varepsilon_k = \pm 1.$$

Thus we obtain the unconditionality of martingale differences in $L^p(\mathcal{M})$, *i.e.* $(BG'_p)$ (defined at the end of Sect. 2) with $\beta'_p \leq C_p$, which, together with the non-commutative Khintchine inequalities, implies easily $(BG_p)$. $\square$

*Remark 3.2.* In this tensor product case (also in the two following) the above transference proof gives better constants $\alpha_p$ and $\beta_p$ in $(BG_p)$ than the general proof in Sect. 2. Indeed, by the argument in [B1-2] or [BGM], one can show that the constant $C_p$ is $O(p^2)$ (resp. $O(1/(p-1)^2)$) as $p \to \infty$ (resp. $p \to 1$). Note that, when $p \geq 2$, the preceding proof yields (in the tensor product case) $\alpha_p \leq C_p$ and $\beta_p \leq C_p\delta_p$, and when $1 < p \leq 2$, $\alpha_p \leq \alpha^{-1}C_p\gamma_p$ and $\beta_p \leq C_p$. Actually, a more careful use of duality yields that for $p \geq 2$, we still have $\beta_p \leq C_p$. Therefore, the preceding sketch of proof yields the following estimates for $\alpha_p$ and $\beta_p$ in $(BG_p)$: $\alpha_p$ and $\beta_p$ are both of order $O(p^2)$ as $p \to \infty$, and respectively of order $O((p-1)^{-6})$ and $O((p-1)^{-2})$ as $p \to 1$.

*II. Clifford algebras.* Our second example concerns Clifford algebras. We take this opportunity to give a brief introduction to von Neumann Clifford algebras and to prepare ourselves for the next section. The reader is referred to [PR, BR, S and C] for more information on this subject.

Let $H$ be a complex Hilbert space with a conjugation $J$. Let $\mathcal{C}(H,J)$ or simply $\mathcal{C}(H)$ denote the von Neumann Clifford algebra associated to the $J$-real subspace of $H$. $\mathcal{C}(H)$ is a finite von Neumann algebra. Let us briefly describe $\mathcal{C}(H)$ via its Fock representation.

Denote by $\Lambda^n(H)$ the $n$-fold antisymmetric product of $H$, equipped with the canonical scalar product:

$$\langle u_1 \wedge \cdots \wedge u_n, v_1 \wedge \cdots \wedge v_n \rangle = \det(\langle u_k, v_j \rangle_{1 \leq k,j \leq n}).$$

$\Lambda^0(H) = \mathbb{C}\mathbb{1}$, where $\mathbb{1}$ is the vacuum vector. The antisymmetric Fock space $\Lambda(H)$ is the direct sum of $\Lambda^n(H)$:

$$\Lambda(H) = \bigoplus_{n \geq 0} \Lambda^n(H).$$

Given any $v \in H$ the associated creator $c(v)$ on $\Lambda(H)$ is linearly defined over antisymmetric tensors by

$$c(v) u_1 \wedge \cdots \wedge u_n = v \wedge u_1 \wedge \cdots \wedge u_n.$$

$c(v)$ is bounded on $\Lambda(H)$ and $\|c(v)\| = \|v\|$. Its adjoint $c(v)^*$ is the annihilator $a(v)$ associated to $v$. The creators and annihilators satisfy the following canonical anticommutation relation (CAR):

$$\{c(u), a(v)\} = \langle u, v \rangle, \qquad \{c(u), c(v)\} = 0, \quad \forall u, v \in H,$$

where $\{S,T\} = ST + TS$ stands for the anticommutator of $S$ and $T$. The Fermion field $\Phi$ is then defined by

$$\Phi(v) = c(v) + a(Jv), \qquad \forall v \in H.$$

$\Phi$ is a linear map from $H$ to $B(\Lambda(H))$. Moreover

$$\{\Phi(u), \Phi(v)\} = 2\langle u, Jv \rangle, \qquad \forall u, v \in H.$$

Therefore, if $u$ and $Jv$ are orthogonal, $\Phi(u)$ and $\Phi(v)$ anticommute. Notice also that $\Phi(v)$ is hermitian for any $J$-real vector $v$ (i.e., $Jv = v$). Then the von Neumann Clifford algebra $\mathcal{C}(H)$ is exactly the subalgebra of $B(\Lambda(H))$ generated by $\{\Phi(v)\colon\ v \in H\}$. Observe that if $\{e_i\colon\ i \in I\}$ is a $J$-real orthonormal basis of $H$, $\{\Phi(e_i)\colon\ i \in I\}$ is a family of anticommuting hermitian unitaries, and it generates $\mathcal{C}(H)$.

The vector state on $B(\Lambda(H))$, given by the vacuum $\mathbb{1}$, induces a trace $\tau$ on $\mathcal{C}(H)$: $\tau(x) = \langle x(\mathbb{1}), \mathbb{1} \rangle$ for any $x \in \mathcal{C}(H)$. Let $L^p(\mathcal{C}(H))$ denote the associated noncommutative $L^p$-space.

If $K$ is a $J$-invariant closed subspace of $H$, $\mathcal{C}(K)$ is naturally identified as a subalgebra of $\mathcal{C}(H)$. Now let $(H_n)_{n\geq 0}$ be an increasing sequence of $J$-invariant closed subspaces of $H$ such that $\overline{\bigcup_{n\geq 0} H_n} = H$. Then the corresponding von Neumann Clifford algebras $(\mathcal{C}(H_n))_{n\geq 0}$ form a filtration of von Neumann subalgebras of $\mathcal{C}(H)$. We will call a non-commutative martingale with respect to $(\mathcal{C}(H_n))_{n\geq 0}$ a Clifford martingale. Therefore, by Theorem 2.1, we have inequalities $(BG_p)$ for Clifford martingales. In fact, this Clifford martingale case can be easily reduced to Theorem 3.1 (the tensor product case) with the help of the classical Jordan-Wigner transformation.

Let us consider only a special case for Clifford martingales, where dim $H_n = n$ for all $n \geq 0$. Fix a $J$-real orthonormal basis $(e_n)_{n\geq 1}$ of $H$ such that $e_n \in H_n \ominus H_{n-1}$ for all $n \geq 1$. Then $\mathcal{C}_n = \mathcal{C}(H_n)$ is the $C^*$-algebra generated by $\{\Phi(e_k)\}_{k=1}^n$ and of dimension $2^n$. For convenience we set $e_0 = 1$ and $e_{-1} = 0$. Let $x = (x_n)_{n\geq 0}$ be a Clifford $L^p$-martingale. Then $dx_n$ can be written as

$$dx_n = \varphi_n(e_1, \dots, e_{n-1})\Phi(e_n),$$

where $\varphi_n = \varphi(e_1, \dots, e_{n-1})$ belongs to $L^p(\mathcal{C}_{n-1})$. Let $\varphi = (\varphi_n)_{n\geq 0}$ and $\mathcal{C} = \mathcal{C}(H)$.

**Proposition 3.3.** *Let $1 \leq p \leq \infty$ and $x = (x_n)_{n\geq 0}$ be a bounded Clifford $L^p$-martingale as above. Then $\|dx\|_{L^p(\mathcal{C};l^2_R)} = \|\varphi\|_{L^p(\mathcal{C};l^2_R)}$ and*

$$\frac{1}{2}\|\varphi\|_{L^p(\mathcal{C};l^2_C)} \leq \|dx\|_{L^p(\mathcal{C};l^2_C)} \leq 2\|\varphi\|_{L^p(\mathcal{C};l^2_C)}.$$

*Proof.* Since $\Phi(e_n)$ is unitary (and hermitian), we have $\|dx\|_{L^p(\mathcal{C};l^2_R)} = \|\varphi\|_{L^p(\mathcal{C};l^2_R)}$. To prove the inequalities on $L^p(\mathcal{C}; l^2_C)$ we need the grading automorphism (or parity) $G$ of $\mathcal{C}$: $G$ is uniquely determined by

$$G\big(\Phi(v_1)\dots\Phi(v_n)\big) = \Phi(-v_1)\dots\Phi(-v_n), \quad \forall\, v_k \in H, 0 \leq k \leq n.$$

This means that $G$ is the automorphism induced by minus the identity of $H$. Recall that $a \in L^p(\mathcal{C})$ is called even (resp. odd) if $G(a) = a$ (resp. $G(a) = -a$). We have the decomposition $L^p(\mathcal{C}) = L^p(\mathcal{C}^+) \oplus L^p(\mathcal{C}^-)$ into even and odd parts; more precisely for any $a \in L^p(\mathcal{C})$

$$a = \frac{a + G(a)}{2} + \frac{a - G(a)}{2} = a^+ + a^-.$$

Since $G$ is isometric on $L^p(\mathcal{C})$,

$$\max(\|a^+\|_p, \|a^-\|_p) \leq \|a\|_p \leq \|a^+\|_p + \|a^-\|_p.$$

Now for $x = (x_n)_{n\geq 0}$ as in the proposition we have

$$G(dx_n) = -G(\varphi_n)\Phi(e_n);$$

so $(dx_n)^+ = \varphi_n^-\Phi(e_n)$. Notice that $\varphi_n^- \in L^p(\mathcal{C}_{n-1}^-)$. Then by the anticommutation of $\Phi(e_n)$ with $\Phi(e_k)$ $(1 \leq k \leq n-1)$ we get $\varphi_n^-\Phi(e_n) = -\Phi(e_n)\varphi_n^-$. Therefore $(dx_n)^+ = -\Phi(e_n)\varphi_n^-$; hence, since $\Phi(e_n)$ is unitary,

$$\|(dx_n^+)_{n\geq 0}\|_{L^p(\mathcal{C};l^2_C)} = \|(\varphi_n^-)_{n\geq 0}\|_{L^p(\mathcal{C};l^2_C)}.$$

Similarly,

$$\|(dx_n^-)_{n\geq 0}\|_{L^p(\mathcal{C};l^2_C)} = \|(\varphi_n^+)_{n\geq 0}\|_{L^p(\mathcal{C};l^2_C)}.$$

Combining the preceding inequalities, we get

$$\frac{1}{2}\|\varphi\|_{L^p(\mathcal{C};L^2_C)} \le \|dx\|_{L^p(\mathcal{C};l^2_C)} \le 2\|\varphi\|_{L^p(\mathcal{C};l^2_C)},$$

proving the proposition. $\square$

Let us record explicitly the following consequence of Theorem 2.1 and Proposition 3.3.

**Corollary 3.4.** *Let $1 < p < \infty$ and $x = (x_n)_{n\ge 0}$ be as in Proposition 3.3. Then if $2 \le p < \infty$ we have*

$$\|x\|_{\mathcal{H}^p(\mathcal{C})} \approx \max\{\|\varphi\|_{L^p(\mathcal{C};l^2_C)},\ \|\varphi\|_{L^p(\mathcal{C};l^2_R)}\},$$

*and if $1 < p < 2$ we have*

$$\|x\|_{\mathcal{H}^p(\mathcal{C})} \approx \inf\{\|\varphi'\|_{L^p(\mathcal{C};l^2_C)} + \|\varphi''\|_{L^p(\mathcal{C};l^2_R)}\},$$

*where the infimum runs over all $\varphi' \in L^p(\mathcal{C}; l^2_C)$, $\varphi'' \in L^p(\mathcal{C}; l^2_R)$ such that $\varphi = \varphi' + \varphi''$ and $\varphi'_n, \varphi''_n \in L^p(\mathcal{C}_{n-1})$ for all $n \ge 0$.*

*III. Free group algebras.* Let $\mathbb{F}_n$ be the free group of $n$ generators. Let $vN(\mathbb{F}_n)$ be the von Neumann algebra of $\mathbb{F}_n$, equipped with its standard normalized trace $\tau$. $vN(\mathbb{F}_n)$ is naturally identified as a subalgebra of $vN(\mathbb{F}_{n+1})$, so that $\big(vN(\mathbb{F}_n)\big)_{n\ge 1}$ is an increasing filtration of von Neumann subalgebras of $vN(\mathbb{F}_\infty)$, which generate $vN(\mathbb{F}_\infty)$. For convenience, we put $vN(\mathbb{F}_0) = \mathbb{C}\mathbb{1}$. Thus we can consider martingales with respect to $\big(vN(\mathbb{F}_n)\big)_{n\ge 0}$. Let $\mathcal{H}^p\big(vN(\mathbb{F}_\infty)\big)$ denote the corresponding Hardy space. Then Theorem 2.1 gives

**Theorem 3.5.** *Let $1 < p < \infty$. Then*

$$\mathcal{H}^p\big(vN(\mathbb{F}_\infty)\big) = L^p\big(vN(\mathbb{F}_\infty)\big) \quad \textit{with equivalent norms}.$$

Let us emphasize that, a priori, the above situation is quite different from the one considered in the tensor product case, since $vN(\mathbb{F}_n)$ is not hyperfinite as soon as $n \ge 2$. However, Theorem 3.5 also admits an alternate proof, which appears as a limit case of the tensor product case: indeed, as Philippe Biane kindly pointed out to us, this can be done via random matrices with the help of Voiculescu's limit theorem [V]. We omit the details. Note that again this argument yields better constants when $p$ tends to infinity, the same ones as indicated in Remark 3.2.

## 4. Applications to the Ito-Clifford Integral

In this section $H$ denotes $L^2(\mathbb{R}_+)$ with its usual Lebesgue measure and complex conjugation; $\mathcal{C} = \mathcal{C}(H)$ is the associated von Neumann Clifford algebra equipped with its normalized trace $\tau$. For $t \ge 0$ let $H_t$ denote the subspace $L^2(0,t)$ and $\mathcal{C}_t = \mathcal{C}(H_t)$. Clearly, $\mathcal{C}_0 = \mathbb{C}$ and $\mathcal{C}_s \subset \mathcal{C}_t$ for $0 \le s \le t$. Let $\mathcal{E}_t = \mathcal{E}(\cdot \mid \mathcal{C}_t)$ be the conditional expectation of $\mathcal{C}$ with respect to $\mathcal{C}_t$. Thus we have a continuous time filtration of von Neumann subalgebras $(\mathcal{C}_t)_{t\ge 0}$ of $\mathcal{C}$, which generate $\mathcal{C}$. All the notions for discrete martingales in Sect. 1 can be transferred to this continuous time setting. Thus a Clifford $L^p$-martingale is a family $X = (X_t)_{t\ge 0}$ such that $X_t \in L^p(\mathcal{C}_t)$ and $\mathcal{E}_s X_t = X_s$ for $0 \le s \le t$; if additionally $\|X\|_p = \sup_{t\geq 0} \|X_t\|_p < \infty$, $X$ is said to be bounded. In this section, unless otherwise stated all martingales are Clifford martingales with respect to $(\mathcal{C}_t)_{t\geq 0}$. The main result here is the analogue of Theorem 2.1 for these Clifford martingales. We will deduce it from Theorem 2.1 by discretizing continuous time Clifford martingales. This reduction from continuous time to discrete time will be done via the Ito-Clifford integral developed by Barnett, Streater and Wilde, who had extended the classical Ito integral theory to Clifford $L^2$-martingales. They showed that any Clifford $L^2$-martingale admits an Ito-Clifford integral representation. The Clifford martingale inequalities below will allow us to extend this Ito-Clifford integral theory from $L^2$-martingales to $L^p$-martingales for any $1 < p < \infty$. As a consequence, we will show that any Clifford $L^p$-martingale $(1 < p < 2)$ has an Ito-Clifford integral representation.

Let us first recall the Ito-Clifford integral defined in [BSW1-2]. For given $t \geq 0$, let $\Phi_t = \Phi(\chi_{[0,t)})$ (recalling that $\Phi$ is the Fermion field defined in Sect. 3). Then $\Phi_t$ is hermitian and belongs to $\mathcal{C}_t$; by the canonical anticommutation relations, $(\Phi_t - \Phi_s)^2 = t - s$ for $0 \leq s \leq t$. $\Phi_t$ is the Fermion analogue of Brownian motion.

Like in the classical Ito integral, Barnett, Streater and Wilde develop their Ito-Clifford integral by first defining the integrals of simple processes. A simple adapted $L^p$-process is a function $f\colon \mathbb{R}_+ \to L^p(\mathcal{C})$ such that $f(t) \in L^p(\mathcal{C}_t)$ for $t \geq 0$ and

$$f(t) = \sum_{k\geq 0} f(t_k)\chi_{[t_k,t_{k+1})}(t),$$

where $(t_k)_{k\geq 0}$ is a subdivision of $\mathbb{R}_+$, i.e., $0 = t_0 < t_1 < \cdots$ increasing to $+\infty$. For such an $f$ we define its Ito-Clifford integral as follows: for $t_k \leq t < t_{k+1}$,

$$X_t = \int_0^t f(s)d\Phi_s = \sum_{j=0}^{k-1} f(t_j)(\Phi_{t_{j+1}} - \Phi_{t_j}) + f(t_k)(\Phi_t - \Phi_{t_k}).$$

Clearly, $X = (X_t)_{t\geq 0}$ is a Clifford $L^p$-martingale; and if $p = 2$,

$$\|X_t\|_2^2 = \int_0^t \|f(s)\|_2^2 ds, \qquad \forall t \geq 0.$$

This identity allows one to define the Ito-Clifford integral of any "adapted $L^2$-process" $f$ belonging to $L^2_{\mathrm{loc}}(\mathbb{R}_+; L^2(\mathcal{C}))$:

$$X_t = \int_0^t f(s)d\Phi_s, \qquad \forall t \geq 0.$$

$(X_t)_{t\geq 0}$ is again a Clifford $L^2$-martingale and the above identity still holds. Conversely, any Clifford $L^2$-martingale admits such an Ito-Clifford integral representation (cf. [BSW1]).

As in the discrete case, for any simple adapted process $f$ we define

$$S_{C,t}(f) = \Big[\int_0^t f^*(s)f(s)ds\Big]^{1/2} \quad \text{and} \quad S_{R,t}(f) = \Big[\int_0^t f(s)f^*(s)ds\Big]^{1/2}.$$

Let $\mathcal{S}^p_{ad}$ be the linear space of all simple adapted $L^p$-processes and $\mathcal{S}^p_{ad}[0,t]$ its subspace of processes vanishing in $(t,\infty)$. Then like in the case of discrete time, $\|S_{C,t}(f)\|_p$ and $\|S_{R,t}(f)\|_p$ define two norms on $\mathcal{S}^p_{ad}[0,t]$. The completions of $\mathcal{S}^p_{ad}[0,t]$ with respect to them are denoted respectively by $\mathcal{H}^p_C[0,t]$ and $\mathcal{H}^p_R[0,t]$ for $1 \le p < \infty$. Let us point out that elements in $\mathcal{H}^p_C[0,t]$ and $\mathcal{H}^p_R[0,t]$ can be regarded as measurable operators in $L^p(\mathcal{C}_t \otimes B(L^2[0,t]))$ (see Sect. 1 about the column and row subspaces). Let $\mathcal{H}^p_{C,\mathrm{loc}}(\mathbb{R}_+)$ (resp. $\mathcal{H}^p_{R,\mathrm{loc}}(\mathbb{R}_+)$) denote the space of all functions $f\colon \mathbb{R}_+ \to L^p(\mathcal{C})$ whose restrictions to $[0,t]$ belong to $\mathcal{H}^p_C[0,t]$ (resp. $\mathcal{H}^p_R[0,t]$) for all $t \ge 0$. We call elements in $\mathcal{H}^p_{C,\mathrm{loc}}(\mathbb{R}_+)$ and $\mathcal{H}^p_{R,\mathrm{loc}}(\mathbb{R}_+)$ (measurable) adapted $L^p$-processes. As in the discrete case, we define

$$\mathcal{H}^p[0,t] = \mathcal{H}^p_C[0,t] + \mathcal{H}^p_R[0,t] \quad \text{for} \quad 1 \le p < 2,$$

and

$$\mathcal{H}^p[0,t] = \mathcal{H}^p_C[0,t] \cap \mathcal{H}^p_R[0,t] \quad \text{for} \quad 2 \le p < \infty.$$

We endow $\mathcal{H}^p[0,t]$ with the corresponding sum or intersection norm. Similarly, we define $\mathcal{H}^p_{\mathrm{loc}}(\mathbb{R}_+)$.

Now we can state the main result of this section.

**Theorem 4.1.** *Let $1 < p < \infty$. Then for any $f \in \mathcal{H}^p_{\mathrm{loc}}(\mathbb{R}_+)$ its Ito-Clifford integral*

$$X_t = \int_0^t f(s)d\Phi_s, \qquad t \ge 0$$

*is a well-defined Clifford $L^p$-martingale and*

$$\alpha_p^{-1}\|f\|_{\mathcal{H}^p[0,t]} \le \|X_t\|_p \le \beta_p\|f\|_{\mathcal{H}^p[0,t]}, \qquad \forall\ t \ge 0.$$

*Remarks.* (i) Carlen and Krée [CK] proved that if $p \le 2$ and if $f$ is a simple adapted process, then the Ito-Clifford integral $(X_t)$ of $f$ satisfies

$$\|X_t\|_p \le \beta_p \min\Big\{\Big\|\big[\int_0^t |f(s)|^2\,ds\big]^{1/2}\Big\|_p,\ \Big\|\big[\int_0^t |f(s)^*|^2\,ds\big]^{1/2}\Big\|_p\Big\}.$$

(This corresponds essentially to the second inequality of Theorem 4.1 for $p \le 2$.) From this they deduced some sufficient conditions for the existence of Ito-Clifford integrals. They also proved Theorem 4.1 for $p = 4$ (and mentioned that the same argument works for $p = 6$ and 8).

(ii) If $2 \le p < \infty$, then

$$\mathcal{H}^p_{\mathrm{loc}}(\mathbb{R}_+) \subset L^2_{\mathrm{loc}}(\mathbb{R}_+; L^2(\mathcal{C}));$$

so adapted $L^p$-processes are adapted $L^2$-processes. Thus the existence of Ito-Clifford integrals of adapted $L^p$-processes ($p \ge 2$) goes back to [BSW1]. Note also that in the case $p = 2$ the inequalities in Theorem 4.1 become equalities (i.e., $\alpha_2 = \beta_2 = 1$). This is the only case already treated in [BSW1]. If $f$ is an adapted $L^1$-process, then its Ito-Clifford integral is also a well-defined Clifford $L^1$-martingale $X = (X_t)_{t\ge 0}$ and we have

$$\|X_t\|_1 \le \beta_1\|f\|_{\mathcal{H}^1[0,t]}, \qquad \forall t \ge 0$$

(see the corollary in the appendix and Remark 2.9). Of course, the reverse inequality fails this time.

We will reduce Theorem 4.1 to simple adapted processes and then apply Theorem 2.1. For this reduction to be successful we have to check two things. The first one is the density of $\mathcal{S}^p_{ad}[0,t]$ in $\mathcal{H}^p[0,t]$ (this is trivial for $1 \le p \le 2$). The second one is that the norm of a simple adapted $L^p$-process $f$ in $\mathcal{H}^p[0,t]$ for $1 < p < 2$ is equivalent to

$$\inf\{\|g\|_{\mathcal{H}^p_C[0,t]} + \|h\|_{\mathcal{H}^p_R[0,t]}\colon\ f = g + h, \quad g, h \in \mathcal{S}^p_{ad}\}.$$

These will be done by the following lemmas.

**Lemma 4.2.** *Let $\sigma = (t_k)_{k=0}^{\infty}$ be a subdivision of $\mathbb{R}_+$. Define the map $Q_\sigma$ over simple adapted processes by*

$$Q_\sigma(f)(t) = \frac{1}{t_{k+1} - t_k}\int_{t_k}^{t_{k+1}} \mathcal{E}_{t_k} f(s)ds, \qquad t_k \le t < t_{k+1}, \quad t \ge 0.$$

*Then for $1 < p < \infty$, $Q_\sigma$ extends to a bounded projection on $\mathcal{H}^p_C[0,t]$ and $\mathcal{H}^p_R[0,t]$ for all $t \ge 0$.*

*Proof.* Suppose $f$ is a simple adapted $L^p$-process:

$$f = \sum_{j\ge 0} f(s_j)\chi_{[s_j, s_{j+1})}.$$

By refining the subdivision $(s_j)_{j\ge 0}$ if necessary we may assume it is finer than $\sigma$. Then

$$Q_\sigma f = \sum_{k\ge 0}\Big[\sum_{j:t_k\le s_j<t_{k+1}} \theta_{k,j}\mathcal{E}_{t_k} f(s_j)\Big]\chi_{[t_k, t_{k+1})},$$

where

$$\theta_{k,j} = \frac{s_{j+1} - s_j}{t_{k+1} - t_k} \quad \text{for} \quad t_k \le s_j < t_{k+1}.$$

Note that

$$\sum_{j:t_k\le s_j<t_{k+1}} \theta_{k,j} = 1, \qquad \forall\ k \ge 0.$$

Observe also the following elementary and well known inequality: for any sequence of operators $(a_j)$ in $B(H)$ ($H$ being a Hilbert space) and for any finitely supported sequence $(\theta_j)$ with $\theta_j \ge 0$ and $\sum\theta_j = 1$, we have (in the order of $B(H)$)

$$|\sum \theta_j a_j|^2 \le \sum \theta_j |a_j|^2.$$

(Indeed, for any $h$ in $H$, by convexity of $\|\cdot\|^2$, we have $\|\sum\theta_j a_j h\|^2 \le \sum\theta_j\|a_j h\|^2$, whence the desired inequality.) Therefore, for all $k \ge 0$,

$$|\sum_{j:t_k\le s_j<t_{k+1}} \theta_{k,j}\mathcal{E}_{t_k} f(s_j)|^2 \le \sum_{j:t_k\le s_j<t_{k+1}} \theta_{k,j}|\mathcal{E}_{t_k}(f(s_j))|^2.$$

Now let $t \ge 0$. Without loss of generality we assume $t = t_{n+1}$ for some $n \ge 0$. Then by Theorem 2.3,

$$\begin{aligned}\|Q_\sigma f\|_{\mathcal{H}_C^p[0,t]} &= \Big\| \Big[ \int_0^t (Q_\sigma f(s))^*(Q_\sigma f(s))ds \Big]^{1/2} \Big\|_p \\ &\le \Big\| \Big[ \sum_{k=0}^n \sum_{t_k \le s_j < t_{k+1}} (s_{j+1} - s_j)|\mathcal{E}_{t_k} f(s_j)|^2 \Big]^{1/2} \Big\|_p \\ &\le \beta_p \Big\| \Big[ \sum_{k=0}^n \sum_{t_k \le s_j < t_{k+1}} (s_{j+1} - s_j)|f(s_j)|^2 \Big]^{1/2} \Big\|_p \\ &= \beta_p \|f\|_{\mathcal{H}_C^p[0,t]}.\end{aligned}$$

Therefore $Q_\sigma$ extends to a bounded map (projection) on $\mathcal{H}_C^p[0,t]$. The same reasoning applies to $\mathcal{H}_R^p[0,t]$. $\square$

**Lemma 4.3.** *Let $1 < p < \infty$ and $f \in \mathcal{H}_{C,\mathrm{loc}}^p(\mathbb{R}_+)$ (resp. $\mathcal{H}_{R,\mathrm{loc}}^p(\mathbb{R}_+)$). Then for all $t \ge 0$,*

$$\lim_\sigma Q_\sigma f = f \quad \text{in} \quad \mathcal{H}_C^p[0,t] \quad (\text{resp. } \mathcal{H}_R^p[0,t]),$$

*where the limit is taken relative to the subdivision $\sigma = (t_k)_{k\ge 0}$ when $\sup_{k\ge 0}(t_{k+1} - t_k)$ goes to zero.*

*Proof.* If $f \in \mathcal{S}_{ad}^p$, then $Q_\sigma f = f$ when $\sigma$ is sufficiently fine; so the lemma is true for simple adapted processes. The general case is proved by Lemma 4.2 and the density of $\mathcal{S}_{ad}^p[0,t]$ in $\mathcal{H}_C^p[0,t]$ and $\mathcal{H}_R^p[0,t]$. $\square$

**Lemma 4.4.** *Let $1 \le p < \infty$. Then $\mathcal{S}_{ad}^p[0,t]$ is dense in $\mathcal{H}^p[0,t]$ for all $t \ge 0$.*

*Proof.* This is trivial for $1 \le p < 2$ because $\mathcal{H}^p[0,t] = \mathcal{H}_C^p[0,t] + \mathcal{H}_R^p[0,t]$ in this case. For $2 \le p < \infty$ and $f \in \mathcal{H}^p[0,t]$ Lemma 4.3 implies that

$$\lim_\sigma Q_\sigma f = f \quad \text{in} \quad \mathcal{H}^p[0,t].$$

Thus $\mathcal{S}_{ad}^p[0,t]$ is also dense in $\mathcal{H}^p[0,t]$. $\square$

**Lemma 4.5.** *Let $1 < p < \infty$ and $f \in \mathcal{S}_{ad}^p$. Then for all $t \ge 0$,*

$$\|f\|_{\mathcal{H}_C^p[0,t]+\mathcal{H}_R^p[0,t]} \approx \inf\{\|g\|_{\mathcal{H}_C^p[0,t]} + \|h\|_{\mathcal{H}_R^p[0,t]}\},$$

*where the infimum is taken over all $g, h \in \mathcal{S}_{ad}^p[0,t]$ such that $f = g + h$.*

*Proof.* Let $f$ be a simple adapted $L^p$-process defined by a subdivision $\sigma = (t_k)_{k\ge 0}$:

$$f = \sum_{k\ge 0} f(t_k)\chi_{[t_k,t_{k+1})}.$$

Let $g \in \mathcal{H}_C^p[0,t]$, $h \in \mathcal{H}_R^p[0,t]$ such that $f = g + h$ and

$$\|g\|_{\mathcal{H}_C^p[0,t]} + \|h\|_{\mathcal{H}_R^p[0,t]} \le 2\|f\|_{\mathcal{H}_C^p[0,t]+\mathcal{H}_R^p[0,t]}.$$

Then $f = Q_\sigma f = Q_\sigma g + Q_\sigma h$, and $Q_\sigma g$, $Q_\sigma h \in \mathcal{S}_{ad}^p$. By Lemma 4.2

$$\begin{aligned}\|Q_\sigma g\|_{\mathcal{H}_C^p[0,t]} &\le \beta_p \|g\|_{H_C^p[0,t]}, \\ \|Q_\sigma h\|_{H_R^p[0,t]} &\le \beta_p \|h\|_{\mathcal{H}_C^p[0,t]};\end{aligned}$$

whence the equivalence in the lemma. $\square$

Now we are ready to show Theorem 4.1.

*Proof of Theorem 4.1.* First consider the case $2 \le p < \infty$. Let $f \in \mathcal{S}^p_{ad}$:

$$f = \sum_{k\ge 0} f(t_k)\chi_{[t_k,t_{k+1})}.$$

Then (assuming $t = t_n$)

$$X_t = \sum_{k=0}^{n-1} f(t_k)[\Phi(t_{k+1}) - \Phi(t_k)].$$

Thus $(X_{t_k})_{k=0}^n$ is a finite Clifford $L^p$-martingale with respect to $(\mathcal{C}(H_{t_k}))_{k=0}^n$. Set

$$d_k = X_{t_{k+1}} - X_{t_k} = f(t_k)[\Phi(t_{k+1}) - \Phi(t_k)].$$

Then by Theorem 2.1,

$$\|X_t\|_p \approx \|\big[\sum_{k=0}^{n-1}|d_k|^2\big]^{1/2}\|_p + \|\big[\sum_{k=0}^{n-1}|d_k^*|^2\big]^{1/2}\|_p.$$

Since $\Phi(t_{k+1}) - \Phi(t_k)$ is hermitian and

$$[\Phi(t_{k+1}) - \Phi(t_k)]^2 = t_{k+1} - t_k,$$

we have

$$\begin{aligned}\sum_{k=0}^{n-1}|d_k^*|^2 &= \sum_{k=0}^{n-1} f(t_k)f(t_k)^*(t_{k+1} - t_k)\\ &= \int_0^t f(s)f(s)^*ds.\end{aligned}$$

On the other hand, since $\chi_{[t_k,t_{k+1})}$ is orthogonal to $L^2(0,t_k)$ and since $f(t_k) \in \mathcal{C}_{t_k}$, by Proposition 3.3 and its proof

$$\begin{aligned}\Big\|\big[\sum_{k=0}^{n-1}|d_k|^2\big]^{1/2}\Big\|_p &\approx \Big\|\big[\sum_{k=0}^{n-1} f(t_k)^*f(t_k)(t_{k+1} - t_k)\big]^{1/2}\Big\|_p\\ &= \Big\|\big[\int_0^t f(s)^*f(s)ds\big]^{1/2}\Big\|_p\\ &= \|f\|_{\mathcal{H}^p_C[0,t]}\,.\end{aligned}$$

Therefore, we finally deduce that

$$\|X_t\|_p \approx \max(\|f\|_{\mathcal{H}^p_C[0,t]}, \|f\|_{\mathcal{H}^p_R[0,t]})\,,$$

proving Theorem 4.1 in the case $2 \le p < \infty$ for simple adapted $L^p$-processes. The general adapted $L^p$-processes are treated by approximation by means of Lemma 4.4.

Now suppose $1 < p < 2$ and $f \in \mathcal{S}^p_{ad}$. Write

$$f = \sum_{k\ge 0} f(t_k)\chi_{[t_k,t_{k+1})}.$$

Since step functions are dense in $L^2[0, t_k]$, by refining $(t_k)_{k\geq 0}$ if necessary we may assume $f(t_k)$ belongs to the von Neumann algebra generated by $\{\Phi(t_{j+1}) - \Phi(t_j)\}_{j=0}^{k-1}$. Let $L_k$ denote the subspace of $H_{t_k}$ spanned by $\{\chi_{[t_j,t_{j+1})}\}_{j=0}^{k-1}$. Then $\dim L_k = k$ and $f(t_k) \in \mathcal{C}(L_k)$. Let $t = t_n$ for some $n \geq 0$. Then

$$X_t = \sum_{k=0}^{n-1} f(t_k)[\Phi(t_{k+1}) - \Phi(t_k)].$$

Thus $(X_{t_k})_{k=1}^n$ is a finite Clifford martingale relative to $(\mathcal{C}(L_k))_{k=1}^n$. Applying Corollary 3.4 to $(X_{t_k})_{k=1}^n$ we get

$$\|X_t\|_p \approx \inf\left\{ \|\big[\sum_{k=0}^{n-1} |a_k|^2(t_{k+1} - t_k)\big]^{1/2}\|_p + \|\big[\sum_{k=0}^{n-1} |b_k^*|^2(t_{k+1} - t_k)\big]^{1/2}\|_p \right\},$$

where the infimum runs over all $(a_k)$ and $(b_k)$ such that $a_k + b_k = f(t_k)$ and $a_k, b_k \in \mathcal{C}(L_k)$ for all $0 \leq k \leq n-1$. Let us show that the last infimum is equivalent to $\|f\|_{\mathcal{H}^p[0,t]}$. By Lemma 4.5 (recall that $f \in \mathcal{S}_{ad}^p$) there are $g, h \in \mathcal{S}_{ad}^p$ such that

$$\|g\|_{\mathcal{H}_C^p[0,t]} + \|h\|_{\mathcal{H}_R^p[0,t]} \leq \beta_p \|f\|_{\mathcal{H}^p[0,t]};$$

moreover, we may assume that $g$ and $h$ are given by the same subdivision as $f$. Therefore

$$\|g\|_{\mathcal{H}_C^p[0,t]} = \|\big[\sum_{k=0}^{n-1} |g(t_k)|^2(t_{k+1} - t_k)\big]^{1/2}\|_p.$$

Applying Theorem 2.3 to the sequence of conditional expectations $\{\mathcal{E}\big(\cdot \mid \mathcal{C}(L_k)\big)\}_{k=1}^n$, we deduce that

$$\|\big[\sum_{k=0}^{n-1} |\mathcal{E}\big(g(t_k)|\mathcal{C}(L_k)\big)|^2(t_{k+1} - t_k)\big]^{1/2}\|_p \leq \beta_p \|g\|_{\mathcal{H}_C^p[0,t]}.$$

The same inequality holds for $h$ and $\mathcal{H}_R^p[0,t]$ in place of $g$ and $\mathcal{H}_C^p[0,t]$. Since $f(t_k) \in \mathcal{C}(L_k)$,

$$f = \sum_{k=0}^{n-1} \Big[\mathcal{E}\big(g(t_k)|\mathcal{C}(L_k)\big) + \mathcal{E}\big(h(t_k)|\mathcal{C}(L_k)\big)\Big]\chi_{[t_k,t_{k+1})}.$$

Set $a_k = \mathcal{E}\big(g(t_k)|\mathcal{C}(L_k)\big)$ and $b_k = \mathcal{E}\big(h(t_k)|\mathcal{C}(L_k)\big)$ for $0 \leq k \leq n-1$. Then $f(t_k) = a_k + b_k$ and

$$\|\big[\sum_{k=0}^{n-1} |a_k|^2(t_{k+1} - t_k)\big]^{1/2}\|_p \leq \beta_p \|g\|_{\mathcal{H}_C^p[0,t]},$$

$$\|\big[\sum_{k=0}^{n-1} |b_k^*|^2(t_{k+1} - t_k)\big]^{1/2}\|_p \leq \beta_p \|h\|_{\mathcal{H}_R^p[0,t]}.$$

Thus the desired equivalence follows, and so

$$\|X_t\|_p \approx \|f\|_{\mathcal{H}^p[0,t]}.$$

Therefore, the inequalities of Theorem 4.1 in the case $1 < p < 2$ have been proved for simple adapted processes. Now let $f \in \mathcal{H}^p_{\text{loc}}(\mathbb{R}_+)$ ($1 < p < 2$). Let $f_n \in \mathcal{S}^p_{ad}[0,t]$ converge to $f$ in $\mathcal{H}^p[0,t]$. Set

$$X^n_t = \int_0^t f_n(s)ds.$$

Then

$$\|X^n_t - X^m_t\|_p \approx \|f_n - f_m\|_{\mathcal{H}^p[0,t]}.$$

Therefore $X^n_t$ converges to some $X_t$ as $n \to \infty$. It is clear that $(X_t)_{t\geq 0}$ is a Clifford $L^p$-martingale and

$$\|X_t\|_p \approx \|f\|_{\mathcal{H}^p[0,t]}, \qquad \forall t \geq 0.$$

Also $(X_t)_{t\geq 0}$ is uniquely determined by $f$. Then we define the Ito-Clifford integral of $f$ to be $(X_t)_{t\geq 0}$. Hence the proof of Theorem 4.1 is complete. $\square$

As a consequence of Theorem 4.1 we get the following Ito-Clifford integral representation for Clifford $L^p$-martingales ($1 < p < \infty$), which extends to any $p \in (1,\infty)$ the Barnett-Streater-Wilde representation theorem for $L^2$-martingales.

**Theorem 4.6.** *Let $1 < p < \infty$. Then for any Clifford $L^p$-martingale $(X_t)_{t\geq 0}$ there exists an adapted $L^p$-process $f \in \mathcal{H}^p_{\text{loc}}(\mathbb{R}_+)$ such that*

$$X_t = X_0 + \int_0^t f(s)d\Phi_s, \qquad \forall t \geq 0.$$

*Proof.* Let $(X_t)_{t\geq 0}$ be a Clifford $L^p$-martingale. Without loss of generality assume $X_0 = 0$. It suffices to construct the required adapted process over any interval $[0,T]$. Thus fix $T > 0$. For any subdivision $\sigma$ of $[0,T]$: $0 = t_0 < \cdots < t_n = T$, let $L_\sigma$ denote the subspace of $H_T = L^2[0,T]$ spanned by $\{\chi_{[t_k,t_{k+1})}\}_{k=0}^{n-1}$. Since the union of all $L_\sigma$ is dense in $H_T$, $\mathcal{C}_T = \mathcal{C}(H_T)$ is generated by the union of all Clifford algebras $\mathcal{C}(L_\sigma)$. It follows that $\bigcup_\sigma \mathcal{C}(H_\sigma)$ is dense in $L^p(\mathcal{C}_T)$. Therefore there exists a sequence $(X^n_T)_{n\geq 0}$ of $L^p(\mathcal{C}_T)$ such that $\lim_{n\to\infty} X^n_T = X_T$ in $L^p(\mathcal{C}_T)$ and such that $X^n_T \in \mathcal{C}(H_{\sigma_n})$ for some subdivision $\sigma_n$ of $[0,T]$. Let $\sigma_n = (t^n_k)_{k=0}^{N_n}$. Then $X^n_T$ can be written as

$$X^n_T = \sum_{k=0}^{N_n-1} a_{n,k}[\Phi(t^n_{k+1}) - \Phi(t^n_k)],$$

where $a_{n,k}$ belongs to the $C^*$-algebra generated by $\{\Phi(t^n_{j+1}) - \Phi(t^n_j)\}_{j=0}^{k-1}$ for all $0 \leq k \leq N_n$ and $n \geq 0$. Put

$$f_n = \sum_{k=0}^{N_n-1} a_{n,k}\chi_{[t^n_k,t^n_{k+1})}.$$

Then $f_n$ is a simple adapted $L^p$-process and

$$X^n_T = \int_0^T f_n(s)ds.$$

Therefore, by Theorem 4.1,

$$\|X_T^n - X_T^m\|_p \approx \|f_n - f_m\|_{\mathcal{H}^p[0,T]},$$

whence $(f_n)_{n\geq 0}$ is a Cauchy sequence in $\mathcal{H}^p[0,T]$, so it converges to some adapted $L^p$-process $f \in \mathcal{H}^p[0,T]$. Then clearly

$$X_T = \int_0^T f(s)ds.$$

This finishes the proof of Theorem 4.6. $\square$

*Remark.* If we identify a Clifford $L^p$-martingale with the integrand (adapted $L^p$-process) in its Ito-Clifford integral representation (this is always possible by Theorem 4.6), then Theorem 4.1 can be reformulated as follows: for any $1 < p < \infty$ and any $t \geq 0$,

$$L_0^p(\mathcal{C}_t) = \mathcal{H}^p[0,t] \quad \text{with equivalent norms},$$

where

$$L_0^p(\mathcal{C}_t) = \{X \in L^p(\mathcal{C}_t) : \tau(X) = 0\}.$$

This equivalence can be extended to the whole of $\mathbb{R}_+$. Let us say that an adapted $L^p$-process $f$ belongs to $\mathcal{H}^p(\mathbb{R}_+)$ if

$$\|f\|_{\mathcal{H}^p(\mathbb{R}_+)} = \sup_{t\geq 0} \|f\|_{\mathcal{H}^p[0,t]} < \infty.$$

Then for $1 < p < \infty$ a Clifford $L^p$-martingale $X = (X_t)_{t\geq 0}$ is bounded iff the associated adapted $L^p$-process $f$ belongs to $\mathcal{H}^p(\mathbb{R}_+)$; moreover, in this case we have

$$\|X\|_p = \sup_{t\geq 0} \|X_t\|_p \approx \|X_0\|_p + \|f\|_{\mathcal{H}^p(\mathbb{R}_+)}.$$

Recall also that $X = (X_t)_{t\geq 0}$ is bounded iff $\lim_{t\to\infty} X_t = X_\infty$ exists in $L^p(\mathcal{C})$. Identifying the three objects $X = (X_t)_{t\geq 0}$ with $X_0 = 0$, $f$ and $X_\infty$, we get that $\mathcal{H}^p(\mathbb{R}_+) = L_0^p(\mathcal{C})$ with equivalent norms.

## Appendix

In this appendix we consider the non-commutative analogue of the classical duality between the Hardy space $H^1$ and BMO of martingales (see [G]). We will show this duality remains valid in the non-commutative case.

Let us go back to the general situation presented in Sect. 1. In all what follows $(\mathcal{M}, \tau)$ denotes a finite von Neumann algebra with a normalized trace $\tau$, and $(\mathcal{M}_n)$ an increasing filtration of von Neumann subalgebras of $\mathcal{M}$, which generate $\mathcal{M}$. Recall that $\mathcal{E}_n$ denotes the conditional expectation of $\mathcal{M}$ with respect to $\mathcal{M}_n$. In Sect. 1 we have introduced the Hardy spaces $\mathcal{H}_C^1(\mathcal{M})$, $\mathcal{H}_R^1(\mathcal{M})$ and $\mathcal{H}^1(\mathcal{M})$ of martingales with respect to $(\mathcal{M}_n)$.

Now let us define the corresponding BMO-spaces. We set

$$\mathcal{BMO}_C(\mathcal{M}) = \{a \in L^2(\mathcal{M}) : \sup_{n\geq 0} \|\mathcal{E}_n|a - \mathcal{E}_{n-1}a|^2\|_\infty < \infty\},$$

where, as usual, $\mathcal{E}_{-1}a = 0$ (recall $|a|^2 = a^*a$). $\mathcal{BMO}_C(\mathcal{M})$ becomes a Banach space when equipped with the norm

$$\|a\|_{\mathcal{BMO}_C(\mathcal{M})} = \big(\sup_{n\geq 0}\|\mathcal{E}_n|a-\mathcal{E}_{n-1}a|^2\|_\infty\big)^{1/2}.$$

Similarly, we define $\mathcal{BMO}_R(\mathcal{M})$, which is the space of all $a$ such that $a^* \in \mathcal{BMO}_C(\mathcal{M})$, equipped with the natural norm. Finally, $\mathcal{BMO}(\mathcal{M})$ is the intersection of these two spaces

$$\mathcal{BMO}(\mathcal{M}) = \mathcal{BMO}_C(\mathcal{M}) \cap \mathcal{BMO}_R(\mathcal{M}),$$

and for any $a \in \mathcal{BMO}(\mathcal{M})$,

$$\|a\|_{\mathcal{BMO}(\mathcal{M})} = \max\{\|a\|_{\mathcal{BMO}_C(\mathcal{M})}, \|a\|_{\mathcal{BMO}_R(\mathcal{M})}\}.$$

Notice that if $a_n = \mathcal{E}_n a$, then

$$\mathcal{E}_n|a-\mathcal{E}_{n-1}a|^2 = \mathcal{E}_n\big(\sum_{k\geq n}|da_k|^2\big).$$

Note also that $\mathcal{E}_n|a|^2 = \mathcal{E}_{n-1}|a|^2 + \mathcal{E}_n|a-\mathcal{E}_{n-1}a|^2$, so that $\mathcal{E}_n|a-\mathcal{E}_{n-1}a|^2 \leq \mathcal{E}_n|a|^2$. Therefore, it follows that

$$\|a\|_{\mathcal{BMO}(\mathcal{M})} \leq \|a\|_\infty\,. \tag{$A_1$}$$

For simplicity we will denote $\mathcal{H}^1_C(\mathcal{M})$, $\mathcal{BMO}_C(\mathcal{M})$, *etc.*, respectively by $\mathcal{H}^1_C$, $\mathcal{BMO}_C$, etc. We will also adapt the identification between a martingale and its limit whenever the latter exists. The result of this appendix is the following duality.

**Theorem.** *We have $(\mathcal{H}^1_C)^* = \mathcal{BMO}_C$ with equivalent norms. More precisely,*

*(i) Every $a \in \mathcal{BMO}_C$ defines a continuous linear functional on $\mathcal{H}^1_C$ by*

$$\varphi_a(x) = \tau(a^*x), \quad \forall\, x \in L^2(\mathcal{M}). \tag{$A_2$}$$

*(ii) Conversely, any $\varphi \in (\mathcal{H}^1_C)^*$ is given as above by some $a \in \mathcal{BMO}_C$. Moreover,*

$$\frac{1}{\sqrt{3}}\|a\|_{\mathcal{BMO}_C} \leq \|\varphi_a\|_{(\mathcal{H}^1_C)^*} \leq \sqrt{2}\|a\|_{\mathcal{BMO}_C}.$$

*The same duality holds between $\mathcal{H}^1_R$, $\mathcal{BMO}_R$ and between $\mathcal{H}^1$, $\mathcal{BMO}$ as well:*

$$(\mathcal{H}^1_R)^* = \mathcal{BMO}_R \quad \textit{and} \quad (\mathcal{H}^1)^* = \mathcal{BMO}.$$

*Remark.* In the duality ($A_2$) we have identified an element $x \in L^2$ with the martingale $(\mathcal{E}_n x)_{n\geq 0}$. It is evident that this martingale is in $\mathcal{H}^1_C$ and

$$\|x\|_{\mathcal{H}^1_C} \leq \|x\|_2.$$

Let us also note that from the discussions in Sect. 1 the family of finite martingales is dense in $\mathcal{H}^1_C$, and so is $L^2$. Of course, the same remark applies to $\mathcal{H}^1_C$ and $\mathcal{H}^1$ as well.

Before proceeding to the proof of the theorem, let us note that the equivalence constants in (ii) above are the same as in [G]. In fact, our proof below is modelled on the one presented in [G], although one should be careful about some difficulties caused by the non-commutativity. However, this time, they are much less substantial than those appearing in the proof of Theorem 2.1. We will frequently use the tracial property of $\tau$ and the following elementary property of expectation:

$$\mathcal{E}_n(abc) = a\mathcal{E}_n(b)c, \quad \forall\, a, c \in \mathcal{M}_n,\ \forall\, b \in \mathcal{M}.$$

*Proof of the theorem.* (i) Let $a \in \mathcal{BMO}_C$. Define $\varphi_a$ by $(A_2)$. We must show that $\varphi_a$ induces a continuous functional on $\mathcal{H}^1_C$. To that end let $x$ be a finite $L^2$- martingale. Then (recalling our identification between a martingale and its limit)

$$\varphi_a(x) = \sum_{n\geq 0} \tau(da_n^* dx_n)\,.$$

Set, as in Sect. 1,

$$S_{C,n} = \Big(\sum_{k=0}^{n} |dx_k|^2\Big)^{1/2} \quad \text{and} \quad S_C = \Big(\sum_{k=0}^{\infty} |dx_k|^2\Big)^{1/2}.$$

By approximation we may assume the $S_{C,n}$'s are invertible elements in $\mathcal{M}$. Then by the Cauchy-Schwarz inequality

$$\begin{aligned}
|\varphi_a(x)| &= |\sum_{n\geq 0} \tau(S_{C,n}^{1/2} da_n^* dx_n S_{C,n}^{-1/2})| \\
&\leq \Big[\tau\Big(\sum_{n\geq 0} S_{C,n}^{-1/2} |dx_n|^2 S_{C,n}^{-1/2}\Big)\Big]^{1/2} \Big[\tau\Big(\sum_{n\geq 0} S_{C,n}^{1/2} |da_n|^2 S_{C,n}^{1/2}\Big)\Big]^{1/2} \\
&= \Big[\tau\Big(\sum_{n\geq 0} S_{C,n}^{-1} |dx_n|^2\Big)\Big]^{1/2} \Big[\tau\Big(\sum_{n\geq 0} S_{C,n} |da_n|^2\Big)\Big]^{1/2} \\
&= I \cdot II.
\end{aligned}$$

We are going to estimate I and II separately. First for I we have

$$\begin{aligned}
I^2 &= \sum_{n\geq 0} \tau\big([S_{C,n}^2 - S_{C,n-1}^2] S_{C,n}^{-1}\big) \\
&= \sum_{n\geq 0} \tau\big([S_{C,n} - S_{C,n-1}][1 + S_{C,n-1} S_{C,n}^{-1}]\big) \\
&\leq \sum_{n\geq 0} \tau\big(S_{C,n} - S_{C,n-1}\big) \|1 + S_{C,n-1} S_{C,n}^{-1}\|_\infty \\
&\leq 2\tau\Big(\sum_{n\geq 0} S_{C,n} - S_{C,n-1}\Big) \\
&= 2\tau(S_C) = 2\|x\|_{\mathcal{H}^1_C},
\end{aligned}$$

where we have used the trivial fact that (noting $S_{C,n-1}^2 \leq S_{C,n}^2$)

$$\|S_{C,n-1} S_{C,n}^{-1}\|_\infty^2 = \|S_{C,n}^{-1} S_{C,n-1}^2 S_{C,n}^{-1}\|_\infty \leq 1.$$

As for II, set $\theta_0 = S_{C,0}$ and $\theta_n = S_{C,n} - S_{C,n-1}$ for $n \geq 1$. Then $\theta_n \in \mathcal{M}_n$, and

$$\begin{aligned} II^2 &= \sum_{n\geq 0} \tau\big(S_{C,n}|da_n|^2\big) \\ &= \sum_{k\geq 0} \tau\big[\theta_k \sum_{n\geq k} |da_n|^2\big] \\ &= \sum_{k\geq 0} \tau\big[\theta_k \mathcal{E}_k\big(\sum_{n\geq k} |da_n|^2\big)\big] \\ &\leq \sum_{k\geq 0} \tau(\theta_k) \|\mathcal{E}_k\big(\sum_{n\geq k} |da_n|^2\big)\|_\infty \\ &\leq \|a\|^2_{\mathcal{BMO}_C} \|x\|_{\mathcal{H}^1_C}. \end{aligned}$$

Combining the preceding estimates on I and II, we obtain, for any finite $L^2$-martingale $x$,

$$|\varphi_a(x)| \leq \sqrt{2}\|a\|_{\mathcal{BMO}_C}\|x\|_{\mathcal{H}^1_C}.$$

Therefore, $\varphi_a$ extends to a continuous functional on $\mathcal{H}^1_C$ of norm$\leq \sqrt{2}\|a\|_{\mathcal{BMO}_C}$.

(ii) Now suppose $\varphi \in (\mathcal{H}^1_C)^*$. Then by the Hahn-Banach theorem, $\varphi$ extends to a continuous functional on $L^1(\mathcal{M}, l^2_C)$ of the same norm. Thus by the duality (see Sect. 1)

$$\big(L^1(\mathcal{M}, l^2_C)\big)^* = L^\infty(\mathcal{M}, l^2_C),$$

there exists a sequence $(b_n) \in L^\infty(\mathcal{M}, l^2_C)$ such that

$$\|\sum_{n\geq 0} |b_n|^2\|_\infty = \|\varphi\|^2 \quad \text{and} \quad \varphi(x) = \sum_{n\geq 0} b_n^* dx_n, \quad \forall\, x \in \mathcal{H}^1_C.$$

Let $a = \sum_{n\geq 0} \big(\mathcal{E}_n b_n - \mathcal{E}_{n-1} b_n\big)$ (and so $da_n = \mathcal{E}_n b_n - \mathcal{E}_{n-1} b_n$). Then $a \in L^2$ and

$$\varphi(x) = \sum_{n\geq 0} da_n^* dx_n = \varphi_a(x), \quad \forall\, x \in \mathcal{H}^1_C.$$

Therefore, $\varphi$ is given by $\varphi_a$ as in (i). It remains to show $a \in \mathcal{BMO}_C$ and to bound $\|a\|_{\mathcal{BMO}_C}$ by $\|\varphi\|$. This is done as follows. If $k-1 \geq n \geq 0$,

$$\mathcal{E}_n\big[\mathcal{E}_k b_k^* \mathcal{E}_{k-1} b_k\big] = \mathcal{E}_n\big[\mathcal{E}_{k-1}(\mathcal{E}_k b_k^* \mathcal{E}_{k-1} b_k)\big] = \mathcal{E}_n\big[\mathcal{E}_{k-1} b_k^* \mathcal{E}_{k-1} b_k\big];$$

similarly,

$$\mathcal{E}_n\big[\mathcal{E}_{k-1} b_k^* \mathcal{E}_k b_k\big] = \mathcal{E}_n\big[\mathcal{E}_{k-1} b_k^* \mathcal{E}_{k-1} b_k\big].$$

It then follows that if $k-1 \geq n \geq 0$,

$$\begin{aligned} \mathcal{E}_n\big[|da_k|^2\big] &= \mathcal{E}_n\big[(\mathcal{E}_k b_k - \mathcal{E}_{k-1} b_k)^*(\mathcal{E}_k b_k - \mathcal{E}_{k-1} b_k)\big] \\ &= \mathcal{E}_n\big[\mathcal{E}_k b_k^* \mathcal{E}_k b_k - \mathcal{E}_{k-1} b_k^* \mathcal{E}_{k-1} b_k\big] \\ &\leq \mathcal{E}_n\big[\mathcal{E}_k b_k^* \mathcal{E}_k b_k\big] \leq \mathcal{E}_n |b_k|^2. \end{aligned}$$

Hence,

$$\begin{aligned}\|\mathcal{E}_n|a-\mathcal{E}_{n-1}a|^2\|_\infty &= \|\mathcal{E}_n\sum_{k\geq n}|da_k|^2\|_\infty \\ &\leq \|\mathcal{E}_n\big[|da_n|^2+\sum_{k\geq n+1}|b_k|^2\big]\|_\infty \\ &\leq 3\|\sum_{k\geq 0}|b_k|^2\|_\infty \leq 3\|\varphi\|^2;\end{aligned}$$

whence

$$a\in\mathcal{BMO}_C \quad \text{and} \quad \|a\|_{\mathcal{BMO}_C}\leq\sqrt{3}\|\varphi\|.$$

Thus we have finished the proof of the theorem concerning $\mathcal{H}^1_C$ and $\mathcal{BMO}_C$. Passing to adjoints yields the part on $\mathcal{H}^1_R$ and $\mathcal{BMO}_R$. Finally, the duality between $\mathcal{H}^1$ and $\mathcal{BMO}$ is obtained by the classical (and easy) fact that the dual of a sum is the intersection of the duals. □

**Corollary.** *Let $x\in\mathcal{H}^1$. Then $x_n$ converges in $L^1$ and*

$$\|x\|_1\leq\sqrt{2}\|dx\|_{L^1(\mathcal{M};l^2_C)+L^1(\mathcal{M};l^2_R)}\leq\sqrt{2}\|dx\|_{\mathcal{H}^1}. \tag{$A_3$}$$

*Proof.* Let $x\in\mathcal{H}^1$. By the discussions in Sect. 1, the finite martingale $(x_0,\cdots,x_n,x_n,\cdots)$ converges to $x$ in $\mathcal{H}^1$. This, together with $(A_3)$, implies the convergence of $x_n$ in $L^1$. Thus it remains to show $(A_3)$; also it suffices to show the first inequality of $(A_3)$ for the second one is trivial. To this end fix $n\geq 0$, and choose $a\in L^1(\mathcal{M}_n)$ such that $\|a\|_\infty\leq 1$ and $\|x_n\|_1=\tau(a^*x_n)$. Put $a_k=\mathcal{E}_k(a)$ for $k\geq 0$. Then $a_k=a$ for all $k\geq n$, and

$$\begin{aligned}\|x_n\|_1 &= \tau\sum_{k=0}^{n}da_k^*dx_k=\tau\sum_{k=0}^{\infty}da_k^*dx_k \\ &\leq \|dx\|_{L^1(\mathcal{M};l^2_C)+L^1(\mathcal{M};l^2_R)}\,\|da\|_{\frac{L^\infty(\mathcal{M};l^2_C)\cap L^\infty(\mathcal{M};l^2_R)}{(\mathcal{H}^1)^\perp}}\,.\end{aligned}$$

However, by the preceding theorem

$$\frac{L^\infty(\mathcal{M};l^2_C)\cap L^\infty(\mathcal{M};l^2_R)}{(\mathcal{H}^1)^\perp}=\big(\mathcal{H}^1\big)^*\cong\mathcal{BMO}\,.$$

Therefore, by $(A_1)$

$$\|da\|_{\frac{L^\infty(\mathcal{M};l^2_C)\cap L^\infty(\mathcal{M};l^2_R)}{(\mathcal{H}^1)^\perp}}\leq\sqrt{2}\|a\|_{\mathcal{BMO}}\leq\sqrt{2}\|a\|_\infty\leq\sqrt{2}.$$

Combining the previous inequalities we obtain $(A_3)$, and thus complete the proof of the corollary. □

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