Submodularity -

1. $$\boxed{\begin{aligned}&P(\beta,\gamma)+P(\alpha,\delta)-P(\beta,\delta)-P(\alpha,\gamma)\geq 0\\ &\beta>\alpha,\ \delta>\gamma\end{aligned}} \quad \text{— (1)}$$

1.1 To show - $P(w_m, w_n) = c(w_m - w_n)^2$ is submodular

$$\boxed{P(w_m, w_n) = c(w_m - w_n)^2} \quad \text{— (2)}$$

Modifying (1) using (2)

$$\Rightarrow c(\beta-\gamma)^2 + c(\alpha-\delta)^2 - c(\beta-\delta)^2 - c(\alpha-\gamma)^2$$

$$\Rightarrow c\{(\beta-\gamma)^2 + (\alpha-\delta)^2 - (\beta-\delta)^2 - (\alpha-\gamma)^2\}$$

$$\Rightarrow c(\cancel{\beta^2}+\cancel{\gamma^2}-2\beta\gamma+\cancel{\alpha^2}+\cancel{\delta^2}-2\alpha\delta-\cancel{\beta^2}-\cancel{\delta^2}+2\beta\delta-\cancel{\alpha^2}-\cancel{\gamma^2}+2\alpha\gamma)$$

$$\Rightarrow 2c(-\beta\gamma-\alpha\delta+\beta\delta+\alpha\gamma)$$

$$\Rightarrow 2c(\beta(\delta-\gamma) - \alpha(-\gamma+\delta))$$

$$\Rightarrow \cancel{2c((\beta-\alpha)}$$

$$\Rightarrow 2c((\beta-\alpha)(\delta-\gamma)) \quad \text{— (3)}$$

as $\beta > \alpha$, $\delta > \gamma$

~~⇒2c(β~~ eqn (3) will always ≥ 0

provided c is positive

$$\Rightarrow P(\beta,\gamma) + P(\alpha,\delta) - P(\beta,\delta) - P(\alpha,\gamma) \geq 0$$

Hence proved.

1.2 To show $P(w_m, w_n) = c(1-\delta(w_m - w_n))$ is not submodular

Submodularity : $P(\beta,\gamma)+P(\alpha,\delta)-P(\beta,\delta)-P(\alpha,\gamma)\geq 0$ (1)

$\forall\ \alpha,\beta,\gamma,\delta \qquad \beta>\alpha\ ,\ \delta>\gamma$

let us define $\delta(p)=1\ ,\ p=0$

$\delta(p)=0\ ,$ otherwise

let $\alpha=\delta$

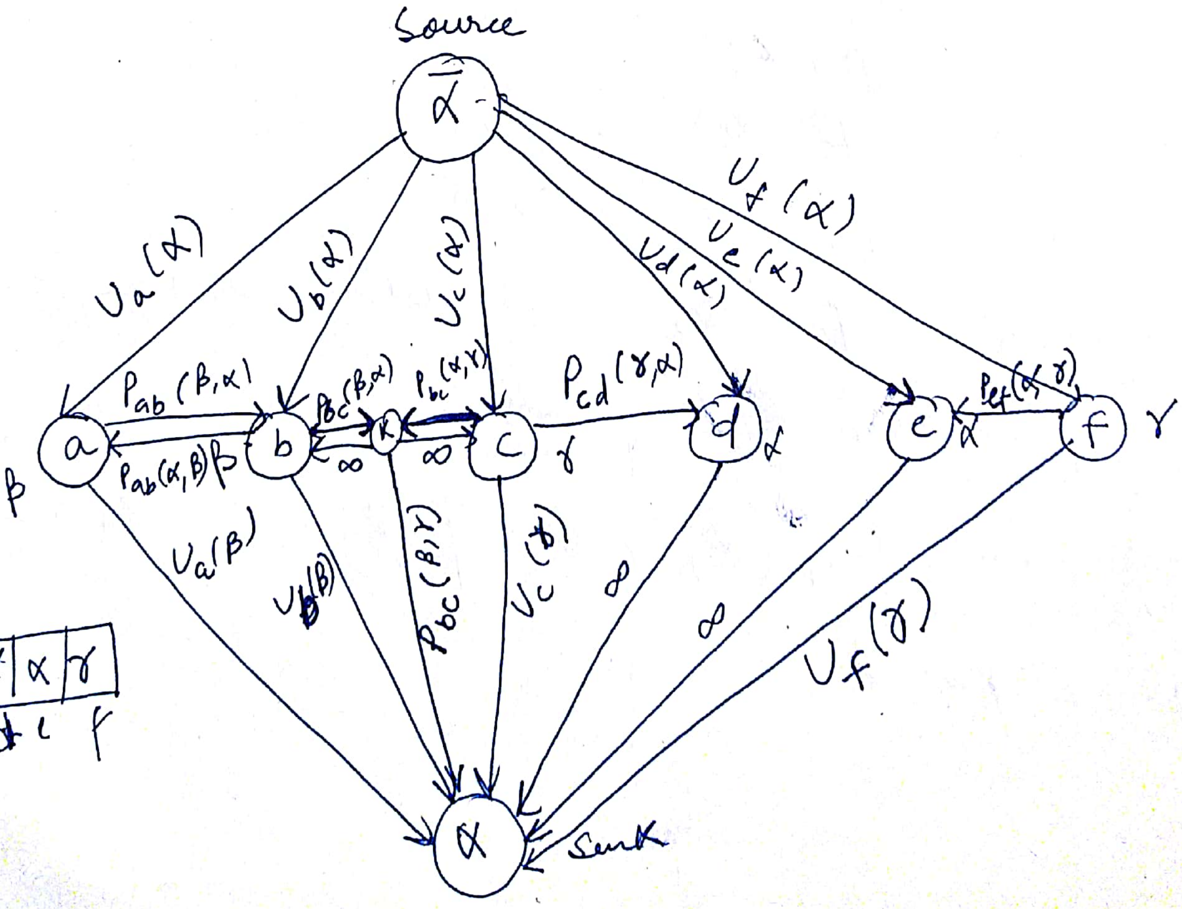
So eq^n (1) becomes

$$\phi = c(1)+c(0)-c(1)-c(1)$$
$$= \cancel{c}-\cancel{c}-c$$
$$= -c < 0$$

violating the submodularity constraint

$\Rightarrow$ Potts model is not submodular.

2.



a	b	c	d	e	f
β	β	γ	α	α	γ