MA-INF 2201 Computer Vision Assignment-1

Tridivraj Bhattacharyya(3035538)

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3. Prove that convolutions are in the continuous case associative.

Solution- Let f and g be two functions, the convolution of which is,

$$(f * g)(t) = \int_0^t f(s)g(t-s)ds$$

Now, expanding this to three functions,

$((f * g) * h)(t) = \int_0^t (f * g)(s)h(t-s)ds = \int_{s=0}^t (\int_{u=0}^t f(s)g(s-u)du)h(t-s)ds$
$= \int_{s=0}^t \int_{u=0}^t f(s)g(s-u)h(t-s)duds$
$= \int_{u=0}^t \int_{s=0}^{t-u} f(u)g(s)h(t-s-u)dsdu$
$= \int_{u=0}^t f(u)(\int_{s=0}^{t-u} g(s)h(t-u-s)ds)du$
$= \int_{u=0}^t f(u)(g * h)(t-u)du = (f * (g * h))(t)$

Therefore, convolutions are associative.

6. Prove that convolution two times with a Gaussian kernel with standard deviation σ is the same as convolution once with a Gaussian kernel with standard deviation $\sqrt{2}\sigma$

Solution-
By definition, a gaussian functuon,

$G_\sigma(x) = \frac{1}{\sqrt{2\pi\sigma^2}} * e^{\frac{-x^2}{2*\sigma^2}}$

Now, $(G_\sigma * G_\sigma)(x) = \int_{-\infty}^{\infty} G(x)G(x-u)du$
$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} * e^{-\frac{x^2}{2*\sigma^2}} * \frac{1}{\sqrt{2\pi\sigma^2}} * e^{\frac{-(x-u)^2}{2*\sigma^2}} du$

$= \frac{1}{2\pi\sigma^2} \int_{-\infty}^{\infty} e^{-\frac{u^2+(x-u)^2}{2*\sigma^2}} du$
$= \frac{1}{2\pi\sigma^2} \int_{-\infty}^{\infty} e^{-\frac{2u^2+x^2-2ux}{2*\sigma^2}} du$
$= \frac{1}{2\pi\sigma^2} * e^{-\frac{x^2}{2*\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{u^2+ux}{\sigma^2}} du$
$= \frac{1}{2\pi\sigma^2} * e^{-\frac{x^2}{2*\sigma^2}} \int_{-\infty}^{\infty} e^{(-\frac{u^2}{\sigma^2}+\frac{ux}{\sigma^2}+0)} du - (1)$

Now, we know, $\int_{-\infty}^{\infty} e^{-Au^2+Bu+C} = \sqrt{\frac{\pi}{A}} * e^{(\frac{B^2}{4*A}+C)}$ –(2)

Therefore, combining (1) and (2) we get,
$(G_\sigma * G_\sigma)(x) = \frac{1}{2\pi\sigma^2} * e^{-\frac{x^2}{2*\sigma^2}} \sqrt{\pi * \sigma^2} * e^{-\frac{x^2*\sigma^2}{2*\sigma^4}}$
$= \frac{1}{\sqrt{2\pi(\sqrt{2}\sigma)^2}} * e^{(-\frac{x^2}{2*\sigma^2})} * e^{-\frac{x^2*}{(\sqrt{2}*\sigma)^2}} = G_{\sqrt{2}\sigma}(x)$