The Impact of Financial Frictions on Innovation, Gibrat's Law, and Growth with Heterogeneous Firms

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Abstract

The firm size-growth relation plays a central role in many economic growth studies. Empirical literature shows that this relation depends on financial market conditions: sometimes small firms grow faster, but growth rates become independent of firm sizes when frictions are high (Gibrat's law). Contrast to most growth models that assume a fixed size-growth relation, I provide a framework in which it varies with financial frictions. In the model, firms of different sizes grow by internal (improving existing products) and external (developing new products) R&D, but R&D expenditure is restricted by profits (or cash flows). This setup allows for rich interactions between size-growth relation and compositional change of R&D types in evaluating how financial frictions affect aggregate growth. The model is consistent with the fact that financially constrained firms switch to internal R&D—a new finding revealed using matched U.S. firm-patent (1996-2016) data—and reduce R&D expenditure. The equilibrium is solved with neural networks. The estimated model suggests significant drop in growth and welfare after the Great Recession. Lastly, I quantify the importance of size-dependent policies that subsidize small firm R&D more.

***Keywords*:** Gibrat's law, innovation, external and internal R&D, financial frictions, Great Recession, neural networks, reinforcement learning, panel regression, heterogeneous firms

1. Introduction

The relationship between firm size and firm growth plays a central role in several classes of growth models and also in many models of firm dynamics. Most models assume a fixed relation: Some are based on Gibrat's law[1] that a firm's growth rate is independent of its size, while others feature a structure in which small firms grow faster than large firms[2]. Empirically, however, the firm size-growth relation is shown to be dependent on financial market conditions: Small firms grow faster[3] unless financial frictions are high, in which case we observe Gibrat's law[4][5]. This paper is among the first, to the best of my knowledge, to develop a unified framework that can account for both sets of observations, in which the relationship between firm growth and size depends critically on financial frictions—how difficult it is for firms to obtain outside financing.

The interaction between firm size-growth relation and financial frictions affects how firms innovate, which in turn, influences economic growth. By considering both firm dynamics and how it affects innovation, this framework allows me to analyze the aggregate implications of financial frictions with a much richer set of economic forces that are previously studied in isolation: How small and large firms react differently to frictions? How would they adjust not only the amount but also the type of R&D they perform? How will firm size distribution evolve?

The model is an extension of Akcigit and Kerr (2016) with financial constraints. Firms grow by performing two types of R&D: internal—improving existing products—and external—developing new products and capturing markets from others[6]. Subject to financial constraints, a firm's R&D expenditure is restricted by its internal funding: R&D costs cannot exceed a multiple (termed "pledgeability") of firm profits (so profits function as collateral).

[1] See Klette and Kortum (2004),Lentz and Mortensen (2008) and Acemoglu and Cao (2015).

[2] See Akcigit and Kerr (2016) and Acemoglu, Akcigit, Alp, Bloom, and Kerr (2018).

[3] See Sutton (1997); Caves (1998); Lentz and Mortensen (2008); Akcigit and Kerr (2016).

[4] See Leitão, Serrasqueiro, and Nunes (2010); Lensink, Steen, and Sterken (2005); Choi (2010); Fujiwara, Di Guilmi, Aoyama, Gallegati, and Souma (2004); Fariñas and Moreno (2000); Acs and Audretsch (1990). For a literature review, see Nassar, Almsafir, and Al-Mahrouq (2014); Santarelli, Klomp, and Thurik (2006); and Chapter 4 of Coad (2009).

[5] The literature provides empirical evidence of the effect of financial frictions/development on firm growth, in both advanced and emerging economies. Some explicitly study Gibrat's law, such as Lensink et al. (2005); Van Biesebroeck (2005); Audretsch and Elston (2006, 2010); Leitão et al. (2010). Others examine the impact of finance on small firm growth more specifically, as in Kim, Lin, and Chen (2016); Beck, Demirguc-Kunt, Laeven, and Levine (2008); Carpenter and Petersen (2002); Becchetti and Trovato (2002); Beck and Demirguc-Kunt (2006); Beck, Demirgüç-Kunt, and Maksimovic (2005); OECD (2014). Santarelli and Vivarelli (2007) also offer a survey on how credit constraints limit the survival and growth rate of small firms.

[6] An example of the first kind is Apple's upgrade of iPhone each year, and the second is Apple's introduction of a new electric vehicle business line (to compete with Tesla).

In the model, financially constrained firms reduce their overall R&D expenditure and increase the share of internal innovation. The latter shift in R&D trajectory stems from the fact that external R&D is more elastic than internal R&D (due to, for instance, higher fixed costs). As a result, external R&D decreases proportionally more than internal R&D when constraints worsen.

The tightness of the financial constraint dictates the size-growth relation. On the one hand, when financial frictions are low, R&D is not (too) restricted by its cost. Assuming that a firm's ability in external R&D does not increase proportionally (linearly) with its number of product lines, small firms will thus be better at introducing new products (relative to their size) and grow faster, even if all firms are equally skilled in internal R&D[7]. One the other hand, R&D can be significantly constrained when financial frictions are high. This has two consequences: First, internal R&D share increases so small firms lose their comparative advantage; Second, the overall level of R&D decreases, especially for small firms. Since external R&D is undirected, its cost depends on the aggregate level of productivity[8], so a less productive firm with lower profit will find R&D hard to afford. Such firms are also likely to be smaller. With these two results combined, small firm growth rates are disproportionally reduced.

The model is consistent with three empirical observations after the Great Recession, revealed using matched Compustat and PatentsView patent data. The first finding shows that small firms' (employment less than 100) real R&D expenditures dropped by more than 20% in the Great Recession, three times more than that of large firms. This confirms the argument that small firms were harder hit by the crisis[9].

The second empirical fact implies that firms will shift toward internal R&D when financial constraints tighten, and more so for small firms. To investigate whether the shift is indeed due to financial friction, I perform a correlated random effect IV Tobit panel regression. It shows that R&D type for firms in sectors more dependent on external finance (defined per Rajan and Zingales (1998)) display higher sensitivity to internal funding availability: A one standard deviation decrease in cash flow ratio is associated with a 6% to 9% increase in internal patent share, or about the medium share. I use the Great Recession as a natural experiment, and find that the sensitivity doubled in 2007–2009. Results are robust to different variable definitions, identification methods, and econometric models.

[7] Intuitively, external R&D is not perfectly scalable because it is hard to use the knowledge/expertise embedded in existing product lines for new product development. For instance, compared to a start-up, Apple may be better at smartphone innovation, but no more capable at electric car engineering.

[8] For example, when Apple wants to develop electric vehicles (EVs), its R&D costs depend on the quality (and cost) of EVs in general, not iPhone's quality or costs.

[9] Also see Kabukcuoglu (2017) for evidence in the U.S., and Schmitz (2018); Garcia-Macia (2017) for Spain.

Third, I document a drastic change in the size-growth pattern for U.S. innovative firms after 2007. Before 2007, there is a negative and significant relation between firm size and growth rate. However, this relation becomes insignificant after 2007. The result persists even after accounting for survival bias (Heckman), measurement error, and firm life cycle. This is consistent with the empirical literature that firm growth rate is affected by financial market conditions.

Before performing quantitative counterfactual analysis, I solve the equilibrium numerically and estimate model parameters by simulated method of moments. The model solution is challenging. Because external R&D is undirected, firm value functions depend on the entire productivity distribution, which is an infinite-dimensional object characterized by a system of integro-differential delayed equations influenced concurrently by firm decisions. The solution method is inspired by reinforcement learning. In essence, I expand the state space for the value function to include firm-specific states, endogenous variables such as growth rate, to-be-estimated parameters, and discretized productivity distribution density: in total, 121 dimensions. The value function is then solved as a neural network by iteration over the Hamilton-Jacobi-Bellman equation.

I consider three adverse scenarios for the economy: a 5% drop in profit pledgeability[10], a 5% drop in aggregate demand (which reduces profits), and both. In the third case, the size-growth relation and labor productivity growth rate match well with their empirical counterparts in 2007–2016, suggesting a 22.8% drop in consumption-equivalent welfare if such counterfactual adverse drops are permanent.

Based on the third scenario with both drops, I analyze the effects of two policies: One is a 10% R&D subsidy to all firms, and the other is size-dependent. The second policy has an ex ante average subsidy rate of 10%, but higher a rate on smaller firms. The uniform subsidy improves the welfare to 82.3% of the baseline economy and the size-dependent subsidy to 88.5%. This suggests that offering more support to smaller firms, which are both more productive at R&D and susceptible to financial market conditions, can be beneficial.

2. Related Literature

This paper provides a unified theoretical framework in which the size-growth relation varies with financial frictions. The Schumpeterian growth literature either predicts proportional growth, such as Klette and Kortum (2004) or a faster growth rate for small firms, as in Akcigit and Kerr (2016) and Acemoglu et al. (2018). This paper offers a reconciliation of

[10]Recall that R&D costs cannot exceed a multiple of profits. I estimate that the latter was reduced by 5% in 2007–2016 compared with 1997–2006.

these two strands of models by introducing a constraint on R&D expenditures, which reacts to changes in demand or financial market conditions[11].

Related to the study on the aggregate impact of financial frictions, this paper contributes by incorporating a rich set of firm heterogeneity: R&D type, firm size, and firm productivity. Earlier papers consider a representative firm setting[12]. More recently, some authors stress the important role of firm heterogeneity in amplifying the impact of shocks. For example, Garcia-Macia (2017) considers how firms with different intangible capital to capital ratios react to asset price shocks in an economy with collateral constraints. However, his model does not feature endogenous growth. More closely related to my paper is Schmitz (2018). We both consider heterogeneous firm and imperfectly scalable R&D technology, but our models differ in two ways. First, he studies a one-time financial shock by which firms are cut out of financial markets and must rely on self-financing, while my definition of friction is more general: The pledgeability of cash flow is a continuous variable. Therefore, in addition to analyzing the aftermaths of a financial crisis, my model is useful for comparing countries with different financial market development. Second, I incorporate both internal and external R&D—another important dimension of heterogeneity—because the two types of R&D make distinct contributions to aggregate growth (Akcigit and Kerr, 2016).

This paper is part of the nascent literature on solving complex continuous time heterogeneous agent models using machine learning techniques. Fernandez-Villaverde, Hurtado, and Nuno (2018) use supervised learning tools to approximate the evolution of aggregate variables. Duarte (2018) applies reinforcement learning to solve value functions as neural networks. Building on Duarte (2018), I extend his method by considering a model in which endogenous distribution of agent types directly affects agents' decisions, and incorporate such a distribution into the domain of value function.

Lastly, my work is related to the empirical literature on the relation between R&D and financial markets[13]. De Ridder (2016) studies the effect of the Great Recession on R&D for U.S. firms and Peia (2017) provides international evidence. Many papers also use natural experiments to analyze the effect of credit constraints on R&D: Chang, Chen, Wang, Zhang, and Zhang (2017) for the initiation of CDS trading; Nanda and Nicholas (2014) concerning the Great Depression; Chava, Oettl, Subramanian, and Subramanian (2013); Cornaggia, Mao, Tian, and Wolfe (2015) and Amore, Schneider, and Žaldokas (2013) for

[11]Empirically, this paper also provides evidence supporting the view that the validity of Gibrat's law is dependent on the economic environment (see footnotes 4, 5, and 3 for a brief review).

[12] Comin and Gertler (2006) and Anzoategui, Comin, Gertler, and Martinez (2017) study R&D in an expanding variety framework, and Barlevy (2007) considers a quality ladder model. Some papers study the effect of shocks and/or frictions on human/physical capital accumulation and growth, including Aghion, Angeletos, Banerjee, and Manova (2010) and Bianchi, Kung, and Morales (2017).

[13]See Hall and Lerner (2010) and Kerr and Nanda (2015) for a survey

banking deregulation; and Mann (2013) with court rulings.

The rest of the paper is organized as follows. Section 3 presents three empirical findings and Section 4 presents the model. Several theoretical results are derived in Section 5. Section 6 quantifies the model and examines counterfactuals and R&D policies, and Section 7 concludes. Detailed proofs and the data description are included in the Appendix.

3. Empirical Analysis

In this section, I document three empirical observations about R&D growth rate, type of R&D (i.e., internal vs. external) and firm size-growth relation that motivate the model.

3.1. Data

I employ the Compustat (firm data) and PatentsView (patents) dataset. The raw data spans Jan 1st, 1976 to May 16th, 2018. Firm and patent data are matched according to firm names. There are 85% exact matches and 15% fuzzy matches[14].

Following the selection method in Akcigit and Kerr (2016), the main sample I use contain for-profit non-financial, non-farm public U.S. firms that are continuously innovating. A firm is considered "continuously innovative" if it has conducted R&D or filed at least one patent in both of the two the 5-year windows, i.e., 1997-2006 and 2007-2016, when it was operational. For patents, I consider only utility patents granted to U.S. corporations. In total, there are 31,895 firm-year observations, 3,187 firms and 569,304 utility patents in the 1997-2016 sample. Detailed data development process is included in the Appendix B.1

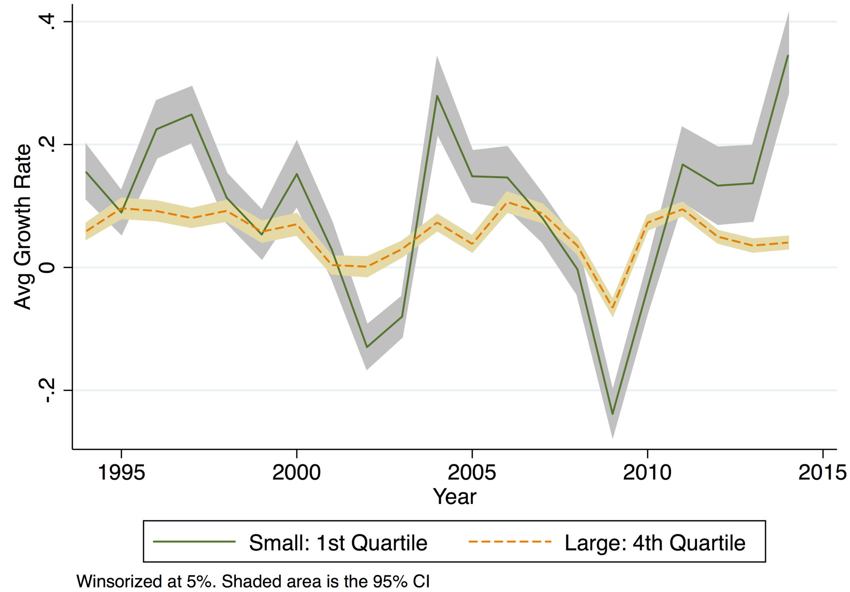
To get a sense of the sampled firms, consider a snapshot in 2015[15]. 57% of firms have more than 500 employees (53% in 1997) and 48% more than 1,000 (45% in 1997). In total, their R&D spending totaled $287 billion, accounting for 81% of total U.S. business R&D in 2015 . Their total sales is $5,033 billion, or 56% of all U.S. for-profit, non-farm companies with more than five employees in 2015[16]. Other summary statistics is in Appendix B.2.

[14]PatentsView is a public data source, which uses data derived from the US Patent and Trademark Office (USPTO) bulk data files. See www.patentsview.org. Due to typos or different abbreviation standards, firm names, even after standaraization, do not always match perfectly from different data sources. For fuzzy matching, I use the package FuzzyWuzzy in Python.

[15]2015 is the year of the most recent Business R&D and Innovation Survey (BRDIS) available as the writing of this paper. See www.nsf.gov/statistics/srvyindustry

[16]The corresponding Compustat entry was REVT.

Fig. 1. Real R&D Exp. Growth Rate by Firm Size



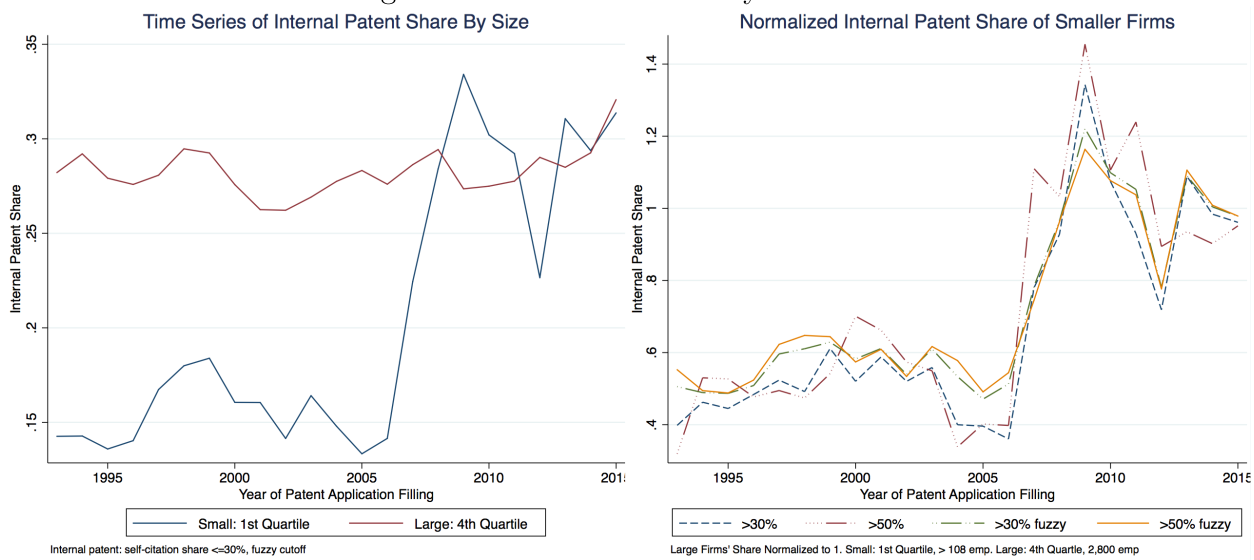
3.2. R&D Expenditure Growth Rate During the Great Recession

The empirical literature has argued that firms reduce the amount of R&D expenditure when financial constraints tighten. For example, Aghion, Askenazy, Berman, Cette, and Eymard (2012) documented the pro-cyclical R&D pattern for French firms with financial constraints (see section 2 for a brief literature review). Since the link between financial constraintsand R&D amount has been studied extensively in the literature, I will only provide some aggregate evidence in this section.

Figure 1 plots the change in level of R&D for firms of different sizes (similar to Barlevy (2007)). Real R&D expenditure growth rates dropped more for smaller firms[17], which are more likely to be financially constrained in the model in section 4. During the trough (2008-2009) of the crisis, for instance, large firms' R&D decreased by 7%, yet small firms slashed their R&D expenditure by more than 20%. Also visible from figure 1 is that small firms R&D spending is more volatile in general, suggesting that their capacity to smooth out their R&D is more limited.

[17]I categorize small and large firms according to the quartiles of firm employment. The quartile cut-offs are set based on the 2003 sample employment distribution: 108 and 2,800 employees respectively. Year 2003 is chosen arbitrarily and will not alter the results.

Fig. 2. Internal Innovation by Firm Size



3.3. *Internal/External R&D and Financial Frictions*

This section explores the relation between financial frictions and the type of R&D—internal vs. external innovation. I will first present aggregate-level evidence and then firm-level panel regressions.

3.3.1. *Aggregate Evidence*

I proxy internal R&D by internal patent share, which is defined based on patent citation patterns. More speficially, I first calculate the share of citations a patent made to its assignee's prior patents, termed self-citation[18], among its total backward citations. If a patent's self-citation share exceeds a certain threshold, it is counted as internal patent. Internal patent share is the ratio of internal patents to total patents applied in a given year[19]. I use four thresholds to define internal patents: 30%, 50%, 30% (fuzzy) and 50% (fuzzy). All four assign a unit weight to a patent (i.e. count as one internal patent) when its self-citation share, x, passes the threshold, $A = 30\%$ or 50%, whereas the latter two *fuzzy* measures also assign fractional weight, x/A , when $x < A$[20].

[18] For example, when a patent in iPhone X cites another Apple's patent, this citation is called self-citation.

[19] For the aggregate measure plotted in figure 2, I sum over all internal patents for small (large) firms by year, then divide it by all filed patents of small (large) firms to arrive at the ratio. Alternatively, I can first calculate the internal patent share for each firm-year observation, and then take the average across firms in a given year. However, this measure of sample mean is biased, because not every firm filed patents every year (Akcigit and Kerr, 2016).

[20] For instance, using 50% fuzzy threshold, a patent with 10% self-citation ratio is counted as 0.2 internal patents, but with 50% thresholdm it is counted as 0 internal patent.

Figure 2 plots the time series of internal patent by small and large firms, showing that small firms drastically increased their share of internal R&D after 2007. The left panel plots the internal patent share using fuzzy 30% threshold. Before 2007, small firms' internal patent shares fluctuated around 15% and large around 30%, suggesting that small firms focus more on external innovation (consistent with Akcigit and Kerr (2016)). After 2007, small firms internal share doubled to 30% (or more) yet, by contrast, that of the large firms remained stable. This distinct response to financial crisis, a period with difficult credit conditions, will be closely reflected in the model, where small firms are more vulnerable to financial market imperfections.

The right panel serves as a robustness check. The conclusion from the left panel holds under various internal patent definitions. To make different time series comparable, I normalized large firms' internal patent shares to one. They all point out that small firms switched toward internal innovation during the Great Recession.

3.3.2. Firm-Level Evidence

To further analyze the effect of financial constraintson R&D type, I employ firm-level panel regression analysis. In particular, I estimate the following equation:

$$\begin{aligned}\text{Internal Patent Share}_{it} = \alpha_0 &+ \alpha_1 \text{EFD}_s \times \text{Cash Flow Ratio}_{i,t-1} \\ &+ \alpha_2 \text{EFD}_s \times \text{Cash Flow Ratio}_{i,t-1} \times \text{Crisis}_t \\ &+ \gamma' \text{Firm and Sector Controls}_{i,t-1} + u_i + v_t + \epsilon_{it}\end{aligned} \tag{1}$$

Internal Patent Share$_{it}$ is a proxy for the internal innovation share. See section 3.3.1 for detailed definitions.

Cash Flow Ratio$_{i,t-1}$ is the lagged cash flow to total asset ratio[21].

Identification

The identification strategy relies on variation along two dimension: (1) firms in sectors that are dependent on external finance v.s. firms in other sectors; and (2) financial crisis (i.e. the Great Recession) v.s. other time periods.

Across Firm Variation

To capture the sector-wise variation in combination with firm-level heterogeneity, I construct the interacted variable

[21] Following Barlevy (2007), I define cash flow = Income Before Extraordinary Items (Compustat item IB) + Depreciation and Amortization (Compustat item DP). I use cash flow to asset rather than to physical capital because R&D is considered intangible capital investment. Nonetheless, I included cash flow to net property, plant and equipment (PP&E) in the robustness check.

$$\text{EFD}_s \times \text{Cash Flow Ratio}_{i,t-1}.$$

This variable measures the financial constraint faced by firm i, scaled by a sector-specific factor. On the one hand, cash flow ratio reflects the cash available to be deployed. One the other, EFD_s measures the importance of such cash for a typical firm in its sector. This is because EFD_s—*sector* (2-digit SIC code) external financial dependence ratio—is calculated as the sector median of *(capital expenditure - operational cash flow)/capital expenditure* using 1986-1992 Compustat data[22], so it is the external funding required as a fraction of capital expenditure. If we observe $\alpha_1 < 0$, it implies that as lower cash flow tightens constraints, firms perform more internal R&D.

The use of EFD_s comes with three advantages. Firstly, estimated over a long period to smooth out cyclical fluctuation, it captures the financing needs of an industry, determined by structural characteristics. Secondly, based on data prior to the sample period, it is free from concurrent compounding factors. Lastly, it is an industry level statistics and not affected by individual firms[23].

I employ four methods to address the endogeneity concern from $\text{Cash Flow Ratio}_{i,t-1}$. Firstly, it is lagged by one period, which precludes the simultaneous feedback from firm innovation choice to financing constraints. Secondly, I include year and firm fixed effects u_i and v_t to account for any time- or firm-invariant unobservables (e.g. the availability of new ideas at a given year and a firm's innate tendency toward internal innovation). Thirdly, multiple firm and sector controls are included for time and firm variant covariates. Lastly, cash flow terms are instrumented with the appropriated lagged cash flow ratios (lag three and further) to control for any remaining endogenous factors[24].

Firm controls are Tobin's Q, firm sale growth rate and log-transformed variables, including total assets, capital expenditure, net PP&E, cash and short-term investment, short-term and long-term debt, net sales and the total number of patents owned by a firm (a.k.a. patent stock). Firm controls are lagged by one year to address endogeneity. Additionally, I add 2-digit SIC real value-added growth rates to reflect the industry wide demand factors. *Across Time Variation*

The interaction of $\text{EFD}_s \times \text{Cash Flow Ratio}_{i,t-1} \times \text{Crisis}_t$ captures the potential change in the impact of financing constraints during the Great Recession, essentially treating the

[22] External financial dependence follows Rajan and Zingales (1998). Aghion et al. (2012) also show that innovative firms dependent on external financing are sensitive to credit conditions.

[23] See also in Laeven and Valencia (2013)

[24] Also note that for any omitted variables that are persistent (high autocorrelation), they are controlled by firm fixed effects u_i. For transitory ones (low autocorrelation), they are accounted by the lagged instruments and additional control variables.

crisis as a natural experiment. Crisis_t is a binary variable, equaling 1 if $t = 2007, 2008$ or 2009. If the crisis exacerbated the financing constraints *even if* cash flow and external financial dependence stayed the same, we would expect $\alpha_2 < 0$. A significant and negative α_2 thus signals deteriorated financial conditions that are not explained by a firm's own characteristics.

Econometric Assumptions

However, there are some complication arriving from the "small N large T " panel data structure and the fact that the dependent variable is left and right censored. I explain my econometric assumptions to cope with these difficulties next.

In the preferred specification, I use correlated random effect Tobit model with instrument variables.

The internal patent share is left-censored at 0 (1/3 to 1/2 of the observations) and right-censored at 100% (1% of the observations), because firms with 0 or 100% internal patent share can still have different degrees of "internalness" as I do not control for the actual contents of patents: some may conduct more exploratative R&D than others. Thus, it is more appropriate to use Tobit regression.

A concern for nonlinear regression like Tobit is that one cannot estimate u_i using fixed-effect models, due to the "incidental parameter bias". I follow the Chamberlain-Mundlak approach of correlated random effect by assuming

$$u_i = a_0 + a_1' \bar{x}_i + a_2' \bar{z}_i + e_i$$

where $\bar{x}_i = \sum x_{it}/T$, $\bar{z}_i = \sum z_{it}/T$, x_i is the independent variable(s), z_i other controls, and T the number of years when a firm is operational.

To further overcome potential endogenity issues, I used the 3-rd to 8th lags of independent variables as instruments[25].

Regression Results

The results are shown in table 1. Following Akcigit and Kerr (2016), I use 50% as the threshold of internal patent definition. The preferred specification, IV Tobit with correlated random effects, is presented in column (5). For robustness check, I also adopted fixed effect static panel regression (model 1), dynamic panel system-GMM estimator developed by Arellano and Bover (1995) and Blundell and Bond (1998) (model 2), random effect Tobit regression (model 3) and correlated random effect Tobit without instrument variables (model 4).

The first coefficient, or α_1 in (1), is negative and statistically significant, which implies

[25]The construction of instrument variables resembles Arellano-Bond. See Roodman (2009)

Table 1: Regression of Internal Patent Share and Financial Constraint

Models		(1) Fixed Effects	(2) System GMM	(3) RE Tobit	(4) CRE Tobit	(5) IV Tobit
Dependent Variable:						
Internal patent share, defined as patents with self-citation share larger than or equal to 50%						
Independent Variable:						
(Lag) Cash Flow/Asset $\times$ (1986-1992) Sector External Financial Dependence						
α_1	...	-0.00220	-0.0191**	-0.0358**	-0.0181*	-0.109***
		(0.0100)	(0.00847)	(0.0181)	(0.0109)	(0.0421)
α_2	$\times$Crisis	-0.0562**	-0.0728***	-0.118***	-0.105***	-0.0886*
		(0.0261)	(0.0166)	(0.0349)	(0.0351)	(0.0519)
Observations		6,239	5,300	6,239	6,217	6,217
Number of firms		1,365	1,084	1,365	1,365	1,343
Firm and sector controls		Y	Y	Y	Y	Y
Year FE		Y	Y	Y	Y	Y
Firm FE		Y	Y	Random	Correlated	Correlated
Instrument variables			Y			Y

a. *** $p < 0.01$, ** $p < 0.05$, * $p < 0.1$. Sample period: 2002-2016. All variables (including controls) are winsorized at 1% and 99%. Model (1) is within estimator with std. err. clustered by firm. Model (2) uses Arellano-Bover/Blundell-Bond system-GMM two-step estimator with the 3-rd till 8-th lags of indep. variables as instruments. Model (3) is a random effect Tobit model. Model (4) applies correlated random effects Tobit model, a.k.a. Chamberlain-Mundlak approach, where unobserved time-invariant effect is estimated as a function of sample-period averages of controls and independent variables (std. err. clustered by firm). Model (5) uses lagged indep. variables (3-rd and deeper) as instruments on top of model (4), and adopts Neweys (1987) two-step estimator.

b. External financial dependence follows Rajan and Zingales (1998) using 1986-1992 Compustat data, defined as the sector (2-digit SIC) median of (capital expenditure - operational cash flow) / capital expenditure. Firm controls are lagged by one year, including Tobin's Q and log-transformed total assets, capital expenditure, net PP&E, cash and short-term investment, short-term and long-term debt, net sales, sale growth rate and patent stock. Sector controls are real value-added growth rates.

c. For (2), the p-value of Hansen test is 0.163 and that of the autocorrelation test is 0.382 (thus valid IVs). For (5), the Wald exogeneity test p-value is 0.0457 (thus justify the use of IVs).

Table 2: Regression with Different Internal Patent Definition

	(1)	(2)	(3)	(4)
Dependent Variable:				
Internal patent share, defined with self-citation share larger than or equal to the following values				
Threshold of internal patent	30% Fuzzy	30%	50% Fuzzy	50%
Independent Variable:				
(Lag) Cash Flow/Asset × (1986-1992) Sector External Financial Dependence				
...	-0.0720***	-0.131***	-0.101***	-0.109***
	(0.0229)	(0.0464)	(0.0322)	(0.0421)
×Crisis	-0.0407**	-0.0574**	-0.0209	-0.0886*
	(0.0206)	(0.0289)	(0.0388)	(0.0519)
Firm and Sector Controls	Y	Y	Y	Y
Year FE	Y	Y	Y	Y
Firm FE	Correlated	Correlated	Correlated	Correlated
Instrument Variables	Y	Y	Y	Y
Observations	6,217	6,217	6,217	6,217

a. *** $p < 0.01$, ** $p < 0.05$, * $p < 0.1$. Sample period: 2002-2016. All variables (including controls) are winsorized at 1% and 99%. All regressions adopt the specification as model (5) in table 1. For details of controls, see footnotes of table 1.

b. Self-citation: when a patent cites its assignee's prior patents. Self-citation share for a patent = self-citations/total citations made (a.k.a. backward citations). I define internal patent using four thresholds: 30%, 50%, 30% (fuzzy) and 50% (fuzzy): all four assign a unit weight to a patent (i.e. count as one internal patent) when its self-citation share, x, passes the threshold, A, whereas the latter two also assign fractional weight, x/A , when $x < A$.

that when financial constraints loosen due to the increased internal funding (i.e. cash flow ratio increases), firms lean toward more external (exploratory) innovation (and thus internal patent ratio decreases), especially for firms in sectors more dependent of external financing (higher EFD_s). The coefficients are also economically significant. One standard deviation decrease in the independent variable (0.85) is associated with 9.3 percentage points increase in internal patent share, more than the median share 5%.

The estimates of α_2 in (1) are presented next. They are mostly negative and significant as well, except for the regression using 50% fuzzy threshold where the estimates are insignificant at 10% confidence level. In terms of magnitude, they are close to that of α_1, meaning that the relation between cash flow and R&D became much more significant during the 2007-2009 financial crisis. Everything else equal, firms are more sensitive to internal funding and more susceptible to credit market conditions. One interpretation is that firms became much more risk averse and thus averted external innovation. Alternatively, firms could find it harder to secure outside funding from banks or financial market and thus were more cautious on their R&D projects. Either interpretation suggests an adverse shock to firm R&D decisions.

Robustness Check

I preformed various robustness checks with different samples, variable definitions and

Table 3: Robustness Check of Internal Innovation and Financial constraintsRegression

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
Dependent Variable:								
Internal patent share, defined as patents with self-citation share larger than or equal to 50%								
Independent Variable:								
(Lag) Cash Flow/Asset × …								
… (86-92) Sector External Financial Dependence	-0.103*	-0.117***						
	(0.0562)	(0.0451)						
… Crisis × (86-92) Sector External Financial Dependence	-0.412**	-0.0849**						
	(0.171)	(0.0424)						
… (86-92) EFD Dummy			-0.266***					
			(0.0833)					
… Crisis × (86-92) EFD Dummy			-1.140**					
			(0.502)					
… (86-01) Sector External Financial Dependence				-0.0559**				
				(0.0234)				
… Crisis × (86-01) Sector External Financial Dependence				-0.0294				
				(0.0222)				
… (86-92) Sector External Financial Dependence					-0.148***			
					(0.0409)			
…						0.0807		
						(0.0611)		
… Crisis						-0.132***		
						(0.0477)		
… Dividend Dummy							-0.0939	
							(0.149)	
… Crisis × Dividend Dummy							-0.497*	
							(0.278)	
(Lag) Cash Flow/Net PP&E × …								
... (86-92) Sector External Financial Dependence								-0.000445*
								(0.000226)
... Crisis × (86-92) Sector External Financial Dependence								-0.0129***
								(0.00328)
Observations	6,217	9,366	6,217	6,217	6,217	6,217	6,217	6217
Number of Firms	1,365	1,708	1,365	1,365	1,365	1,365	1,365	1,365
Sample Period	2002-2016	1992-2016	2002-2016	2002-2016	2002-2016	2002-2016	2002-2016	2002-2016

[a] *** $p < 0.01$, ** $p < 0.05$, * $p < 0.1$. All variables (including controls) are winsorized at 1% and 99%. All regressions apply correlated random effects IV Tobit model as model (5) in table 1. Model (1) includes the full interaction; (2) expands the sample to 1992-2016; (3) uses external financial dependence (EFD) dummy and DID; (4) uses 1986-2001 EFD; (5) drops the interaction with crisis dummy; (6) drops the interaction with EFD; (7) uses dividend dummy; (8) uses cash flow to net pp&e ratio. The detailed explanation is in the main text.

[b] Self-citation: when a patent cites its assignee's prior patents. Self-citation share of a patent = self-citations/total citations made (a.k.a., backward citations). External financial dependence follows Rajan and Zingales (1998), defined as the sector (2-digit SIC) median of (capital expenditure - operational cash flow) / capital expenditure. The dividend payout dummy is constructed per Bond and Meghir (1994) where it equals zero when dividends are positive in both adjacent periods, and one otherwise. EFD Dummy = 1 for EFD ¿ median EFD across sectors.

[c] Firm controls are lagged by one year, including Tobin's Q and log-transformed total assets, capital expenditure, net PP&E, cash and short-term investment, short-term and long-term debt, net sales, sale growth rate and patent stock. Sector controls are real value-added growth rates.

identification methods. All robustness check regressions are based on IV correlated random effect Tobit (model 5 in table 1).

The first set of regressions adopt different definitions of internal patents, shown in table 2. As evident in the table, both estimates of α_1 and α_2 remain negative and significant.

The second set of results employ the 50% threshold and econometric assumptions as model (5) in table 1. The results are listed in table 3. In column (1), I include the full set of interaction terms[26] among EFD_s, Crisis_t and Cash Flow $\text{Ratio}_{i,t-1}$. The regression equation is now

$$\begin{aligned}\text{Internal Patent Share}_{it} =& \alpha_0 + \alpha_1 \text{EFD}_s \times \text{Cash Flow Ratio}_{i,t-1} \\ &+ \alpha_2 \text{EFD}_s \times \text{Cash Flow Ratio}_{i,t-1} \times \text{Crisis}_t \\ &+ \alpha_3 \text{Cash Flow Ratio}_{i,t-1} \times \text{Crisis}_t + \alpha_4 \text{Cash Flow Ratio}_{i,t-1} \\ &+ \gamma' \text{Firm and Sector Controls}_{i,t-1} + u_i + v_t + \epsilon_{it}\end{aligned}$$

Both α_1 and α_2 show similar conclusions as table 1.

I expand the sample by another ten years (prior to 2002) to be 1992-2016, shown in column (2).

For column (3), I replace the continuous variable EFD_s by a binary variable IEFD_s. $\text{IEFD}_s = 1$ if $\text{EFD}_s >$ median EFD_s across all industries. Firms with $\text{IEFD}_s = 1$ are considered as the treatment group, and Crisis_t is a natural experiment. I them adopt regression discontinuity by estimating

$$\begin{aligned}\text{Internal Patent Share}_{it} =& \alpha_0 + \alpha_1 \text{IEFD}_s \times \text{Cash Flow Ratio}_{i,t-1} \\ &+ \alpha_2 \text{IEFD}_s \times \text{Crisis}_t \times \text{Cash Flow Ratio}_{i,t-1} \\ &+ \alpha_3 \text{Cash Flow Ratio}_{i,t-1} \times \text{Crisis}_t + \alpha_4 \text{Cash Flow Ratio}_{i,t-1} \\ &+ \gamma' \text{Firm and Sector Controls}_{i,t-1} + u_i + v_t + \epsilon_{it}\end{aligned}$$

Similarly, α_1 and α_2 are reported.

I then change the calculation of EFD_s using 1986 to 2001 data instead of 1986 to 1992 as the main specification. α_1 is still significant at 5% level.

In column (5), I drop the Crisis_t from the interaction term in regression equation (1), essentially forcing the coefficient of $\text{EFD}_s \times$ Cash Flow $\text{Ratio}_{i,t-1}$, i.e. α_1 to be time-invariant. As expected, the estimate is negative and statistically significant.

Turning to (6), I drop the EFD_s in the interaction and only interact Cash Flow $\text{Ratio}_{i,t-1}$

[26] Common in the literature using external financial dependence, the full interaction set is not necessary. See Rajan and Zingales (1998); Laeven and Valencia (2013). Note that here it is not necessary to include the term EFD_s separately, as the sector time-invariant effect is alrealy captured by firm fixed effects.

with Crisis_t. In this way, I do not differentiate firms according to their external financial dependence. Instead, I analyze how the Great Recession, as a natural experiment, altered the relation between financing constraints and R&D for a given firm. The coefficient of $\text{Cash Flow Ratio}_{i,t-1}$ is insignificant, suggesting that on average, there is not strong evidence on financial friction's effect on R&D trajectory. This justifies controlling for external financing dependence in the main specification to identify their link (as mentioned in Aghion et al. (2012)). The coefficient of $\text{Cash Flow Ratio}_{i,t-1} \times \text{Crisis}_t$ is still negative and significant at 1% level, indicating that the financial condition worsened during the crisis.

For (7), I use the dividend payout pattern of a firm to determine when a firm is financially constrained or not. Following Bond and Meghir (1994), I construct a dummy variable, $\text{Div}_{i,t}$, that equals zero when dividends are positive in both adjacent periods, and one otherwise. $\text{Div}_{i,t} = 1$ therefore signals an abrupt cessation in dividend payout, suggesting financial hardship. The regression equation is

$$
\begin{aligned}
\text{Internal Patent Share}_{it} =& \alpha_0 + \alpha_1 \text{Div}_{i,t} \times \text{Cash Flow Ratio}_{i,t-1} \\
&+ \alpha_2 \text{Div}_{i,t} \times \text{Cash Flow Ratio}_{i,t-1} \times \text{Crisis}_t \\
&+ \alpha_3 \text{Cash Flow Ratio}_{i,t-1} \times \text{Crisis}_t + \alpha_4 \text{Cash Flow Ratio}_{i,t-1} \\
&+ \gamma' \text{Firm and Sector Controls}_{i,t-1} + u_i + v_t + \epsilon_{it}
\end{aligned}
$$

As shown in table 3, α_2 is negative and significant. Financially stressed firms during the crisis indeed changed their R&D to be more internally oriented.

One advantage of (7) is that $\text{Div}_{i,t}$ is now firm and time specific, allowing firms within the same sector to be categorized differently in terms of financial constraint, and enabling firms to switch in and out of constraints. Yet a major drawback is that most innovative small firms do not pay dividend at all in the whole sampling period (so $\text{Div}_{i,t} = 0$ for all t), and many firms that stopped dividend payout (so $\text{Div}_{i,t} = 1$ for all t) are mature and less innovative. In addition, firms may reduce dividend payout rather than stop it completely. Due to these drawbacks, the main specification applies external financial dependence.

Lastly in (8), I change the definition of cash flow ratio. It is now calculated cash flow to net PP&E ratio. The conclusion stays unchanged.

3.4. Firm Size-Growth Relation and the Great Recession

The last empirical fact pertains to the firm size-growth relation before and after the Great Recession in the U.S.. I find a negative relation between size and growth rate for all time periods but after 2007.

Table 4: Size-Growth Relation Before and After 2007

	Pre-Crisis		Post-Crisis	
	(1)	(2)	(3)	(4)
	2002-2006	1997-2006	2007-2011	2007-2016
	5-year	10-year	5-year	10-year
Log Employment	-0.0322**	-0.0336***	0.00142	-0.00668
	(0.0126)	(0.00749)	(0.00387)	(0.00457)
Constant	0.0137*	0.0188***	-0.0253***	0.0183***
	(0.00701)	(0.00650)	(0.00543)	(0.00588)
Observations	8,519	18,209	6,978	13,208
Sector-Year Controls	Y	Y	Y	Y

a. *** $p < 0.01$, ** $p < 0.05$, * $p < 0.1$ Windsorized at 1% and 99%. Standard errors are clustered by firm.

3.4.1. Baseline Regression

Following Akcigit and Kerr (2016), I run the regression:

$$g_{f,t} = \alpha + \eta_{\text{Sector}\times\text{Year}} + \beta \log(\text{Employment}_{f,t}) + \epsilon_{f,t} \tag{2}$$

where $g_{f,t} = \frac{\text{Employment}_{t+1}}{\text{Employment}_t} - 1$ and $g_{f,t} = -1$ if firm f exits at time $t + 1$.

Table 4 contains the regression results. Column (1) and (2) refer to the estimates based on the 5-year or 10-year periods before 2007, while (3) and (4) for the post 2007 counterparts. The estimates of β are negative and significant[27] prior to the Great Recession, indicating that small innovative firms grow faster than larger ones. However, such negative size-growth relation became statistically insignificant after 2007. In other words, we cannot reject Gibrat's law ($\beta = 0$) after the Great Recession.

The regression results echo the argument that size-growth relation is dependent on economic environment (see footnote 3), as firms found it harder to secure outside financing and suffered from diminished profits (inside financing) during the Great Recession. Both adverse development—tightened credits and dampened aggregate demand—will be reflected in the model as possible drivers of increase in financial frictions.

[27] Akcigit and Kerr (2016) estimate β to be -0.0351 with standard error 0.0013 over the 1982-1997 period, which is very close to my estimates.

3.4.2. Robustness Check

To test the robustness of the baseline results, I will control for firm survival bias, measurement errors and sample selections[28].

Table 5 presents the results. Column (1) is the baseline regression using 2002-2011 data with minor adjustment, given by

$$\begin{aligned} g_{f,t} = \alpha &+ \eta_{\text{Sector}\times\text{Year}} + \beta_1 \log(\text{Employment}_{f,t}) + \beta_2 \log(\text{Employment}_{f,t}) \times \text{Post 2007}_t \\ &+ \beta_3 \text{Post 2007}_t + \epsilon_{f,t} \end{aligned} \tag{3}$$

where indicator variable $\text{Post 2007}_t = 1$ for $t \geq 2007$. β_1 measures the pre-2007 size-growth relation, $\beta_1 + \beta_2$ the post-2007 relation and β_2 the difference between the two. Their estimates are the same as those in (1) and (3) in table 4. $\beta_2 > 0$ highlights the diminished size-growth relation after 2007.

I correct for the survival bias using two-step Heckman selection model. In the baseline regression, when a firm exits the sample data, I set its employment growth rate $g_{f,t}$ to -1. However, this overestimates the actual growth rate employment before a firm exits, as -1 is the lower bound. Since small firms have higher exit rates, the overestimation penalize small firm growth rates, and thus β_1 is biased upward (closer to 0). To apply the Heckman selection model, I defined selected firms as remaining (survived) firms. The selection equation is

$$\begin{aligned} 0 < &b_0 + b_1 \log(\text{Employment}_{f,t}) + b_2 \log(\text{Employment}_{f,t}) \times \text{Post 2007}_t + b_3 \text{Post 2007}_t \\ &+ b_4 \log(\text{Total Long Term Debt})_{f,t} + b_5 \log(\text{Debt Due in 1 Year})_{f,t} + u_{f,t} \end{aligned} \tag{4}$$

Model (2) shows the Heckman regression results. The right sub-column presents the selection equation. The negative coefficient of $\log(\text{Debt Due in 1 Year})_{f,t}$ shows that, after controlling for the total indebtedness by $\log(\text{Total Long Term Debt})_{f,t}$, firms with more short-term debts are associated with lower survival probability. Also, the positive b_1 estimate implies that large firms are more likely to survive.

From the left sub-column in (2), we see consistent results with the baseline regression. It shows a more negative β_1 as expected. In addition, positive β_2 estimate indicates that small firms growth rates were more negatively impacted after 2007.

Column (3) accounts for possible measurement error in the proxy for firm size, i.e. firm employment. Since employment appears both in the regressors and denominator of the left-hand size variables, a measurement error can generate spurious negative association between

[28] I follow similar robustness check as in Akcigit (2008)

Table 5: Robustness Check of Firm Size-Growth Relation

Model	(1) Baseline	(2) Heckman	Select	(3) IV	(4) Age
Dependent variable: Employment growth rate					
Independent variables					
Log employment	-0.0322**	-0.0394***	0.0700***	-0.0292**	-0.0283**
	(0.0126)	(0.0102)	(0.0104)	(0.0132)	(0.0129)
Log employment × Post 2007	0.0336**	0.0384***	0.0254*	0.0316**	0.0339**
	(0.0132)	(0.00770)	(0.0135)	(0.0137)	(0.0132)
Log total long term debt			0.00716**		
			(0.00354)		
Log debt due in 1 year			-0.0131***		
			(0.00394)		
Age					-0.00262***
					(0.000611)
Constant	-0.00388	0.0816	1.463***	-0.160*	0.0411***
	(0.00465)	(0.248)	(0.0242)	(0.0957)	(0.0114)
Observations	15,497	15,296 (1,162 censored)		15,497	15,497
Sector-Year FE	Y	Y		Y	Y
Instruments				Y	

[a] *** $p < 0.01$, ** $p < 0.05$, * $p < 0.1$. Sample period: 2002-2011. All variables (including controls) are winsorized at 1% and 99%. Robust std. err. clustered by firm in the parenthesis. "Post 2007" is an indicator variable.

[b] Model (1) is the baseline OLS with sector-year fixed effects. Model (2) uses Heckman selection model, with the selection equation in the "select" column. Model (3) is 2SLS with lagged log employment as instruments. In model (4), "age" is the year since a firm first appeared in Compustat (from 1970/1).

size and growth rate and thus downward bias on β_2 (away from 0). I address this concern using lagged log(Employment as instruments and apply two-stage least square model (2SLS). As expected, β_1 is closer to zero than model (1). Nonetheless, the main conclusion remains unchanged[29].

Lastly, I address the sample selection problem in column (4). Due to data limitation (Compustat), I only include public U.S. firms, so only 25% of the sample firms are considered typical "small" firms (less than 100 employees). This selection problem is solved by including age, a proxy of firm life-cycle, in the regression[30]. The negative coefficient of age indicates that firm growth rate decreases with age. The estimates of β_1 and β_2 remain relatively unchanged[31].

4. Model

In this section, I will build a model that can generate different size-growth pattern depending on the tightness of financial frictions. This model is consistent with the empirical facts summarized in section 3.

4.1. Environment and Preferences

I consider a closed economy in a continuous time setting, admitting a representative household with CRRA utility function

$$U = \int_0^\infty e^{-\rho t} \frac{c(t)^{1-\theta} - 1}{1-\theta} dt \tag{5}$$

where $\rho > 0$ is the discount factor, θ is the constant relative risk aversion parameter and $c(t)$ is consumption at time t. The household is populated by a continuum of individuals with measure one. Each one is endowed with one unit of labor, $L = 1$, that is supplied inelastically at wage rate $w(t)$. The household also owns all the firms in the economy, which generates a risk-free flow rate of return $r(t)$. Therefore, the household maximizes (5) subject to the intertemporal budget constraint

$$c(t) + \dot{a}(t) \leq r(t)a(t) + w(t) \tag{6}$$

[29] The 2SLS model controlled for firm-clustered standard errors (SEs), which makes standard overidentification test difficult to implement. I refit the model using heteroskedasticity-robust SEs (I obtain similar results as the cluster-robust SEs) and perform Wooldridges score test of overidentifying restrictions. The p-value for the test is 0.2588, suggesting that the instruments are valid.

[30] Age is calculated as the number of years since a firm first appeared in Compustat database.

[31] In addition, I also run the regression with firm fixed effects to control for unobserved firm characteristics. The conclusions from the baseline model still hold.

where $a(t)$ is the asset holdings of the household.

Individuals consume a final good $Y(t)$, which is also used for R&D (to be discussed later). The final good market is perfectly competitive. The final good is produced using a continuum of intermediate goods $j \in [0,1]$ according to the CES production technology:

$$Y(t) = A\left(\int_0^1 y_j(t)^{\frac{\epsilon-1}{\epsilon}} dj\right)^{\frac{\epsilon}{\epsilon-1}} \tag{7}$$

where $y_j(t)$ is the quantity of intermediate good j, $\epsilon > 1$ is the elasticity of substitution between products, and A is the *aggregate demand parameter*—for reasons that will be made clear in Section 4.2. The final good $Y(t)$ is chosen as the numeraire.

4.2. Intermediate Goods Production and R&D

Measure M (endogenously determined) of firms produce intermediate goods. Each firm, indexed by $f \in M \in (0,1)$, owns a countable set of product lines $j \in \mathbf{J}_f \subset [0,1]$. A firm is characterized by the number of product lines n_f it operates (n_f is the cardinality of the set $\mathbf{J}_f$) and its labor productivity z_f. Each good $j \in \mathbf{J}_f$ is produced according to the linear technology

$$y_j = z_f l_j \tag{8}$$

Compared with models such as Akcigit and Kerr (2016) and Acemoglu et al. (2018), I simplify the model by assuming the same productivity z_f for all $j \in \mathbf{J}_f$, instead of making the productivity good-specific (i.e., $z_j \neq z_i$ for $i, j \in \mathbf{J}_f$). With this simplification, I do not need to keep track of the multiset of productivity or the product portfolio of firm f. This will not affect the Schumpeterian (creative destruction) nature of the model, which will be explained in Section 4.2.2.

I define the aggregate labor productivity (or aggregate productivity for short) as

$$\bar{z} \equiv \left[\int_0^M \sum_{j\in\mathbf{J}_f} z_f^{\epsilon-1} d_f\right]^{\frac{1}{\epsilon-1}} = \left[\int_0^1 z_j^{\epsilon-1} d_j\right]^{\frac{1}{\epsilon-1}} \tag{9}$$

and the relative productivity by $\hat{z} \equiv z/\bar{z}$.

I omit the firm index f from now on when there is no confusion.

4.2.1. Profit Maximization

Each intermediate good producer is a monopolist[32]. The inverse demand function for each good j is

$$p_j = (Y/y_j)^{\frac{1}{\epsilon}} A^{\frac{\epsilon-1}{\epsilon}} = A \left(\frac{\bar{z}}{y_j} \right)^{1/\epsilon} \tag{10}$$

Therefore, I interpret A as an *aggregate demand parameter*.

As shown in Appendix A.1, in equilibrium, a firm characterized by (n, z) earns profits

$$\Pi(n, z) = \max_{\{y_j\},\{l_j\}_{j\in\mathbf{J}}} \sum_{j\in\mathbf{J}} [p_j y_j - w l_j] = n\bar{z}A\hat{z}^{\epsilon-1}/\epsilon \tag{11}$$

4.2.2. Innovation

As common in Schumpeterian models, the producer of each intermediate good j will be replaced by another firm (i.e., through creative destruction) unless it can keeps innovating and successfully maintain its monopoly. There are two ways for a firm to innovate: internal R&D and external R&D. The former improves a firm's productivity and the later expands its product lines by capturing markets from another incumbent.

Internal Innovation

Incumbents undertake internal innovation (a.k.a., R&D) to improve their labor productivity. Successful internal R&D arrives at the instantaneous Poisson rate

$$I = F_I(k_I) = \alpha_I k_I^{\beta_I} \tag{12}$$

where k_I is the internal research intensity, $0 < \beta_I < 1$ is the internal innovation elasticity, and $\alpha_I > 0$ is a scalar. Conditional on a successful internal innovation, firms' n and z update according to

$$n \to n \text{ and } z \to (1 + \lambda)z$$

where $\lambda > 0$ is a multiplicative factor for productivity improvement.

External Innovation

Firms can also conduct external R&D to develop products that they do not currently produce[33]. External innovations are realized with the Poisson flow rate

[32] One can rationalize this assumption using a two-stage bidding game as Akcigit and Kerr (2016).

[33] In Schumpeterian models such as this one, the economy-wise variety of products is constant. Firms compete to obtain market from other incumbents.

$$X \equiv nx_n = nF_x(k_x, n) = n\alpha_x k_x^{\beta_x} n^{\gamma} \tag{13}$$

where k_x can be interpreted as external research intensity, $\alpha_x > 0, 0 < \beta_I < \beta_x < 1$ and $-1 \leq \gamma \leq 0$.

The condition $\beta_I < \beta_X$ implies that the success rate of external R&D is more elastic to R&D intensity (R&D inputs). One interpretation is that external R&D will only succeed when its research intensity is higher than some thresholds[34], making the innovation outcome very sensitive around these thresholds.

Another key assumption is $-1 \leq \gamma \leq 0$, following Akcigit and Kerr (2016). As $\gamma \geq -1$, the rate of external R&D is increasing in n—the number of product lines—because firms with more product lines are likely to have expertise in multiple fields, raising their odds of developing new products[35]. In addition, $\gamma \leq 0$ suggests that external R&D production function has *diminishing return to* n (a.k.a., *imperfect scalablility*), because a firm's skill in *new* product development does not increase *linearly* with its expertise in its *current* product lines [36] (see also in footnote 7.).

Conditional on a successful external innovation,

$$n \to n+1 \text{ and } z \to \frac{nz}{n+1} + \frac{z'(1+\eta)}{n+1}$$

where z' is the new product line's original producer's labor productivity (recall that a firm has to capture market from an incumbent) and $\eta > 0$.

It should be noted that the firm's new productivity is a weighted average between its original z and new $z'(1+\eta)$. A firm's *total labor productivity* nz always increases after external R&D[37], now becoming $nz + z'(1+\eta)$. As a result, the economy-wise aggregate productivity $\bar{z}$ is always increasing. However, the *average labor productivity* z for a firm can decrease if $z' \ll z$. This allows the model to capture a *dilution effect* of introducing new product line, which is neglected by other Schumpeterian models[38].

Because external R&D is undirected across of all product lines $j \in [0, 1]$, a firm has equal

[34] This is quite reasonable considering one needs to develop products in a completely new field/market. A firm will have to spend a certain amount in R&D, such as hiring new researchers, conducting market survey, and general administrative adjustments, before they can develop a new product line.

[35] As suggested by Klette and Kortum (2004), n captures the human capital embedded in product lines.

[36] This is also supported by data. See Cohen (2010) for a literature review. I also provide some empirical evidences in 7.

[37] The incumbent who loses this product line has total labor productivity $(n'-1)z'$, decreased from $n'z'$.

[38] For a firm with very high productivity, it is reasonable to consider the possibility of lowered average productivity as a result of new product development. For example, one can think of Amazon's Fire Phone, Google's Google Glass or Google Plus, and Apple's Newton. The cessation of these product developments all increased corresponding firm's stock price.

probability to capture the market from any other incumbents. As a result, z' is a random draw from the current (endogenous) distribution of labor productivity. Furthermore, since each firm's product portfolio is of measure zero, firms will not innovate over their own product lines through external innovation.

4.2.3. R&D Cost Function

From equation (11) and Section 4.2.2, it is clear that we can equivalently characterize a firm by n and its relative labor productivity $\hat{z}$.

The cost of internal innovation in unit of final goods is $\bar{z}n\hat{z}^{\epsilon-1}k_I$. It is proportional to n and k_I, but convex in relative productivity $\hat{z}$ (strictly convex when $\epsilon > 2$).

To derive the cost of external innovation, we take expectation over firm specific productivity $\hat{z}$ and obtain $\bar{z}n\mathbb{E}(\hat{z}^{\epsilon-1})k_x = \bar{z}nk_x$, because external R&D is undirected (by construction, $\mathbb{E}(\hat{z}^{\epsilon-1}) = \int_0^1 \hat{z}_j^{\epsilon-1}dj = 1$). This implies that external R&D cost only depends on the *aggregate productivity*[39].

In total, a firm choosing innovation intensity k_I and k_x pays

$$R(k_I, k_x; n, z) \equiv \bar{z}n\hat{R}(k_I, k_x; \hat{z}) \equiv \bar{z}n[\hat{z}^{\epsilon-1}k_I + k_x] \tag{14}$$

units of final goods for R&D expenditure.

4.3. Financial Frictions

In this model, R&D expenditure cannot exceed a multiple of profits, or equivalently cash flow. This constraint is formally given by

$$R(k_I, k_x; n, z) \leq \iota\Pi(n, z) \tag{15}$$

where $\iota > 0$ is the *pledgeability* of profits. $\iota = \infty$ corresponds to a perfect capital market.

Empirical studies have stressed the importance of internal finance (e.g., cash flow or retailed earnings, both are related to profits), which has remained as the majority ($> 60\%$) source of funds for firms in all sizes (Fazzari, Hubbard, Petersen, Blinder, and Poterba, 1988). It is less affected by asymmetric information concerns common in external financing[40], and thus becomes essential for R&D—a type of intangible investment with high uncertainty[41].

[39] In standard Schumpeterian models as Klette and Kortum (2004), there is only external R&D and its cost is only dependent on aggregate productivity. See also in footnote 8.

[40] See Myers and Majluf (1984); Leland and Pyle (1977); Myers (1984) and many other related papers.

[41] See Kamien and Schwartz (1978); Brown, Fazzari, and Petersen (2009); Hall and Lerner (2010); Brown and Petersen (2011); Kerr and Nanda (2015).

Constraint (15) reflects the financial frictions (financial market imperfections) in the economy. Without any friction, firms can borrow indefinitely and thus afford any amount of R&D expenditure (i.e. $\iota = \infty$). When there is friction, a firm's debt capacity is limited. This limit is linked with firm profits in this model, because profits convey information of a firm's ability to generate cash flow and honor debt repayment[42].

Constraint (15) is analogous to the collateral constraints in the literature[43], which states that total debt cannot exceed a multiple of capital (i.e. collateral). Firms do not possess capital in this model, so profit flows serve the purpose of collateral.

With (11) and (14), (15) can be simplified into $\bar{z}n[\hat{z}^{\epsilon-1}k_I + k_x] \leq \iota n\bar{z}A\hat{z}^{\epsilon-1}/\epsilon$. In other words,

$$k_I + k_x/\hat{z}^{\epsilon-1} \leq \iota A/\epsilon \tag{16}$$

Immediate from (16), we can see that there are three ways for the financial constraint to tighten: (1) decrease in aggregate demand A; (2) decrease in profit pledgeability ι; and (3) decrease in relative productivity $\hat{z}$. The first two will affect every firm, while the third only small firms (in $\hat{z}$).

4.4. Discussion of Size-Growth Relation

There are two counteracting forces affecting firm size-growth relation: one from the innovation function (13) and the other from the financial constraint (16).

If financial friction is low (due to high A or ι), R&D expenditure will not be restricted by (16). Because of the diminishing return of n in (13), small firms (in n) grow faster and thus the size-growth relation is negative. This is the case analyzed in Akcigit and Kerr (2016).

In contrast, if the constraint (16) is tight, then small firms' (in $\hat{z}$) innovation endeavor will be severely contained, while large firms (in $\hat{z}$) will be relatively unrestricted. As a result, small firms can no longer grow faster, and the economy features an independent (and even positive) size-growth pattern.

Which of (13) or (16) prevails then depends on the severity of financial frictions. I will explore this result further in Section 5 and 6.

[42] Also, the limit is only linked with profits, rather than a direct function of the sufficient statistics of a firm: n and $\hat{z}$. One justification is that n and $\hat{z}$ are not observable (or verifiable) due to information frictions.

[43] Jermann and Quadrini (2012); Moll (2014); Midrigan and Xu (2014); Garcia-Macia (2017).

4.5. Value Functions

To summarize, a firm can be characterized by the pair $(n, \hat{z})$: the number of product lines and relative labor productivity. A firm's value will also depends on the aggregate productivity. Denote a firm's value function by $\mathbf{V}(n, \hat{z}, \bar{z})$. It follows the following lemma

Lemma 1 (Value Function)**.** *$\mathbf{V}(n, \hat{z}, \bar{z}) = \bar{z}V(n, \hat{z})$ and*

$$\begin{aligned}(r-g)V(n,\hat{z}) = \max_{k_I,k_x} & \underbrace{An\hat{z}^{\epsilon-1}/\epsilon}_{\textit{profit}} - \underbrace{n(\hat{z}^{\epsilon-1}k_I + k_x)}_{\textit{R\&D cost}} \\ & + \underbrace{F_I(k_I)[V(n,\hat{z}(1+\lambda)) - V(n,\hat{z})]}_{\textit{return from internal R\&D}} \\ & + \underbrace{nF_x(k_x,n)\left[\mathbb{E}_{\hat{z}'}V(n+1,\frac{n\hat{z}+\hat{z}'(1+\eta)}{n+1}) - V(n,\hat{z})\right]}_{\textit{return from external R\&D}} \\ & + \underbrace{n\tau[V(n-1,\hat{z}) - V(n,\hat{z})]}_{\textit{creative destruction}} \\ & - \underbrace{\frac{\partial V}{\partial \hat{z}}(n,\hat{z})g\hat{z}}_{\textit{change in } \hat{z}} + \underbrace{\varphi\left[\iota A\hat{z}^{\epsilon-1}/\epsilon - k_I\hat{z}^{\epsilon-1} - k_x\right]n}_{\textit{financial constraint}}\end{aligned} \tag{17}$$

where τ is the equilibrium creative destruction rate, $g = \dot{\bar{z}}/\bar{z}$ and φ the Lagrangian multiplier of the financial constraint.

Proof. See Appendix A.2 ■

Each incumbent firm maximizes its value by choosing R&D intensity k_I and k_x. The first line of the right hand side of (17) is the operating profit over currently held product lines net of R&D costs. The second line is the product of internal innovation Poisson arrival rate $F_I(k_I)$ and the change in firm value following internal innovation. The third line denotes the return from external innovation. The fourth line shows the change in firm value due to losing its product lines through creative destruction at Poisson rate τ. The first item in the final line represents the change in firm value due to the change in $\hat{z} \equiv z/\bar{z}$, as $\bar{z}$ grows at rate g. Lastly, we have the financial constraint faced by incumbents.

4.6. Entry and Exit

There is a unit measure of potential entrants. Each entrant has access to the external innovation technology similar to (13), different only up to a scalar. An entrant chooses an innovation flow rate $x_e > 0$ with cost (in terms of final goods) $\nu x_e^{\frac{1}{\beta_x}} \mathbb{E}[\hat{z}^{\epsilon-1}]\bar{z} = \nu x_e^{\frac{1}{\beta_x}} \bar{z}$, where $\nu > 0$ captures the entry costs. Upon a successful innovation, the entrant replaces an incumbent and start producing an intermediate goods. Denote the incumbent's (relative) labor productivity as $\hat{z}'$, the new entrant's productivity is $(1+\eta)\hat{z}'$. Since innovation is random, $\hat{z}'$ is drawn from the current (endogenous) distribution $\phi(\hat{z})$.

Denote $V(n, \hat{z})$ as the value of having n product lines with relative productivity $\hat{z}$, then entrants' maximization problem is

$$\max_{x_e} \bar{z} \left\{ x_e \mathbb{E}_{\hat{z}'} V(1, \hat{z}'(1+\eta)) - \nu x_e^{\frac{1}{\beta_x}}] \right\}$$

Entry rate x_e then satisfies the free-entry condition

$$\mathbb{E}_{\hat{z}'} V(1, \hat{z}'(1+\eta))/\nu = x_e^{\frac{1}{\beta_x}-1} \tag{18}$$

As in all Schumpeterian models, incumbents can lose some of their current product lines to other firms through competition (a.k.a., creative destruction). A firm that loses all product lines, i.e., $n = 0$, exits the economy.

4.7. Market Clearing and Stationary Distributions

I now close the model by specifying the market clearing conditions.

Final goods are used in consumption and innovation by incumbents and entrants. The market of final goods satisfies

$$Y = C + \bar{z} \int_{f \in M} n_f \hat{R}(k_I, k_x; \hat{z}_f) df + \bar{z}\nu x_e^{\frac{1}{\beta_x}} \tag{19}$$

In terms of labor market,

$$\int_{f \in M} l_f df = \int_0^1 l_j dj = 1 \tag{20}$$

The equilibrium is also characterized by the stationary joint distribution of relative productivity $\hat{z}$ and firm size n. Denote the joint distribution by $H(n, \hat{z}) = Prob(\tilde{n} = n, q \leq \hat{z})$. Its density $h(n, \hat{z})$ (and also the measure of firms M) satisfies the Kolmogorov forward equation listed in lemma 5 in the Appendix A.2. The marginal density of relative productivity $\hat{z}$ is then $\phi(\hat{z}) \equiv \sum_{n=1}^{\infty} h(n, \hat{z})$.

4.8. Aggregate Growth and Creative Destruction Rate

As shown in Appendix A.1, the standard Euler equation states that

$$g = \frac{\dot{C}}{C} = \frac{\dot{Y}}{Y} = \frac{\dot{\bar{z}}}{\bar{z}} = \frac{r-\rho}{\theta}$$

We can also characterize the growth rate g and creative destruction rate.

Lemma 2 (Aggregate Growth and Creative Destruction)**.**

$$g = \frac{\overbrace{\tau\left[(1+\eta)^{\epsilon-1}-1\right]}^{\text{external R\&D}} + \overbrace{\mathbb{E}_{\hat{z}}\left\{I(\hat{z})\left[(\hat{z}(1+\lambda))^{\epsilon-1}-\hat{z}^{\epsilon-1}\right]\right\}}^{\text{internal R\&D}}}{\epsilon-1} \tag{21}$$

where $I(\hat{z}) = M\sum_{n=1}^{\infty} F_I(k_I(n,\hat{z}))h(n,\hat{z})/\phi(\hat{z})$ is the internal innovation rate conditional on being type $\hat{z}$ firms, and M is the measure of incumbents.

The aggregate creative destruction rate is

$$\tau = \underbrace{M\sum_{n=1}^{\infty}\int_0^{\infty} nF_x(k_x(n,\hat{z}))h(n,\hat{z})d\hat{z}}_{\text{agg. external R\&D rate}} + \underbrace{x_e}_{\text{entry}} \tag{22}$$

Proof. See the Appendix A.2. ■

Lemma 2 shows that growth comes from both internal and external innovation. External innovation is affected by the rate of creative destruction τ, which is jointly determined by both incumbents' and entrants' R&D efforts. Equation (21) also implies that the step sizes of R&D, η and τ, and the composition of internal/external R&D alter the economic growth.

4.9. Stationary Equilibrium and Welfare

Finally, we can summarize the equilibrium of the economy.

Definition 1. *A stationary equilibrium consists of*

$$\{y_j, p_j, l_j, V(n,\hat{z}), k_x(n,\hat{z}), k_i(n,\hat{z}), x_e, M, h(n,\hat{z}), g, \tau, r, w\}$$

where

- *y_j, p_j and l_j maximize profit as in (11)*
- *$k_x(n,\hat{z})$ and $k_l(n,\hat{z})$ maximize $V(n,\hat{z})$. x_e solves entrants' problem as in (18)*
- *wage w is consistent with (19) and (20)*
- *the interest rate r satisfies the Euler equation $r = \rho + \theta g$*
- *The stationary distribution $h(n,\hat{z})$ and measure of firms M satisfy (30)*

- *g is given by (21)*
- *The creative destruction rate τ is given by (22)*

Normalize the initial aggregate productivity level to one, the welfare is given by the life-time utility of the household

$$U(c_0, g) = \int_0^\infty e^{-\rho t} \frac{c(t)^{1-\theta} - 1}{1-\theta} dt = \frac{1}{1-\theta} \left[\frac{c_0^{1-\theta}}{\rho - (1-\theta) g} - \frac{1}{\rho} \right] \tag{23}$$

where

$$c_0 = \bar{z} \left[A - M \int_0^\infty \sum_{n=1}^\infty n(\hat{z}^{\epsilon-1} k_I(n, \hat{z}) + k_x(n, \hat{z})) dH(n, \hat{z}) - \nu x_e^{1/\beta_x} \right]$$

As in Acemoglu et al. (2018), I compare the consumption-equivalent change ξ along the balanced growth path for two economies with g^1, c_0^1 and g^2, c_0^2. ξ is defined as

$$U(\xi c_0^2, g^2) = U(c_0^1, g^1)$$

The equilibrium[44] is rather complex and has no analytic solution. In the next section, I will derive some theoretical results to clarify the intuition of the model.

5. Theoretical Results

This section provides some theoretical results from the model and show that the model is consistent with the three empirical facts documented in Section 3: (1) R&D growth rate decreases when financial constraints tighten, more severely for small firms (Section 3.2); (2) Constrained firms switch toward internal R&D (Section 3.3); and (3) The relation between size and growth rate will approach zero (i.e. independent relation) from negative when financial frictions increase (Section 3.4).

5.1. *Firm Innovation and Financial Frictions*

We have the following result regarding a firm's innovation intensity.

[44] Note that the competitive equilibrium is inefficient, as in all Shumpeterian models, because firms do not fully internalize the positive externality of innovation. In addition, there is another source of inefficiency from the business stealing effect of external innovation (Aghion, Akcigit, and Howitt, 2014). However, these inefficiencies are not the focus of this paper.

Lemma 3 (Firm Innovation Strategies)**.**

$$k_I^* = \left[\frac{\alpha_I \beta_I [V(n, \hat{z}(1+\lambda)) - V(n, \hat{z})]}{n\hat{z}^{\epsilon-1}(1+\varphi)}\right]^{\frac{1}{1-\beta_I}} \quad \textit{and } k_x^* = \left[\frac{\alpha_x \beta_x [\mathbb{E}_{\hat{z}'} V(n+1, \frac{n\hat{z}+\hat{z}'(1+\eta)}{n+1}) - V(n, \hat{z})]}{n^{-\gamma}(1+\varphi)}\right]^{\frac{1}{1-\beta_x}}$$

where φ is the shadow price of the constraint

$$\iota A \hat{z}^{\epsilon-1}/\epsilon \geq k_I \hat{z}^{\epsilon-1} + k_x$$

Proof. See Appendix A.2 ■

The $1+\varphi$ term in the denominator indicates that, everything else constant, the tighter the financial constraint (thus higher φ), the lower innovation intensity k_I and k_x. This is related to the empirical fact analyzed in Section 3.2.

The next result pertains to Section 3.3.

Proposition 1 (Shift toward Internal Innovation)**.** *The ratio of internal to external innovation intensity $\frac{k_I}{k_x}$ is increasing in φ (decreasing in ι) when $\varphi > 0$.*

Proof. From lemma 3, we have $\frac{k_I}{k_x} \propto (1+\varphi)^{\frac{1}{1-\beta_x} - \frac{1}{1-\beta_I}}$. Then the proposition follows naturally from $\beta_x > \beta_I$. See Appendix A.2 for details. ■

Recall that β_I and β_x are the elasticities of R&D production functions from equation (12) and (13). $\beta_x > \beta_I$ indicates that the success rate of developing new products is more sensitive to R&D inputs than internal innovation (the intuition is included in the texts after equation (13)). Therefore, when total R&D expenditures drop, internal R&D will decrease more than that of external R&D. The ratio k_I/k_x will then decrease.

The shift in the composition of R&D types has implications on the aggregate growth rate (see lemma 2) and firm size-growth relation. The latter arises from the fact that internal R&D has constant return to n, so small firms are deprived of their advantages in innovation when they are forced to perform more internal innovation.

5.2. Firm Growth Rate and Financial Frictions

The section shows that the size-growth relation depends on financial frictions, relating to Section 3.4. From the equilibrium characterization in Appendix A.1, a firm's size measured by employment is proportional to $n\hat{z}^{\epsilon-1} \equiv Q_f$. Accordingly, firm growth is equivalent to the growth of Q_f. We have the following definition:

Definition 2 (Firm Growth Rate). *$g_f \equiv \mathbf{E}\left(\dot{Q}_f/Q_f\right)$ is the average growth rate of a firm with $n\hat{z}^{\epsilon-1} \equiv Q_f$. g_f is characterized by lemma 4 in Appendix A.2.*

The relation between g_f and size Q_f (i.e., $\frac{\partial g_f}{\partial Q}$) depends on two opposing forces, as discussed in Section 4.4. On the one hand, the diminishing return of external R&D with respect to n (a.k.a., *imperfect scalablility*) works in favor of smaller firms (small in n). On the other, the financial constraint works against smaller firms (small in $\hat{z}$). Which force prevails depends on the severity of financial frictions (ι and A).

To clarify the intuition, consider myopic firms as in Akcigit, Hanley, and Serrano-Velarde (2017), where firms only consider the current and immediate next period's profits, $\Pi(n, \hat{z})$. Also set $\epsilon = 2$ (linear profit function), $\gamma = -1$ (extreme inscalability) and normalize $A = 1$.

As shown in Appendix A.3, when the financial constraint is not binding, i.e. $\varphi = 0$, $g_f = \frac{C_x}{n^{\frac{2-\beta_x}{1-\beta_x}}\hat{z}} + C_I - \tau$, where C_x and C_I are two constants[45]. Therefore $\frac{\partial g_f}{\partial Q} < 0$. This is the case analyzed in Akcigit and Kerr (2016), in which smaller firms grow faster where only the first force, i.e. imperfect scalability, is at work.

However, when $\varphi > 0$, the sign of $\frac{\partial g_f}{\partial Q}$ is ambiguous (dependent on φ). When $\frac{\partial g_f}{\partial Q} = 0$, we will return to the case Klette and Kortum (2004) that the growth rate follows Gibrat's law and independent of size.

The above result is summarized in the following proposition:

Proposition 2 (Firm Growth Rate and Financial Frictions). *Consider myopic firms with $\epsilon = 2, \gamma = -1$.*

- *When the financial constraint is not binding, i.e. $\varphi = 0$, $g_f = \frac{C_x}{n^{\frac{2-\beta_x}{1-\beta_x}}\hat{z}} + C_I - \tau$, where C_x and C_I are two constants. The relation between size and growth rate is negative.*

$$\frac{\partial g_f}{\partial Q} < 0$$

- *When $\varphi > 0$, the size-growth relation is ambiguous.*

Proof. See Appendix A.3. ■

6. Quantitative Analysis

This section estimates the parameters in the model using simulated method of moments (SMM) and perform counterfactual and policy analysis.

[45] $C_x = \alpha_x(1+\eta)(\alpha_x\beta_x(1+\eta))^{\frac{\beta_x}{1-\beta_x}}$ and $C_I = \lambda\alpha_I(\alpha_I\beta_I\lambda)^{\frac{\beta_I}{1-\beta_I}}$

6.1. *Equilibrium Solution*

Solving this model is challenging. The introduction of financial constraint (15) significantly increases the complexity of the model. For Schumpeterian models[46], a firm's value function is usually separable in terms of firms' different product lines. In other words, production (and innovation) on each product line is independent. However, the financial constraint is global, because it is determined by *firm-wide* profits and R&D expenditures.

In addition, methods such as finite-difference are typically computationally inefficient in this model. The main reason is that in external innovation, $\hat{z}$ is not contained to movement around its neighbourhood[47]: it spans the whole distribution support according to $\hat{z} \rightarrow \frac{n\hat{z}}{n+1} + \frac{\hat{z}'(1+\eta)}{n+1}$.

The value function $V(n, \hat{z})$ (17) directly depends on the distribution of relative productivity $\phi(\hat{z})$, instead of a single-dimensional equilibrium variable (e.g. prices). Furthermore, the joint distribution of number of product lines and relative productivity $h(n, \hat{z})$ is determined by a complex system of delay integral-differential equations (see lemma 5 in Appendix A.2), which, in turn, are affected by firm value functions (firm choices).

To calculate the joint distribution, I discretize the state-space of n and $\hat{z}$ into a grid[48]. The discretized version of density $h(n, \hat{z})$ is the stationary distribution of a discrete Markov chain, of which the transition matrix depends on the firm innovation choices: Firms move up and down along n or $\hat{z}$ dimension within the grid with reflective boundaries. Given the joint distribution, $\phi(\hat{z})$ is represented by a discretized probability distribution (I use 108 grid points for $\hat{z}$). The details are in Appendix C.1.

The value function is approximated by a 3-layer neural network $V(n, \hat{z}, g, \tau, \phi(\hat{z}), \Omega)$. Ω are the 9 to-be-estimated parameters (explained later). The total input dimension is 121. The calculation of value functions is similar to Duarte (2018). It explicitly requires the convergence of Hamiltonian-Jacobian-Bellman (HJB) equation. Note the solution of $V(n, \hat{z}, g, \tau, \phi(\hat{z}), \Omega)$ is a function, so we can easily compute the value given any combination of its inputs without performing the value function iteration again.

I employ the Python packages TensorFlow and Keras to train the neural network (reinforcement learning). The details are given in Appendix C.2.

The overall algorithm is summarized below.

[46]Such as Lentz and Mortensen (2008); Akcigit et al. (2017) and Akcigit and Kerr (2016).
[47]Technically, the HJB equation will not produce a sparse matrix. See Achdou, Han, Lasry, Lions, and Moll (2017).
[48]As in Akcigit et al. (2017), I impose upper bounds on n and $\hat{z}$ (10 and 6) in the numerical solution.

Table 6: Calibrated Parameters

	Description	Value	Notes
ρ	Discount rate	0.02	Annual discount factor 0.97
θ	Inverse of the inter-temporal elasticity of substitution	2.00	Acemoglu et al. (2018)
ϵ	Elasticity of substitution between products	2.90	Acemoglu et al. (2018)
A	Production function scalar	1.00	Normalized

Algorithm Overview

I. Calculate the neural network approximiation of value function $V(n, \hat{z}, g, \tau, \phi(\hat{z}), \Omega)$ and thus innovation rates $I(n, \hat{z}) = F_I(k_I(n, \hat{z}))$ and $X(n, \hat{z}) = nx(n, \hat{z}) = nF_x(k_x(n, \hat{z}), n)$

II. Find the equilibrium for any given parameters Ω

  (i.) Guess equilibrium g and τ
  - Guess relative productivity distribution $\phi(\hat{z})$
  - Calculate innovation rates using the value functions
  - Calculate the stationary joint distribution $h(n, \hat{z}; g, \tau, \Omega)$
  - Update $\phi(\hat{z})$
  - Repeat until $\phi(\hat{z})$ converges

  (ii.) Update g' and τ' according to (21) and (22)

  (iii.) Solve the fixed points of $g' = F_g(g, \tau; \Omega)$ and $\tau' = F_\tau(g, \tau; \Omega)$, where F_g and F_τ denote the functions that return the updated value g' and τ'. This step is solved using Python non-linear function solver.

6.2. Estimation

This section presents the estimation procedure of model parameters. Four parameters listed in table 6 are externally calibrated to values common in the literature. The rest nine parameters in table 7, or Ω, are estimated by matching data moments.

I use simulated method of moments (SMM) to estimate the parameters. Define $\Gamma(\mathbf{Y})$ and $\Gamma(\Omega)$ to be the data ($\mathbf{Y}$) moments and simulation moments. The estimator minimizes

$$\hat{\Omega} = \underset{\Omega}{\operatorname{argmin}} [\Gamma(\mathbf{Y}) - \Gamma(\Omega)]^T \mathbf{W} [\Gamma(\mathbf{Y}) - \Gamma(\Omega)]$$

where $\mathbf{W}$ is a diagonal weighting matrix with $\mathbf{W}_{ii} = 1/\Gamma_i(\mathbf{Y})^2$ and $\mathbf{W}_{ij} = 0$ for $i \neq j$. The aggregate growth rate's weight is increased by a factor of three (Akcigit et al., 2017). The simulated moments $\Gamma(\Omega)$ are calculated by Monte Carlo with 5,000 firms.

All parameters are identified jointly. The next subsection provides a heuristic discussion

Table 7: Estimated Parameters

Parameter	Description	Estimated value
β_I	Internal innovation elasticity	0.408
β_x	External innovation elasticity	0.471
α_x	External innovation scalar	0.501
α_I	Internal innovation scalar	6.483
η	External innovation step size	0.099
λ	Internal innovation step size	0.018
γ	External innovation scalability	-0.463
ν	Entry cost scalar	2.482
ι	Profit pledgeability scalar	5.76e-4

on identification.

6.2.1. Empirical Moments and Identification

In the model, size is jointly determined by n and $\hat{z}$. Revenue, profit and employment are linear in $Q_f = n\hat{z}^{\epsilon-1}$. Since total employment $l_f = Q_f$ in equilibrium, I will use employment as the size measure.

Growth Regression Coefficient Γ_1

The empirical regression is given by table 4. In simulation, it is the regression of firm growth rate $g_f(Q_f)$ on $\log(Q_f) \equiv \log(n\hat{z}^{\epsilon-1})$. As mentioned in the theoretical section 5, Γ_1 is particularly sensitive to γ and ι.

Firm Entry Rate Γ_2

Due to data limitation (no Census data), I will use the annual average firm death rate from Business Dynamics Statistics[49]. Entry rate is estimated as the annual average of employment share of age 0 & 1 firms. The model equivalent is $x_e\mathbb{E}(\hat{z}^{1-\epsilon})$. Entry rate will discipline ν, η, ι, α_x and β_x.

Aggregate Labor Productivity Growth Rate Γ_3

The aggregate labor productivity is calculated as real total value-added per worker. Value-added is the difference between net sales and cost of goods sold. Growth rates are computed as geometric averages. It is particularly affected by τ and innovation step sizes η and λ, as shown in equation (21).

Firm Growth Rate by Size Γ_4, Γ_5

They are defined as the averaged annual growth rates of total employment for firms larger

[49]https://www.census.gov/ces/dataproducts/bds/data_firm.html. I exclude sectors 00-09 (AGR Agriculture, Forestry, and Fishing) and 60 (FIRE Finance, Insurance, and Real Estate). The time series plot is included in the Appendix B.2

Table 8: Model and Data Moments

Moments	Data	Model
Growth regression coefficient	-0.034	-0.040
Entry employment share	0.055	0.049
Aggregate labor productivity growth rate	0.030	0.027
Small firm employment growth	0.076	0.087
Large firm employment growth	0.063	0.045
Small firm productivity growth	0.069	0.091
Large firm productivity growth	0.009	0.006
Small/large firm internal patent share ratio	0.542	0.671
Small/large firm R&D intensity ratio	2.474	2.968

or smaller than median size in the sample. They are informative in determining γ, τ, η and λ and innovation production function. The expression of $g_f(Q_f)$ is given by 29 in lemma 4.

Firm Relative Productivity Growth Rate by Size Γ_6, Γ_7

The model equivalent is $\mathbb{E}[I\lambda + nx\frac{(1+\eta)\hat{z}'/\hat{z}-1}{n+1}] - g$, where I and x are (firm-specific) Poisson arrival rates of internal and external R&D.

Internal Innovation Ratio by Size Γ_8

I define Γ_8 as the ratio of internal innovation share between large and small firms, where internal innovation share is proxy by the share of internal patents using fuzzy 30% cutoff. It corresponds to $\mathbb{E}(\frac{F_I}{nF_x+F_I})$ in the model.

Innovation Intensity by Size Γ_9

Innovation intensity is measure as R&D expenditure to net sales ratio, or $\epsilon(k_I + k_x\hat{z}^{1-\epsilon})/A$. Γ_9 is informative on innovation production function parameters, ι and γ

6.2.2. Model Fit

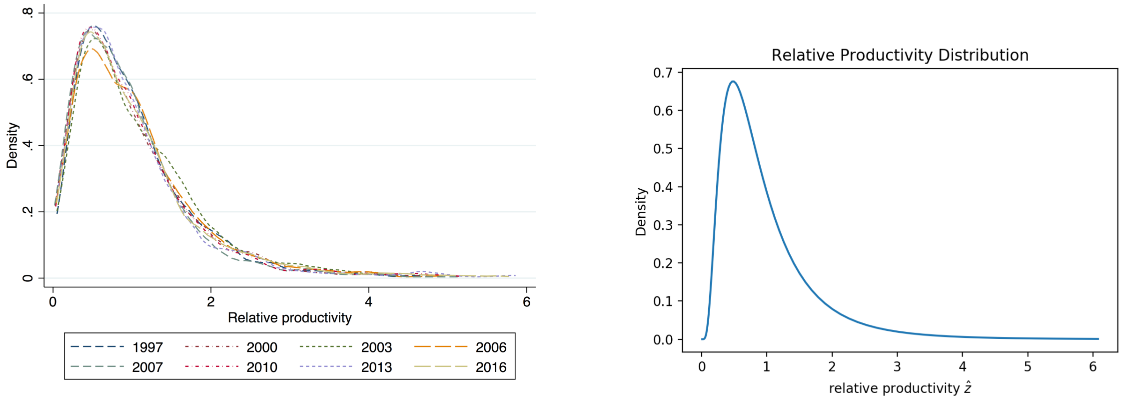
The estimation results are shown in table 6.2.2. The simulated moments match the empirical counterparts relatively well.

Besides targeted moments, I also compare the untargeted moments, namely relative productivity distribution and firm size distribution.

The model produces similar relative productivity[50] distribution as the empirical one, shown in figure 3. On the left panel, I plot the distribution for each three years throughout the sample periods. It shows a stable distribution over time, even considering that the sample spans both the early 2000s Dot-Com bubble and the Great Recession. The model, however, cannot fully match the left tail of the empirical distribution. One explanation is

[50] Relative productivity $\hat{z}_i = z_i/\bar{z}$. $\bar{z}$ = Total real value added/Total employment. Same for z_i

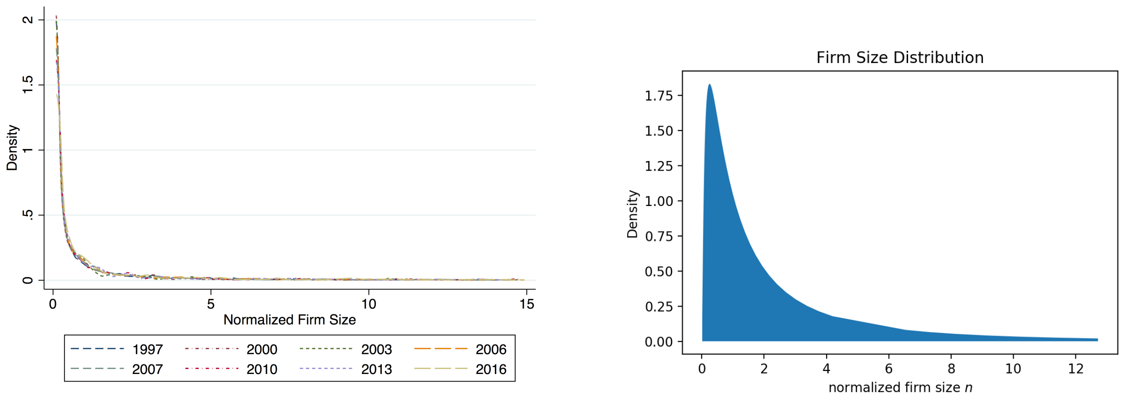
Fig. 3. Relative Productivity Distribution



(a) Data

(b) Model

Fig. 4. Firm Size Distribution (Normalized to Unit Mean)



(a) Data

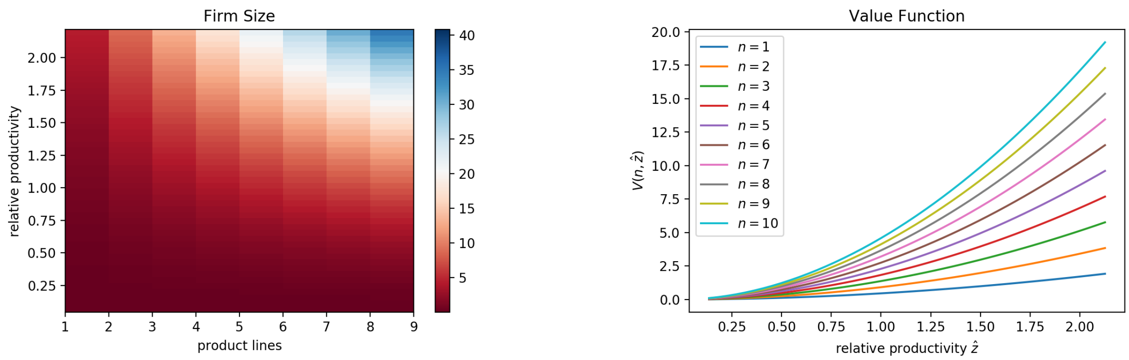
(b) Model

that the Compustat dataset does not contain enough small firms. Also, the model does not feature fixed operation cost or exogenous exits, so the survival rates (and growth rates) of small firms can be higher from the simulation[51].

The simulated firm size distribution also resembles its empirical counterpart as in figure 4. In both the data and model, firm size is defined as total employment, or $n\hat{z}^{\epsilon-1}$. To make the two comparable, I normalize both distribution to have unit mean. The empirical distribution takes the shape of a Pareto distribution,consistent with the literature (e.g. Klette and Kortum (2004)). The simulated one has less mass concerntrated on the left tail, which is inherited from the relative productivity distribution in figure 4(b).

[51]/This is also apparent in table 6.2.2. Compared to the empirical moments, model generates higher (lower) growth rates (in both employment and productivity) for small (large) firms. The simulated growth coefficient, -0.040, is also more negative.

Fig. 5. Firm Size and Value Function



(a) Firm Size (b) Value Function

6.2.3. Characterization of the Economy

In this section, I will discuss some properties from the simulated economy.

Value Function

For illustration, I plot the relation between size and $(n, \hat{z})$ in figure 5, which is determined by $Q = n\hat{z}^{\epsilon-1}$.

A firm's value function, $V(n, \hat{z})$ in equation (17), is plotted in figure 5. $V(n, \hat{z})$ is increasing in both the number of product lines n and relative productivity $\hat{z}$.

Innovation Rate

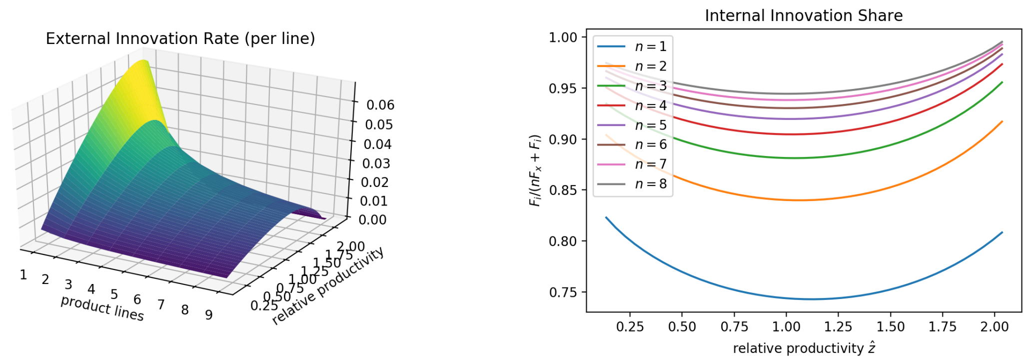
Figure 6 depicts the innovation policy function for different n and $\hat{z}$. External innovation rate per product line[52], $F_x(n, \hat{z}) = \alpha_x k_x^{\beta_x} n^{\gamma}$, is shown in the left panel. Across the n-axis, F_x is decreasing as the number of product line increases. This reflects the comparative advantage in external innovation for firms with less product lines, captured by the n^{γ} term in the model.

Along the $\hat{z}$-axis, we see an inverted-U shape of F_x. The upwar-sloping fraction results from the financial frictions that R&D expenses cannot exceed a multiple of profits, $\iota A\hat{z}^{\epsilon-1}/\epsilon \geq k_I\hat{z}^{\epsilon-1} + k_x$. Therefore, given n, higher $\hat{z}$ will relax the constraint, enabling more R&D and higher innovation rate. This is a countervailing force against the inscalability of external innovation. It works in favor of firms large in terms of $\hat{z}$.

The downward-sloping part comes from the fact that after a successful external innovation, a firm's productivity $\hat{z}$ updates according to $\hat{z} \to \frac{n\hat{z}}{n+1} + \frac{\hat{z}'(1+\eta)}{n+1}$. For a firm with

[52] The external innovation intensity $k_x(n, \hat{z})$ shows similar pattern.

Fig. 6. Innovation Rate



(a) External Innovation Rate

(b) Internal Innovation Share

very high $\hat{z}$, it will be unlikely for it to obtain a higher $\hat{z}'$, and thus the benefit of external innovation diminishes. This is the *dilution effect* analyzed in Section 4.2.2.

The internal innovation share, $\frac{F_I(n,\hat{z})}{F_I(n,\hat{z})+nF_x(n,\hat{z})}$, is shown on the right panel of figure 6. Consistent with observations from external innovation rate, smaller firms in terms of n has higher internal innovation share, similar to Akcigit and Kerr (2016). We also see a U-shaped relation between $\hat{z}$ and internal innovation share.

6.3. *Counterfactual and Policy Analysis*

In this section, I will perform several counterfactual analyses and policy experiments to quantify the implications of financial frictions, and how to design appropriate policies.

6.3.1. *Quantifying the Financial Constraint*

The Great Recession is a period with sizable change in financial market and aggregate economy. In Section 3, I have shown evidences that firm innovation and growth rate were affected by the crisis. I will now first quantify the change in financial constraints after the Great Recession, and then evaluate its implications.

From (16), the constraint can become more severe for all firms when A or ι decreases. I refer the changes to ι and A as profit pledgeability and aggregate demand shocks[53].

[53] Even though I use the term "shocks", I do not perform impulse response analysis. Instead, I compare stationary equilibria (balanced growth paths) after permanent change in parameter values.

Table 9: Estimation of Change in ι_t

	1997-2006	2007-2016	% Change
Average	0.325	0.311	-4.6%
SD	0.037	0.018	

a. R&D expenses$_{it}$/Cash Flow$_{i,t-1}$ are winsorized at 5% and 95% by year. Cash Flow$_{i,t-1}$ is calculated as income before extraordinary items plus depreciation and amortization.

Frictionless Counterfactual

As a benchmark, I consider the case when the financial constraint never binds, i.e. $\iota = \infty$. The result is shown in table 10. The productivity growth rate g is increased to 0.035, and the negative relation between firm size and growth rate (β) is more significant. The total welfare (in consumption-variation terms, see equation (23)) increases slightly by 10.2% compared to the baseline case[54].

Decrease in Pledgeability ι

In the model, the financial friction is given by

$$R_t(k_I, k_x; n, z) \leq \iota \Pi_t(n, z)$$

In other words, $\frac{R_t(k_I,k_x;n,z)}{\Pi_t(n,z)} \leq \iota$. Motivated by the model, I estimate the ι for each sample period (1997-2006, 2007-2016) by

$$\bar{\iota} = \frac{1}{T}\sum_t \frac{1}{N_t}\sum_i \text{R\&D expenses}_{it}/\text{Cash Flow}_{i,t-1} \tag{24}$$

Cash Flow$_{i,t-1}$ is defined the same as in the empirical regression (1). The estimated ι are listed in table 9. So in the counterfactual analysis, I consider the case where ι is decreased by 5%.

Table 10 shows the equilibrium outcome after a 5% decrease in ι. The aggregate labor productivity growth rate drops by 0.6 percentage points compared to the baseline scenario. The relation between growth rates and firm sizes is also weakens to -0.020. Welfare also suffers a 8.6% decrease. However, both the growth rate g and coefficient β are still higher than those from 2007-2016 data.

[54]Note that higher growth rates from more innovation efforts also entail higher R&D costs and thus less to consume

Table 10: Counterfactual Analysis

	iota	A	Beta	g	Welfare
Baseline	$\bar{\iota}$	1	-0.040	0.027	100
frictionless	∞	1	-0.060	0.035	110.2
Decrease in profit pledgeability	$0.95\bar{\iota}$	1	-0.020	0.021	91.4
Decrease in aggregate demand	$\bar{\iota}$	0.95	-0.007	0.018	87.0
Decrease in both	$0.95\bar{\iota}$	0.95	-0.006	0.010	77.2
1997-2006 Data			-0.034	0.030	
2007-2016 Data			-0.009	0.009	

a. Baseline model are estimated using 1997-2006 data. In the frictionless case, $\iota = \infty$.
b. Beta refers to the regression coefficient of firm log employment on firm employment growth rate. g is the aggregate labor productivity growth rate. The baseline welfare is normalized to 100.

Decrease in aggregate demand A

I now consider a drop in A by the same magnitude 5%[55]. The consequence now is more severe than 5% ι drop. Growth rate is now 0.9 percentage points lower and welfare 23.0% less. Also, the growth rate difference between large and small firms, measured by β, is significantly smaller at -0.007. One possible explanation is that A not only affect the financing constraint, but also the profit flow $\pi = n\bar{z}A\hat{z}^{\epsilon-1}/\epsilon$ and thus firm values. Reduced firm values depress both innovation, internal and external, and firm entry. Furthermore, smaller A also directly reduces final output from equation (7). This result signifies the importance of boosting aggregate demand in recessions in addition to restoring a well-functioning financial system, albeit both reinforce each other.

Decrease in both ι and A

More likely than not, both the financial market and aggregate demand worsened during the Great Recession. When both ι and A are reduced to 95% of their baseline levels, the aggregate productivity growth g is now 0.010, very close to the actual 0.009 measured in the data. Similarly, *beta* is -0.006, within the 95% confidence interval, $(-0.014, -0.004)$, of that estimated from the data. The welfare loss is approximately 22.8% compared to the baseline economy.

6.4. R&D Subsidy

Given the 5% decrease in both ι and A, I consider two types of R&D subsidies aiming for relaxing the financing constraint. Denote the (firm-specific) subsidy rate by s_f. The

[55] To put this into perspective, during the Great Recession, employment dropped by 6.7%, output 7.2% and consumption 5.4% in the United States from 2007 Q4 to 2009 Q3.

financing constraint with subsidy is now

$$\frac{\iota A\hat{z}^{\epsilon-1}}{(1-s_f)\epsilon} \geq k_I\hat{z}^{\epsilon-1} + k_x \tag{25}$$

The contemporaneous after-R&D profit flow is $A\hat{z}^{\epsilon-1}/\epsilon - n(1-s_f)(k_I\hat{z}^{\epsilon-1}+k_x)$. The subsidy is financed by a lump-sum tax T to the shareholders , i.e., the representative household where $T = \bar{z}\int_f s_f n_f \hat{R}(k_I, k_x; \hat{z}_f)df$.

Uniform Subsidy

As both ι and A decrease by 5%, I consider a simple policy that every firm receives 10% subsidy to their R&D. Table 11 presents the results. Both the welfare and aggregate growth rate g are now much higher than before the subsidy. The size-growth relation also favors smaller firms more with $\beta = -0.012$, compared to -0.006 without subsidies. Intuitively, this is because now the financing constraint is essentially restored to baseline case[56]. Nevertheless, all metrics are still worse than the baseline economy, since deteriorated aggregate demand will negatively affect firm values and welfare.

Size-dependent Subsidy

As advocated in the literature as well as implemented in practice (see for example Acemoglu et al. (2018)), smaller firms should receive higher subsidy, as they are more susceptible to demand fluctuation and financial market imperfections. Here I consider a size-dependent policy where the average subsidy is still 10%. In particular, let

$$s_f = s(Q_f) = 1 - 1/(a + b\frac{Q_{min}}{Q_f}) \tag{26}$$

where $Q_f = n\hat{z}^{\epsilon-1}$ and $Q_{min} = \min_{f\in[0,M]} Q_f$. Note that $\frac{1}{1-s(Q_f)} = a + b\frac{Q_{min}}{Q_f}$. Also,

$$\sum_{n=1}^{\infty}\int_0^{\infty} s(Q)dH(n,\hat{z}) = 10\%$$

$$s(Q_{max}) = 0$$

Although in the theoretical model, $Q_{max} = \infty$, in the numerical solution, I set $Q_{max} = n_{max}\hat{z}_{max}^{\epsilon-1}$. Also, I set $H(n,\hat{z})$ to be the distribution of the no-subsidy economy, so a more

[56] To see this, note that $0.95 * 0.95/0.9 \approx 1.0028$

Table 11: Policy Analysis

	Beta	g	Welfare
Uniform subsidy	-0.012	0.015	82.3
Size-dependent subsidy	-0.019	0.019	88.5

[a] In both cases, ι and A decrease by 5% compared to the baseline case in table 10.

appropriate interpretation is that the ex-ante subsidy is 10%. a and b are two parameters, calculated to be 1.099980 and 35.132840 respectively.

As shown in table 11, there is a modest improvement compared to the uniform subsidy case, albeit still less than the baseline scenario for the same reason discussed before. Consistent with the literature, size-dependent subsidy is more effective than a uniform one, considering smaller firms are associated with higher innovation capacity (Akcigit and Kerr, 2016). The improvement is less pronounced than other estimates, as they typically further differentiate innovation efficiency among firms within the same size group (e.g. by age), or taxing large but inefficient incumbents (Acemoglu and Cao, 2015; Acemoglu et al., 2018), or relating subsidy directly with productivity (Akcigit, 2008). In this paper, large firm is not necessarily more productive, as firm size Q is a function of both the number of product lines n and productivity $\hat{z}$. Interestingly, even with this crude measure of size, size-dependent R&D policy still yields better results.

7. Conclusions

In this paper, I build a Schumpeterian growth model with financial frictions. Firms are financially constrained that their R&D spending cannot exceed a multiple of profits. Depending on the severity of the constraint, size-growth relation can be negative or independent. Two opposite forces are at work. One the one hand, the diminishing return of R&D to the number of product lines generates higher growth for small firms when the financial constraint is slack. On the other, firms with low productivity, which also makes them small in size, find it more difficult to conduct R&D, for R&D expenditure is increasing in the aggregate level of productivity. Which force dominates depends on the tightness of the financial constraint.

The model is also consistent with three observations from the data. Smaller firm R&D growth rates dropped more precipitously during the Great Recession. Financially constrained firms switch to internal innovation, measured by higher share of internal patents. In addition, the relation between size and growth rate for U.S. public innovative firms changed from negative to statistically insignificant after 2007. The latter two are analyzed by panel regression, and are robust to various econometric models, variable definitions, identification strategies and sample selection.

Due to the complexity of the equilibrium, the model is solved using reinforced learning techniques. In particular, the state space of the value function is expanded to include firm-specific state variables, aggregate endogenous variable, to-be-estimated parameters and discretized aggregate productivity distribution. Then I approximate the value function by a neural network.

The financial constraint can tighten because of either reduction in profit pledgeability or drop in aggregate demand. Accordingly, I test three adverse scenarios: 5% drop in pledgeability, 5% drop in aggregate demand or both. The second case shows a larger decrease in growth rate and welfare, while the third one matches the empirical growth rate and size-growth relation, suggesting that the Great Recession is likely a combination of financial and demand shocks.

Lastly, I consider two R&D policies. In one every firm receives the same rate of R&D subsidy (uniform) while in the other small firms obtain higher subsidies, but both policies have the same average subsidy rate of 10%. The size-dependent policy yields higher growth rate and welfare improvement.

There are several extensions possible for future research. First, I only consider permanent change in model parameters. Since the Great Recession left a long-lasting impact on the economy and firms were unclear of the persistence of shocks at the time, it is not inappropriate to simplify the analysis and consider a permanent shift, but it will be interesting to study the transition dynamics as well. Second, if firm-level data are available, one can also examine how financial development affects growth of small firms in a cross-country setting. Lastly, as in Acemoglu et al. (2018), a more comprehensive analysis should incorporate fixed operating costs, resources (mis)allocation and more complex R&D policies.

Appendix A. Theoretical Appendix

A.1. *Equilibrium Prices and Profits*

A.1.1. *Final Goods Production*

Omitting t

$$\max_{\{y_j\}} A\left(\int_0^1 y_j(t)^{\frac{\epsilon-1}{\epsilon}} dj\right)^{\frac{\epsilon}{\epsilon-1}} - \int_0^1 p_j y_j dj$$

F.O.C. w.r.t. y_j

$$p_j = A^{1-1/\epsilon}\left(\frac{Y}{y_j}\right)^{1/\epsilon}$$

A.1.2. *Intermediate goods production*

$$\Pi(n,z) = \max_{\{y_j\}_{j\in\mathbf{J}}} \sum_{j\in\mathbf{J}} [p_j y_j - \frac{w}{z} y_j] = \max_{\{y_j\}_{j\in\mathbf{J}}} \sum_{j\in\mathbf{J}} [A^{1-1/\epsilon} Y^{1/\epsilon} y_j^{1-1/\epsilon} - \frac{w}{z} y_j]$$

Therefore

$$y_j = \left[\frac{(\epsilon-1)z}{\epsilon w}\right]^{\epsilon} Y A^{\epsilon-1} \tag{27}$$

Substitute (27) into final good production (7), we have

$$w = A\frac{\epsilon-1}{\epsilon}\bar{z} \tag{28}$$

where $\bar{z} \equiv \left[\int_0^M \sum_{j\in\mathbf{J}_f} z_f^{\epsilon-1} d_f\right]^{\frac{1}{\epsilon-1}} = \left[\int_0^1 z_j^{\epsilon-1} d_j\right]^{\frac{1}{\epsilon-1}}$.
Use (28) to rewrite y_j in terms of $\hat{z} = z/\bar{z}$, we have

$$y_j = \hat{z}^{\epsilon} A^{-1} Y \text{ and } p_j = A\hat{z}^{-1} \text{ and } l_j = y_j/z = \hat{z}^{\epsilon-1} A^{-1} Y/\bar{z}$$

Therefore net sales is $p_j y_j = \hat{z}^{\epsilon-1} Y$ and profit $\pi_j = p_j y_j - w l_j = \hat{z}^{\epsilon-1} Y/\epsilon$ and

$$\Pi(n,z) = n\pi_j = nY\hat{z}^{\epsilon-1}/\epsilon$$

A.1.3. *Labor Market Clearing*

From labor market clearing $\int_0^1 l_j dj = 1$, we have

$$Y = A\bar{z}$$

Thus,

$$\Pi(n,z) = n\pi_j = nY\hat{z}^{\epsilon-1}/\epsilon = n\bar{z}A\hat{z}^{\epsilon-1}/\epsilon$$

The profit share of total output is, using the fact that $\int_0^1 \hat{z}_j^{\epsilon-1}dj = 1$

$$\frac{\int_0^1 \pi_j dj}{Y} = \frac{Y/\epsilon}{Y} = 1/\epsilon$$

and labor income share is $1 - 1/\epsilon$

A.1.4. Euler Equation

From the maximization problem of households, we have the standard Euler equation

$$g = \frac{\dot{C}}{C} = \frac{\dot{Y}}{Y} = \frac{\dot{\bar{z}}}{\bar{z}} = \frac{r-\rho}{\theta}$$

A.2. Lemma and Proof

Lemma 1

Proof. The firm's value function $\mathbf{V}(n,\hat{z},\bar{z})$ is defined as

$$\begin{aligned}
r\mathbf{V}(n,\hat{z},\bar{z}) = \max_{k_I,k_x} & \underbrace{n\bar{z}A\hat{z}^{\epsilon-1}/\epsilon}_{\text{profit}} - \underbrace{n(k_I\hat{z}^{\epsilon-1}+k_x)\bar{z}}_{\text{R\&D cost}} \\
& + \underbrace{F_I(k_I)[\mathbf{V}(n,\hat{z}(1+\lambda),\bar{z}) - \mathbf{V}(n,\hat{z},\bar{z})]}_{\text{return from internal R\&D}} \\
& + \underbrace{n\tau[\mathbf{V}(n-1,\hat{z},\bar{z}) - \mathbf{V}(n,\hat{z},\bar{z})]}_{\text{creative destruction}} \\
& + \underbrace{nF_x(k_x,n)\left[\mathbb{E}_{\hat{z}'}\mathbf{V}(n+1,\frac{n\hat{z}+\hat{z}'(1+\eta)}{n+1},\bar{z}) - \mathbf{V}(n,\hat{z},\bar{z})\right]}_{\text{return from external R\&D}} \\
& - \frac{\partial \mathbf{V}}{\partial \hat{z}}(n,\hat{z},\bar{z})g\hat{z} + \frac{\partial \mathbf{V}}{\partial \bar{z}}(n,\hat{z},\bar{z})g\bar{z} \\
& + \underbrace{\varphi\left[\iota A\hat{z}^{\epsilon-1}/\epsilon - k_I\hat{z}^{\epsilon-1} - k_x\right]n\bar{z}}_{\text{financial constraint}}
\end{aligned}$$

where the second and third to last terms use $\dot{\bar{z}} = g\bar{z}$ and $\dot{\hat{z}} = -g\hat{z}$ in balance growth path equilibrium.

We can easily verify that $\mathbf{V}(n,\hat{z},\bar{z}) = \bar{z}V(n,\hat{z})$, where $V(n,\hat{z})$ is defined in (17).

■

Lemma 2

Proof. For a some time interval Δ, the evolution of the aggregated productivity $\bar{z}$ is given by

$$\bar{z}^{\epsilon-1}(t+\Delta) = \int_0^\infty z^{\epsilon-1}\tilde{\phi}_{t+\Delta}(z)dz$$
$$= \int_0^\infty \left[\Delta\tau(z(1+\eta))^{\epsilon-1} + \Delta\tilde{I}(z)(z(1+\lambda))^{\epsilon-1} + (1-\Delta\tau-\Delta\tilde{I}(z))z^{\epsilon-1}\right]\tilde{\phi}_t(z)dz$$

where $\tilde{\phi}_t(z)$ is the marginal distribution of z at time t, and $\tilde{I}(z)$ is the internal innovation rate by firms with productivity z.

The differential becomes

$$\frac{\bar{z}^{\epsilon-1}(t+\Delta)-\bar{z}^{\epsilon-1}(t)}{\Delta} = \int_0^\infty \left[\tau((z(1+\eta))^{\epsilon-1}-z^{\epsilon-1}) + \tilde{I}(z)((z(1+\lambda))^{\epsilon-1}-z^{\epsilon-1})\right]\tilde{\phi}_t(z)dz$$

Normalize by dividing both sides by $\bar{z}^{\epsilon-1}(t)$, it becomes

$$\frac{\bar{z}^{\epsilon-1}(t+\Delta)-\bar{z}^{\epsilon-1}(t)}{\Delta\bar{z}^{\epsilon-1}(t)} = \int_0^\infty \left[\tau((\hat{z}(1+\eta))^{\epsilon-1}-\hat{z}^{\epsilon-1}) + I(\hat{z})((\hat{z}(1+\lambda))^{\epsilon-1}-\hat{z}^{\epsilon-1})\right]\phi_t(\hat{z})d\hat{z}$$

where $\phi(q) = \sum_{n=1}^\infty h(n,q)$ is the unconditional distribution of $\hat{z}$

and $I(\hat{z}) = M\sum_{n=1}^\infty F_I(k_I(n,\hat{z}))h(n,\hat{z})/\phi(\hat{z})$ is the internal innovation rate conditional on being type $\hat{z}$ firms.

Take Δ to 0. We have

$$g = \frac{\overbrace{\tau\mathbb{E}_{\hat{z}}\left[(\hat{z}(1+\eta))^{\epsilon-1}-\hat{z}^{\epsilon-1}\right]}^{\text{agg. external R\&D and entry}} + \overbrace{\mathbb{E}_{\hat{z}}\left\{I(\hat{z})\left[(\hat{z}(1+\lambda))^{\epsilon-1}-\hat{z}^{\epsilon-1}\right]\right\}}^{\text{agg. internal R\&D}}}{\epsilon-1}$$

Simplify and using $\mathbb{E}_{\hat{z}}(\hat{z}^{\epsilon-1}) = 1$, we have equation (21).

The aggregate creative destruction rate τ is derived from definition.

∎

Lemma 3

Proof. When $k_x > 0$, from the FOC of intermediate goods producers' value function

$$n\hat{z}^{\epsilon-1}(1+\varphi) = \alpha_I \beta_I k_I^{\beta_I - 1}[V(n, \hat{z}(1+\lambda)) - V(n,\hat{z})]$$
$$n(1+\varphi) = \alpha_x \beta_x k_x^{\beta_x - 1} n^{\gamma+1}\left[\mathbb{E}_{\hat{z}'} V(n+1, \frac{n\hat{z}+\hat{z}'(1+\eta)}{n+1}) - V(n,\hat{z})\right]$$
$$\iota A \hat{z}^{\epsilon-1}/\epsilon \geq k_I \hat{z}^{\epsilon-1} + k_x + \Phi \text{ with equality when } \varphi > 0$$

Then we have

$$k_I^* = \left[\frac{\alpha_I \beta_I [V(n,\hat{z}(1+\lambda)) - V(n,\hat{z})]}{n\hat{z}^{\epsilon-1}(1+\varphi)}\right]^{\frac{1}{1-\beta_I}} \text{ and } k_x^* = \left[\frac{\alpha_x \beta_x [\mathbb{E}_{\hat{z}'} V(n+1, \frac{n\hat{z}+\hat{z}'(1+\eta)}{n+1}) - V(n,\hat{z})]}{n^{-\gamma}(1+\varphi)}\right]^{\frac{1}{1-\beta_x}}$$

■

Lemma 4 (Firm Growth Rate)**.** *Let $g_f \equiv \mathbf{E}\left(\dot{Q}_f/Q_f\right)$ be the average growth rate of a firm with $n\hat{z}^{\epsilon-1} \equiv Q_f$. Then in equilibrium,*

$$g_f = \frac{n x(n,\hat{z})\left\{(n+1)\mathbb{E}\left[\left(\frac{n\hat{z}+(1+\eta)\hat{z}'}{n+1}\right)^{\epsilon-1}\right] - Q_f\right\}}{Q_f} + I(n,\hat{z})\lambda - \tau \tag{29}$$

where $x(n,\hat{z}) = F_x(k_x(n,\hat{z}))$ and $I(n,\hat{z}) = F_I(k_I(n,\hat{z}))$

Proof. Consider a small time interval Δ, then $Q_f = n\hat{z}^{\epsilon-1}$ follows

$$\begin{aligned} Q_f(t+\Delta) =& n x(n,\hat{z})\Delta(n+1)\mathbb{E}\left\{\left[\frac{n\hat{z}+(1+\eta)\hat{z}'}{n+1}\right]^{\epsilon-1}\right\} + I(n,\hat{z})\Delta[Q_f(t)(1+\lambda)] + n\tau\Delta[Q_f(t) - \frac{Q_f(t)}{n}] \\ &+ (1 - n x(n,\hat{z})\Delta - I(n,\hat{z})\Delta - n\tau\Delta)Q_f(t) \end{aligned}$$

where $x(n,\hat{z}) = F_x(k_x(n,\hat{z}))$ and $I(n,\hat{z}) = F_I(k_I(n,\hat{z}))$

Subtract $Q_f(t)$ from both sides of the equation, and then divide both sides by $\Delta Q_f(t)$, then take Δ to 0, we have equation (29), using the fact that $g_f = \lim_{\Delta\to 0}\frac{Q_f(t+\Delta)-Q_f(t)}{\Delta Q_f(t)}$

■

Lemma 5 (Stationary Distribution)**.** *Denote the joint distribution by $H(n,\hat{z}) = Prob(q \leq \hat{z}, \tilde{n} = n)$. Its density $h(n,\hat{z})$ satisfies the Kolmogorov forward equation (for $n \geq 2$)*

$$\dot{h}(n,\hat{z}) =$$
$$- \underbrace{\left[I(n,\hat{z})h(n,\hat{z}) - \frac{1}{1+\lambda} I(n, \frac{\hat{z}}{1+\lambda})h(n, \frac{\hat{z}}{1+\lambda}) \right]}_{\text{outflow due to internal R\&D}} - \underbrace{nx(n,\hat{z})h(n,\hat{z})}_{\text{n-sized firm to n+1}} - \underbrace{n\tau h(n,\hat{z})}_{\text{n-size firm to n}-1}$$
$$+ \underbrace{\int_0^{\frac{n\hat{z}}{1+\eta}} (n-1)x(n-1, \frac{n\hat{z}-q(1+\eta)}{n-1})h(n-1, \frac{n\hat{z}-q(1+\eta)}{n-1})\phi(q)dq}_{\text{inflow due to external R\&D by size n}-1\text{ firms}} \tag{30}$$
$$+ \underbrace{(n+1)\tau h(n+1,\hat{z})}_{\text{creative destruction of size-n+1 firms}} + \underbrace{\frac{\partial[g\hat{z}h(n,\hat{z})]}{\partial \hat{z}}}_{\text{extensive margin}}$$

where $x(n,q) = F_x(k_X(n,q),n)$, $I(n,q) = F_I(K_I(n,q))$ *and* $q \sim \phi(q) = \sum_{n=1}^{\infty} h(n,q)$

One can derive a similar equation for $n = 1$.

The total measure of firms M *satisfy*

$$1 = M \sum_{n=1}^{\infty} n \int_0^{\infty} h(n,\hat{z})d\hat{z}$$

Proof. for $n \geq 2$

Consider banks with $= n$ and $\leq \hat{z}$

Outflow

(i) n-sized firms internally innovate to $> \hat{z}$

(ii) n-sized firms externally innovate to $n+1$

(iii) n-sized firm being creatively destructed to $n-1$

Inflow

(i) $(n-1)$-sized firms externally innovate to $= n$ but $\leq \hat{z}$

(ii) Firms with $= n+1, \leq \hat{z}$ being creatively destructed to $= n$

(iii) firm with $= \hat{z}$ becomes $< \hat{z}$ when it fails to innovate as $\bar{z}$ grows

Let $x(n,q) = F_x(k_X(n,q),n)$ and $I(n,q) = F_I(K_I(n,q))$. Consider a small time interval from t to $t+\Delta$

$$
\begin{aligned}
H_{t+\Delta}(n,\hat{z}) &= H_t(n,\hat{z}(1+g\Delta)) \\
&- \underbrace{\int_{\hat{z}(1+g\Delta)/(1+\lambda)}^{(1+g\Delta)\hat{z}} \Delta I(n,q)h(n,q)dq}_{\text{outflow due to internal R\&D}} - \underbrace{\int_0^{\hat{z}(1+g\Delta)} \Delta n x(n,q)h(n,q)dq}_{\text{outflow: n-sized firm external R\&D to } n+1} \\
&- \underbrace{\Delta n\tau H_t(n,\hat{z})}_{\text{out: creative destruction of n-sized firm}} + \underbrace{\Delta X(n-1,\hat{z})}_{\text{inflow due to external R\&D by size }=n-1\text{ firms}} + \underbrace{(n+1)\Delta\tau H(n+1,\hat{z}(1+g\Delta))}_{\text{creative destruction of size-}n+1\text{ firms}}
\end{aligned}
\tag{31}
$$

where

$$
X(n-1,\hat{z}) = \int_0^{\hat{z}}\int_0^{\infty}(n-1)x(n-1,\frac{n\tilde{z}-q'(1+\eta)}{n-1})h(n-1,\frac{n\tilde{z}-q'(1+\eta)}{n-1})\phi(q')dq'd\tilde{z}
$$

$$
q \sim \phi(q) = \sum_{n=1}^{\infty} h(n,q) \text{ the marginal distribution of } \hat{z}
$$

Note that the second to the last line comes from the convolution formula:

(i) After a successful external innovation of a firm of size $= n-1$, its post-innovation productivity should be $\tilde{z} \leq \hat{z}$

(ii) Let its original productivity be q; the productivity of the product line it creatively destructs be q'; therefore, we need $n\tilde{z} \equiv (n-1)q + q'(1+\eta) < n\hat{z}$

(iii) q' is an random draw from the overage relative productivity distribution $H(\hat{z})$, so q' is independent of q

(iv) The pdf of the new productivity $\tilde{z}$ follows

$$
\int_0^{\infty}(n-1)x(n-1,\frac{n\tilde{z}-q'(1+\eta)}{n-1})h(n-1,\frac{n\tilde{z}-q'(1+\eta)}{n-1})\phi(q')dq'
$$

(v) Therefore, the total outflow due to external innovation is

$$
\int_0^{\hat{z}}\int_0^{\infty}(n-1)x(n-1,\frac{n\tilde{z}-q'(1+\eta)}{n-1})h(n-1,\frac{n\tilde{z}-q'(1+\eta)}{n-1})\phi(q')dq'd\tilde{z}
$$

Now subtract both sides of (31) by $H_t(n,\hat{z})$, then divide both sides by Δ and take Δ to 0, using the fact that

$$
\lim_{\Delta\to 0}\frac{H_{t+\Delta}(n,\hat{z}) - H_t(n,\hat{z})}{\Delta} \equiv \dot{H}(n,\hat{z})
$$

and

$$\lim_{\Delta \to 0} \frac{H_t(n, \hat{z}(1+g\Delta)) - H_t(n, \hat{z})}{\Delta} = g\hat{z}h(n, \hat{z})$$

we can derive the flow equation.

$$\begin{aligned}
&\dot{H}(n, \hat{z}) = \\
&- \int_{\hat{z}/(1+\lambda)}^{\hat{z}} I(n, q)h(n, q)dq - \int_0^{\hat{z}} nx(n, q)h(n, q)dq \\
&- \tau n H(n, \hat{z}) + X(n-1, \hat{z}) + (n+1)\tau H(n+1, \hat{z}) + g\hat{z}h(n, \hat{z})
\end{aligned}$$

Now differentiate w.r.t. $\hat{z}$ using Leibniz integral rule

$$\begin{aligned}
&\dot{h}(n, \hat{z}) = \\
&- \left[I(n, \hat{z})h(n, \hat{z}) - \frac{1}{1+\lambda} I(n, \frac{\hat{z}}{1+\lambda})h(n, \frac{\hat{z}}{1+\lambda}) \right] - n(x(n, \hat{z}) + \tau)h(n, \hat{z}) \\
&+ \varrho(n-1, \hat{z}) + (n+1)\tau h(n+1, \hat{z}) + \frac{\partial [g\hat{z}h(n, \hat{z})]}{\partial \hat{z}}
\end{aligned}$$

$$\begin{aligned}
\varrho(n-1, \hat{z}) &= \int_0^{\frac{n\hat{z}}{1+\eta}} (n-1)x(n-1, \frac{n\hat{z} - q(1+\eta)}{n-1})h(n-1, \frac{n\hat{z} - q(1+\eta)}{n-1})\phi(q)dq \\
q \sim \phi(q) &= \sum_{n=1}^{\infty} h(n, q) \text{ the marginal distribution of } \hat{z}
\end{aligned}$$

For $n = 1$, we have

$$\begin{aligned}
\underbrace{\dot{H}(1, \hat{z})}_{\text{net flow}} = &- \underbrace{\int_{\hat{z}/(1+\lambda)}^{\hat{z}} I(1, q)h(1, q)dq}_{\text{outflow due to internal R\&D}} - \underbrace{\left[\int_0^{\hat{z}} x(1, q)h(1, q)dq + \tau H(1, \hat{z}) \right]}_{\text{external R\&D and creative destruction}} \\
&\underbrace{+ x_e \int_0^{\hat{z}/(1+\eta)} \phi(q)dq}_{\text{inflow from entrants}} + \underbrace{2\tau H(2, \hat{z})}_{\text{creative destruction of size 2 firms}} + \underbrace{g\hat{z}h(1, \hat{z})}_{\text{extensive margin}}
\end{aligned}$$

So the Kolmogorov forward equation is

$$\dot{h}(1,\hat{z}) = -\left[I(1,\hat{z})h(1,\hat{z}) - \frac{1}{1+\lambda}I(1,\frac{\hat{z}}{1+\lambda})h(1,\frac{\hat{z}}{1+\lambda})\right] - x(1,\hat{z})h(1,\hat{z}) - \tau h(1,\hat{z})$$
$$+ x_e\phi(\hat{z}/(1+\eta)) + 2\tau h(2,\hat{z}) + \frac{\partial[g\hat{z}h(1,\hat{z})]}{\partial\hat{z}}$$

■

Proposition 1

Proof. From lemma 3, a firm with given$(n,\hat{z})$ will have

$$\frac{k_I}{k_x} \propto (1+\varphi)^{\frac{1}{1-\beta_x}-\frac{1}{1-\beta_I}}$$

$\frac{k_I}{k_x}$ is increasing in φ if and only if $\beta_x > \beta_I$.

■

A.3. Myopic Firm Problem

This section lays out the analysis for a myopic profit maximizing firm to derive closed-form solutions.

Consider a small time interval Δ, a myopic firm maximizes one-period ahead payoffs

$$\begin{aligned}
\max_{k_I,k_x} \; & \Delta[\Pi(n,\hat{z}) - n(k_I\hat{z}^{\epsilon-1} + k_x)] \\
& + (1-r\Delta)\Delta F_I(k_I)[\Pi(n,\hat{z}(1+\lambda)) - \Pi(n,\hat{z})] \\
& + (1-r\Delta)\Delta n F_x(k_x,n)\left\{\mathbf{E}_{\hat{z}'}\Pi\left[n+1,\frac{n\hat{z}+\hat{z}'(1+\eta)}{n+1}\right] - \Pi(n,\hat{z})\right\} \\
& + (1-r\Delta)\Delta n\tau[\Pi(n-1,\hat{z}) - \Pi(n,\hat{z})] \\
& + (1-r\Delta)\Pi(n,\hat{z})
\end{aligned}$$

s.t.

$$\iota A n\hat{z}^{\epsilon-1}/\epsilon \geq n(k_I\hat{z}^{\epsilon-1} + k_x)$$

Only keep terms related to k_I and k_x, factor out Δ and take $\Delta \to 0$, we have

$$\begin{aligned}\max_{k_I,k_x} \; & -n(k_I\hat{z}^{\epsilon-1}+k_x) \\ & + F_I(k_I)[\Pi(n,\hat{z}(1+\lambda))-\Pi(n,\hat{z})] \\ & + nF_x(k_x,n)\left\{\mathbf{E}_{\hat{z}'}\Pi\left[n+1,\frac{n\hat{z}+\hat{z}'(1+\eta)}{n+1}\right]-\Pi(n,\hat{z})\right\}\end{aligned}$$

Using $\Pi(n,\hat{z}) = nA\hat{z}^{\epsilon-1}/\epsilon \equiv n\pi(\hat{z})$ where $\pi(\hat{z}) = A\hat{z}^{\epsilon-1}/\epsilon$, we have

$$\begin{aligned}\max_{k_I,k_x} \; & -(k_I\hat{z}^{\epsilon-1}+k_x+\Phi\cdot 1_{k_x>0}) \\ & + F_I(k_I)[\pi(\hat{z}(1+\lambda))-\pi(\hat{z})] \\ & + F_x(k_x,n)\left\{(n+1)\mathbf{E}_{\hat{z}'}\pi\left[\frac{n\hat{z}+\hat{z}'(1+\eta)}{n+1}\right]-n\pi(\hat{z})\right\} \\ & + \varphi[\iota\pi(\hat{z})-(k_I\hat{z}^{\epsilon-1}+k_x]\end{aligned} \tag{32}$$

When $\epsilon = 2$, we then have linear profit functions. Furthermore, set $A = 1$. Then the maximization problem (32) becomes

$$\max_{k_I,k_x} F_I(k_I)\hat{z}\lambda + F_x(k_x,n)(1+\eta) - (k_I\hat{z}+k_x) + \varphi[\iota A\hat{z}/2 - (k_I\hat{z}+k_x)]$$

Note that now $\bar{\hat{z}} = 1$ when $\epsilon = 2$

$$\max_{k_I,k_x} \alpha_I k_I^{\beta_I}\hat{z}\lambda + \alpha_x k_x^{\beta_x} n^{\gamma}(1+\eta) - (k_I\hat{z}+k_x) + \varphi[\iota\hat{z}/2 - (k_I\hat{z}+k_x)]$$

F.O.Cs for firms with $n \geq 1$

$$k_I^* = \left[\frac{\alpha_I\beta_I\lambda}{1+\varphi}\right]^{\frac{1}{1-\beta_I}} \quad \text{and } k_x^* = \left[\frac{\alpha_x\beta_x(1+\eta)n^{\gamma}}{1+\varphi}\right]^{\frac{1}{1-\beta_x}}$$

where φ is the shadow price of the financial constraint

$$\iota\hat{z}/2 \geq k_I\hat{z} + k_x$$

Applying Lemma 4, the firm growth rate when $\epsilon = 2$ (linear profit function) is

$$g_f(Q_f) = g_f(n\hat{z}) = \frac{F_x(k_x(n,\hat{z}))(1+\eta)}{\hat{z}} + F_I(k_I(n,\hat{z}))\lambda - \tau \tag{33}$$

Disproportional Growth Rate: Deviation from Gibrat's Law

When ι is sufficiently large, $\varphi = 0$. In this case, $k_I^* = (\alpha_I\beta_I\lambda)^{\frac{1}{1-\beta_I}}$ and $F_I(k_I) = \alpha_I(\alpha_I\beta_I\lambda)^{\frac{\beta_I}{1-\beta_I}}$. Also, $k_x^* = (\alpha_x\beta_x(1+\eta)n^\gamma)^{\frac{1}{1-\beta_x}}$ and $F_x(k_x, n) = \alpha_x(\alpha_x\beta_x(1+\eta))^{\frac{\beta_x}{1-\beta_x}} n^{-\frac{2-\beta_x}{1-\beta_x}}$

Accordingly, $g_f = \frac{C_x}{n^{\frac{2-\beta_x}{1-\beta_x}}\hat{z}} + C_I - \tau$ is decreasing in both n and $\hat{z}$, where C_x and C_I are two constants. $C_x = \alpha_x(1+\eta)(\alpha_x\beta_x(1+\eta))^{\frac{\beta_x}{1-\beta_x}}$ and $C_I = \lambda\alpha_I(\alpha_I\beta_I\lambda)^{\frac{\beta_I}{1-\beta_I}}$. Thus,

$$\frac{\partial g_f}{\partial Q} = \frac{\partial g_f}{\partial n}\frac{\partial n}{\partial Q} + \frac{\partial g_f}{\partial \hat{z}}\frac{\partial \hat{z}}{\partial Q} < 0$$

This is the case analyzed in Akcigit and Kerr (2016) where smaller firms grow faster.

Proportional Growth Rate: Gibrat's Law

Consider $\varphi > 0$.

Then $g_f = \frac{C_x}{n^{\frac{2-\beta_x}{1-\beta_x}}\hat{z}(1+\varphi)^{\frac{\beta_x}{1-\beta_x}}} + C_I/(1+\varphi)^{\frac{\beta_I}{1-\beta_I}} - \tau$. It is easy to see that $\frac{\partial g_f}{\partial \varphi} < 0$

From the financial constraint we can see φ is defined by the implicit function

$$G(\varphi, \hat{z}, n) = \hat{z} - c_1\hat{z}(1+\varphi)^{\frac{-1}{1-\beta_I}} - c_2[n(1+\varphi)]^{\frac{-1}{1-\beta_x}} \tag{34}$$

where $c_1 = (\alpha_I\beta_I\lambda)^{\frac{1}{1-\beta_I}}/(0.5\iota)$ and $c_2 = (\alpha_x\beta_x(1+\eta))^{\frac{1}{1-\beta_x}}/(0.5\iota)$ are two positive constants.

It is straightforward to show that $\frac{\partial G}{\partial \varphi} = \frac{c_1}{1-\beta_I}\hat{z}(1+\varphi)^{\frac{-2+\beta_I}{1-\beta_I}} + \frac{c_2}{1-\beta_x}n^{\frac{-1}{1-\beta_x}}(1+\varphi)^{\frac{-2+\beta_x}{1-\beta_x}} > 0$ and $\frac{\partial G}{\partial n} = \frac{c_2}{1-\beta_x}(1+\varphi)^{\frac{-1}{1-\beta_x}} n^{\frac{-2+\beta_x}{1-\beta_x}}$. Also, $\frac{\partial G}{\partial \hat{z}} = 1 - c_1(1+\varphi)^{\frac{-1}{1-\beta_I}} > 0$, because $\hat{z} \geq c_1\hat{z}(1+\varphi)^{\frac{-1}{1-\beta_I}} + c_2[n(1+\varphi)]^{\frac{-1}{1-\beta_x}} > c_1\hat{z}(1+\varphi)^{\frac{-1}{1-\beta_I}}$.

By implicit function theorem,

$$\frac{\partial \varphi}{\partial \hat{z}} = -\frac{\partial G}{\partial \hat{z}}/\frac{\partial G}{\partial \varphi} < 0 \qquad \frac{\partial \varphi}{\partial n} = -\frac{\partial G}{\partial n}/\frac{\partial G}{\partial \varphi} < 0$$

In addition,

$$\frac{\partial g_f}{\partial \hat{z}} = -\frac{C_x}{n^{\frac{2-\beta_x}{1-\beta_x}}\hat{z}^2(1+\varphi)^{\frac{\beta_x}{1-\beta_x}}} + \frac{\partial g_f}{\partial \varphi}\frac{\partial \varphi}{\partial \hat{z}}$$
$$\frac{\partial g_f}{\partial n} = -\frac{2-\beta_x}{1-\beta_x}\frac{C_x}{n^{\frac{3-2\beta_x}{1-\beta_x}}\hat{z}(1+\varphi)^{\frac{\beta_x}{1-\beta_x}}} + \frac{\partial g_f}{\partial \varphi}\frac{\partial \varphi}{\partial n}$$

Because the first term is negative while the second positive, the sign of $\frac{\partial g_f}{\partial \hat{z}}$ and $\frac{\partial g_f}{\partial n}$ become ambiguous. When $\frac{\partial g_f}{\partial Q} = 0$, we will return to the case Klette and Kortum (2004) where the unconditional growth rate follows Gibrat's law: growth rate is independent of size.

Appendix B. Additional Empirical Evidence

B.1. Data Development and Selection

For compustat, see cleanCompustat.do and compustat_clean.dta

Table 12: Sample Selection for Compustat Data

	Selection	SIC	Obs	Firms
	Compustat		378,983	32,467
–	Financials	6000-6999	287,321	23,888
–	Argricultural	0-999	285,954	23,761
–	Public Service	9000-9999	280,291	23,225
–	R&D=0 for its entire operation periods		148,290	11,395

For PatentsView, see cleanPatent.do and al_pat_clean.dta. For type: Classification of assignee (2 - US Company or Corporation, 3 - Foreign Company or Corporation, 4 - US Individual, 5 - Foreign Individual, 6 - US Government, 7 - Foreign Government, 8 - Country Government, 9 - State Government (US). Note: "A" or "1" appearing before any of these codes signifies part interest) .

Table 13: Sample Selection for Unmatched Patent Data

Selection	Criteria	Obs	Selection	Criteria	Obs
Patent		6,408,293	Application		6,422,963
Country	=US	6,408,291	Series code	$\neq$29	5,993,792
Type	=Utility	5,807,528			
Kind	=A,B1,B2	5,807,519			
Selection	Criteria	Obs	Selection	Criteria	Obs
Assignee		379,565	Merged Patents Files		6,162,134
Type	=2	167,986	Assignee	Missing Info	5,354,027
Organization	Missing name	167,741	Type	=2	2,657,303
			Citation data	Matched	2,560,204
			Application year	< 1975 or > 2017	2,536,585

Match names. see matchName.do. Note that in step 3, the number of matches with ratio

≥ 95 is 468, which I keep them with certainty. For matches with ratio < 95 but ≥ 93, I check them one by one manually. I only kept 201 of them (524).

Table 14: Name Matching Process

Step	Criteria	Fuzzy Match	Obs	Cumulative Matches	File
0			11,398		getCompName.do
1	standard_name	N		5,278	
2	stem_name	N		6,635	
3	stem_name	Y	992	7,304	match_ass.py
4	merged with cleaned patent data			6,705	clean_patent.do

By matching application year-firm (gvkey) pairs and corresponding fiscal year-firm (gvkey) pairs data from Compustat, we have the total number of matched patent is 810,451 (unmatched 1,749,753), firm 4,630 and firm-year 70,004.

Table 15: Selection for Matched PatentsView-Compustat Data

	Criteria	Obs	Firms	Patents
total	firm-patent-year obs	929,003	11,395	862,038
total	firm-year obs	147,850	11,395	862,038
Patent=0 for its entire operation periods		70,004	4630	862,038
	1997-2016 data	35,691	3,482	604,193
	Continuous innovationg from 97-16	31,895	3,187	569,304

B.2. Additional Statistics

Total number of observation is 1,040.

Table 16: Summary Statistics (Fiscal Year 2015)

	Mean	Median	St. Dev.
Employee	10,071	660	32,508
Revenue (mil \$)	4,320	200	18,113
R&D expense (mil \$)	253	28	1,019
R&D Exp/Net sales	9.74%	13.20%	79.56%
Pre-tax income/Net sales	-20.98	0.00	232.48
Growth rate of net sales	3.93%	2.77%	62.50%
Patents applied	13.36	1.00	130.03
Self-citation ratio	14.28%	8.67%	18.17%

Firm size distribution over time. We can see firms got bigger, especially after 2011 and for the top half of firms.

Here is a a table summarizing the size.

Table 17: Percentiles of Size Distribution

Percentile	1997	2002	2007	2011	2016
10	44	40	38	27	26
20	87	93	91	80	73
30	143	152	165	165	131
40	235	250	290	346	290
50	372	431	496	614	630
60	716	793	951	1300	1475
70	1444	1577	2000	2735	3085
80	3360	3522	4700	5672	6966
90	9100	8600	13355	16000	17678

Fig. 7. Innovation Capacity by Firm Size

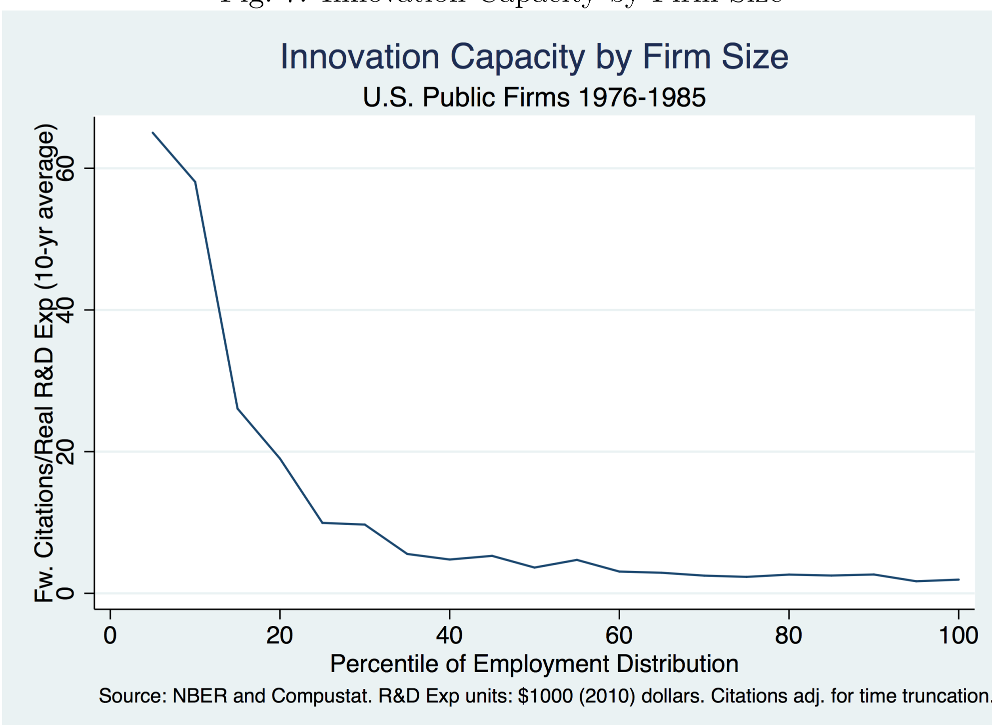
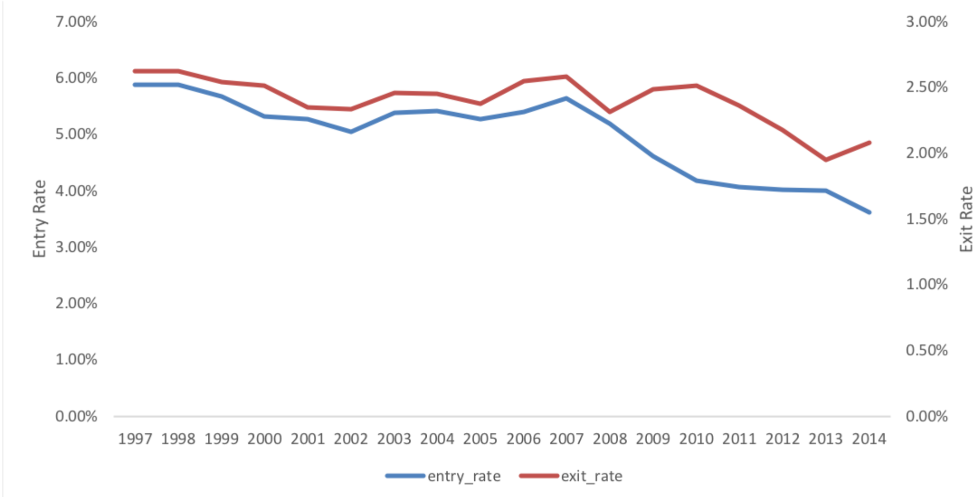


Fig. 8. Firm Entry and Exit Rates by Year



Appendix C. Computational Details

C.1. Discretized Joint Distribution $h(n, \hat{z})$

(1) **Discretize** the $n - \hat{z}$ grid

(2) Calculate the **transition matrix P**

For each $n - \hat{z}$ pair, it can transit to

I. n and $\hat{z}\frac{1+\lambda}{1+g\Delta t}$ with probability $F_I \Delta t = \alpha_I k_I^{\beta_I} \Delta t$

II. $n+1$ and $\frac{n\hat{z}}{(n+1)(1+g\Delta t)} + \frac{q(1+\eta)}{(n+1)(1+g\Delta t)}$ with probability $nF_x \phi(q)\Delta t = n^{\gamma+1}\alpha_x k_x^{\beta_x}\phi(q)\Delta t$ where $\phi(\hat{z})$ is the marginal distribution of $\hat{z}$

III. $n-1$ and $\frac{\hat{z}}{1+g\Delta t}$ with probability $\tau \Delta t$

IV. n and $\frac{\hat{z}}{1+g\Delta t}$ with probability Δt

V. n and $\hat{z}$ with probability $1 - F_I\Delta t - nF_x\Delta t - \tau\Delta t - \Delta t$

Because updated $\hat{z}$ will not land on the grid points exactly, I use the following approximation: Suppose $a < \hat{z} < b$ where a, b are adjacent grid points. Then the probability of hitting a is $\frac{\hat{z}-a}{b-a}$ and b is $\frac{b-\hat{z}}{b-a}$. Also, the index of a, b on the grid is $\left\lfloor \frac{\log(\hat{z}/z)}{\log(1+\Delta z)} \right\rfloor$ and $\left\lceil \frac{\log(\hat{z}/z)}{\log(1+\Delta z)} \right\rceil$

Also, boundaries are reflective.

(3) Calculate the stationary distribution of $n - \hat{z}$, **h**

$$\mathbf{h} = \mathbf{hP}$$

h is the left eigenvector of **P**. Note that the stationary distribution of this Markov Chain exists and it is unique, since **P** is aperiodic and irreducible.

C.2. Value Functions

Given function inputs $n, \hat{z}, g, \tau, \phi(\hat{z})$ and Ω, we have the following iteration algorithm

$$V^{i+1}(\cdot) = \mathbf{T}_V \left\{V^i(\cdot)\right\} \equiv V^i(\cdot) + \Delta HJB(V^i(\cdot))$$

where $\mathbf{T}_V$ describe the Bellman operator and the HJB equation is defined as

$$\begin{aligned}
HJB[V(\cdot)] = (g-r)V(n,\hat{z}) + \max_{k_I,k_x}\{&An\hat{z}^{\epsilon-1}/\epsilon - n(\hat{z}^{\epsilon-1}k_I + k_x) \\
&+ F_I(k_I)[V(n,\hat{z}(1+\lambda)) - V(n,\hat{z})] \\
&+ n\tau[V(n-1,\hat{z}) - V(n,\hat{z})] \\
&+ nF_x(k_x,n)\left[\mathbb{E}_{\hat{z}'}V(n+1,\frac{n\hat{z}+\hat{z}'(1+\eta)}{n+1}) - V(n,\hat{z})\right] \\
&- \frac{\partial V}{\partial \hat{z}}(n,\hat{z})g\hat{z} + \varphi\left[\iota A\hat{z}^{\epsilon-1}/\epsilon - k_I\hat{z}^{\epsilon-1} - k_x\right]n\}
\end{aligned}$$

Neural Network Representation

Denote the state space for neural network by $\mathbf{X}_V = \{n, \hat{z}, g, \tau, \phi(\hat{z}), \Omega\}$. Denote the respective neural network representation by $V(\mathbf{X}_V;\Theta_V)$.

Start

1. A random draw of inputs (I use 20,000 draws) $\mathbf{X}_V$
2. Initial guess of value function

In each iteration:

1. Calculate the value function according to last iteration's output $V^i(\mathbf{X}_V)$
2. Calculate the results from HJB equation (which also uses the last round's value function in the calculation) $HJB^i(\mathbf{X}_V)$
3. Calculate the "target" for the neural network as $V^i + \Delta HJB^i$
4. Train the neural network to fit the target
5. The trained network is a representation of value function, call it V^{i+1}
6. Repeat until convergence

The neural network is calibrated with the following hyper-parameters in table 18.

The machine-learning package, TensorFlow and Keras, do the most heavy-lifting of training the neural network. I only need to write out the discretized HJB equation.

Algorithm 1: Value Function Algorithm

Input : Uniform random draw of $\mathbf{S} = \{(\mathbf{s}_i) | \mathbf{s}_i = n, \hat{z}; \phi(\hat{z}), g, \tau, \Omega\}$

Output: Converged value functions $V(n, \hat{z}; \phi(\hat{z}), g, \tau, \Omega\}$

$i \leftarrow 0$, $\varepsilon \leftarrow 10$

while $i < max$ *and* $\epsilon > tol$ **do**

  target $\leftarrow \Delta t \cdot HJB(V^i(\mathbf{S})) + V^i(\mathbf{S})$; `/* When converged,` $HJB(V^i(\mathbf{S})) = 0$ `*/`

  $V^{i+1}(\cdot) \leftarrow$ model.fit($\mathbf{S}$,target)

  $\varepsilon = \sum_{\mathbf{s}_i} |HJB(V^{i+1}(\mathbf{s}_i))|$

  $i \leftarrow i + 1$

end

Table 18: Neural Network Hyper Parameters

Parameters	Value
Hidden Layers	3
Hidden Units	[100,100,100]
Activation Function	Leaky ReLu
Batch Normalization	No
Gradient Descent	ADAM
Learning Rate	1e-4 to 1e-5
Batch Size	100 - 2000
Epoch	10 - 200
dt	0.07, 0.02, 0.01

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