Dark Matter Self-Scattering

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Contents

1 Self-interaction cross section of Dirac and Majorana DM **1**
1.1 Scattering amplitude . . . 1
1.2 Distinguishbale DM . . . 2
1.2.1 Partial wave expansion . . . 2
1.2.2 Born approximation . . . 2
1.3 Indistiguishable DM . . . 3
1.3.1 Partial wave expansion . . . 3
1.3.2 Memo for calculation . . . 3
1.3.3 Born approximation . . . 4
1.4 Velocity average . . . 5

2 Resonant self-scattering **6**
2.1 Scattering amplitude beyond the Born approximation . . . 6
2.2 The differential cross section . . . 8
2.3 Width . . . 9
2.4 Transfer cross section . . . 10
2.5 Physical interpretation . . . 13

1 Self-interaction cross section of Dirac and Majorana DM

1.1 Scattering amplitude

We consider the non-relativistic DM-DM scattering by the Yukawa potential

$$V(r) = \pm \frac{\alpha_X}{r} e^{-m_\phi r}, \tag{1.1}$$

where $+$ $(-)$ corresponds to the repulsive (attractive) interaction and m_ϕ is the mass of the mediator. If the mediator is scalar, the interaction is always attractive while if the mediator is vector, it contains both repulsive and attractive force. The scattering amplitude and cross section depend on 4-parameters:

$$m_\chi, m_\phi, \alpha_X, v, \tag{1.2}$$

where m_χ is DM mass and v is the relative velocity of colliding DMs.

The amplitude of non-relativistic t-channel scattering, $f(\theta)$, is calculated by solving the Schrodinger equation. By the partial wave expansion, the amplitude is given by [1]

$$f(\theta) = \frac{1}{k}\sum_{\ell=0}^{\infty}(2\ell+1)e^{i\delta_\ell}\sin\delta_\ell P_\ell(\cos\theta), \tag{1.3}$$

where δ_ℓ is the phase shift and P_ℓ is the Legendre polynomials. When the incoming particles have the same masses m_χ, the momentum is written as $k = m_\chi v/2$.

1.2 Distinguishbale DM

Let us assume the classical distinguishability. Transfer cross section is defined by

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= |f(\theta)|^2, \\ \sigma_T &= 2\pi\int_{-1}^{1} d\cos\theta(1-\cos\theta)\frac{d\sigma}{d\Omega},\end{aligned} \tag{1.4}$$

where θ is the scattering angle in the center of mass frame. Usual cross section $\int d\sigma$ receives strong enhancement at $\theta = 0$ for the light mediator case, but this is spurious because nothing has changed after collision. Instead the transfer cross section is often used in the literature to regulate the forward scattering divergence.

1.2.1 Partial wave expansion

The transfer cross section is calculated by the partial wave expansion

$$\frac{\sigma_T k^2}{4\pi} = \sum_{\ell=0}^{\infty}(\ell+1)\sin^2(\delta_{\ell+1}-\delta_\ell), \tag{1.5}$$

where again $k = m_\chi v/2$.

1.2.2 Born approximation

If the one-particle exchange is the good approximation, namely $\alpha_X m_\chi/m_\phi \ll 1$, we can use the Born approximation:

$$\sigma_T \simeq \frac{8\pi\alpha_X^2}{m_\chi^2 v^4}\left[\log\left(1+\frac{m_\chi^2 v^2}{m_\phi^2}\right) - \frac{m_\chi^2 v^2}{m_\phi^2 + m_\chi^2 v^2}\right] \tag{1.6}$$

1.3 Indistiguishable DM

Let us consider the scattering of identical particles. Because of the quantum indistiguishability, the scattering amplitude is the sum of t- and u- channels, $f(\theta) \pm f(\pi - \theta)$. The relative sign depends on the parity of orbital wave function; $+$ for spin singlet and $-$ for spin triplet.

1.3.1 Partial wave expansion

Let us consider the scattering of two identical spin 1/2 fermions. When the initial state is unpolarized, the differential cross section is given by the average of spin singlet and triplet contributions as

$$\frac{d\sigma}{d\Omega} = \left[\frac{1}{4}|f(\theta) + f(\pi - \theta)|^2 + \frac{3}{4}|f(\theta) - f(\pi - \theta)|^2.\right] \tag{1.7}$$

The factor 1/2 takes care of double counting. When we expand the amplitude by partial wave, the even ℓ contributes to the first term, and the odd ℓ contributes to the last term.

Taking quantum indistinguishably into account, Eq. (1.4) is not valid because the backward scattering is equivalent to the forward scattering. One way to treat the effect is to limit the integration interval of $\cos\theta$ from $[-1, 1]$ to $[0, 1]$ and multiplying an additional factor 2 [2] (see also appendix C in Ref. [3])

$$\begin{aligned} \sigma_T &= \frac{1}{2} \times 2\pi \int_{-1}^{1} d\cos\theta (1 - |\cos\theta|) \frac{d\sigma}{d\Omega} \\ &= \frac{1}{2} \times 4\pi \int_{0}^{1} d\cos\theta (1 - \cos\theta) \frac{d\sigma}{d\Omega}. \end{aligned} \tag{1.8}$$

There is no simple formula to resum the partial waves.

There is another way to regulate the forward and backward scattering. The viscosity cross section is defined by

$$\sigma_V = \frac{1}{2} \times 2\pi \int_{-1}^{1} d\cos\theta \sin^2\theta \frac{d\sigma}{d\Omega}. \tag{1.9}$$

Substituting Eq. (1.7), this is expressed as

$$\begin{aligned} \frac{\sigma_V k^2}{4\pi} &= \frac{1}{2} \sum_{\ell:\text{even}} \frac{(\ell+1)(\ell+2)}{2\ell+3} \sin^2(\delta_{\ell+2} - \delta_\ell) \\ &\quad + \frac{3}{2} \sum_{\ell:\text{odd}} \frac{(\ell+1)(\ell+2)}{2\ell+3} \sin^2(\delta_{\ell+2} - \delta_\ell). \end{aligned} \tag{1.10}$$

1.3.2 Memo for calculation

When we truncate the sum over ℓ at a certain $\ell_{\max}$, the transfer cross section for identical particles is written as

$$
\begin{aligned}
\frac{\sigma_T k^2}{4\pi} &\simeq \frac{1}{2}\sum_{\ell:\text{even}}^{\ell_{\max}}\sum_{\ell':\text{even}}^{\ell_{\max}}(2\ell+1)(2\ell'+1)e^{i\delta_\ell}e^{-i\delta'_\ell}\sin\delta_\ell\sin\delta_{\ell'}\int_0^1 dx(1-x)P_\ell(x)P_{\ell'}(x)\\
&+\frac{3}{2}\sum_{\ell:\text{odd}}^{\ell_{\max}}\sum_{\ell':\text{odd}}^{\ell_{\max}}(2\ell+1)(2\ell'+1)e^{i\delta_\ell}e^{-i\delta'_\ell}\sin\delta_\ell\sin\delta_{\ell'}\int_0^1 dx(1-x)P_\ell(x)P_{\ell'}(x)\\
&=\frac{1}{2}\sum_{\ell:\text{even}}^{\ell_{\max}}\sum_{\ell':\text{even}}^{\ell_{\max}}e^{i\delta_\ell}e^{-i\delta'_\ell}\sin\delta_\ell\sin\delta_{\ell'}\\
&\times\left[(2\ell+1)(2\ell'+1)F_{\ell\ell'}-(\ell+1)(2\ell'+1)F_{\ell+1\ell'}-\ell(2\ell'+1)F_{\ell-1\ell'}\right]\\
&+\frac{3}{2}\sum_{\ell:\text{odd}}^{\ell_{\max}}\sum_{\ell':\text{odd}}^{\ell_{\max}}e^{i\delta_\ell}e^{-i\delta'_\ell}\sin\delta_\ell\sin\delta_{\ell'}\\
&\times\left[(2\ell+1)(2\ell'+1)F_{\ell\ell'}-(\ell+1)(2\ell'+1)F_{\ell+1\ell'}-\ell(2\ell'+1)F_{\ell-1\ell'}\right],
\end{aligned}
\tag{1.11}
$$

where $F_{\ell\ell'} \equiv \int_0^1 P_\ell(x)P_{\ell'}(x)$. We can find the analytical expression of $F_{m,n}$ here. For even m and odd n,

$$
\begin{aligned}
F_{m,n} &= f_{m,n}\\
&= \frac{(-1)^{(m+n+1)/2}m!n!}{2^{m+n-1}(m-n)(m+n+1)\left[\left(\frac{m}{2}\right)!\right]^2\left[\left(\frac{n-1}{2}\right)!\right]^2}.
\end{aligned}
\tag{1.12}
$$

For large m or n, the factorial may cause overflow. In such a case, the following asymptotic expressions are useful:

$$
\frac{m!}{2^m\left[\left(\frac{m}{2}\right)!\right]^2} \simeq \sqrt{\frac{2}{\pi}}\frac{1}{m^{1/2}}-\frac{1}{2\sqrt{2\pi}}\frac{1}{m^{3/2}}+\cdots \tag{1.13}
$$

$$
\frac{n!}{2^{n-1}\left[\left(\frac{n-1}{2}\right)!\right]^2} \simeq \sqrt{\frac{2}{\pi}}n^{1/2}-\frac{1}{2\sqrt{2\pi}}\frac{1}{n^{1/2}}+\frac{1}{16\sqrt{2\pi}}\frac{1}{n^{3/2}}\cdots. \tag{1.14}
$$

The deviation from the true value is below 1% for $m, n > 5$.

1.3.3 Born approximation

In the Born limit ($\alpha_X m_\chi/m_\phi \ll 1$), the transfer cross section is approximated as

$$
\sigma_T \simeq \frac{4\pi\alpha_X^2}{m_\chi^2 v^4}\left[6\ln\left(1+\frac{m_\chi^2v^2}{2m_\phi^2}\right)-\frac{4m_\chi^2v^2+6m_\phi^2}{m_\chi^2v^2+2m_\phi^2}\ln\left(1+\frac{m_\chi^2v^2}{m_\phi^2}\right)\right]. \tag{1.15}
$$

Let me define $a = v/2\alpha_X$ and $b = \alpha_X m_\chi/m_\phi$. If $ab \ll 1$, the following expansion is useful for numerical calculation:

$$\begin{aligned}\frac{\sigma_T k^2}{4\pi} &= \frac{1}{16a^2}\left[6\ln(1+2a^2b^2) - \frac{3+8a^2b^2}{1+2a^2b^2}\ln(1+4a^2b^2)\right]\\ &\simeq \frac{a^2b^4}{4} - a^4b^6 + \frac{23a^6b^8}{6} + \cdots .\end{aligned} \tag{1.16}$$

1.4 Velocity average

We take the velocity average, assuming Maxwellian distribution [1]:

$$f(\boldsymbol{v}) = \frac{1}{(2\pi\sigma_0^2)^{3/2}} e^{-\frac{v^2}{2\sigma_0^2}}. \tag{1.17}$$

The thermally averaged cross section (times relative velocity) is written as

$$\begin{aligned}\langle \sigma v\rangle &= \int d^3v_1 d^3v_2 f(\boldsymbol{v}_1) f(\boldsymbol{v}_2)\sigma\left(|\boldsymbol{v}_2-\boldsymbol{v}_1|\right)|\boldsymbol{v}_2-\boldsymbol{v}_1|\\ &= \int \frac{d^3v}{(2\pi v_0^2)^{3/2}} e^{-\frac{v^2}{2v_0^2}}\sigma(v)v\\ &= 2\sqrt{\frac{2}{\pi}}v_0\int_0^\infty dx e^{-x} x\sigma(v = v_0\sqrt{2x}),\end{aligned} \tag{1.18}$$

where $v = |\boldsymbol{v}_2 - \boldsymbol{v}_1|$ is the relative velocity, and $v_0 = \sqrt{2}\sigma_0$. This integration is evaluated by, e.g., Gauss-Laguerre quadrature. The average of relative velocity is given by

$$\langle v\rangle = \frac{2\sqrt{2}}{\sqrt{\pi}}v_0 = \frac{4}{\sqrt{\pi}}\sigma_0. \tag{1.19}$$

In Fig. 1.1, we have checked the velocity averaged cross sections at the benchmark point of Ref. [4]. Comparing distinguishable and indistinguishable DMs, we see the somewhat large discrepancy for small velocity.

For small velocity, the scattering is dominated by s-wave; $\ell = 0$. From Eq. (1.5), (1.10)

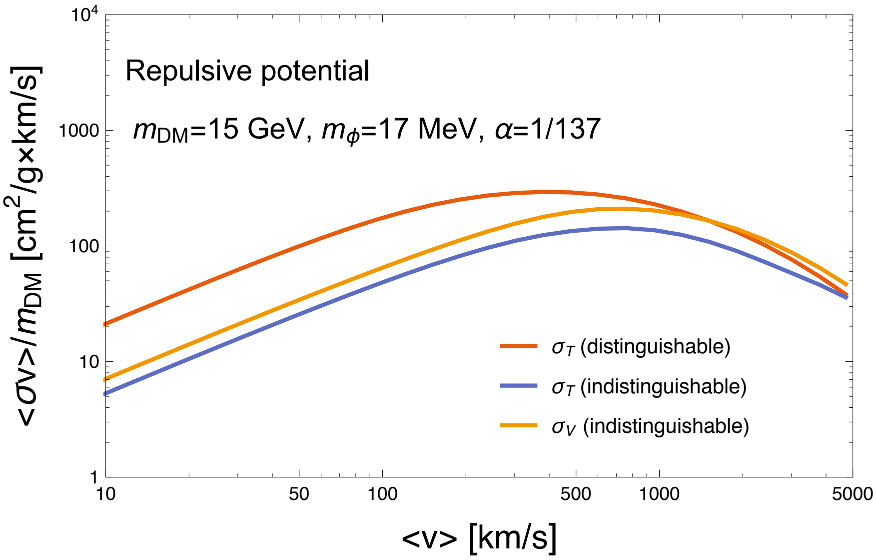

Figure 1.1: Comparison of different cross sections.

and (1.11), the s-wave cross sections are estimated as

$$\sigma_T \sim \frac{4\pi}{k^2}\sin^2\delta_0 \;\text{ , for distinguishable DM,} \tag{1.20}$$

$$\sigma_T \sim \frac{\pi}{k^2}\sin^2\delta_0 \;\text{ , for indistinguishable DM,} \tag{1.21}$$

$$\sigma_V \sim \frac{4\pi}{3k^2}\sin^2\delta_0. \tag{1.22}$$

In Fig. 1.1, we see $\sigma_T(\text{distinguishableDM}) \sim 4\sigma_T(\text{indistinguishableDM}) \sim 3\sigma_V$ at $\langle v\rangle \sim 10\,\text{km/s}$.

2 Resonant self-scattering

In the field theory, s-channel exchange of a light mediator generates the resonance for $m_\chi \sim m_\phi/2$. In this section, we discuss this resonance and the interference between s-and t, u-channel. We focus on the scattering between Majorana fermions χ:

$$\chi_{s_1}(p_1) + \chi_{s_2}(p_2) \to \chi_{s_1'}(p_1') + \chi_{s_2'}(p_2') \tag{2.1}$$

2.1 Scattering amplitude beyond the Born approximation

The scattering amplitude in the field theory is written as

$$\mathcal{M} = \mathcal{M}_t + \mathcal{M}_u + \mathcal{M}_s\,. \tag{2.2}$$

Using the scattering amplitude of the non-relativistic quantum mechanics (QM), $f(\theta)$, we write each term as

$$\mathcal{M}_t = \frac{4\pi}{m_\chi} f(\theta) \bar{u}_{1'} u_1 \bar{u}_{2'} u_2 \tag{2.3}$$

$$\mathcal{M}_u = -\frac{4\pi}{m_\chi} f(\pi - \theta) \bar{u}_{2'} u_1 \bar{u}_{1'} u_2 \tag{2.4}$$

$$\mathcal{M}_s = \frac{c_s^2}{-s + m_\phi^2} \bar{v}_2 u_1 \bar{u}_{1'} v_{2'} \,. \tag{2.5}$$

We will see the validity of these expressions in the following subsections.

Kinematic variables We use center of mass frame, where

$$s = -(p_1 + p_2)^2 = 4(m_\chi^2 + k^2) \tag{2.6}$$

$$t = -(p_1' - p_1)^2 = -4k^2 \sin^2 \frac{\theta}{2} \equiv -K^2 \tag{2.7}$$

$$u = -(p_2' - p_1)^2 = -4k^2 \cos^2 \frac{\theta}{2} \tag{2.8}$$

where $k \equiv |\boldsymbol{p}_1| = |\boldsymbol{p}_2| = |\boldsymbol{p}_1'| = |\boldsymbol{p}_2'|$, and θ is angle between $\boldsymbol{p}_1$ and $\boldsymbol{p}_1'$.

Born limit For the potential

$$V(r) = -\frac{c_s^2}{4\pi} \frac{e^{-m_\phi r}}{r} \,, \tag{2.9}$$

the usual Born approximation formula of QM reads

$$f(\theta) = -m_\chi \int_0^\infty dr V(r) r \frac{\sin Kr}{K} = \frac{c_s^2}{4\pi} \frac{m_\chi}{-t + m_\phi^2} \,, \tag{2.10}$$

which reproduces the ordinary quantum filed theory (QFT) result (see, e.g., Eq. (49.7) in Srednicki).

Spin-averaged amplitude We take the spin average of initial state particles and spin sum of final state particles, which is denoted by

$$\langle \dots \rangle \equiv \frac{1}{4} \sum_{s_1, s_2, s_1', s_2'} \dots \,. \tag{2.11}$$

The squared amplitudes are calculated as

$$\left\langle|\mathcal{M}_t|^2\right\rangle = \frac{16\pi^2}{m_\chi^2}(t-4m_\chi^2)^2\,|f(\theta)|^2 \ , \tag{2.12}$$

$$\left\langle|\mathcal{M}_u|^2\right\rangle = \frac{16\pi^2}{m_\chi^2}(u-4m_\chi^2)^2\,|f(\pi-\theta)|^2 \ , \tag{2.13}$$

$$\left\langle|\mathcal{M}_s|^2\right\rangle = c_s^4\frac{(s-4m_\chi^2)^2}{(s-m_\phi^2)^2+m_\phi^2\Gamma^2} \ , \tag{2.14}$$

$$\tag{2.15}$$

and the interference terms are

$$\langle\mathcal{M}_u^*\mathcal{M}_t+\text{c.c.}\rangle = \frac{16\pi^2}{m_\chi^2}(tu-4m_\chi^2 s)\text{Re}\left[f(\theta)f(\pi-\theta)^*\right] \ , \tag{2.16}$$

$$\langle\mathcal{M}_t^*\mathcal{M}_s+\text{c.c.}\rangle = -\frac{4\pi}{m_\chi}c_s^2\frac{st-4m_\chi^2 u}{(s-m_\phi^2)^2+m_\phi^2\Gamma^2}\left[(s-m_\phi^2)\text{Re}f(\theta)-m_\phi\Gamma\text{Im}f(\theta)\right] \ , \tag{2.17}$$

$$\langle\mathcal{M}_u^*\mathcal{M}_s+\text{c.c.}\rangle = -\frac{4\pi}{m_\chi}c_s^2\frac{su-4m_\chi^2 t}{(s-m_\phi^2)^2+m_\phi^2\Gamma^2}\left[(s-m_\phi^2)\text{Re}f(\pi-\theta)-m_\phi\Gamma\text{Im}f(\pi-\theta)\right] \ . \tag{2.18}$$

Consistency check Let us focus on only t- and u-channel, and take the non-relativistic limit: $t\simeq u\simeq 0$ and $s\simeq 4m_\chi^2$. Then the differential cross section becomes

$$\left(\frac{d\sigma}{d\Omega}\right)_{t+u} \simeq \frac{1}{64\pi^2(2m_\chi)^2}\left\langle|\mathcal{M}_t+\mathcal{M}_u|^2\right\rangle \simeq \frac{1}{4}\left|f(\theta)+f(\pi-\theta)\right|^2+\frac{3}{4}\left|f(\theta)-f(\pi-\theta)\right|^2 \ ,$$

which reproduces Eq. (1.7). By taking $\Gamma\to 0$ and using the QM Born formula (2.10), we can also reproduce the Born limit formula (see, e.g., Eq. (49.8) in Srednicki):

$$\begin{aligned}\left\langle\left|\mathcal{M}^{\text{Born}}\right|^2\right\rangle = c_s^4\Bigg[&\frac{(s-4m_\chi^2)^2}{(-s+m_\phi^2)^2}+\frac{st-4m_\chi^2 u}{(-s+m_\phi^2)(-t+m_\phi^2)}\\ &+\frac{(t-4m_\chi^2)^2}{(-t+m_\phi^2)^2}+\frac{tu-4m_\chi^2 s}{(-t+m_\phi^2)(-u+m_\phi^2)}\\ &+\frac{(u-4m_\chi^2)^2}{(-u+m_\phi^2)^2}+\frac{us-4m_\chi^2 t}{(-u+m_\phi^2)(-s+m_\phi^2)}\Bigg] \ .\end{aligned} \tag{2.19}$$

2.2 The differential cross section

For notational simplicity, we introduce

$$\epsilon\equiv\frac{k^2}{m_\chi^2}\,,\alpha_X\equiv\frac{c_s^2}{4\pi}\,,g\equiv\frac{m_\chi(s-m_\phi^2)}{(s-m_\phi^2)^2+m_\phi^2\Gamma^2}\,,G\equiv\frac{m_\chi^2}{(s-m_\phi^2)^2+m_\phi^2\Gamma^2}\,. \tag{2.20}$$

The differential cross section at center of mass frame is

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s} \left\langle |\mathcal{M}|^2 \right\rangle = \frac{1}{256\pi^2 m_\chi^2 (1+\epsilon)} \left\langle |\mathcal{M}|^2 \right\rangle . \tag{2.21}$$

Then the differential cross sections at t- and u-channel are calculates as

$$\begin{aligned}(1+\epsilon)\left(\frac{d\sigma}{d\Omega}\right)_{t+u} &= \frac{1}{256\pi^2 m_\chi^2} \left\langle |\mathcal{M}_t + \mathcal{M}_u|^2 \right\rangle \\ &= |f(\theta)|^2 \left(1+\epsilon \sin^2\frac{\theta}{2}\right)^2 + |f(\pi-\theta)|^2 \left(1+\epsilon\cos^2\frac{\theta}{2}\right)^2 \\ &\quad - \mathrm{Re}\left[f(\theta) f(\pi-\theta)^*\right] \left(1+\epsilon - \frac{\epsilon^2}{4}\sin^2\theta\right) . \end{aligned} \tag{2.22}$$

The s-channel resonance part is

$$(1+\epsilon)\left(\frac{d\sigma}{d\Omega}\right)_s = \frac{1}{256\pi^2 m_\chi^2} \left\langle |\mathcal{M}_s|^2 \right\rangle = \alpha_X^2 \epsilon^2 G \,. \tag{2.23}$$

The interference term is

$$\begin{aligned}(1+\epsilon)\left(\frac{d\sigma}{d\Omega}\right)_{s,t+u} &= \frac{1}{256\pi^2 m_\chi^2} \left\langle \mathcal{M}_s^* \left(\mathcal{M}_t + \mathcal{M}_u\right) + \mathrm{c.c.} \right\rangle \\ &= -\epsilon\left(1+\frac{\epsilon}{2}\right)\alpha_X \cos\theta \left\{ g\mathrm{Re}\left[f(\theta) - f(\pi-\theta)\right] - \frac{m_\phi \Gamma G}{m_\chi} \mathrm{Im}\left[f(\theta) - f(\pi-\theta)\right] \right\} \\ &\quad + \frac{\epsilon^2}{2}\alpha_X \left\{ g\mathrm{Re}\left[f(\theta) + f(\pi-\theta)\right] - \frac{m_\phi \Gamma G}{m_\chi} \mathrm{Im}\left[f(\theta) + f(\pi-\theta)\right] \right\} . \end{aligned} \tag{2.24}$$

2.3 Width

The width in the denominator of s-channel comes from the self-energy of ϕ, which is generally dependent on the velocity. The propagator of ϕ is written at loop level by the self energy function Π as

$$\frac{1}{p^2 + m_\phi^2 - \Pi(p^2)} \,. \tag{2.25}$$

In the Yukawa interaction with the Majorana fermion χ, the 1-loop self-energy is

$$\Pi(p^2) = -\frac{6c_s^2}{16\pi^2} \int_0^1 dx \, \left(x(1-x)p^2 + m_\chi^2\right) \left[\frac{2}{\varepsilon} + \frac{1}{3} - \ln\left(\frac{x(1-x)p^2 + m_\chi^2 - i\epsilon}{\mu^2}\right)\right] , \tag{2.26}$$

where we adopt the dimensional regularization with $d = 4-\varepsilon$ and μ is a renormalization parameter.

Suppose $p^2 = -s < 0$, since we are interested in the real ϕ production at s-channel. The imaginary part of the self-energy is calculated as[1]

$$\mathrm{Im}\Pi(-s) = \Theta(s - 4m_\chi^2)\frac{c_s^2}{16\pi\sqrt{s}}\left(s - 4m_\chi^2\right)^{\frac{3}{2}} . \tag{2.27}$$

Thus, to the lowest order of v in $s = 4m_\chi^2 + m_\chi^2 v^2$, running width is defined as follows

$$m_\phi \Gamma(v) = \mathrm{Im}\Pi(-s) \simeq \frac{\alpha_X}{8} m_\chi^2 v^3 . \tag{2.28}$$

For the case of our interest, ϕ can also decay to SM particles by the mixing with the Higgs. In such a case, the velocity dependence is weakened as the running width is proportional to some power of $s - 4m_\chi^2 = 4(m_\chi^2 - m^2) + m_\chi^2 v^2$, where m is the mass of a particle in the SM. In practice we use

$$\Gamma_\phi(v) = \frac{\alpha_X}{8}\frac{m_\chi^2}{m_\phi} v^3 + \Gamma(\phi \to \mathrm{SM}) , \tag{2.29}$$

where the last term is the decay width of ϕ to SM particles at its rest frame, and thus does not depend on v.

2.4 Transfer cross section

$t + u$ channel In the non-relativistic limit, $\epsilon \to 0$, we can use the usual formula (1.11).

s-channel Using Eq. (2.23) and (2.28), we write the transfer cross section as

$$(1 + \epsilon)\left(\sigma_T\right)_s = \frac{\pi\alpha_X^2}{16 m_\chi^2}\frac{v^4}{(v^2 - v_R^2)^2 + (\alpha_X v^3/8 + m_\phi \Gamma^{\mathrm{SM}}/m_\chi^2)^2} , \tag{2.30}$$

where $\Gamma^{\mathrm{SM}} \equiv \Gamma(\phi \to \mathrm{SM})$, and

$$v_R^2 = \frac{m_\phi^2}{m_\chi^2} - 4 , \tag{2.31}$$

[1] The imaginary part comes from $x(1-x)p^2 + m_\chi^2 - i\epsilon < 0$. This corresponds to the integration over $x_- < x < x_+$, where $x_\pm = 1/2 \pm 1/2 \cdot \sqrt{1 + 4m_\chi^2/p^2}$. For this x, $\ln(-1)$ gives $-\pi$.

which is positive for $m_\phi > 2m_\chi$.

Near the resonance, $m_\phi \simeq 2m_\chi$, we can also write it as

$$(\sigma_T)_s = \frac{4\pi S}{m_\chi E(v)} \frac{\Gamma(v)^2/4}{(E(v) - E(v_R))^2 + \Gamma(v)^2/4}\,, \tag{2.32}$$

where $S = 1/4$, $E(v) = m_\chi v^2/4$ and $E(v_R) = (m_\phi^2 - 4m_\chi^2)/(4m_\chi)$ and we neglect $\Gamma^{\rm SM}$. Equation (2.32) is consistent with the second term of Eq. (2) in Ref. [5].

To have qualitative understating for $v_R^2 > 0$, we use the narrow width approximation for the s-channel propagator:

$$\frac{w}{(v^2 - v_R^2)^2 + w^2} \xrightarrow{w \to 0} \frac{\pi}{2v_R}\delta(v - v_R)\,. \tag{2.33}$$

The transfer cross section is written as

$$(\sigma_T)_s \simeq \frac{\pi^2 \alpha_X^2 v_R^3}{32 m_\chi^2 w}\delta(v - v_R)\,, \tag{2.34}$$

where $w = \alpha_X v^3/8 + m_\phi \Gamma^{\rm SM}/m_\chi^2$. With this approximation, the velocity averaged cross section is easily calculated as

$$\frac{\langle (\sigma_T)_s\, v\rangle}{m_\chi} \simeq \frac{\pi^{3/2}\alpha_X^2 v_R^6}{16\sqrt{2} m_\chi^3 w v_0^3} \exp\left(-\frac{v_R^2}{2v_0^2}\right)\,. \tag{2.35}$$

Interference term Using the partial wave expansion (1.3), we obtain

$$\begin{aligned}
&(1+\epsilon)\,(\sigma_T)_{s,t+u} \\
&= 2\pi \cdot \frac{1}{k}\sum_\ell (2\ell+1)\sin\delta_\ell \cdot \alpha_X \left(g\cos\delta_\ell - \frac{m_\phi \Gamma G}{m_\chi}\sin\delta_\ell\right) \\
&\times \left[-\epsilon\left(1+\frac{\epsilon}{2}\right)\left(1-(-1)^\ell\right)\left(F_{1,\ell} - \frac{2}{3}F_{2,\ell} - \frac{1}{3}F_{0,\ell}\right) + \frac{\epsilon^2}{2}\left(1+(-1)^\ell\right)\left(F_{0,\ell} - F_{1,\ell}\right)\right] \\
&= \frac{\pi\alpha_X}{m_\chi^2} v \sum_\ell (2\ell+1)\sin\delta_\ell \cdot \frac{\cos\delta_\ell \cdot (v^2 - v_R^2) - \sin\delta_\ell \cdot (\alpha_X v^3/8 + m_\phi \Gamma^{\rm SM}/m_\chi^2)}{(v^2 - v_R^2)^2 + (\alpha_X v^3/8 + m_\phi\Gamma^{\rm SM}/m_\chi^2)^2} \\
&\times \left[-\left(1+\frac{\epsilon}{2}\right)\left(1-(-1)^\ell\right)\left(\frac{1}{3}\delta_{1,\ell} - \frac{2}{3}F_{2,\ell} - \frac{1}{3}F_{0,\ell}\right) + \frac{\epsilon}{2}\left(1+(-1)^\ell\right)\left(\delta_{0,\ell} - F_{1,\ell}\right)\right]\,,
\end{aligned} \tag{2.36}$$

where $F_{\ell,\ell'}$ is the same as defined in Sec. 1.3.2. As it turns out, our interest is $\epsilon \lesssim \mathcal{O}(10^{-2})$.

Thus we leave the lowet order of ϵ as

$$(\sigma_T)_{s,t+u} \simeq \frac{2\pi\alpha_X}{m_\chi^2} \cdot v \sum_{\ell:\mathrm{odd}} (2\ell+1)\sin\delta_\ell \cdot \frac{\cos\delta_\ell \cdot (v^2 - v_R^2) - \sin\delta_\ell \cdot (\alpha_X v^3/8 + m_\phi \Gamma^{\mathrm{SM}}/m_\chi^2)}{(v^2 - v_R^2)^2 + (\alpha_X v^3/8 + m_\phi \Gamma^{\mathrm{SM}}/m_\chi^2)^2} \times \left(-\frac{1}{3}\delta_{1,\ell} + \frac{2}{3}F_{2,\ell} + \frac{1}{3}F_{0,\ell}\right) \tag{2.37}$$

For $\alpha_X m_\chi/m_\phi \ll 1$, we can use the Born approximation, which reads

$$(\sigma_T)_{s,t+u} \simeq -\epsilon^2 \frac{\pi\alpha_X}{m_\chi^2} \frac{v^2 - v_R^2}{(v^2 - v_R^2)^2 + (\alpha_X v^3/8 + m_\phi \Gamma^{\mathrm{SM}}/m_\chi^2)^2} \frac{2-3r^2}{3r^4} + \mathcal{O}(\epsilon^3)\,, \tag{2.38}$$

where $r = m_\phi/m_\chi$.

Using the narrow width approximation, we can write the interference term as

$$(\sigma_T)_{s,t+u} \simeq -\frac{\pi^2\alpha_X}{2v_R m_\chi^2} \sum_\ell (2\ell+1)\sin^2\delta_\ell \cdot \delta(v - v_R) \times \left[-\left(1+\frac{\epsilon}{2}\right)\left(1-(-1)^\ell\right)\left(\frac{1}{3}\delta_{1,\ell} - \frac{2}{3}F_{2,\ell} - \frac{1}{3}F_{0,\ell}\right) + \frac{\epsilon}{2}\left(1+(-1)^\ell\right)\left(\delta_{0,\ell} - F_{1,\ell}\right)\right]\,, \tag{2.39}$$

and the velocity averaged cross section is written as

$$\frac{\left\langle (\sigma_T)_{s,t+u}\, v\right\rangle}{m_\chi} \simeq -\frac{\pi^{3/2}\alpha_X v_R^3}{\sqrt{2} m_\chi^3 v_0^3} \exp\left(-\frac{v_R^2}{2v_0^2}\right) \sum_\ell (2\ell+1)\sin^2\delta_\ell \times \left[-\left(1+\frac{\epsilon}{2}\right)\left(1-(-1)^\ell\right)\left(\frac{1}{3}\delta_{1,\ell} - \frac{2}{3}F_{2,\ell} - \frac{1}{3}F_{0,\ell}\right) + \frac{\epsilon}{2}\left(1+(-1)^\ell\right)\left(\delta_{0,\ell} - F_{1,\ell}\right)\right]\,, \tag{2.40}$$

where we use $v = v_R$ for calculation of the phase shift. Note that the Born approximated interference term vanishes with narrow width approximation.

2.5 Physical interpretation

Let us understand the results in the last section more intuitively. We take the non-relativistic limit in Eqs. (2.3) and (2.4), and leave only the lowest order term in each channel:[2]

$$\mathcal{M}_t \simeq 16\pi m_\chi f(\theta)\delta_{s_1 s_1'}\delta_{s_2 s_2'}\,, \tag{2.42}$$

$$\mathcal{M}_u \simeq -16\pi m_\chi f(\pi-\theta)\delta_{s_1 s_2'}\delta_{s_2 s_1'}\,, \tag{2.43}$$

$$\begin{aligned}\mathcal{M}_s &= \frac{4c_s^2 k^2}{-s+m_\phi^2-im_\phi\Gamma}\cdot \eta_{s_2}^\dagger(\hat{p}_1\cdot\boldsymbol{\sigma})\xi_{s_1}\cdot\xi_{s_1'}^\dagger(\hat{p}_1'\cdot\boldsymbol{\sigma})\eta_{s_2'} \\ &= -\frac{4c_s^2 k^2}{-s+m_\phi^2-im_\phi\Gamma}\cdot\delta_{s1,-s2}\cdot\xi_{s_1'}^\dagger(\hat{p}_1'\cdot\boldsymbol{\sigma})\eta_{s_2'}\,,\end{aligned} \tag{2.44}$$

where we take the initial state momentum along z-axis: $\hat{p}=\hat{z}$. Note that we do not use any approximation at s-channel.

We can readily understand how the s-channel resonance appears:

- $\mathcal{M}_s$ contributes only to total spin 0 state: $S_z = s_{1z}+s_{2z}=0$. This is because the intermediate particle ϕ is a scalar.
- $\mathcal{M}_s$ is p-wave ($\propto k^2$), so only the total spin 1 ($S=1$) state contributes. This is consistent with the CP conservation of the Yukawa interaction; $\chi\chi$ state has $CP=(-1)^{S+1}$, and the intermediate scalar has $CP=+1$.

In the following, we focus on $S_z=0$ state for incoming particles. The amplitude for each spin combination is read from these equations, which is summarized as follows:

$$\mathcal{M}\left(+-\to+-\right) = 16\pi m_\chi f(\theta) + \frac{4c_s^2k^2}{-s+m_\phi^2-im_\phi\Gamma}\cos\theta\,, \tag{2.45}$$

$$\mathcal{M}\left(+-\to-+\right) = -16\pi m_\chi f(\pi-\theta) + \frac{4c_s^2k^2}{-s+m_\phi^2-im_\phi\Gamma}\cos\theta\,, \tag{2.46}$$

$$\mathcal{M}\left(+-\to++\right) = -\frac{4c_s^2k^2}{-s+m_\phi^2-im_\phi\Gamma}e^{-i\phi}\sin\theta\,, \tag{2.47}$$

$$\mathcal{M}\left(+-\to--\right) = \frac{4c_s^2k^2}{-s+m_\phi^2-im_\phi\Gamma}e^{i\phi}\sin\theta\,, \tag{2.48}$$

$$\tag{2.49}$$

[2] For the spinors, we adopt the convention used in the textbook of Srednicki. The 2-component spinor ξ_s and η_s is defined by

$$\xi_+ = \begin{pmatrix}1\\0\end{pmatrix},\xi_- = \begin{pmatrix}0\\1\end{pmatrix},\eta_+ = \begin{pmatrix}0\\1\end{pmatrix},\eta_- = \begin{pmatrix}-1\\0\end{pmatrix}. \tag{2.41}$$

$$\mathcal{M}\left(-+\to+-\right)=-16\pi m_\chi f(\pi-\theta)+\frac{4c_s^2k^2}{-s+m_\phi^2-im_\phi\Gamma}\cos\theta\,, \tag{2.50}$$

$$\mathcal{M}\left(-+\to-+\right)=16\pi m_\chi f(\theta)+\frac{4c_s^2k^2}{-s+m_\phi^2-im_\phi\Gamma}\cos\theta\,, \tag{2.51}$$

$$\mathcal{M}\left(-+\to++\right)=-\frac{4c_s^2k^2}{-s+m_\phi^2-im_\phi\Gamma}e^{-i\phi}\sin\theta\,, \tag{2.52}$$

$$\mathcal{M}\left(-+\to--\right)=\frac{4c_s^2k^2}{-s+m_\phi^2-im_\phi\Gamma}e^{i\phi}\sin\theta\,, \tag{2.53}$$

We can calculate the scattering amplitude of $(S, S_z) = (1, 0)$ state by

$$\langle s_1's_2'\,|\,1,0\rangle=\frac{1}{\sqrt{2}}\left(\langle s_1's_2'\,|+-\rangle+\langle s_1's_2'\,|-+\rangle\right)=\frac{1}{\sqrt{2}}\left(\mathcal{M}\left(+-\to s_1's_2'\right)+\mathcal{M}\left(-+\to s_1's_2'\right)\right)\,. \tag{2.54}$$

We can also check that for $\langle s_1's_2'\,|\,0,0\rangle$, the s-channel contribution trivially cancels out. The differential cross section of $(1,0)$ state is written as

$$\begin{aligned}(1+\epsilon)\left(\frac{d\sigma}{d\Omega}\right)_{(1,0)} &= \frac{1}{256\pi^2m_\chi^2}\sum_{s_1's_2'}\left|\langle s_1's_2'\,|\,1,0\rangle\right|^2 \\ &= \left|f(\theta)-f(\pi-\theta)\right|^2+4\alpha^2\epsilon^2G \\ &\quad -4\alpha\epsilon\cos\theta\left\{g\mathrm{Re}\left[f(\theta)-f(\pi-\theta)\right]-\frac{m_\phi\Gamma G}{m_\chi}\mathrm{Im}\left[f(\theta)-f(\pi-\theta)\right]\right\}\,,\end{aligned} \tag{2.55}$$

where the last term correctly reproduces Eq. (2.24) at $\mathcal{O}(\epsilon)$.

I leave some comments:

- The starting point, Eq. (2.3), seems incorrect when we consider higher order of k^2/m_χ^2. This is because $f(\theta)$ is obtained in the non-relativistic limit, where the spin-orbital interaction does not exist. In the spinor product $\bar{u}_{1'}u_1$, the non-trivial spin term such as $\xi_{s_1'}^\dagger\boldsymbol{\sigma}\xi_{s_1}$ will appear at higher order. If we calculate amplitudes at such an order, Eq. (2.3) will be inconsistent.
- We have used the non-relativistic expansion only for the spinor product at t- and u-channel. $\mathcal{O}(\epsilon^2)$ terms in Eq. (2.24) come only from the higher order terms of this spinor product, not from s-channel counterpart. Although such higher order terms are practically not important, we may use only $\mathcal{O}(\epsilon)$ result in the interference term for the whole consistency.

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