MA-INF 2201 Computer Vision Assignment-7

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Using the equations for multivariate Gaussian Distribution

$$\int Norm_x[a, A] Norm_x[b, B] = \int \frac{1}{\sqrt{(2\pi)^k |A||B|}} \exp[-\frac{1}{2}(x-a)^T A^{-1}(x-a)] \exp[-\frac{1}{2}(x-b)^T B^{-1}(x-b)] dx$$

Expanding and taking the common terms

$$= \frac{1}{(2\pi)^k \sqrt{|A||B|}} \int \exp[-\frac{1}{2}(x^T(A^{-1}+B^{-1})x - x^T(A^{-1}a + B^{-1}b)$$
$$-(A^{-1}a+B^{-1}b)x + (a^T A^{-1}a + b^T B^{-1}b)))]dx$$

Plugging in, $(A^{-1}a+B^{-1}b)^T(A^{-1}+B^{-1})^{-1}(A^{-1}a+B^{-1}b)$

$$= \frac{1}{(2\pi)^k \sqrt{|A||B|}} \exp[-\frac{1}{2}((a^T A^{-1}a + b^T B^{-1}b) - (A^{-1}a+B^{-1}b)^T(A^{-1}+B^{-1})^{-1}(A^{-1}a+B^{-1}b))]$$
$$\int \exp[-\frac{1}{2}(x^T(A^{-1}+B^{-1})x - x^T(A^{-1}a+B^{-1}b) - (A^{-1}a+B^{-1}b)x)$$
$$+ (A^{-1}a+B^{-1}b)^T(A^{-1}+B^{-1})^{-1}(A^{-1}a+B^{-1}b)]dx$$

Plugging in $(A^{-1}+B^{-1})^{-1}(A^{-1}+B^{-1})$
with the 2nd and 3rd terms inside the second exponential above and solving,

$$= \frac{\sqrt{(2\pi)^k |A^{-1}+B^{-1}|}}{(2\pi)^k \sqrt{|A||B| \sqrt{(2\pi)^k |A^{-1}+B^{-1}|}}}$$
$$\exp[-\frac{1}{2}((a^T A^{-1}a + b^T B^{-1}b) - (A^{-1}a+B^{-1}b)^T(A^{-1}+B^{-1})^{-1}(A^{-1}a+B^{-1}b))]$$
$$\int \exp[-\frac{1}{2}(x-(A^{-1}+B^{-1})^{-1}(A^{-1}a+B^{-1}b)^T(A^{-1}+B^{-1})(x-(A^{-1}+B^{-1})^{-1}(A^{-1}a+B^{-1}b))]dx$$
$$= C \int Norm_x[(A^{-1}+B^{-1})^{-1}(A^{-1}a+B^{-1}b), (A^{-1}+B^{-1})^{-1}]dx$$

Now solving the constant term C,

$$C = \frac{\sqrt{(2\pi)^k|A^{-1}+B^{-1}|}}{(2\pi)^k\sqrt{|A||B|}} \exp[-\frac{1}{2}((a^TA^{-1}a+b^TB^{-1}b)-(A^{-1}a+B^{-1}b)^T(A^{-1}+B^{-1})^{-1}(A^{-1}a+B^{-1}b))]$$

We know, $A^{-1}+B^{-}-1^{-1} = \dfrac{AB}{A+B}$ Therefore,

$$C = \frac{1}{\sqrt{(2\pi)^k|A+B|}} \exp[-\frac{1}{2}((a^T(A+B)^{-1}(A+B)A^{-1}a+b^T(A+B)^{-1}(A+B)B^{-1}b)-$$
$$a^T(A^{-1}+B^{-1})^{-1}A^{-1}a-2a^TA^{-1}(A^{-1}+B^{-1})^{-1}B^{-1}b-b^TB^{-1}(A^{-1}+B^{-1})^{-1}B^{-1}b)]$$
$$= \frac{1}{\sqrt{(2\pi)^k|A+B|}} \exp[-\frac{1}{2}(a^T(A+B)^{-1}a+a^T(A+B)^{-1}BA^{-1}a$$
$$+b^T(A+B)^{-1}AB^{-1}b+b^T(A+B)^{-1}B)b-a^T(A^{-1}+B^{-1})^{-1}A^{-1}a$$
$$-2a^T(A+B)^{-1}b-b^TB^{-1}(A^{-1}+B^{-1})^{-1}B^{-1}b)]$$

Cancelling out the common terms above,

$$C = \frac{1}{\sqrt{(2\pi)^k|A+B|}} \exp[-\frac{1}{2}(a^T(A+B)^{-1}a+b^T(A+B)^{-1}B)b-2a^T(A+B)^{-1}b)]$$
$$= \frac{1}{\sqrt{(2\pi)^k|A+B|}} \exp[-\frac{1}{2}((a-b)^T(A+B)^{-1}(a-b))]$$
$$= Norm_a[b, A+B]$$

From the above,

$$\int Norm_x[a,A]Norm_x[b,B]dx = Norm_a[b,A+B]\int Norm_x[\Sigma_*(A^{-1}a+B^{-1}b),\Sigma_*]dx$$

where, $\Sigma_* = (A^{-1}+B^{-1})^{-1}$ (Proved)