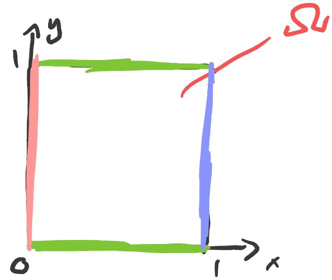
Exercise 1



(1) $$-\mu \Delta u + u_x = 0 \qquad \forall x \in \Omega$$

$$u = 0 \qquad x = 0$$

$$u = 1 \qquad x = 1$$

$$\frac{\partial u}{\partial n} = 0 \qquad y = 0,\ y = 1$$

$\mu \in \mathbb{R}$

<u>Ansatz:</u>

$$u(x,y) = X(x)\,Y(y)$$

$$-\mu X''Y - \mu X Y'' + X'Y = 0$$

$$(-\mu X'' + X')\,Y - \mu X Y'' = 0$$

$$\frac{(-\mu X'' + X')}{X} = \mu \frac{Y''}{Y} = -k_n^2$$

Look only at Y:

We have that $Y'' = -\frac{K_n^2}{\mu} Y$, so

$$Y(y) = A_n^{(y)} \sin\left(\frac{K_n}{\sqrt{\mu}}\, y\right) + B_n^{(y)} \cos\left(\frac{K_n}{\sqrt{\mu}}\, y\right)$$

We have dirichlet boundary conditions:

$$Y'(y) = A_n^{(y)} \frac{K_n}{\sqrt{\mu}} \cos\left(\frac{K_n}{\sqrt{\mu}} y\right) + B_n^{(y)} \frac{K_n}{\sqrt{\mu}} \sin\left(\frac{K_n}{\sqrt{\mu}} y\right)$$

Use $Y'(0)=0$:

$$Y'(0) = \underbrace{A_n^{(y)}}_{=0} \frac{K_n}{\sqrt{\mu}} \underbrace{\cos\left(\frac{K_n}{\sqrt{\mu}} 0\right)}_{=1} - \underbrace{B_n^{(y)}}_{\to B_n} \frac{K_n}{\sqrt{\mu}} \underbrace{\sin\left(\frac{K_n}{\sqrt{\mu}} 0\right)}_{=0}$$

So, we have:

$$Y(y) = B_n \cos\left(\frac{K_n}{\sqrt{\mu}} y\right)$$

From $Y'(1)=0$, we have:

$$Y'(1) = -\cancel{B_n \frac{K_n}{\sqrt{\mu}}} \sin\left(\frac{K_n}{\sqrt{\mu}} y\right) = 0$$

So, we have

$$K_n = \sqrt{\mu}\, n\pi, \quad n = 0, 1, 2, \dots$$

Look only at X:

$$\cancel{\mu} X'' - \cancel{\tfrac{X'}{\mu}} = \cancel{\mu} n^2\pi^2 X$$

Use substitution $Z = X'$

$$Z' = \frac{1}{\mu} Z + n^2\pi^2 X$$

$$X' = Z$$

On matrix form

$$\begin{bmatrix} X' \\ Z' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ n^2\pi^2 & \frac{1}{\mu} \end{bmatrix} \begin{bmatrix} X \\ Z \end{bmatrix}$$

Find eigenvalues:

$$\hookrightarrow \begin{vmatrix} -\lambda & 1 \\ n^2\pi^2 & \frac{1}{\mu}-\lambda \end{vmatrix} = (-\lambda)\left(\frac{1}{\mu}-\lambda\right) - n^2\pi^2$$

$$= \lambda^2 - \frac{1}{\mu}\lambda - n^2\pi^2$$

Solve quadratic equation for λ:

$$\sqrt{\frac{1}{\mu^2} + 4n^2\pi^2} = \sqrt{\frac{1 + 4\mu^2 n^2\pi^2}{\mu^2}} = \frac{\sqrt{1+(2\mu n\pi)^2}}{\mu} = \frac{\alpha_n}{\mu}$$

$$\lambda_n^{(1)} = \frac{1+\alpha_n}{2\mu} \qquad \lambda_n^{(2)} = \frac{1-\alpha_n}{2\mu}$$

Use solution for linear ODEs:

$$X(x) = A_n^{(x)} \exp\left(\lambda_n^{(1)} x\right) + B_n^{(x)} \exp\left(\lambda_n^{(2)} x\right)$$

$$X(0) = A_n^{(x)} + B_n^{(x)} = 0$$

$$B_n^{(x)} = -A_n^{(x)} = -\tilde{A}_n$$

$$X(x) = \tilde{A}_n \left(\exp\left(\lambda_n^{(1)} x\right) - \exp\left(\lambda_n^{(2)} x\right)\right)$$

Use *

$$X(x) = \tilde{A}_n \exp\left(\frac{x}{2\mu}\right)\left(\exp\left(\frac{\alpha_n}{2\mu} x\right) - \exp\left(-\frac{\alpha_n}{2\mu} x\right)\right) \Big| \; A_n = \frac{\tilde{A}_n}{2}$$

$$= A_n \exp\left(\frac{x}{2\mu}\right) \sinh\left(\frac{\alpha_n}{2\mu} x\right)$$

So, by superposition, we have:

$$u(x,y) = \sum_n \underbrace{A_n B_n}_{C_n} \cos(n\pi y) \exp\left(\frac{x}{2\mu}\right) \sinh\left(\frac{\alpha_n}{2\mu} x\right)$$

Using the boundary conditions

$$u(1,y) = \sum_{n=1}^{\infty} C_n \cos(n\pi y) \underbrace{\exp\left(\frac{1}{2\mu}\right) \sinh\left(\frac{\alpha_n}{2\mu}\right)}_{D_n C_n^{-1}}$$

$$\sum_{n=0}^{\infty} D_n \cos(n\pi y) = 1,$$

Use orthogonality of cos

$$\int_0^1 \cos(k\pi y) \sum_n D_n \cos(n\pi y)\, dy = \int_0^1 1 \cos(k\pi y)\, dy$$

$$\int_0^1 D_k \cos^2(k\pi y)\, dy = \int_0^1 \cos(k\pi y)\, dy$$

$$D_k \int_0^1 \cos^2(k\pi y)\, dy = \underbrace{\int_0^1 \cos(k\pi y)\, dy}_{=0\ \forall k \neq 0}$$

From this, we have

$$D_k = \begin{cases} 0, & k \neq 0 \\ 1, & k = 0 \end{cases}$$

Using the expression for D_0 & α_0:

$$C_0 = \frac{D_0}{\exp\left(\frac{1}{2\mu}\right)\sinh\left(\frac{\alpha_0}{2\mu}\right)}, \qquad \alpha_0 = \sqrt{1+(2\mu\pi \times 0)^2} = 1$$

So we obtain this expression for C_0:

$$C_0 = \frac{1}{\exp\left(\frac{1}{2\mu}\right)\sinh\left(\frac{1}{2\mu}\right)}$$

Combining everything we get

$$u(x,y) = u(x) = \frac{\exp\left(\frac{x}{2\mu}\right)\sinh\left(\frac{x}{2\mu}\right)}{\exp\left(\frac{1}{2\mu}\right)\sinh\left(\frac{1}{2\mu}\right)}$$