QUIZ

Ynupc

I give you this file to make you train the TEXwriting skill of mathematical expressions. Try to write some of the following expressions (at least two expressions) in TEXon your Overleaf or ShareLaTeX. It also includes mathematical quizzes regarding combinatorics. Have fun with the quizzes if you want to try to solve them. The following expressions in the quizzes are full of memories for me since I came up with them when I was a senior in senior high school. It was one of the results of my act of escapism against the competition to get in university. I created many mathematical expressions right before the entrance exams and zero university allowed me to get in sadly.

QUIZ 1

Consider two functions, S(x, n) and I(x, n) defined as follows.

$$\begin{aligned} S(x,n) &= \left.\frac{\sum_{0\le k_{n-1}}^{k_n}\sum_{0\le k_{n-2}}^{k_{n-1}}\cdots\sum_{0\le k_0}^{k_1}k_0^2}{\sum_{0\le k_{n-1}}^{k_n}\sum_{0\le k_{n-2}}^{k_{n-1}}\cdots\sum_{0\le k_0}^{k_1}k_0}\right|_{k_n=x} \\ I(x,n) &= \left.\frac{\int_{0\le k_{n-1}}^{k_n}\int_{0\le k_{n-2}}^{k_{n-1}}\cdots\int_{0\le k_0}^{k_1}k_0^2 dk_0\cdots dk_{n-2}dk_{n-1}}{\int_{0\le k_{n-1}}^{k_n}\int_{0\le k_{n-2}}^{k_{n-1}}\cdots\int_{0\le k_0}^{k_1}k_0 dk_0\cdots dk_{n-2}dk_{n-1}}\right|_{k_n=x} \end{aligned}$$

Simplify the following expression.

$$\left\{\prod_{1\le n}^{N}(S(x,n)-I(x,n))\right\}\cdot(1+2+3+\ldots+(N+1))$$

HINT 1-1

$$\left.\sum_{0\le k_{n-1}}^{k_n}\sum_{0\le k_{n-2}}^{k_{n-1}}\cdots\sum_{0\le k_0}^{k_1} 1\right|_{k_n=x} = \left(\!\!\left(\begin{array}{c} x \\ n \end{array}\right)\!\!\right) \tag{1}$$

$\left(\!\!\left(\begin{array}{c} x \\ n \end{array}\right)\!\!\right)$ is the maximal number of nonzero terms in a homogeneous polynomial of degree n in x variables. It is also called the homogeneous product. Also, it is equal to the following binomial coefficient.

$$\left(\!\!\left(\begin{array}{c} x \\ n \end{array}\right)\!\!\right) = \left(\begin{array}{c} x+n-1 \\ n \end{array}\right)$$

In Japan, the homogeneous product and the binomial coefficient are expressed as follows.

$$\left(\!\!\left(\begin{array}{c} x \\ n \end{array}\right)\!\!\right) = {}_xH_n = {}_{x+n-1}C_n$$

Probe the expression (1) with mathematical induction's two steps, basis and inductive step.

(BASIS) Show that the expression holds for $n = 0$.

$$\begin{aligned}(LHS) &= 1 \\ (RHS) &= \left.\left(\!\!\left(\begin{array}{c} x \\ n \end{array}\right)\!\!\right)\right|_{n=0} = \left(\!\!\left(\begin{array}{c} x \\ 0 \end{array}\right)\!\!\right) = \left(\begin{array}{c} x+0-1 \\ 0 \end{array}\right) = 1 \\ Therefore, (LHS) &= (RHS)\end{aligned}$$

The two sides are equal, so the expression (1) is true for $n = 0$. Thus it has been shown that the expression (1) holds for $n = 0$.

(INDUCTIVE STEP) Show that if the expression holds for $n = m$, then also it holds for $n = m + 1$.
Assume the expression (1) holds for $n = m$. It must then be shown that the expression (1) holds for $n = m + 1$, that is:

$$\left.\sum_{0\le k_m}^{k_{m+1}}\sum_{0\le k_{m-1}}^{k_m}\cdots\sum_{0\le k_0}^{k_1} 1\right|_{k_{m+1}=x} = \left(\!\!\left(\begin{array}{c} x \\ m+1 \end{array}\right)\!\!\right)$$

Using the induction hypothesis that the expression (1) holds, the left-hand side can be rewritten to:

$$
\begin{aligned}
(LHS) = \left. \sum_{0 \le k_m}^{k_{m+1}} \sum_{0 \le k_{m-1}}^{k_m} \cdots \sum_{0 \le k_0}^{k_1} 1 \right|_{k_{m+1}=x} &= \left. \sum_{0 \le k_m}^{k_{m+1}} \left(\sum_{0 \le k_{m-1}}^{k_m} \sum_{0 \le k_{n-2}}^{k_{m-1}} \cdots \sum_{0 \le k_0}^{k_1} 1 \right) \right|_{k_{m+1}=x} \\
&= \left. \sum_{0 \le k_m}^{k_{m+1}} \left(\binom{k_m}{m} \right) \right|_{k_{m+1}=x}
\end{aligned}
$$

The summation of the maximal numbers of nonzero terms in homogeneous polynomials of degree m in k_m variables where k_m is an integer under $0 \le k_m \le k_{m+1}$ is obviously the maximal number of nonzero terms in a homogeneous polynomial of degree $m+1$ in k_{m+1} variables because the homogeneous polynomial of degree $m+1$ in k_{m+1} can be expressed as an inner product of a degree $m+1$ vector of degree 1 monomials and a degree $m+1$ vector which consists of $m+1$ homogeneous polynomials of degree m. If you need the proof, it is easily proved with expansion and I write the proof in Appendix A. Here, however, I would like to show an example to understand it intuitively. When the $m = 3$ and $k_{m+1} = 3$, then the polynomial of degree 3 $(= m)$ in 3 $(= k_{m+1})$ variables can be expressed $(a + b + c)^3$. It has 10 nonzero terms. $(a + b + c)^3$ can be expressed as inner product of a degree 3 vector (a, b, c) and a degree 3 vector which each component is a polynomial. The latter vector has three polynomials as its components. The first polynomial consists of same monomials as $(a+b+c)^2$, the second polynomial consists of same monomials as $(b+c)^2$ and the last polynomial consists of same monomials as $(c)^2$. Expand the $(a + b + c)^3$ and make the components of the vectors indeed. First, when the $m = 2$ and $k_m = 0$, then the polynomial can be expressed $()^2$ and it is ignored due to 0 nonzero term. Second, when the $m = 2$ and $k_m = 1$, then the polynomial can be expressed $(c)^2$ and it has 1 nonzero term. Third, then the $m = 2$ and $k_m = 2$, then the polynomial can be expressed $(b+c)^2$ and it has 3 nonzero terms. Lastly, when the $m = 2$ and $k_m = 3 (= k_{m+1})$, then the polynomial can be expressed $(a + b + c)^2$ and it has 6 nonzero terms. The summation of them: 1, 3, 6 is equal to 10. It is equal to the number of nonzero terms of $(a+b+c)^3$ because $(a+b+c)^3$ is expanded to 3 polynomials of polynomial A which every monomial includes a variable, polynomial B which every monomial does not include a variable but includes b variable and polynomial C which every monomial includes neither a variable nor b variable but only c variable. The polynomial A is $a(a^2 + b^2 + c^2 + 3ab + 3ac + 3bc)$. The polynomial B is $b(b^2 + 3bc + 3c^2)$. The polynomial C is $c(c^2)$. That is $(a+b+c)^3 = (a, b, c)\cdot(a^2+b^2+c^2+3ab+3ac+3bc, b^2+3bc+3c^2, c^2)$. Each component of the latter vector consists of same homogeneous monomials as $(a + b + c)^2$, $(b+c)^2$ and $(c)^2$ respectively. Conversely, it holds that the number of the nonzero terms of the direct sum of the monomials of $(a+b+c)^2$, $(b+c)^2$ and $(c)^2$ is equal to the number of the nonzero terms of $(a + b + c)^3$. Thus, the following transformation is intuitively clear and I guess that such ideas are important to make sense of combinatorics.

$$
\begin{aligned}
(LHS) &= \left. \sum_{0 \leq k_m}^{k_{m+1}} \left(\!\!\binom{k_m}{m}\!\!\right) \right|_{k_{m+1}=x} \\
&= \left. \left\{ \left(\!\!\binom{0}{m}\!\!\right) + \left(\!\!\binom{1}{m}\!\!\right) + \cdots + \left(\!\!\binom{k_{m+1}-1}{m}\!\!\right) + \left(\!\!\binom{k_{m+1}}{m}\!\!\right) \right\} \right|_{k_{m+1}=x} \\
&= \left. \left(\!\!\binom{k_{m+1}}{m+1}\!\!\right) \right|_{k_{m+1}=x} \\
&= \left(\!\!\binom{x}{m+1}\!\!\right) = (RHS)
\end{aligned}
$$

Thereby showing that the expression (1) holds for $n = m + 1$.
Since both the basis and the inductive step have been performed, by mathematical induction, the expression (1) holds for all natural numbers n. Q.E.D.

In addition, the expression (1) obviously means the following expression at the same time.

$$
\left. \sum_{0 \leq k_{n-1}}^{k_n} \sum_{0 \leq k_{n-2}}^{k_{n-1}} \cdots \sum_{0 \leq k_0}^{k_1} \left(\!\!\binom{k_0}{s}\!\!\right) \right|_{k_n=x} = \left(\!\!\binom{x}{s+n}\!\!\right)
$$

HINT 1-2

$$\sum_{0\le k_{n-1}}^{k_n}\sum_{0\le k_{n-2}}^{k_{n-1}}\cdots\sum_{0\le k_0}^{k_1} k_0^p\cdot\left(\!\!\binom{k_0}{s}\!\!\right)\Bigg|_{k_n=x} = f(p,s,x,n)\cdot\left(\!\!\binom{x}{s+n}\!\!\right) \tag{2}$$

$$\begin{aligned} f(0,s,x,n) &= 1 \qquad (3)\\ f(p+1,s,x,n) &= \frac{(s+1)(x+s+n)}{s+n+1}\cdot f(p,s+1,x,n) - s\cdot f(p,s,x,n) \qquad (4)\end{aligned}$$

When $s = 0$,

$$\sum_{0\le k_{n-1}}^{k_n}\sum_{0\le k_{n-2}}^{k_{n-1}}\cdots\sum_{0\le k_0}^{k_1} k_0^p\Bigg|_{k_n=x} = f(p,0,x,n)\cdot\left(\!\!\binom{x}{n}\!\!\right) \tag{5}$$

$$f(p+1,0,x,n) = \frac{x+n}{n+1}\cdot f(p,1,x,n) \tag{6}$$

ANSWER 1

The answer is 1.

$$
\begin{aligned}
S(x,n) &= \left.\frac{\sum_{0\le k_{n-1}}^{k_n}\sum_{0\le k_{n-2}}^{k_{n-1}}\cdots\sum_{0\le k_0}^{k_1} k_0^2}{\sum_{0\le k_{n-1}}^{k_n}\sum_{0\le k_{n-2}}^{k_{n-1}}\cdots\sum_{0\le k_0}^{k_1} k_0}\right|_{k_n=x} \\
&= \frac{\frac{2x+n}{n+2}\cdot\left(\!\!\binom{x}{n+1}\!\!\right)}{\left(\!\!\binom{x}{n+1}\!\!\right)} = \frac{2x+n}{n+2}
\end{aligned}
$$

$$
\begin{aligned}
I(x,n) &= \left.\frac{\int_{0\le k_{n-1}}^{k_n}\int_{0\le k_{n-2}}^{k_{n-1}}\cdots\int_{0\le k_0}^{k_1} k_0^2 dk_0\cdots dk_{n-2}dk_{n-1}}{\int_{0\le k_{n-1}}^{k_n}\int_{0\le k_{n-2}}^{k_{n-1}}\cdots\int_{0\le k_0}^{k_1} k_0 dk_0\cdots dk_{n-2}dk_{n-1}}\right|_{k_n=x} \\
&= \frac{\frac{1}{(x+1)!}\cdot x^{n+1}}{\frac{2}{(x+2)!}\cdot x^{n+2}} = \frac{2x}{n+2}
\end{aligned}
$$

$$
\begin{aligned}
S(x,n)-I(x,n) &= \frac{2x+n}{n+2}-\frac{2x}{n+2} \\
&= \frac{n}{n+2}
\end{aligned}
$$

$$
\begin{aligned}
\prod_{1\le n}^{N}(S(x,n)-I(x,n)) &= \prod_{1\le x}^{N}\frac{N}{N+2} \\
&= \frac{2}{(N+1)(N+2)}
\end{aligned}
$$

$$
\begin{aligned}
\left\{\prod_{1\le n}^{N}(S(x,n)-I(x,n))\right\}\cdot(1+2+3+\ldots+(N+1)) &= \frac{2}{(N+1)(N+2)}\cdot\frac{(N+1)(N+2)}{2} \\
&= 1
\end{aligned}
$$

QUIZ 2

Prove the following equation.

$$\sum_{0 \le k_{n-1}}^{k_n} \sum_{0 \le k_{n-2}}^{k_{n-1}} \cdots \sum_{0 \le k_0}^{k_1} k_0^{-1} \Bigg|_{k_n = x} = \left(\!\!\left(\begin{array}{c} x+1 \\ n-1 \end{array} \right)\!\!\right) \sum_{1 \le k}^{x} \frac{1}{k+n-1}$$

HINT 2-1

$$\sum_{0\le k_{n-1}}^{k_n}\sum_{0\le k_{n-2}}^{k_{n-1}}\cdots\sum_{0\le k_0}^{k_1} k_0^{-1}\left(\!\!\left(\begin{array}{c} x \\ s \end{array}\right)\!\!\right)\Bigg|_{k_n=x} = \frac{\left(\!\!\left(\begin{array}{c} x+1 \\ s+n-1 \end{array}\right)\!\!\right) - \left(\!\!\left(\begin{array}{c} x+1 \\ n-1 \end{array}\right)\!\!\right)}{s} \tag{7}$$

This expression (7) also satisfies the expressions: (2)-(4).

ANSWER 2

$$
\begin{aligned}
(LHS) &= \left.\sum_{0\le k_{n-1}}^{k_n}\sum_{0\le k_{n-2}}^{k_{n-1}}\cdots\sum_{0\le k_0}^{k_1} k_0^{-1}\right|_{k_n=x} \\
&= \left.\lim_{s\to 0}\sum_{0\le k_{n-1}}^{k_n}\sum_{0\le k_{n-2}}^{k_{n-1}}\cdots\sum_{0\le k_0}^{k_1} k_0^{-1}\left(\!\!\binom{x}{s}\!\!\right)\right|_{k_n=x} \\
&= \lim_{s\to 0}\frac{\left(\!\!\binom{x+1}{s+n-1}\!\!\right)-\left(\!\!\binom{x+1}{n-1}\!\!\right)}{s} \\
&= \frac{\partial}{\partial n}\left(\!\!\binom{x+1}{n-1}\!\!\right) \\
&= \frac{\partial}{\partial n}\frac{(n+x-1)!}{x!(n-1)!} \\
&= \frac{1}{x!}\frac{\partial}{\partial n}\frac{\Gamma(n+x)}{\Gamma(n)} \\
&= \frac{1}{x!\Gamma^2(n)}\left\{\Gamma'(n+x)\Gamma(n)-\Gamma(n+x)\Gamma'(n)\right\} \\
&= \frac{\Gamma(n+x)}{x!\Gamma(n)}\left\{\frac{\Gamma'(n+x)}{\Gamma(n+x)}-\frac{\Gamma'(n)}{\Gamma(n)}\right\} \\
&= \frac{(n+x-1)!}{x!(n-1)!}\left\{\psi(n+x)-\psi(n)\right\} \\
&= \left(\!\!\binom{x+1}{n-1}\!\!\right)\sum_{0\le k}^{\infty}\left(\frac{1}{k+n}-\frac{1}{k+n+x}\right) \\
&= \left(\!\!\binom{x+1}{n-1}\!\!\right)\sum_{0\le k}^{x-1}\frac{1}{k+n} \\
&= \left(\!\!\binom{x+1}{n-1}\!\!\right)\sum_{1\le k}^{x}\frac{1}{k+n-1} = (RHS)
\end{aligned}
$$

QUIZ 3

Simplify the following expression.

$$\sum_{0\le k_{n-1}}^{k_n} \sum_{0\le k_{n-2}}^{k_{n-1}} \cdots \sum_{0\le k_0}^{k_1} k_0^{-p} \Bigg|_{k_n=x}$$

Appendix A

Prove the following expression.

$$\sum_{0\le k_m}^{k_{m+1}} \left(\!\!\left(\begin{array}{c} k_m \\ m \end{array} \right)\!\!\right) = \left(\!\!\left(\begin{array}{c} k_{m+1} \\ m+1 \end{array} \right)\!\!\right)$$

$$\begin{aligned}
(LHS) &= \sum_{0\le k_m}^{k_{m+1}} \left(\!\!\left(\begin{array}{c} k_m \\ m \end{array} \right)\!\!\right) \\
&= \sum_{0\le k_m}^{k_{m+1}} \frac{(k_m+m-1)!}{(k_m-1)!m!} \\
&= \frac{1}{(k_{m+1}-1)!} \left\{ \begin{array}{r}
1\cdot 2\cdot 3\cdot\dots\cdot(k_{m+1}-2)\cdot(k_{m+1}-1) \\
+(1+m)\cdot 2\cdot 3\cdot\dots\cdot(k_{m+1}-2)\cdot(k_{m+1}-1) \\
+(1+m)\cdot(2+m)\cdot 3\cdot\dots\cdot(k_{m+1}-2)\cdot(k_{m+1}-1) \\
+(1+m)\cdot(2+m)\cdot(3+m)\cdot\dots\cdot(k_{m+1}-2)\cdot(k_{m+1}-1) \\
\vdots \\
+(1+m)\cdot(2+m)\cdot\dots\cdot(k_{m+1}-2+m)\cdot(k_{m+1}-1) \\
+(1+m)\cdot(2+m)\cdot\dots\cdot(k_{m+1}-2+m)\cdot(k_{m+1}-1+m)
\end{array} \right\} \\
&= \frac{1}{(k_{m+1}-1)!} \left\{ \begin{array}{r}
(2+m)\cdot 2\cdot 3\cdot\dots\cdot(k_{m+1}-2)\cdot(k_{m+1}-1) \\
+(1+m)\cdot(2+m)\cdot 3\cdot\dots\cdot(k_{m+1}-2)\cdot(k_{m+1}-1) \\
+(1+m)\cdot(2+m)\cdot(3+m)\cdot\dots\cdot(k_{m+1}-2)\cdot(k_{m+1}-1) \\
\vdots \\
+(1+m)\cdot(2+m)\cdot\dots\cdot(k_{m+1}-2+m)\cdot(k_{m+1}-1) \\
+(1+m)\cdot(2+m)\cdot\dots\cdot(k_{m+1}-2+m)\cdot(k_{m+1}-1+m)
\end{array} \right\} \\
&= \frac{1}{(k_{m+1}-1)!} \left\{ \begin{array}{r}
(3+m)\cdot(2+m)\cdot 3\cdot\dots\cdot(k_{m+1}-2)\cdot(k_{m+1}-1) \\
+(1+m)\cdot(2+m)\cdot(3+m)\cdot\dots\cdot(k_{m+1}-2)\cdot(k_{m+1}-1) \\
\vdots \\
+(1+m)\cdot(2+m)\cdot\dots\cdot(k_{m+1}-2+m)\cdot(k_{m+1}-1) \\
+(1+m)\cdot(2+m)\cdot\dots\cdot(k_{m+1}-2+m)\cdot(k_{m+1}-1+m)
\end{array} \right\} \\
&= \frac{1}{(k_{m+1}-1)!} \left\{ \begin{array}{r}
(4+m)\cdot(2+m)\cdot(3+m)\cdot\dots\cdot(k_{m+1}-2)\cdot(k_{m+1}-1) \\
\vdots \\
+(1+m)\cdot(2+m)\cdot\dots\cdot(k_{m+1}-2+m)\cdot(k_{m+1}-1) \\
+(1+m)\cdot(2+m)\cdot\dots\cdot(k_{m+1}-2+m)\cdot(k_{m+1}-1+m)
\end{array} \right\}
\end{aligned}$$

$$
\begin{aligned}
&= \cdots \\
&= \frac{1}{(k_{m+1}-1)!} \left\{ \begin{array}{c} (k_{m+1}-1+m)\cdot(2+m)\cdot\cdots\cdot(k_{m+1}-2+m)\cdot(k_{m+1}-1) \\ +(1+m)\cdot(2+m)\cdot\cdots\cdot(k_{m+1}-2+m)\cdot(k_{m+1}-1+m) \end{array} \right\} \\
&= \frac{1}{(k_{m+1}-1)!} \left\{ (k_{m+1}+m)\cdot(2+m)\cdot\cdots\cdot(k_{m+1}-2+m)\cdot(k_{m+1}-1+m) \right\} \\
&= \frac{(2+m)\cdot(3+m)\cdot\cdots\cdot(k_{m+1}-1+m)\cdot(k_{m+1}+m)}{(k_{m+1}-1)!} \\
&= \frac{(k_{m+1}+m)!}{(k_{m+1}-1)!(m+1)!} \\
&= \left(\binom{k_{m+1}}{m+1} \right) = (RHS)
\end{aligned}
$$

Q.E.D.

Appendix B

$$f(p,s,x,n)=\sum_{0\leq k}^{p}\left[(-1)^{p-k}\left\{\prod_{1\leq j}^{k}\frac{(s+j)(x+s+n+j-1)}{s+n+j}\right\}\sum_{C}\prod_{0\leq b}^{k}(s+b)^{a_b}\right] \tag{8}$$

C: $\sum_{0\leq b}^{k} a_b = p-k, 0 \leq a_b \in \mathbb{Z}$

$$f(p,0,x,n)=\sum_{0\leq k}^{p}\left[(-1)^{p-k}\left\{\prod_{1\leq j}^{k}\frac{j(x+n+j-1)}{n+j}\right\}\sum_{D}\prod_{1\leq b}^{k}b^{a_b}\right] \tag{9}$$

D: $\sum_{1\leq b}^{k} a_b = p-k, 0 \leq a_b \in \mathbb{Z}$