Locking

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April 28, 2021

Linear elasticity

- Linear elasticity is a model of how objects deform
- Equation from Hooke's Law (small deformations, isotropic media)
  - $-2\mu\nabla \cdot \epsilon(u) + \lambda\nabla(\nabla \cdot u) = f$
- u is the deformation field we want to find
- λ and μ are the material-dependent Lamé parameters
  - Loosly said:
  - λ is tied to compressibility, larger λ means harder to change the volume ratio
  - μ is tied to stiffness, larger μ means harder to deform keeping the same volume ratio
- ϵ is the symmetric gradient

Simplest form

- Homogenous Dirichlet boundary conditions
- The elasticity equation is given by:

$$\begin{aligned} -2\mu\nabla\cdot\epsilon(u)-\lambda\nabla\nabla\cdot u &= f \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega \end{aligned}$$

Weak form

Multiply by test function and integrate:

$$-2\mu\int_\Omega(\nabla\cdot\epsilon(u))\cdot v\mathrm{dx}-\lambda\int_\Omega\nabla(\nabla\cdot u)\cdot v\mathrm{dx} \quad = \int_\Omega f\cdot v\mathrm{dx}$$

Integration by parts:

$$2\mu\int_\Omega\epsilon(u)\cdot\nabla v\mathrm{dx}+\lambda\int_\Omega(\nabla\cdot u)(\nabla\cdot v)\mathrm{dx} \quad = \int_\Omega f\cdot v\mathrm{dx}$$

$A:B=A:B_S$ if $A=A^T$:

$$2\mu\int_\Omega\epsilon(u)\cdot\epsilon(v)\mathrm{dx}+\lambda\int_\Omega(\nabla\cdot u)(\nabla\cdot v)\mathrm{dx} \quad = \int_\Omega f\cdot v\mathrm{dx}$$

Weak form cont.

Find $u \in H_0^1$ such that

$$a(u, v) = f(v), \forall v \in H_0^1$$

where

$$a(u, v) = 2\mu(\epsilon(u), \epsilon(v)) + \lambda(\nabla \cdot u, \nabla \cdot v),$$

$$f(v) = (f, v)$$

Locking

- Locking arises from numerical errors when $\lambda \gg \mu$
- Difficulty in optimizing both $\nabla(u - u_h)$ and $\nabla \cdot (u - u_h) = 0$ at the same time

Locking from an optimisation standpoint

Solving the elasticity equation with the finite element method is equivalent to solving

$$\min_{u_h \in U_h} \|u - u_h\|_a^2 = \min_{u_h \in U_h} \left(\|\nabla (u - u_h)\|_{L^2}^2 + \lambda \|\nabla \cdot (u - u_h)\|_{L^2}^2 \right). \tag{1}$$

To understand locking, we let $\lambda \to \infty$ and obtain

$$\begin{aligned} &\min_{u_h \in U_h} \|\nabla (u - u_h)\|_{L^2}^2 \\ &\text{s.t. } (\nabla \cdot (u - u_h), v_h) = 0 \qquad \text{for all } v_h \in U_h. \end{aligned} \tag{2}$$

We can introduce a solid pressure, p, to circumvent locking

We introduce the solid pressure variable

$$p = \lambda \nabla \cdot u, \tag{3}$$

which, when used in the linear elastic equations becomes

$$\begin{aligned} -\mu \Delta u - \nabla p &= f, \\ \nabla \cdot u - \frac{1}{\lambda} p &= 0, \end{aligned}$$

which, as $\lambda \to \infty$ becomes Stokes equation.

We want to use the finite element method and need the weak form

Find $u \in H_0^1$ and $p \in L^2$ such that

$$\begin{aligned} a(u,v) + b(p,v) &= (f,v) && \forall v \in H^1 \\ b(q,u) - c(p,q) &= 0 && \forall q \in \mathbb{R} \end{aligned}$$

where

$$\begin{aligned} a(u,v) &= (\epsilon(u), \epsilon(v)) \\ b(p,v) &= (p, \nabla \cdot v) \\ c(p,q) &= \frac{1}{\lambda}(p,q) \end{aligned}$$

To understand why c is stabilising, we use the saddle point formulation

Solving the weak form of the two-field linear elastic equations is equivalent to solving the saddle point problem

$$\max_{p \in Q_h} \min_{u \in V_h} \{ \overbrace{a(u,u) - (f,u) + b(p,u)}^{\text{Bounded below}} \mathllap{\underbrace{\phantom{b(p,u) - \lambda^{-1}(p,p)}}_{\text{Bounded above}}} - \lambda^{-1}(p,p) \}$$

where λ^{-1} works as a regulariser which means that we can circumvent the inf-sup condition.

Numerical experiments

Let's consider a concrete example, where we have a true u given by

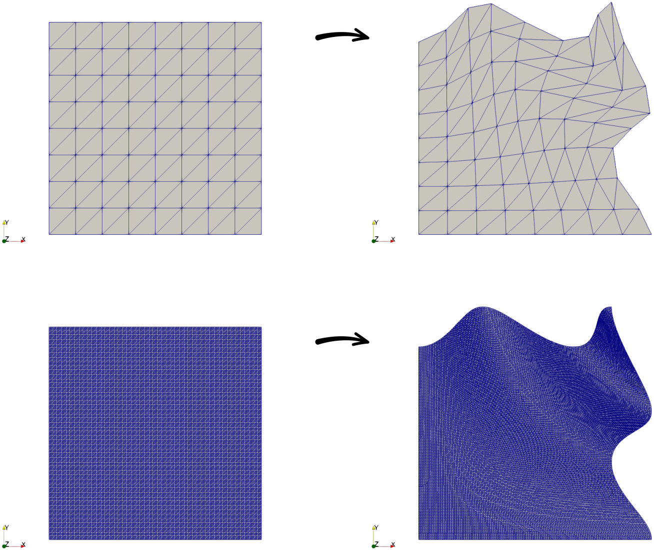
$$u = (\frac{\partial\phi}{\partial y}, -\frac{\partial\phi}{\partial x})$$

where

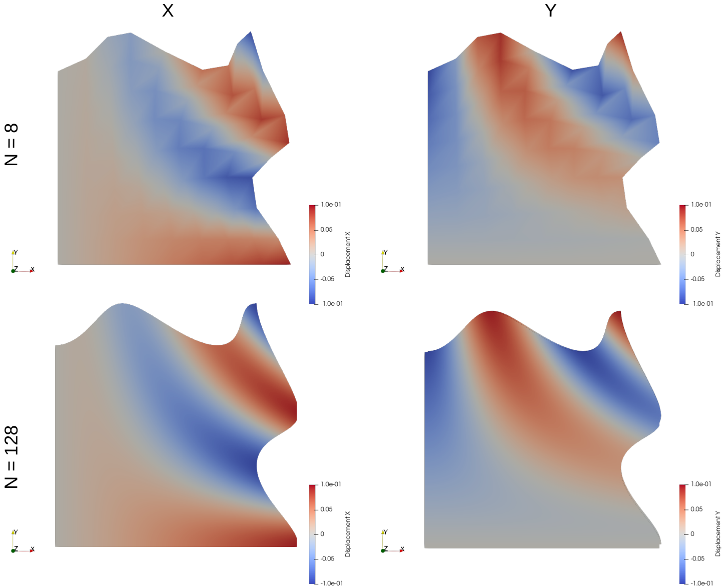
$$\phi = \frac{1}{100}\sin(3xy\pi)$$

implemented in FEniCS with assigned Dirichlet boundary conditions. Here the true divergence should be 0, meaning our solution should be dependent on μ only.

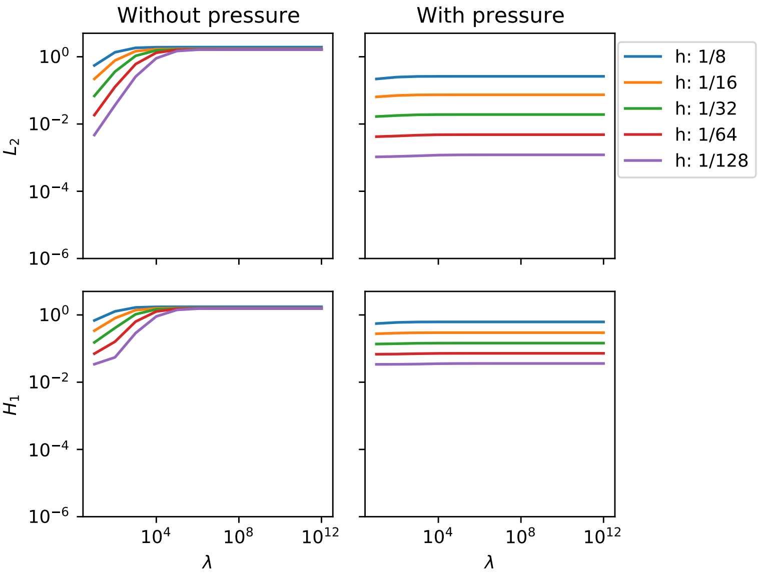
Deformation and meshes



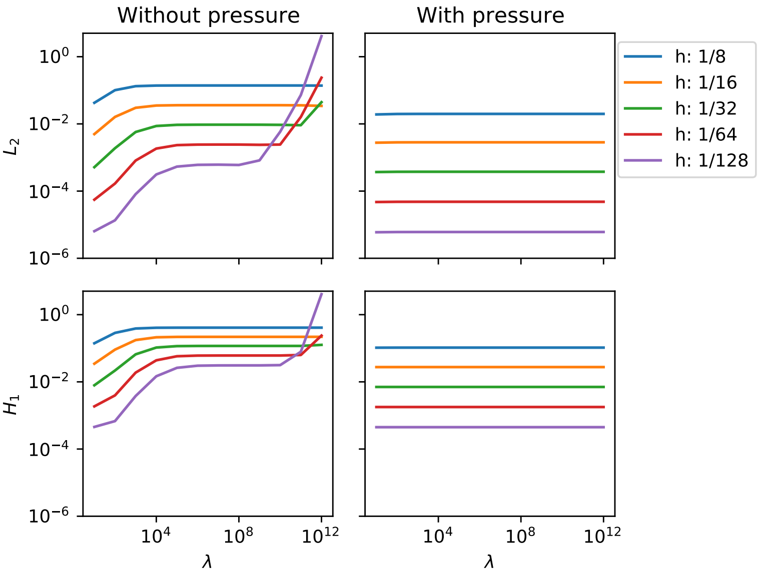
Deformation, component-wise



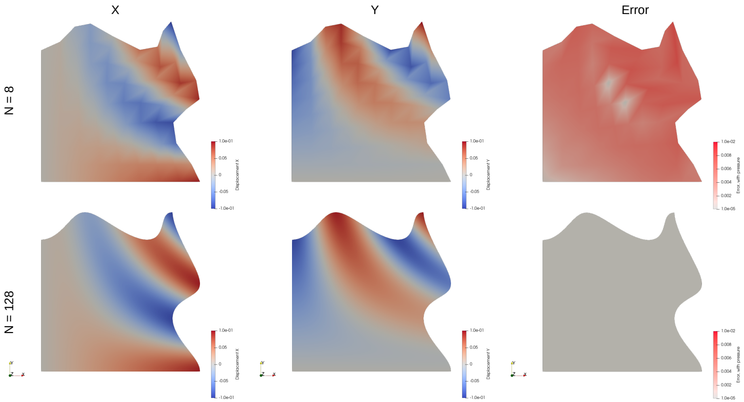
Relative Error, both u and p in CG-1



Relative Error, u in CG-2 and p in CG-1



Spatial plot of u, $\lambda = 10$ without solid pressure



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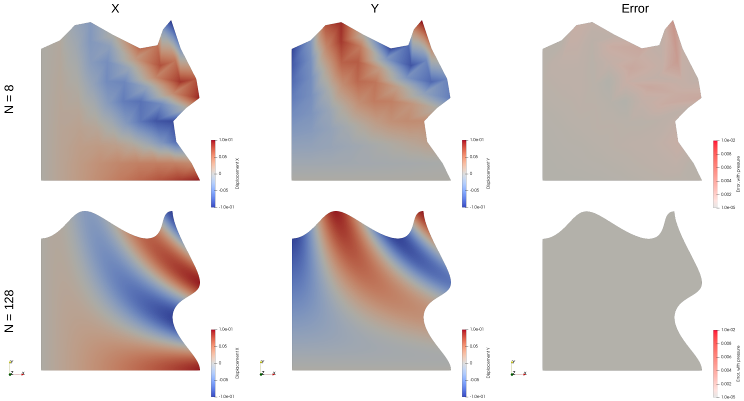
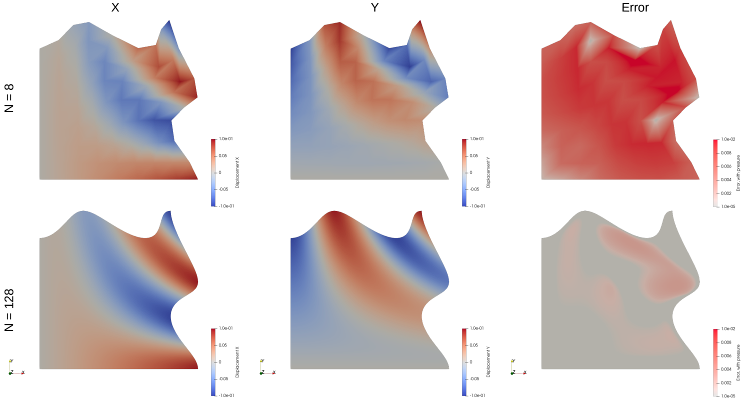
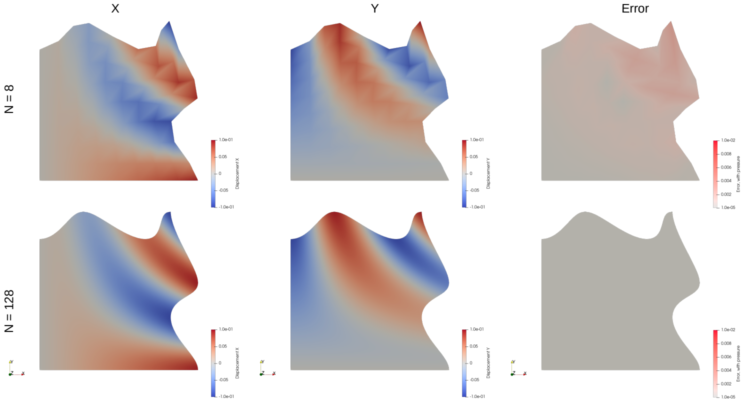


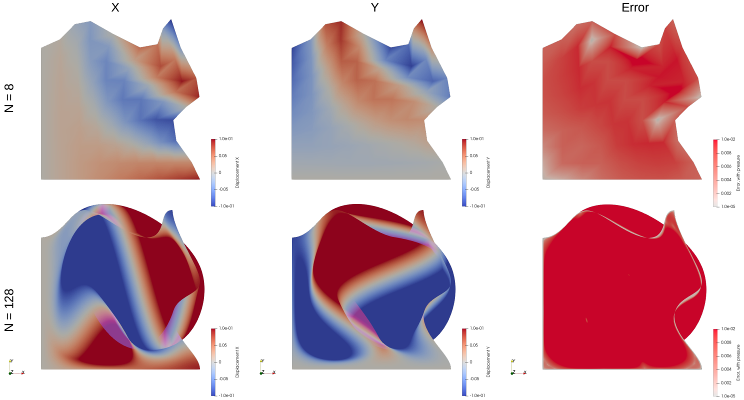
Spatial plot of u, $\lambda = 10^6$, without solid pressure



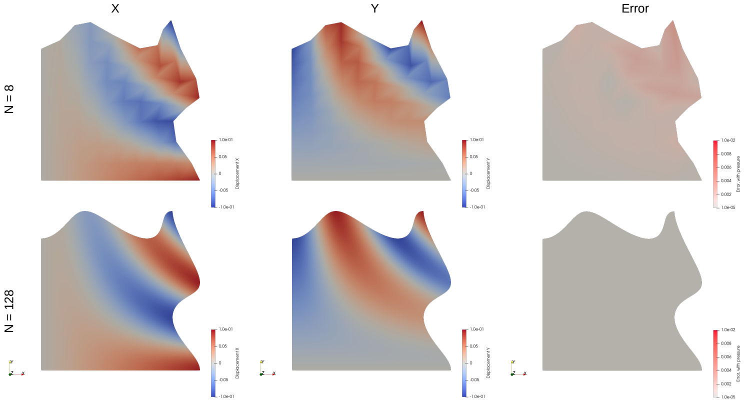
Spatial plot of u, $\lambda = 10^6$, with solid pressure



Spatial plot of u, $\lambda = 10^{12}$, without solid pressure



Spatial plot of u, $\lambda = 10^{12}$, with solid pressure



Conclusions

If in doubt, use a solid pressure formulation.
Code link?