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Multiplant location for profit maximisation

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Abstract. Since many important firms operate several plants, the problem of the choice of an optimal pattern of locations for these plants is of interest. In this paper, multiplant location without interaction is studied. It is shown that it is sufficient to consider a finite number of particular sites to obtain an optimal pattern of plants. The distribution network of the output is then analysed, taking into account the technologies of the plants. Finally, a numerical method of resolution for the multiplant location problem, with increasing returns to scale in the production, is proposed.

Introduction

Until now, economic theory has devoted little attention to the location problems of the multiplant firm; among the main contributions, one may mention Manne (1967), Teitz (1968) and Beckenstein (1975). On the contrary, in operations research, many models and algorithms for the simultaneous optimal location of several plants, or warehouses have been proposed. Some of these models (Cooper, 1967; 1972; Kuenne and Soland, 1972) assume that the plants may be located anywhere in the Euclidean plane; most assume that the plants may only be set up in locations belonging to a finite set. In this case a general formulation in which flows between plants, as well as between plants and customers, can be taken into account has been defined as the *quadratic-assignment problem* (Koopmans and Beckmann, 1957). This problem is difficult and can only be solved by heuristic methods when more than about fifteen locations are considered (Pierce and Crowston, 1971; Hanan and Kurtzberg, 1972; Lawler, 1975). When flows between plants are negligible, the problem is easier. A formulation due to Balinski (1964) and Manne (1964) has become popular under the name of the *simple-plant location problem* (SPLP). The SPLP consists in determining the number, sizes, and locations of plants to be set up in order to satisfy a given demand, while minimising construction, operating, and transportation costs. An efficient algorithm has been proposed by Efroymson and Ray (1966) to solve it. See also Zimmerman (1967), Spielberg (1969a; 1969b), Hansen (1972), Khumawala (1972). If constraints on the capacity of production are imposed, the problem becomes a *capacitated-plant location problem*, for which efficient algorithms also exist (Spielberg, 1970; Geoffrion and Graves, 1974). The simultaneous location of plants and warehouses, or of several levels of warehouses, has also been recently studied (Scott, 1971; Kaufman et al, 1977).

Usually, the models proposed assume that the quantities demanded are given. Thus it appears that the demand functions are perfectly inelastic and that the choice of a price policy is neglected. A first generalisation, due to Wagner and Falkson (1975), integrates linear demand functions into a model of location of public facilities. The purpose of this paper is to study the general case where the demand is expressed by a given set of elastic demand functions on different markets, and where the maximisation of joint profit is sought through the determination of the optimal number of plants, their sizes and locations, as well as through the quantities sold—or, equivalently, the pricing policy—on all markets. We call this problem the *profit-maximising multiplant location problem* (PMMPLP). The analysis of the PMMPLP shows that the pattern of the output distribution depends on the assumptions that are made about the plant's technology. Two cases must be distinguished: nondecreasing, and decreasing returns to scale in the production. The first case corresponds to many location problems, given the probable existence of large indivisibilities in setting up the plants. We shall derive some theorems for both cases and propose a numerical method of resolution for the first one. This particular version of the PMMPLP is called the *profit-maximising simple-plant location problem* (PMSPLP).

In order to make an accurate representation of the transportation space, we model the transport network by using graph theory and some simple topological tools. Our presentation resembles those by Dearing and Francis (1974) and by Thisse (1975). Within this space, we show that, assuming concave transportation costs, the locations of the optimal pattern belong to a finite set of particular sites of the space, that is, supply and demand points, points where external economies occur, and crossroads of the network. We then prove some properties of the optimal productions of the plants. Some of these results allow us to formulate the PMSPLP as a Boolean program and to show it can be reduced, in a mathematical but not in an economical sense, to the SPLP, thus providing efficient algorithms for the PMSPLP. A similar reduction to the SPLP was performed by Wagner and Falkson (1975) for their problem on the location of public facilities. Finally, we give an example and some suggestions about different interpretations and possible extensions of the PMSPLP.

The network metric space

Let X denote the set of all geographical sites considered. We shall assume that a transportation network is given on X. From the previous actions of the agents, we deduce the existence in X of a certain number of particular sites. These sites correspond to points of supply or of demand (plants, towns, markets, etc), to sources of external economies, or to crossroads of the transportation network. To these should be added sites where the firm may benefit from advantages granted by public authorities. The set of all these points is denoted by L. Clearly L is a subset of X and may be assumed to be finite. Between some sites of L there are lines of communication of the transportation network considered. From direct observations of any transportation network, it can be inferred that all lines of communication possess certain characteristics. First, a line of communication is in a single piece; as such, it must have a finite length for the Euclidean metric. Second, if it contains loops these may be neglected, since passing through them would imply a useless lengthening of the journey. Last, if there are several lines of communication between the same two points of L, then, for a similar reason, we only consider the shortest one.

Given these characteristics, let us see how a particular transportation network may be formalised. To do this, we let X be an infinite continuous subset of the Euclidean plane R^2. The elements of L are $l_1, ..., l_n$. From the set L and the configuration of the transportation network, we shall construct a graph G. From this graph we shall construct a network, N, which is a mathematical representation of the real transportation network. Consider therefore a set R defined from L by $(x_1, x_2) \in R \subseteq L^2$ if and only if there exists at least one line of communication which joints the sites x_1 and x_2 without passing by any other point of L; the relation defining R is assumed to be symmetrical. Neglecting bridges and tunnels, it follows that the graph $G = (L, R)$ is planar. Moreover, for obvious reasons of accessibility, we assume that G is connected. Let l_i and l_j denote any two vertices of

G such that $(l_i, l_j) \in R$. Let f_{ij} denote a mapping from $[0, 1]$ to X such that, (a) $f_{ij}(0) = l_i$ and $f_{ij}(1) = l_j$, and (b) to each point x of the line of communication joining l_i and l_j there corresponds one and only one value $t \in [0, 1]$ such that $f_{ij}(t) = x$. The image set $f_{ij}([0, 1]) = \gamma_{ij}$ is called a *route*, with endpoints l_i and l_j. First, the route must be connected in R^2; for this, it is sufficient that the mapping f_{ij} be continuous. Second, for the route to have a finite length, it is sufficient that f_{ij} has bounded variations. Third, the absence of loops along the route is satisfied if f_{ij} is a one-to-one mapping. If the mapping f_{ij} satisfies these three properties, the *route* γ_{ij} *is a continuous rectifiable arc*[1]. As the relation defining R is symmetrical, we may assume $\gamma_{ij} = \gamma_{ji}$. The *network* is then defined as the union of all the routes obtained from the graph G, that is

$$N = \bigcup_{(l_i, l_j) \in R} \gamma_{ij} \, .$$

A distance may be associated with the transportation network. For this, let us introduce the concept of subroute. Let n_i and n_j denote any two points on a route γ_{ij}. The subroute with endpoints n_i and n_j is, by definition, the image of the restriction of f_{ij} to a subinterval $[t_i, t_j]$ of $[0, 1]$ such that $f_{ij}(t_i) = n_i$ and $f_{ij}(t_j) = n_j$. Two (sub)routes will be called adjacent if and only if they possess a common endpoint. Finally a chain with endpoints $n' \in N$ and $n'' \in N$ is a sequence of adjacent routes and subroutes going from n' to n''; then, let n' and n'' be any two points of N. As the graph G is connected, there exists at least one chain joining n' and n''. By definition the length of a chain is equal to the sum of the lengths of the routes and subroutes of that chain. The distance $\delta(n', n'')$ between n' and n'' is equal to the length of the shortest chain joining n' and n''. It is easy to show that δ possesses the properties of a metric. It is called the *network metric*; the couple (N, δ) is called the *network-metric space.*

Thus it appears that a real transportation network induces on the set of accessible sites[2] a precise mathematical structure. As far as possible, the analysis of location problems should be based on the network-metric space, which seems to be a more realistic mathematical representation of the real transportation network than the Euclidean space.

The profit-maximising multiplant location problem

The PMMPLP may be defined as follows: given a finite set of sites where the demand function is known and a finite set of sites where each input of production can be bought according to a given supply function, determine the number, the locations, and the size of the plants to be set up, the flows of goods between the sources of inputs and the plants on the one hand, and between the plants and the sites of demand on the other hand, in order to maximise the joint profit function of the firm.

The PMMPLP so defined subsumes, as particular cases, the single-plant location problem, the simple-plant or warehouse location problem, and the discrete location-allocation problem. Let us now state precisely the required data, and the results that are being sought.

(1) There are v commodities in the economy. The first commodity is the output of the firm considered and the other ones are the possible inputs of production.

(2) Each plant of the firm has a function of production which is assumed to be independent of the chosen location, except perhaps in a subset of L. These sites are sources of external economies.

(1) It may be worth stressing that we do not use the concept of edge as it is usually used in graph theory, but a concept of route (or continuous rectifiable arc) which comes from algebraic topology.

(2) The extension of the metric structure on X may be done through the concept of branching on the network.

(3) For the output of the firm, s_l sites belonging to L exist, where the inverse demand function $\pi_k(q_k)$, $k = 1, ..., s_l$, is given.
(4) For the input of production j, $j = 2, ..., v$, s_j sites belonging to L exist, where the inverse supply function $\pi_{jk}(q_{jk})$, $k = 1, ..., s_j$, is given.
(5) The transportation rate of commodity j, $j = 1, ..., v$, is a function of distance and of the transported quantity; it is denoted by r_j.
(6) The set of possible locations is N.
(7) The objective of the firm is the joint profit maximisation.
The following unknowns must be determined.
(a) the number of plants to be set up,
(b) the location of each plant,
(c) the production technique and the size of each plant, and
(d) the distribution of buying and selling between the supply points and the demand points, as well as the distribution of production between the plants.

Let us define the average transportation cost between the plant located in $n_i \in N$ and the demand point $m_k \in L$ by

$$T_{ik}(q_{ik}, n_i) = r_i[q_{ik}, \delta(n_i, m_k)]\delta(n_i, m_k) ,$$

where q_{ik} is the quantity of output sold by the plant located in n_i to the demand point m_k.
Regardless of the agent who pays the transportation costs, the function of demand considered by the plant located in n_i is not π_k but another one which integrates the average transportation cost, that is,

$$p_{ik}(\pi_k, T_{ik}) = \pi_k - T_{ik} .$$

Similarly, we have for the input of production j,

$$p_{ijk}(\pi_{jk}, T_{ijk}) = \pi_{jk} + T_{ijk} .$$

The functions p are said to be in *firm-prices*.

The joint profit function of the firm may then be written

$$P = \sum_{i=1}^{s_0} \sum_{k=1}^{s_l} p_{ik}\left[\pi_k\left(\sum_{h=1}^{s_0} q_{hk}\right),\ T_{ik}(q_{ik}, n_i)\right] q_{ik} - \sum_{i=1}^{s_0} \sum_{j=2}^{v} \sum_{k=1}^{s_j} p_{ijk}\left[\pi_{jk}\left(\sum_{h=1}^{s_0} q_{hjk}\right),\ T_{ijk}(q_{ijk}, n_i)\right] q_{ijk} ,$$

where q_{hk} (q_{hjk}) is the quantity of output (of input j) exchanged between the plant located in $n_h \in N$ and the demand point (supply point) $m_k \in L$, and s_0 denotes the number of plants.

Properties

The following result will allow us to obtain an important characterisation of the optimal pattern of the plants.

Theorem 1: Let us consider a PMMPLP. *If the average transportation costs can be expressed as increasing and concave functions of the distance, it is sufficient to take the sites of L as possible locations to obtain an optimal pattern of plants. Moreover, if for each plant and for at least one commodity the average transportation cost is a strictly concave function of the distance, all optimal patterns contain only points of L.*

Proof: Assume there exists an optimal pattern such that some plants, denoted by $e_1, ..., e_f$, are located in sites of $N-L$, denoted by $n_1^*, ..., n_f^*$. For each plant e_i, the optimal combination of production may be realised in all sites of N. Indeed, the production function of e_i is assumed to be identical on $N-L$. This is no longer true for the sites of L where the possibilities of production may be modified, given the existence of external economies. The nature of these is such that the optimal combination of production feasible in n_i^* can also be realised in any site of L. Hence, as there are no exchanges between the plants, the location of each plant e_i can be studied independently of the locations of the others, the quantities transported remaining unchanged. Then, by a method similar to that of Thisse and Perreur (1977), it can be shown that the location of e_i minimises, ex post, the total transportation cost associated with the optimal quantities exchanged by e_i. Therefore, the sum of total transportation costs for $e_1, ..., e_f$ is minimised, ex post. We may use a theorem of Levy (1967) to show that, in this problem, it is sufficient to consider the sites of L to obtain an optimal pattern for $e_1, ..., e_f$. Taking into account only the sites of L, we are sure to obtain a pattern as good as $n_1^*, ..., n_f^*$.

The second part of the theorem is a direct consequence of a result due to Wendell and Hurter (1973).

Before considering the consequences of theorem 1, let us briefly discuss the meaning of the hypothesis of the concavity of the average transportation costs. For each commodity, the marginal transportation cost is a nonincreasing function of the distance. If we take into account the probable existence of important indivisibilities in the transportation activity, this assumption does not appear to be very restrictive. Moreover, let us note that concavity is a sufficient but not a necessary condition for the obtention of Levy's (1967) theorem. In his proof a characterisation of the functions T, which is between concavity and quasiconcavity, yields the desired result.

Let us now examine the implications of the theorem. First, a specific characterisation of the location problem has been obtained. Although the set of possible actions is infinitely continuous, we have shown that it is necessary and sufficient to consider, a priori, a finite subset of N in order to obtain an optimal pattern of the plants. Thus the multilocation problem, without exchanges between plants, is finite in nature. Consequently, any process of location (delocation) cannot be realised marginally but must be done in a discontinuous manner. Hence the use of differential methods must be rejected even if the differentiability hypotheses are verified. Second, the sites that must be considered are not chosen at random; they are all particular sites of the space considered. It is clear that it is that choice which is made by managers in most cases. Third, in view of the theorem, we can justify the preferential use of certain resolution methods. Indeed, it is well known that operational models can be classified into two distinct families according to whether the possibilities of location are finite in number or not. We can now say that in most cases the finite approach is the only correct one, and so justify the use of graph theory and Boolean programming in the resolution of location problems.

Let us study some properties of the optimum of production. First of all, we shall make four additional assumptions.

(1) transportation costs are paid by the firm,

(2) average transportation costs are nonincreasing with respect to the quantities,

(3) the supply functions of all inputs of production are perfectly elastic,

(4) for all $k = 1, ..., s_l$, $\pi_k(0)$ is finite.

It is now an easy matter to prove the following result:

Proposition 1: At the optimum, it is sufficient to consider solutions such that each input of production is bought by each plant in one point of supply.

In general a similar result does not hold for the points of demand. It is well known that the same plant can supply several markets. However, it is possible to characterise the reciprocal relation between the markets and the plants.

Theorem 2: *If the production cost of each plant is a concave function, at the optimum it is sufficient to consider solutions such that each point of demand is supplied by at most one plant.*

Proof: Let m_k denote any point of demand. To simplify the notation let us assume that the firm is composed of two plants. If $\overline{C}_1$ and $\overline{C}_2$ denote their average costs of production then, without loss of generality, we may assume that

$$\overline{C}_1(0) + T_{1k}(0) \geqq \overline{C}_2(0) + T_{2k}(0) \; .$$

If

$$\pi_k(0) \leqq \overline{C}_2(0) + T_{2k}(0) \; ,$$

then none of the plants supplies the market m_k. If

$$\overline{C}_2(0) + T_{2k}(0) < \pi_k(0) \leqq \overline{C}_1(0) + T_{1k}(0) \; ,$$

then only the second plant supplies m_k. Finally, let us consider the case where

$$\pi_k(0) > \overline{C}_1(0) + T_{1k}(0) \; ,$$

and let us assume that both plants supply m_k, that is, $q^*_{1k} > 0$ and $q^*_{2k} > 0$. We then have

$$\begin{aligned} P(q^*_{1k}, q^*_{2k}, ...) &= (q^*_{1k} + q^*_{2k})\pi(q^*_{1k} + q^*_{2k}) + ... - q^*_{1k}[\overline{C}_1(q^*_{1k}) + T_{1k}(q^*_{1k})] \\ &\qquad - q^*_{2k}[\overline{C}_2(q^*_2) + T_{2k}(q^*_{2k})] - ... \\ &\leqq (q^*_{1k} + q^*_{2k})\pi(q^*_{1k} + q^*_{2k}) + ... \\ &\qquad - (q^*_{1k} + q^*_{2k}) \inf [\overline{C}_1(q^*_1 + q^*_{2k}) + T_{1k}(q^*_{1k} + q^*_{2k}) \, , \; \overline{C}_2(q^*_2 + q^*_{1k}) \\ &\qquad\qquad + T_{2k}(q^*_{1k} + q^*_{2k})] \; ... \; , \end{aligned}$$

since the average costs are nonincreasing. Then the firm can secure a profit at least equal to the one given by the initial solution by supplying m_k from a single plant.

From this result, we deduce that, in the PMSPLP, each demand point may be assigned to one plant at most. This implies that the optimal market areas define a partition of the set of supplied sites, hence, for each class of the partition, the problem reduces to the case of single location. Furthermore, for each plant and for all markets supplied by that plant, the equality between the marginal revenues and the marginal cost, all evaluated in firm-prices, is preserved. Finally, at the optimum, it is therefore sufficient to consider solutions such that the number of plants is less than or equal to the number of demand sites.

Theorem 2 is no longer valid if the production cost functions are strictly convex and if the average transportation costs are nondecreasing with respect to the quantities. Indeed, as the average costs of production are increasing, the same market may be supplied by several plants[3]. However, in this case theorem 2 may be replaced by the following, less strong, result due to Lantner (1973), of which we give a new proof.

Theorem 3: *At the optimum, if a plant supplies k points of demand, it is sufficient to consider solutions such that no other plant supplies more than one of these markets.*

Proof: Given the optimal quantities produced by each plant, and the optimal quantities sold on each market, the problem reduces to the ex post minimisation of the transportation costs. Clearly, this problem is a transportation problem in the Hitchcock–Koopmans–Kantorovich sense. Thus there is an optimal solution without cycles in the exchanges graph.

[3] A more complete discussion of the convex cost functions case will be given in a paper in preparation.

Mathematical formulation of the PMSPLP

Let us specify the production cost function of each plant. We assume that, in each site $l_i \in L$, the firm disposes of a finite number, μ_i, of production techniques. The data on these techniques are given in two matrices $\mathbf{A}^{(i)}$ and $\mathbf{B}^{(i)}$, of dimensions $\mu_i \times (v-1)$, which are, respectively, the input–output matrix and the setup inputs matrix(4). Thus if the technique h is used, the production cost of the plant located in l_i is

$$\begin{aligned} C_{hi}(q_i) &= q_i A_h^{(i)}\overline{p} + B_h^{(i)}\overline{p} \\ &= a_{hi}q_i + b_{hi} \;, \end{aligned} \tag{1}$$

where

$A_h^{(i)}$ and $B_h^{(i)}$ denote the hth column of the matrices $\mathbf{A}^{(i)}$ and $\mathbf{B}^{(i)}$ respectively,

$q_i = \left(\sum_{k=1}^{s_i} q_{ik}\right)$ is the output of that plant, and

$\overline{p}$ is the $(v-1)$ vector of the minimum firm-prices of the inputs (see proposition 1).

Finally, the cost of production is given by

$$C_i(q_i) = \min_{h=1,\ldots,\mu_i} C_{hi}(q_i) \; .$$

Clearly C_i is a concave and piecewise linear function of q_i; the equations of the lines are given by equation (1), and, hence, theorem 2 can be applied. Moreover, as the cost function is concave it is easily seen that a single technique will be used in each plant. Formally, the PMSPLP, with μ_i techniques in each site of L, is equivalent to a problem with $\sum_{i=1}^{n} \mu_i$ sites and a single technique in each of these sites.

It is also possible to integrate economies of scale into the model in a more general way. This can be done by post multiplying the input–output matrix by a diagonal matrix of concave functions, $[g_{hi}(q_i)]$.

If the technique h is used, the production cost of the plant located in l_i is given by

$$\begin{aligned} C_{hi}(q_i) &= g_{hi}(q_i)A_h^{(i)}\overline{p} \;, \\ &= a_{hi}g_{hi}(q_i) \;, \end{aligned}$$

where $A_h^{(i)}$ is the hth column of $\mathbf{A}^{(i)}$.

Finally, the cost of production is given by

$$C_i(q_i) = \min_{h=1,\ldots,\mu_i} C_{hi}(q_i) \; .$$

As C_{hi} is a concave function of q_i, so is C_i. It is possible to linearise by pieces, the functions C_i, which reduces the problem to that of the previous case. Therefore the conclusions obtained above can still be applied.

The PMSPLP may be formulated mathematically. Consider the case where $p_{ik}(0) \geqq a_{hi}$. If the market m_k is supplied by the plant located in l_i and if the technique h is used, it is possible to compute the optimal production q_{hik}^* and the corresponding firm-price $p_{ik}(q_{hik}^*)$. Let

$$e_{hik} = q_{hik}^*[p_{ik}(q_{hik}^*) - a_{hi}] \geqq 0 \;,$$

where

$$e_{hik} = 0 \;, \qquad \text{for} \quad p_{ik}(0) < a_{hi} \; .$$

(4) Note that the set of production techniques may change from location to location because of the existence of external economies.

The following Boolean program corresponds to the PMSPLP:

$$\text{maximise} \sum_{k=1}^{s_l} \sum_{i=1}^{n} \sum_{h=1}^{\mu_i} e_{hik} \lambda_{hik} - \sum_{i=1}^{n} \sum_{h=1}^{\mu_i} b_{hi} \theta_{hi} \,, \tag{2}$$

in our case e_{hik} will depend
on whole lambda vector
here also the case

$$\text{subject to} \sum_{i=1}^{n} \sum_{h=1}^{\mu_i} \lambda_{hik} \leqq 1 \,, \qquad k = 1, ..., s_l \,, \tag{3}$$

$$0 \leqq \lambda_{hik} \leqq \theta_{hi} \leqq 1 \,, \qquad h = 1, ..., \mu_i \,; \quad i = 1, ..., n \,; \quad k = 1, ..., s_l \,, \tag{4}$$

$$\lambda_{hik} \in \{0, 1\}, \, \theta_{hi} \in \{0, 1\} \,, \qquad h = 1, ..., \mu_i \,; \quad i = 1, ..., n \,; \quad k = 1, ..., s_l \,. \tag{5}$$

The variable θ_{hi} is equal to unity if a plant that uses technique h is set up in l_i, and is equal to zero otherwise. The variable λ_{hik} is equal to unity if the market m_k is supplied from a plant that uses technique h and is set up in l_i, and is equal to zero otherwise.

The constraints (3) and (5) express the fact that, as a result of theorem 2, every market m_k is supplied by one plant at most, whereas constraints (4) show that, if the technique j is used, any market may be supplied by a plant located in l_i only if such a plant exists.

Reduction to the SPLP

The PMSPLP, as expressed by the program (2) to (5), can be formally reduced to the simple plant-location problem. The PMSPLP differs from the SPLP because the objective function is to be minimized with nonnegative coefficients and because the constraints (3) are equalities instead of inequalities. To obtain equalities let us introduce a fictive plant. This plant is located in a fictive site l_{n+1} and uses one technique ($\mu_{n+1} = 1$); it has no setup cost and may supply any market with no profit ($e_{l,\, n+1,\, k} = 0$). The s_l Boolean variables $\lambda_{l,\, n+1,\, k}$ associated with this plant play the role of slack variables in constraint (3). Furthermore let

$$\bar{e} = \max_{h,\, i,\, k} e_{hik} \,, \quad \text{and} \quad e'_{hik} = \bar{e} - e_{hik} \,, \qquad h = 1, ..., \mu_i \,; \quad i = 1, ..., n \,; \quad k = 1, ..., s_l \,.$$

It is easy to see that the optimal solution of the PMSPLP is given by the optimal solution of the following program:

$$\text{minimise} \sum_{k=1}^{s_l} \sum_{i=1}^{n+1} \sum_{h=1}^{\mu_i} e'_{hik} \lambda_{hik} + \sum_{i=1}^{n+1} \sum_{h=1}^{\mu_i} b_{hi} \theta_{hi} \,, \tag{6}$$

$$\text{subject to} \sum_{i=1}^{n+1} \sum_{h=1}^{\mu_i} \lambda_{hik} = 1 \,, \qquad k = 1, ..., s_l \,, \tag{7}$$

$$0 \leqq \lambda_{hik} \leqq \theta_{hi} \leqq 1 \,, \qquad h = 1, ..., \mu_i \,; \quad i = 1, ..., n+1 \,; \quad k = 1, ..., s_l \,, \tag{8}$$

$$\lambda_{hik} \in \{0, 1\}, \quad \theta_{hi} \in \{0, 1\} \,, \qquad h = 1, ..., \mu_i \,; \quad i = 1, ..., n+1 \,; \quad k = 1, ..., s_l \,. \tag{9}$$

The value of the optimal solution of constraints (2) to (5) is obtained by subtracting from $s_l \bar{e}$ the value of the optimal solution of constraints (6) to (9).

The problem (6) to (9) is an SPLP. When the number, $\sum_{i=1}^{n+1} \mu_i$, of variables θ_{hi} is not too large (that is, about one hundred or less), the problem may be solved by a branch-and-bound technique. If $\sum_{i=1}^{n+1} \mu_i$ is larger, efficient heuristic methods quickly provide a near-optimal solution. [See Hansen and Kaufman (1972) for a comparison of heuristic methods for the SPLP.] Both in the exact and the heuristic algorithms, the resolution time is only a linear function of the number of markets, s_l.

Example

Let us consider a network with ten vertices and fourteen routes, as illustrated in figure 1. The lengths of the routes are written alongside the lines to which they correspond. We assume that there are four markets, located at l_1, l_4, l_7, and l_{10} respectively, with the following inverse demand functions:

$$\pi_1(q) = 60-q \ , \quad \pi_4(q) = 64-2q \ , \quad \pi_7(q) = 70-2q \ , \quad \text{and} \quad \pi_{10}(q) = 80-q \ .$$

The particular sites at which plants may be set up are the ten vertices of the network. In order to simplify the problem, we assume that the matrices $\mathbf{A}^{(i)}$ and $\mathbf{B}^{(i)}$ are the same everywhere and that a single production technique is available: $\boldsymbol{A}_1 = (5, 2, 1)$ and $\boldsymbol{B}_1 = (4, 2, 2)$, with three inputs of production. Input 1 is available in l_1 at a price of 1 and in l_8 at a price of 2, input 2 is only available in l_6 at a price of 4, and input 3 is available everywhere at a price of 5. Finally, the transportation rate of each input is equal to 2, whereas that of the output is 4.

Table 1 provides all the necessary information for solving the problem.

It is easy to see that two plants are set up in l_1 and l_8 respectively. The optimal quantities are $q^*_{111} = 13$, $q^*_{114} = 5\cdot5$, $q^*_{117} = 3$, and $q^*_{1\,8\,10} = 15$.

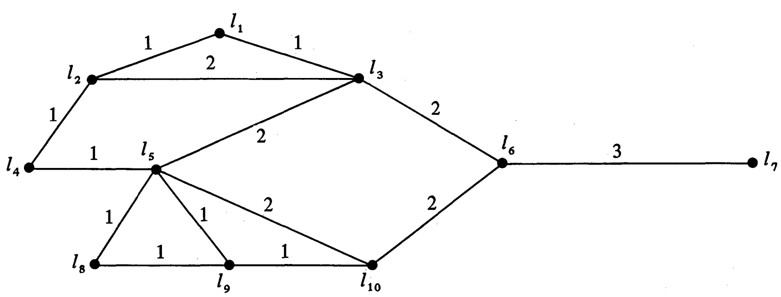

Figure 1. The network.

Table 1. Intermediate results for the example.

Possible locations	1	2	3	4	5	6	7	8	9	10
Minimum firm-price of input										
1	1	3	3	5	4	7	13	2	4	6
2	10	12	8	10	12	4	10	12	10	8
3	5	5	5	5	5	5	5	5	5	5
Marginal production cost	34	36	38	50	50	46	82	42	46	50
Setup cost	30	44	36	50	49	48	90	39	45	51
Optimal quantity sold in										
1	13	5	9	1	0	1	0	1	0	0
4	5·5	3·5	3·5	3·5	2·5	0	0	3·5	2·5	0·5
7	3	0	3	0	0	3	0	0	0	0
10	13	9	13	9	11	13	0	15	15	15
e_{hik}										
1	169	25	81	1	0	1	0	1	0	0
4	60·5	24·5	24·5	24·5	12·5	0	0	24·5	12·5	0·5
7	18	0	18	0	0	18	0	0	0	0
10	169	81	169	81	121	169	0	225	225	225

Conclusions

In summary, the main results of the paper are: first, we have obtained an important characterisation of the problem of multilocation, without exchanges between plants, by showing that it is sufficient to consider a finite set of sites in order to get an optimal pattern of locations; second, we have shown that, in the PMSPLP, each market is assigned to one plant at most. Thus it appears that PMSPLP is the most straightforward extension of single location. Because of these two results, we have been able to express formally the PMSPLP as an SPLP; this allows the use of well-known methods of resolution. We may conclude that the PMSPLP is now completely solvable provided that the required information on the demand functions is available.

By slightly modifying the structure of the model it would be possible to study the multilocation of public facilities. To do this we shall assume that the transportation costs of the output are paid by the customers, and in the objective function replace the profit of each plant by the sum of the profit and the customer's surplus. In that case, we obtain the *welfare-maximising simple-facility location problem*, the analysis of which is similar to that of the PMSPLP. The results derived here confirm and extend those of Wagner and Falkson (1975).

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References

Balinski M L, 1964 "On finding integer solutions to linear programs" *IBM Data Processing Division* 225-248

Beckenstein A R, 1975 "Scale economies in the multiplant firm: theory and empirical evidence" *The Bell Journal of Economics* **6** 644-657

Cooper L, 1967 "Solutions of generalized locational equilibrium problems" *Journal of Regional Science* **7** 1-18

Cooper L, 1972 "The transportation-location problem" *Operations Research* **20** 94-108

Dearing D M, Francis R L, 1974 "A minimax location problem on a network" *Transportation Science* **8** 333-343

Efroymson M A, Ray T L, 1966 "A branch-and-bound algorithm for plant location" *Operations Research* **14** 361-368

Geoffrion A M, Graves G W, 1974 "Multicommodity distribution system design by Benders decomposition" *Management Science* **20** 822-844

Hanan M, Kurtzberg J M, 1972 "A review of the placement and quadratic assignment problem" *SIAM Review* **14** 324-342

Hansen P, 1972 "Two algorithms for the simple plant location problem using additive penalties" paper presented at the European Meeting of the Econometric Society, Budapest, Hungary

Hansen P, Kaufman L, 1972 "Comparaison d'algorithmes pour le problème de la localisation des entrepôts" in *Operational Research in Industrial Systems* Ed. J Brennan (English Universities Press, London) pp 281-294

Kaufman L, Van den Eede M, Hansen P, 1977 *Operational Research Quarterly* forthcoming

Khumawala B M, 1972 "An efficient branch-and-bound algorithm for the warehouse location problem" *Management Science* **18** B-718-B-733

Koopmans T C, Beckmann M, 1957 "Assignment problems and the location of economic activities" *Econometrica* **25** 53-76

Kuenne R E, Soland R M, 1972 "Exact and approximate solutions to the multi-source Weber problem" *Mathematical Programming* **3** 193-209

Lantner R, 1973 "Un modèle de localisation de la firme au sein de la branche d'activité" thèse complémentaire, University of Dijon, Dijon, France

Lawler E L, 1975 "The quadratic assignment problem: a brief review" in *Combinatorial Programming: Methods and Applications* Ed. B Roy (Reidel, Dordrecht-Holland) pp 351-360

Levy J, 1967 "An extended theorem for location on a network" *Operational Research Quarterly* **18** 433-443

Manne A S, 1964 "Plant location under economies of scale-decentralization and computation" *Management Science* **11** 213-235

Manne A S, Ed., 1967 *Investments For Capacity Expansion: Size, Location and Time Phasing* (MIT Press, Cambridge, Mass)
Pierce J F, Crowston W, 1971 "Tree-search algorithms for quadratic assignment problems" *Naval Research Logistics Quarterly* **18** 1-36
Scott A J, 1971 "Operational analysis of nodal hierarchies in network systems" *Operational Research Quarterly* 22 25-37
Spielberg K, 1969a "Algorithms for the simple plant location problem with some side-conditions" *Operations Research* **17** 85-111
Spielberg K, 1969b "Plant location with generalized search origin" *Management Science* **16** 165-178
Spielberg K, 1970 "On solving plant location problems" in *Applications of Mathematical Programming* Ed. E M L Beale (English Universities Press, London) pp 216-234
Teitz M, 1968 "Locational strategies for competitive systems" *Journal of Regional Science* 8 135-148
Thisse J, 1975 *Contribution à la Théorie Microéconomique Spatiale* PhD thesis, University of Liège, Liège, Belgium
Thisse J, Perreur J, 1977 "Relations between the point of maximum profit and the point of minimum total transportation cost: a restatement" *Journal of Regional Science* forthcoming
Wagner J L, Falkson L M, 1975 "The optimal nodal location of public facilities with price-sensitive demand" *Geographical Analysis* 7 69-83
Wendell R E, Hurter A P, 1973 "Optimal locations on a network" *Transportation Science* 7 18-33
Zimmerman R, 1967 "A branch and bound algorithm for depot location" *Metra* **6** 661-674